Gauss and the History of the Fast Fourier Transform

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Introduction

The fast FOURIER transform (FFT) has become well known as a very efficient algorithm for calculating the discrete FOURIER transform (DFT)—a formula for evaluating the N FOURIER coefficients from a sequence of N numbers. The DFT is used in many disciplines to obtain the spectrum or frequency content of a signal and to facilitate the computation of discrete convolution and correlation. Indeed, the publication of the FFT algorithm as a means of calculating the DFT by J. W. COOLEY & J. W. TUKEY in 1965 [1] was a turning point in digital signal processing and in certain areas of numerical analysis. They showed that the DFT, which was previously thought to require N^2 arithmetic operations, could be calculated by the new FFT algorithm using a number of operations proportional to $N \log N$. This algorithm had a revolutionary effect on the way much of digital processing was done and the FFT remains the most widely used method of computing FOURIER transforms [2].

In their original paper COOLEY & TUKEY referred only to the work of I. J. GOOD [3] published in 1958 as influencing their development. However, it was soon discovered there are major differences between the COOLEY-TUKEY FFT and the algorithm described by GOOD, which is now commonly referred to as the prime factor algorithm (PFA). Soon after the appearance of the COOLEY-TUKEY paper, RUDNICK [4] demonstrated a similar algorithm based on the work of DANIELSON & LANCZOS [5] which had appeared in 1942. This discovery prompted an investigation into the history of the FFT algorithm by COOLEY, LEWIS, & WELCH [6]. They discovered that the DANIELSON-LANCZOS paper referred to the works of RUNGE published at the turn of the century [7, 8]. While not influencing their work directly, the algorithm developed by COOLEY & TUKEY clearly had roots in the early twentieth century.

In a recently published history of numerical analysis [9], H. H. GOLDSTINE attributes to CARL FRIEDRICH GAUSS, the eminent German mathematician, an algorithm similar to the FFT for the computation of the coefficients of a finite FOURIER series. GAUSS' treatise describing the algorithm was not published in his lifetime; it appeared only in his collected works [10] as an unpublished manuscript. The presumed year of the composition of this treatise is 1805, thereby suggesting that efficient algorithms for evaluating coefficients of FOURIER series were developed at least a century earlier than had been previously known. If this year is accurate, it predates FOURIER's first presentation on harmonic analysis in 1807. A second reference to GAUSS' algorithm is found in an article in the *Encyklopädie der Mathematischen Wissenschaften* [11] which was written by H. BURKHARDT in 1904. BURKHARDT comments that GAUSS' method was general, but seemingly not known by practitioners. It is interesting to note that GOLDSTINE'S and BURKHARDT'S works went almost as unnoticed as GAUSS' work itself.

Because of the importance of the FFT, its history is again open to question. Is GAUSS' method indeed equivalent to a modern FFT algorithm? If so, which type and why is this work by one of the greatest mathematicians not known by engineers and physicists even after the publication of GOLDSTINE'S book? What influenced GAUSS' work and who developed the DFT? How firmly established is the date of writing? To answer these questions and to trace the history of FOURIER series coefficient calculation into the eighteenth and nineteenth centuries, we undertook our own historical investigation,[1] dealing primarily with the original texts and concentrating on GAUSS' work. What follows is a summary of our work, with historical references and evidence provided for the reader to pursue the history as he or she wishes. Another class of efficient DFT algorithms, called prime factor algorithms, which include work from THOMAS [13], GOOD [3], WINOGRAD [14], and others is not included in this investigation.

The Twentieth Century

COOLEY, LEWIS, & WELCH [6] discovered that the DANIELSON-LANCZOS paper referred to the work of CARL DAVID TOLMÉ RUNGE (1856–1927) [7, 8] as the inspiration for their algorithm. In these two papers and the book by RUNGE & KÖNIG [15], a doubling algorithm is decribed which computes the FOURIER transform of two N-point subsequences to obtain a $2N$-point FOURIER transform using approximately N auxiliary operations. This algorithm is not as general as the COOLEY-TUKEY FFT algorithm because it only allows doubling of the original sequence length whereas the COOLEY-TUKEY approach efficiently computes the DFT for any multiple of the original length. The work of RUNGE also influenced STUMPFF who, in his book on harmonic analysis and periodograms [16], gives a doubling and tripling algorithm for the evaluation of harmonic series. Furthermore, on page 142 of that book, he suggests a generalization to an arbitrary multiple. All of this historical information was exposed by COOLEY, LEWIS, & WELCH [6] in much greater detail and has been repeated here to provide a background on the knowledge of the history of the FFT circa 1967.

