ON MATRICES AND LINEAR TRANSFORMATIONS

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Abstract. Everyone who has taken an introductory level linear algebra course should be aware of the correlation between matrices and linear transformations. In this paper, which should be viewed as an educational resource rather than an academic paper with groundbreaking original ideas, we aim to approach to this correlation in a way that is as formal (consequently nonsense) as possible, and define the matrix associated with a linear transformation together with proving the isomorphism between $\mathcal{L}_{\mathbb{F}}(V, W)$, the space of linear transformations from V to W where V and W are vector spaces over a field $\mathbb{F}$ with $\dim V = n$ and $\dim W = m$; and $\mathcal{M}_{m\times n}(\mathbb{F})$, the space of $m \times n$ matrices with coefficients in $\mathbb{F}$. The prerequisites to read this paper are just having taken an introductory linear algebra course and basic familiarity with some fundamental mathematical notations.

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1. Introduction

Before entering the theorem-definition loop, we would like to emphasize a question whose answer will determine the progress of this paper. Which matrix qualifies to be the matrix of a given linear transformation? Assume $\mathbb{F}^n$ and $\mathbb{F}^m$ be vector spaces over the field $\mathbb{F}$ and let $f\colon \mathbb{F}^n \to \mathbb{F}^m$ be a linear transformation from $\mathbb{F}^n$ to $\mathbb{F}^m$. As one would already know, there exists a (unique) matrix A such that

$$f(x) = A \cdot x$$

for all $x \in \mathbb{F}^n$, which we will prove in section 3. Therefore, performing the transformation f on the vector $x \in \mathbb{F}^n$ is equivalent to multiplying this vector by the matrix

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A. Then, one can see that the matrix A perfectly fits to be the associated matrix of the linear transformation f.

Now, let $\varphi\colon V \to W$ be a linear transformation where V and W are arbitrary vector spaces over a field $\mathbb{F}$ with $\dim V = n$ and $\dim W = m$. Can we find a matrix A such that

$$\varphi(v) = A \cdot v$$

for all $v \in V$? Unfortunately, it won't be that easy since multiplying a matrix with an element of an arbitrary vector space is not defined. Instead, we will use the fact that every vector $v \in V$ has a representation as an element of the vector space $\mathbb{F}^n$, that is, the (unique) coordinate matrix of v with respect to a basis of V.

Let $\mathcal{B}$ and $\mathcal{C}$ be bases for V and W respectively. Consider the following functions

$$\begin{array}{rccc} \gamma_{(V;\mathcal{B})}\colon & V & \longrightarrow & \mathbb{F}^n \\ & \Psi & & \Psi \\ & v & \longmapsto & [v]_{\mathcal{B}} \end{array} \qquad \text{and} \qquad \begin{array}{rccc} \gamma_{(W;\mathcal{C})}\colon & W & \longrightarrow & \mathbb{F}^m \\ & \Psi & & \Psi \\ & w & \longmapsto & [w]_{\mathcal{C}} \end{array}$$

that maps the vectors v, w from V and W to their coordinate matrices $[v]_{\mathcal{B}}$ and $[w]_{\mathcal{C}}$ with respect to bases $\mathcal{B}$ and $\mathcal{C}$ of V and W respectively. Therefore, we have

$$\gamma_{(V;\mathcal{B})}(v) = [v]_{\mathcal{B}} \qquad \text{and} \qquad \gamma_{(W;\mathcal{C})}(\varphi(v)) = [\varphi(v)]_{\mathcal{C}}$$

for all $v \in V$ and $\varphi(v) \in W$. Now, consider the function

$$\begin{array}{rccc} \phi\colon & \mathbb{F}^n & \longrightarrow & \mathbb{F}^m \\ & \Psi & & \Psi \\ & [v]_{\mathcal{B}} & \longmapsto & [\varphi(v)]_{\mathcal{C}} \end{array}$$

in which we will see that is indeed a linear transformation from $\mathbb{F}^n$ to $\mathbb{F}^m$. Therefore, by the theorem we mentioned, there exists a unique matrix A such that

$$\phi([v]_{\mathcal{B}}) = A \cdot [v]_{\mathcal{B}}$$

and since $\phi([v]_{\mathcal{B}}) = [\varphi(v)]_{\mathcal{C}}$, it concludes that

$$[\varphi(v)]_{\mathcal{C}} = A \cdot [v]_{\mathcal{B}}$$

for all $v \in V$. Thus, since $[v]_{\mathcal{B}}$ and $[\varphi(v)]_{\mathcal{C}}$ are uniquely determined by v and $\varphi(v)$ and the matrix A is unique, this matrix is the perfect candidate to be the matrix associated with the linear transformation $\varphi\colon V \to W$ and provide an isomorphism between the space of linear transformations from V to W and the space of $m \times n$ matrices where $\dim V = n$ and $\dim W = m$, as we will see in the section 3.

2. Preliminaries

In this section, we will have a more formal and rigorous approach to what we have discussed in section 1. We will give the definition of the coordinate map of a vector space, the $\mathbb{F}^n \to \mathbb{F}^m$ representation of a linear transformation and at the end of the section, we will define an isomorphism from the space of linear transformations from V to W to the space of linear transformations from $\mathbb{F}^n$ to $\mathbb{F}^m$ where $\dim V = n$ and $\dim W = m$.

2.1. **Coordinate Map of a Vector Space.** Before moving on to our first theorem, let us restate some useful facts.

Remark 2.1. Let V be a vector space over the field $\mathbb{F}$ and let $\mathcal{B} = (v_1, v_2, \ldots, v_n)$ be an ordered basis for V. Then there exist unique scalars $\lambda_i \in \mathbb{F}$ with $1 \leq i \leq n$ such that

$$v = \sum_{i=1}^{n} \lambda_i v_i.$$

Remark 2.2. Let V be a vector space over the field $\mathbb{F}$ and let $\mathcal{B} = (v_1, v_2, \ldots, v_n)$ be an ordered basis for V. Then the **coordinate matrix** of a vector $v \in V$ with respect to basis $\mathcal{B}$, denoted by $[v]_{\mathcal{B}}$, is defined as

$$[v]_{\mathcal{B}} = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{pmatrix} \iff v = \sum_{i=1}^{n} \lambda_i v_i$$

with $\lambda_i \in \mathbb{F}$, $1 \leq i \leq n$.

Remark 2.3. Let V and W be vector spaces over the same field $\mathbb{F}$. We call that a function $\varphi\colon V \to W$ is a **linear transformation** iff the following conditions hold:

(a) For all $v, w \in V$,
$$\varphi(v + w) = \varphi(v) + \varphi(w).$$
(b) For all $v \in V$, $c \in \mathbb{F}$,
$$\varphi(c \cdot v) = c \cdot \varphi(v).$$

Moreover, if this linear transformation is bijective then it is called an **isomorphism** (or **vector space isomorphism**). If that is the case, we say that the vector spaces V and W are isomorphic.

Now, we give our first fundamental theorem:

Theorem 2.1. *Let V be a vector space over the field $\mathbb{F}$. Let $\mathcal{B} = (v_1, v_2, \ldots, v_n)$ be an ordered basis for V and let $\{e_1, e_2, \ldots, e_n\}$ be the standard basis of $\mathbb{F}^n$. Then there exists an isomorphism $\phi\colon V \to \mathbb{F}^n$ that satisfies the equality*

$$\phi(v_i) = e_i$$

for all $1 \leq i \leq n$.

