Design and Analysis of Parallel Programs

TDDD56 Lecture 4

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Outline

Lecture 1: Multicore Architecture Concepts

Lecture 2: Parallel programming with threads and tasks

Lecture 3: Shared memory architecture concepts

Lecture 4: Design and analysis of parallel algorithms

- Parallel cost models
- Work, time, cost, speedup
- Amdahl's Law
- Work-time scheduling and Brent's Theorem

Lecture 5: Parallel Sorting Algorithms

…

Parallel Computation Model = Programming Model + Cost Model

+ abstract from hardware and technology
+ specify basic operations, when applicable
+ specify how data can be stored

→ analyze algorithms before implementation independent of a particular parallel computer $\rightarrow T = f(n, p, \ldots)$

→ focus on most characteristic (w.r.t. influence on exec. time) features of a broader class of parallel machines

Programming model
- shared memory / message passing,
- degree of synchronous execution

Cost model
- key parameters
- cost functions for basic operations
- constraints

Parallel Computation Models

Shared-Memory Models

- PRAM (Parallel Random Access Machine) [Fortune, Wyllie '78]
  including variants such as Asynchronous PRAM, QRQW PRAM
- Data-parallel computing
- Task Graphs (Circuit model; Delay model)
- Functional parallel programming
- …

Message-Passing Models

- BSP (Bulk-Synchronous Parallel) Computing [Valiant'90]
  including variants such as Multi-BSP [Valiant'08]
- LogP
- Synchronous reactive (event-based) programming e.g. Erlang
- …

Cost Model

Cost model: should

+ explain available observations
+ predict future behaviour
+ abstract from unimportant details → generalization

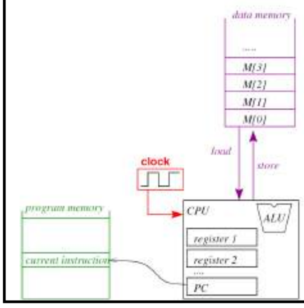
Simplifications to reduce model complexity:

- use idealized multicomputer model
  ignore hardware details: memory hierarchies, network topology, ...
- use scale analysis
  drop insignificant effects
- use empirical studies
  calibrate simple models with empirical data
  rather than developing more complex models

Flashback to DALG, Lecture 1:
The RAM (von Neumann) model for sequential computing

RAM (Random Access Machine) [PPP

programming and cost model for the analysis of sequential algorithms



Basic operations (instructions):
- Arithmetic (add, mul, …) on registers
- Load
- Store
- Branch

op

op1

op2

Simplifying assumptions for time analysis:
- All of these take 1 time unit
- Serial composition adds time costs
  T(op1;op2) = T(op1)+T(op2)

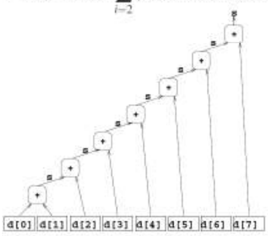
Analysis of sequential algorithms: RAM model (Random Access Machine)

Algorithm analysis: Counting instructions

Example: Computing the global sum of N elements

```
s = d(0)
do i = 1, N-1
     s = s + d(i)
end do
```

$$t = t_{load} + t_{store} + \sum_{i=2}^{N} (2t_{load} + t_{add} + t_{store} + t_{branch}) = 5N - 3 \in \Theta(N)$$



$\rightarrow$ arithmetic circuit model, directed acyclic graph (DAG) model

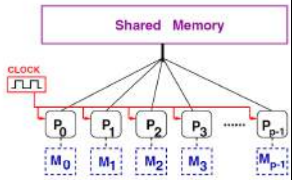
The PRAM Model – a Parallel RAM

Parallel Random Access Machine [Fortune/Wyllie'78]

p processors
- MIMD
- common clock signal
- arithm./jump: 1 clock cycle

shared memory
- uniform memory access time
- latency: 1 clock cycle (!)
- concurrent memory accesses
- sequential consistency

private memory (optional)
- processor-local access only



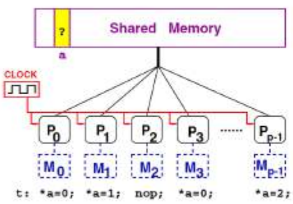
PRAM Variants

Exclusive Read, Exclusive Write (EREW) PRAM
- concurrent access only to different locations in the same cycle

