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What are Irrational Numbers? - Definition & Examples

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What is an irrational number? Learn about these numbers, their definition, what makes them irrational, and different types of irrational numbers with examples.

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What Is an Irrational Number?

In everyday speech, the word *irrational* means illogical or even insane. In math, however, it has a different, more technical definition. The word *rational* comes from the word *ratio*, so a **rational number** is a number that can be written as a ratio, or fraction, of two integers. For instance, $\frac{2}{5}$ is a rational number.

An **irrational number**, on the other hand, is a number that *cannot* be written as a fraction of two integers. For instance, $\sqrt{2}$ is an irrational number because, try as you might, you will never be able to write it as a fraction of two integers.

Interestingly, irrational numbers may have had a somewhat perilous beginning. Legend has it that the ancient Greek philosopher Hippasus was the first person to discover irrational numbers, and for his discovery, the Pythagoreans (the followers of Pythagoras) drowned him in order to keep this 'dangerous'

discovery a secret. As exciting as this story is, there is actually little reliable evidence to believe that it is anything more than a myth.

An imaginary engraving of the philosopher Hippasus of Metapontum

Properties of Irrational Numbers

Let's look at the definition of an irrational number and use it to examine some of its properties. Remember, an irrational number is a number that cannot be written as a fraction of two integers. The first example above was $\sqrt{2}$, which is equal to 1.41421356237... . Notice that the digits in this number go on forever without a repeating pattern. This is actually a general property of all irrational numbers: their decimal representations will always go on forever without a repeating pattern.

Right away, this means that whole numbers like 3 or 17 are *not* irrational numbers since they have no digits after the decimal point. The numbers 3 and 17 can also be written as the fractions $\frac{3}{1}$ and $\frac{17}{1}$, respectively, which is further proof that they are not irrational numbers.

As another example, the decimal representation of $\frac{7}{9}$ is 0.7777777777... . Although the digits go on forever, there is a clear repeating pattern, so this is *not* an irrational number either.

Finally, a terminating decimal, like 4.789, is *not* an irrational number because its digits do not go on forever. It can be written as a fraction by writing the decimal with 1,000 as the denominator: $4.789 = 4\frac{789}{1{,}000} = \frac{4{,}789}{1{,}000}$.

Common Types of Irrational Numbers

Now let's look at some common types of irrational numbers as examples.

Roots

One of the most common types of irrational numbers you will encounter is roots. For instance, the square roots, $\sqrt{2}$, $\sqrt{3}$, and $\sqrt{5}$, are all irrational numbers. Irrational numbers can also be negative. For instance, $-\sqrt{2}$ is an irrational number. Furthermore, cube roots like $\sqrt[3]{2}$, $\sqrt[3]{3}$, and $\sqrt[3]{10}$ are irrational numbers, too.

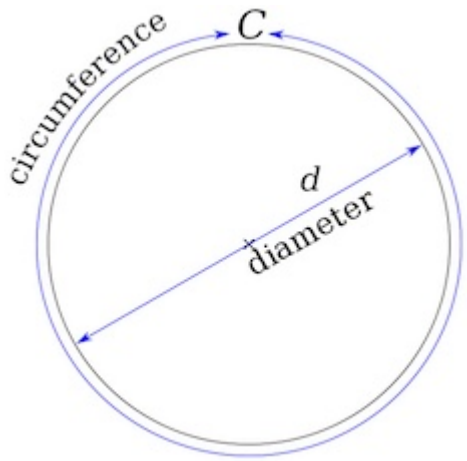
Not all roots are irrational however. Some roots, like $\sqrt{16}$ and $\sqrt[3]{27}$ can be written as whole numbers (in this case, 4 and 3, respectively), so they are *not* irrational numbers.

Finally, in certain cases, multiple irrational numbers can be multiplied together to get a rational number. For instance, $\sqrt{7}$ is an irrational number, but $\sqrt{7} \times \sqrt{7}$ is equal to 7, which is a whole number and therefore a rational number.

