Blow-Ups and 3264 Conics

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Introduction : Problem : Strategy

In the 19$^{\text{th}}$ century, Jakob Steiner asked–and answered incorrectly–how many conics are there in the plane tangent to five fixed conics? His answer was a naive application of Bezout's theorem. The set of conics in $\mathbb{P}^2$ is parametrized by $\mathbb{P}^5$ and the set of conics tangent to a fixed conic defines a degree 6 hypersurface H_Q in $\mathbb{P}^5$. Intersecting the five hypersurfaces corresponding to any five conics then gives the locus of conics tangent to all 5. If the five hypersurfaces meet transversely, then by Bezout's theorem, this locus consists of 6^5 points, which was the answer given by Jakob Steiner. However, no arrangement of conics gives a pair-wise transverse collection of hypersurfaces. In fact, the locus is always infinite. This is because we have nonreduced conics whose equation is simply the square of the equation of a line. If one of these double lines meets a conic at a point, then because it is cut out by the square of the line, the intersection multiplicity is automatically at least two, so they are counted as tangent, no matter if the underlying reduced line is tangent to the conic. Since every line meets every conic in $\mathbb{P}^2$, any double line is tangent to all five conics, so we have a copy of $\mathbb{P}^2$ in the intersection on $\mathbb{P}^5$.

It is easy to get a handle on what this excess intersection looks like. The embedding takes the line corresponding to the equation $ax+by+cz=0$, ie. the point $[a\colon b\colon c]\in\mathbb{P}^2$ to the double conic defined by $(ax+by+cz)^2=a^2x^2+abxy+acxz+b^2y^2+bcyz+c^2z^2=0$, ie. the point $\left[a^2:ab:ac:b^2:bc:c^2\right]\in\mathbb{P}^5$. This is the Veronese embedding, and it is a smooth surface in $\mathbb{P}^5$, call it V. We don't care so much about these double lines–Jakob Steiner probably did not have them in mind when he asked for conics–so we would like to remove V from the intersection of our five hypersurfaces. Intuitively, the moduli space of conics we are after is $\mathbb{P}^5-V$, and we have chosen the wrong compactification. We may try to obtain another compactification of this moduli space by forming the blow-up $\mathrm{Bl}_V\mathbb{P}^5$. The projection $\pi\colon \mathrm{Bl}_V\mathbb{P}^5\to\mathbb{P}^5$ is an isomorphism away from the exceptional divisor E, so $\mathrm{Bl}_V\mathbb{P}^5-E$ is another copy of the moduli space of reduced conics. Then we want to count the intersection of the proper transforms of the five hyperplanes. It turns out that the intersection of the proper transforms is indeed transverse, and we obtain the answer to Jakob Steiner's puzzle.

There is a less philosophical argument that leads us to the blow-up. This is based on the notion of the dual conic. Notice that the space of lines in $\mathbb{P}^2$ is a dual copy of $\mathbb{P}^2$. If we have a conic C in $\mathbb{P}^2$, we can ask for the locus of lines tangent to this conic. If the conic is integral, then this locus will be an integral conic $\hat{C}$ in the dual $\mathbb{P}^2$. If the conic is a union of two distinct lines, its dual will be the union of the duals of these lines. If the conic is a nonreduced line, then its dual will be all of $\mathbb{P}^2$. Thus, away from the Veronese, we have a birational map $\mathbb{P}^5\to\mathbb{P}^5$ by taking conics to their duals. Blowing up the Veronese is a universal way to pass to a case where this duality is an honest isomorphism.

Let us attack the general question of calculating the Chow ring of a blow-up. Let V be a smooth, regularly embedded subscheme of a smooth projective scheme X with normal bundle $N=N_{V/X}$. Let $\tilde{X}=\mathrm{Bl}_VX$, $\pi\colon\tilde{X}\to X$ the projection, $\tilde{V}=\pi^{-1}(V)$ the exceptional divisor. π realises $\tilde{V}$ as a projective bundle over V, which we see is in fact the projectivization of the normal bundle N. This allows us to compute the Chow ring of $\tilde{V}$ from the Chern classes of N using the following result.

Theorem 1. *Let E be a vector bundle of rank $r+1$ over a scheme X, $\pi\colon\mathbb{P}E\to X$ its projectivization, and $\zeta=c_1\mathcal{O}_{\mathbb{P}E}(1)$. The pullback map $\pi^*\colon A^*(X)\to A^*(\mathbb{P}E)$ is injective, and, identifying $A^*(X)$ with its image, we have*

$$A^*(\mathbb{P}E)=A^*(X)[\zeta]/\big(\zeta^{r+1}+c_1(E)\zeta^r+\cdots+c_{r+1}(E)\big).$$

Then, knowing further the Chow rings of V and X, we can deduce the additive structure of $A^*(\tilde{X})$ with the folowing geometrically intuitive exact sequence.