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[1] This has resulted in a bibliography of over 2000 entries [12].

The Nineteenth and Early Twentieth Centuries

The work of RUNGE was well-known in the early part of the Twentieth Century and is even referred to in the popular textbook written by WHITTAKER & ROBINSON [17], which was originally published in 1924. WHITTAKER & ROBINSON state that RUNGE'S method has been widely published and cite a paper by SILVANUS PHILLIPS THOMPSON (1851–1916) [18]. THOMPSON, who was also a biographer of Sir WILLIAM THOMSON, Lord KELVIN (1824–1907), was apparently trying to popularize RUNGE'S method in Great Britain through his articles [18, 19]. THOMPSON'S second paper [18] does not actually use an FFT method to obtain computational savings, but is interesting because of the discussion included at the end of the paper. The discussion includes comments by GEORGE HOWARD DARWIN (1845–1912), son of the more famous CHARLES, who claims to have used efficient techniques for the harmonic analysis of tides in 1883 [20], which he attributes to ARCHIBALD SMITH (1813–1872) in 1874 [21] and Sir RICHARD STRACHEY (1917–1908) in 1884 [22]. In DARWIN'S paper [20], reference is made to a paper by JOSEPH DAVID EVERETT (1831–1904) published in 1860 [23], and credit is also given to ARCHIBALD SMITH. EVERETT was working with Lord KELVIN on harmonic analysis of daily temperature variations and gives a method for harmonic analysis using 12 samples which he claims is an extension of the method used by Lord KELVIN in [24]. Lord KELVIN used a method based on 32 samples which was due to ARCHIBALD SMITH and was originally published in 1846 [25] and presented in more detail in 1850 [26] and 1855 [27]. Apparently S. P. THOMPSON, Lord KELVIN'S biographer, was unaware of the use of efficient techniques of harmonic analysis in this paper.

The British discovery of efficient techniques for harmonic analysis can be reliably traced to ARCHIBALD SMITH in 1846. Other algorithms had been developed independently by various researchers in the Nineteenth Century. These techniques are tabulated on pp. 686–687 of the article by BURKHARDT [11]. The earliest method referenced by BURKHARDT is that of FRANCESCO CARLINI (1783–1862) in 1828 [28] for $n = 12$. The only other method found by BURKHARDT which predates ARCHIBALD SMITH is that of PETER ANDREAS HANSEN (1795–1874) in 1835 [29] for $n = 64$. These works have no apparent influence on the work by the British. HANSEN was heavily influenced by GAUSS in his astronomical work, but does not mention GAUSS in the development of his algorithms for harmonic analysis for reasons which shall be made clear later.

An important detail that should not be overlooked is that most of the methods preceding RUNGE were not intended for computing harmonics above the fourth. For most applications of harmonic analysis in the Nineteenth Century this was adequate because the measurement quantization was generally on the same order of magnitude as the contributions of the higher-order harmonics. These methods are, therefore, similar in style to what are now called 'pruned' FFT's [30]. Most of these methods were described by computational tables for a fixed number of samples and were not presented as general techniques for computing harmonics for an arbitrary number of samples. These algorithms consisted of grouping terms in the trigonometric series having the same multiplicative coefficient. Thus, a group of n multiplies and $n - 1$ additions was reduced to a single multiply and $n - 1$

additions. However, such an approach will not result in what is termed today an efficient algorithm, such as FFT. Efficient algorithms are derived by decomposing a DFT into a sequence of shorter length DFTs. As will be seen, this procedure was that used by GAUSS; consequently, the nineteenth century work was unrelated to GAUSS' and did not foreshadow the twentieth century work on efficient DFT algorithms.

