Study.com

What are Real Numbers? - Definition & Properties

Contributors: Riley Kench, Jennifer Beddoe, Kathryn Boddie

What are real numbers? Learn about the different types of real numbers that occur in mathematics as well as how to identify them. In this lesson, you will learn the definition of real numbers, examples of real numbers, and what real numbers mean in mathematics.

Table of Contents

Introduction to Real Numbers

In this lesson, the primary concern will be the classification of numbers, particularly the distinction between real and imaginary numbers. There are a myriad of ways in which numbers can be classified in mathematics. There are several terms such as *natural* numbers, *positive* numbers, *negative* numbers, *integers*, *rational* numbers, *irrational* numbers, and *imaginary* numbers that appear in mathematics. These are all classifications of numbers that occur in mathematics.

The classification of numbers as *real* versus *imaginary* is the topic of this lesson. First, we will present the definition of real numbers and how it is distinguished from imaginary numbers, as well as other number classifications. Then, examples of real numbers, rules related to real numbers, and the properties of real numbers will be explored.

What are Real Numbers?

In mathematics, **real numbers** are defined as the combination of rational and irrational numbers. Rational numbers are any numbers that can be represented by a fraction: {eq}\frac{a}{b} {/eq} where both {eq}a, b {/eq} are integers and {eq}b \neq 0 {/eq}. Irrational numbers are simply not rational numbers in that they cannot be represented as a fraction of two integers. Numbers like pi and e are irrational.

If the sets of rational and irrational numbers are combined, we will get the set of real numbers. Real numbers are informally any number that can be expressed as an infinite decimal. They arise in measurement and counting as well.

Here are some examples:

- {eq}\pi {/eq} is a real number.
- {eq}-13 {/eq} is a real number.
- {eq}\sqrt{-1} {/eq} is *not* a real number.

The first two examples are of real numbers. Notice how pi is an irrational number but still is a real number. In the second example, the number is negative, but since it is still an integer (and a rational number), it is also classified as a real number. The number {eq}i=\sqrt{-1} {/eq} is an **imaginary number**. An imaginary number is any non-real number; that is, any number that is not real is therefore imaginary. An imaginary number is any number of the form {eq}a+bi {/eq} where a and b are real numbers and {eq}i=\sqrt{-1} {/eq}. If the b term in this expression is zero, then the number is wholly real.

This gives rise to another definition of a real number. Namely, a real number is any number that contains no imaginary part.

Imaginary numbers are useful in solving complex equations, or equations involving imaginary variables. For instance, instead of using the variable x in a real algebraic equation, one would use z to indicate that we are now dealing with complex (or imaginary) numbers. The equation would behave similarly to real algebraic equations, but exploring that is beyond the scope of this lesson.

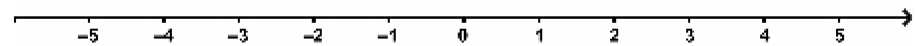
Another non-real number would be {eq}\infty {/eq}. Since infinity cannot be classified as a rational or irrational number, it is not real.

Now, let's examine a number line:

Any real number can be found on a real number line so long as the line is continued enough to find the given number. This gives rise to the questions: can real numbers be negative? Can real numbers be decimals? The answer to both of these questions is yes.

On this number line, each of the labeled numbers is a real number. Moreover, all of the numbers represented on the continuum are real; that is, numbers with decimal value, as represented on this line, are also real numbers.

The entire line itself contains only real numbers, although only the integers are labeled. Notice again that negative numbers are included on the number line because negative numbers are also real numbers.



A Real Number Line

Rules of Real Numbers

There are three rules of real numbers that all real numbers must follow:

- The first rule is that real numbers must be measurable. That is, given any two real numbers {eq}a, b {/eq} such that {eq}b>a {/eq}, we can *measure* the difference between them by {eq}b-a {/eq}.

- The second rule is that all real numbers must have real value. This is intrinsic to real numbers as if they did not have real value, they would not be classified as real. Consider an imaginary number that contains i: {eq}a+bi {/eq} where {eq}a, b {/eq} are real numbers. Since this number has i in its expression, it does not have a real value and is therefore not real.

