Anomalous Magnetic Moment of Charged Leptons

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Abstract

In this report, we investigate the theoretical calculation of the anomalous magnetic moment of charged leptons within the framework of Quantum Electrodynamics (QED). We begin by revisiting the classical concept of magnetic moments and the g-factor as predicted by the Dirac equation. However, experimental observation suggests a deviation from the value predicted by Dirac. QED explains the anomaly as suggested by the experiments by taking radiative corrections into account. We specifically calculate the one-loop correction to the magnetic moment, famously known as Schwinger's term, which contributes an amount of α/π to the anomaly. The anomalous magnetic moment, defined as $a_l = \frac{g_l - 2}{2}$, quantifies this deviation from the Dirac prediction of g equal to 2.

Contents

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1 Introduction

We begin by examining a classical scenario in which a charged particle orbits a closed path with radius r and velocity v. The magnetic moment is described by,

$$\vec{\mu} = \frac{e}{2m}\vec{L},$$

where e represents the charge of the particle, m its mass, and $\vec{L}$ its angular momentum. This magnetic moment determines the torque and energy of the charged particle within a magnetic field. However, early 20th-century experiments by Stern and Gerlach introduced a new quantum number, the spin, indicating that the classical calculation of the magnetic moment underestimates the actual value for electrons by a factor of 2. This leads us to propose a general expression for the magnetic moment of a charged particle with spin S, represented as

$$\vec{\mu} = \frac{ge}{2m}\vec{S},$$

with $g = 2$, in accordance with experimental findings. The Dirac equation, formulated in 1932, accurately predicted the g-factor to be 2, matching excellently with experimental observations. The Dirac equation for a charged spinor in an electromagnetic field is given by,

$$(i\not{D} - m)\psi = 0 \tag{1}$$

Where, $D_\mu = \partial_\mu + ieA_\mu$ is the covariant derivative and A_μ is the photon field. Multiplying equation (1) by $(i\not{D} + m)$ gives, $(\not{D}^2 + m^2)\psi = 0$. Where,

$$\not{D}^2 = D_\mu^2 + \frac{e}{2m}F_{\mu\nu}\sigma^{\mu\nu} \tag{2}$$

Here, $F_{\mu\nu}$ is the field strength tensor and $\sigma_{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$. The term $\frac{e}{2m}F_{\mu\nu}\sigma^{\mu\nu}$ encodes the magnetic moment of charged spinors. In the momentum space, $(\not{D}^2 + m^2)\psi = 0$ can be written as, [5]

$$\frac{(H - e\phi_0)^2}{2m}\psi = \left(\frac{m}{2} + \frac{(\vec{p} - e\vec{A})^2}{2m} - 2\frac{e}{2m}\vec{B}\cdot\vec{S} \pm i\frac{e}{m}\vec{E}\cdot\vec{S}\right)\psi \tag{3}$$

In this context, the magnetic moment term $\vec{\mu} = 2\frac{e}{2m}\vec{B}$ obtained from the above equation predicts $g = 2$. It is crucial to highlight that the term associated with the magnetic moment is expressed as $F_{\mu\nu}\sigma^{\mu\nu}$. In 1947, the experimental results from P. Kusch and H.M. Foley on the Zeeman spectra of gallium atoms challenged the Dirac theory's prediction of $g = 2$. While Dirac's theory predicted a gyromagnetic ratio exactly equal to 2, the measured value of g was approximately $2(1.00119 \pm 0.00005)$. Dirac's theory fails to explain this anomaly which can be quantified as $a_e = (g-2)/2 = 0.00119 \pm 0.00005$. In the next section, we will see if Quantum Electrodynamics (QED) can solve or rescue us from this anomaly. [4]

2 Radiative Corrections

In QED scattering problems, our primary focus is typically on tree-level diagrams, which involve no loops and are derived from leading-order perturbation theory. For instance, consider the process, $e^-(p_1)A_\mu(q) \longrightarrow e^-(p_2)$, with $A_\mu(x)$ as a fixed classical potential and two spinor states $\bar{u}(p_2)$ and $u(p_1)$. At tree level, the S-matrix element describing the scattering from this field can be expressed as,

$$i\mathcal{M}_0(2\pi)\delta(p_2^0 - p_1^0) = -ie\bar{u}(p_2)\gamma^\mu u(p_1)\tilde{A}_\mu(q) \tag{4}$$