Concurrent Read, Exclusive Write (CREW) PRAM
- simultaneous reading from *or* writing to same location is possible:

Concurrent Read, Concurrent Write (CRCW) PRAM
- simultaneous reading from *or* writing to same location is possible:
  - Weak CRCW
  - Common CRCW
  - Arbitrary CRCW
  - Priority CRCW
  - Combining CRCW
    (global sum, max, etc.)

No need for ERCW ...

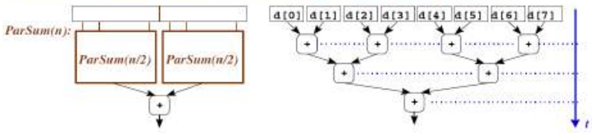


Divide&Conquer Parallel Sum Algorithm in the PRAM / Circuit (DAG) cost model

Given n numbers $x_0, x_1, \ldots, x_{n-1}$ stored in an array.

The global sum $\sum_{i=0}^{n-1} x_i$ can be computed in $\lceil \log_2 n \rceil$ time steps on an EREW PRAM with n processors.

Parallel algorithmic paradigm used: Parallel Divide-and-Conquer

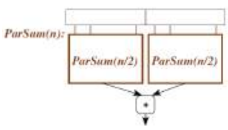


Divide phase: trivial, time $O(1)$
Recursive calls: parallel time $T(n/2)$
with base case: load operation, time $O(1)$ $\rightarrow$ $T(n) = T(n/2) + O(1)$
Combine phase: addition, time $O(1)$

Use induction or the master theorem [Cormen+'90 Ch.4] $\rightarrow T(n) \in O(\log n)$

Recursive formulation of DC parallel sum algorithm in EREW-PRAM model

Fork-Join execution style: single thread starts, threads spawn child threads for independent subtasks, and synchronize with them



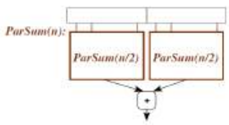
Implementation in Cilk:

```
cilk int parsum ( int *d, int from, int to )
{
   int mid, sumleft, sumright;
   if (from == to)  return d[from];    // base case
   else {
      mid = (from + to) / 2;
      sumleft = spawn parsum ( d, from, mid );
      sumright = parsum( d, mid+1, to );
      sync;
      return sumleft + sumright;
   }
}
```

11

Recursive formulation of DC parallel sum algorithm in EREW-PRAM model

SPMD (single-program-multiple-data) execution style: code executed by all threads (PRAM procs) in parallel, threads distinguished by thread ID $

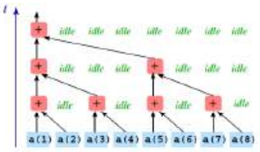


in the PRAM programming language Fork [Keller, K., Träff'01]

```
sync int parsum( sh int *d, sh int n)
{
  // calling group's processor ranks $ in [0...#-1]
  sh int s1, s2;
  sh int nd2 = n / 2;
  if (n==1) return d[0]; // base case
  if ($<nd2)          // split processor group:
     s1 = parsum( d, nd2 );
  else                  s2 = parsum( &(d[nd2]), n-nd2 );
  // subgroups merge here, barrier synchronization
  return s1 + s2;
}
```

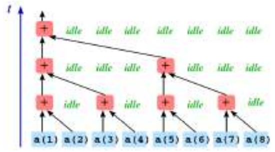
Iterative formulation of DC parallel sum in EREW-PRAM model

```
int sum(sh int a[], sh int n)
{
  int d, dd;
  int ID = rerank();
  d = 1;
  while (d<n) {
    dd = d; d = d*2;
    if (ID%d==0)  a[ID] = a[ID] + a[ID+dd];
  }
}
```



Circuit / DAG model

- Independent of <u>how</u> the parallel computation is expressed, the resulting (unfolded) task graph looks the same.