- The final rule for real numbers is that they must be able to be manipulated or changed to other forms via mathematical operations. That is, given a real number, it is possible to mathematically manipulate it to give a different expression without changing its underlying value. For instance, consider the infinite decimal {eq}\overline{0.333} {/eq}. Another representation for this would be {eq}\frac{1}{3} {/eq}. Although these two expressions are different, they still possess the same real value.

Finally, there is another rule that the real numbers must follow: it is that all real numbers except zero must be either positive or negative. This is apparent since zero has no value while all other numbers must be either positive or negative.

These rules are essential to classifying real numbers and determining which numbers are in fact real numbers.

Properties of Real Numbers

There are several mathematical properties that apply to real numbers. These properties dictate how real numbers operate with each other. In this instance, consider only the operations of addition and multiplication. The first property is called the closure property:

- The **closure property** for real numbers states that for any pair of real numbers {eq}a, b {/eq}, the sum and products of {eq}a, b {/eq} are also real numbers. That is, {eq}{a*b} {/eq} and {eq}a+b {/eq} will give real numbered results when calculated.

- The **associative property** of real numbers works for both addition and multiplication. They state that for any real numbers {eq}a, b, {/eq} and {eq}c {/eq}, we have that {eq}a+(b+c)=(a+b)+c {/eq} and {eq}a*(b*c)=(a*b)*c {/eq}. In general, the associative property for real numbers guarantees that no matter how the terms are grouped or associated in an expression involving real numbers, the result will be the same.

- The **commutative property** for real numbers states that for any real numbers {eq}a, b {/eq}, we have that {eq}a*b=b*a {/eq} and {eq}a+b=b+a {/eq}. This property states that no matter the order of the real numbers, the result after the operation will be the same.

- The **distributive property** for real numbers governs the product of a real number with the sum of two other real numbers. For any real numbers {eq}a, b {/eq} and {eq}c {/eq}, we have that:{eq}a*(b+c)=a*b+a*c {/eq}.

Types of Real Numbers

There are several subsets of real numbers that have their own unique relations. It is useful to look at real numbers as a set as well as the various subsets of the real numbers.

- The **real numbers** are the set of all non-imaginary numbers; or, they are the set of the irrational and rational numbers. The sign used to represent real numbers in mathematics is {eq}\mathbb{R} {/eq}.

- The next set is the **whole numbers**. These are defined as the counting numbers when counting from zero to infinity. The symbol for whole numbers is {eq}\mathbb{W} {/eq}. The set would look like {eq}\mathbb{W}=\lbrace 0, 1, 2, ... \rbrace {/eq}.

- The set of **positive numbers** is the set of all whole numbers that are positive and excluding zero. It is symbolized by {eq}\mathbb{Z}^{+} {/eq}. The set would look like {eq}\mathbb{Z}^{+}= \lbrace 1, 2, 3, ... \rbrace {/eq}.

- The set of **negative numbers** is the set of all whole numbers that are not positive and excluding zero. It is also considered the complement of the whole numbers. Its symbol is {eq}\mathbb{Z}^{-} {/eq}. This set would look like {eq}\mathbb{Z}^{-}= \lbrace ..., -2, -1 \rbrace {/eq}.

- The set of **integers** is the set of all whole numbers including all of the negative whole numbers. It is represented by {eq}\mathbb{Z} {/eq}. This set would look like {eq}\mathbb{Z}= \lbrace ..., -3, -2, -1, 0, 1, 2, 3, ... \rbrace {/eq}.

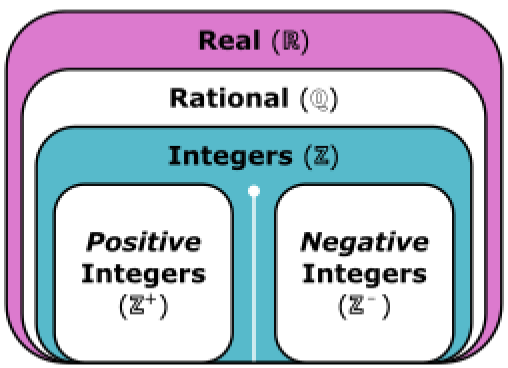
- The set of **natural numbers** are a subset of the whole numbers. Instead of starting from zero, the natural numbers start from one. They are represented by the symbol {eq}\mathbb{N} {/eq}. This set would look like {eq}\mathbb{N}= \lbrace 1, 2, 3, ... \rbrace {/eq}.