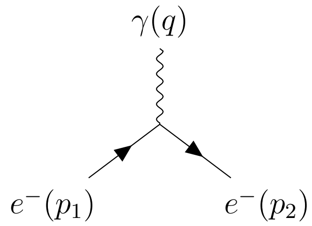

Figure 1: Tree-level Feynman diagram [3]

where photon momentum is, $q^\mu = p_2^\mu - p_1^\mu$ and $\tilde{A}_\mu(q)$ is the Fourier transform of $A_\mu(x)$. The leading order Feynman diagram 3 contains g = 2. We anticipated a term like $F_{\mu\nu}\sigma^{\mu\nu}$ to appear for contribution to the magnetic moment. In momentum space, the term would look like, $q_\nu\sigma^{\mu\nu}$. Considering the photon to be off-shell and spinors on-shell, we use Gordon's identity (see Appendix) to manipulate the term γ_μ, appearing at the tree level.

$$\mathcal{M}_0 = -e\left(\frac{p_1^\mu + p_2^\mu}{2m}\right)\bar{u}\left(p_2\right)u\left(p_1\right) - \frac{e}{2m}i\bar{u}\left(p_2\right)q_\nu\sigma^{\mu\nu}u\left(p_1\right) \tag{5}$$

We indeed get a spin dependent term, $q_\nu\sigma^{\mu\nu}$ which gives the magnetic moment with g = 2. However, as mentioned earlier, we concluded the introduction by highlighting the deviation of g from 2. Hence, it becomes necessary to include higher-order corrections to investigate such deviations. Therefore, we must determine how the coefficient of $q_\nu\sigma^{\mu\nu}$ is altered at the loop level.

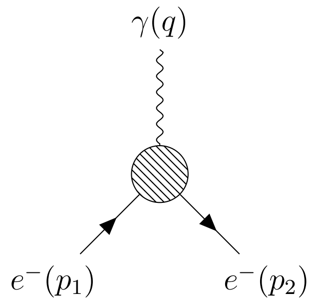

Figure 2: The most general Vertex correction [3]

The blob in the Feynman diagram 2 denotes loop corrections at all orders. This contribution results in a term proportional to Γ_μ, altering the matrix element,

$$i\mathcal{M}(2\pi)\delta(p_2^0 - p_1^0) = -ie\bar{u}(p_2)\Gamma^\mu u(p_1)\tilde{A}_\mu(q) \tag{6}$$

We know that 4 transforms like a vector under Lorentz transformation. Therefore, we can pinpoint the form of Γ_μ as a linear combination of the following terms:

$$\Gamma_\mu = A\gamma_\mu + B(p_{2\mu} + p_{1\mu}) + C(p_{2\mu} - p_{1\mu}) \tag{7}$$

Where, the scalar coefficients A, B, and C are a function of non-trivial sclar, $q^2 = 2m^2 - 2p_1 \cdot p_2$. Next, we use the Ward identity, $q_\mu\Gamma^\mu = 0$ along with the fact that $\not{p_1}u(p_1) = mu(p_1)$ and $\bar{u}(p_2)\not{p_2} = m\bar{u}(p_2)$.

$$\bar{u}(p_2)q^\mu\Gamma_\mu u(p_1) = \bar{u}(p_2)\left[A(p_2 - p_1)^\mu\gamma_\mu + B(p_2 - p_1)^\mu(p_2 + p_1)_\mu + C(p_2 - p_1)^\mu(p_2 - p_1)_\mu\right]u(p) = 0 \tag{8}$$

Where the terms involving A and B vanishes implying that the coefficient C must be set to zero as $q^2 = (p_2 - p_1)^2 \neq 0$. The equation (7) is then modified by using Gordon's identity by replacing the term containing $p_{2\mu} + p_{1\mu}$ by $q_\nu\sigma^{\mu\nu}$. The final form of Γ_μ can be written as,

$$\Gamma_\mu(q^2) = \gamma^\mu F_1(q^2) + i\frac{\sigma^{\mu\nu}}{2m}q^\nu F_2(q^2) \tag{9}$$