- **Task graph** is a directed acyclic graph (DAG) G=(V,E)
  - Set V of vertices: elementary tasks (taking time 1 resp. O(1))
  - Set E of directed edges: dependences (partial order on tasks) (v1,v2) in E → v1 must be finished before v2 can start
- **Critical path** = longest path from an entry to an exit node
  - Length of critical path is a lower bound for parallel time complexity
- **Parallel time** can be longer if number of processors is limited
  - → *schedule tasks* to processors such that dependences are preserved (by programmer (SPMD execution) or run-time system (fork-join exec.))

Parallel Time, Work, Cost

problem size n

processors p

time $t(p,n)$

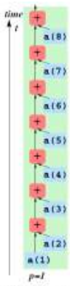
work $w(p,n)$

cost $c(p,n) = t \cdot p$

Example:
seq. sum algorithm

```
s = a(1)
do i = 2, n
  s = s + a(i)
end do
```

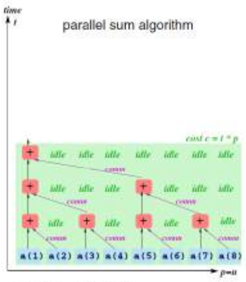
$n-1$ additions
n loads
$O(n)$ other



$t(1,n) = t_{seq}(n) = O(n)$

$w(1,n) = O(n)$

$c(1,n) = t(1,n) \cdot 1$
$= O(n)$



$t(n,n) = O(\log n)$

$w(n,n) = O(n)$

$c(n,n) = O(n \log n)$

par. sum alg. *not* cost-effective!

Parallel work, time, cost

parallel work $w_A(n)$ of algorithm A on an input of size n
= max. number of instructions performed by all procs during execution of A, where in each (parallel) time step as many processors are available as needed to execute the step in constant time.

parallel time $t_A(n)$ of algorithm A on input of size n
= max. number of parallel time steps required under the same circumstances

parallel cost $c_A(n) = t_A(n) * p_A(n)$ $\rightarrow c_A(n) \le w_A(n)$
where $p_A(n) = \max_i p_i(n)$ = max. number of processors used in a step of A

Work, time, cost are thus *worst-case* measures.

$t_A(n)$ is sometimes called the depth of A
(cf. circuit model of (parallel) computation)

$p_i(n)$ = number of processors needed in time step i, $0 \le i < t_A(n)$, to execute the step in constant time. Then, $w_A(n) = \sum_{i=0}^{t_A(n)} p_i(n)$

Work-optimal and cost-optimal

A parallel algorithm A is asymptotically work-optimal iff $w_A(p,n) = O(t_{seq}(n))$

A parallel algorithm A is asymptotically cost-optimal iff $c_A(p,n) = O(t_{seq}(n))$

Making the parallel sum algorithm cost-optimal:

Instead of n processors, use only $n/\log_2 n$ processors.

First, each processor computes sequentially the global sum of "its" $\log n$ local elements. This takes time $O(\log n)$.

Then, they compute the global sum of $n/\log n$ partial sums using the previous parallel sum algorithm.

Time: $O(\log n)$ for local summation, $O(\log n)$ for global summation

Cost: $n/\log n \cdot O(\log n) = O(n)$ *linear!*

This is an example of a more general technique based on Brent's theorem.

