MATH 2352 Solution Sheet 05

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Created on March 24, 2015

[Problems] 4.4: 5, 13; 5.1: 7, 13, 23, 25; 5.2: 9, 19; 5.4: 6, 19, 20, 35.

4.4 - 5. Use the method of variation of parameters to determin the general solution of the given differntial equation.

$$y''' - y'' + y' - y = 2e^{-t}\sin t.$$

Solution. The fundamental solution of the homogeneous equation is

$$\begin{aligned} y_1(t) &= e^t, \\ y_2(t) &= \sin t, \\ y_3(t) &= \cos t. \end{aligned}$$

Let $y(t) = a(t)\, y_1(t) + b(t)\, y_2(t) + c(t)\, y_3(t)$, then

$$y'(t) = [a'(t)\, y_1(t) + b'(t)\, y_2(t) + c'(t)\, y_3(t)] + [a(t)\, y_1'(t) + b(t)\, y_2'(t) + c(t)\, y_3'(t)].$$

Let

$$a'(t)\, y_1(t) + b'(t)\, y_2(t) + c'(t)\, y_3(t) = 0,$$

then

$$y'(t) = a(t)\, y_1'(t) + b(t)\, y_2'(t) + c(t)\, y_3'(t),$$

and

$$y''(t) = a'(t)\, y_1'(t) + b'(t)\, y_2'(t) + c'(t)\, y_3'(t) + a(t)\, y_1''(t) + b(t)\, y_2''(t) + c(t)\, y_3''(t).$$

Again let

$$a'(t)\, y_1'(t) + b'(t)\, y_2'(t) + c'(t)\, y_3'(t) = 0,$$

then

$$y''(t) = a(t)\, y_1''(t) + b(t)\, y_2''(t) + c(t)\, y_3''(t),$$

and

$$y'''(t) = a'(t)\, y_1''(t) + b'(t)\, y_2''(t) + c'(t)\, y_3''(t) + a(t)\, y_1'''(t) + b(t)\, y_2'''(t) + c(t)\, y_3'''(t).$$

Plugging in,

$$y''' - y'' + y' - y = a'(t)\, y_1''(t) + b'(t)\, y_2''(t) + c'(t)\, y_3''(t).$$

Therefore, we have three equations for a, b, c

$$\begin{cases} a'(t)\, y_1(t) + b'(t)\, y_2(t) + c'(t)\, y_3(t) &= 0 \\ a'(t)\, y_1'(t) + b'(t)\, y_2'(t) + c'(t)\, y_3'(t) &= 0 \\ a'(t)\, y_1''(t) + b'(t)\, y_2''(t) + c'(t)\, y_3''(t) &= 2\, e^{-t} \sin t \end{cases}.$$

Plugging solutions of homogeneous equation,

$$\begin{cases} a'(t)\, e^t + b'(t) \sin t + c'(t) \cos t &= 0 \\ a'(t)\, e^t + b'(t) \cos t - c'(t) \sin t &= 0 \\ a'(t)\, e^t - b'(t) \sin t - c'(t) \cos t &= 2\, e^{-t} \sin t \end{cases}.$$

Add the first to the third and get

$$2a'(t)\, e^t = 2\, e^{-t} \sin t.$$

So $a'(t) = e^{-2t} \sin t$ and

$$\begin{cases} b'(t) \sin t + c'(t) \cos t &= -e^{-t} \sin t \\ b'(t) \cos t - c'(t) \sin t &= -e^{-t} \sin t \end{cases}.$$

Then

$$\begin{aligned} b'(t) &= b'(t)\, (\sin^2 t + \cos^2 t) \\ &= -e^{-t} \sin^2 t - e^{-t} \sin t \cos t, \end{aligned}$$

$$\begin{aligned} c'(t) &= c'(t)\, (\sin^2 t + \cos^2 t) \\ &= -e^{-t} \sin t \cos t + e^{-t} \sin^2 t. \end{aligned}$$

Solving these a, b, c yields

$$\begin{aligned} a(t) &= c_1 - \frac{2}{5} e^{-2t} \sin(t) - \frac{1}{5} e^{-2t} \cos(t), \\ b(t) &= c_2 + \frac{1}{2} e^{-t} + \frac{3}{10} e^{-t} \sin(2t) + \frac{1}{10} e^{-t} \cos(2t), \\ c(t) &= c_3 - \frac{1}{2} e^{-t} - \frac{1}{10} e^{-t} \sin(2t) + \frac{3}{10} e^{-t} \cos(2t). \end{aligned}$$

Then we have

$$\begin{aligned} y(t) &= a(t)\, y_1(t) + b(t)\, y_2(t) + c(t)\, y_3(t) \\ &= c_1\, e^t + c_2 \sin t + c_3 \cos t - \frac{2}{5}\, e^{-t} \cos t. \end{aligned}$$

4.4 - 13. Given that x, x^2 and $1/x$ are solutions of the homogeneous equation corresponding to

$$x^3 y''' + x^2 y'' - 2x\, y' + 2y = 2x^4, \quad x > 0,$$

determine a particular solution.