F_1 and F_2 are known as Form factors and are functions of q^2. The leading order diagram implies, $F_1 = 1$ and $F_2 = 0$. The F_1 term modifies the coupling that is it renormalizes the electric charge. On the other hand, the F_2 term, which contains $\sigma^{\mu\nu}$, corresponds to the anomalous magnetic moment part, which is of particular interest. As $F_2 = 0$ yields $g = 2$ (tree level diagram), we infer that F_2 modifies the moment at the scale associated with q, resulting in $g \to 2 + 2F_2(q^2)$. To verify this let us consider the vector potential of the form, $A_\mu(x) = (0, \vec{A}(x))$. The scattering amplitude therefore becomes,

$$iM = ie\bar{u}\left(p_2\right)\left[\gamma^i F_1(q^2) + i\frac{\sigma^{i\nu}q_\nu}{2m}F_2(q^2)\right]u(p_1)\tilde{A}^i(\vec{q}) \tag{10}$$

Let us evaluate this term by explicitly putting the non-relativistic form of spinors $u(p)$, $(E_2 \approx E_1 \approx m)$

$$u(p) = \begin{pmatrix} \sqrt{\frac{E+m}{2m}}\phi \\ \frac{\vec{\sigma}\cdot\vec{p}}{\sqrt{2m(E+m)}}\phi \end{pmatrix} \approx \begin{pmatrix} \phi \\ \frac{\vec{\sigma}\cdot\vec{p}}{2m}\phi \end{pmatrix} \tag{11}$$

Following this substitution and applying the identity $\sigma^i\sigma^j = \delta^{ij} + i\epsilon^{ijk}\sigma^k$ to the F_1 term,

$$\bar{u}\left(p_2\right)\gamma^i u(p_1) = 2m\phi^{2\dagger}\left[\sigma^i\frac{\vec{p_2}\cdot\vec{\sigma}}{2m} + \frac{\vec{p_1}\cdot\vec{\sigma}}{2m}\sigma^i\right]\phi^1 \tag{12}$$

$$\bar{u}\left(p_2\right)\gamma^i u(p_1) = 2m\phi^{2\dagger}\left[(p_2 + p_1)^i - i\epsilon^{ijk}q^j\frac{\sigma^k}{2m}\right]\phi^1 \tag{13}$$

The first term is a spin-independent term with a linear combination of p_1 and p_2 while the later term contains the spin-dependent term and is of our interest. In the limit $q \to 0$, the F_1 term boils down to,

$$\bar{u}\left(p_2\right)\gamma^i F_1(q^2)u(p_1) = 2m\phi^{2\dagger}\left[-i\epsilon^{ijk}q^j\frac{\sigma^k}{2m}F_1(0)\right]\phi^1 \tag{14}$$

The second term F_2 in the scattering amplitude in the limit $q \to 0$ can be written as,

$$\bar{u}\left(p_2\right)\left[i\frac{\sigma^{i\alpha}}{2m}q_\alpha F_2(0)\right]u(p_1) = 2m\phi^{2\dagger}\left[-i\epsilon^{ijk}q^j\frac{\sigma^k}{2m}F_2(0)\right]\phi^1 \tag{15}$$

Combining both these terms gives,

$$iM = 2ime\phi^{2\dagger}\left[\frac{-i}{2m}\epsilon^{ijk}q^j\sigma^k\left(F_1(0) + F_2(0)\right)\right]\phi^1\tilde{A}^i(\vec{q}) \tag{16}$$

In position space, the magnetic field associated with a vector potential $\vec{A}$ can be expressed as $\vec{B} = \vec{\nabla} \times \vec{A}$. Its Fourier transform yields the magnetic field in momentum space, given by $\tilde{B}^k(\vec{q}) = -i\epsilon^{ijk}q^j\tilde{A}^k(\vec{q})$. Therefore,

$$iM = 2ime\phi^{2\dagger}\left[\frac{-1}{2m}\sigma^k\left(F_1(0) + F_2(0)\right)\right]\phi^1\tilde{B}^k(\vec{q}) \tag{17}$$