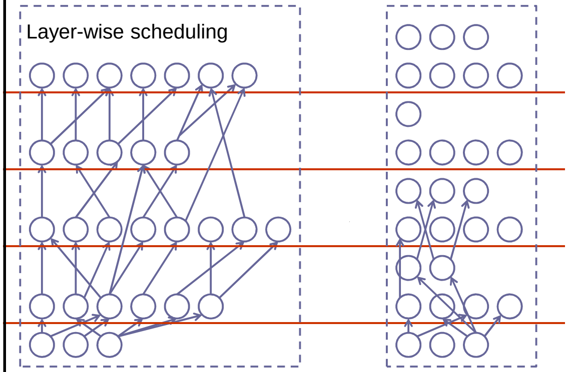
Some simple task scheduling techniques

Greedy scheduling
(also known as ASAP, as soon as possible)
Dispatch each task as soon as
- it is data-ready (its predecessors have finished)
- and a free processor is available

Critical-path scheduling
Schedule tasks on critical path first,
then insert remaining tasks where dependences allow,
inserting new time steps if no appropriate free slot available

Layer-wise scheduling
Decompose the task graph into layers of independent tasks
Schedule all tasks in a layer before proceeding to the next

Work-Time (Re)scheduling



19

Brent's Theorem [Brent 1974]

Any PRAM algorithm A
which runs in $t_A(n)$ time steps and performs $w_A(n)$ work
can be implemented to run on a p-processor PRAM in

$$O\left(t_A(n) + \frac{w_A(n)}{p}\right)$$

time steps.

Proof: see [PPP p.41]

Algorithm design issue: Balance the terms for cost-effectiveness:
$\rightarrow$ design A with $p_A(n)$ processors such that $w_A(n)/p_A(n) = O(t_A(n))$

Note: Proof is non-constructive!
$\rightarrow$ How to determine the active processors for each time step?
$\rightarrow$ language constructs, dependence analysis, static/dynamic scheduling, ...

Speedup

Consider problem $\mathcal{P}$, parallel algorithm A for $\mathcal{P}$

T_s = time to execute the best serial algorithm for $\mathcal{P}$ on one processor of the parallel machine

$T(1)$ = time to execute parallel algorithm A on 1 processor
$T(p)$ = time to execute parallel algorithm A on p processors

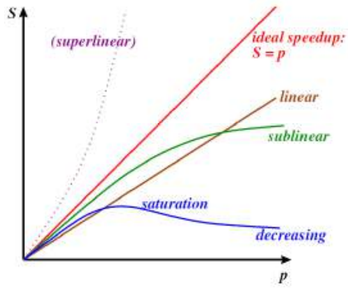
Absolute speedup $S_{abs} = \frac{T_s}{T(p)}$

Relative speedup $S_{rel} = \frac{T(1)}{T(p)}$

$S_{abs} \leq S_{rel}$

21

Speedup



trivially parallel
(e.g., matrix product, LU decomposition, ray tracing)
$\rightarrow$ close to ideal $S = p$

work-bound algorithms
$\rightarrow$ linear $S \in \Theta(p)$, work-optimal

tree-like task graphs
(e.g., global sum / max)
$\rightarrow$ sublinear $S \in \Theta(p/\log p)$

communication-bound
$\rightarrow$ sublinear $S = 1/fn(p)$

Most papers on parallelization show only relative speedup
(as $S_{abs} \leq S_{rel}$, and best seq. algorithm needed for S_{abs})

Amdahl's Law: Upper bound on Speedup

Consider execution (trace) of parallel algorithm A:
sequential part A^s where only 1 processor is active
parallel part A^p that can be sped up perfectly by p processors

$\rightarrow$ total work $w_A(n) = w_{A^s}(n) + w_{A^p}(n)$, time $T = T_{A^s} + \frac{T_{A^p}}{p}$,

