A Survey on Type Theory and Logic

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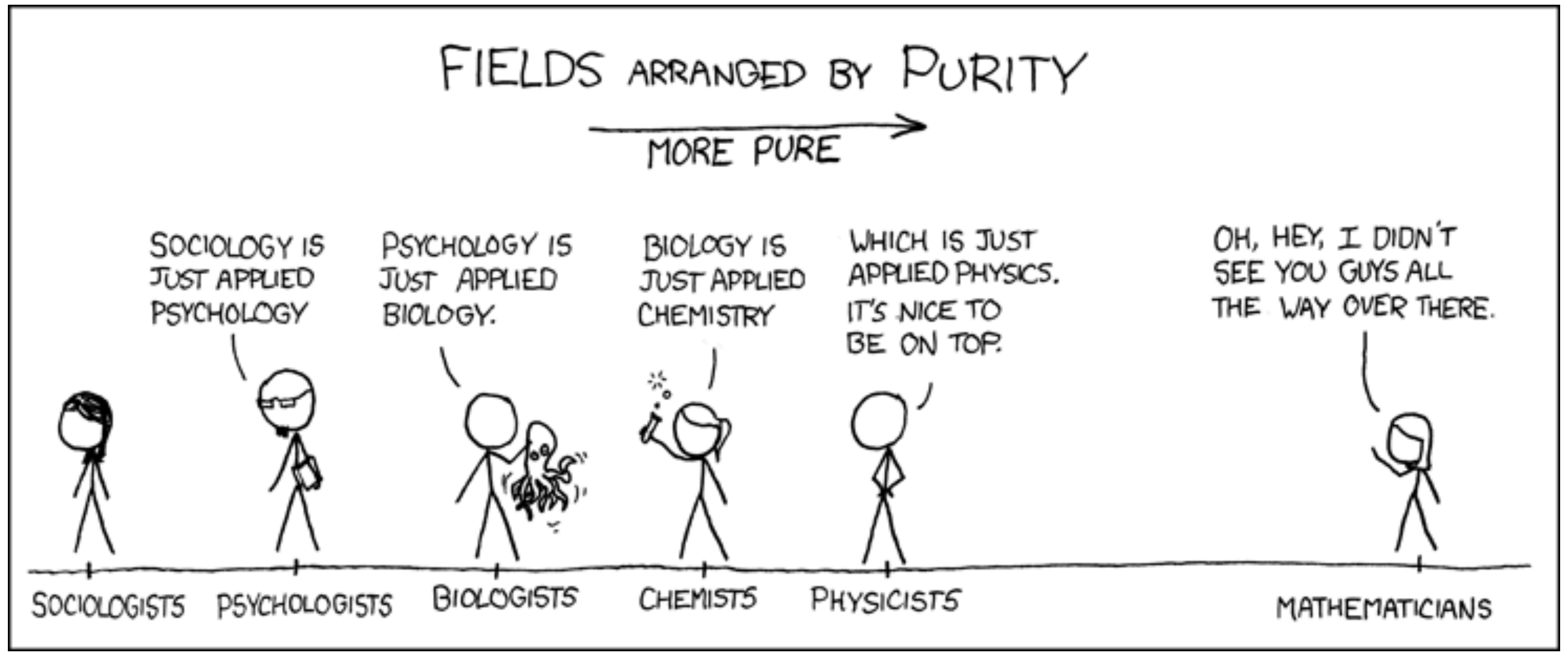
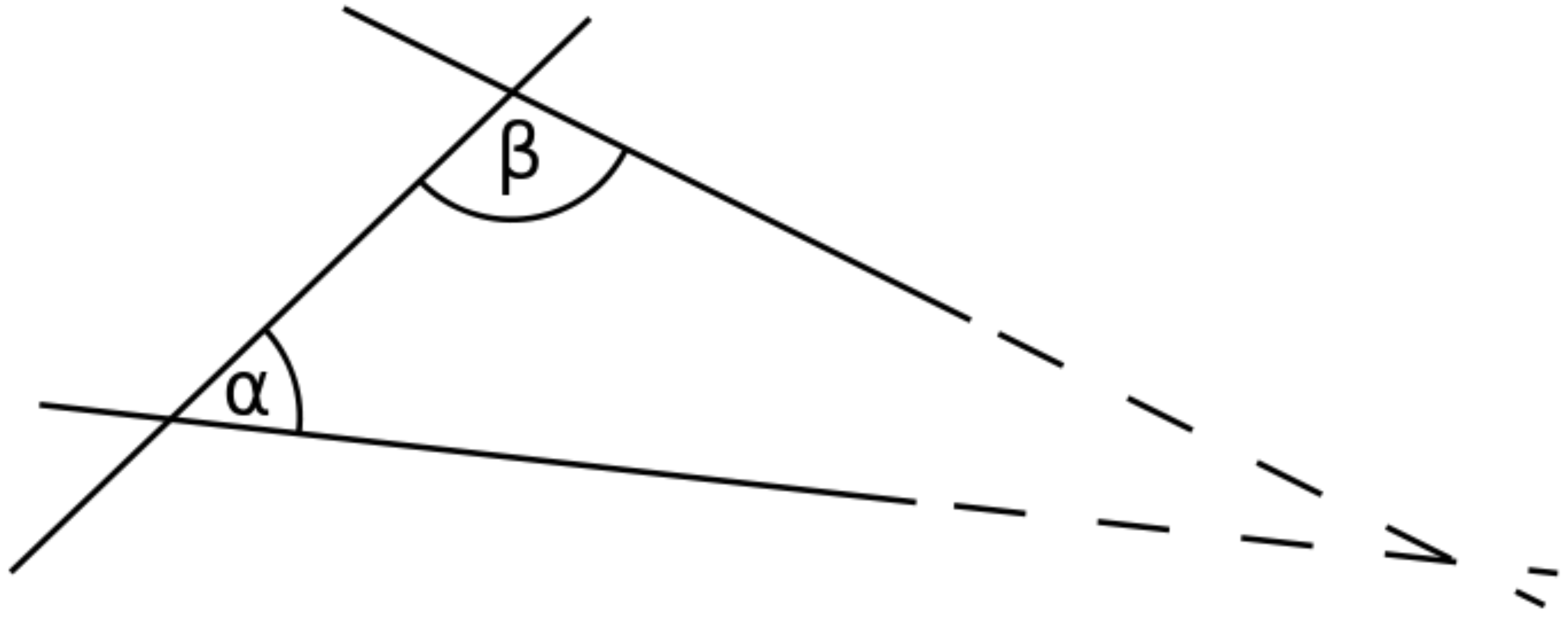
Why Study Logic/Type Theory?

