Parallel Sorting Algorithms

TDDD56 Lecture 6/7

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TDDD56 Outline

2

Motivation

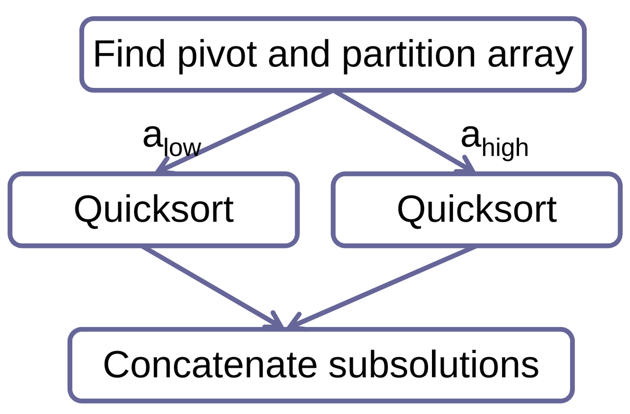
- Sorting is one of the most important subroutines
- Large data sets,
  little computation, mainly control and data movement
- Design of *scalable* parallel sorting algorithms is not straightforward
  - As opposed to many numerical kernel problems
- Parallel sorting algorithms heavily investigated since 40+ years
- Many parallel sorting algorithms known today
- Consider **3 representatives**, here for shared-memory:
  - Parallel Quicksort + variants
  - Bitonic Sort
  - Parallel Mergesort
  - All based on parallel divide-and-conquer principle

3

Simple Parallel Quicksort

Recall: Sequential Quicksort

- Algorithmic design pattern: Divide-and-Conquer
- **void** quicksort (int a[*n*])
  {
  // divide phase:
  choose pivot a[*i*] for some *i* in 0…*n*-1;
  // partition array a into a_{low}, a_{high}:
  a_{low} = { a[j] with a[j] <= a[i] };
  a_{high} = { a[j] with a[j] >= a[i] };
  // recursive calls:
  quicksort (a_{low});
  quicksort (a_{high});
  // combine phase:
  a = concat (a_{low}, a_{high});
  }



- Time complexity: O(*n* log *n*) on average, O(n^2) worst-case

Remarks on Pivot choice

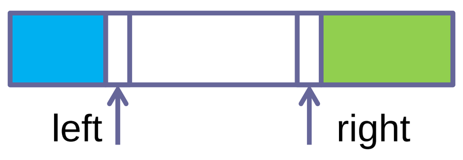
- First or last element in subsequence???
  → Worst-case time with pre-sorted data
- Pivot chosen randomly…
  → still not unlikely to generate unbalanced subproblems
- Better: draw random sample e.g. of size O(sqrt(*n*)) and choose pivot as the median of these
  → improves balance of a_{low} to a_{high}
- Pivot randomly attached to one of the two partitions
  - Randomly to avoid continued disbalance,
  - Attachment avoids separate treatment, e.g. in concat

6

Remarks on Partitioning

- **In-place partitioning** (reordering)
  - See your algorithms course or textbook for proof of correctness

```
left = 0;  right = n-1;
do {
    while ( a[left] < a[i] )   left++;
    while ( a[right] > a[i] )   right--;
    exchange ( a[left++], a[right--] );
} while (left < right);
```



☺ Avoids separate arrays for a_{low}, a_{high}

☺ Pointers suffice, concat implicit

☺ Cache friendly   (why?)

Background: Divide-and-Conquer

General strategy:

Divide-and-conquer(Problem P of size n):

If P is trivial (i.e., n very small), solve P directly.

otherwise:

divide P into $q \geq 1$ $\boxed{\textit{independent}}$ subproblems $P_1,\ldots,P_q$ of smaller size $n_1,\ldots,n_q$

solve these recursively:

$S_1 \leftarrow$ Divide-and-conquer(P_1, n_1),

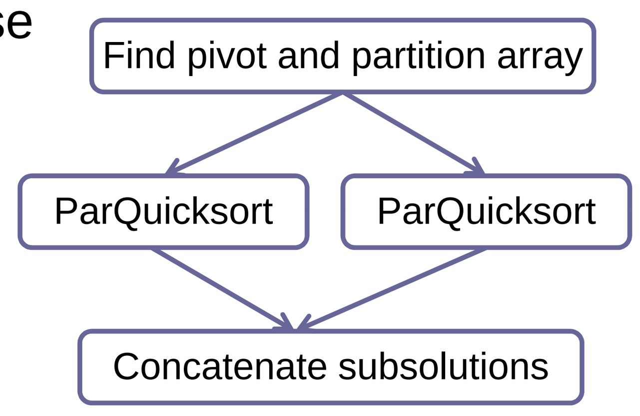
⋮

$S_q \leftarrow$ Divide-and-conquer(P_q, n_q)

combine the subsolutions $S_1,\ldots,S_q$ to a solution S for P.

Parallel Divide-and-Conquer
Simple Parallel Quicksort

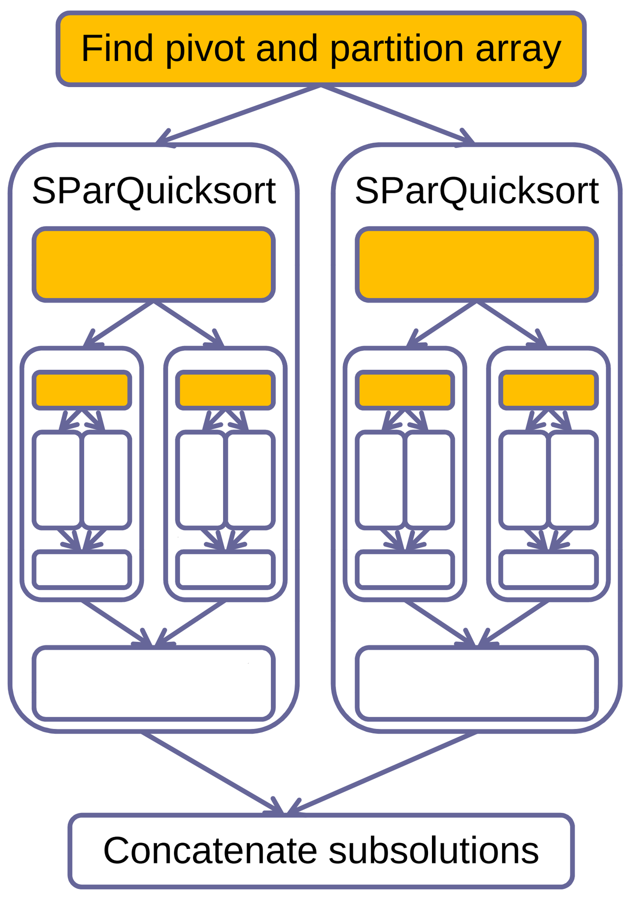
- The q subproblems (of size < *n*) are independent of each other
  - → can execute the recursive calls in parallel
  - → defines *q* independent subtasks
  - → **SimpleParQuicksort** algorithm
- Achievable speedup is quite limited if the *divide* and the *combine* phase are not parallelized (cf. Amdahl's Law!)



