Combinatorial Applications of Cohomology of Toric Varieties

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1 f-vector for convex polytopes

Given a convex polytope P of full dimension in $\mathbb{R}^d$, we can consider the f-vector, f_k = number of k-faces, $0 \leq k \leq d-1$. A natural question is

Question 1. Given an abstract tuple f of d integers, what are necessary and sufficient conditions for f to be the f-vector of a convex polytope?

It turns out to be quite difficult to give sufficient conditions, so let's determine some necessary ones. First, since P is convex, its boundary is a $(d-1)$-sphere, so we have Euler's relation

$$\chi(\partial P) = \sum_k (-1)^k f_k = 1 - (-1)^d.$$

Theorem 1. This is the only linear relation satisfied by all convex polytopes.

To see this, first consider $d = 1, 2$. Then suppose we've shown the result up to some $d \geq 2$. Let

$$\sum_k \alpha_k f_k = \beta$$

be a linear relation for f-vectors of $(d+1)$-polytopes. Given a d-polytope P, we can form the pyramid $P*$ and bipyramid $*P*$, both convex $(d+1)$-polytopes. These have face vectors

$$f(P*) = (1 + f_0, f_0 + f_1, f_1 + f_2, ..., f_{d-2} + f_{d-1}, f_{d-1} + 1)$$

$$f(*P*) = (2 + f_0, 2f_0 + f_1, ..., 2f_{d-2} + f_{d-1}, 2f_{d-1}).$$

By assumption, we then have

$$\alpha_0(1 + f_0) + \sum_k \alpha_k(f_{k-1} + f_k) + \alpha_d(f_{d-1} + 1) = \beta$$

$$\alpha_0(2 + f_0) + \sum_k \alpha_k(2f_{k-1} + f_k) + 2\alpha_d f_{d-1} = \beta.$$

The difference between the second and the first is

$$\alpha_0 - \alpha_d + \sum_k \alpha_k f_{k-1} + \alpha_d f_{d-1} = 0.$$

By the induction hypothesis, we then obtain $\alpha_k = (-1)^k \alpha_0$. Considering the simplex, we also find $\beta = (1 - (-1)^d)\alpha_0$. Thus, our new relation is proportional to Euler's relation.

It turns out that going further than linear relations is quite hard for arbitrary simplicial complexes. Thus, we restrict our focus to simplicial polytopes.

2 simplicial polytopes

For simplicial polytopes we have more linear relations. One way to see this is to consider that the incidence relation on faces is rather simple. For instance, every k-face is adjacent to exactly

$$\binom{k}{j}$$

j-faces for $j \leq k$ and

$$\binom{d-k}{d-j}$$

j-faces for $j \geq k$. Thus if we consider the sum over k-faces F_k and j-faces F_j, $j \leq k$,

$$I = \sum_{j \leq k} \sum_{F_k, F_j} (-1)^j \text{ if } F_k \text{ and } F_j \text{ are adjacent, 0 otherwise,}$$

we can compute this:

$$I = \sum_{j \le k} \sum_{F_j} \binom{d-j}{d-k} (-1)^j = \sum_{j \le k} \binom{d-k}{d-j} (-1)^j f_j.$$

On the other hand,

$$I = \sum_{F_k} \sum_{j \le k} \binom{k}{j} (-1)^j = \sum_{F_k} 1 = f_k,$$

since F_k is a simplex. Thus, we've derived

$$f_k = \sum_{j \le k} \binom{d-j}{d-k} (-1)^j f_j.$$

These are called the Dehn-Sommerville relations.

Theorem 2. These are the only linear relations satisfied by all convex simplicial polytopes.

We will not prove this theorem. It also turns out that these conditions, though powerful, are still not sufficient. We need to construct more refined invariants.

3 simplicial projective toric varieties

Given a full dimensional lattice polytope P in $\mathbb{R}^d$, we can construct a projective complex variety $X(P)$ of complex dimension d in a way that the combinatorics of P is realized geometrically. $X(P)$ is furthermore equipped with an action of a torus $(\mathbb{C}^\times)^d$ having a dense orbit isomorphic to this torus. In fact, all the orbits are tori, and the closure of any orbit is a union of orbits. The incidence data of these orbits is captured by the incidence data of the faces of P in what is called the orbit-cone correspondence.

Theorem 3. To each k-face σ of P corresponds a complex k-dimensional orbit $O_\sigma \subset X(P)$ such that $\bar{O_\sigma} = \bigcup_{\tau \le \sigma} O_\tau$, where the union is over faces τ of σ.

The singularities of toric varieties are rather controlled. For instance, one can prove that all toric varieties are Cohen-Macauly, so Serre duality holds for all coherent sheaves on $X(P)$. Further, the singularities are rational, so when we resolve them minimally by blow-ups, the higher pushforwards of the projection are all zero. This lets us understand the cohomology of $X(P)$ in terms of its resolution.