Proof. Define the following function

$$\begin{array}{rccc} \phi\colon & V & \longrightarrow & \mathbb{F}^n \\ & \cup\!\!\!\!\!\!\;\shortmid & & \cup\!\!\!\!\!\!\;\shortmid \\ & v & \longmapsto & \phi(v) = [v]_{\mathcal{B}} \end{array}$$

that takes a vector from V and maps it to its coordinate matrix with respect to basis $\mathcal{B}$. We claim that this function is an isomorphism. Moreover, it satisfies the equality $\phi(v_i) = e_i$ for all $1 \leq i \leq n$.

(1) **Linearity.**
   (a) For all $v, w \in V$,
$$\gamma_{(V;\mathcal{B})}(v + w) = [v + w]_{\mathcal{B}} = [v]_{\mathcal{B}} + [w]_{\mathcal{B}} = \gamma_{(V;\mathcal{B})}(v) + \gamma_{(V;\mathcal{B})}(w).$$

(b) For all $v \in V$, $c \in \mathbb{F}$,
$$\gamma_{(V;\mathcal{B})}(c \cdot v) = [c \cdot v]_{\mathcal{B}} = c \cdot [v]_{\mathcal{B}} = c \cdot \gamma_{(V;\mathcal{B})}(v).$$

Therefore, $\gamma_{(V;\mathcal{B})}$ is a linear transformation. Now, it is left to be shown that it is a bijection:

(2) **Injectivity.** Let $v, w \in V$ and let $\gamma_{(V;\mathcal{B})}(v) = \gamma_{(V;\mathcal{B})}(w)$. We need to show that $v = w$. Since $\mathcal{B}$ is a basis for V, there exist unique scalars $\lambda_i, \mu_i \in \mathbb{F}$ such that
$$v = \sum_{i=1}^{n} \lambda_i v_i \quad \text{and} \quad w = \sum_{i=1}^{n} \mu_i v_i.$$
Therefore,
$$\gamma_{(V;\mathcal{B})}(v) = [v]_{\mathcal{B}} = (\lambda_1, \lambda_2, \ldots, \lambda_n) \quad \text{and} \quad \gamma_{(V;\mathcal{B})}(w) = [w]_{\mathcal{B}} = (\mu_1, \mu_2, \ldots, \mu_n)$$
by the definition of coordinate matrices. Because $\gamma_{(V;\mathcal{B})}(v) = \gamma_{(V;\mathcal{B})}(w)$ by our assumption, we have
$$(\lambda_1, \lambda_2, \ldots, \lambda_n) = (\mu_1, \mu_2, \ldots, \mu_n).$$
By the definition of matrix equality, it concludes that $\lambda_i = \mu_i$ for all $1 \leq i \leq n$. Thus, $v = w$.

(3) **Surjectivity.** Let $x = (x_1, x_2, \ldots, x_n) \in \mathbb{F}^n$. We need to find a $v \in V$ such that $\gamma_{(V;\mathcal{B})}(v) = x$. Simply choose
$$v = x_1 v_1 + x_2 v_2 + \cdots + x_n v_n \in V$$
and therefore
$$\gamma_{(V;\mathcal{B})}(v) = [v]_{\mathcal{B}} = (x_1, x_2, \ldots, x_n) = x.$$

Thus, the linear transformation $\gamma_{(V;\mathcal{B})}$ is bijective and hence is an isomorphism. Now, let $v_i \in \mathcal{B}$ be any basis vector of V. Then,
$$v_i = 0 \cdot v_1 + \cdots + 1 \cdot v_i + \cdots + 0 \cdot v_n$$
and therefore
$$[v_i]_{\mathcal{B}} = (0, \ldots, 0, 1, 0, \ldots, 0) = e_i$$
Thus $\gamma_{(V;\mathcal{B})}(v_i) = [v_i]_{\mathcal{B}} = e_i$, which completes our proof. □

Notice that since $\gamma_{(V;\mathcal{B})}$ is an isomorphism, therefore invertible, one can also write $\gamma^{-1}_{(V;\mathcal{B})}(e_i) = v_i$ for all $1 \leq i \leq n$.

Definition 2.1. Let V be a vector space over the field $\mathbb{F}$. Let $\mathcal{B} = (v_1, v_2, \ldots, v_n)$ be an ordered basis for V and let $\{e_1, e_2, \ldots, e_n\}$ be the standard basis of $\mathbb{F}^n$. The preceeding theorem guarantees that there exists an isomorphism
$$\begin{array}{cccc} \phi\colon & V & \longrightarrow & \mathbb{F}^n \\ & \cup\!\!\!\!\!\!\;\; \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ & v & \longmapsto & \phi(v) = [v]_{\mathcal{B}} \end{array}$$
with $\phi(v_i) = e_i$. We define this isomorphism as the **coordinate map of V with respect to basis $\mathcal{B}$** and denote it by $\gamma_{(V;\mathcal{B})}$.

This transformation will play an important role in defining the $\mathbb{F}^n \to \mathbb{F}^m$ representation of a linear transformation by providing an isomorphism between V and $\mathbb{F}^n$.

2.2. $\mathbb{F}^n \to \mathbb{F}^m$ Representation of a Linear Transformation.

Remark 2.4. The set $\mathcal{L}_{\mathbb{F}}(V, W)$ denotes the space of linear transformations from V to W, where V and W are vector spaces over the field $\mathbb{F}$.

As we discussed in section 1, if we can find a linear transformation $\phi\colon \mathbb{F}^n \to \mathbb{F}^m$ such that $\phi([v]_{\mathcal{B}}) = [\varphi(v)]_{\mathcal{C}}$, we will be able to find a matrix such that $[\varphi(v)]_{\mathcal{C}} = A \cdot [v]_{\mathcal{B}}$. In the next theorem, we will prove the existence of such a transformation.

Theorem 2.2. *Let V and W be vector spaces over the same field $\mathbb{F}$ with ordered bases $\mathcal{B} = (v_1, v_2, \ldots, v_n)$ and $\mathcal{C} = (w_1, w_2, \ldots, w_m)$, respectively. Let $\gamma_{(V;\mathcal{B})}$ be the coordinate map of V with respect to basis $\mathcal{B}$ and let $\gamma_{(W;\mathcal{C})}$ be the coordinate map of W with respect to basis $\mathcal{C}$. Then there exists a unique linear transformation $\phi\colon \mathbb{F}^n \to \mathbb{F}^m$ that satisfies the equality*

$$\phi\gamma_{(V;\mathcal{B})} = \gamma_{(W;\mathcal{C})}\varphi$$

for all $\varphi \in \mathcal{L}_{\mathbb{F}}(V, W)$.

Proof. To visualize what we are going to do, consider the following diagram

$$\begin{tikzcd}
\mathbb{F}^n \arrow[rr, "\gamma_{(W;\mathcal{C})}\varphi\gamma^{-1}_{(V;\mathcal{B})}"] \arrow[d, shift left, "\gamma^{-1}_{(V;\mathcal{B})}"] & & \mathbb{F}^m \arrow[d, shift left, "\gamma^{-1}_{(W;\mathcal{C})}"] \\
V \arrow[u, shift left, "\gamma_{(V;\mathcal{B})}"] \arrow[rr, "\varphi"] & & W \arrow[u, shift left, "\gamma_{(W;\mathcal{C})}"]
\end{tikzcd}$$

and using this diagram, we can define the function

$$\begin{array}{rccc}
\gamma_{(W;\mathcal{C})}\varphi\gamma^{-1}_{(V;\mathcal{B})}\colon & \mathbb{F}^n & \longrightarrow & \mathbb{F}^m \\
& \Cup & & \Cup \\
& x & \longmapsto & (\gamma_{(W;\mathcal{C})}\varphi\gamma^{-1}_{(V;\mathcal{B})})(x)
\end{array}$$

which we claim is the function we are looking for. But first, we need to show that it is a linear transformation in the most terrifying way possible:

(a) For all $x, y \in \mathbb{F}^n$,

$$\begin{aligned}
(\gamma_{(W;\mathcal{C})}\varphi\gamma^{-1}_{(V;\mathcal{B})})(x+y) &= \gamma_{(W;\mathcal{C})}(\varphi(\gamma^{-1}_{(V;\mathcal{B})}(x+y))) \\
&= \gamma_{(W;\mathcal{C})}(\varphi(\gamma^{-1}_{(V;\mathcal{B})}(x) + \gamma^{-1}_{(V;\mathcal{B})}(y))) && (\gamma^{-1}_{(V;\mathcal{B})} \text{ is linear}) \\
&= \gamma_{(W;\mathcal{C})}(\varphi(\gamma^{-1}_{(V;\mathcal{B})}(x)) + \varphi(\gamma^{-1}_{(V;\mathcal{B})}(y))) && (\varphi \text{ is linear}) \\
&= \gamma_{(W;\mathcal{C})}(\varphi(\gamma^{-1}_{(V;\mathcal{B})}(x))) + \gamma_{(W;\mathcal{C})}(\varphi(\gamma^{-1}_{(V;\mathcal{B})}(y))) && (\gamma_{(W;\mathcal{C})} \text{ is linear}) \\
&= (\gamma_{(W;\mathcal{C})}\varphi\gamma^{-1}_{(V;\mathcal{B})})(x) + (\gamma_{(W;\mathcal{C})}\varphi\gamma^{-1}_{(V;\mathcal{B})})(y).
\end{aligned}$$

(b) For all $x \in \mathbb{F}^n$, $c \in \mathbb{F}$,

$$\begin{aligned}
(\gamma_{(W;\mathcal{C})}\varphi\gamma^{-1}_{(V;\mathcal{B})})(c \cdot x) &= \gamma_{(W;\mathcal{C})}(\varphi(\gamma^{-1}_{(V;\mathcal{B})}(c \cdot x))) \\
&= \gamma_{(W;\mathcal{C})}(\varphi(c \cdot \gamma^{-1}_{(V;\mathcal{B})}(x))) && (\gamma^{-1}_{(V;\mathcal{B})} \text{ is linear}) \\
&= \gamma_{(W;\mathcal{C})}(c \cdot \varphi(\gamma^{-1}_{(V;\mathcal{B})}(x))) && (\varphi \text{ is linear}) \\
&= c \cdot \gamma_{(W;\mathcal{C})}(\varphi(\gamma^{-1}_{(V;\mathcal{B})}(x))) && (\gamma_{(W;\mathcal{C})} \text{ is linear}) \\
&= c \cdot (\gamma_{(W;\mathcal{C})}\varphi\gamma^{-1}_{(V;\mathcal{B})})(x).
\end{aligned}$$

Now, let us show that this linear transformation satisfies the equality

$$(\gamma_{(W;\mathcal{C})}\varphi\gamma^{-1}_{(V;\mathcal{B})})\gamma_{(V;\mathcal{B})} = \gamma_{(W;\mathcal{C})}\varphi.$$

It can be easily verified as follows: Let $v \in V$. Then

$$(\gamma_{(W;\mathcal{C})}\varphi\gamma^{-1}_{(V;\mathcal{B})})\gamma_{(V;\mathcal{B})} = \gamma_{(W;\mathcal{C})}\varphi(\gamma^{-1}_{(V;\mathcal{B})}\gamma_{(V;\mathcal{B})}) = \gamma_{(W;\mathcal{C})}\varphi.$$

Finally, for the uniqueness part, let ϕ' and ϕ be two linear transformations that satisfy this equality. Then

$$\begin{aligned}
(\phi'\gamma_{(V;\mathcal{B})} = \gamma_{(W;\mathcal{C})}\varphi \quad \& \quad \phi\gamma_{(V;\mathcal{B})} = \gamma_{(W;\mathcal{C})}\varphi) &\implies \phi'\gamma_{(V;\mathcal{B})} = \phi\gamma_{(V;\mathcal{B})} \\
&\implies (\phi'\gamma_{(V;\mathcal{B})})\gamma^{-1}_{(V;\mathcal{B})} = (\phi\gamma_{(V;\mathcal{B})})\gamma^{-1}_{(V;\mathcal{B})} \\
&\implies \phi'(\gamma^{-1}_{(V;\mathcal{B})}) = \phi(\gamma^{-1}_{(V;\mathcal{B})}) \\
&\implies \phi' = \phi
\end{aligned}$$

which completes our proof. $\square$

Notice that for $v \in V$, we have $\phi(\gamma_{(V;\mathcal{B})}(v)) = \gamma_{(W;\mathcal{C})}(\varphi(v))$ and since $\gamma_{(V;\mathcal{B})}(v) = [v]_{\mathcal{B}}$ and $\gamma_{(W;\mathcal{C})}(\varphi(v)) = [\varphi(v)]_{\mathcal{C}}$ this equality becomes

$$\phi([v]_{\mathcal{B}}) = [\varphi(v)]_{\mathcal{C}}.$$

as we wanted.

Definition 2.2. Let V and W be vector spaces over a field $\mathbb{F}$. Let $\mathcal{B}$ and $\mathcal{C}$ be bases for V and W respectively. The preceding theorem guarantees that there exists a unique linear transformation

$$\begin{aligned}
\gamma_{(W;\mathcal{C})}\varphi\gamma^{-1}_{(V;\mathcal{B})} \colon \quad \mathbb{F}^n &\longrightarrow \mathbb{F}^m \\
\Uparrow \qquad &\qquad \Uparrow \\
[v]_{\mathcal{B}} &\longmapsto [\varphi(v)]_{\mathcal{C}}
\end{aligned}$$

for all $\varphi \in \mathcal{L}_{\mathbb{F}}(V, W)$, where $\gamma_{(V;\mathcal{B})}$ and $\gamma_{(W;\mathcal{C})}$ are the coordinate maps of V and W respectively. We call this transformation the **representation of φ as a transformation from $\mathbb{F}^n$ to $\mathbb{F}^m$ with respect to bases $\mathcal{B}$ and $\mathcal{C}$** and denote it by $\mathbf{R}_{(\varphi;\mathcal{B},\mathcal{C})}$. In other words,

$$\mathbf{R}_{(\varphi;\mathcal{B},\mathcal{C})}(x) := (\gamma_{(W;\mathcal{C})}\varphi\gamma^{-1}_{(V;\mathcal{B})})(x)$$

for all $x \in \mathbb{F}^n$.

Lemma 2.1. *Let V and W be vector spaces over a field $\mathbb{F}$ with $\dim V = n$ and $\dim W = m$. Let $\mathcal{B}$ and $\mathcal{C}$ be bases for V and W respectively. Then*

(1) *For all $\varphi, \psi \in \mathcal{L}_{\mathbb{F}}(V, W)$,*

$$\mathbf{R}_{(\varphi+\psi;\mathcal{B},\mathcal{C})} = \mathbf{R}_{(\varphi;\mathcal{B},\mathcal{C})} + \mathbf{R}_{(\psi;\mathcal{B},\mathcal{C})}.$$

(2) *For all* $\varphi \in \mathcal{L}_{\mathbb{F}}(V,W)$, $c \in \mathbb{F}$,

$$\mathbf{R}_{(c\cdot\varphi;\mathcal{B},\mathcal{C})} = c \cdot \mathbf{R}_{(\varphi;\mathcal{B},\mathcal{C})}.$$