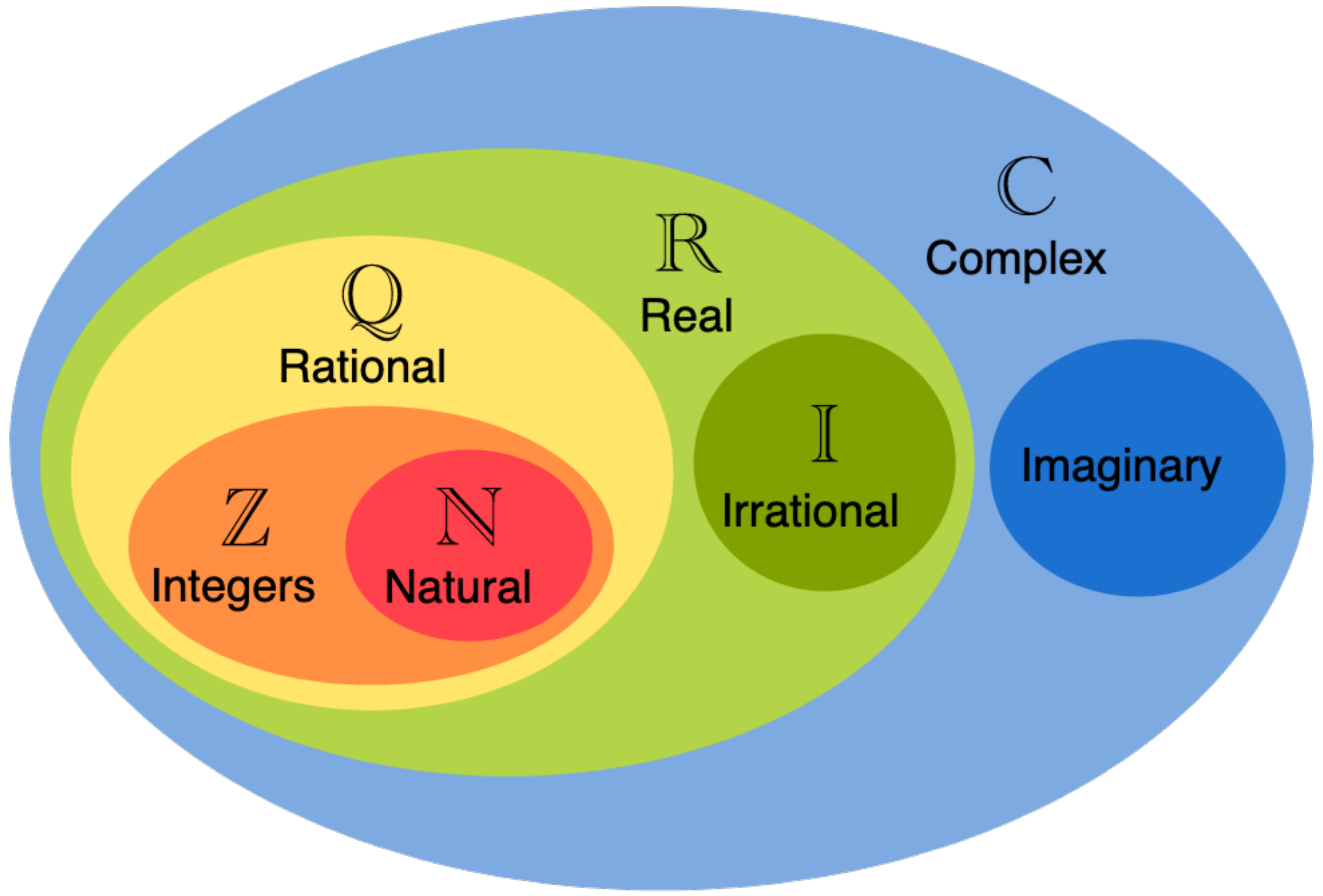
- Proof assistants/automated theorem proving
- Programming language design
- Compilers and domain specific languages
- Ties with mathematics
- Pedagogical bridge



