Abstract

July 2023

1 Abstract

In the 1960s and 70s Vladimir Vapnik and Alexey Chervonenkis were developing new ideas in statistical learning theory. They introduced VC-Dimension, a measure of the expressive power of sets. Since then, the notion has been thoroughly applied in areas such as extremal combinatorics, additive combinatorics, group theory, model theory. In particular, lots of exploration has been done in finite fields and $\mathbb{R}^n$. Some elementary examples in $\mathbb{R}^n$ include the VC-dimension of spheres, axis-parallel boxes in $\mathbb{R}^n$. These sets can be thought of as generalizations of arithmetic progressions. However notions of VC-dimension in non-abelian group structures are much less understood.

And so our work has been focused on free groups and nilpotent groups. The main kinds of sets we have been investigating within these groups are (non-abelian) progressions. To bound the VC-dimension of a progression in a free group, we have examined the properties of Cayley graphs and translated the problem to a graph theoritic one about relative distances of end points of a tree. Alternatively, we have formulated the problem of VC-dimension of progressions in the Heisenberg group into one of sets of polynomials. In both cases, we have also used computational evidence to gain insights on the behavior of different sets in these groups. We have also employed well known results in the VC-dimension and model theory literature: the Sauer-Shelah lemma and the model theory of semi-algebraic sets. Consequently, we have two main theorems:

Theorem 1 *The VC-Dimension of progressions in the free group, F_k, on k generators has the following bounds:*

$$k \leq VC_{F_k} \leq O(k \log k) \tag{1}$$

Theorem 2 *The VC-Dimension of progressions in the Heisenberg group with respect to its canonical generators, is bounded by a constant independent of the size of the progressions.*