Linear–Gaussian Bayesian Update
(Kalman Measurement Update)

Setup

Let the state be $x \in \mathbb{R}^n$ and the measurement be $y \in \mathbb{R}^p$.

Prior. Assume a Gaussian prior distribution

$$p(x) = \mathcal{N}(x; \hat{x}', P'), \tag{1}$$

where $\hat{x}' \in \mathbb{R}^n$ is the prior mean and $P' \in \mathbb{R}^{n \times n}$ is the prior covariance ($P' \succ 0$).

Measurement model. Assume the linear sensor model

$$y = Cx + v, \qquad v \sim \mathcal{N}(0, Q), \tag{2}$$

where $C \in \mathbb{R}^{p \times n}$ and $Q \in \mathbb{R}^{p \times p}$ is the measurement-noise covariance ($Q \succ 0$).

Likelihood. Conditioning on x, we have $y \mid x = Cx + v$, hence

$$p(y \mid x) = \mathcal{N}(y; Cx, Q) \ \propto\ \exp\Big(- \tfrac{1}{2}(y - Cx)^\top Q^{-1}(y - Cx)\Big). \tag{3}$$

Bayesian Update

By Bayes' rule,

$$p(x \mid y) \propto p(y \mid x)\, p(x). \tag{4}$$

Taking logs and dropping constants independent of x,

$$\log p(x \mid y) = -\tfrac{1}{2}(x - \hat{x}')^\top (P')^{-1}(x - \hat{x}') - \tfrac{1}{2}(y - Cx)^\top Q^{-1}(y - Cx) + \text{const}. \tag{5}$$

Complete the Square

Expand and collect terms in x.

Prior term.

$$(x - \hat{x}')^\top (P')^{-1}(x - \hat{x}') = x^\top (P')^{-1} x - 2\hat{x}'^\top (P')^{-1} x + \hat{x}'^\top (P')^{-1} \hat{x}'. \tag{6}$$

Likelihood term.

$$(y - Cx)^\top Q^{-1}(y - Cx) = y^\top Q^{-1}y - 2y^\top Q^{-1}Cx + x^\top C^\top Q^{-1}Cx. \tag{7}$$

Substituting back and collecting the quadratic and linear terms in x yields

$$\log p(x \mid y) = -\tfrac{1}{2}x^\top \Big((P')^{-1} + C^\top Q^{-1}C\Big)x + x^\top \Big((P')^{-1}\hat{x}' + C^\top Q^{-1}y\Big) + \text{const.} \tag{8}$$

A Gaussian in canonical form satisfies

$$-\tfrac{1}{2}(x - \hat{x})^\top P^{-1}(x - \hat{x}) = -\tfrac{1}{2}x^\top P^{-1}x + x^\top P^{-1}\hat{x} + \text{const.} \tag{9}$$

Matching coefficients gives the posterior covariance and mean.

Posterior Covariance (Information Form)

$$\boxed{P^{-1} = (P')^{-1} + C^\top Q^{-1}C.} \tag{10}$$

Posterior Mean

From the linear term,

$$P^{-1}\hat{x} = (P')^{-1}\hat{x}' + C^\top Q^{-1}y, \tag{11}$$

hence

$$\boxed{\hat{x} = P\Big((P')^{-1}\hat{x}' + C^\top Q^{-1}y\Big).} \tag{12}$$

Equivalently, rearranging into the *innovation* form:

$$\hat{x} = \hat{x}' + PC^\top Q^{-1}(y - C\hat{x}'). \tag{13}$$

Kalman Gain Form

Define the gain

$$\boxed{K \triangleq PC^\top Q^{-1}.} \tag{14}$$

Then the posterior mean update is

$$\boxed{\hat{x} = \hat{x}' + K\big(y - C\hat{x}'\big).} \tag{15}$$

Notes

- The quantity $r \triangleq y - C\hat{x}'$ is called the *innovation* or *residual*.
- The update above is the closed-form Bayesian posterior for a Gaussian prior and linear measurement with Gaussian noise.
- If you prefer the more common gain expression, combine $P^{-1} = (P')^{-1} + C^\top Q^{-1}C$ with matrix identities to obtain

  $$K = P'C^\top(CP'C^\top + Q)^{-1}, \qquad P = (I - KC)P'(I - KC)^\top + KQK^\top,$$

  which is algebraically equivalent.