- Example:
  Parallel Quicksort, general case:

$T(n) \geq T(n/2) + T_{partition}(n) + T_{concat}(n) + O(1) \;\; = T(n/2) + O(n)$

Simple Parallel Quicksort, Analysis



← Time O(*n*)

NB: Partitioning (linear in input size) does all the heavy work in Quicksort

← Time O(*n*/2)   (best case)

← Time O(*n*/4)

…

Parallel Time: (simplified analysis)

$T(n) \geq T(n/2) + T_{partition}(n) + T_{concat}(n) + O(1)$
$= T(n/2) + O(n)$
$= O(n) + O(n/2) + O(n/4) + \ldots + O(1)$
$= O(n)$ ☹

Parallel Work: $O(n \log n)$ ☺

Simple Parallel Quicksort,   Discussion

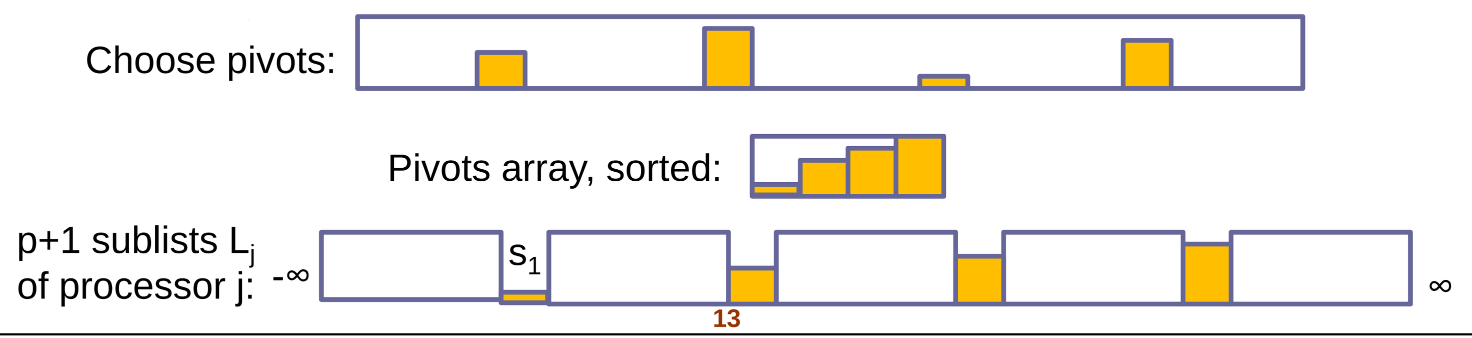
- Parallel time $\Omega(n)$
  → speedup O(log *n*) ☹
  - ok for *p* in O(log *n*)   i.e. for small processor count
- Task granularity control
  - Avoid too small tasks (overhead of spawning…)
  - Spawn SimpleParQuicksort (a_{high}) to different processor if size > *n*/*p*, otherwise do it sequentially
  - Parallel time: still O(*n*)
    sequence of partitions,   time *n*+*n*/2+*n*/4+… = O(*n*)
    plus seq. sorts of size ≤ *n*/*p*,   time O(*n*/*p**log(*n*/*p*))

Parallel Sample Sort

How to partition in parallel?

Approach 1: parallel *(p+1)*-way partitioning (**Parallel Sample Sort**)

- choose p-1 pivots (splitters) $s_1 \ldots s_{p-1}$ from array a ($s_0=-\infty$, $s_p=\infty$) and sort them (sequentially)
- each processor i partitions n/p elements of array a into p partial lists $L_{i,j}$, j=0…p-1, according to pivots: a[k] into list $L_{i,j}$ iff s_j < a[k] < s_{j+1}, for k=i*n/p…(i+1)*n/p-1
- each processor j gathers all partial lists $L_{i,j}$ into list L_j
- each processor j sorts its list L_j sequentially



Example for Parallel Samplesort

- Here, *p*=3

Input (*n*=15): 9, 3, 17, 4, 5, 20, 19, 11, 1, 8, 7, 2, 15, 14, 6
Pivot candidates: 17, 19, 7, 6
Choose *p*-1 pivots and sort them: 7, 17

Proc0: list L0,0=(3,4,5)	L0,1=(9)	L0,2=(17)
Proc1: list L1,0=(1)	L1,1=(8,11)	L1,2=(19,20)
Proc2: list L2,0=(2,6,7)	L2,1=(14,15)	L2,2= --

Proc0 sorts: 3,4,5,1,2,6,7
Proc1 sorts: 9,8,11,14,15
Proc2 sorts: 17,19,20

Parallel Sample Sort, Analysis

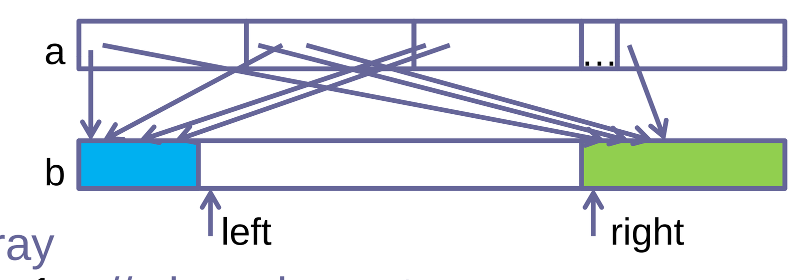
- Sequential sorting of the *p*-1 pivots: time O(*p* log *p*)
- Parallel (*p*-1)-way-partitioning in time O ((*n/p*) * log *p*) because of binary search in the sorted Pivots array
- Simultaneous sequential sorting of *p* lists L_j in time O (*n/p* * log(*n/p*))
- Total: parallel time O ((*n/p* + *p*) * log (*n/p* + *p*))
  - time-optimal for *p* in O(*n/p*), i.e. *p* in O(sqrt(*n*)).
- Advantages:
  - no recursive calls
  - can also be used on message-passing machines (one all-to-all communication)
- Disadvantage: not in-place, lists $L_{i,j}$ need separate array

Fully Parallel Quicksort

How to partition in parallel? (2)