- The **rational numbers** are any number that can be represented as a fraction such that the denominator of that fraction is nonzero. The symbol for the rational numbers is {eq}\mathbb{Q} {/eq}. This set would contain an uncountable amount of fractions so enumerating it would be tedious.

- The **irrational numbers** are simply the numbers that are not rational. That is, they cannot be represented as a fraction. This includes numbers like e and pi. The symbol for the irrational numbers is {eq}\mathbb{Q}^{c} {/eq}. This set is also uncountable so it would be useless to try to write it down.

These sets of numbers are all subsets of the real numbers. Many of these subsets are complements or even subsets of each other, so it is likely that a real number will fit into multiple subsets listed above. For example, the set of integers is a subset of the set of rational numbers.

Below is a visual representation of some of the sets described above. This image shows how certain sets are subsets of other sets of numbers.



The sets of numbers shown visually

Examples of Real Numbers

Below are several examples of real numbers and explanations as to why they are considered real numbers using the Rule of Real Numbers from above.

- Is the number {eq}5 {/eq} is a real number? It has a real value as it has no imaginary parts. Moreover, since it is nonzero, it must be positive or negative. In this case, five is positive, making it a positive number. Moreover, five is a whole number since it has no decimal parts. It is also an integer because it is a whole number and also a positive number. Overall, five must be a real number.

- Is the number {eq}\overline{0.666} {/eq} a real number? It has a real value since there is no imaginary part to it. It is nonzero, so it must be either positive or negative. Since it is greater than zero, it must be positive, making it a positive number. However, if operations are performed on this decimal, it would show that it is equal to {eq}\frac{2}{3} {/eq}. This makes it a rational number since it is the division of two integers. Moreover, since it was manipulated into another form with equal value, this again demonstrates that it is a real number. In general, all numbers with decimal values are real numbers provided they do not contain any imaginary parts.

- Is zero a real number? Operating under the assumption that it is, let {eq}a {/eq} be a real number. Since every real number has a negative, {eq}-a {/eq} is also a real number. Finally, since the real numbers are closed under addition, we have that {eq}a+(-a) {/eq} must also be a real number. Therefore, {eq}a+(-a)=0 {/eq} implies that zero must be a real number. {eq}\square {/eq}.

Lesson Summary

In this lesson, the definition of a **real number** was presented in terms of two of its underlying subsets, rational and irrational numbers. It was also defined by first examining imaginary numbers. The rules of real numbers were presented and it was shown that they help us determine whether a number is real or not. The properties of real numbers were then examined. These properties will always hold for real numbers under the given operations. Finally, the different subsets of the real numbers, their definitions, and how they are symbolically represented in mathematics were explored. Below are several important points from this lesson:

- The **real numbers** are defined to be the set of numbers formed by combining the rational numbers and the irrational numbers.
- If the **imaginary numbers** are defined as any number having the form {eq}a+bi {/eq}, then the real numbers are all imaginary numbers such that the b term is always zero.
- Several **rules of the real numbers** were presented. They were: the real numbers are measurable, the real numbers have real value, and the real numbers can be mathematically manipulated into other forms that still possess the same value.
- The properties of real numbers were also examined in this lesson. The main properties examined were the **closure property**, the **associative property**, the **commutative property**, and the **distributive property**.
- The subsets of the real numbers were also defined and their symbols shown. The subsets presented were the real numbers as a set, the whole numbers, the positive and negative numbers, the integers, the natural numbers, and the rational and irrational numbers.
- Finally, several examples of real numbers were presented.

Video Transcript

What are Real Numbers?

Numbers can be grouped in many different ways. There are rational and irrational numbers, positive and negative numbers, integers, natural numbers and real or imaginary numbers. It can be difficult to keep them all straight. For this lesson, we will define **real numbers** and give some examples. Real numbers can be defined in many different ways; here are a few of the different types of ways to describe the set of real numbers.

3 Rules Describing Real Numbers

One way you can define real numbers is through the rules that govern them. There are three main rules:

First, real numbers are **measurable**.

This means that the set of real numbers are those numbers that can be mapped on a number line. The number line has three parts: a negative side, a positive side and the zero in between. Each side of the number line goes on infinitely; there is no end to the positive or negative numbers that make up the set of real numbers.

