MATH 2352 Solution Sheet 04

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[Problems] 3.6: 14, 17; 3.7: 7, 10; 3.8: 6, 8; 4.1: 5, 6, 10, 19, 20; 4.2: 25, 34; 4.3: 12.

3.6 - 14. Verify that the given functions y_1 and y_2 satisfy the corresponding homogeneous equation; then find a particular solution of the given nonhomogeneous equation.

$$\begin{aligned} t^2 y'' - t(t+2) y' + (t+2) y &= 6t^3, \quad t > 0; \\ y_1(t) &= t, \\ y_2(t) &= t e^t. \end{aligned}$$

Solution. $y(t) = c_1 t + c_2 e^t t - 6 t^2$.

3.6 - 17. Verify that the given functions y_1 and y_2 satisfy the corresponding homogeneous equation; then find a particular solution of the given nonhomogeneous equation.

$$\begin{aligned} x^2 y'' - 3x y' + 4y &= x^2 \ln x, \quad x > 0; \\ y_1(x) &= x^2, \\ y_2(x) &= x^2 \ln x. \end{aligned}$$

Solution. $y(x) = c_1 x^2 + c_2 x^2 \ln x + \frac{1}{6} x^2 \ln^3 x$.

3.7 - 7. A mass weighting 3 lb stretches a spring 3 in. If the mass is pushed upward, contracting the spring a distance of 1 in, and then set in motion with a downward velocity of 4 ft/s, and if there is no damping, find the position u of the mass at any time t. Determine the frequency, period, amplitude, and phase of the motion.

Solution. (note that 1 ft = 12 in)

Since the motion of mass attached to an ideal spring satisfies the equation (Newton's 2nd Law)

$$\ddot{x} = -\frac{k}{m} x,$$

where x is the displacement from equilibrium position, m is the weight of the mass, and k is a constant.

Consider the force balance when the mass is static, we have

$$\begin{aligned} -m g &= k u_s \\ k &= -\frac{m g}{u_s}, \end{aligned}$$

where g is the gravity acceleration constant, $u_s = -3$ is the static position of the mass. Thus $x = u - u_s$, and

$$\begin{aligned} \ddot{x} &= \frac{g}{u_s} x, \\ x(0) &= u(0) - u_s = 1 - (-3) = 4, \\ \dot{x}(0) &= -48. \end{aligned}$$

Note that the equation above has nothing to do with m. Take $g=10$ for simplicity, we have

$$x(t) \;=\; 4\cos\left(\sqrt{\frac{10}{3}}\,t\right) - 24\sqrt{\frac{6}{5}}\sin\left(\sqrt{\frac{10}{3}}\,t\right).$$

And by $x=u-u_s$,

$$\begin{aligned} u(t) &= x(t)+u_s \\ &= 4\cos\left(\sqrt{\frac{10}{3}}\,t\right) - 24\sqrt{\frac{6}{5}}\sin\left(\sqrt{\frac{10}{3}}\,t\right) - 3. \end{aligned}$$

Then the angular speed $\omega=\sqrt{\frac{10}{3}}$, frequency

$$f \;=\; \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{10}{3}} \quad (s^{-1}).$$

The period

$$T \;=\; \frac{1}{f} = 2\pi\sqrt{\frac{3}{10}} \quad (s).$$

If the canonical form is $x(t)=A\cos(\omega t+\Phi_0)$, the amplitude

$$A \;=\; \sqrt{4^2+\left(24\sqrt{\frac{6}{5}}\right)^2} = 4\sqrt{\frac{221}{5}} \quad (\text{in}).$$

The phase of motion

$$\Phi(t) \;=\; \sqrt{\frac{10}{3}}\,t + \arctan\left(\frac{24\sqrt{\frac{6}{5}}}{4}\right) = \sqrt{\frac{10}{3}}\,t + \arctan\left(6\sqrt{\frac{6}{5}}\right) \quad (\text{rad}).$$

Remark 1. These quantities can also be obtained using energy conservation law.

3.7 - 10. A mass weighting 16 lb stretches a spring 3 in. The mass is attached to a viscous damper with a damping constant of 2 lb·s/ft. If the mass is set in motion from its equilibrium position with a downward velocity of 6 in/s, find its position u of the mass at any time t. Plot u versus t. Determine when the mass first returns to its equilibrium position. Also find the time τ such that $|u(t)|<0.01$ in for all $t>\tau$.

Solution. ?? dimension of damping constant should be newton-seconds per meter, or kilograms per second.

For the case with damping,

$$\ddot{x} \;=\; -\frac{k}{m}\,x - \frac{c}{m}\,\dot{x},$$

The corresponding quadratic polynomial

$$m\,\lambda^2 + c\,\lambda + k \;=\; 0.$$

Then $\Delta=c^2-4\,\lambda\,k$, and

- If $\Delta>0$, the system is called overdamped. An over-damped door-closer takes longer to close than a critically damped door does.

- If $\Delta=0$, it is called critical damping. A critically damped system converges to zero as fast as possible without oscillating (although overshoot can occur). An example of critical damping is the door-closer seen on many hinged doors in public buildings. The recoil mechanisms in most guns are also critically damped so that they return to their original position, after the recoil due to firing, in the least possible time.