Approach 2: Using Atomic Fetch&Increment

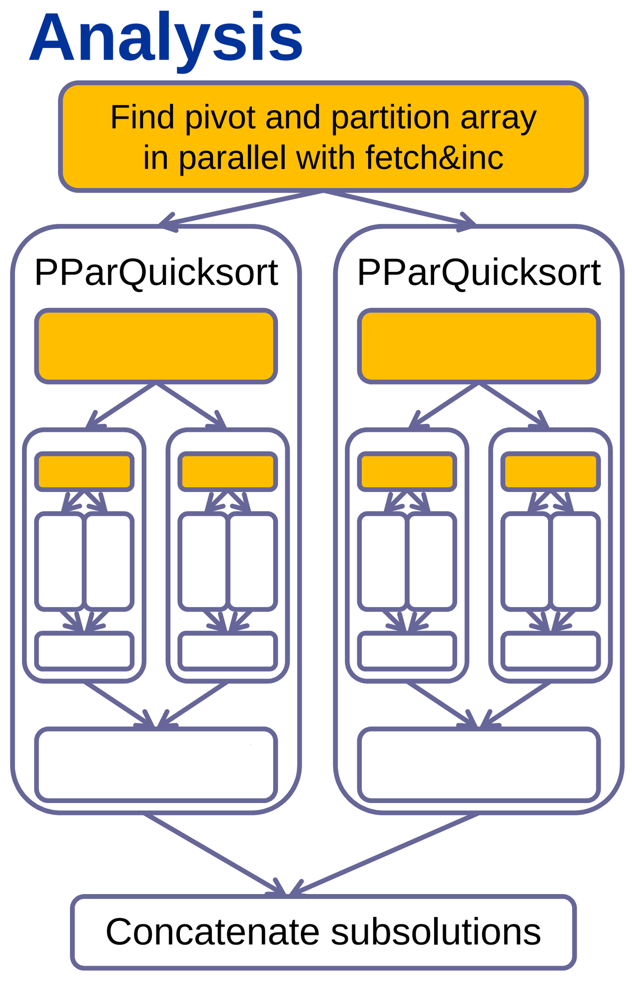
- If the hardware provides it …
- Keep degree-2 divide-and-conquer, parallelize partition loop
- Each processor partitions part of array of size *n/p*
  Then re-order partial partitions
- Needs a temporary array



- ParPartition(**int** a[n])

```
{
   shared int b[n]; // temp. array
   shared int left = 0, right = n-1;   // shared counters
   forall i in 0...n-1 do in parallel
      if  ( a[i] <= pivot )    b[ fetch&inc( left, +1 )] = a[i];
      else                     b[ fetch&inc( right, -1 )] = a[i];
   // … and copy back b to a in parallel for in-place partitioning
}
```


Fully Parallel Quicksort with Fetch&Inc, Analysis



Time O(*n/p*) on *p* processors, up to *p*≤*n* procs could be used!

NB: Partitioning (linear in input size) does all the heavy work in Quicksort

Time O(*n/p*) (best case) on *p* processors

Time O(*n/p*)
…

Parallel Time: (simplified analysis)

$T(n) \geq T(n/2) + T_{ParPartition}(n) + T_{concat}(n) + O(1)$
$= T(n/2) + O(n/p)$
$= O(n/p) + O(n/p) + \ldots + O(n/p)$
$= O((n \log n) / p)$ ☺

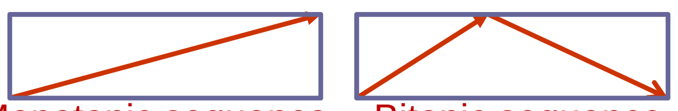
Parallel Work: O(*n* log *n*) ☺

Fully Parallel Quicksort with Fetch&Inc, Remarks

- Could utilize up to n processors in parallel on each level, then have O(n/p) i.e. constant time per level ☺
  - **IF** the hardware implementation does **not serialize** all Fetch&Inc ops!
    - p concurrent calls to Fetch&Inc done per level
  - Not a realistic assumption for large p
  - Requires a Combining CRCW PRAM (i.e., a very strong shared memory interface)
  - Only few parallel systems (such as SBPRAM) support this.
- Only applicable if the hardware provides atomic fetch-and-inc
  - Software implementation of f&i with mutex lock would serialize anyway
- **Alternative to hardware Fetch&Inc:** Implement scalable parallel Fetch&Inc over p processors in software with *Parallel Prefix Sums* (see Lecture 6)
  - Adds factor log p to the time complexity: O(n/p log n log p), i.e., expected parallel time O((log n)^2) with n processors

Bitonic Sort

Bitonic Sort



Monotonic sequence Bitonic sequence

- A sequence of numbers a=(a_1,…,a_n) is called *bitonic* if either there is a k in {1,..,n} such that $a_1 \leq \ldots \leq a_k \geq \ldots \geq a_n$ or the sequence can be rotated to that form

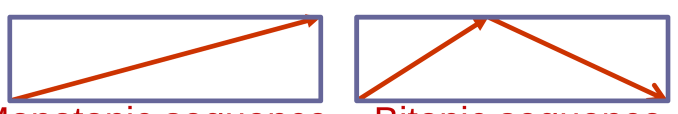
Example for Bitonic Sort

- a= (4, 1, 2, 7, 6, 5) is a bitonic sequence
  (could be rotated left to 1, 2, 7, 6, 5, 4)

  k=3, a_3=7 is largest element, 1, 2, 7 is ascending part,
  6, 5, 4 is descending part

- 4, 1, 7, 6, 5, 2 is not a bitonic sequence
  (cannot be rotated to have a unique „peak")
  1, 7, 6, 5, 2, 4 7, 6, 5, 2, 4, 1
  6, 5, 2, 4, 1, 7 5, 2, 4, 1, 7, 6 2, 4, 1, 7, 6, 5

Bitonic Sort (cont.)



Monotonic sequence Bitonic sequence

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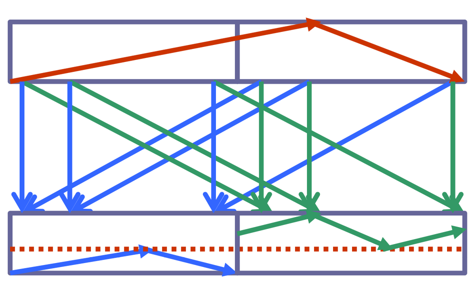
- **Lemma** (Batcher, 1968):
  If a is bitonic, then
  a' = min(a_1, $a_{n/2+1}$),…,min($a_{n/2}$, a_n)
  a'' = max(a_1, $a_{n/2+1}$),…,max($a_{n/2}$, a_n)
  are both bitonic and max(a') ≤ min(a'')



Proof not shown here, see e.g. Cormen et al.

- Kind of divide step for bitonic sequences

Example for Bitonic Sort

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  (could be rotated left to 1, 2, 7, 6, 5, 4)

  k=3, a_3=7 is largest element, 1, 2, 7 is ascending part,
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- 4, 1, 7, 6, 5, 2 is not a bitonic sequence
  (cannot be rotated to have a unique „peak")
  1, 7, 6, 5, 2, 4 7, 6, 5, 2, 4, 1
  6, 5, 2, 4, 1, 7 5, 2, 4, 1, 7, 6 2, 4, 1, 7, 6, 5

- Batcher's lemma for a = (4, 1, 2, 7, 6, 5) from above:
  a' = min(4,7),min(1,6),min(2,5) = 4, 1, 2 is bitonic
  a'' = max(4,7),max(1,6),max(2,5) = 7, 6, 5 is bitonic

BitonicMerge

- Consequence of Batcher's Lemma:

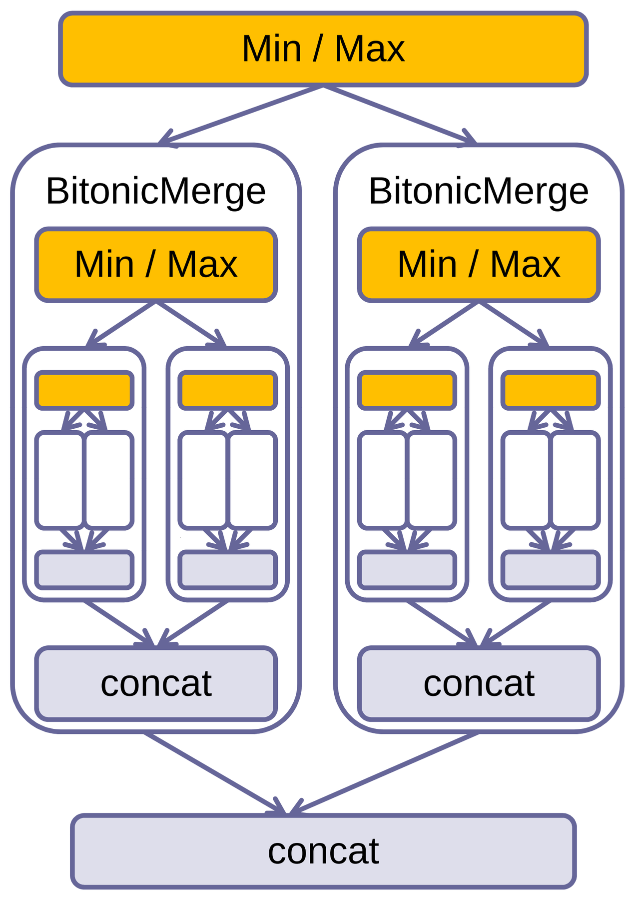
  BitonicMerge (**int** a[n], **int** *order*)
  { // a must be bitonic
  compute a', a'' according to Lemma if *order* == ascending
  (exchange min and max if *order* == descending)
  return concat (BitonicMerge(a', *order*),
  BitonicMerge(a'', *order*))
  } // now sorted in *order*

Parallel divide+conquer,
base case is n=2
(trivial to merge)

- Analysis:
  bitonic sequence can be b.-merged in parallel time O(log n)
  with n processors (n/2 doing min, n/2 doing max)

- Note: ascending/descending order needed in a minute

Bitonic Merge, Analysis



Parallel Time:

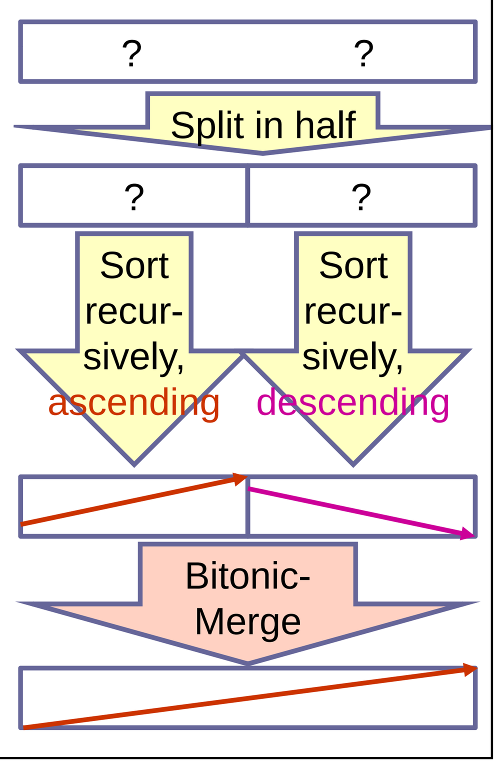
$T_{BitonicMerge}(n)$
$= T_{BitonicMerge}(n/2) + T_{MinMax}(n) + T_{concat}(n) + O(1)$
$= T_{BitonicMerge} (n/2) + O(1)$
$= O(1) + O(1) + \ldots + O(1)$
$= O(\log n)$

Parallel Work:

$W_{BitonicMerge} (n)$
$= 2\, W_{BitonicMerge} (n/2) + O(n)$
$= \ldots$
$= O(n \log n)$

BitonicMerge – and then?

- BitonicMerge brings any bitonic sequence into sorted form.
- But how do we derive, from unsorted input data, a bitonic sequence?
- Idea: combine both problems, sort recursively and flip the *order*
- Parallel Divide-and-Conquer, again!



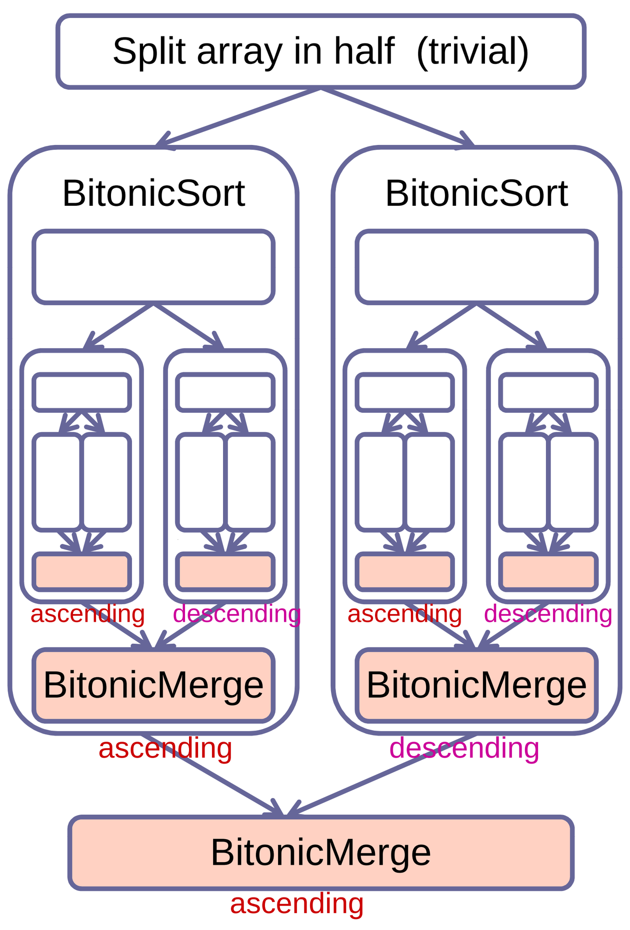
Bitonic Sort

- Idea: Turn arbitrary sequence into bitonic sequence by sorting its halves in ascending and descending order:

  BitonicSort (**int** a[n], **int** *order*)
  { // a is an arbitrary sequence
  BitonicSort(a[1..n/2], *ascending*);
  || BitonicSort(a[n/2+1..n], *descending*); // in parallel
  // a is now bitonic
  BitonicMerge (a, *order*);
  }

- Analysis for n processors:
  $T(n) = T(n/2) + O(\log n) = O((\log n)^2)$
  - Not time-optimal but the constant factor is very small

Bitonic Sort, Analysis



NB: BitonicMerge (logarithmic in input size) does all the heavy lifting work in BitonicSort

Parallel Time:

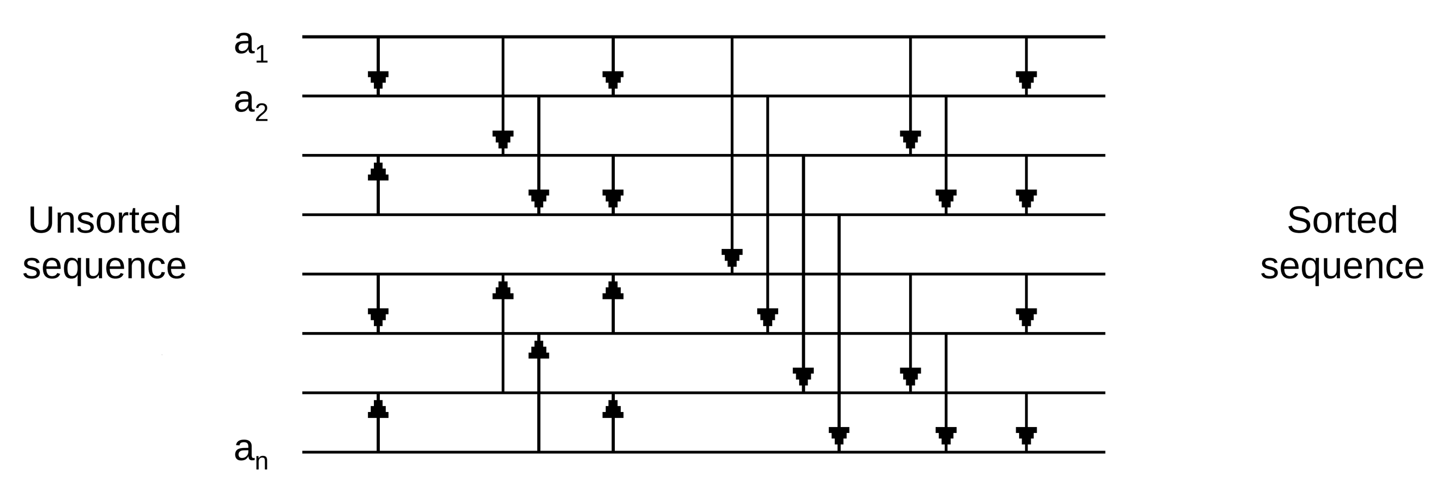
$T(n) = T(n/2) + T_{split}(n) + T_{BitonicMerge}(n) + O(1)$
$= T(n/2) + O(\log n)$
$= O(\log n) + O(\log n/2) + \ldots + O(1)$
$= O(\log^2 n)$ 😐

Parallel Work:

$W(n) = 2\, W(n/2) + O(n \log n)$
$= \ldots$
$= O(n \log^2 n)$ 😐

Bitonic Sort

- Unfolded task graph for n=8:



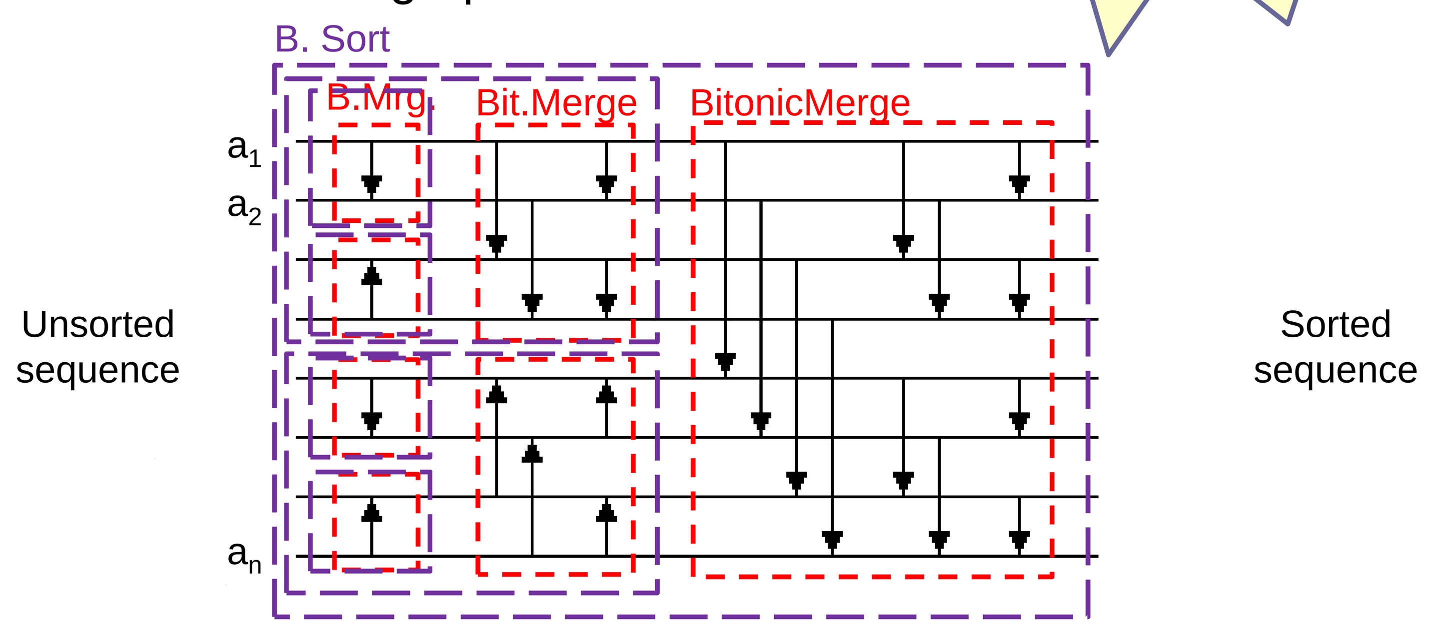
Legend: Compare and reorder in arrow order (asc)

Compare and reorder in arrow order (desc)

Bitonic Sort

Nested recursion /
Nested parallel DC

- Unfolded task graph for n=8:



Legend: ↓ Compare and reorder in arrow order (asc)

↑ Compare and reorder in arrow order (desc)

Remarks on Bitonic Sort

- Bitonic Sort is a ***sorting network***, designed as massively parallel hardware algorithm (comparator network) from the beginning
- Bitonic Sort is an ***oblivious algorithm***, i.e., control flow only depends on input *size*, not on input *contents*
  - Stable, well predictable performance, good for realtime computing
- Add granularity control:
  - Stop BitonicSort recursion at problem size *n*/*p* and sort sequentially instead

Parallel Mergesort

Mergesort (1)