Amdahl's Law

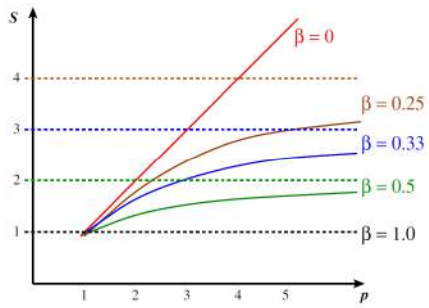
If the sequential part of A is a *fixed* fraction of the total work irrespective of the problem size n, that is, if there is a constant β with

$$\beta = \frac{w_{A^s}(n)}{w_A(n)} \leq 1$$

the relative speedup of A with p processors is limited by

$$\frac{p}{\beta p + (1-\beta)} < 1/\beta$$

Amdahl's Law



$$S(p) = \frac{p}{\beta p + (1-\beta)} < 1/\beta$$

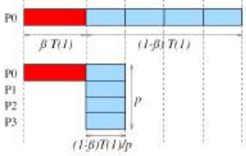
24

Proof of Amdahl's Law

$$S_{rel} = \frac{T(1)}{T(p)} = \frac{T(1)}{T_{A^s} + T_{A^p}(p)}$$

Assume perfect parallelizability of the parallel part A^p,
that is, $T_{A^p}(p) = (1-\beta)T(p) = (1-\beta)T(1)/p$:

$$S_{rel} = \frac{T(1)}{\beta T(1) + (1-\beta)T(1)/p} = \frac{p}{\beta p + 1 - \beta} < 1/\beta$$



Remark:
For most parallel algorithms the sequential part is *not* a fixed fraction.

Remarks on Amdahl's Law

Not limited to speedup by parallelization only!

Can also be applied with other optimizations
e.g. SIMDization, instruction scheduling, data locality improvements, ...

Amdahl's Law, general formulation:

If you speed up a fraction $(1-\beta)$ of a computation by a factor p,
the overall speedup is $\frac{p}{\beta p + (1-\beta)}$, which is $< \frac{1}{\beta}$.

Implications

- Optimize for the common case.
  If $1-\beta$ is small, optimization has little effect.
- Ignored optimization opportunities (also) limit the speedup.
  As $p \longrightarrow \infty$, speedup is bound by $\frac{1}{\beta}$.

Speedup Anomalies

Speedup anomaly:
An implementation on p processors may execute faster than expected.

Superlinear speedup
speedup function that grows faster than linear, i.e., in $\omega(p)$

Possible causes:
- cache effects
- search anomalies

Real-world example: move scaffolding

Speedup anomalies may occur only for fixed (small) range of p.

Theorem:
There is no absolute superlinear speedup for arbitrarily large p.

Search Anomaly Example: Simple string search

Given: Large unknown string of length *n*,
pattern of constant length *m* << *n*
Search for *any* occurrence of the pattern in the string.

Simple sequential algorithm: Linear search

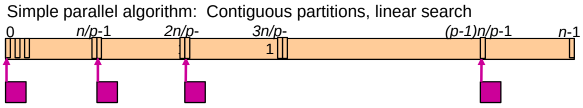
Pattern found at first occurrence at position *t* in the string after *t* time steps
or not found after *n* steps

28

Parallel Simple string search

Given: Large unknown shared string of length *n*,
pattern of constant length *m* << *n*
Search for *any* occurrence of the pattern in the string.

Simple parallel algorithm: Contiguous partitions, linear search



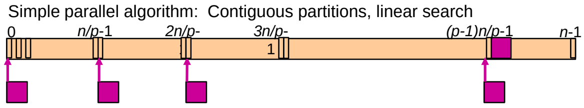
Case 1: Pattern not found in the string
→ measured parallel time *n/p* steps
→ speedup = *n* / (*n/p*) = *p* ☺

29

Parallel Simple string search

Given: Large unknown shared string of length *n*,
pattern of constant length *m* << *n*
Search for *any* occurrence of the pattern in the string.

Simple parallel algorithm: Contiguous partitions, linear search



Case 2: Pattern found in the first position scanned by the last processor
→ measured parallel time 1 step, sequential time *n-n/p* steps
→ observed speedup *n-n/p*, "superlinear" speedup?!?