Theorem 2. *For each k, we have an exact sequence*

$$0 \to A_k(V) \xrightarrow{\alpha} A_k(\tilde{V}) \oplus A_k(X) \xrightarrow{\beta} A_k(\tilde{X}) \to 0,$$

where $\alpha(v) = (c_1(M) \cap \pi^* v, -i_* v)$ and $\beta(\tilde{v}, x) = j_* \tilde{v} + \pi^* x$, where i and j are the closed immersions $V \to X$ and $\tilde{V} \to \tilde{X}$, respectively, and M is the excess normal bundle $\pi^* N / N_{\tilde{V}/\tilde{X}} = \pi^* N / \mathcal{O}_{\mathbb{P}N}(-1)$.

Finally, we can deduce the multiplicative structure of $A^*(\tilde{X})$ by decomposing the pullback of any cycle in X in terms of its proper transform. Recall that the proper transforms of the degree six hypersurfaces considered in Steiner's problem are precisely what we're after.

Theorem 3. *Let Y be a k-dimensional subvariety of X, $\tilde{Y}$ its proper transform, then*

$$\pi^*[Y] = \left[\tilde{Y}\right] + j_* \{c(M) \cap \pi^* s(Y \cap V, Y)\}_k,$$

where $s(Y \cap V, Y)$ is the total Segre class of the normal cone of $Y \cap V$ in Y, and $\{-\}_k$ denotes projection onto A^k.

Proof of Theorem 1

Injectively of π^* follows from the fact that $\pi_*(\zeta^r \pi^* \alpha) = \alpha$, so π^* has a left inverse. In fact, this shows the map $\varphi_j : A^*(\mathbb{P}E) \to A^*(X)$ sending $\alpha \mapsto \pi_*(\zeta^j \alpha)$ sends $\zeta^{r-j} A^*(X) \subset A^*(\mathbb{P}E)$ isomorphically to $A^*(X)$. The

$$\begin{array}{ccc} X'' & \overset{i''}{\to} & Y'' \\ q\downarrow & & \downarrow p \\ X' & \overset{i'}{\to} & Y' \\ g\downarrow & & \downarrow f \\ X & \overset{i}{\to} & Y \end{array}$$

inverses thus assemble into an inclusion $\bigoplus_{j=0}^r \zeta^j A^*(X) \to A^*(\mathbb{P}E)$. In fact, we will show this inclusion is surjective.

The strategy will be to deform an arbitrary cycle in $A^*(\mathbb{P}E)$ to one whose fibers over X are linear subspaces. Notice that for any line bundle L on X, $\mathcal{O}_{\mathbb{P}(E \otimes L)}(1) = \mathcal{O}_{\mathbb{P}E}(1) \otimes \pi^* L^*$, so by tensoring with a negative enough power of an ample line, we can reduce to the case when E^* is generated by global sections. Let $Y \subset \mathbb{P}E$ be an irreducible dimension k subvariety. There are a couple cases to consider.

∘ Suppose first that the general fiber of Y over its image, $Z \subset X$, has full dimension r, in other words, is $\mathbb{P}^r$. This implies $[Y] = \pi^*[Z]$, so there is nothing to prove in this case.

∘ Now assume that the general fiber of Y over Z has dimension $r-1$. Let $p \in Y$ be in the fiber above some $q \in Z$. E^* is globally generated, so there are transverse sections σ_0, ..., σ_r linearly independent in some open neighborhood U of q. Perhaps choosing U a bit smaller, we can choose a trivialization Φ: $E|_U \to U \times \mathbb{C}^{r+1}$. Now let $\omega \neq 0$ be a point in the fiber of E over q not contained in either the fiber of Y over q (pulled back to $E_q - 0$) or the zero locus of σ_0. We can extend ω to a section of E over U.

