Continuity

In broad terms, a **continuous function** is a function that has a “smooth,” unbroken curve at every point on the interval. The graph of a **discontinuous function** is “broken” at one or more points. The graphs below illustrate these points.

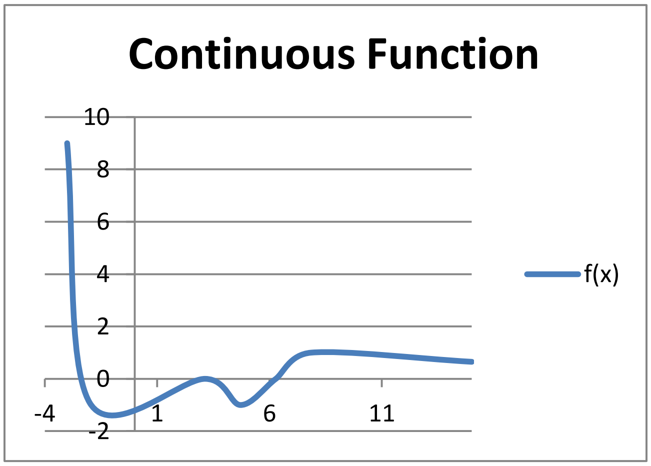


Figure 1: A “smooth” continuous function at every point.

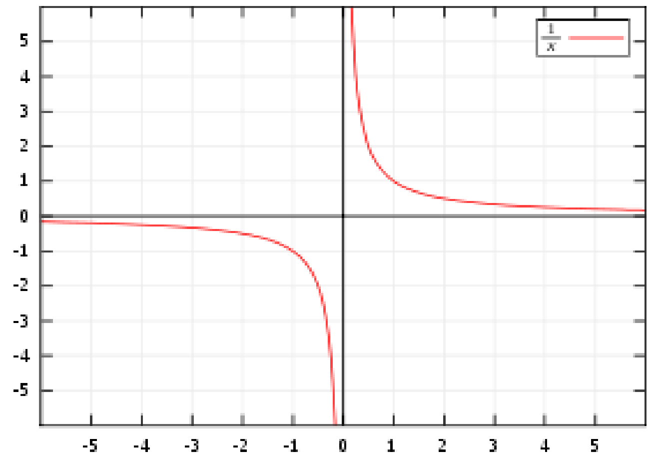


Figure 2: A discontinuous function at x = 0.

In more formal terms, a function f(x) is said to be continuous at a point x = a if

$$\lim_{x\to a} f(x) = f(a)$$

Given the definition of continuity, the following must be true:

1. $\lim_{x\to a} f(x)$ exists
2. f(a) exists
3. $\lim_{x\to a} f(x)\ is\ equal\ to\ f(a)$

__Note:__ The definition of continuity allows us to directly plug in the point “a” into a continuous function to evaluate the limit.

A point on a function may be discontinuous in the following three ways:

1. Infinite discontinuity
2. Removable discontinuity
3. Jump discontinuity

1. **Infinite Discontinuity**

Figure 2 above is an example of an infinite discontinuity at the point x = 0. In this case we have a vertical asymptote at x = 0 and evaluating the limit at that point would result in ∞ or $-\infty$. Since the limit does not exist at x = 0, the function is not continuous at that point.

2. **Removable Discontinuity**

A removable discontinuity is when there is a hole on the curve of a function.