Comparing the above equation to the classical expression for electron scattering from a potential $V(\vec{x}) = -\langle\vec{\mu}\rangle \cdot \vec{B}(\vec{x})$, where $\langle\vec{\mu}\rangle$ represents the average magnetic moment we get,

$$< \vec{\mu} >= \frac{e}{m}\left[F_1(0) + F_2(0)\right]\phi^{2\dagger}\vec{\sigma}\phi^1 \equiv g\frac{e}{2m}\vec{S} \tag{18}$$

Where, $\phi^{2\dagger}\vec{\sigma}\phi^1 = \vec{S}$. The formula for gyromagnetic ratio or Lande g factor (g) reduces to,

$$g = 2\left[F_1(0) + F_2(0)\right] \tag{19}$$

The leading order contribution sets $F_1(0)$ to 1 as already discussed. Therefore we have,

$$\boxed{g = 2 + 2F_2(0)} \tag{20}$$

The Dirac equation and the tree-level term in QED lead to the prediction $g = 2$, implying $F_2 = 0$. However, higher-order diagrams in QED show a non-zero value for F_2, causing a deviation from $g = 2$. In the upcoming section, we will compute the one-loop contribution to $F_2(0)$, also referred to as the Schwinger term, to address the anomalous magnetic moment.

3 Schwinger's term

The three one-loop Feynman diagrams shown below give terms proportional to γ^μ merely correcting the propagators. Therefore these diagrams contribute to F_1 but do not affect the F_2 term.

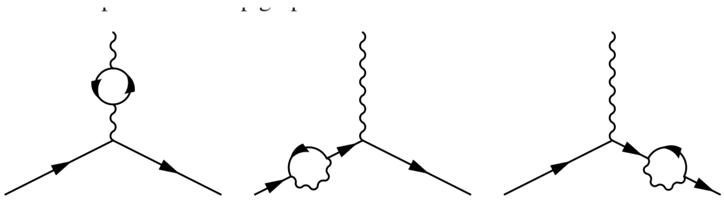

Figure 3: 1 - loop Feynman diagrams

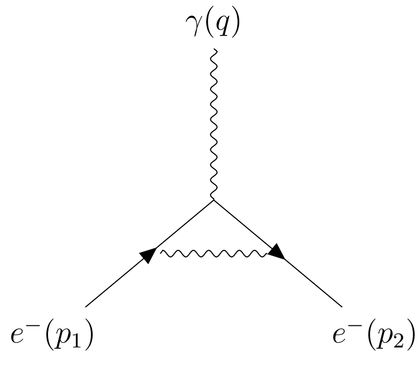

Figure 4: One loop vertex correction

The above Feynman diagram gives a non-zero contribution to the F_2 term and hence is evaluated further. Let us consider The vertex correction is denoted as $\Gamma_1^\beta = \gamma^\beta + \delta\Gamma_1^\beta$, indicating an additional contribution of $\delta\Gamma_1^\beta$. In this context, the subscript "1" signifies the one-loop vertex correction.

$$\begin{aligned}
&\bar{u}\left(p_2\right)\delta\Gamma^\beta u(p_1) \\
&= \int \frac{d^4h}{(2\pi)^4}\frac{-ig_{\eta\alpha}}{(h-p_1)^2 + i\epsilon}\bar{u}\left(p_2\right)\left(-i\epsilon\gamma^\alpha\right)\frac{i(\not{h}' + m)}{h'^2 - m^2 + i\epsilon}\gamma^\beta\frac{i(\not{h} + m)}{h^2 - m^2 + i\epsilon}\left(-i\epsilon\gamma^\eta\right)u(p_1) \\
&= 2ie^2\int\frac{d^4h}{(2\pi)^4}\bar{u}\left(p_2\right)\frac{\gamma^\alpha(\not{h}' + m)\gamma^\beta(\not{h} + m)\gamma^\eta}{\left[(h-p_1)^2 + i\epsilon\right]\left[h'^2 - m^2 + i\epsilon\right]\left[h^2 - m^2 + i\epsilon\right]}u(p_1) \\
&= 2ie^2\int\frac{d^4h}{(2\pi)^4}\bar{u}\left(p_2\right)\frac{\not{h}'\gamma^\beta\not{h} + m^2\gamma^\beta - 2m\left(h + h'\right)^\beta}{\left[(h-p_1)^2 + i\epsilon\right]\left[h'^2 - m^2 + i\epsilon\right]\left[h^2 - m^2 + i\epsilon\right]}u(p_1)
\end{aligned} \tag{21}$$