Proof. Let $\gamma_{(V;\mathcal{B})}$ and $\gamma_{(W;\mathcal{C})}$ be the coordinate maps of V and W respectively. Let $x \in \mathbb{F}^n$. Then, clearly

$$\begin{aligned}
\mathbf{R}_{(\varphi+\psi;\mathcal{B},\mathcal{C})}(x) &= (\gamma_{(W;\mathcal{C})}(\varphi+\psi)\gamma^{-1}_{(V;\mathcal{B})})(x) && \text{(definition 2.2)}\\
&= \gamma_{(W;\mathcal{C})}((\varphi+\psi)(\gamma^{-1}_{(V;\mathcal{B})}(x)))\\
&= \gamma_{(W;\mathcal{C})}(\varphi(\gamma^{-1}_{(V;\mathcal{B})}(x)) + \psi(\gamma^{-1}_{(V;\mathcal{B})}(x)))\\
&= \gamma_{(W;\mathcal{C})}(\varphi(\gamma^{-1}_{(V;\mathcal{B})}(x))) + \gamma_{(W;\mathcal{C})}(\psi(\gamma^{-1}_{(V;\mathcal{B})}(x))) && (\gamma_{(W;\mathcal{C})} \text{ is linear})\\
&= (\gamma_{(W;\mathcal{C})}\varphi\gamma^{-1}_{(V;\mathcal{B})})(x) + (\gamma_{(W;\mathcal{C})}\psi\gamma^{-1}_{(V;\mathcal{B})})(x)\\
&= \mathbf{R}_{(\varphi;\mathcal{B},\mathcal{C})}(x) + \mathbf{R}_{(\psi;\mathcal{B},\mathcal{C})}(x).
\end{aligned}$$

and thus we proved the first part of our theorem. Similarly,

$$\begin{aligned}
\mathbf{R}_{(c\cdot\varphi;\mathcal{B},\mathcal{C})}(x) &= (\gamma_{(W;\mathcal{C})}(c\cdot\varphi)\gamma^{-1}_{(V;\mathcal{B})})(x) && \text{(definition 2.2)}\\
&= \gamma_{(W;\mathcal{C})}((c\cdot\varphi)(\gamma^{-1}_{(V;\mathcal{B})}(x)))\\
&= \gamma_{(W;\mathcal{C})}(c\cdot\varphi(\gamma^{-1}_{(V;\mathcal{B})}(x)))\\
&= c\cdot\gamma_{(W;\mathcal{C})}(\varphi(\gamma^{-1}_{(V;\mathcal{B})}(x))) && (\gamma_{(W;\mathcal{C})} \text{ is linear})\\
&= c\cdot(\gamma_{(W;\mathcal{C})}\varphi\gamma^{-1}_{(V;\mathcal{B})})(x)\\
&= c\cdot\mathbf{R}_{(\varphi;\mathcal{B},\mathcal{C})}(x)
\end{aligned}$$

as we stated. □

2.3. The Isomorphism Between $\mathcal{L}_{\mathbb{F}}(V,W)$ and $\mathcal{L}_{\mathbb{F}}(\mathbb{F}^n,\mathbb{F}^m)$.

Theorem 2.3. *Let V and W be vector spaces over the same field $\mathbb{F}$. Then there exists an isomorphism between $\mathcal{L}_{\mathbb{F}}(V,W)$ and $\mathcal{L}_{\mathbb{F}}(\mathbb{F}^n,\mathbb{F}^m)$.*

Proof. We claim that the function

$$\begin{array}{cccc}
\Theta\colon & \mathcal{L}_{\mathbb{F}}(V,W) & \longrightarrow & \mathcal{L}_{\mathbb{F}}(\mathbb{F}^n,\mathbb{F}^m)\\
& \cup\!\!\!\shortmid & & \cup\!\!\!\shortmid\\
& \varphi & \longmapsto & \mathbf{R}_{(\varphi;\mathcal{B},\mathcal{C})} = \gamma_{(W;\mathcal{C})}\varphi\gamma^{-1}_{(V;\mathcal{B})}
\end{array}$$

that maps a linear transformation to its $\mathbb{F}^n \to \mathbb{F}^m$ representation (with respect to bases $\mathcal{B}$ and $\mathcal{C}$) is an isomorphism from $\mathcal{L}_{\mathbb{F}}(V,W)$ to $\mathcal{L}_{\mathbb{F}}(\mathbb{F}^n,\mathbb{F}^m)$.

(1) **Linearity.**

(a) For all $\varphi,\psi \in \mathcal{L}_{\mathbb{F}}(V,W)$,

$$\begin{aligned}
\Theta(\varphi+\psi) &= \mathbf{R}_{(\varphi+\psi;\mathcal{B},\mathcal{C})}\\
&= \mathbf{R}_{(\varphi;\mathcal{B},\mathcal{C})} + \mathbf{R}_{(\psi;\mathcal{B},\mathcal{C})} && \text{(lemma 2.1)}\\
&= \Theta(\varphi)+\Theta(\psi).
\end{aligned}$$

(b) For all $\varphi \in \mathcal{L}_{\mathbb{F}}(V,W)$, $c \in \mathbb{F}$,

$$\begin{aligned}
\Theta(c\cdot\varphi) &= \mathbf{R}_{(c\cdot\varphi;\mathcal{B},\mathcal{C})}\\
&= c\cdot\mathbf{R}_{(\varphi;\mathcal{B},\mathcal{C})} && \text{(lemma 2.1)}\\
&= c\cdot\Theta(\varphi).
\end{aligned}$$

Now, let us show that this linear transformation is a bijection.

(2) **Injectivity.** Let $\varphi, \psi \in \mathcal{L}_{\mathbb{F}}(V, W)$ and let $\Theta(\varphi) = \Theta(\psi)$. We need to show that $\varphi = \psi$.

$$
\begin{aligned}
\Theta(\varphi) = \Theta(\psi) &\implies \mathbf{R}_{(\varphi;\mathcal{B},\mathcal{C})} = \mathbf{R}_{(\psi;\mathcal{B},\mathcal{C})} \\
&\implies (\gamma_{(W;\mathcal{C})}\varphi\gamma^{-1}_{(V;\mathcal{B})})(x) = (\gamma_{(W;\mathcal{C})}\psi\gamma^{-1}_{(V;\mathcal{B})})(x) \quad \forall x \in \mathbb{F}^n && \text{(definition 2.2)} \\
&\implies (\gamma_{(W;\mathcal{C})}\varphi\gamma^{-1}_{(V;\mathcal{B})})(x) - (\gamma_{(W;\mathcal{C})}\psi\gamma^{-1}_{(V;\mathcal{B})})(x) = 0 \\
&\implies \gamma_{(W;\mathcal{C})}(\varphi(\gamma^{-1}_{(V;\mathcal{B})}(x))) - \gamma_{(W;\mathcal{C})}(\psi(\gamma^{-1}_{(V;\mathcal{B})}(x))) = 0 \\
&\implies \gamma_{(W;\mathcal{C})}(\varphi(\gamma^{-1}_{(V;\mathcal{B})}(x)) - \psi(\gamma^{-1}_{(V;\mathcal{B})}(x))) = 0 && (\gamma_{(W;\mathcal{C})} \text{ is linear}) \\
&\implies \varphi(\gamma^{-1}_{(V;\mathcal{B})}(x)) - \psi(\gamma^{-1}_{(V;\mathcal{B})}(x)) = 0 && (\gamma_{(W;\mathcal{C})} \text{ is an isomorphism}) \\
&\implies \varphi(\gamma^{-1}_{(V;\mathcal{B})}(x)) = \psi(\gamma^{-1}_{(V;\mathcal{B})}(x)).
\end{aligned}
$$

Now, let $v \in V$ be arbitrary. Since the linear transformation $\gamma^{-1}_{(V;\mathcal{B})}$ is an isomorphism and therefore surjective, there exists a $y \in \mathbb{F}^n$ such that $\gamma^{-1}_{(V;\mathcal{B})}(y) = v$. Thus,

$$
\varphi(\gamma^{-1}_{(V;\mathcal{B})}(y)) = \psi(\gamma^{-1}_{(V;\mathcal{B})}(y)) \implies \varphi(v) = \psi(v)
$$

and since $v \in V$ was arbitrary, it concludes that $\varphi = \psi$.