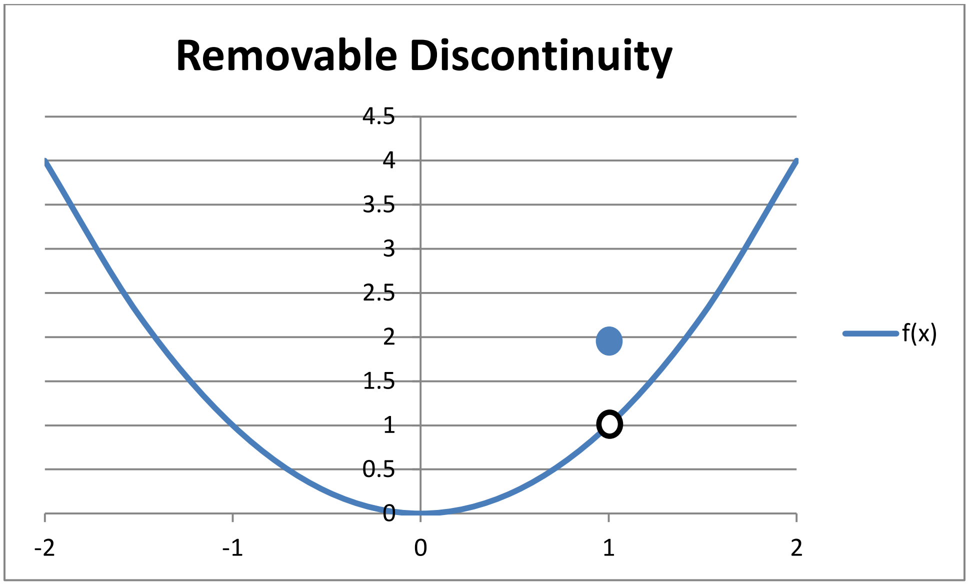


Figure 2: Removable discontinuity at x=1

In **Figure 3**, we see a hole at the point x = 1. Evaluating the limit from the left and right hand side of x = 1, we have

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = \lim_{x\to 1} f(x) = 1$$

However, evaluating $f(1)$ we notice $f(1) = 2$.

Since, $\lim_{x\to 1} f(x) \neq f(1)$ the function is not continuous at the point x = 1.

3. **Jump Discontinuity**

As the name implies, a jump discontinuity is when we have a "jump" in the function.

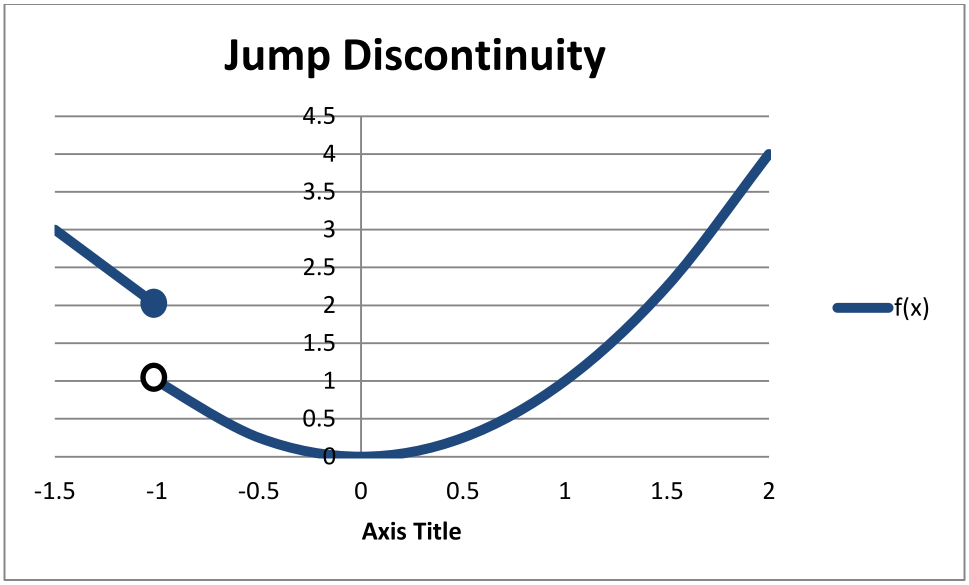


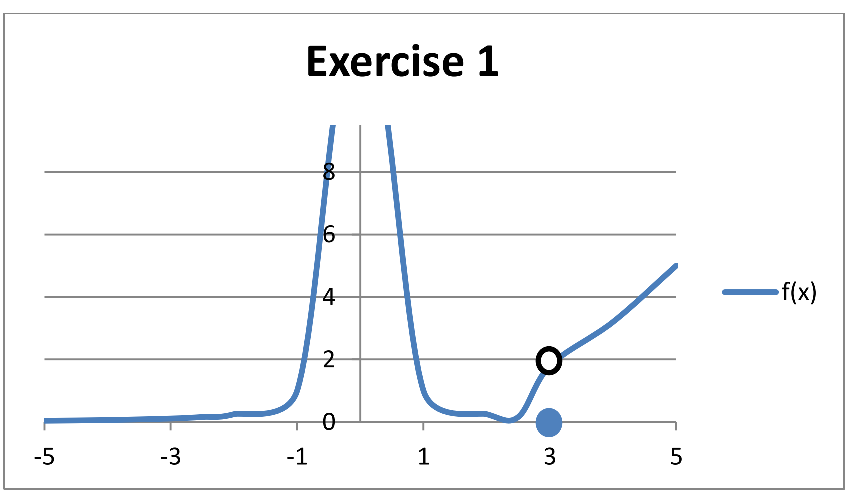
Figure 3: Jump discontinuity at x= −1

In **Figure 4**, we can see that there is a break in the function at the point x = −1. Since the left-hand side and right-hand side limits do not equal as x approaches -1, the limit does not exist at that point. Thus, the function is discontinuous at the point x = -1.