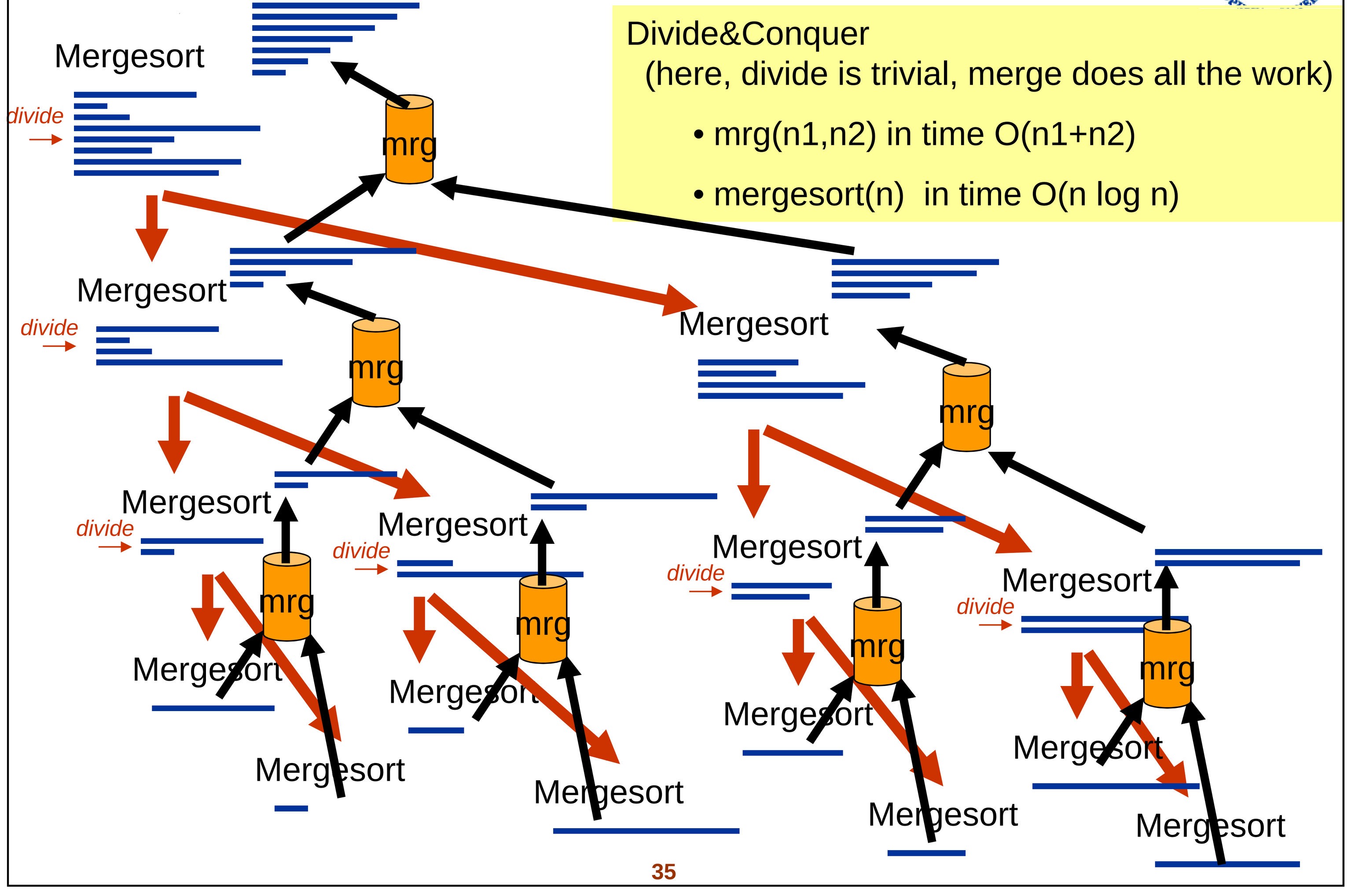
- Known from sequential algorithm design
- **Merge**: take two sorted blocks of length *k* and combine into one sorted block of length 2*k*

```
SeqMerge ( int a[k], int b[k], int c[2k] )
{
    int ap=0, bp=0, cp=0;
    while ( cp < 2k ) {        // assume a[k] = b[k] = ∞
        if (a[ap]<b[bp])  c[cp++] = a[ap++];
        else              c[cp++] = b[bp++];
    }
}
```

- Sequential time: O(*k*)
- Can also be formulated for in-place merging (copy back)

Sequential Mergesort

Divide&Conquer
(here, divide is trivial, merge does all the work)
- mrg(n1,n2) in time O(n1+n2)
- mergesort(n) in time O(n log n)

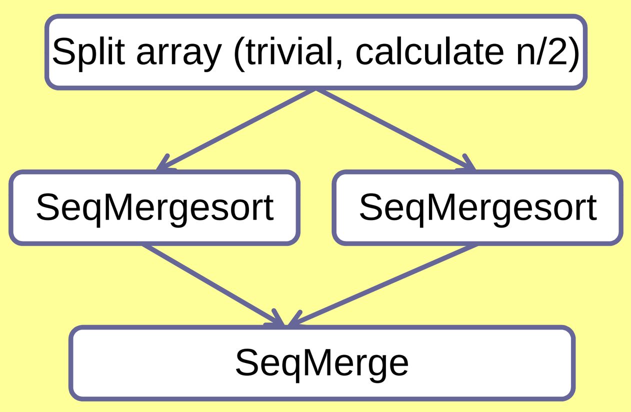


Sequential Mergesort

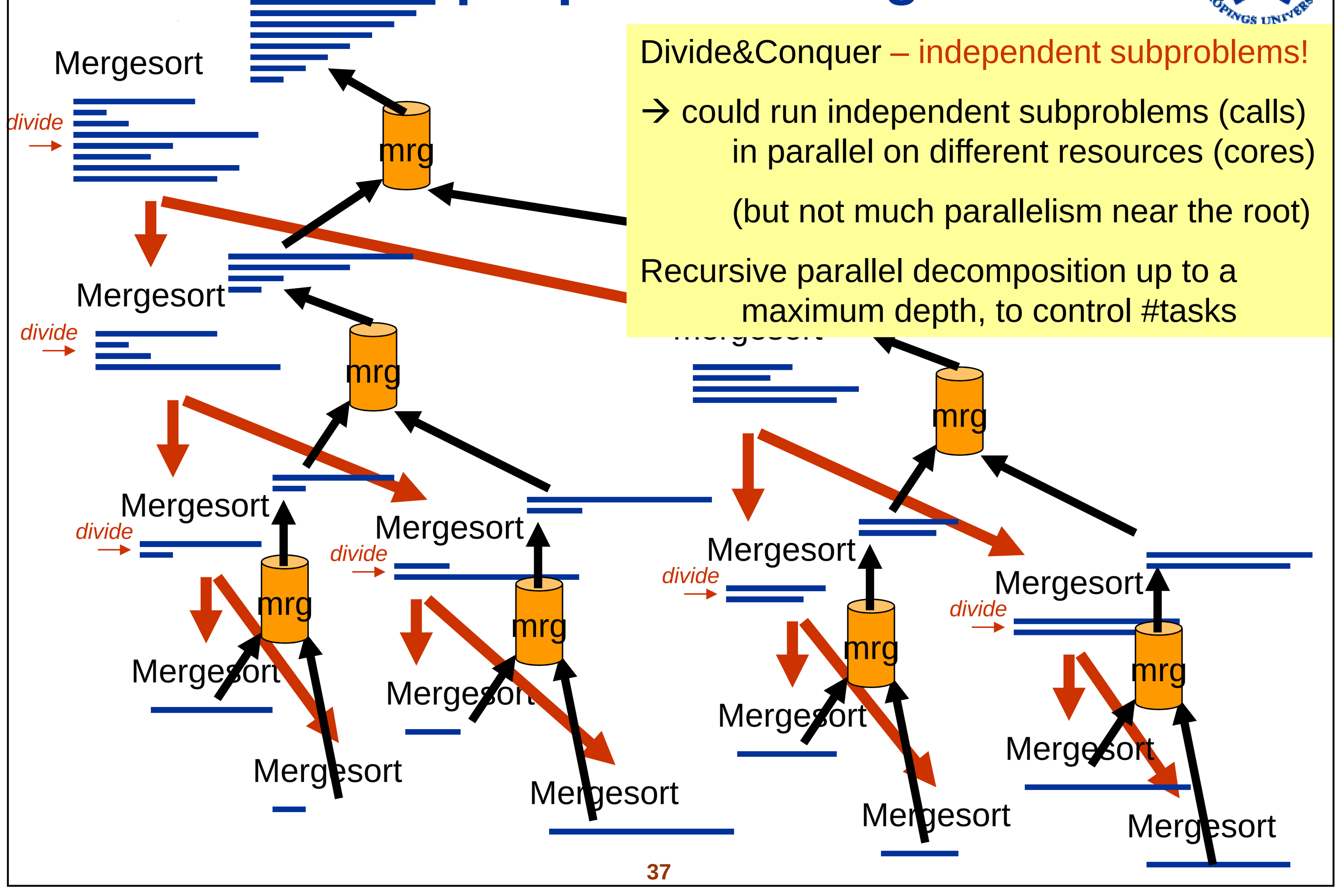
Time: O(n log n)

```
void SeqMergesort ( int *array, int n )   // in place
{
  if (n==1) return;
  // divide and conquer:
  SeqMergesort ( array, n/2);
  SeqMergesort ( array + n/2, n-n/2 );
  // now the subarrays are sorted
  SeqMerge ( array, n/2, n-n/2 );
}

void SeqMerge ( int array, int n1, int n2 )   // sequential merge in place
{
  ... ordinary 2-to-1 merge in O(n1+n2) steps ...
}
```