Formal Systems and “Objectivity” of math

- Mathematics is built on consistent axioms
- Examples:
  - Euclid’s postulates
  - Gödel’s Theorems
  - ZFC





Classical vs Constructive Logic

- Propositional/First Order Logic
  - Propositions, predicates, quantification
  - Used to formalize modern mathematics
- Constructive Logic
  - Not as canonical
  - Law of Excluded Middle/Double negation

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overline{\forall x\ A(x)}^{\,1}}{A(y)} \qquad \dfrac{\overline{\forall x\ B(x)}^{\,2}}{B(y)}}{A(y) \wedge B(y)}}{\forall y\ (A(y) \wedge B(y))}}{\forall x\ B(x) \to \forall y\ (A(y) \wedge B(y))}\,{}^{2}}{\forall x\ A(x) \to (\forall x\ B(x) \to \forall y\ (A(y) \wedge B(y)))}\,{}^{1}$$

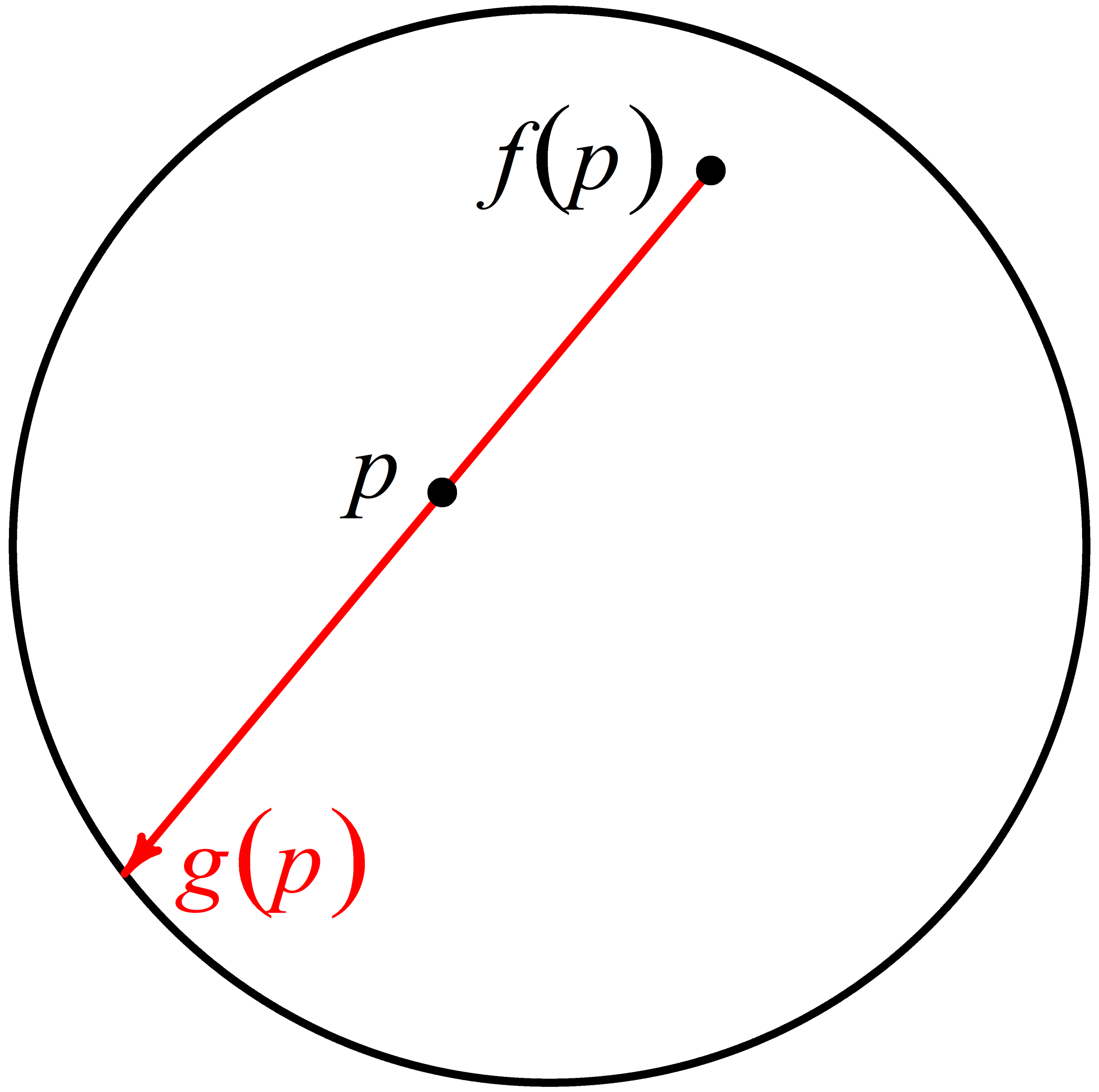
$$\forall P, \quad P \vee \neg P$$

The “Classic” Example

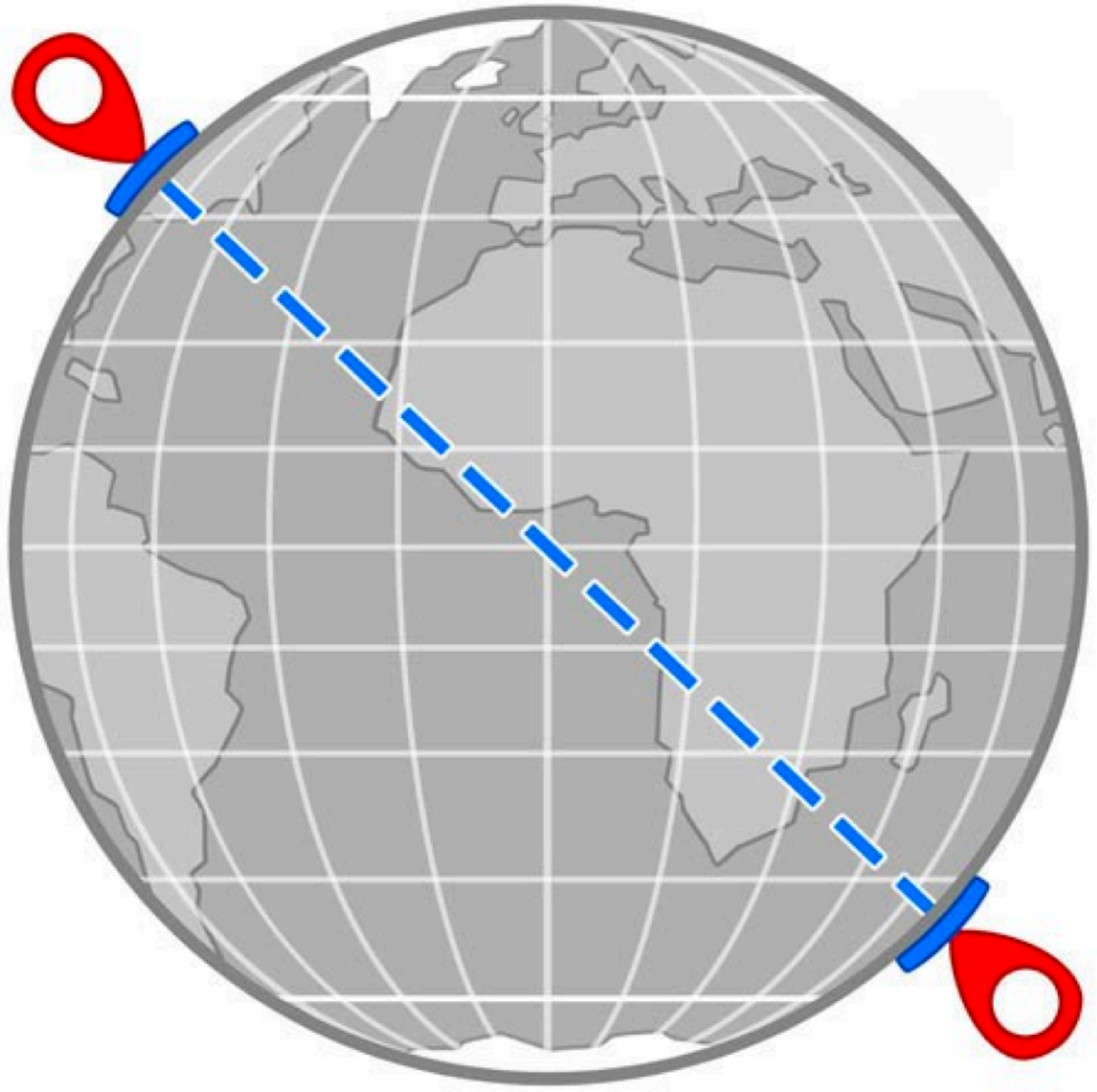
Prove that there exists rational a^b, for irrational a and b

- Consider $\sqrt{2}^{\sqrt{2}}$
    - If it is rational, then we are done
    - If it is not rational, then consider this number to the $\sqrt{2}$ power
- Have we actually found such a number?

