הקדמה לספר הגיון

Draft: YSO

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Introduction to Propositional Logic (revised ג׳ אלול תשפ״ד)

Abstract

Summary

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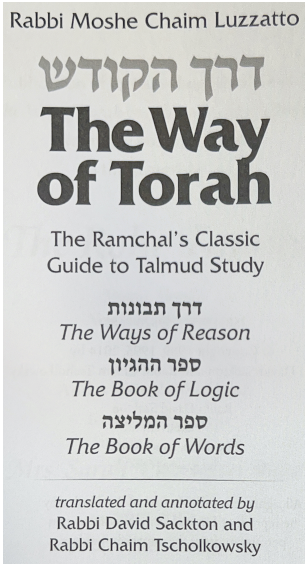


1 Introduction

Reasoning is the rational but mysterious movement of the mind from premises to conclusion. For example:

- Either the butler or the gardener did it; The butler didn't do it; *therefore* the gardener did it.
- If G-d does not exist objective moral values do not exist. Objective moral values do exist. *Therefore* G-d exists.[1]
- Given $(x = 4)$ and also $(x = 4) \Rightarrow (2x - y = 0)$ *it follows* that $y = 8$.[2]

Propositional logic was not part of classical logic discussed in the Rambam's *Treatise on Logic* and the Ramchal's ספר הגיון. Propositional logic is a type of "baby logic" compared to the classical but it is at the foundation of modern mathematical logic and an easy introduction to classical logic

We can translate the first example into the language of propositional logic as follows. Let B stands for "the butler did it" and G stand for "the gardener did it".

Premise 1	$B \vee G$	(the butler did it) **or** (the gardener did it);
Premise 2	$\neg B$	**not** (the butler did it);
Conclusion	G	*therefore* the gardener did it.

In each example we have a premise or series of premises and a conclusion. If the premises are true and rules of inference valid, then the argument is sound and the conclusion is necessarily true.

The argument appears to be valid and we will show that it is so in the sequel. We have used capital letters to represent atomic statements and connectives such as **not** ($\neg$) and **or** ($\vee$) to join atomic statements into compound statements.

Connectives	Meaning
$\neg$	**not** P
$\wedge$	P **and** Q
$\vee$	P **or** Q
$\Rightarrow$	P **implies** Q
$\equiv$	P **equivalent to** Q

[1] See the appendix for the proof that this argument is valid.

[2] If P is a proposition such as "x has the value 4" and Q is another proposition then the conditional proposition $P \Rightarrow Q$ is read "P **implies** Q" and its meaning is the conditional: *if* P is true *then* Q is true. Nothing can be inferred from the conditional proposition if P is not true.

2 Inference Rules

1. If G-d does not exist, objective moral values do not exist.
2. Objective moral values do exist.
3. Therefore, G-d exists.

That is a very simple argument for the existence of G-d. It is just three steps. It is logically valid. If the two premises are true then the conclusion follows necessarily and logically. The argument is valid if we can prove the conclusion using truth preserving inference rules, given the premises.

Convert informal language to a symbolic representation. Logic has its own language which is precise and unambiguous. So our first step is to convert the informal statements into *formal* ones. *Form* (or formal) means to capture the form of the argument but not yet its meaning.

G: G-d exists.

M: Objective moral values exist.

So, the formal version of the argument is:

Premise 1	$\neg G \Rightarrow \neg M$
Premise 2	M
Conclusion	G

Recall that $\neg M$ is "**not** (Objective moral values exist)" and $P \Rightarrow Q$ is the conditional "**if** P is true **then** Q is true". Here is the proof:

1	$\neg G \Rightarrow \neg M$	Premise
2	M	Premise
3	$\neg\neg M$	Double Negation (DN), 2
4	$\neg\neg G$	Modus Tollens (MT), 1, 3
5	G	Double Negation (DN), 4

Note that every single line of the proof is **justified**. The inference rules used in the proof are provided in Table 1. One of the inference rules is MT (Modus Tollens) as shown below. A valid inference rule must be truth preserving no matter what substitutions are made for the propositional symbols R and C.