We can use this data to define a one parameter family of automorphisms of $E|_U$, ϕ_t, such that–perhaps after shrinking U yet again so that ω is never in the zero locus of σ_0 or in the pullback of Y–the zero locus of σ_0 has eigenvalue 1 and the span of ω has eigenvalue t. This induces a 1-parameter family of automorphisms of $\mathbb{P}E$, and hence a 1 parameter family of cycles $\mathcal{Y} \subset \mathbb{P}E_U \times \mathbb{C}^\times$ generated by Y. Taking the closure in $\mathbb{P}E \times \mathbb{C}$, we get a family of cycles Y_t such that the fiber over $t = 1$ is our original cycle and the fiber at $t = 0$ is supported on the union of two loci : the intersection of $\pi^{-1}(Z)$ with the zero locus of σ_0 and the preimage of $Z - U \cap Z$. Note that these each have dimension k. The first of these is $\zeta \pi^*[Z]$, so we've expressed the class $[Y]$ as a sum of things in $\zeta A^*(X)$ and $A^*(X)$ (identified as subgroups of $A^*(\mathbb{P}E)$). This finishes this case.

∘ Now we can proceed by induction. Suppose we have already proved that when Y has general fibers of dimension $r-s$, $[Y]\in\bigoplus_{j=0}^{s}\zeta^j A^*(X)$. We can take $s\geqslant 1$ by the first two cases. Suppose Y has general fiber dimension $r-s-1$. We perform the same trick as in the second case. Namely, we choose $s+1$ sections $\omega_0,\dots,\omega_s$ which span the complement of the zero locus $\sigma_0=\dots=\sigma_s=0$ and are disjoint from Y. We obtain likewise a one parameter family of cycles Y_t such that $Y_1=Y$ and Y_0 is the sum of something supported on $\pi^{-1}(Z-Z\cap U)$ and a multiple of $\zeta^{s+1}\pi^*[Z]$. Thus, $[Y]\in\bigoplus_{j=0}^{s+1}\zeta^j A^*(X)$. This finishes off the lemma.

In particular, $\zeta^{r+1}\in\bigoplus_{j=0}^{r}\zeta^r A^*(X)$, so ζ satisfies a monic degree $r+1$ polynomial $f\in A^*(X)[z]$. The lemma states $A^*(\mathbb{P}E)=A^*(X)[z]/(f)$. To prove the theorem, we need only show that the polynomial is the one claimed. Consider the tautological line bundle S over $\mathbb{P}E$. This includes into π^*E. Let Q denote the quotient. S is in fact the dual of $\mathcal{O}_{\mathbb{P}E}(1)$, whose Chern class we called ζ. Thus, $c(Q)=c(\pi^*E)c(S)^{-1}=c(\pi^*E)(1-\zeta)^{-1}=c(\pi^*E)\big(1+\zeta+\zeta^2+\cdots\big)$. We also have $c(\pi^*E)=\pi^*c(E)$, so degree $r+1$ part of the above equation reads

$$\zeta^{r+1}+\zeta^r\pi^*c_1(E)+\cdots+\pi^*c_{r+1}(E)=c_{r+1}(Q)=0,$$

since Q has rank r. Identifying $A^*(X)$ with its image in $A^*(\mathbb{P}E)$ under π^*, we see that f is as claimed, proving theorem 1.

Proof of Theorem 2

We begin with a useful lemma :

Lemma 4. *(Excess Intersection Formula) Suppose we have a cartesian diagram*

$$\begin{array}{ccc} X'' & \xrightarrow{i''} & Y'' \\ q\downarrow & & \downarrow p \\ X' & \xrightarrow{i'} & Y' \\ g\downarrow & & \downarrow f \\ X & \xrightarrow{i} & Y \end{array}$$

with i and i' regular closed imbeddings of codimension d and d' and normal bundles N and N'. There is a canonical embedding of N' into g^*N. The quotient $M=g^*N/N'$ is a vector bundle of rank $e=d-d'$ on X' we call the excess normal bundle. For any $\alpha\in A_k(Y'')$,

$$i^!(\alpha)=c_e(q^*M)\cap i'^!(\alpha)$$

in $A_{k-d}(X'')$.

The proof is a simple calculation. We consider the projectivizations $E'=\mathbb{P}(q^*N'\oplus 1)$ and $E=\mathbb{P}(q^*g^*N\oplus 1)$, each bundles on X'' carrying universal quotient bundles Q' and Q. It is not so hard to see that there is a canonical embedding of E' in E so that the tautological line bundle of E restricts to that of E', since N' canonically is a subbundle of g^*N. We thus obtain an exact sequence

$$0\to Q'\to Q|_{E'}\to r^*(q^*M)\to 0,$$

where r is the bundle projection $E'\to X''$. We may assume α is represented by a subvariety $V\subset Y'$. Then set $P=\mathbb{P}(C_{V\cap X'}V\oplus 1)$. By definition, we have $i^!(\alpha)=r_*(c_d(Q)\cap[P])$. The exact sequence above tell us this equals $r_*(c_{d'}(Q')\cap c_e(r^*q^*M)\cap[P])$, which is $c_e(q^*M)\cap r^*\big(c_{d'}(Q')\cap[P]\big)=c_e(q^*M)\cap i'^!(\alpha)$, and we're done.