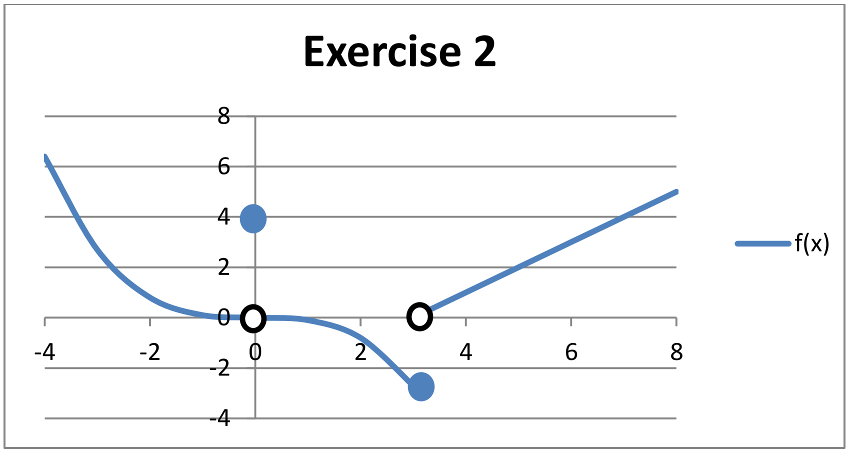
Exercises:

In each case, determine where the function is discontinuous and identify the type of discontinuity.

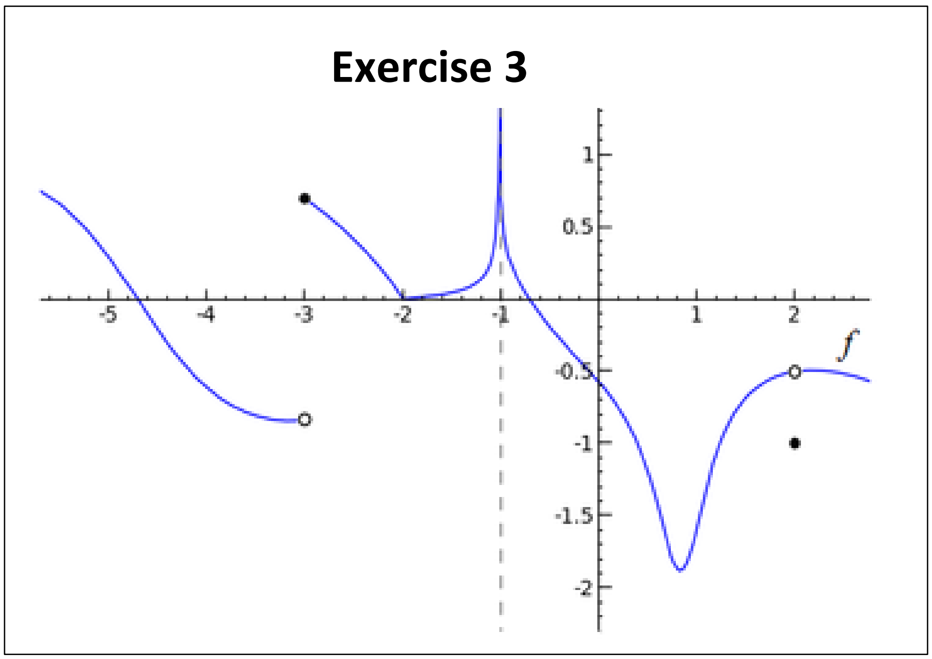
a)



b)



c)



Solutions:

a) Discontinuous at x=0, Infinite discontinuity
Discontinuous at x=3, Removable discontinuity

b) Discontinuous at x=0, Removable discontinuity
Discontinuous at x=3, Jump discontinuity

c) Discontinuous at x=-3, Jump discontinuity
Discontinuous at x=2, Removable discontinuity