The Background of Gauss' Work

The use of trigonometric series in analysis originates in the work of LEONHARD EULER (1707–1783) [31–35]. In [35], EULER gives the formulas for the coefficients of the FOURIER series representation of a function of a real variable. In another work [34], EULER uses a trigonometric series to describe the motions of a discrete approximation to sound propagation in an elastic medium. By example, he derives a formula for the coefficients of a series of sines given *samples* of the function, which can be interpreted as the DFT for a series consisting only of sines [36]. The stature of EULER in his own time caused his works to be read by his contemporaries, particularly the French mathematicians CLAIRAUT, D'ALEMBERT, and LAGRANGE. ALEXIS-CLAUDE CLAIRAUT (1713–1765) published in 1754 [37] what we currently believe to be the earliest explicit formula for the DFT (the computation for series coefficients from equally spaced samples of the function), but it was restricted to a cosine FOURIER series. DANIEL BERNOULLI (1700–1782) expressed the form of a vibrating string as a series of sine and cosine terms with arguments of both time and distance in 1753 [38]. Perhaps the most influential work in the latter portion of the nineteenth century on the DFT is that of JOSEPH LOUIS LAGRANGE (1736–1813). Extending the work of EULER, he published a DFT-like formula for finite FOURIER series containing only sines, in 1759 [39] and in 1762 [40]. This work was referred to, for example, by CARLINI in his paper of 1828. The most authoritative compilation of the early history of trigonometric series is an article of 536 pages by H. BURKHARDT [41].

CLAIRAUT and LAGRANGE were concerned with orbital mechanics and the problem of determining from a finite set of observations the details of an orbit. Consequently, their data was periodic and they used an interpolation approach to orbit determination: in modern terminology and notation, an even periodic function $f(x)$ having a normalized period of one is represented as a finite trigonometric series by

$$f(x) = \sum_{k=0}^{N-1} a_k \cos 2\pi kx, \qquad 0 \leqq x \leqq 1. \tag{1}$$

The problem is to find the coefficients $\{a_k\}$ from the N values of $f(x)$ for values of $x_n = \frac{n}{N}$ with $n = 0, 1, \ldots, N-1$. By forcing $f(x)$ to equal the observed values at the abscissas $\{x_n\}$, one can easily show that the coefficients $\{a_k\}$ are given by the cosine DFT of the observed values of $f(x)$. GAUSS presumably knew of the papers of LAGRANGE [39, 40], for while a student, from 1795 to 1798,[2] he borrowed from the library at Göttingen the volumes containing them.

[2] DUNNINGTON by searching the University library records [42] has compiled a list of books borrowed by GAUSS at Göttingen.

Gauss' Algorithm for Computing the DFT

The treatise of interest was written by CARL FRIEDRICH GAUSS (1777–1855) and entitled "Theoria Interpolationis Methodo Nova Tractata". It was published only posthumously in Volume 3 of his collected works in 1866 [10], but was originally written, most likely, in 1805. GOLDSTINE [9] gives, on pages 249–253, an English translation of parts of the sections of GAUSS' paper related to trigonometric interpolation algorithms, which are Articles 25 through 28 of the original Latin text. In this treatise, GAUSS extended the work on trigonometric interpolation to periodic functions which are not necessarily odd or even while considering the problem of determining the orbit of certain asteroids from sample locations. These functions are expressed by a FOURIER series of the form

$$f(x) = \sum_{k=0}^{m} a_k \cos 2\pi kx + \sum_{k=1}^{m} b_k \sin 2\pi kx, \tag{2}$$

where $m = (N-1)/2$ for N odd, or $m = N/2$ for N even. GAUSS showed in Articles 19–20 that if one were given the values of $f(x_n)$, $x_n = \frac{n}{N}$ $(n = 0, 1, \ldots,$ $\ldots, N-1)$, that the coefficients a_k and b_k are given by the now well-known formulas for the DFT [43]. This set of equations is the earliest explicit formula for the general DFT that we have found.

GAUSS develops his efficient algorithm by using N_1 equally spaced samples over one period of the signal. This set of N_1 samples is a subset of N total samples, where $N = N_1 N_2$. GAUSS computes the finite FOURIER series which passes through these samples using m harmonics where m is as defined in (2). He then assumes that another subset of N_1 equally spaced samples of the signal are measured which are offset from the original set of samples by a fraction, $1/N_2$, of the original sample interval where N_2 is a positive integer. A finite FOURIER series with m harmonics is computed which passes through this new set of samples and it is discovered that these coefficients are quite different from those computed for the original N_1 samples. GAUSS realized what the problem was and proceeded to develop a method for correcting the coefficients he had already calculated and to determine additional coefficients for the higher frequency harmonics. Using modern terminology, we would say that the waveform was undersampled; therefore the coefficients were in error because of aliasing of the high frequency harmonics [2].