(3) **Surjectivity.** Let $f \in \mathcal{L}_{\mathbb{F}}(\mathbb{F}^n, \mathbb{F}^m)$. We need to find a $\varphi \in \mathcal{L}_{\mathbb{F}}(V, W)$ such that $f = \mathbf{R}_{(\varphi;\mathcal{B},\mathcal{C})}$. By considering the diagram below

$$
\begin{tikzcd}
\mathbb{F}^n \arrow[rrr, "f"] \arrow[d, shift left, "\gamma^{-1}_{(V;\mathcal{B})}"] & & & \mathbb{F}^m \arrow[d, shift left, "\gamma^{-1}_{(W;\mathcal{C})}"] \\
V \arrow[u, shift left, "\gamma_{(V;\mathcal{B})}"] \arrow[rrr, "\gamma^{-1}_{(W;\mathcal{C})} f \gamma_{(V;\mathcal{B})}"] & & & W \arrow[u, shift left, "\gamma_{(W;\mathcal{C})}"]
\end{tikzcd}
$$

simply choose the function

$$
\begin{aligned}
\varphi\colon \quad V &\longrightarrow W \\
v &\longmapsto \varphi(v) = (\gamma^{-1}_{(W;\mathcal{C})} f \gamma_{(V;\mathcal{B})})(v)
\end{aligned}
$$

where $\gamma_{(V;\mathcal{B})}\colon V \to \mathbb{F}^n$ and $\gamma_{(W;\mathcal{C})}\colon W \to \mathbb{F}^m$ are the coordinate maps of V and W, with respect to bases $\mathcal{B}$ and $\mathcal{C}$ respectively. Let us easily verify that this function is a linear transformation.

(a) For all $v, w \in V$,

$$\begin{aligned}
\varphi(v+w) &= (\gamma^{-1}_{(W;\mathcal{C})} f \gamma_{(V;\mathcal{B})})(v+w) \\
&= \gamma^{-1}_{(W;\mathcal{C})}(f(\gamma_{(V;\mathcal{B})}(v+w))) \\
&= \gamma^{-1}_{(W;\mathcal{C})}(f(\gamma_{(V;\mathcal{B})}(v) + \gamma_{(V;\mathcal{B})}(w))) && (\gamma_{(V;\mathcal{B})} \text{ is linear}) \\
&= \gamma^{-1}_{(W;\mathcal{C})}(f(\gamma_{(V;\mathcal{B})}(v)) + f(\gamma_{(V;\mathcal{B})}(w))) && (f \text{ is linear}) \\
&= \gamma^{-1}_{(W;\mathcal{C})}(f(\gamma_{(V;\mathcal{B})}(v))) + \gamma^{-1}_{(W;\mathcal{C})}(f(\gamma_{(V;\mathcal{B})}(w))) && (\gamma^{-1}_{(W;\mathcal{C})} \text{ is linear}) \\
&= (\gamma^{-1}_{(W;\mathcal{C})} f \gamma_{(V;\mathcal{B})})(v) + (\gamma^{-1}_{(W;\mathcal{C})} f \gamma_{(V;\mathcal{B})})(w) \\
&= \varphi(v) + \varphi(w).
\end{aligned}$$

(b) For all $v \in V$, $c \in \mathbb{F}$,

$$\begin{aligned}
\varphi(c \cdot v) &= (\gamma^{-1}_{(W;\mathcal{C})} f \gamma_{(V;\mathcal{B})})(c \cdot v) \\
&= \gamma^{-1}_{(W;\mathcal{C})}(f(\gamma_{(V;\mathcal{B})}(c \cdot v))) \\
&= \gamma^{-1}_{(W;\mathcal{C})}(f(c \cdot \gamma_{(V;\mathcal{B})}(v))) && (\gamma_{(V;\mathcal{B})} \text{ is linear}) \\
&= \gamma^{-1}_{(W;\mathcal{C})}(c \cdot f(\gamma_{(V;\mathcal{B})}(v))) && (f \text{ is linear}) \\
&= c \cdot \gamma^{-1}_{(W;\mathcal{C})}(f(\gamma_{(V;\mathcal{B})}(v))) && (\gamma^{-1}_{(W;\mathcal{C})} \text{ is linear}) \\
&= c \cdot (\gamma^{-1}_{(W;\mathcal{C})} f \gamma_{(V;\mathcal{B})})(v) \\
&= c \cdot \varphi(v).
\end{aligned}$$

Now, it follows that

$$\begin{aligned}
\Theta(\varphi) &= \mathbf{R}_{(\varphi;\mathcal{B},\mathcal{C})} \\
&= \gamma_{(W;\mathcal{C})} \varphi \gamma^{-1}_{(V;\mathcal{B})} && (\text{definition 2.2}) \\
&= \gamma_{(W;\mathcal{C})} (\gamma^{-1}_{(W;\mathcal{C})} f \gamma_{(V;\mathcal{B})}) \gamma^{-1}_{(V;\mathcal{B})} \\
&= (\gamma_{(W;\mathcal{C})} \gamma^{-1}_{(W;\mathcal{C})}) f (\gamma_{(V;\mathcal{B})} \gamma^{-1}_{(V;\mathcal{B})}) \\
&= f.
\end{aligned}$$

Therefore, since the linear transformation Θ is bijective, it is an isomorphism. Thus, the space of linear transformations from V to W is isomorphic to the space of linear transformations from $\mathbb{F}^n$ to $\mathbb{F}^m$, as we stated. $\square$

3. Matrices and Linear Transformations

Remark 3.1. The set $\mathcal{M}_{m\times n}(\mathbb{F})$ denotes the space of $m\times n$ matrices with coefficients in $\mathbb{F}$.

In this section, we will define the matrix associated with a linear transformation and define an isomorphism between $\mathcal{L}_{\mathbb{F}}(V, W)$ and $\mathcal{M}_{m\times n}(\mathbb{F})$. In the last subsection, we will be developing a way to numerically calculate the matrix of a linear transformation.

3.1. Matrix of a Linear Transformation from $\mathbb{F}^n$ to $\mathbb{F}^m$.

Theorem 3.1. *Let $\mathbb{F}^n$ and $\mathbb{F}^m$ be vector spaces over the same field $\mathbb{F}$. Let $f \in \mathcal{L}_{\mathbb{F}}(\mathbb{F}^n, \mathbb{F}^m)$ be a linear transformation. Then there exists a unique matrix $A \in$*

$\mathcal{M}_{m\times n}(\mathbb{F})$ such that

$$f(x) = A \cdot x$$

for all $x \in \mathbb{F}^n$.