Even nicer is when P is simplicial. In this case, the singularities of $X(P)$ are Abelian quotient singularities. These have minimal resolutions with finite preimages. Such a resolution is called small.

There is an improvement of cellular homology for stratified spaces such as $X(P)$ called intersection homology, whose chains are required to be transverse to the singular locus in codimension at most have the dimension of the chain. The resulting complex IC_* and its homology IH_* are self-dual under Poincare duality. Further, they play quite nicely with Abelian quotient singularities.

Theorem 4. If $\tilde{X} \to X$ is a small resolution, then $IH_*(X) = H_*^{BM}(\tilde{X})$, where BM denotes Borel-Moore homology, ie. homology with locally finite chains.

However, if we are content with using $\mathbb{Q}$-coefficients, we can get away with using the rational Chow ring. This will be enough for our purposes. We define $A^k(X)$ to be free $\mathbb{Q}$-vector space generated by dimension k subvarieties of X modulo rational equivalence, a sort of algebraic cobordism parametrized by $\mathbb{P}^1$.

Theorem 5. If X is projective and has at worst Abelian quotient singularities then $A_*(X)$ is a ring given by intersection when the subvarieties are transverse.

If X isn't projective but still with only Abelian quotient singularities, then we can consider locally finite subvarieties and get at least a vector space and sometimes also a ring.

For toric varieties we can even say that A^k is generated by torus invariant subvarieties. These are of course unions of closures of dimension k orbits, and relations are given by the orbits of higher dimension.

Theorem 6. For $k \geq 1$,

$$\dim A_k = \sum_{j \geq k} (-1)^{j-k} \binom{j}{k} f_{d-j-1}.$$

Let us order the vertices of P by their inner product with a general vector in $\mathbb{R}^d$. Let σ_i denote the corresponding top dimensional cones. Let $\tau_i \subset \sigma_i$ be the intersection of all σ_j with $j \geq i$. Since the ordering is induced by a linear function, it has several nice properties.

Lemma 1. The following hold.

- If $\tau_i \subset \sigma_j$, then $i \leq j$.
- For each cone γ, there is a unique $i = i(\gamma)$ such that $\tau_i \subset \gamma \subset \sigma_i$.
- $i(\gamma)$ is the least i with $\gamma \subset \sigma_i$
- If γ is a face of γ', then $i(\gamma) \leq i(\gamma')$

Now let $Y_i = \bigcup_{\tau_i \subset \gamma \subset \sigma_i} O_\gamma = V(\tau_i) \cap U_{\sigma_i}$, where O_γ is the orbit corresponding to γ, $V(\tau_i)$ is the closure of O_{τ_i}, and U_{σ_i} is the affine open given by the attracting set of O_{σ_i} (under the torus action), which is a point. Also let $Z_i = \bigcup_{j \geq i} Y_j$.

Notice that $X(P)$ is a disjoin union of the Y_i's by the lemma. Then by the orbit-cone correspondence, the closure of O_γ is the union of all $O_{\gamma'}$ for γ' containing γ, and for these $i(\gamma) \leq i(\gamma')$, so $O_{\gamma'}$ appears in $Z_{i(\gamma)}$. Thus, Z_i is closed for all i.

Further, Y_i is a quotient of $\mathbb{C}^{d-\dim \tau_i}$ by an abelian group, so $A_*(Y_i)$ is freely generated by the class of Y_i.

We also have $Z_i - Z_{i+1} = Y_i$, so we get an exact sequence

$$A_k(Z_{i+1}) \to A_k(Z_i) \to A_k(Y) \to 0.$$

Starting from $Z_m = Y_m$, where m is the number of vertices, it follows that $A_*(Z_i)$ is freely generated by the classes of $V(\tau_j)$ for $j \geq i$. In particular, $Z_1 = X$, so $A_*(X)$ is freely generated by all the $V(\tau_j)$'s.

There are exactly

$$\binom{d-j}{d-k}$$

k-dimensional cones γ with $\tau_j \leq \gamma \leq \sigma_j$. Since every γ occurs this way, and there are f_{d-k-1} k-dimensional cones (we set $f_{-1} = 1$).

$$f_{d-k-1} = \sum_{j \leq k} \binom{d-j}{d-k} \dim A_j.$$

Equivalently,

$$\dim A_k = \sum_{j\geq k}(-1)^{j-k}\binom{j}{k}f_{d-j-1}.$$

These dimensions are often collected into a vector called the h-vector with $h_k = \dim A_k$.

Theorem 7. The Dehn-Sommerville relations are equivalent to Poincare duality for $X(P)$, ie. $h_k = h_{d-k}$.