Premise 1	$R \Rightarrow C$	**If** its raining **then** there are some clouds
Premise 2	$\neg C$	There are **no** clouds
Conclusion	$\neg R$	Therefore, it is **not** raining

To make an unanswerable argument we must be sure of three things:

1. Are all our terms clear and unambiguous?
2. Are the premises all true?

3. Is the reasoning logically valid?

Let's look at the first premise: if God does not exist, objective moral values do not exist. What is key to this premise is understanding what is meant by the word "objective." To say that moral values are objective means that these moral values are **real** and binding independently of whether any human being believes in them or not. For example, to say that the Holocaust was objectively evil is to say that it was evil even though the Nazis who carried it out thought that it was good, and it would still have been evil even if the Nazis had won World War II and succeeded in brainwashing or exterminating everybody who disagreed with them so that everyone thought the Holocaust was good. To say that the Holocaust was objectively evil is to say it was evil regardless of whether anybody thinks that it was or not. That is what we mean by objective moral values. They hold independently of whether any human being happens to agree with them or not.

Most atheists agree that if G-d does not exist then moral values are not objective in that sense. What about the second premise? Relativism has been inculcated into most over the years through their secondary and university education. So it is the second premise that is controversial. I leave that up to the reader to defend.

2.1 Axioms, Inference Rules, Theorems

A proof system consists of axioms, inference rules and theorems derived from the axioms via the inference rules.

Propositional logic is the simplest form of logic and is the foundation for more complex arguments that require **Predicate logic**. It is so simple that all of it can be based on three axioms and one inference rule. We will use 10 truth preserving inference rules to make reasoning more efficient. Each of those inference rules should be consistent with your common sense.

An axiom is usually considered to be "self evident" (מושכל ראשון). A finite part cannot be bigger than the whole. That is not something one can prove, or that we take on blind faith, as anything like that would just confuse the issue. It is planted in the rational intellect as a starting point of common sense that is at the foundation of דעת.[3]. Logic itself requires that there must be a starting point that is a acceptable to all rational beings.

Another example of an axiom is *principle of the excluded middle* which has the following form: $P \vee \neg P$. Either the lamp is my study is either really there or it is *not*. There is no other possibility. There is an "in-between" state of becoming a lamp, but that is not yet an actual lamp. A lamp in the process of being made is not yet a lamp, but the parts may already be there and partially assembled. So, again, either the lamp really is there or it is not. Another example:

[3]See Sefer HaIkrim, I,16-17

either the light of the lamp is fully on or it is *not* fully on. Either Felix over there is a cat or he is *not* a cat.

In a Hilbert proof, such as the one above, the propositions above the line may be axioms, premises, and derived lines. Below the line is the conclusion that follows.

Below are the inference rules for propositional logic. All these rules are valid, which means they are truth preserving. If the premises are true then the conclusion necessarily follows. We may use the axioms and inference rules to derive theorems such as the *de Morgan* theorem (T1) $\neg(P \wedge Q) \equiv (\neg P \vee \neg Q)$. Theorems do not depend on any premises and are in this sense like axioms.

Table 1. Axioms and Inference Rules

Valid Inference Rules		Some Axioms/Theorems
Modus Ponens (MP)	Modus Tollens (MT)	(A1) $P \vee \neg P$
Disjunctive Elimination (DE)	Double Negation (DN)	(A2) $P \equiv P$
Rule of Simplification (RS)	Rule of Adjunction (RA)	(A3) $\neg(P \wedge \neg P)$
Hypothetical Syllogism (HS)	Rule of Weakening (RW)	(A4) $\neg\neg P \equiv P$
Rule of Contradiction (RC)	Disjunctive Syllogism (DS)	(A5) $P \wedge Q \equiv Q \wedge P$
(A6) $(P \equiv Q) \stackrel{\text{def}}{=} (P \Rightarrow Q) \wedge (Q \Rightarrow P)$	(T1) $\neg(P \wedge Q) \equiv (\neg P \vee \neg Q)$	(T2) $\neg(P \vee Q) \equiv (\neg P \wedge \neg Q)$