To tackle this integral, it is essential to simplify both the numerator and the denominator. Let us begin by addressing the denominator and employing a technique called Feynman parametrization (see Appendix) for this purpose. We can write,

$$\frac{1}{\left[(h-p_1)^2+i\epsilon\right]\left[h'^2-m^2+i\epsilon\right]\left[h^2-m^2+i\epsilon\right]}=\int_0^1 dxdydz\delta(x+y+z-1)\frac{2}{F^3} \tag{22}$$

Where,

$$F=x\left(h^2-m^2\right)+y\left(h'^2-m^2\right)+z(h-p_1)^2+(x+y+z)i\epsilon \tag{23}$$

From the delta function in (22), $x+y+z=1$ and $h'=h+q$ from four momentum conservation. Therefore,

$$F=h^2+2h\cdot(yq-zp_1)+yq^2+zp_1^2-(x+y)m^2+i\epsilon \tag{24}$$

We can write F in terms of two parameters defined as follows, $r\equiv h+yq-zp_1$ and $\Delta=-xyq^2+(1-z)^2m^2$.

$$F=r^2-\Delta+i\epsilon \tag{25}$$

In scattering processes, where $q^2<0$ therfore Δ becomes positive. After addressing the denominator, we rewrite the numerator of the equation using r^2 terms where applicable. Since F relies solely on the magnitude of r, symmetries yield the following identities.

$$\begin{gathered}\int\frac{d^4r}{(2\pi)^4}\frac{r^\mu}{F^3}=0\\ \int\frac{d^4r}{(2\pi)^4}\frac{r^\mu r^\nu}{QF^3}=\int\frac{d^4r}{(2\pi)^4}\frac{\eta^{\mu\nu}r^2}{F^3}\end{gathered} \tag{26}$$

Using these symmetry arguments, the numerator simplifies to,

$$\begin{aligned}&\bar{u}\left(p_2\right)\left[\not{h}'\gamma^\beta\not{h}+m^2\gamma^\beta-2m\left(h+h'\right)^\beta\right]u(p_1)\\ &=\bar{u}\left(p_2\right)\left[-\frac{1}{2}\gamma^\beta r^2+(-y\not{q}+z\not{p}_1)\gamma^\beta((1-y)\not{q}+z\not{p}_1)+m^2\gamma^\beta-2m\left((1-2y)q^\beta+2zp_1^\beta\right)\right]u(p_1)\end{aligned} \tag{27}$$

We further simplify the numerator by using the following identities, $\not{p}_1\gamma^\beta+\gamma^\beta\not{p}_1=2p_1^\beta, \not{p}_1u(p_1)=mu(p_1)$ and $\bar{u}\left(p_2\right)\not{p}_2=m\bar{u}\left(p_2\right)$ and $\{\gamma^\alpha,\gamma^\beta\}=2\eta^{\alpha\beta}$

$$\begin{aligned}&\bar{u}\left(p_2\right)\left[\not{h}'\gamma^\beta\not{h}+m^2\gamma^\beta-2m\left(h+h'\right)^\beta\right]u(p_1)\\ =&\bar{u}\left(p_2\right)\left[\gamma^\beta\left(-\frac{1}{2}r^2+(1-x)(1-y)q^2\right]u(p_1)\\ &+\bar{u}\left(p_2\right)\left[\left(\left(1-2z-z^2\right)m^2+m\left((p_2+p_1)^\beta\right)z(z-1)+m(z-2)(x-y)q^\beta\right)\right]u(p_1)\end{aligned} \tag{28}$$

The term with q^β vanishes by symmetry argument. We see a linear combination of initial and final state moments hence we make use of Gordon's identity to get terms involving $\sigma^{\mu\nu}$. The integral can be written as,

$$\begin{aligned}&\bar{u}\left(p_2\right)\delta\Gamma_1^\beta u(p_1)\\ &=2ie^2\int\frac{d^4r}{(2\pi)^4}\int_0^1 dxdydz\delta(x+y+z-1)\frac{2}{F^3}\\ &\quad\times\bar{u}\left(p_2\right)\left[\gamma^\beta\left(-\frac{1}{2}r^2+(1-x)(1-y)q^2+\left(\left(1-4z+z^2\right)m^2+i\frac{\sigma^{\beta\nu}}{2m}q_\nu\left(2m^2z(1-z)\right)\right)\right)u(p_1)\right]\end{aligned} \tag{29}$$