We show that the Dehn-Sommerville relations imply Poincare duality.

$$h_{d-k} = \sum_{j\geq d-k}(-1)^{j-k}\binom{j}{d-k}f_{d-j-1} = \sum_{i\leq k}(-1)^{j-k}\binom{d-i}{d-k}f_{i-1}.$$

Then, using the Dehn-Sommerville relation,

$$h_{d-k} = \sum_{i\leq j\leq k}\binom{d-i}{d-j}\binom{d-j}{d-k}(-1)^{i-k}f_{i-1} = \sum_{i\leq k}\binom{d-i}{k-i}(-1)^{k-i}f_{i-1} = h_k.$$

There is a strengthening of Poincare duality for smooth complex projective d-dimensional varieties X. Since X is projective, we can embed X in $\mathbb{P}^n$ and consider the intersection of X with a hyperplane H. The first step is, using a suitable Morse function on $\mathbb{C}^n = \mathbb{P}^n - H$ one can show that $Y = X - \omega$ has the homotopy type of a (real!) d-dimensional CW complex, and so we can prove

Theorem 8. (Lefschetz Hyperplane Theorem) The restriction map $H^k(X) \to H^k(Y)$ is an isomorphism for $k \leq d-2$ and surjective for $k = d-1$.

We can make this better by considering Y as a class $\omega \in H^2(X)$ and considering how this class interacts with the Hodge structure on X.

Theorem 9. (Hard Lefschetz Theorem) The map $H^{d-k}(X) \to H^{d+k}(X)$ given by intersecting with ω^k is an isomorphism.

It turns out that intersection homology, and therefore the rational Chow ring A^* for $X(P)$ satisfy the hard Lefschetz theorem as well, using a relative version of the small resolution theorem called the decomposition theorem. An immediate corollary is

Theorem 10. The h-vector of P is "unimodal":

$$h_0 \leq h_1 \leq ... \leq h_{[d/2]} \geq h_{[d/2]+1} \geq ... \geq h_d.$$

4 more conditions

For concreteness, consider the case $d = 4$. Euler's relation says

$$f_0 - f_1 + f_2 - f_3 = 0.$$

Also, since every 3-simplex has four 2-faces, each on two 3-simplices,

$$f_2 = 2f_3.$$

To bound a 4-dimensional solid we need

$$f_0 \geq 5.$$

Since two vertices can be joined by at most one edge,

$$f_1 \leq \binom{f_0}{2}.$$

One can also derive the lower bound

$$f_1 \geq 4f_0 - 10.$$

The first and second conditions are the two independent Dehn-Sommerville relations. The second condition is equivalent to

$$h_{d-1} \geq 1 = h_0.$$

The third condition is equivalent to

$$h_2 - h_1 \leq \binom{h_1 - h_0 + 1}{2}.$$

Let us define $g_k = h_k - h_{k-1}$. We can now phrase the generalizations of the above conditions as McMullen did. Stanley proved these conditions are necessary and Billera and Lee proved they are sufficient.

Theorem 11. f is the face vector of a convex simplicial polytope if and only if the following conditions hold:

- (Dehn-Sommerville) $h_k = h_{d-k}$
- $g_k \geq 0$ for $1 \leq k \leq d/2$

- If one writes
$$g_k = \binom{n_k}{k} + \binom{n_{k-1}}{k-1} + ... + \binom{n_r}{r},$$
with $n_k > n_{k-1} > ... > n_r \geq r \geq 1$, then
$$g_{k+1} \leq \binom{n_k+1}{k+1} + \binom{n_{k-1}+1}{k} + ... + \binom{n_r+1}{r+1}.$$

We will only prove necessity. The other direction is given by a construction [1].

The first condition is Poincare duality and the second is unimodality from the Lefschetz thereom.

For the third, note that g_k always has a unique expression of the desired form by taking n_k to be the biggest such that $\binom{n_k}{k} \leq g_k$ and continuing this way. Then consider the commutative graded $\mathbb{Q}$-algebra
$$R = H^*(X(P))/\omega.$$
The dimension of the kth graded piece is g_k. Macaulay was the first to characterize the sequences which can arise from such algebras, and the third condition is necessary for the existence of R. See [2] for a proof.

5 general convex polytopes revisited

Simplicial is really a necessary condition for controlling the singularities of $X(P)$ and for being able to use intersection homology with impunity. However, simplicial is also an important condition for a polytope being equivalent to a lattice polytope, which is the only case when we can form $X(P)$.

Theorem 12. (Mnev Universality) The realization space of d-polytopes with $\geq d+4$ vertices can have arbitrary stable equivalence type.

Suppose P has n vertices $p_1, ..., p_n \in \mathbb{R}^d$. The realization space of P is the subset of $\mathbb{R}^{d\times n}$ given by tuples $(p_1, ..., p_{d+1}, q_{d+2}, ..., q_n)$ of elements in $\mathbb{R}^d$ whose convex hull is combinatorially equivalent to P. For simplicial P, these sets are parametrized by edge lengths, so the theorem cannot distinguish between P with rational vertices or irrational vertices. Stable equivalence however is enough to remember whether a general P has rational or irrational vertices. See [3].

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