Proof. Let $\{e_1, e_2, \dots, e_n\}$ and $\{e'_1, e'_2, \dots, e'_m\}$ be the standard bases of $\mathbb{F}^n$ and $\mathbb{F}^m$ respectively. Also let $x = (x_1, x_2, \dots, x_n) \in \mathbb{F}^n$. Therefore, one can write

$$x = x_1e_1 + x_2e_2 + \cdots + x_ne_n = \sum_{j=1}^{n} x_je_j.$$

and hence

$$f(x) = f\Big(\sum_{j=1}^{n} x_je_j\Big) = \sum_{j=1}^{n} x_j \cdot f(e_j)$$

since f is linear. Now, because $f(e_j) \in \mathbb{F}^m$ for $1 \le j \le n$, there exist scalars $a_{ij} \in \mathbb{F}$ with $1 \le j \le n$, $1 \le i \le m$ such that

$$f(e_j) = \sum_{i=1}^{m} a_{ij}e'_i. \tag{1}$$

Thus, it follows that

$$\begin{aligned}
f(x) = f\Big(\sum_{j=1}^{n} x_je_j\Big) &= \sum_{j=1}^{n} x_j \cdot f(e_j) \\
&= \sum_{j=1}^{n} x_j\Big(\sum_{i=1}^{m} a_{ij}e'_i\Big) && \text{(eq. (1))} \\
&= \sum_{j=1}^{n}\Big(\sum_{i=1}^{m} x_ja_{ij}e'_i\Big) && \text{(summation property)} \\
&= \sum_{i=1}^{m}\Big(\sum_{j=1}^{n} x_ja_{ij}e'_i\Big) && \text{(summation property)} \\
&= \sum_{i=1}^{m}\Big(\sum_{j=1}^{n} x_ja_{ij}\Big)e'_i. && \text{(summation property)}
\end{aligned}$$

If we write the last sum in its full form, we have

$$\begin{aligned}
f(x) &= \Big(\sum_{j=1}^{n} x_j a_{1j}\Big)e'_1 + \Big(\sum_{j=1}^{n} x_j a_{2j}\Big)e'_2 + \cdots + \Big(\sum_{j=1}^{n} x_j a_{mj}\Big)e'_m \\
&= \begin{pmatrix} \sum_{j=1}^{n} x_j a_{1j} \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \sum_{j=1}^{n} x_j a_{2j} \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \cdots + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \sum_{j=1}^{n} x_j a_{mj} \end{pmatrix} && (e'_i \text{ standard basis vector}) \\
&= \begin{pmatrix} \sum_{j=1}^{n} x_j a_{1j} \\ \sum_{j=1}^{n} x_j a_{2j} \\ \vdots \\ \sum_{j=1}^{n} x_j a_{mj} \end{pmatrix} \\
&= \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} && (\text{matrix multiplication})
\end{aligned}$$

and if we let $A = (a_{ij})_{\substack{1\le i\le m \\ 1\le j\le n}} \in \mathcal{M}_{m\times n}(\mathbb{F})$ it concludes that $f(x) = A \cdot x$. Now, for the uniqueness part, let $A_1, A_2 \in \mathcal{M}_{m\times n}(\mathbb{F})$ be two matrices such that

$$f(x) = A_1 \cdot x \qquad \text{and} \qquad f(x) = A_2 \cdot x$$

for all $x \in \mathbb{F}^n$. Therefore

$$\begin{aligned}
A_1 \cdot x = A_2 \cdot x &\implies A_1 \cdot x - A_2 \cdot x = 0 \\
&\implies (A_1 - A_2) \cdot x = 0
\end{aligned}$$

and since the last equality is true for any $x \in \mathbb{F}^n$, it must be the case that $A_1 - A_2 = 0$ and thus $A_1 = A_2$. □

3.2. **Matrix of a Linear Transformation.** Now, we are ready to define the matrix associated with a linear transformation.

Definition 3.1 (Matrix of a Linear Transformation)**.** Let V and W be vector spaces over the same field $\mathbb{F}$ and let $\mathcal{B}$ and $\mathcal{C}$ be bases for V and W respectively, with $\dim V = n$ and $\dim W = m$. Let $\varphi \in \mathcal{L}_{\mathbb{F}}(V, W)$ be a linear transformation and let $\mathbf{R}_{(\varphi;\mathcal{B},\mathcal{C})}$ be its representation as a transformation from $\mathbb{F}^n$ to $\mathbb{F}^m$, with respect to bases $\mathcal{B}$ and $\mathcal{C}$. By theorem 3.1, there exists a unique matrix $A \in \mathcal{M}_{m\times n}(\mathbb{F})$ such that

$$\mathbf{R}_{(\varphi;\mathcal{B},\mathcal{C})}(x) = A \cdot x$$

for all $x \in \mathbb{F}^n$. We define the **matrix associated with the linear transformation** φ **with respect to bases** $\mathcal{B}$ **and** $\mathcal{C}$ as A and denote it by $\mathbf{M}(\varphi; \mathcal{B}, \mathcal{C})$. In other words,

$$\mathbf{R}_{(\varphi;\mathcal{B},\mathcal{C})}(x) = \mathbf{M}(\varphi; \mathcal{B}, \mathcal{C}) \cdot x$$

for all $x \in \mathbb{F}^n$. Notice that for $x = [v]_{\mathcal{B}} \in \mathbb{F}^n$, we have

$$[\varphi(v)]_{\mathcal{C}} = \mathbf{M}(\varphi; \mathcal{B}, \mathcal{C}) \cdot [v]_{\mathcal{B}}$$

for all $\varphi \in \mathcal{L}_{\mathbb{F}}(V, W)$, as we discussed.

Thus, the matrix of an arbitrary linear transformation $\varphi\colon V \to W$ is the matrix satisfying

$$\mathbf{R}_{(\varphi;\mathcal{B},\mathcal{C})}(x) = A \cdot x$$

for all $x \in \mathbb{F}^n$, where $\mathbf{R}_{(\varphi;\mathcal{B},\mathcal{C})}$ is the representation of φ from $\mathbb{F}^n$ to $\mathbb{F}^m$. The uniqueness of such a matrix (by theorem 3.1), indeed, guarantees that the matrix associated with a linear transformation is well-defined.

Lemma 3.1. *Let V and W be vector spaces over a field $\mathbb{F}$ and let $\mathcal{B}$ and $\mathcal{C}$ be bases for V and W respectively. Let $\varphi, \psi \in \mathcal{L}_{\mathbb{F}}(V, W)$ and $c \in \mathbb{F}$. If φ has the matrix $\mathbf{M}(\varphi;\mathcal{B},\mathcal{C})$ and ψ has the matrix $\mathbf{M}(\psi;\mathcal{B},\mathcal{C})$, then*

(1) *The linear transformation $\varphi + \psi$ has the matrix*

$$\mathbf{M}(\varphi + \psi;\mathcal{B},\mathcal{C}) = \mathbf{M}(\varphi;\mathcal{B},\mathcal{C}) + \mathbf{M}(\psi;\mathcal{B},\mathcal{C}).$$

(2) *The linear transformation $c \cdot \varphi$ has the matrix*

$$\mathbf{M}(c \cdot \varphi;\mathcal{B},\mathcal{C}) = c \cdot \mathbf{M}(\varphi;\mathcal{B},\mathcal{C}).$$