We can now extract the form factors $F_1(q^2)$ and $F_2(q^2)$ from (29).

$$\begin{aligned} F_1\left(q^2\right) =& 1 + 2ie^2 \int_0^1 dxdydz\delta(x+y+z-1) \\ &\times \int_{-\infty}^{\infty} \frac{d^4r}{(2\pi)^4}\frac{2}{F^3}\left(-\frac{1}{2}r^2 + (1-x)(1-y)q^2 + (1-4z+z^2)m^2\right) \end{aligned} \tag{30}$$

$$\boxed{F_2\left(q^2\right) = 2ie^2 \int_0^1 dxdydz\delta(x+y+z-1)\int_{-\infty}^{\infty}\frac{d^4r}{(2\pi)^4}\frac{2}{F^3}2m^2z\left(1-z\right)} \tag{31}$$

We notice that the integrals of $F_1(q^2)$ and $F_2(q^2)$ involve two kinds of integrands, with poles as $r_0 = -\sqrt{|\vec{r}|^2 + \Delta} + i\epsilon$ and $r_0 = \sqrt{|\vec{r}|^2 - \Delta} + i\epsilon$. This allows us to rotate the contour counterclockwise by 90° in the complex plane,

$$r^0 = ir_E^0; \vec{r} = \vec{r}_E$$

The above transformation is known as Wick's rotation (see Appendix). We now evaluate the term $F_2(q^2)$ at $q^2 = 0$.

$$F_2\left(q^2=0\right) = 4\frac{e^2}{(4\pi)^2}\int_0^1 dxdydz\delta(x+y+z-1)\frac{z(1-z)}{(1-z)^2} \tag{32}$$

$$\boxed{F_2\left(0\right) = \frac{\alpha}{\pi}\int_0^1 dz\int_0^{1-z} dy\frac{z}{1-z} = \frac{\alpha}{2\pi}} \tag{33}$$

Therefore recalling the expression of g we derived (20),

$$\boxed{g = 2 + 2F_2(0) = 2 + \frac{\alpha}{\pi} = 2.0023228} \tag{34}$$

It is worth noting that the QED contribution of g remains the same for all leptons. The significance of this outcome is such that it has been immortalized on Julian Schwinger's tombstone.

Before presenting the experimental results for the anomalous magnetic moment of electron and muon let us briefly discuss the contributions to the anomaly from different sectors of the Standard Model. The anomalous part is defined as, $a_l^{SM} = \frac{g_l - 2}{2}$, where l denotes lepton.

$$a_l^{SM} = a_l^{QED} + a_l^{EW} + a_l^{QCD} \tag{35}$$

The dominant and most accurately calculated contribution to the magnetic anomaly arises from quantum electrodynamics (QED) interactions involving only leptons (electrons, μ, and τ particles) and photons γ. The contribution from the electroweak interaction arises from the interaction with gauge bosons: Z, $W^{\pm}$, and γ. It is suppressed by a factor approximately proportional to the square of the lepton mass (m_l^2) divided by the mass of the W boson (M_W), compared to the QED contribution. The contribution from Quantum Chromodynamics (QCD) offers the highest uncertainty.

4 The experimental status: Triumph of QED

In the previous section, we computed the anomalous magnetic moment of charged leptons up to order α. However, calculations have been conducted up to order α^5, providing precise values for the electron's anomalous magnetic moment.

$$a_e^{th} = 0.001159652181643(764)$$

With the current measured experimental value, [2]

$$a_e^{exp} = 0.00115965218059(13)$$

The agreement between the quantum electrodynamics (QED) prediction and the experimentally measured value exceeds 10 significant figures for the magnetic moment of the electron. This level of agreement establishes QED as one of the most precisely confirmed predictions in the history of physics.