With the fiber diagram of the blow-up:

$$\begin{array}{ccc} \tilde{V} & \xrightarrow{j} & \tilde{X} \\ \downarrow\pi & & \downarrow\pi \\ V & \xrightarrow{i} & X \end{array}$$

we obtain the formula

$$\pi^* i_*(x) = j_* \pi^!(x) = j_*(c_{d-1}(M) \cap \pi^* x),$$

where d is the codimension of V. *This proves $\beta\alpha = 0$ in the sequence of theorem 2.*

Now let $x \in A_k(X)$. I will show $\pi_*\pi^* x = x$. We may assume $x = [W]$. If $W \subset V$, then $x = i_* v$, $v = [W] \in A_k(V)$, and so $\pi_*\pi^* x = \pi_* j_*(c_{d-1}(M) \cap \pi^* v) = i_*\pi_*(c_{d-1}(M) \cap \pi^* v) = i_* v = x$. If $W \not\subset V$, then let $\tilde{W}$ be the blow-up of W along $W \cap V$, ie. the proper transform $\tilde{W} = \overline{\pi^{-1}(W - V)}$. Then we can write $\pi^*[W] = \left[\tilde{W}\right] + j_*(\tilde{x})$, where $\tilde{x}$ is a class in dimension k, supported on the exceptional divisor $\tilde{W} \cap \tilde{V}$. It follows $\pi_*\pi^* x = \pi_*\left[\tilde{W}\right] + \pi_* j_*(\tilde{x}) = x + i_*\pi_*(\tilde{x})$, but $\pi_*(\tilde{x})$ is supported on $W \cap V$, which has dimension $< k$, so $\pi_*(\tilde{x}) = 0$, so again $\pi_*\pi^* x = x$.

Suppose $(\tilde{v}, x) \in \ker \beta$, ie. $j_*\tilde{v} + \pi^* x = 0$. By what we have just proven, $x = \pi_*\pi^* x = -\pi_* j_* \tilde{v} = -i_*\pi_*\tilde{v}$. Set $\tilde{v}' = \tilde{v} - c_{d-1}(M) \cap \pi^*\pi_*\tilde{v}$. Then $\pi_*\tilde{v}' = \pi_*\tilde{v} - \pi_*(c_{d-1}(M) \cap \pi^*\pi_*\tilde{v}) = 0$ and $j_*\tilde{v}' = j_*\tilde{v} - \pi^* i_*\pi_*\tilde{v} = j_*\tilde{v} + \pi^* x = 0$. $\tilde{V}$ is a projective bundle over V of rank $d-1$, so by theorem 1, $\tilde{v}' = \sum_{j=0}^{d-1} \zeta^{j+1} \cap \pi^* v_j$, for some $v_0, \ldots, v_{d-1} \in A_*(V)$. We have $0 = \pi_*\tilde{v}' = v_{d-1}$, and $0 = j^* j_* \tilde{v}' = -\sum_{j=0}^{d-2} \zeta^{j+1} \cap \pi^* v_j$, so by theorem 1, $v_j = 0$ for $0 \leqslant j < d-1$. Together, we obtain $\tilde{v}' = 0$. Thus, $\tilde{v} = c_{d-1}(M) \cap \pi^*\pi_*\tilde{v}$, so $(\tilde{v}, x) = \alpha(\pi_*\tilde{v}) \in \operatorname{Im}\alpha$. *This proves the sequence of theorem 2 is exact in the middle.*

Now let $\tilde{x} \in A_k\left(\tilde{X}\right)$. Notice that $\tilde{x} - \pi^*\pi_*\,\tilde{x}$ restricts to 0 on $\tilde{X} - \tilde{V}$, where π is an isomorphism, and hence is of the form $j_*\tilde{v}$ for some $\tilde{v} \in A_k\left(\tilde{V}\right)$. *This proves the surjectivity of β.*

Finally, consider the map γ: $A_k(\tilde{V}) \oplus A_k(X) \to A_k(V)$ sending $(\tilde{v}, x) \mapsto \pi_*\tilde{v}$. One sees $\gamma(\alpha(v)) = \pi_*(c_{d-1}(M) \cap \pi^* v) = v$, *so α is injective*. This finishes off theorem 2.

Proof of Theorem 3

First, if $Y \subset V$, then $\tilde{Y}$ is empty, and $s(Y \cap V, Y) = s(Y, Y) = [Y]$, since the normal cone of Y in Y is $\underline{\operatorname{Spec}}\, \mathcal{O}_Y = Y$. The claimed formula then reduces to the first corollary we derived from the excess intersection formula, so we can assume $Y \not\subset V$.