But, …
… this is not the worst case (but the best case) for the parallel algorithm;
… and we could have achieved the same effect in the sequential algorithm, too, by altering the string traversal order

30

Further fundamental parallel algorithms

Parallel prefix sums

Parallel list ranking

Data-Parallel Algorithms

- One task (virtual processor) associated with each data element
  Agglomeration + mapping to hardware processors by the compiler
- Problems of size N
  solved usually in time $O(1)$ or $O(\log N)$ using N processors

Some data-parallel algorithms (see Hillis/Steele):

- Parallel sum $\surd$
- Prefix sums (partial sums)
- Radix sort
- Parsing a regular language
- Parallel combinator reduction
- List ranking (finding the end of a parallel linked list, list prefix sums etc.)
- Matching components of two lists

Read the article by Hillis and Steele (see Further Reading)

The Prefix-Sums Problem

Given: a set S (e.g., the integers)
a binary associative operator $\oplus$ on S,
a sequence of n items $x_0, \ldots, x_{n-1} \in S$

compute the sequence y of *prefix sums* defined by

$$y_i = \bigoplus_{j=0}^{i} x_j \text{ for } 0 \leq i < n$$

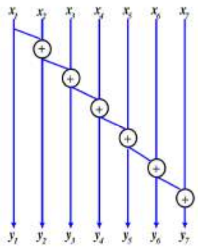
An important building block of many parallel algorithms! [Blelloch'89]

typical operations $\oplus$:
integer addition, maximum, bitwise AND, bitwise OR

Example:
bank account: initially 0\$, daily changes x_0, x_1, ...
$\rightarrow$ compute daily balances: (0,) x_0, $x_0 + x_1$, $x_0 + x_1 + x_2$, ...

Sequential prefix sums algorithm

```
void seq_prefix( int x[], int n, int y[] )
{
  int i;
  int ps;   // i'th prefix sum
  if (n>0) ps = y[0] = x[0];
  for (i=1; i<n; i++) {
    ps += x[i];
    y[i] = ps;
  }
}
```



if run in parallel on n virtual processors:
time $\Theta(n)$, work $\Theta(n)$, cost $\Theta(n^2)$

Task dependence graph: linear chain of dependences
$\rightarrow$ seems to be inherently sequential — how to parallelize?

Parallel prefix sums algorithm 1
A first attempt…

Naive parallel implementation:

apply the definition,

$$y_i = \bigoplus_{j=0}^{i} x_j \text{ for } 0 \leq i < n$$

and assign one processor for computing each y_i

$\rightarrow$ parallel time $\Theta(n)$, work and cost $\Theta(n^2)$

But we observe:
a lot of redundant computation (common subexpressions)

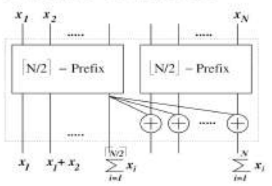
Parallel Prefix Sums Algorithm 2:
Upper-Lower Parallel Prefix

Algorithmic technique: parallel divide&conquer

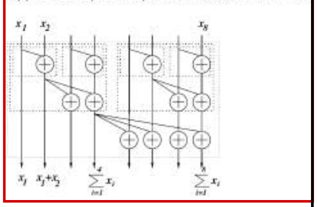
We consider the simplest variant, called Upper/lower parallel prefix:

recursive formulation:

N–prefix is computed as



Upper/lower parallel prefix, unfolded for $N = 8$:

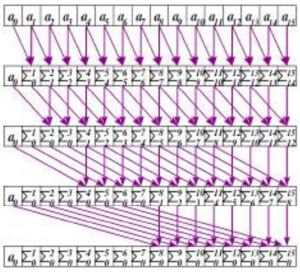


Parallel time: $\log n$ steps, work: $n/2 \log n$ additions, cost: $\Theta(n \log n)$

Not work-optimal! And needs concurrent read...