Brouwer's Fixed Point theorem



- Every continuous function $f : B^2 \to B^2$ has a fixed point, i.e. f(x) = x for some x.
- Implies law of excluded middle
- "Earth Theorem"
- "Map theorem"

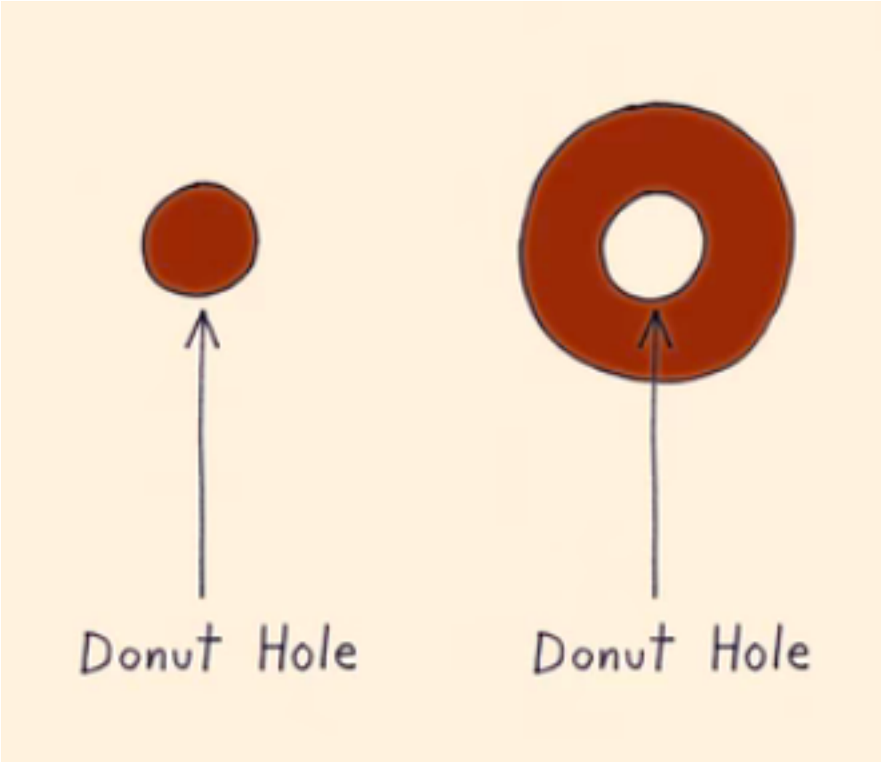


Functions and Computability

- Mathematical functions are just abstractions:
  - Pairs of input(s) and output(s)
  - $f : \mathbb{R} \to \mathbb{R}$
    - $f(x) = x^2$
  - $h : \mathbb{R} \to \mathbb{R}$
    - $h(x) = 1$ if P = NP, 0 if P $\neq$ NP
  - Functions are computable without “choice” or LEM

“Usefulness”