Towards a simple parallel Mergesort…



Divide&Conquer – independent subproblems!

→ could run independent subproblems (calls) in parallel on different resources (cores)

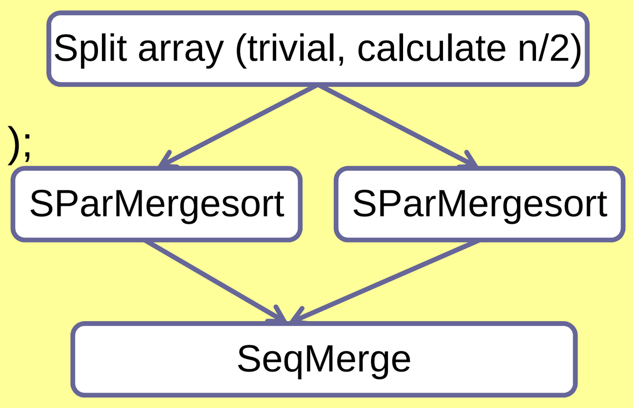
(but not much parallelism near the root)

Recursive parallel decomposition up to a maximum depth, to control #tasks

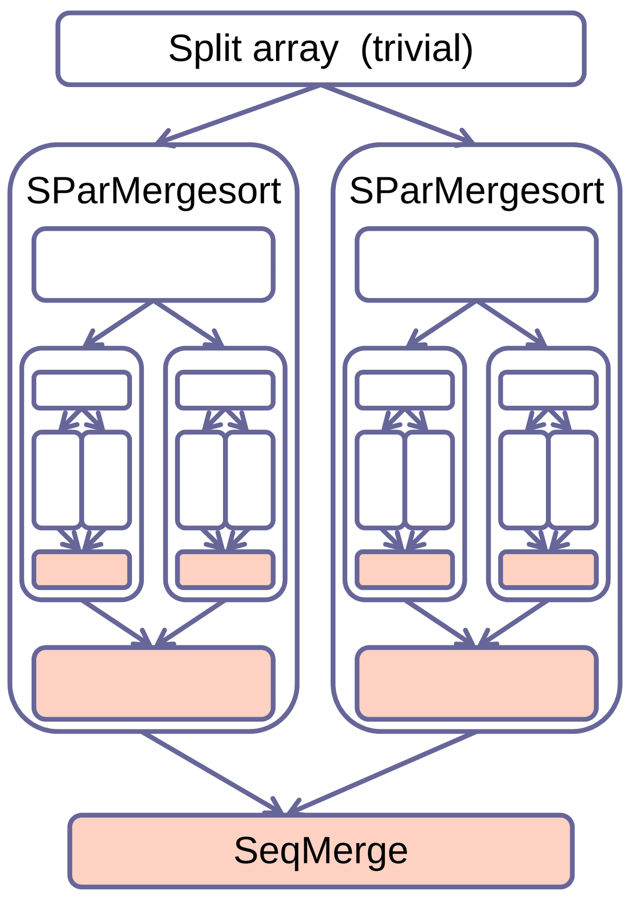
Simple Parallel Mergesort

```
void SParMergesort ( int *array, int n )   // in place
{
   if (n==1) return;   // nothing to sort
   if (depth_limit_for_recursive_parallel_decomposition_reached())
        SeqMergesort( array, n );   // switch to sequential
   // parallel divide and conquer:
   in parallel do {
         SParMergesort ( array,  n/2);
         SParMergesort ( array + n/2, n-n/2 );
   }
   // now the two subarrays are sorted
   seq  SeqMerge ( array, n/2, n-n/2 );
}

void SeqMerge ( int *array, int n1, int n2 )   // sequential merge in place
{
   // ... merge in O(n1+n2) steps ...
}
```



Simple Parallel Mergesort, Analysis



NB: SeqMerging (linear in input size) does all the heavy work in Mergesort

…

Parallel Time:

$T(n) = T(n/2) + T_{split}(n) + T_{SeqMerge}(n) + O(1)$
$= T(n/2) + O(n)$
$= O(n) + O(n/2) + O(n/4) + … + O(1)$
$= O(n)$ ☹

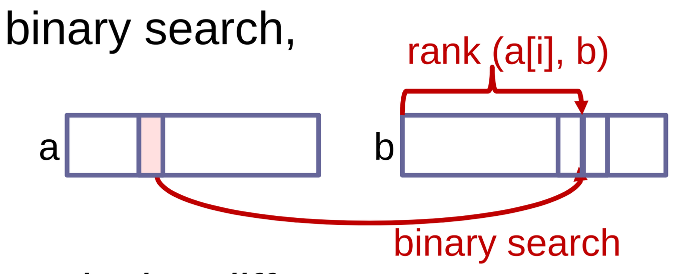
Parallel Work: O(n log n) ☺

Simple Parallel Mergesort, Discussion

- Structure is symmetric to Simple Parallel Quicksort
- Here, all the heavy work is done in the SeqMerge() calls
  - The counterpart of SeqPartition in Quicksort
  - Limits speedup and scalability
- Parallel time O(n),
  parallel work O(n log n),
  speedup limited to O(log n)
- (Parallel) Mergesort is an oblivious algorithm
  - could be used for a sorting network like bitonic sort
- Exercise:
  Iterative formulation   (use a **while** loop instead of recursion)

How to merge in parallel?