Parallel Prefix Sums Algorithm 3: Recursive Doubling (for EREW PRAM)

EREW (exclusive read, exclusive write) prefix sums algorithm:



iterative formulation
in data-parallel pseudocode:

real a : **array**$[0..N-1]$;
int stride;

stride $\leftarrow$ 1;
while stride $< N$ **do**
  forall i : $[0..N-1]$ **in parallel do**
    if $i \geq$stride **then**
      $a[i] \leftarrow a[i-\text{stride}] + a[i]$;
  stride := stride * 2;
(* finally, sum in $a[N-1]$ *)

Work: $\Theta(n \log n)$:-(

Parallel Prefix Sums Algorithms Concluding Remarks

There are improved algorithms for parallel prefix sums:

- Odd-even parallel prefix sums (EREW, work $\Theta(n)$)
- Ladner-Fischer parallel prefix sums [Ladner/Fischer 1980] cost-optimal (cost $\Theta(n)$) if using $\Theta(n/\log n)$ virtual processors only

38

Parallel List Ranking (1)

Parallel list: (unordered) array of list items (one per proc.), singly linked

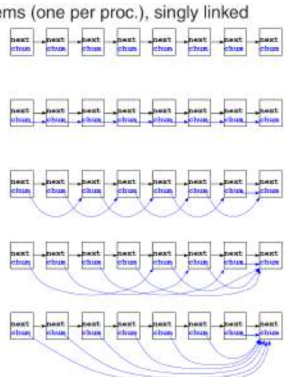
Problem: for each element, find the end of its linked list.

Algorithmic technique:
recursive doubling, here:
"pointer jumping" [Wyllie'79]

The algorithm in pseudocode:

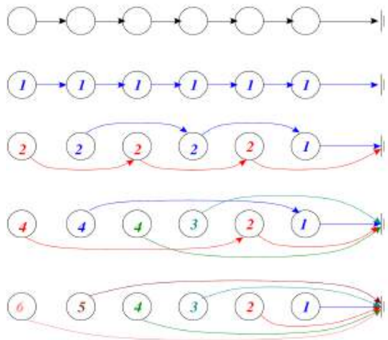
forall k in $[1..N]$ **in parallel do**
  chum$[k] \leftarrow$ next$[k]$;
  while chum$[k] \neq$ null
    and chum[chum$[k]] \neq$ null **do**
    chum$[k] \leftarrow$ chum[chum$[k]]$;
  od
od

lengths of chum lists halved in each step
$\Rightarrow \lceil \log N \rceil$ pointer jumping steps



Parallel List Ranking (2)

Extended problem: compute the rank = distance to the end of the list



By pointer jumping:
in each step:
to my own distance value,
I add the distance of my $\rightarrow$chum that I splice out of the list

Every step
+ doubles #lists
+ halves lengths

$\rightarrow \lceil \log_2 n \rceil$ steps

Not work-efficient!

40

Parallel List Ranking (3)

NULL-checks can be avoided by marking list end by a self-loop.

Pointer jumping algorithm for list ranking, implementation in Fork:

```
wyllie( sh LIST list[], sh int length )
{
  LIST *e; // private pointer
  int nn;

  e = list[$$];    // $$ is my processor index
  if (e->next != e)  e->rank = 1;  else e->rank = 0;
  nn = length;
  while (nn>1) {
    e->rank = e->rank + e->next->rank;
    e->next = e->next->next;
    nn = nn>>1;  // division by 2
  }
}
```

41

Questions?

Further Reading

On PRAM model and Design and Analysis of Parallel Algorithms

- J. Keller, C. Kessler, J. Träff: ***Practical PRAM Programming.*** Wiley Interscience, New York, 2001.
- J. JaJa: ***An introduction to parallel algorithms.*** Addison-Wesley, 1992.
- D. Cormen, C. Leiserson, R. Rivest: ***Introduction to Algorithms,*** Chapter 30. MIT press, 1989.
- H. Jordan, G. Alaghband: ***Fundamentals of Parallel Processing.*** Prentice Hall, 2003.
- W. Hillis, G. Steele: Data parallel algorithms. *Comm. ACM* **29**(12), Dec. 1986. Link on course homepage.
- Fork compiler with PRAM simulator and system tools http://www.ida.liu.se/chrke/fork (for Solaris and Linux)