The anomalous magnetic moment of the muon (a_μ) is determined by three main components: (a_μ^{QED}), (a_μ^{EW}), and (a_μ^{hadron}).

$$a_\mu^{th} = 0.00116591804(51)$$

The E821 Experiment at Brookhaven National Laboratory reported an average value of $a_\mu = 0.0011659209(6)$, while the ongoing "Muon g-2" experiment at Fermilab, with results announced on August 10, 2023, from the first three years of data-taking, revealed a new world average of $a_\mu = 0.00116592059(22)$. The experimental result deviates from the Standard Model theory prediction by 5.1 σ suggesting tantalizing hints of Physics Beyond the Standard Model. [1]

The tau lepton, being the most short-lived of all leptons, presents challenges in accurately measuring its experimental value. According to the Standard Model, the anomalous magnetic dipole moment of the tau particle is predicted to be $a_\tau = 0.00117721(5)$, with experimental constraints placing the most stringent bounds on a_τ as $-0.052 < a_\tau < +0.013$.

5 BSM implications

The 5 σ discrepancy observed between the theoretical and experimental predictions for the muon $g-2$ hints at physics Beyond the Standard Model (BSM). To explore this, we turn to a supersymmetric model where each fermion is paired with a scalar partner and each gauge boson has a corresponding fermionic partner. For example, the electron's partner is the selectron ($\tilde{e}$), the muon's partner is the smuon ($\tilde{\mu}$), and the photon's partner is the photino ($\tilde{\gamma}$). Introducing additional terms into the Lagrangian ($\mathcal{L}_{\mathrm{SUSY}}$), which include terms for the scalar partners, enables the inclusion of kinetic and mass terms for the selectron and smuon, as well as interaction terms with gauge fields. Both the smuon and selectron carry an electric charge of -1. The inclusion of one-loop Feynman diagrams involving smuon and photino can contribute to the anomaly offering a plausible explanation for the deviation.

6 Conclusion

Our exploration began with the classical prediction of a gyromagnetic ratio g of 1, transitioning to Dirac's theory which proposed g = 2. However, Dirac's theory was unable to explain the experimental results at the time. The radiative corrections from QED then rescued us by explaining the anomaly as observed in the experiments. We looked at the scattering of electron from an electromagnetic field and calculated the one-loop correction term contributing to the anomalous part α/π. The derivation of this anomalous moment by Schwinger, Feynman, and Tomonaga in 1948, along with its confirmation through experimental data, marked a notable achievement for QED. The anomalous magnetic moment of the electron aligns with experimental values to more than ten significant figures. However, the anomalous magnetic moment of the muon deviates from the Standard Model (SM) predictions by a 5 σ margin, offering tantalizing hints of potential physics Beyond the Standard Model.

A Appendix A

A.1 Gordon's Identity

For on-shell spinors, we show that,

$$\bar{u}(p_2)\gamma^\mu u(p_1) = \bar{u}(p_2)\left[\frac{p_2^\mu + p_1^\mu}{2m} + \gamma^\mu - \frac{p_2^\mu + p_1^\mu}{2m}\right]u(p_1) \tag{36}$$

$$\sigma^{\mu v} = \frac{i}{2}\left[\gamma^\mu, \gamma^v\right]$$

The second term in RHS of the Gordon identity can be evaluated as follows:

$$\begin{aligned}
&\bar{u}(p_2)\frac{i\sigma^{\mu v}q_v}{2m}u(p_1) \\
&= \frac{i}{2m}\cdot\frac{i}{2}\bar{u}(p_2)\left[\gamma^\mu\gamma^v - \gamma^v\gamma^\mu\right]\left[p_{2v} - p_1\right]u(p_1) \\
&= \frac{i}{2m}\cdot\frac{i}{2}\bar{u}(p_2)\left[\left[\gamma^\mu\gamma^v - \gamma^v\gamma^\mu\right]p_{2v} - \left[\gamma^\mu\gamma^v - \gamma^v\gamma^\mu\right]p_1\right]u(p_1) \\
&= \frac{i}{2m}\cdot\frac{i}{2}\bar{u}(p_2)\left[\left[2g^{\mu v} - 2\gamma^v\gamma^\mu\right]p_{2v} - \left[2\gamma^\mu\gamma^v - 2g^{\mu v}\right]p_1\right]u(p_1)
\end{aligned} \tag{37}$$

