MATH 2352 Solution Sheet 03

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[Problems] 2.7: 16; 3.1: 4, 11; 3.3: 14, 19; 3.4: 6, 11; 3.2: 9, 10, 14, 25; 3.5: 6, 20;

2.7 - 16. Consider the initial value problem

$$y' = t^2 + y^2, \quad y(0) = 1.$$

Use Euler's method with $h = 0.1, 0.05, 0.025$, and 0.01 to explore the solution of this problem for $0 \leqslant t \leqslant 1$. What is your best estimate of the value of the solution at $t = 0.8$? At $t = 1$? Are your results consistent with the direction field in Problem 9?

Solution.

The following Matlab code solves the problem:

```
matlab>>hh = [ 0.1, 0.05, 0.025, 0.01 ];
        f = @(t,y)(t.^2+y.^2);
        N1 = floor(0.8./hh);
        N2 = floor(1./hh);
        v = zeros(4, 2);
        for i = 1:4
           y = 1; h = hh(i);
           for j = 1:N2(i)
              y = y + f(j*h-h,y)*h;
              if j == N1(i)
                 v(i,1) = y;
              end;
           end;
           v(i,2) = y;
        end;
matlab>>disp(v);
          3.5078    7.1895
          4.2013   12.3209
          4.8004   23.9260
          5.3428   90.7555
```

3.1 - 4. Find the general solution of the given differential equation:

$$3\,y'' - 4\,y' + y \;=\; 0.$$

Solution. Since the equation can be written as

$$\begin{aligned} 3\,y'' - 3\,y' - y' + y &= 0 \\ 3(y'' - y') - (y' - y) &= 0 \\ (y' - y)' &= (y' - y)/3, \end{aligned}$$

we have

$$y' - y \;=\; C_0\, e^{x/3}.$$

One solution is

$$y(x) \;=\; -\frac{3}{2}C_0\, e^{x/3}.$$

Let $C_1 = -\frac{3}{2}C_0$, and notice that the solution can differ by $C_2\, e^x$,

$$y(x) \;=\; C_1\, e^{x/3} + C_2\, e^x.$$

3.1 - 11. Find the solution of the given initial value problem. Sketch the graph of the solution and describe its behavior as t increases.

$$\begin{aligned} 12y'' - 7y' + y &= 0, \\ y(0) &= 4, \\ y'(0) &= 0. \end{aligned}$$

Solution. The equation can be written as

$$\begin{aligned} 12y'' - 3y' - 4y' + y &= 0 \\ (4y' - y)' &= (4y' - y)/3. \end{aligned}$$

From this equation $4y' - y$ can be solved.

On the other hand,

$$\begin{aligned} 12y'' - 4y' - 3y' + y &= 0 \\ (3y' - y)' &= (3y' - y)/4. \end{aligned}$$

So the general solution is

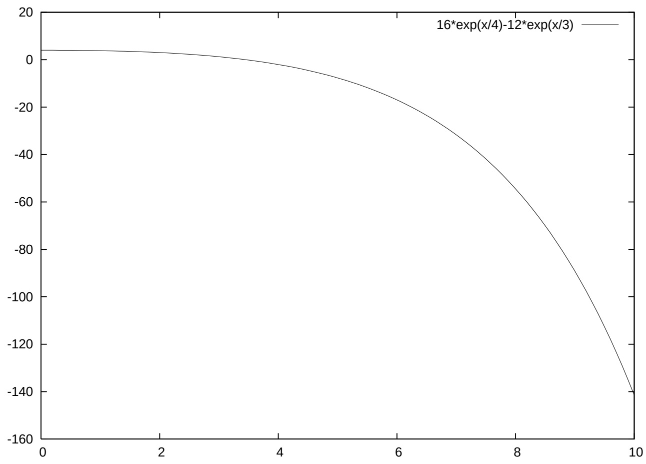
$$y(x) \;=\; C_1 e^{x/3} + C_2\, e^{x/4}.$$

The solution to the I.V.P. is

$$y(x) \;=\; 16e^{x/4} - 12\, e^{x/3}.$$

The solution is shown in below:

```
GNUplot] plot [0:10] 16*exp(x/4)-12*exp(x/3)
```



3.3 - 14. Find the general solution of given differential equation:

$$9y'' + 3y' - 2y \;=\; 0.$$

Solution. Assume λ is a real constant, then

$$9y'' + \lambda\, y + (3 - \lambda) y - 2y \;=\; 0.$$

Let

$$\frac{9}{3 - \lambda} \;=\; \frac{\lambda}{-2}.$$

This equation has two roots $\lambda_1 = -3$, $\lambda_2 = 6$.