Pi

The Greek letter π (which is written as **pi** and pronounced "pie") is used to represent the ratio of the circumference of a circle divided by its diameter. Though it looks like a variable, π is actually a constant, that is, it is always equal to the same decimal, about 3.1415926535... . Like all irrational numbers, it is impossible to write all the digits of π because they go on forever and have no repeating pattern.

The number π is commonly used in geometry and trigonometry, especially in problems that involve circles. For instance, π is vital to finding the area and circumference of a circle. When making calculations with π, it is often approximated as 3.14 or by the fraction $\frac{22}{7}$. Of course, like all irrational numbers, there is no terminating decimal or fraction that is exactly equal to π.



The number pi is the ratio of the circumference of a circle divided by

Phi

The Greek letter ϕ (which is written as **phi** and pronounced "fie") is sometimes referred to as the "**golden ratio**" or "**golden mean**." Similar to π, it is a constant that is approximately equal to 1.6180339887..., though 1.618 is a common approximation. The value of ϕ can actually be written in an exact form: $\frac{1+\sqrt{5}}{2}$. Though this is a fraction, it is not the fraction of two *integers* since $1 + \sqrt{5}$ is not an integer, so ϕ is definitely *not* a rational number. Because its decimal representation goes on forever without a repeating pattern, ϕ is an irrational number.

its diameter

Euler's Number

In math, the letter e is reserved for the mathematical constant that is sometimes referred to as **Euler's number**. Because of this, you will almost never see *e* used as a variable. It is named after Leonhard Euler (pronounced "Oiler"), a very important 18th-century Swiss mathematician.

e is approximately equal to 2.718281828... . Like the other constants we have looked at, its decimal representation goes on forever without a repeating pattern, so it is also an irrational number. The number e often comes up in situations involving compound interest or exponential growth or decay.

Transcendental Numbers

As strange as irrational numbers are, there is an even more bizarre type of number than irrational numbers: **transcendental numbers**. Transcendental numbers are numbers that are not a solution to any polynomial equation. For instance, $\sqrt{2}$ is a solution to the polynomial equation $x^2 = 2$ (sometimes written as $x^2 - 2 = 0$ if it is required that the right side be zero). That means that $\sqrt{2}$ is *not* a transcendental number.

In fact, roots are never transcendental number. Neither are rational numbers. In other words, transcendental numbers are *always* irrational numbers, but they are even more exclusive than that since many irrational numbers (like roots) are not transcendental numbers.

Actually, we have seen two examples of transcendental numbers already: π and e are both transcendental numbers, though ϕ is not. That is because ϕ is a solution to the equation $x^2 - x - 1 = 0$, which is a polynomial equation.

Lesson Summary

An **irrational number** is a number that cannot be written as a fraction of two integers. The decimal representation of irrational numbers will always go on forever without a repeating pattern. Numbers like 3, 17, $\frac{7}{9}$, and 4.789 are *not* irrational numbers.

Common irrational numbers include:

- Most roots, except those like $\sqrt{16}$ and $\sqrt[3]{27}$ which can be simplified to rational numbers
- The constant π (written as **pi**), which is the ratio of the circumference of a circle divided by its diameter, and is approximately equal to 3.1415926535...

- The constant ϕ (written as **phi**), which is also called the "**golden ratio**," and is equal to $\frac{1+\sqrt{5}}{2}$ or approximately 1.6180339887...
- **Euler's number** e, which is approximately equal to 2.718281828...

Transcendental numbers are number that are not a solution to any polynomial equation. π and e are both transcendental numbers.

Video Transcript

What are Irrational Numbers?

Outside of mathematics, we use the word 'irrational' to mean crazy or illogical; however, to a mathematician, **irrational** refers to a kind of number that cannot be written as a fraction (ratio) using only positive and negative counting numbers (integers). For example, you can write the rational number 2.11 as 211/100, but you cannot turn the irrational number 'square root of 2' into an exact fraction of any kind.