Solution. $\frac{x^4}{15}$.

5.1 - 7. Determine the radius of convergence of the given power series.

$$S(x) = \sum_{n=1}^{\infty} \frac{(-1)^n n^2 (x+2)^n}{4^n}.$$

Solution. Using ratio test

$$\begin{aligned}\frac{\frac{(-1)^{n+1}(n+1)^2(x+2)^{n+1}}{4^{n+1}}}{\frac{(-1)^n n^2(x+2)^n}{4^n}} &= \frac{(-1)^{n+1}(n+1)^2(x+2)^{n+1}}{(-1)^n n^2(x+2)^n}\frac{4^n}{4^{n+1}} \\ &= -\frac{(n+1)^2(x+2)}{4n^2},\end{aligned}$$

If $|x+2|<4$, the limiting absolute value of ratio is less than 1. So the convergence radius is 4.

5.1 - 13. Determine the Taylor series about the point x_0 for the given function. Also determine the radius of convergence of the series.

$$\ln x, \quad x_0 = 1.$$

Solution. Taylor series is

$$\ln x = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(x-1)^n,$$

and the convergence radius is 1.

5.1 - 23. Rewrite the given expression as a sum whose generic term involves x^n.

$$S(x) = x^2 \sum_{n=1}^{\infty} n\, a_n\, x^{n-1} + \sum_{k=0}^{\infty} a_k\, x^k.$$

Solution. Define $a_{-1} = 0$,

$$\begin{aligned}S(x) &= x^2 \sum_{n=1}^{\infty} n\, a_n\, x^{n-1} + \sum_{k=0}^{\infty} a_k\, x^k \\ &= \sum_{n=1}^{\infty} n\, a_n\, x^{n+1} + \sum_{k=0}^{\infty} a_k\, x^k \\ &= \sum_{n=2}^{\infty} (n-1)\, a_{n-1}\, x^n + \sum_{k=0}^{\infty} a_k\, x^k \\ &= \sum_{n=0}^{\infty} (n-1)\, a_{n-1}\, x^n + \sum_{k=0}^{\infty} a_k\, x^k \\ &= \sum_{n=0}^{\infty} [(n-1)\, a_{n-1} + a_n]\, x^n.\end{aligned}$$

5.1 - 25. Rewrite the given expression as a sum whose generic term involves x^n.

$$S(x) = \sum_{m=2}^{\infty} m(m-1)a_m\, x^{m-2} + x^3 \sum_{k=1}^{\infty} k\, a_k\, x^{k-1}.$$

Solution. Define $a_{-1} = a_{-2} = 0$,

$$\begin{aligned} S(x) &= \sum_{m=0}^{\infty} (m+2)(m+1)a_{m+2}\, x^m + \sum_{k=1}^{\infty} k\, a_k\, x^{k+2} \\ &= \sum_{m=0}^{\infty} (m+2)(m+1)a_{m+2}\, x^m + \sum_{k=3}^{\infty} (k-2)\, a_{k-2}\, x^k \\ &= \sum_{m=0}^{\infty} (m+2)(m+1)a_{m+2}\, x^m + \sum_{k=0}^{\infty} (k-2)\, a_{k-2}\, x^k \\ &= \sum_{m=0}^{\infty} [(m+2)(m+1)a_{m+2} + (m-2)\, a_{m-2}]\, x^m \end{aligned}$$

5.2 - 9. For

$$\begin{aligned} (1+x^2)y'' - 4\,x\,y' + 6y &= 0, \\ x_0 &= 0. \end{aligned}$$

(a) Seek power series solutions of the given differential equation about the given point x_0; find the recurrence relation.

(b) Find the first four terms in each of two solutions y_1 and y_2 (unless the series terminates sooner).

(c) By evaluating the Wronskian $W(y_1, y_2)(x_0)$, show that y_1 and y_2 form a fundamental set of solutions.

(d) If possible, find the general term in each solution.

Solution.

(a) Let $y(x) = \sum_{n=0}^{\infty} a_n\, x^n$, then when the power series is uniformly convergent,

$$\begin{aligned} x\,y'(x) &= x \sum_{n=0}^{\infty} n\, a_n\, x^{n-1} \\ &= \sum_{n=0}^{\infty} n\, a_n\, x^n, \\ x^2\, y''(x) &= \sum_{n=0}^{\infty} n(n-1)\, a_n\, x^n, \\ y''(x) &= \sum_{n=0}^{\infty} n(n-1)\, a_n\, x^{n-2} \\ &= \sum_{n=0}^{\infty} (n+2)(n+1)\, a_{n+2}\, x^n. \end{aligned}$$