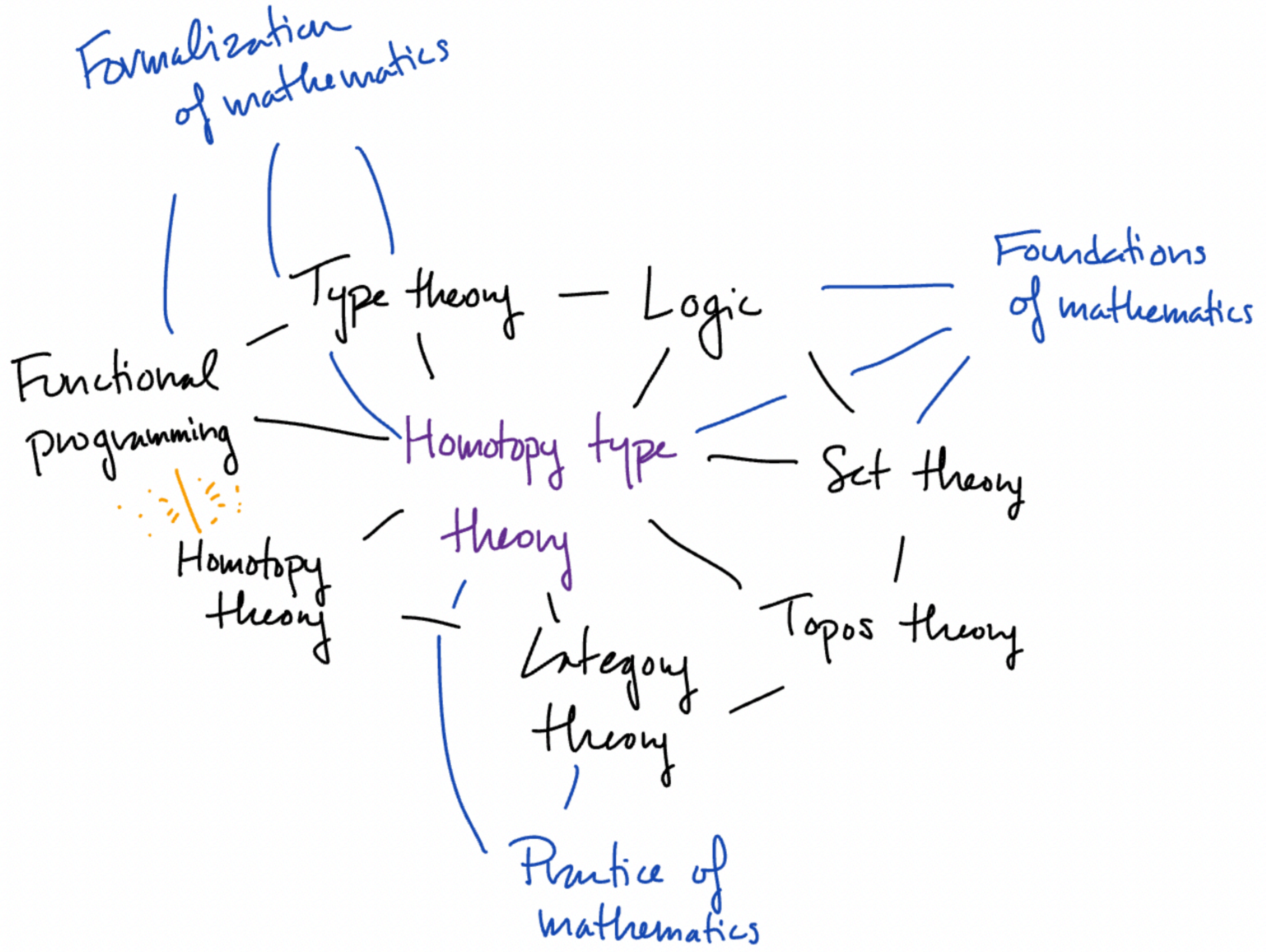
- Is it the case that all nonconstructive proofs/ results are useless?
- Existence of Nash equilibrium, algebraic topology
- “The Topological Structure of Asynchronous Computation” by Maurice Herlihy and Nir Shavit



	A	B
C	(1,1)	(2,3)
D	(4,1)	(2,2)

What is Type Theory?

- Again no “canonical” type theory
- Generally “constructive”
- Homotopy Type Theory
    - Ideas align with Category Theory
- A formal language



Judgements

- 4 kinds
  - Type A is well defined in a context:
    - $\Gamma \vdash A$ type
  - a is an “element” of type A:
    - $\Gamma \vdash a : A$
  - Judgemental equalities:
    - $\Gamma \vdash A \doteq B$ type
    - $\Gamma \vdash a \doteq b : A$