Table 2. Inference Rules for Hilbert Proofs

(MP)	$\dfrac{P,\ P \Rightarrow Q}{Q}$	(MT)	$\dfrac{P \Rightarrow Q,\ \neg Q}{\neg P}$
(DE)	$\dfrac{P \vee Q,\ \neg Q}{P}$	(DN)	$\dfrac{\neg\neg P}{P}$
(RS)	$\dfrac{P \wedge Q}{P}$	(RA)	$\dfrac{P,\ Q}{P \wedge Q}$
(HS)	$\dfrac{P \Rightarrow Q,\ Q \Rightarrow R}{P \Rightarrow R}$	(RW)	$\dfrac{P}{P \vee Q}$
(RC)	$\dfrac{P \Rightarrow \mathit{false}}{\neg P}$	(DS)	$\dfrac{P \vee Q,\ P \Rightarrow S,\ Q \Rightarrow R}{R \vee S}$

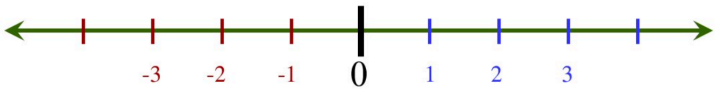
3 Proof by Contradiction

We are used to seeing a set of numbers such as $\{1, 2, 3, 4, 5\}$ like the five fingers of our hand. The cardinality of such as set is the number of elements in it so we write $|\{1, 2, 3, 4, 5\}| = 5$. Until a few hundred

years ago the set of *natural numbers* $\{0, 1, 2, 3, 4, \ldots\}$ would have been unusual because the concept of zero had not yet been discovered.

> At dinner recently, with non-scientists, discussion revolved about ages and a very young child, not even able to read yet, volunteered about his forthcoming little brother that "when he comes out his age will be **zero**". An adult remarked "indeed, and right now his age is minus five months", which everyone young and old seemingly found self-evident. How remarkable!

What is even more remarkable is negative numbers, such as being "minus five months old".



Negative numbers were considered "absurd" until recently. The famous logician De Morgan (the one who came up with the two famous laws of logic duality) is cited as follows:

> $8 - 3$ is easily understood; 3 can be taken from 8 and the remainder is 5; but $3-8$ is an impossibility; it requires you to take from 3 more than there is in 3, which is absurd. If such an expression as $3 - 8$ should be the answer to a problem, it would denote either that there was some absurdity inherent in the problem itself, or in the manner of putting it into an equation.

Having overcome our fears about negative numbers, we can now put them to good use. See Table 3 for a conjecture that we prove by contradiction.

We will prove the truth of Conjecture 1 with a proof by contradiction.

Even numbers are $\{0, 2, 4, 6, \ldots\}$ and odd numbers are $\{1, 3, 5, \ldots\}$. An integer n is even, denoted $even(i)$, if there is an integer i such that $n = 2i$. So $even(6)$ is true because $2 \times 3 = 6$. An integer n is odd, denoted $odd(n)$, if there is an integer j so that $n = 2j + 1$. So 7 is odd because $(2 \times 3) + 1 = 7$.

See Table 4 for a proof that there is no integer that is both even and odd. Each step is justified by appeal to the definitions of odd and even numbers and algebra.

Exercise 1. Prove that $even(n^2) \Rightarrow even(n)$. Hint: use proof by contradiction.

We will use the exercise in the next proof that $\sqrt{2}$ is irrational. The Talmud approximates the $\sqrt{2}$ by 1.4 which is the fraction $\frac{14}{10}$ where $= \frac{14}{10} = 1\frac{2}{5}$. However, as Rambam points out, $\sqrt{2}$ is irrational, i.e. it cannot be represented by a fraction. $\sqrt{2} = 1.41421356237309\ldots$ has been calculated to hundreds of trillions of decimal places, but of

course the part of the decimal never terminates. So we always must use an approximation.

The fraction $\frac{14}{10}$ has 2 as a common divisor as both the numerator (14) and the denominator (10) are divisible by 2 so we can reduce the fraction to a simpler form $\frac{7}{5}$ (because $14 \div 10 = 7 \div 5$). In fact $\gcd(14, 10) = 2$ meaning the greatest common divisor of 14 and 10 is 2. We will use simple fractions with no common factors other than 1. Thus $\gcd(7, 5) = 1$.