Then we can solve the two 1-st order equations and get the general solution

$$y(x) \;=\; C_1\, e^{-2/3} + C_2\, e^{1/3}.$$

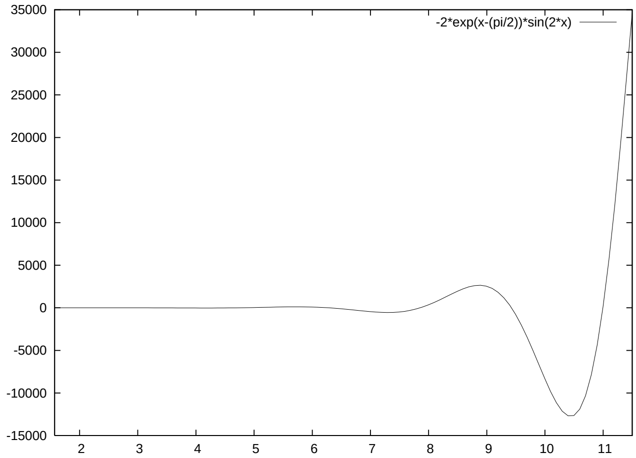
3.3 - 19. Find the solution of the given initial value problem. Sketch the graph of the solution and describe its behavior as t increases.

$$\begin{aligned} y'' - 2y' + 5y &= 0, \\ y\left(\frac{\pi}{2}\right) &= 0, \\ y'\left(\frac{\pi}{2}\right) &= 4. \end{aligned}$$

Solution. The solution is

$$y(x) \;=\; -2\, e^{x - \frac{\pi}{2}} \sin(2\, x).$$

```
GNUplot] plot [pi/2:11.5] -2*exp(x-(pi/2))*sin(2*x)
```



3.4 - 6. Find the general solution of given differential equation:

$$y'' - 10y' + 25y \;=\; 0.$$

Solution. This is the degenerate case, meaning only one reduced equation can be written

$$(y' - 5y)' \;=\; 5\,(y' - 5y).$$

And the solution is $y' - 5\,y = C_0\,e^{5x}$.

Solving this yields

$$y(x) \;=\; C_1 e^{5x} + C_2\,x\,e^{5x}.$$

3.4 - 11. Find the solution of the given initial value problem. Sketch the graph of the solution and describe its behavior as t increases.

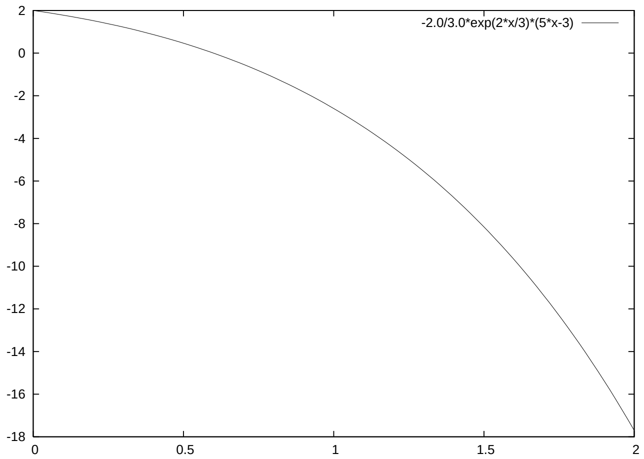
$$\begin{aligned} 9y'' - 12y' + 4y &= 0, \\ y(0) &= 2, \\ y'(0) &= -2. \end{aligned}$$

Solution. The solution is

$$y(x) \;=\; -\frac{2}{3}e^{2x/3}(5x-3).$$

Plot the solution:

```
GNUplot] plot [0:2] -2.0/3.0*exp(2*x/3)*(5*x-3)
```



3.2 - 9. Determin the longest interval in which the given initial value problem is certain to have a unique twice-differentiable solution. Do not attempt to find the solution.

$$\begin{aligned} t(t-4)y'' + 3ty' + 5y &= 2, \\ y(3) &= 0, \\ y'(3) &= -1. \end{aligned}$$

Solution. (3,4).

3.2 - 10. Determin the longest interval in which the given initial value problem is certain to have a unique twice-differentiable solution. Do not attempt to find the solution.

$$\begin{aligned} y'' + (\sin t)y' + 3(\ln|t|)y &= 0, \\ y(1) &= 3, \\ y'(1) &= 1. \end{aligned}$$

Solution. $(1, +\infty)$.

3.2 - 14. Verify that $y_1(t)=1$ and $y_2(t)=t^{1/2}$ are solution of the differential equation $y\,y''+(y')^2=0$ for $t>0$. Then show that $y=c_1+c_2\,t^{1/2}$ is not, in general, a solution of theis equation. Explain why this result does not contradict Theorem 3.2.2.

Solution. Nonlinearity.

3.2 - 25. Verify that the functions y_1 and y_2 are solutions of the given differential equation. Do they constitute a fundamental set of solutions?

$$\begin{aligned} y''-4\,y'+4y &= 0; \\ y_1(t) &= e^{2t}, \\ y_2(t) &= t\,e^{2t}. \end{aligned}$$

Solution. Yes.