Don't assume, however, that irrational numbers have nothing to do with insanity. Legend suggests that, around 500 B.C., a guy named Hippasus was thrown overboard from a ship by the Pythagoreans, a group of Greek philosophers, as punishment for proving that the square root of 2 is irrational.

A mental trick you can use to help you visualize whether a number is rational or irrational is to think of the number in terms of cutting pizzas. For instance, if a number is rational, you can imagine cutting pizzas into equal-sized slices described by the denominator of a fraction and then eating the number of slices described by the numerator. For example, 6/8 can be found by cutting a pizza into 8 slices and then consuming 6 of those slices.

For a number like 3.95, you imagine cutting pizzas into a hundred slices each and then taking 395 slices. While you'll probably never be quite that hungry, you can imagine it. A negative number like -3/10 is a little tougher, but you could still visualize it if you slice pizzas into tenths and then give back 3 slices.

It's impossible to think of the square root of 5 that way. While there might be some other way to figure out how to get exactly the square root of 5 pizzas, you can't do it by cutting the pizza into any set number of equal slices and then taking the correct share of those.

In most cases, the best we can do to visualize an irrational number is approximate it with a decimal number.

Let's look at some common irrational numbers.

Roots

Some of the most common irrational numbers are roots, such as the square root of 5 or the cube root of 7. Square roots, cube roots, and roots of any higher power are often irrational, as long as they can't be simplified in a way that the radical (square root) symbol vanishes.

Sometimes we write irrational numbers approximately as decimal numbers, but we can never do it exactly because the decimal places go on forever and never fall into a repeating pattern.

As the unlucky Hippasus demonstrated, there is no way to write the square root of 2 as an exact fraction. It is irrational. (Square root of 2 = 1.41421356...).

On the other hand, -5.2 can be written as -52/10, which means that it's a rational number, and even the Pythagoreans wouldn't issue a death sentence over it.

Pi

The circumference of a circle divided by its diameter is always a little more than 3. In fact, the result of this division is an irrational number that we commonly refer to as pi.

Pi is part of a group of special irrational numbers that are sometimes called **transcendental numbers**. These numbers cannot be written as roots, like the square root of 11.

Many people remember the first few digits of pi: 3.14. Remembering those digits can be helpful, but it is not exact since pi goes on indefinitely (pi = 3.141592...). As of 2011, people

have discovered more than 5 trillion digits of pi, but we'll never get to the end of it, because there is no end!

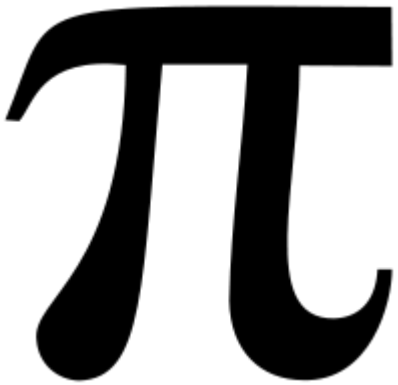
Sometimes you might see pi written as 22/7; however, be aware that, like 3.14, 22/7 is only an approximation. It is close to pi, but it's not equal. There is no fraction that exactly equals pi.

Euler's Number (*e*)

To mathematicians, *e* is more than just a letter in the alphabet. The irrational number ***e*** is formally named **Napier's constant**, but it is commonly called **Euler's number**, after Leonhard Euler (pronounced 'Oiler'). Just like pi, *e* occurs commonly in the real world.

Briefly, *e* is the result of adding a tiny bit to 1 and then raising that to a really big power. The resulting value (2.7182818284...) is irrational. The decimals go on forever without falling into a repeating pattern.

E is a very useful number in the worlds of science and business. It helps us calculate how things grow over time - the number of bacteria in a petri dish, the size of rabbit populations, or the interest your money earns in a savings account.