GAUSS' solution to this problem is to measure a total of N_2 sets of N_1 equally spaced samples which together form the overall set of N equally spaced samples. The finite FOURIER series for the entire set of N samples is computed by first computing the coefficients for each of the N_2 sets of length N_1 all shifted relative to a common origin, and then computing coefficients of the N_1 series of length N_2 which are formed from the coefficients of corresponding terms in the N_2 sets of coefficients originally computed. A final trigonometric identity is used to convert these coefficients into the finite FOURIER series coefficients for the N samples.

In modern terminology, the DFT of the samples of $f(x)$ is defined by

$$C(k) = \sum_{n=0}^{N-1} X(n)\, W_N^{nk} \tag{3}$$

where, if $f(x)$ has a period of one, $X(n) = f(n/N)$ are the N equally spaced samples, $W_N = e^{-j2\pi/N}$ and $k = 0, 1, \ldots, N-1$ are the indices of the FOURIER coefficients. This DFT can be rewritten in terms of N_2 sets of N_1 subsamples by the change of index variables [44]:

$$n = N_2 n_1 + n_2,$$

$$k = k_1 + N_1 k_2,$$

for $n_1, k_1 = 0, 1, \ldots, N_1 - 1$ and $n_2, k_2 = 0, 1, \ldots, N_2 - 1$. Each subsequence is a function of n_1 and which subsequence it is, is denoted by n_2. The DFT in (3) becomes

$$C(k_1 + N_1 k_2) = \sum_{n_2=0}^{N_2-1} \left[\sum_{n_1=0}^{N_1-1} X(N_2 n_1 + n_2)\, W_{N_1}^{n_1 k_1} W_N^{n_2 k_1} \right] W_{N_2}^{n_2 k_2}, \tag{4}$$

where the inner sum calculates the N_2 length-N_1 DFTs corrected by a power of W_N, and the outer sum calculates the N_1 length-N_2 DFTs. This is exactly the exponential form of GAUSS' algorithm where the W_N term accounts for the shifts from the origin of the N_2 length-N_1 sequences. This is also exactly the FFT algorithm derived by COOLEY & TUKEY in 1965 [1] where W_N is called a *twiddle factor* [2], a factor to correct the DFT of the inner sum for the shifted samples of $X(n)$. The equivalence of GAUSS' algorithm and the COOLEY-TUKEY FFT is not obvious due to the notation and trigonometric formulation of GAUSS. One can easily verify the results by calculating the inner sum of (4) and comparing the numerical results with the intermediate calculation in Article 28 of [10] after converting from exponential to trigonometric form and correcting for factors of $1/N_1$. The example of $N = 12$ for the orbit of the asteroid Pallas was worked out in Article 28 of [10] for $N_1 = 4, N_2 = 3$ and for $N_1 = 3, N_2 = 4$ and an example was given in Article 41 for $N = 36$ with $N_1 = N_2 = 6$ and for the special case of odd symmetry.

In Article 27 GAUSS states that his algorithm can be generalized to the case where N has more than two factors, although no examples are given. This and the observed efficiency are seen in the following translation from Article 27.

> "And so for this case, where most of the proposed values of the function X, an integral period of the arrangement, the number is composite and $= \pi = \mu\nu$, in articles 25, 26, we learned that through the division of that period into ν periods of μ terms, it produces, when all values are given, the same satisfactory function, which by the immediate application of the general theory applies to the whole period; truly, that method greatly reduces the tediousness of mechanical calculations, success will teach the one who tries it. Now the work will be no greater than the explanation of how that division can be extended still further and can be applied to the case where the majority of all proposed values are composed of three or more factors, for example, if the number μ would again be composite, in that case clearly each period of μ terms can be subdivided into many lesser periods."

He did not, however, go on to quantify the computational requirements of his method to obtain the now familiar $N \Sigma N_i$ or $N \log N$ expressions for its computational complexity. From this short excerpt, GAUSS clearly developed his procedure because it was computationally efficient and because it could be applied to a select, but interesting, set of sequence lengths. Thus, GAUSS' algorithm is as general and powerful as the COOLEY-TUKEY algorithm and is, in fact, equivalent to an algorithm called decimation-in-frequency adapted to a real data sequence.