Suppose $\pi_*\tilde{x} = j^*\tilde{x} = 0$. Since β of theorem 2 is surjective, we can write $\tilde{x} = j_*\tilde{v} + \pi^*\pi_*\tilde{x} = j_*\tilde{v}$. We can define $\tilde{v}' = \tilde{v} - (c_{d-1}(M) \cap \pi^*\pi_*\tilde{v})$ as before. By the first corollary to the excess intersection formula, $j_*\tilde{v}' = j_*\tilde{v} - j_*(c_{d-1}(M) \cap \pi^*\pi_*\tilde{v}) = j_*\tilde{v} - \pi^* i_*\pi_*\tilde{v} = j_*\tilde{v} = \tilde{x}$. However, $\pi_*\tilde{v}' = \pi_*\tilde{v} - \pi_*(c_{d-1}(M) \cap \pi^*\pi_*\tilde{v}) = 0$, so, as above, $\tilde{v}' = 0$, and hence $\tilde{x} = j_*\tilde{v}' = 0$. Thus, it suffices to prove the formula after applying π_* and j^*.

∘ As we showed above, $\pi_*\pi^*[Y] = [Y]$, so we need to show $\pi_*[\tilde{Y}] + \pi_* j_*\{c(M) \cap \pi^* s(Y \cap V, Y)\}_k = [Y]$. By the projection formula and the fact that $\pi_*[\tilde{Y}] = [Y]$, the left hand side becomes $[Y] + \pi_*\{c(M) \cap \pi^* s(Y \cap V, Y)\}_k$. $s(Y \cap V, Y)$ is a class on $Y \cap V$, which has dimension $< k$, so the second term is zero, so this case is done.

∘ Applying j^* to the first term, we obtain $j^*\pi^*[V] = \pi^* i^*[V] = \pi^*\{c(N) \cap s(Y \cap V, Y)\}_{k-d} = \{c(\pi^* N) \cap \pi^*\pi_*(\sum_{j \geqslant 0} \zeta^j \cap [\widetilde{V \cap Y}])\}_{k-1}$, where ζ, as above, is $c_1(\mathcal{O}_{\mathbb{P}N}(1))$. For the second term, $j^*[\tilde{Y}] = [\tilde{V}][\tilde{Y}] = [\widetilde{V \cap Y}]$.

- ∘ For the last, since $c(M)=c(\pi^*N)c(\mathcal{O}_{\mathbb{P}N}(-1))^{-1}$, M being the quotient $\pi^*N/\mathcal{O}_{\mathbb{P}N}(-1)$, we obtain

$$j^*j_*\{c(M)\cap\pi^*s(Y\cap V,Y)\}_k$$

$$=-\{\zeta c(M)\cap\pi^*s(Y\cap V,Y)\}_{k-1}$$

$$=-\{c(\pi^*N)\sum_{j\geqslant 1}\zeta^j\cap\pi^*\pi_*(\sum_{i\geqslant 0}\zeta^i\cap[\tilde{Y}\cap\tilde{V}])\}_{k-1}.$$

Combining these three results, we see it suffices to prove

$$\{c(\pi^*N)\sum_{j\geqslant 0}\zeta^j\,\pi^*\pi_*(\sum_{i\geqslant 0}\zeta^i\cap\tilde{v})\}_{k-1}=\tilde{v}$$

for all $\tilde{v}\in A_{k-1}(\tilde{V})$. By theorem 1, we can assume $\tilde{v}=\zeta^q\cap\pi^*v$, $q\leqslant d-1$, $v\in A_*(V)$. $\pi_*(\sum_{j\geqslant 0}\zeta^j\cap\tilde{v})=\pi_*(\sum_{j\geqslant 0}\zeta^j\cap\pi_*v)=s(N)\cap v$. The left hand side above is then

$$\{c(\pi^*N)\sum_{j\geqslant 0}\zeta^j\,\pi^*(s(N)\cap v)\}_{k-1}=\{\sum_{j\geqslant 0}\zeta^j c(\pi^*N)s(\pi^*N)\cap\pi^*v\}_{k-1}=\{\sum_{j\geqslant 0}\zeta^j\cap\pi^*v\}_{k-1}=\tilde{v},$$

as we wanted. This finishes off theorem 3.