See Table 5 for a proof that $\sqrt{2}$ is irrational.

This proof that $\sqrt{2}$ is irrational was known to Aristotle and is similar to the proof in Euclid's *Elements of Geometry* (tenth book).

4 Natural Deduction

Below are the rules for Natural Deduction that overlap with the rules of the earlier table. Ten inference rules are given for propositional logic, two for each of the logical connectives.

$H \vdash G$ is a sequent—meaning there is a proof from hypothesis H to a goal G. In a Hilbert proof, this describes a situation where H occurs earlier in the proof and G is derived on a later line. We thus have the deduction theorem: $H \vdash G$ if and only $H \Rightarrow G$.

- $\wedge$-I: this is the conjunction introduction rule. If P and Q occur higher in the proof, then the conjunction $P \wedge Q$ can be introduced lower in the proof.
- $\wedge$-E: this is the conjunction elimination rule. If $P \wedge Q$ occurs higher in the proof then P can be introduced lower in the proof. Note that conjunction is symmetric: $P \wedge Q \equiv Q \wedge P$. Disjunction and equivalence are also symmetric.

Table 6. Natural Deduction Inference Rules

	Introduction Rules		**Elimination Rules**
$\wedge$-I	$\dfrac{P,\ Q}{P \wedge Q}$	$\wedge$-E	$\dfrac{P \wedge Q}{P}$
$\vee$-I	$\dfrac{P}{P \vee Q}$	$\vee$-E	$\dfrac{P \vee Q,\ P \Rightarrow R,\ Q \Rightarrow R}{R}$
$\Rightarrow$-I	$\dfrac{P_1,\ P_2 \vdash Q}{P_1 \wedge P_2 \Rightarrow Q}$	$\Rightarrow$-E	$\dfrac{P, P \Rightarrow Q}{Q}$
$\equiv$-I	$\dfrac{P \Rightarrow Q,\ Q \Rightarrow P}{P \equiv Q}$	$\Rightarrow$-E	$\dfrac{P \equiv Q}{P \Rightarrow Q}$
$\neg$-I	$\dfrac{P \vdash \mathit{false}}{\neg P}$	$\neg$-E	$\dfrac{\neg P \vdash \mathit{false}}{P}$

Table 7. Quantifier Inference Rules

c is a fresh constant not occurring anywhere else. t is any term.

	Introduction Rules		**Elimination Rules**
$\forall$-I	$\dfrac{H \vdash P[x := c]}{H \vdash \forall x : P(x)}$	$\forall$-E	$\dfrac{H \vdash \forall x : P(x)}{H \vdash P(t)}$
$\exists$-I	$\dfrac{H \vdash P(t)}{H \vdash \exists x : P(x)}$	$\exists$-E	$\dfrac{H,\ \exists x : P(x),\ P[x := c]\ \vdash G}{H \vdash G}$

Table 3. Father will not be twice son's age

Conjecture 1. A father is 56 years old and his son is 29 years old. Even if the the father and son live a very long life there will **not** be a positive number of years x so that the father's age will be twice his son's age.

Proof: We prove Conjecture 1 to be true by **contradiction**. We take the negation of the conjecture and suppose it to be true. Suppose there is a positive number of years x so that the father (current age 56) will be twice the age of his son (currently age 29). Such an x must satisfy the following equation:

$$56 + x = 2(29 + x)$$

Using algebra:

$$\begin{array}{ll} & 56 + x = 2(29 + x) \\ \equiv & \quad \langle \text{ by algebra } 2(29 + x) = 58 + 2x \rangle \\ & 56 + x = 58 + 2x \\ \equiv & \quad \langle \text{ algebra: } 56 - 58 = 2x - x, \text{ i.e. } -2 = x \rangle \\ & x = -2 \end{array}$$

We can now reason as follows:

1	$x > 0$	Premise, years x is positive
2	$56 + x = 2(29 + x)$	Premise, problem description
3	$x = -2$	algebra, 2
4	$\neg(x > 0)$	$(x = -2) \Rightarrow \neg(x > 0)$, arithmetic, 3
5	*false*	$(x > 0) \wedge \neg(x > 0) \equiv false$, 1, 4

Since the negation of the conjecture results in a contradiction (*false*), Conjecture 1 is true.

