確率統計論

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1　P59 の続き

$$
\begin{aligned}
&E\left[\exp\left\{i\sum_{j=1}^{n}\lambda_j(B_{t_j}-Bt_{j-1})\right\}\right]\\
&=E\left[\exp\left\{i\left(\lambda_1(B_{t_1}-B_{t_0})+\lambda_2(B_{t_2}-B_{t_1})+\cdots+\lambda_n(B_{t_n}-B_{t_{n-1}})\right)\right\}\right]\\
&=E\left[\exp\left\{-i\sum_{j=1}^{n}(\lambda_{j+1}-\lambda_j)B_{t_j}\right\}\right]\ (\because B_{t_0}\text{は}\,0\,\text{なので消える})\\
&=\exp\left\{-\sum_{j=1}^{n-1}\sum_{i=j+1}^{n}(\lambda_{j+1}-\lambda_j)(\lambda_{i+1}-\lambda_i)t_j-\frac{1}{2}\sum_{j=1}^{n}(\lambda_{j+1}-\lambda_j)^2t_j\right\}\\
&=\exp\left\{-\sum_{j=1}^{n-1}(\lambda_{j+1}-\lambda_j)(-\lambda_{j+1})t_j-\frac{1}{2}\sum_{j=1}^{n}(\lambda_{j+1}-\lambda_j)^2t_j\right\}\\
&=\exp\left\{-\sum_{j=1}^{n-1}(\lambda_{j+1}-\lambda_j)(-\lambda_{j+1})t_j-\frac{1}{2}\sum_{j=1}^{n-1}(\lambda_{j+1}-\lambda_j)^2-\frac{1}{2}(\lambda_{n+1}-\lambda_n)^2t_n\right\}\\
&=\exp\left\{\frac{1}{2}\sum_{j=1}^{n-1}\left\{2(\lambda_{j+1}-\lambda_j)(\lambda_{j+1})-(\lambda_{j+1}-\lambda_j)^2\right\}t_j-\frac{1}{2}(\lambda_{n+1}-\lambda_n)^2t_n\right\}\\
&=\exp\left\{\frac{1}{2}\sum_{j=1}^{n-1}\left\{2(\lambda_{j+1})^2-2(\lambda_{j+1}\lambda_j)-(\lambda_{j+1})^2+2(\lambda_{j+1}\lambda_j)-(\lambda_j)^2\right\}t_j-\frac{1}{2}(\lambda_n)^2t_n\right\}\ (\because \lambda_{n+1}=0)\\
&=\exp\left\{\frac{1}{2}\sum_{j=1}^{n-1}({\lambda_{j+1}}^2-{\lambda_j}^2)t_j-\frac{1}{2}{\lambda_n}^2t_n\right\}\\
&=\exp\left\{\frac{1}{2}({\lambda_2}^2-{\lambda_1}^2)t_1+\frac{1}{2}({\lambda_3}^2-{\lambda_2}^2)t_2+\cdots+\frac{1}{2}({\lambda_n}^2-{\lambda_{n-1}}^2)t_{n-1}-\frac{1}{2}{\lambda_n}^2t_n\right\}\\
&=\prod_{j=1}^{n}\exp\left\{-\frac{1}{2}{\lambda_j}^2(t_j-t_{j-1})\right\}
\end{aligned}
$$

($\because$ 展開して括り方を変えて、exp の指数部分の和なので、exp 自体の積の形になる)