Resolution of Steiner's Problem

Recall the set up we discussed in the introduction. By blowing up $X=\mathbb{P}^5$ along the Veronese surface V, we obtain a new compactification of the moduli space of conics : $\tilde{X}=\mathrm{Bl}_V X$. In $\mathbb{P}^5$, the set of conics tangent to some fixed conic forms a degree 6 hypersurface H_Q. In $\tilde{X}$, the corresponding subvariety is the proper transform $\tilde{H}_Q$, ie. H_Q blown-up along $V\cap H_Q$ (I admit culpritude in overloaded notation for the exceptional divisor $\tilde{V}$). Since we now have the technology to handle the intersection theory of the blow-up, we can try to calculate $[\tilde{H}_Q]^5$, which will give us the solution to Steiner's problem for 5 general conics provided the corresponding 5 general such $\tilde{H}_Q$'s intersect transversely (we will show that this is the case).

Luckily for us, we know the Chow ring of $\mathbb{P}^n$, so we know $A^*(X)=\mathbb{Z}[\omega]/\omega^6$, $A^*(V)=\mathbb{Z}[\eta]/\eta^3$, since $V\cong\mathbb{P}^2$, where ω and η are the classes of hyperplanes in X and V, respectively. Further, the Veronese map $\mathbb{P}^2\to\mathbb{P}^5$ defining V is the map corresponding to the complete linear series on the line bundle $\mathcal{O}_{\mathbb{P}^2}(2)$, so a hyperplane section of V in X pulls back to a degree 2 hypersurface on $\mathbb{P}^2$. Thus, the map $i^*\colon A^*(X)\to A^*(V)$ sends $\omega\mapsto 2\eta$.

We can calculate the Chern classes of the normal bundle $N_{V/X}=N$ by means of the exact sequence

$$0\to T_{\mathbb{P}^5}|_V\to T_V\to N_{V/X}\to 0.$$

We also know the tangent bundle of $\mathbb{P}^n$ quite well, and we know their Chern classes:

$$c(T_{\mathbb{P}^5})=(1+\omega)^6=1+6\omega+15\omega^2+\text{higher order terms}$$

$$c(T_V)=c(T_{\mathbb{P}^2})=(1+\eta)^3=1+3\eta+3\eta^2.$$

The Chern class of $T_{\mathbb{P}^5}|_V$ will be the image $i^*c(T_{\mathbb{P}^5})=1+12\omega+30\omega^2$. From the exact sequence, then

$$c(N)=c(T_{\mathbb{P}^5}|_V)/c(T_V)=1+9\eta+30\eta^2.$$

By theorem 1, then, $A^*(\tilde{V})=A^*(\mathbb{P}N)=\mathbb{Z}[\eta,\zeta]/(\zeta^3+9\zeta^2\eta+30\zeta\eta^2)$.

Our next step is to get a handle on the class $[\tilde{H}_Q] \in A^1(\tilde{X})$. By theorem 2, since $A_4(V)=0$, V being 2 dimensional, $A_4(\tilde{X}) = A_4(\tilde{V}) \oplus A_4(X) = \mathbb{Z} j_*[\tilde{V}] \oplus \mathbb{Z}\pi^*\omega$. We can thus write $[\tilde{H}_Q] = n j_*[\tilde{V}] + d\pi^*\omega$. As we discussed earlier, $\pi_* j_*[\tilde{V}] = 0$, $\pi_*\pi^*\omega = \omega$. Thus, $[H_Q] = \pi_*[\tilde{H}_Q] = d\omega$. In the introduction, I said H_Q has degree 6, so $d=6$. We can prove this by intersecting H_Q with a general line in $\mathbb{P}^5$, ie. a general pencil of conics. The intersection will be those conics in the pencil who are tangent to a fixed conic C. Any conic in this pencil defines a divisor of degree 4 on C. In other words, we can pull back the linear series in $\mathcal{O}_{\mathbb{P}^2}(2)$ defining the pencil along the degree 2 map $\mathbb{P}^1 \to \mathbb{P}^2$ defining C to get a linear series in $\mathcal{O}_{\mathbb{P}^1}(4)$. This defines a degree 4 map $\mathbb{P}^1 \to \mathbb{P}^1$, which by Riemann-Hurwitz must have 6 branch points. These branch points are precisely the conics in the pencil which are tangent to C (where the Kahler differential of the map $\mathbb{P}^1 \to \mathbb{P}^1$ vanishes). Thus, the intersection is at 6 points, so indeed $d=6$.