Second, real numbers have a concrete **value**.

You know what 15 marbles looks like, or half a cake. You can even know how many pieces are in the square root of 5 pizzas (if you have a calculator). The negative side works for this as well, especially when you are working with money. If your checking account balance is $-2.27, you probably should start to worry.

Finally, real numbers can be **manipulated**.

For example, all real numbers can be rewritten as a decimal. This just means that even if a number looks odd, or has a strange symbol or Greek letter associated with it, it's still a real number. Let's look at a couple of examples:

- 1/4 = 0.25
- The square root of 7 = 2.645751311
- pi = 3.14159...

Another way in which real numbers can be manipulated is by mathematical operations. All of the set of real numbers can be added, subtracted, multiplied or divided with each other, and the result will be another real number, which can also be written as a decimal. For example: 5 + 2 = 7, 3/4 * 1/8 = 0.09375, or the square root of 17 - pi = 0.98

In addition to our three rules, another way to understand the definition of **real numbers** is to remember that they are not imaginary. This might seem obvious, but the need to define a number as real was not necessary until imaginary numbers were discovered. Imaginary numbers are used in mathematics to describe numbers that do not actually exist, but can be used in high level calculations.

Types of Real Numbers

The set of real numbers can be divided into many different groups. Each of these groups has their own special set of characteristics, but they are all still real numbers.

- **Whole numbers** are positive numbers that are not fractions or decimals. They include zero.
- **Positive numbers** include all numbers that are greater than zero. They can be fractions, decimals or whole numbers.
- **Negative numbers** include all numbers that are less than zero. They can also be fractions or decimals.
- **Integers** are whole numbers and their opposite negative numbers (without fractions or decimals).
- **Natural numbers** are the counting numbers (1,2,3,...). They are whole numbers that exclude zero.
- **Rational numbers** are fractions that when written as a decimal either have an end point (0.5) or repeat (1.3333333. . .).
- **Irrational numbers** are numbers that when written as a decimal have no end point (2.5463489762547. . .). Pi is a well-known irrational number. Although it is often abbreviated as 3.14, the numbers after the decimal go on forever.

Lesson Summary

Let's review. Even though there are many definitions that point to the different properties of real numbers, they all describe the same thing. A set of **real numbers** obeys these three rules:

1. The numbers are measurable
2. The numbers have concrete value
3. The numbers can be manipulated

And, of course, the numbers are real, not imaginary. Types of real numbers include **whole** numbers, **rational** numbers and **irrational** numbers.

Learning Outcomes

When you are finished, you should be able to:

- Determine if a number is a real number using the three rules of real numbers
- Name the types of real numbers

Activities FAQs

Discovering π as a Real Number by Measuring

In this activity, we will discover the real number π by taking real world measurements. As discussed in the lesson, π is a real number because it can be measured, has a concrete value, and can be manipulated (for example, it can be written as a decimal). You will need the following materials:

Materials

1. Objects that contain circles of different sizes - these should be perfect circles. Some possibilities include different sized food cans, mugs, etc.

2. String - twine is preferred - it works best if the string cannot be stretched very far.

3. Ruler

4. Calculator

Directions

1. Take the string and wrap it exactly once around the circular part of the object. For example, if it is a food can, set the can on a table and wrap the string around the base of the can directly on top of the table. Mark the spot on the string where you finish wrapping once.

2. Measure the distance marked on the string by straightening the string out along the ruler. Write down this measurement and label it as the circumference for this object.

3. Take the ruler and measure the diameter of the circle on your object. The diameter is the straight line dividing the circle directly in half. Try to get as close to dividing the circle in half as possible. Write down this measurement and label it as the diameter for this object.

4. Repeat steps 1-3 for any other objects. Clearly label which numbers are from step 2 and which are from step 3 for each object.

5. Now it is time to discover π. From your measurements, take the circumference for each object divided by the diameter for each object C/d. Type this into the calculator. Write the result.

Results and Discussion

1. Are the results after typing into the calculator similar for all of the objects?

2. The real number π is equal to the circumference of a perfect circle divided by the diameter of that circle. π is approximately 3.14159. Are your results similar?

The results may not be exact, due to human error in precise measurements. If your answers are far from 3.14159, try measuring again very carefully. π is a real number that can be found by measuring real objects.