- For each element of the two arrays to be merged, calculate its final position in the merged array by cross-ranking
  - rank(x, $(a_0,…,a_{n-1})$) = #elements a_i < x
  - Compute rank by a sequential binary search, time O(log n)



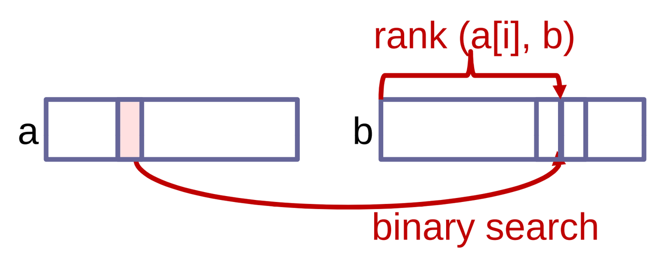
- ParMerge (**int** a[n1], **int** b[n2])
  // simplifying assumption:
  // All elements in both a and b are pairwise different

```
{
   for all i in 0…n1-1 in parallel
      rank_a_in_b[i]  =  compute_rank( a[i], b, n2 );
   for all i in 0…n2-1 in parallel
      rank_b_in_a[i]  = compute_rank( b[i], a, n1 );
   for all i in 0…n1-1 in parallel
      c[ i + rank_a_in_b[i]  ]  =  a[i];
   for all i in 0…n2-1 in parallel
      c[ i + rank_b_in_a[i]  ]  =  b[i];
}
```

Time for one binary search: O(log n)
Par. Time for ParMerge: O(log n)
Par. Work for parMerge: O(n log n)

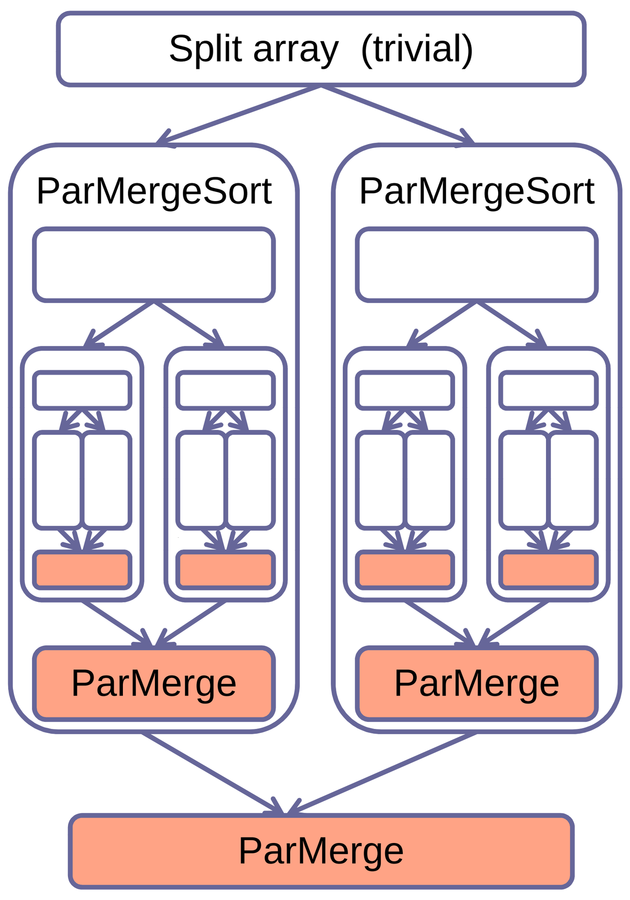
Example: ParMerge

- a = (2, 3, 7, 9),   b = (1, 4, 5, 8),   indices start at 0
- rank_a_in_b = (1, 1, 3, 4)
  rank_b_in_a = (0, 2, 2, 3)
- a[0] to pos. c[0+1] = 1
  a[1] to pos. c[1+1] = 2
  a[2] to pos. c[2+3] = 5
  a[3] to pos. c[3+4] = 7
  b[0] to pos. c[0+0] = 0
  b[1] to pos. c[1+2] = 3
  b[2] to pos. c[2+2] = 4
  b[3] to pos. c[3+3] = 6



- After copying,
  c = (1, 2, 3, 4, 5, 7, 8, 9)

Fully Parallel Mergesort, Analysis



NB: ParMerge (time logarithmic in input size) does all the heavy lifting work in ParMergeSort

Parallel Time:

$T(n) = T(n/2) + T_{split}(n) + T_{ParMerge}(n) + O(1)$
$= T(n/2) + O(\log n)$
$= O(\log n) + O(\log n/2) + \ldots + O(1)$
$= O(\log^2 n)$ ☺

Parallel Work:

$W(n) = 2\,W(n/2) + O(n \log n)$
$= \ldots$
$= O(n \log^2 n)$ ☺

Summary – Parallel Sorting

- We considered a few common parallel sorting algorithms
  - Simple Parallel Quicksort
  - Parallel Samplesort
  - Fully Parallel Quicksort with Fetch&Inc / with ParallelPrefixSums
  - Bitonic Sort
  - Simple Parallel Mergesort
  - Fully Parallel Mergesort
- Many more exist:
  - e.g. parallel rank sort, parallel radix sort, …
- There exist time-optimal ($O(\log n)$) parallel sorting algorithms for n processors, but these are very complex
  - AKS-network (1981), Cole's pipelined parallel mergesort (1988)
- Algorithm engineering necessary to adapt textbook algorithms to perform efficiently on real parallel systems
  - Example: use SIMD hardware, run on GPUs, on distributed memory, on special network topologies, …

Self-Assessment: Fill in the table!

Parallel Sorting Algorithm	Uses up to how many procs	Parallel time	Parallel work	Parallel cost	Restri ctions
Simple parallel quicksort					
Parallel samplesort					
Fully parallel quicksort, f&i					
Fully par. quicks. with par. prefix s.					
Bitonic sort					
Simple parallel mergesort					
Fully parallel mergesort					

Questions?

Further Reading

- J. Keller, C. Kessler, J. Träff: ***Practical PRAM Programming.*** Wiley Interscience, New York, 2001.
- J. JaJa: ***An introduction to parallel algorithms.*** Addison-Wesley, 1992.
- D. Cormen, C. Leiserson, R. Rivest: ***Introduction to Algorithms,*** Chapter 30. MIT press, 1989.
- H. Jordan, G. Alaghband: ***Fundamentals of Parallel Processing.*** Prentice Hall, 2003.
- W. Hillis, G. Steele: Data parallel algorithms. *Comm. ACM* **29**(12), Dec. 1986. Link on course homepage.
- Fork compiler with PRAM simulator and system tools http://www.ida.liu.se/chrke/fork (for Solaris and Linux)

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