3.5 - 6. Find the general solution of given differential equation:

$$y''+2y' = 5+4\sin 2t.$$

Solution. Let $p=y'$,

$$p' = -2p+5+4\sin 2t.$$

Then $p=C_1\,e^{-2t}+\sin(2t)-\cos(2t)+\frac{5}{2}$.

And $y=p+C_2$.

3.5 - 20. Find the solution of the given initial value problem:

$$\begin{aligned} y''+2y'+5y &= 4\,e^{-t}\cos 2t, \\ y(0) &= 0, \\ y'(0) &= 0. \end{aligned}$$

Solution. $y=e^{-t}\,t\sin(2t)$.

Remark 1. The general solution for the homogenuous problem is $y(t)=C\,e^{-t}\sin\,(2t)$. (Only real solution is considered here).

Remark 2. By assuming the solution is $y(t)=C(t)\,e^{-t}\sin\,(2t)$, one can derive the initial value problem for C as

$$\begin{aligned} (\tan 2t)\,C''+4\,C'-4 &= 0, \\ C(0) &= 0. \end{aligned}$$

To solve this, let $z(t)=C'$, and solve the separable equation

$$\begin{aligned} (\tan 2t)\,z'+4\,z-4 &= 0 \\ (\tan 2t)\,\frac{dz}{dt} &= 4-4z \\ \frac{dz}{4-4z} &= \frac{dt}{\tan 2t}, \end{aligned}$$

which gives $z(t) = C_1 \csc^2(2x) + 1$. Therefore,

$$C(t) = C_1 \cot 2t + t + C_2.$$

Since $C(0)$ is well-defined and equals to zero, $C_1 = C_2 = 0$, we have

$$C(t) = t.$$

Remark 3. If you start from the general form $y(t) = c_1(t)\ \phi_1(t) + c_2(t)\ \phi_2(t)$, where $\phi_1(t) = e^{-t}\ \sin(2t)$, $\phi_2(t) = e^{-t}\cos(2t)$. you will need to assume

$$c_1'(t)\, \phi_1(t) + c_2'(t)\, \phi_2(t) = 0 \tag{1}$$

and get

$$c_1'(t)\, \phi_1'(t) + c_2'(t)\, \phi_2'(t) = 4\, e^{-t} \cos 2t = 4\, \phi_2(t) \tag{2}$$

You can use the general formula from Cramer's Rule to get this solution directly, or you can make it simpler to calculate by noting that

$$\begin{aligned} \phi_1'(t) &= -\phi_1(t) + 2\phi_2(t), \\ \phi_2'(t) &= -\phi_2(t) - 2\phi_1(t), \end{aligned}$$

and add (1) to (2) to yield

$$c_1'(t)\, \phi_2(t) - c_2'(t)\, \phi_1(t) = 2\, \phi_2(t).$$

Then using $\phi_1 / \phi_2 = \tan(2t)$ we have two simple equations

$$\begin{aligned} c_1'(t) \tan(2t) + c_2'(t) &= 0, \\ c_1'(t) - c_2'(t) \tan(2t) &= 2. \end{aligned}$$

So

$$\begin{aligned} c_1'(t)\, (1 + [\tan(2t)]^2) &= 2 \\ c_1'(t) &= 2\cos^2(2t) = \cos(4t) + 1 \\ c_1(t) &= \frac{1}{4}\sin(4t) + t + D_1, \end{aligned}$$

and

$$\begin{aligned} c_2'(t)\, (1 + [\tan(2t)]^2) &= -2\tan(2t) \\ c_2'(t) &= -2\tan(2t)\cos^2(2t) = -\sin(4t) \\ c_2(t) &= \frac{1}{4}\cos(4t) + D_2. \end{aligned}$$

Therefore,

$$\begin{aligned} y(t) &= c_1(t)\, \phi_1(t) + c_2(t)\, \phi_2(t) \\ &= \left(\frac{1}{4}\sin(4t) + t + D_1\right) e^{-t} \sin(2t) + \left(\frac{1}{4}\cos(4t) + D_2\right) e^{-t} \cos(2t) \\ &= t\, e^{-t} \sin(2t) + \frac{1}{4}(\sin(4t)\sin(2t) + \cos(4t)\cos(2t)) e^{-t} + D_1 e^{-t} \sin(2t) + D_2 e^{-t} \cos(2t) \\ &= t\, e^{-t} \sin(2t) + \frac{1}{2}\cos(2t) e^{-t} + D_1 e^{-t} \sin(2t) + D_2 e^{-t} \cos(2t) \\ &= t\, e^{-t} \sin(2t) + D_1 e^{-t} \sin(2t) + \left(D_2 + \frac{1}{2}\right) e^{-t} \cos(2t) \end{aligned}$$

By initial condition,

$$
\begin{aligned}
y(0) &= D_2 + \frac{1}{2} = 0, \\
y'(0) &= -y(0) + 2\,D_1 = 0.
\end{aligned}
$$

Therefore,

$$
y(t) = t\,e^{-t}\sin(2t).
$$