The treatise by GAUSS was not published during his lifetime and was not explicitly dated. The hints used by the biographers of GAUSS [42, 45] and by us to establish a date for this work are summarized in the accompanying table. From these facts, we infer that GAUSS wrote this treatise in October–November, 1805. This work predates the work of JEAN BAPTISTE JOSEPH FOURIER (1768–1830) in 1807 on representations of functions as infinite series of harmonics. FOURIER did not publish his results until 1822 [46] because his presentation to the Academy of Sciences in Paris on December 21, 1807 was not well received by LAGRANGE and was refused publication in the Memoirs of the Academy. An earlier manuscript of FOURIER'S dates back to 1804–1805 and includes research which he may have started as early as 1802 [47].

Discussion

The DFT approach to solving the problem of orbital mechanics was one of several. Approaches related to Newtonian mechanics gave alternative solutions to the problem, and, in the end, came to be preferred, even by GAUSS. Mathematicians concerned with orbital mechanics who would have read his posthumous treatise at the time of its publication in 1866 probably would not have found the technique described therein of much interest. For the modern technical reader, the treatise is difficult to read because of the language and the notation adopted by GAUSS to describe his method. Examples of this notation are the use of π as the length of a sequence (instead of N), the use of the symbols $a, b, c, d, \ldots, a', b', c', d', \ldots, a'', b'', c'', d''$, *etc.* as the indices of the time series, and the use of capital letters to refer to the values of a function at a point whose index is the corresponding small letter (*e.g.*, $f(a) = A$). GAUSS' method was also derived using real trigonometric functions rather than complex exponentials, making it more difficult to relate his method to current FFT techniques. Thus, the dated nature of the publication, its publication in Latin, and the lack of notice of GOLDSTINE'S and BURKHARDT'S work contributed to the "loss" of GAUSS' FFT technique until now.

The DFT appears to have been originated by GAUSS, although that work can be considered a simple extension of eighteenth century work. However, GAUSS' derivation of the general DFT formula in 1805 implies that the genesis of modern efficient DFT algorithms presumably could not have occurred prior to that date. Although it appears that the discrete FOURIER transform should have really been named after GAUSS, it is obviously not practical to rename it. The term "GAUSS-FOURIER transform (GFT)" has already been coined by HOPE [48] in 1965 and the term "discrete GAUSS transform (DGT)" has also been previously used [49].

T. S. HUANG was unknowingly correct when he satirically remarked in 1971 that the FFT was GAUSS' 1001st algorithm [50].

This investigation has once again demonstrated the virtuosity of CARL FRIEDRICH GAUSS. In addition, it has shown how certain problems can be timeless, but that their solution can be rediscovered again and again. BURKHARDT pointed out this algorithm in 1904 and GOLDSTINE suggested the connection between GAUSS and the FFT in 1977, but both of these went largely unnoticed, presumably because they were published in books dealing primarily with history. It was shown that various attempts at efficient algorithms were used in Great Britain and elsewhere in the 19th Century, but were unrelated to the work of GAUSS and were, in fact, not as general or well-formulated as GAUSS' work. Almost one hundred years passed between the publication of GAUSS' algorithm and the modern rediscovery of this approach by COOLEY & TUKEY.

Principal Discoveries of Efficient Methods of Computing the DFT

Researcher(s)	Date	Lengths of Sequence	Number of DFT Values	Application
C. F. GAUSS [10]	1805	Any composite integer	All	Interpolation of orbits of celestial bodies
F. CARLINI [28]	1828	12	7	Harmonic analysis of barometric pressure variations
A. SMITH [25]	1846	4, 8, 16, 32	5 or 9	Correcting deviations in compasses on ships
J. D. EVERETT [23]	1860	12	5	Modeling underground temperature deviations
C. RUNGE [7]	1903	2^nK	All	Harmonic analysis of functions
K. STUMPFF [16]	1939	2^nK, 3^nK	All	Harmonic analysis of functions
DANIELSON & LANCZOS [5]	1942	2^n	All	X-ray diffraction in crystals
L. H. THOMAS [13]	1948	Any integer with relatively prime factors	All	Harmonic analysis of functions
I. J. GOOD [3]	1958	Any integer with relatively prime factors	All	Harmonic analysis of functions
COOLEY & TUKEY [1]	1965	Any composite integer	All	Harmonic analysis of functions
S. WINOGRAD [14]	1976	Any integer with relatively prime factors	All	Use of complexity theory for harmonic analysis