What about n? We can calculate this multiplicity using theorem 3, reducing it to an intersection problem on the exceptional divisor $\tilde{V}$ which we know quite well from theorem 1. We first need the Chern classes of the excess normal bundle $M = \pi^*N/\mathcal{O}_{\mathbb{P}N}(-1)$. $c(\mathcal{O}_{\mathbb{P}N}(1)) = 1/c(\mathcal{O}_{\mathbb{P}N}(-1)) = 1-\zeta$ and $c(\pi^*N) = \pi^*c(N) = \pi^*(1+9\eta+30\eta^2) = 1+9\eta+30\eta^2$, where, as in the proof of theorem 1, I identify classes in $A^*(V)$ with their image in $A^*(\tilde{V})$ under π^*, so I write η for $\pi^*\eta$. Thus, $c(M) = (1-\zeta)(1+9\eta+30\eta^2)$. Next, we need the Segre class of the normal bundle of $H_Q \cap V$ in H_Q. Every double line is tangent to every fixed conic, so $H_Q \cap V = V$. We can get the Segre classes from the Chern classes of N_{V/H_Q}. We have exact sequences

$$0 \to T_{H_Q}|_V \to T_V \to N_{V/H_Q} \to 0$$

$$0 \to T_{H_Q} \to T_{\mathbb{P}^5}|_{H_Q} \to N_{H_Q/\mathbb{P}^5} \to 0.$$

We know $N_{H_Q} = \mathcal{O}(6)|_{H_Q}$, so $c(N_{H_Q/\mathbb{P}^5}) = (1+6\omega)|_{H_Q} = 6\omega(1+6\omega)$ in $A^*(\mathbb{P}^5)$. Here we can work in the ambient space, $\mathbb{P}^5$, to avoid having to understand the inclusion $V \to H_Q$. $c(T_{\mathbb{P}^5}|_{H_Q}) = c(T_{\mathbb{P}^5})|_{H_Q}$, so $c(T_{H_Q}|_V) = i^*c(T_{H_Q}) = i^*(c(T_{\mathbb{P}^5})|_{H_Q}/(1+6\omega)|_{H_Q}) = i^*c(T_{\mathbb{P}^5})/(1+6\omega) = (1+2\eta)^6/(1+12\eta) = 1+60\eta^2$. From this, $s(H_Q \cap V, H_Q) = 1-60\eta^2$. Now we simply crank out the 4 dimensional component of the product $(1-\zeta)(1+9\eta+30\eta^2)(1+60\eta^2)$, which we find is 2, so $[\tilde{H}_Q] = -2j_*[\tilde{V}] + 6\pi^*\omega$. We can also calculate n by determining the order of vanishing of the defining equation for H_Q along V, which is also 2.

Now we calculate the expected intersection $[\tilde{H}_Q]^5 = (2j_*[\tilde{V}] + 6\pi^*\omega)^5$. Write for simplicity $\omega = \pi^*\omega$, $\epsilon = j_*[\tilde{V}]$. Five general (proper transforms of) hyperplanes in $\tilde{X}$ intersect at a point, so $\omega^5 = 1$. Four general hyperplanes intersect in a general line, which does not meet the Veronese, which has dimension 2, so $\omega^4\epsilon = 0$, and likewise $\omega^3 e^2 = 0$. Now note that $\epsilon|_{\tilde{V}} = j^*\epsilon = j^*j_*[\tilde{V}]$, the self-intersection of $\tilde{V}$ in $\tilde{X}$, is the first Chern class of the normal bundle $N_{\tilde{V}/\tilde{X}} = \mathcal{O}_{\mathbb{P}N}(-1)$, ie. $-\zeta$. Thus, using the intersection in the Chow ring of the exception divisor,

$$\omega^2\epsilon^3 = \omega^2\epsilon^2|_{[\tilde{V}]} = (2\eta)^2\zeta^2 = 4,$$

$$\omega\epsilon^4 = \omega\epsilon^3|_{[\tilde{V}]} = -2\eta\zeta^3 = 18,$$

$$\epsilon^5 = \epsilon^4|_{[\tilde{V}]} = \zeta^4 = 51.$$

Thus, $[\tilde{H}_Q]^5 = 2^5(3\omega - \epsilon)^5 = 2^5(3^5 - \binom{5}{3}3^2 \cdot 4 + \binom{5}{4}3 \cdot 18 - 51) = 32 \cdot 102 = 3264$. This is our expected solution to Steiner's problem. We need only show that the intersection of five general $\tilde{H}_Q$'s is transverse.

It is clear that the set of 5-tuples of conics whose corresponding H_Q's intersect transversely is a Zariski open subset. 5-tuples of conics are parametrized by the variety $(\mathbb{P}^5)^5$, which is irreducible, so to show the intersection is transverse for 5 general conics, we need only exhibit one such 5-tuple. This can be done on a computer and is quite unenlightening. Instead, I present an heuristic modern construction (which can be made rigorous) by William Fulton that exhibits 3264 conics in a special case, in which it is also easy to see that there are no more conics, so it gives an example of a transverse intersection, proving that 3264 is indeed the correct answer to Steiner's problem.