Therefore,

$$\begin{aligned} (1+x^2)y'' - 4\,x\,y' + 6y &= \sum_{n=0}^{\infty} [n(n-1)\, a_n + (n+2)(n+1)\, a_{n+2} - 4\,n\,a_n + 6a_n]\, x^n \\ &= 0. \end{aligned}$$

Then

$$\begin{aligned} 0 &= n(n-1)\, a_n + (n+2)(n+1)\, a_{n+2} - 4\,n\,a_n + 6a_n \\ &= (n^2 - 5n + 6)\, a_n + (n+2)(n+1)\, a_{n+2}, \end{aligned}$$

so the recurrence relation is

$$a_{n+2} = \frac{(n-2)(n-3)}{(n+2)(n+1)} a_n$$

(b) The solutions can be written as

$$\begin{aligned} y(x) &= a_0 + a_1 x + \frac{6}{2} a_0 x^2 + \frac{2}{6} a_1 x^3 \\ &= a_0(1+3x^2) + a_1\left(x + \frac{1}{3}x^3\right) \end{aligned}$$

(c) The Wronskian is

$$\begin{aligned} \begin{vmatrix} 1+3x^2 & x+\frac{1}{3}x^3 \\ 6x & 1+x^2 \end{vmatrix} &= 1+4x^2+3x^4-(6x^2+2x^4) \\ &= x^4-2x^2+1 = (x^2-1)^2 \neq 0 \end{aligned}$$

(d) $a_2 = 3a_0,\ a_3 = \frac{1}{3}a_1,\ a_n = 0,\ \forall n > 3.$

5.2 - 19.

(a) By making the change of variable $x - 1 = t$ and assuming that y has a Taylor series in power of t, find two series solutions of

$$y'' + (x-1)^2 y' + (x^2-1) y = 0$$

in power series of $x - 1$.

(b) Show that you obtain the same result by assuming that y has a Taylor series in power of $x-1$ and also expressing the coefficient x^2-1 in powers of $x-1$.

Solution. Note that

$$x^2 - 1 = (x-1)^2 + 2(x-1).$$

All points are regular, one can directly solve for a series solution.

5.4 - 6. Determine the general solution of the given differential equation that is valid in any interval not including the singular point.

$$(x-1)^2 y'' + 8(x-1) y' + 12 y = 0.$$

Solution. $x = 1$ is a regular singular point. Under translation $t = x - 1$, the translated equaion is equidimensional. The solution is

$$y(x) = \frac{c_1}{(x-1)^4} + \frac{c_2}{(x-1)^3}.$$

5.4 - 19. Find all singular points of the given equation and determine whether each one is regular or irregular.

$$x^2(1-x) y'' + (x-3) y' - 3x y = 0.$$

Solution. As shown below

Singular point	Type
0	irregular
1	regular

Table 1.

5.4 - 20. Find all singular points of the given equation and determine whether each one is regular or irregular.

$$x^2(1-x^2)y'' + (2/x)y' + 4y = 0.$$

Solution. As shown below

Singular point	Type
0	irregular
1	regular
-1	regular

Table 2.

5.4 - 35. Find all values of α for which all solutions of $x^2y'' + \alpha\, x\, y' + (5/2)y = 0$ approach zero as $x \to 0$.

Solution. This equation is equidimensional, which solves

$$y(x) = c_1 x^{\lambda_1} + c_2 x^{\lambda_2},$$

where

$$\lambda_1 = \frac{1 - a - \sqrt{a^2 - 2a - 9}}{2},$$

$$\lambda_2 = \frac{1 - a + \sqrt{a^2 - 2a - 9}}{2}.$$

Since the solution approach zero as $x \to 0$, both $\lambda_{1,2}$ should be positive, which means

$$1 - a - \sqrt{a^2 - 2a - 9} > 0.$$

This is equivalent to

$$\begin{cases} 1 - a > 0 \\ a^2 - 2a - 9 > 0 \\ (1-a)^2 > \left(\sqrt{a^2 - 2a - 9}\right)^2 \end{cases}.$$

Because the third inequality holds for any a, we only need to specify

$$a < \frac{2 - \sqrt{40}}{2} = 1 - \sqrt{10}.$$

Now we consider the case of 2-fold root λ, which means $a^2-2a-9=0$, then the solution is

$$y(x) \;=\; c_1\,x^{\lambda}+c_2\,x^{\lambda}\ln x.$$

This goes to zero when λ is positive. So $a=1-\sqrt{10}$ is acceptable.

In the end we consider the case of complex roots, where the solution is

$$y(x) \;=\; x^{\mathrm{Re}(\lambda)}\,(c_1\cos\ln x+c_2\sin\ln x).$$

Since the solution approach zero as $x\rightarrow 0$, $\mathrm{Re}(\lambda)$ should be positive, which means

$$1-a \;>\; 0,$$

and by assumption $a^2-2a-9<0$, which means

$$1-\sqrt{10}< \;a\; <1+\sqrt{10}.$$

In summary,

$$a \;\in\; (-\infty,1).$$