Table 4. There is no integer that is both even and odd

Example of proof by contradiction.
Prove that there is no integer that is both even and odd.

Proof: We take the negation of the statement and suppose it to be true.

1	n is an arbitrary integer	Premise
2	$\quad even(n) \wedge odd(n)$	assume n is both even and odd
3	$\quad (n = 2i) \wedge (n = 2j + 1)$	substitute in 2 their definitions
4	$\quad 2i = 2j + 1$	Leibniz, 3
5	$\quad i - j = 1/2$	algebra, 4
6	$\quad false$	$i - j$ cannot be a fraction, 5
7	$even(n) \wedge odd(n) \;\Rightarrow\; false$	discharge assumption, 2–6
8	$\neg(even(n) \wedge odd(n))$	proof by contradiction, 7

Step 4 might seem mysterious so we can expand it as follows.

$$
\begin{array}{ll}
 & (n = 2i) \wedge (n = 2j + 1) \\
\Rightarrow & \quad \langle \text{ Leibniz: } n = x \;\wedge\; n = y \text{ implies } x = y \rangle \\
 & 2i = 2j + 1 \\
= & \quad \langle \text{ algebra: subtract } 2j \text{ from both sides} \rangle \\
 & 2i - 2j = 1 \\
= & \quad \langle \text{ algebra, factoring} \rangle \\
 & 2(i - j) = 1 \\
= & \quad \langle \text{ algebra: divide both sides by } 2 \rangle \\
 & i - j = 1/2 \\
\Rightarrow & \quad \langle\, i, j \text{ assumed integers so } i - j \text{ cannot be a fraction} \rangle \\
 & false
\end{array}
$$

Table 5. Proof that $\sqrt{2}$ is irrational

Theorem 1. $\sqrt{2}$ is irrational.

Proof by contradiction: We take the negation of the statement and suppose it to be true, i.e. suppose there are integers m and n with no common so that $(\sqrt{2} = \frac{m}{n}) \wedge \gcd(m, n) = 1$.

$$
\begin{array}{ll}
 & \sqrt{2} = \dfrac{m}{n} \\
\equiv & \quad \langle \text{ squaring both sides} \rangle \\
 & 2 = \dfrac{m^2}{n^2} \\
\equiv & \quad \langle \text{ multiply both sides by } n^2 \rangle \\
 & m^2 = 2n^2 \\
\equiv & \quad \langle\ m^2 = 2n^2 \text{ implies that } m^2 \text{ is even by definition of even} \rangle \\
 & (m^2 = 2n^2) \ \wedge\ even(m^2) \\
\Rightarrow & \quad \langle \text{ see exercise for } even(m^2) \Rightarrow even(m) \rangle \\
 & (m^2 = 2n^2) \ \wedge\ even(m) \\
\equiv & \quad \langle\ even(m) \Rightarrow m = sk \text{ for some } k\ \rangle \\
 & (m^2 = 2n^2) \ \wedge\ even(m); \wedge; (m = 2k)
\end{array}
$$

1	$\sqrt{2} = \frac{m}{n}$	Premise
2	$\gcd(m, n) = 1$	Premise
3	$m^2 = 2n^2$	equational proof, 1
4	$even(m)$	equational proof, 1
5	$m = 2k$	equational proof, 1
6	$(2k)^2 = 2n^2$	substituting 5 in 3
7	$n^2 = 2k^2$	algebra $(2k)^2 = 2n^2 \ \equiv\ (n^2 = 2k^2)$, 6
8	$even(n^2)$	n^2 is even, 7
9	$even(n)$	by exercise $even(n^2) \Rightarrow even(n)$, 8
10	$gcd(m, n) \neq 1$	$\gcd(m, n) = 2$ both even, 4, 9
11	$false$	contradicts premise, 10, 2

We assumed $\sqrt{2}$ is rational and can thus be represented by a simple fraction. That assumption lead to a contradiction. So we conclude that $\sqrt{2}$ is irrational.