Proof. Let $\mathbf{R}_{(\varphi;\mathcal{B},\mathcal{C})}$ be the representation of φ from $\mathbb{F}^n$ to $\mathbb{F}^m$ and $\mathbf{R}_{(\psi;\mathcal{B},\mathcal{C})}$ be the representation of ψ from $\mathbb{F}^n$ to $\mathbb{F}^m$. Therefore, one can write

$$\mathbf{R}_{(\varphi;\mathcal{B},\mathcal{C})}(x) = \mathbf{M}(\varphi;\mathcal{B},\mathcal{C}) \cdot x \qquad \text{and} \qquad \mathbf{R}_{(\psi;\mathcal{B},\mathcal{C})}(x) = \mathbf{M}(\psi;\mathcal{B},\mathcal{C}) \cdot x$$

for all $x \in \mathbb{F}^n$. For the first part, let $\mathbf{R}_{(\varphi+\psi;\mathcal{B},\mathcal{C})}$ be the representation of $\varphi + \psi$ from $\mathbb{F}^n$ to $\mathbb{F}^m$. It follows that

$$\begin{aligned}
\mathbf{R}_{(\varphi+\psi;\mathcal{B},\mathcal{C})}(x) &= (\mathbf{R}_{(\varphi;\mathcal{B},\mathcal{C})} + \mathbf{R}_{(\psi;\mathcal{B},\mathcal{C})})(x) && \text{(lemma 2.1)}\\
&= \mathbf{R}_{(\varphi;\mathcal{B},\mathcal{C})}(x) + \mathbf{R}_{(\psi;\mathcal{B},\mathcal{C})}(x)\\
&= \mathbf{M}(\varphi;\mathcal{B},\mathcal{C}) \cdot x + \mathbf{M}(\psi;\mathcal{B},\mathcal{C}) \cdot x\\
&= (\mathbf{M}(\varphi;\mathcal{B},\mathcal{C}) + \mathbf{M}(\psi;\mathcal{B},\mathcal{C})) \cdot x.
\end{aligned}$$

and thus, by definition 3.1, the transformation $\varphi + \psi$ has the matrix $\mathbf{M}(\varphi;\mathcal{B},\mathcal{C}) + \mathbf{M}(\psi;\mathcal{B},\mathcal{C})$. Now, for the second part, let $\mathbf{R}_{(c\cdot\varphi;\mathcal{B},\mathcal{C})}$ be the representation of $c \cdot \varphi$ from $\mathbb{F}^n$ to $\mathbb{F}^m$. Then

$$\begin{aligned}
\mathbf{R}_{(c\cdot\varphi;\mathcal{B},\mathcal{C})}(x) &= (c \cdot \mathbf{R}_{(\varphi;\mathcal{B},\mathcal{C})})(x) && \text{(lemma 2.1)}\\
&= c \cdot \mathbf{R}_{(\varphi;\mathcal{B},\mathcal{C})}(x)\\
&= c \cdot (\mathbf{M}(\varphi;\mathcal{B},\mathcal{C}) \cdot x)\\
&= (c \cdot \mathbf{M}(\varphi;\mathcal{B},\mathcal{C})) \cdot x.
\end{aligned}$$

and similarly, by definition 3.1, the transformation $c\cdot\varphi$ has the matrix $c\cdot\mathbf{M}(\varphi;\mathcal{B},\mathcal{C})$ as we stated. □

3.3. **The Isomorphism Between $\mathcal{L}_{\mathbb{F}}(V, W)$ and $\mathcal{M}_{m\times n}(\mathbb{F})$.** Now by using the lemma, we will prove that the spaces $\mathcal{L}_{\mathbb{F}}(V, W)$ and $\mathcal{M}_{m\times n}(\mathbb{F})$ are isomorphic, which is the main theorem of this paper.

Theorem 3.2 (The Isomorphism Theorem)**.** *Let V and W be vector spaces over the same field $\mathbb{F}$. Then there exists an isomorphism between $\mathcal{L}_{\mathbb{F}}(V, W)$ and $\mathcal{M}_{m\times n}(\mathbb{F})$, i.e., these two spaces are isomorphic.*

Proof. Let $\mathcal{B}$ and $\mathcal{C}$ be bases for V and W respectively. We claim that the function

$$\begin{array}{rccc} \Omega\colon & \mathcal{L}_{\mathbb{F}}(V,W) & \longrightarrow & \mathcal{M}_{m\times n}(\mathbb{F}) \\ & \Cup & & \Cup \\ & \varphi & \longmapsto & \mathbf{M}(\varphi;\mathcal{B},\mathcal{C}) \end{array}$$

that maps a linear transformation to its matrix (with respect to bases $\mathcal{B}$ and $\mathcal{C}$) is an isomorphism from $\mathcal{L}_{\mathbb{F}}(V,W)$ to $\mathcal{M}_{m\times n}(\mathbb{F})$. First, let us show that it is a linear transformation:

(1) **Linearity.**
   (a) For all $\varphi,\psi \in \mathcal{L}_{\mathbb{F}}(V,W)$,

$$\begin{aligned} \Omega(\varphi+\psi) &= \mathbf{M}(\varphi+\psi;\mathcal{B},\mathcal{C}) \\ &= \mathbf{M}(\varphi;\mathcal{B},\mathcal{C}) + \mathbf{M}(\psi;\mathcal{B},\mathcal{C}) && \text{(lemma 3.1)} \\ &= \Omega(\varphi)+\Omega(\psi). \end{aligned}$$

   (b) For all $\varphi \in \mathcal{L}_{\mathbb{F}}(V,W)$, $c \in \mathbb{F}$,

$$\begin{aligned} \Omega(c\cdot\varphi) &= \mathbf{M}(c\cdot\varphi;\mathcal{B},\mathcal{C}) \\ &= c\cdot\mathbf{M}(\varphi;\mathcal{B},\mathcal{C}) && \text{(lemma 3.1)} \\ &= c\cdot\Omega(\varphi). \end{aligned}$$

Now, we need to show that this linear transformation is bijective.

(2) **Injectivity.** Let $\varphi,\psi \in \mathcal{L}_{\mathbb{F}}(V,W)$ and let $\Omega(\varphi)=\Omega(\psi)$. We need to show that $\varphi=\psi$. Then

$$\Omega(\varphi)=\Omega(\psi) \implies \mathbf{M}(\varphi;\mathcal{B},\mathcal{C}) = \mathbf{M}(\psi;\mathcal{B},\mathcal{C})$$

and since $\mathbf{M}(\varphi;\mathcal{B},\mathcal{C})$ is the matrix of φ and $\mathbf{M}(\psi;\mathcal{B},\mathcal{C})$ is the matrix of ψ, one can write

$$\mathbf{R}_{(\varphi;\mathcal{B},\mathcal{C})}(x) = \mathbf{M}(\varphi;\mathcal{B},\mathcal{C})\cdot x \qquad \text{and} \qquad \mathbf{R}_{(\psi;\mathcal{B},\mathcal{C})}(x) = \mathbf{M}(\psi;\mathcal{B},\mathcal{C})\cdot x$$

for all $x \in \mathbb{F}^n$ by definition 3.1. Therefore

$$\begin{aligned} \mathbf{M}(\varphi;\mathcal{B},\mathcal{C}) = \mathbf{M}(\psi;\mathcal{B},\mathcal{C}) &\implies \mathbf{M}(\varphi;\mathcal{B},\mathcal{C})\cdot x = \mathbf{M}(\psi;\mathcal{B},\mathcal{C})\cdot x \quad \forall x \in \mathbb{F}^n \\ &\implies \mathbf{R}_{(\varphi;\mathcal{B},\mathcal{C})}(x) = \mathbf{R}_{(\psi;\mathcal{B},\mathcal{C})}(x) \\ &\implies \varphi=\psi && \text{(theorem 2.3)} \end{aligned}$$

and thus, Ω is injective.

(3) **Surjectivity.** Let $A \in \mathcal{M}_{m\times n}(\mathbb{F})$. We need to find a $\varphi \in \mathcal{L}_{\mathbb{F}}(V,W)$ such that A is the matrix of φ with respect to bases $\mathcal{B}$ and $\mathcal{C}$. Simply choose

$$\begin{array}{rccc} \varphi\colon & \mathbb{F}^n & \longrightarrow & \mathbb{F}^m \\ & \Cup & & \Cup \\ & x & \longmapsto & \varphi(x) = A\cdot x \end{array}$$

with $V=\mathbb{F}^n$ and $W=\mathbb{F}^m$ so that we have $\mathbf{R}_{(\varphi;\mathcal{B},\mathcal{C})}(x) = \varphi(x) = A\cdot x$ for all $x \in \mathbb{F}^n$. Let us show that this function is indeed a linear transformation.