Curry Howard Correspondence

Types	Logic	Sets	Homotopy
A	proposition	set	space
$a : A$	proof	element	point
$B(x)$	predicate	family of sets	fibration
$b(x) : B(x)$	conditional proof	family of elements	section
$\mathbf{0}, \mathbf{1}$	$\bot, \top$	$\emptyset, \{\emptyset\}$	$\emptyset, *$
$A + B$	$A \vee B$	disjoint union	coproduct
$A \times B$	$A \wedge B$	set of pairs	product space
$A \to B$	$A \Rightarrow B$	set of functions	function space
$\sum_{(x:A)} B(x)$	$\exists_{x:A} B(x)$	disjoint sum	total space
$\prod_{(x:A)} B(x)$	$\forall_{x:A} B(x)$	product	space of sections
Id_A	equality $=$	$\{ (x, x) \mid x \in A \}$	path space A^I

Table 1: Comparing points of view on type-theoretic operations

Basic Types/Rules

Product Type

$$\frac{\Gamma \vdash a : A \qquad \Gamma \vdash b : B}{\Gamma \vdash (a, b) : A \times B}$$

$$\frac{\Gamma \vdash x : A \times B}{\Gamma \vdash pr_2(x) : B} \qquad \frac{\Gamma \vdash x : A \times B}{\Gamma \vdash pr_1(x) : A}$$

Basic Types/Rules

Implication Type

$$\frac{\Gamma \vdash a : A \vdash b : B}{\Gamma \vdash \lambda(a : A).b : A \to B}$$

$$\frac{\Gamma \vdash f : A \to B \qquad \Gamma \vdash a : A}{\Gamma \vdash f(a) : B}$$

Basic Types/Rules

Pi Type

$$\frac{\Gamma \vdash x : A \vdash b : B(x)}{\Gamma \vdash \lambda(x : A).b : \prod_{x:A} B(x)}$$

$$\frac{\Gamma \vdash f : \prod_{x:A} B(x) \qquad \Gamma \vdash a : A}{\Gamma \vdash f(a) : B(x)}$$

A Formal Proof

$$\dfrac{\dfrac{x : A \times B \vdash pr_2(x) : B \qquad x : A \times B \vdash pr_1(x) : A}{x : A \times B \vdash (pr_2(x), pr_1(x)) : B \times A}}{\lambda(x : A \times B).(pr_2(x), pr_1(x)) : \prod_{x:A\times B} B \times A}$$

Inductive Types

Booleans

$$\frac{}{\vdash \text{bool type}}$$

$$\frac{}{\text{true} : \text{bool}}$$

$$\frac{}{\text{false} : \text{bool}}$$

$$\frac{\Gamma, x : \text{bool} \vdash D(x)\ \text{type} \qquad \Gamma \vdash a : D(\text{true}) \qquad \Gamma \vdash b : D(\text{false})}{\Gamma, x : \text{bool} \vdash \text{ind}_{bool}(a, b, x) : D(x)\ \text{type}}$$

Example

“Not” function

$$\dfrac{\dfrac{\begin{array}{r}\Gamma, x : \mathrm{bool} \vdash \mathrm{bool}(x) \;\; \mathrm{type} \\ \Gamma \vdash \mathrm{false} : \mathrm{bool}(\mathrm{true}) \\ \Gamma \vdash \mathrm{true} : \mathrm{bool}(\mathrm{false})\end{array}}{\Gamma, x : \mathrm{bool} \vdash \mathrm{ind}_{bool}(\mathrm{false}, \mathrm{true}, x) : \mathrm{bool}(x) \;\; \mathrm{type}}}{\Gamma \vdash \lambda(x : \mathrm{bool}).\mathrm{ind}_{bool}(\mathrm{false}, \mathrm{true}, x) : \mathrm{bool} \to \mathrm{bool}(x) \;\; \mathrm{type}}$$

Universes and Homotopy Type Theory

- Turns out more machinery is needed
  - Does 0 = 1?
- Universes: Types with “types as terms”
  - No more interpretation into sets
  - Streamlined category theory
- Interpretation of types as spaces