- If $\Delta < 0$, the system is underdamped. In this situation, the system will oscillate at the natural damped frequency $\omega_1 = \omega_0 \sqrt{1-\zeta^2}$, where $\zeta = \frac{c}{2\sqrt{m\,k}}$ is called the damping ratio. To continue the analogy, an underdamped door closer would close quickly, but would hit the door frame with significant velocity, or would oscillate in the case of a swinging door.

3.8 - 6. A mass weighting 5 kg stretches a spring 10 cm. The mass is acted on by an external force of $10\sin(t/2)$ N (newtons) and moves in a medium that imparts a viscous force of 4 N when the speed of the mass is 8 cm/s. If the mass is set in motion from its equilibrium position with an initial velocity of 3 cm/s, formulate the initial value problem describing the motion of the mass.

Solution. The initial value problem is ($g = 10m/s^2$)

$$\begin{aligned} 5\,\ddot{x} &= -500\,x + 10\sin(t/2) - 50\,\dot{x}, \\ x(0) &= 0, \\ \dot{x}(0) &= 0.03. \end{aligned}$$

Remark 2. For a driven harmonic oscillator, Newton's second law takes the form

$$m\,\ddot{x} = -k\,x + F(t) - c\,\dot{x}.$$

This can be solved using variation of parameters from the undriven equation, which is homogeneous.

If $\zeta \ll 1$, the solution can be expressed as damped sinusoidal oscillations,

$$z(t) = A\,e^{-\zeta\omega_0 t}\sin\Big(\sqrt{1-\zeta^2}\omega_0 t + \Phi\Big).$$

3.8 - 8. (Continued from 3.8 - 6.)

(a) Find the solution of the initial value problem in Problem 6.

(b) Identify the transient and steady state parts of the solution.

(c) Plot the graph of the steady state solution.

(d) If the given external force is replaced by a force of $2\cos\omega t$ of frequency ω, find the value of ω for which the amplitude of the forced response is maximum.

Solution. From the equation

$$\ddot{x} = -100\,x + 2\sin(t/2) - 10\,\dot{x},$$

one can solve the general solution

$$x(t) = -\frac{160}{159601}\cos\Big(\frac{t}{2}\Big) + \frac{3192}{159601}\sin\Big(\frac{t}{2}\Big) + C_1\,e^{-5t}\cos\big(5\sqrt{3}t\big) + C_2\,e^{-5t}\sin\big(5\sqrt{3}t\big).$$

Then using initial conditions, the solution of the IVP can be obtained.

For another external force, the equation is

$$\ddot{x} = -100\,x + \frac{2}{5}\cos\omega t - 10\,\dot{x},$$

and the general solution is

$$x(t) = -\frac{2\,(\omega^2-100)}{5(\omega^4-100\omega^2+10000)}\cos(\omega t) + \frac{4\omega}{\omega^4-100\omega^2+10000}\sin(\omega t) + C_1\,e^{-5t}\cos\big(5\sqrt{3}t\big) + C_2\,e^{-5t}\sin\big(5\sqrt{3}t\big).$$

The limiting amplitude is

$$\begin{aligned} A_\infty &= \left[\left(\frac{2\,(\omega^2-100)}{5(\omega^4-100\omega^2+10000)}\right)^2+\left(\frac{4\omega}{\omega^4-100\omega^2+10000}\right)^2\right]^{1/2} \\ &= \frac{2\sqrt{\omega^4-100\omega^2+10000}}{5(\omega^4-100\omega^2+10000)} \\ &= \frac{2}{5\sqrt{\omega^4-100\omega^2+10000}} \end{aligned}$$

which attains it maximum at

$$\omega^2 = 50.$$

4.1 - 5. Deternim intervals in which solutions are sure to exist.

$$(x-1)\,y^{(4)}+(x+1)y''+3(\tan x)y = 0.$$

Solution. $\left(-\frac{\pi}{2}+k\pi, \frac{\pi}{2}+k\pi\right)$ for $k\neq 0$, and $\left(-\frac{\pi}{2}, 1\right)$, $\left(1, \frac{\pi}{2}\right)$.

4.1 - 6. Deternim intervals in which solutions are sure to exist.

$$(x^2-4)y^{(6)}+x^3y'''+9y = 0.$$

Solution. $(-\infty, -2)$, $(-2, 2)$, $(2, +\infty)$.

4.1 - 10. Determin whether the given functions are linearly dependent or linearly independent. If they are linearly dependent, find a linear relation among them.

$$\begin{aligned} f_1(t) &= 2t-3, \\ f_2(t) &= t^3+2, \\ f_3(t) &= 2\,t^2-t, \\ f_4(t) &= t^2+t+1. \end{aligned}$$

Solution. Note: you only need to examin the 1,3,4

$$\begin{vmatrix} 0 & 2 & -3 \\ 2 & -2 & 0 \\ 1 & 1 & 1 \end{vmatrix} \neq 0.$$

So they are linearly independent.