Symbol for pi

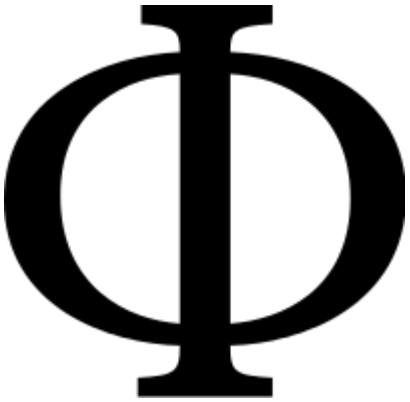
Phi

Another transcendental irrational number is derived from the ratios of the sides of certain geometric shapes. It is sometimes called the **golden ratio**, **golden mean**, or **divine proportion**, and it's represented by the Greek letter **phi**.

The ratio of longer to shorter sides of a five-pointed star

(**pentagram**) represent phi in several ways as shown by the colors in the picture below. If you divide any colored side by the next shorter colored side, you'll get phi.

Just like the other irrational numbers we've discussed, phi's decimal places go on forever (phi = 1.618033988...). In spite of the fact that it is based on a ratio, phi is not based on a ratio of integers, so you wouldn't be able to make exact pizza slices out of it.



Symbol for phi

determined by the denominator and then eat the number of slices determined by the numerator. It's an irrational number if you cannot. Common irrational numbers include **roots, pi, phi, and Euler's number**.

Learning Outcomes

Upon completing this lesson, you should be able to:

- Define irrational numbers
- Recall what a transcendental number is

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$$

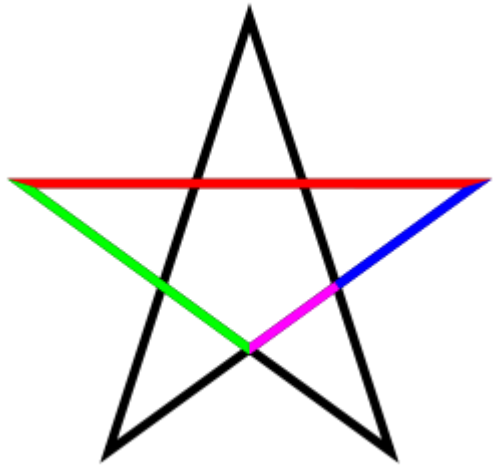
Formula for e

There have been many claims of the golden ratio appearing in nature, the human body, art, and architecture. The golden ratio is considered very pleasing to the human eye, as shown by the Mona Lisa, our galaxy, and the Egyptian pyramids, all of which have dimensions that are close to phi.

Lesson Summary

Let's review. **Irrational numbers** are those that can't be written as a fraction comprised of only integers. Think of a pizza - it's a rational number if you can cut the pizza into equal-sized slices

- Describe four types of common irrational numbers



Pentagram

Frequently Asked Questions

What are 3 irrational numbers?

Three examples of irrational numbers are $\sqrt{2}$, π, and e. Most roots are irrational numbers and irrational numbers can be positive or negative.

Is 4 an irrational number?

No, 4 is not an irrational number. It can be written as the fraction 4/1, which means that it is rational, not irrational.

Is 7/9 an irrational number?

No, 7/9 is not an irrational number. It is the ratio of two integers, so it is rational, not irrational. In addition, it is equal to 0.7777777777..., which clearly has a repeating pattern.

Is 4.789 an irrational number?

No, 4.789 is not an irrational number. It is a terminating decimal, and terminating decimals are never irrational. In addition, it can be written as a fraction of two integers: 4.789 = 4,789/1,000.

What is an irrational number in math?

An irrational number is a number that cannot be written as a fraction of two integers. By looking at the decimal representation of a number, you can tell whether it is rational or irrational. For an irrational number, the decimal representation will always go on forever without a repeating pattern.