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Dating of Gauss' Work on the FFT
Theoria interpolationis methodo nova tractata
Volume III, Werke

April 30, 1777	GAUSS is born in Brunswick.
September 1795	GAUSS arrives at Göttingen. Throughout his stay there, he checked out many books from the university library. In particular, he continually read *Miscellanea Taurinensia*, the proceedings of the academy located in Turin. When these proceedings were being published, LAGRANGE was there and this journal served as his exclusive outlet. In Volumes I and III, LAGRANGE'S DFT (sine only) appears.
November 25, 1796	Date given for diary entry 44,[3] which reads *Formula interpolationis elegans*. Translated, this entry means "Elegant formula for interpolation". The editor of the diary connected this with the LAGRANGE interpolation formula. No specific library books can be readily connected to this entry.
December 1796	Date given for diary entry 46, which reads *Formulae trigonometricae per series expressae*. Translated, this entry means "Trigonometric formulas expressed with series". The editor of the diary made no comment about this entry.
September 28, 1798	GAUSS returns home to Brunswick after finishing his studies at Göttingen.
December, 1804–1805	Correspondence between GAUSS and BESSEL indicates their concern with the interpolation problem. No mention is made, however, of the trigonometric interpolation problem.[4]

[3] The entries of GAUSS' mathematical diary are published with accompanying comments by the editors in Volume X,1 of the *Werke*, pp. 488–571.

[4] *Briefwechsel zwischen Gauss und Bessel*, Leipzig, 1880.

March 25, 1805 — GAUSS, in a letter to the astronomer OLBERS,[5] provides his latest elements for the orbit of the asteroid Juno, among which is the value of the eccentricity 0.254236.[6] This number is used by GAUSS in an example in his FFT writings. Thus the treatise must have been completed after this date. These elements were published later in *Monatliche Correspondenz*, a collection of unreviewed notes containing astronomical observations and information, in May, 1805.

November, 1805 — Diary entry 124, which reads *Theoriam interpolationis ulterius excoluimus*. Translated, this entry means "We have worked out further a theory of interpolation". The editor takes this entry to mean that his treatise on interpolation could not have been written before November, 1805. He refers to a notebook of GAUSS consisting of short mathematical notes (*Mathematische Brouillons*), which was begun in October, 1805. Volume 18 of the notebook contains an opening note on interpolation. The editor takes this note to be a first draft of the treatise. However, the collected work of GAUSS does not contain this paper.

January, 1806 — In correspondence to OLBERS[7], GAUSS mentions his work on interpolation, which he says was done "earlier". He stressed the novelty of the second half of the work. He enclosed a copy of it with the letter, asking OLBERS for criticism. In reply, OLBERS encouraged publication, but admitted not being able to follow the second half.

July 30, 1806 — Date attached to a letter sent from GAUSS to BODE, in which the value of the eccentricity for Juno of 0.2549441 is given. Presumably, this means that the FFT treatise must have been written prior to this date. This letter appeared in *Monatliche Correspondenz* later in 1806.[8]

November 8, 1808 — A letter from SCHUMACHER, a former student of GAUSS, to GAUSS mentions that SCHUMACHER's mother has a handwritten copy of his work on interpolation.[9] It is unclear whether this letter is referring to *Mathematische Brouillons* or the *Theoria interpolationis*.

June 8, 1816 — SCHUMACHER writes GAUSS that he has a handwritten version of GAUSS' work on interpolation, which he hopes GAUSS will publish soon.[9] Thus, GAUSS did not keep this work secret, but presumably was not interested in publishing it.

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[5] *Briefwechsel zwischen Olbers und Gauss*, Vol. 1, Julius Springer, p. 255.

[6] Volume VI, *Werke*, p. 262.

[7] *Briefwechsel zwischen Olbers und Gauss*, Vol. 1, Julius Springer, pp. 281, 286.

[8] Volume VI, *Werke*, p. 279.

[9] Volume X.2, *Werke*, p. 125.

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