Consider 5 general lines L_i with one marked point P_i on each. We will consider all possible enumerative questions of the form "how many conics are tangent to n of the L_i and pass through $5-n$ of the P_i?" Write $\tilde{H}_P$ for the hypersurface of conics passing through the point P, $\tilde{H}_L$ the hypersurface of conics tangent to the line L. The total number of conics satisfying any of these enumerative questions (which one can easily see will all have disjoint solutions for general lines L_i and points P_i) is then

$$[\tilde{H}_P]^5+5[\tilde{H}_P]^4[\tilde{H}_L]+10[\tilde{H}_P]^3[\tilde{H}_L]^2+10[\tilde{H}_P]^2[\tilde{H}_L]^3+5[\tilde{H}_P][\tilde{H}_L]^4+[\tilde{H}_L]^5.$$

Call this number N.

Now we deform each (L_i, P_i) into a hyperbola with origin P_i. Any conic through P_i is deformed into two conics tangent to the new hyperbola, as is every conic tangent to L_i. The figure below is from [3].

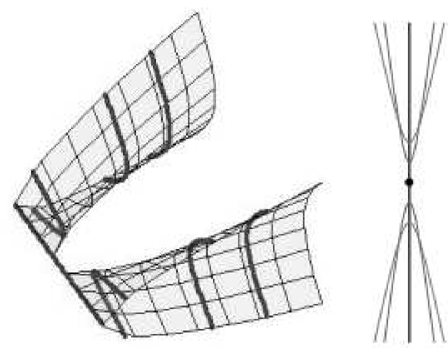

Figure 12. Left: Holding y constant gives cross sections of the surface $x^2y = y^2 + z^2$ that deform to a double line. Right: Superimposed snapshots of the deformation in the plane.

Figure 1.

Thus, we obtain 2^5N conics simultaneously tangent to the five resulting hyperbolas for a general choice of hyperbolas in each deformation family.

We just need to solve the 6 enumerative problems defining N to show $N=102$. Luckily, these problems are much easier than Steiner's problem. First of all, let's re-examine the duality transformation defined on conics in the introduction. Recall that the space of lines in $\mathbb{P}^2$ is another copy of $\mathbb{P}^2$. We defined the dual of a conic in $\mathbb{P}^2$ to be the set of lines tangent to that conic. When the original conic was integral, this set was itself a conic in $\mathbb{P}^2$: the dual conic. We can extend this so that it sends points to the set of lines through that point, which is itself a line in $\mathbb{P}^2$; and sends lines to the set of lines tangent to that line, which is a single line (the line itself!) hence defines a point in $\mathbb{P}^2$. The key is that if a conic passes through a point, then the dual conic is tangent to the dual line, and vice versa. Thus, duality, a priori a birational automorphism of $\mathbb{P}^5$ defined away from the Veronese, lifts to an automorphism of $\tilde{X}=\mathrm{Bl}_V\mathbb{P}^5$ sending $[\tilde{H}_L]\leftrightarrow[\tilde{H}_P]$. As a consequence, $[\tilde{H}_P]^a[\tilde{H}_L]^b=[\tilde{H}_P]^b[\tilde{H}_L]^a$, so we only need to solve 3 of the enumerative problems defining N. These are all easy, so I won't give details. In fact, the intersections of the corresponding hypersurfaces in $\mathbb{P}^5$ are all generically transverse.

How many conics are there passing through 5 general points? One : $[\tilde{H}_P]^5=[\tilde{H}_L]^5=1$.

How many conics are there passing through 4 general points and tangent to a general line? Two : $[\tilde{H}_P]^4[\tilde{H}_L]=[\tilde{H}_P][\tilde{H}_L]^4=2$.

How many conics are there passing through 3 general points and tangent to 2 general lines? Four : $[\tilde{H}_P]^3[\tilde{H}_L]^2=[\tilde{H}_P]^2[\tilde{H}_L]^3=4$.

Thus,

$$N=1+5\cdot 2+10\cdot 4+10\cdot 4+5\cdot 2+1=102.$$

References

[1] William Fulton *Intersection Theory*

[2] David Eisenbud and Joe Harris *3264 & All That: Intersection Theory in Algebraic Geometry*

[3] Andrew Bashelor, Amy Ksir, and Will Traves *Enumerative Algebraic Geometry of Conics*