Using the Dirac equation:

$$\begin{aligned}
&= \frac{i}{2m}\frac{i}{2}\bar{u}(p_2)\,2\left[\left[p^\mu - m\gamma^\mu\right] - \left[m\gamma^\mu - p^\mu\right]\right]u(p_1) \\
&= -\frac{1}{2m}\bar{u}(p_2)\left[\left[p_2^\mu + p_1^\mu\right] - 2m\gamma^\mu\right]u(p_1) \\
&= \bar{u}(p_2)\left[\gamma^\mu - \frac{p_2^\mu + p_1^\mu}{2m}\right]u(p_1)
\end{aligned}$$

Inserting this into the Gordon identity proves it:

$$\begin{aligned}
\bar{u}(p_2)\gamma^\mu u(p_1) &= \bar{u}(p_2)\left[\frac{p_2^\mu + p_1^\mu}{2m} + \frac{i\sigma^{\mu v}q}{2m}\right]u(p_1) \\
&= \bar{u}(p_2)\left[\frac{p_2^\mu + p_1^\mu}{2m} + \gamma^\mu - \frac{p_2^\mu + p_1^\mu}{2m}\right]u(p_1) = \bar{u}(p_2)\gamma^\mu u(p_1)
\end{aligned}$$

A.2 Feynman Parameters

Feynman Parametrization: We can express a fraction of type $\frac{1}{CD}$, in the following way,

$$\frac{1}{CD} = \int_0^1 \frac{dx}{(Cx + D(1-x))^2} = \int_0^1 dxdy\delta(x+y-1)\frac{1}{(Cx+Dy)^2}$$

Where x and y are called Feynman parameters and are imposed to the Dirac delta condition. This can be generalized to

$$\frac{1}{C_1C_2\ldots C_n} = \int_0^1 dx_1dx_2\ldots dx_n\delta\left(\Sigma_i x_i - 1\right)\frac{(n-1)!}{\left(C_1x_1\ldots C_nx_n\right)^n}$$

Where $x_1, x_2, \ldots, x_n$ are the Feynman parameters.

A.3 Wick Rotation

Wick rotation is called a rotation because multiplying a complex number by i corresponds to rotating the vector representing that number counterclockwise by an angle of $\pi/2$ around the origin. To show the importance of Wick rotation we present an example from relativity. The Minkowski metric in natural units is given by,

$$ds^2 = -dt^2 + dr^2 (Minkowski) \tag{38}$$

$$ds^2 = d\tau^2 + dr^2 (Eucledian) \tag{39}$$

By allowing the time coordinate t to assume imaginary values, the Minkowski metric transforms into an Euclidean metric. Substituting $t = -i\tau$ in a problem formulated in Minkowski space sometimes transforms into a problem in real Euclidean, simplifying the solution process. The Euclidean metric therefore has a "spherical" symmetry.

The integrals $F_1(q^2)$ and $F_2(q^2)$ contain two kinds of r integrands. Taking the most general form of F in the denominator as, $F = (r^2 - \Delta + i\epsilon)^j$ where j = 1, 2, 3 ...We obtain solutions to both of these integrals after performing Wick rotation ($r^0 = ir_E^0; \vec{r} = \vec{r}_E$),

$$\begin{aligned}\int \frac{d^4r}{(2\pi)^4}\frac{1}{(r^2-\Delta)^j} &= \frac{i}{(2\pi)^4(-1)^j}\int d^4r_E \frac{1}{(r_E^2+\Delta)^j}\\ &= \frac{i(-1)^j}{(2\pi)^4}\int d\Omega_4 \int_0^\infty dr_E \frac{r_E^3}{(r_E^2+\Delta)^j}\\ &= \frac{i(-1)^j}{(4\pi)^2}\frac{1}{(j-2)(j-1)\Delta^{j-2}}\end{aligned}$$

Similiarly, the second integral involving r^2 term in numerator can be evaluated,

$$\int \frac{d^4r}{(2\pi)^4}\frac{r^2}{(r^2-\Delta)^j} = \frac{i(-1)^{j-1}}{(4\pi)^2}\frac{2}{(j-3)(j-2)(j-1)\Delta^{j-3}}$$