(a) For all $x, y \in \mathbb{F}^n$,

$$\begin{aligned} \varphi(x+y) &= A \cdot (x+y) \\ &= A \cdot x + A \cdot y \\ &= \varphi(x) + \varphi(y). \end{aligned}$$

(b) For all $x \in \mathbb{F}^n$, $c \in \mathbb{F}$,

$$\begin{aligned} \varphi(c \cdot x) &= A \cdot (c \cdot x) \\ &= c \cdot (A \cdot x) \\ &= c \cdot \varphi(x). \end{aligned}$$

Therefore, since φ is a linear transformation with $\mathbf{R}_{(\varphi;\mathcal{B},\mathcal{C})}(x) = A \cdot x$ for all $x \in \mathbb{F}^n$, by definition 3.1, A is the matrix of φ and hence $\Omega(\varphi) = A$, meaning Ω is surjective.

Thus, we showed that Ω is a bijective linear transformation, meaning it is an isomorpism. Therefore, the space of linear transformations from V to W is isomorphic to the space of $m \times n$ matrices with coefficients in $\mathbb{F}$. $\square$

We can visualize the final result by using the following diagram:

$$\begin{tikzcd}
\mathbb{F}^n \arrow[rrr, "\mathbf{R}_{(\varphi;\mathcal{B},\mathcal{C})}"] \arrow[ddd, shift left, "\gamma^{-1}_{(V;\mathcal{B})}"] & & & \mathbb{F}^m \arrow[ddd, shift left, "\gamma^{-1}_{(W;\mathcal{C})}"] \\
& {[v]_{\mathcal{B}}} \arrow[r, mapsto] & {[\varphi(v)]_{\mathcal{C}} = \mathbf{M}(\varphi;\mathcal{B},\mathcal{C}) \cdot [v]_{\mathcal{B}}} & \\
& v \arrow[r, mapsto] \arrow[u, mapsto] & \varphi(v) \arrow[u, mapsto] & \\
V \arrow[rrr, "\varphi"] \arrow[uuu, shift left, "\gamma_{(V;\mathcal{B})}"] & & & W \arrow[uuu, shift left, "\gamma_{(W;\mathcal{C})}"]
\end{tikzcd}$$

3.4. **Finding the Matrix of a Linear Transformation.** The theory aside, is it possible to find the matrix of a linear transformation without struggling with bunch of calculations? In other words, is there a method to numerically find the matrix of a given linear transformation? In this section, we will be developing such a method to achieve this.

Assume the following: Let V and W be vector spaces over the field $\mathbb{F}$. Let $\mathcal{B} = (v_1, v_2, \ldots, v_n)$ and $\mathcal{C} = (w_1, w_2, \ldots, w_m)$ be ordered bases for V and W respectively. Let $\{e_1, e_2, \ldots, e_n\}$ be the standard basis of $\mathbb{F}^n$ and let $\{e'_1, e'_2, \ldots, e'_m\}$ be the standard basis of $\mathbb{F}^m$. Finally, let $\varphi \in \mathcal{L}_{\mathbb{F}}(V, W)$ be a linear transformation with its representation $\mathbf{R}_{(\varphi;\mathcal{B},\mathcal{C})} \in \mathcal{L}_{\mathbb{F}}(\mathbb{F}^n, \mathbb{F}^m)$ from $\mathbb{F}^n$ to $\mathbb{F}^m$ and its matrix $\mathbf{M}(\varphi;\mathcal{B},\mathcal{C})$ with respect to bases $\mathcal{B}$ and $\mathcal{C}$.

Now, let $x = (x_1, x_2, \dots, x_n) \in \mathbb{F}^n$. Then, it follows that

$$\begin{aligned}
\mathbf{R}_{(\varphi;\mathcal{B},\mathcal{C})}(x) &= \mathbf{R}_{(\varphi;\mathcal{B},\mathcal{C})}\Big(\sum_{j=1}^{n} x_j e_j\Big) \\
&= \sum_{j=1}^{n} x_j \cdot \mathbf{R}_{(\varphi;\mathcal{B},\mathcal{C})}(e_j) && (\mathbf{R}_{(\varphi;\mathcal{B},\mathcal{C})} \text{ is linear}) \\
&= \sum_{j=1}^{n} x_j \cdot (\gamma_{(W;\mathcal{C})}\varphi\gamma^{-1}_{(V;\mathcal{B})})(e_j) && (\text{definition 2.2}) \\
&= \sum_{j=1}^{n} x_j \cdot \gamma_{(W;\mathcal{C})}(\varphi(\gamma^{-1}_{(V;\mathcal{B})}(e_j))) \\
&= \sum_{j=1}^{n} x_j \cdot \gamma_{(W;\mathcal{C})}(\varphi(v_j)) && (\text{theorem 2.1})
\end{aligned}$$

and since $\varphi(v_j) \in W$, there exist unique scalars $a_{ij} \in \mathbb{F}$ such that

$$\varphi(v_j) = \sum_{i=1}^{m} a_{ij} w_i \tag{2}$$

and therefore, we have

$$\begin{aligned}
\mathbf{R}_{(\varphi;\mathcal{B},\mathcal{C})}(x) &= \sum_{j=1}^{n} x_j \cdot \gamma_{(W;\mathcal{C})}(\varphi(v_j)) \\
&= \sum_{j=1}^{n} x_j \cdot \gamma_{(W;\mathcal{C})}\Big(\sum_{i=1}^{m} a_{ij} w_i\Big) && (\text{eq. (2)}) \\
&= \sum_{j=1}^{n} x_j \Big(\sum_{i=1}^{m} a_{ij} \cdot \gamma_{(W;\mathcal{C})}(w_i)\Big) && (\gamma_{(W;\mathcal{C})} \text{ is linear}) \\
&= \sum_{j=1}^{n} x_j \Big(\sum_{i=1}^{m} a_{ij} e'_i\Big) && (\text{theorem 2.1}) \\
&= \sum_{i=1}^{m} \Big(\sum_{j=1}^{n} x_j a_{ij} e'_i\Big) && (\text{summation property}) \\
&= \sum_{i=1}^{m} \Big(\sum_{j=1}^{n} x_j a_{ij}\Big) e'_i. && (\text{summation property})
\end{aligned}$$

As we seen in theorem 3.1, this concludes to

$$\mathbf{R}_{(\varphi;\mathcal{B},\mathcal{C})}(x) = A \cdot x$$

where $A = (a_{ij})_{\substack{1\le i\le m\\ 1\le j\le n}} \in \mathcal{M}_{m\times n}(\mathbb{F})$. Indeed, the matrix A is nothing but the matrix of the transformation φ by definition 2.2, i.e.,

$$\mathbf{M}(\varphi;\mathcal{B},\mathcal{C}) = (a_{ij})_{\substack{1\le i\le m\\ 1\le j\le n}}.$$

To make this calculation more useful and practical, notice that the coordinate matrix of $\varphi(v_j) \in W$ is

$$[\varphi(v_j)]_{\mathcal{C}} = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix}$$

by the eq. (2). As one can clearly see, this column matrix is the j^{th} column of the matrix $\mathbf{M}(\varphi; \mathcal{B}, \mathcal{C})$. Thus, it is possible to write

$$\mathbf{M}(\varphi; \mathcal{B}, \mathcal{C}) = \Big[[\varphi(v_1)]_{\mathcal{C}} \;\Big|\; [\varphi(v_2)]_{\mathcal{C}} \;\Big|\; \cdots \;\Big|\; [\varphi(v_n)]_{\mathcal{C}} \Big]$$

which provides a relatively easy way to numerically calculate the matrix of a linear transformation. As always, the examples are trivial and is left to the reader as an exercise.

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