4.1 - 19. Let the linear differential operator L be defined by

$$L[y] = a_0y^{(n)}+a_1y^{(n-1)}+\cdots+a_ny,$$

where $a_0a_1, \ldots, a_n$ are real constants.

(a) Find $L[t^n]$.

(b) Find $L[e^{\mathrm{rt}}]$.

(c) Determine four solutions of the equation $y^{(4)}-5y''+4y=0$. Do you think the four solutions form a fundamental set of solutions? Why?

Solution. For c, one has a polynomial

$$\begin{aligned}\lambda^4 - 5\lambda^2 + 4 &= 0\\ (\lambda-1)(\lambda-2)(\lambda+1)(\lambda+2) &= 0.\end{aligned}$$

The corresponding solutions form a fundamental set of solutions because they are linearly independent.

4.1 - 20. In this problem we show how to generalize Theorem 3.2.7 (Abel's theorem) to higher order equations. We first outline the procedure for the third order equation.

$$y''' + p_1(t)y'' + p_2(t)y' + p_3(t)y = 0.$$

Let y_1, y_2 and y_3 be solutions of this equation on an interval I.

(a) If $W = W(y_1, y_2, y_3)$, show that

$$W' = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1''' & y_2''' & y_3''' \end{vmatrix}.$$

Hint: The derivative of a 3-by-3 determinant is the sum of three 3-by-3 determinants obtained by differentiating the first, second, and third rows, respectively.

(b) Substitude for y_1''', y_2''' and y_3''' from the differential equation; multiply the first row by p_3, multiply the second row by p_2, and add these to the last row to obtain

$$W' = -p_1(t)\, W.$$

(c) Show that

$$W(y_1, y_2, y_3)(t) = c \exp\left[-\int p_1(t)\, dt\right].$$

It follows that W is either always zero or nowhere zero on I.

(d) Generalize this argument to the nth order equation

$$y_{(n)} + p_1(t)y^{(n-1)} + \cdots + p_n(t)y = 0.$$

with solutions $y_1, ..., y_n$. That is, establish Abel's formula

$$W(y_1, ..., y_n)(t) = c \exp\left[-\int p_1(t)\, dt\right]$$

for this case.

Solution. This is a derivation of Liouville's formula. You can find the proof on its Wikipedia page.

4.2 - 25. Find the general solution of the given differential equation.

$$18y''' + 21y'' + 14y' + 4y = 0.$$

Solution. $y(x) = c_1\, e^{-x/2} + c_2\, e^{-x/3} \sin\left(\frac{x}{\sqrt{3}}\right) + c_3\, e^{-x/3} \cos\left(\frac{x}{\sqrt{3}}\right)$.

Hint: use the following rational root theorem to find a root to help you solve a high order polynomial equation: If $x = p\,/\,q$ is a root, then p is an integer factor of the constant term and q is an integer factor of the leading coefficient. (The greast common divisor of p and q should be 1).

4.2 - 34. Find the solution of the given initial value problem, and plot its graph. How does the solution behave as $t \to \infty$?

$$\begin{aligned} 4y''' + y' + 5y &= 0; \\ y(0) &= 2, \\ y'(0) &= 1, \\ y''(0) &= -1. \end{aligned}$$

Solution. $y(t) = \frac{1}{13} e^{-t}\left(3\, e^{3t/2} \sin(t) + 24\, e^{3t/2} \cos(t) + 2\right)$.

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GNUplot] plot [0:20] (1.0/13)*exp(-x)*(3*exp(3*x/2)*sin(x)+24*exp(3*x/2)*cos(x)+2)
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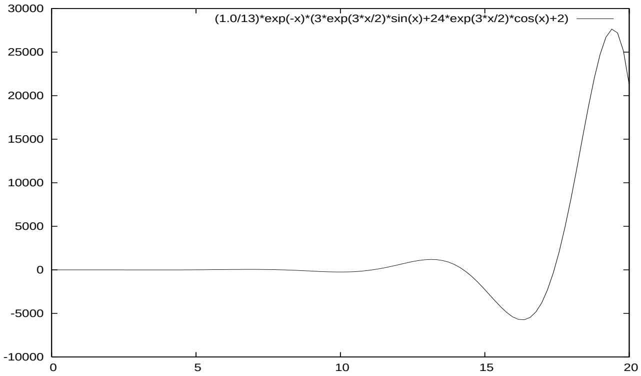

4.3 - 12. Find the solution of the given initial value problem. Then plot a graph of the solution.

$$\begin{aligned} y^{(4)} + 2y''' + y'' + 8y' - 12y &= 12 \sin t - e^{-t}; \\ y(0) &= 3, \\ y'(0) &= 0, \\ y''(0) &= -1, \\ y'''(0) &= 2. \end{aligned}$$

Solution. $y(t) = \frac{1}{520}(73e^{-3t} + 26e^{-t} + 1053\, e^{t} - 416 \sin(t) - 196 \sin(2t) - 208 \cos(t) + 616 \cos(2t))$.

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GNUplot] plot [0:5] (73*exp(-3*x)+26*exp(-x)+1053*exp(x)-416*sin(x)-196*sin(2*x)-
         208*cos(x)+616*cos(2*x))/520
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