Second Compulsory,INF5620

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Mathematical problem

$$\frac{\partial^2 u}{\partial t^2} + b\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(q(x,y)\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(q(x,y)\frac{\partial u}{\partial y}\right) + f(x,y,t) \tag{1}$$

And the bounary condition:

$$\frac{\partial u}{\partial n} = 0 \tag{2}$$

Where $\Omega = [0, L_x] \times [0, L_y]$
The initial conditions are

$$\begin{aligned} u(x,y,0) &= I(x,y) \\ u_t(x,y,0) &= V(x,y) \end{aligned} \tag{3}$$

Discretization

Derive the discrete set of equations to be implemented in a program

General scheme

$$\begin{aligned} W = \frac{\partial u}{\partial n} \mid_{x=0} &\simeq \frac{u^n_{0,j+1} - u^n_{0,j-1}}{2\Delta x} = 0 \\ E = \frac{\partial u}{\partial n} \mid_{x=L_x} &\simeq \frac{u^n_{L_x,j+1} - u^n_{L_x,j-1}}{2\Delta x} = 0 \\ N = \frac{\partial u}{\partial n} \mid_{y=0} &\simeq \frac{u^n_{j+1,0} - u^n_{j-1,0}}{2\Delta y} = 0 \\ S = \frac{\partial u}{\partial n} \mid_{y=L_y} &\simeq \frac{u^n_{i+1,L_y} - u^n_{j-1,L_y}}{2\Delta y} = 0 \end{aligned} \tag{4}$$

Working with the equation on the right side

$$
\begin{aligned}
&\frac{\partial}{\partial x}\left(q(x,y)\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(q(x,y)\frac{\partial u}{\partial y}\right)\\
\simeq\ &\frac{q_{i+\frac{1}{2},j}(u^n_{i+1,j}-u^n_{i,j})-q_{i-\frac{1}{2},j}(u^n_{i,j}-u^n_{i-1,j})}{\Delta x^2}\\
+&\frac{q_{i,j+\frac{1}{2}}(u^n_{i,j+1}-u^n_{i,j})-q_{i,j-\frac{1}{2}}(u^n_{i,j}-u^n_{i,j-1})}{\Delta y^2}
\end{aligned}
\tag{5}
$$

Discretication of the whole equation

$$
[D_tD_tu]^n_{i,j}+[bD_{2t}]^n_{i,j}=[D_xqD_xu]^n_{i,j}+[D_yqD_yu]^n_{i,j}+f^n_{i,j} \tag{6}
$$

$$
\begin{aligned}
\frac{u^{n+1}_{i,j}-2u^n_{i,j}+u^{n-1}_{i,j}}{\Delta t^2}+b\frac{u^{n+1}_{i,j}-u^{n-1}_{i,j}}{2\Delta t}&=\\
\frac{q_{i+\frac{1}{2},j}(u^n_{i+1,j}-u^n_{i,j})-q_{i-\frac{1}{2},j}(u^n_{i,j}-u^n_{i-1,j})}{\Delta x^2}&\\
+\frac{q_{i,j+\frac{1}{2}}(u^n_{i,j+1}-u^n_{i,j})-q_{i,j-\frac{1}{2}}(u^n_{i,j}-u^n_{i,j-1})}{\Delta y^2}+f^n_{i,j}&
\end{aligned}
\tag{7}
$$

If we then put $u^{n+1}_{i,j}$ on the left side, we get the equation:

$$
\begin{aligned}
u^{n+1}_{i,j}=\frac{1}{1+\frac{b\Delta t}{2}}\Bigg[2u^n_{i,j}+(\frac{b\Delta t}{2}-1)u^{n-1}_{i,j}+&\\
\Delta t^2\Bigg(\frac{q_{i+\frac{1}{2},j}(u^n_{i+1,j}-u^n_{i,j})-q_{i-\frac{1}{2},j}(u^n_{i,j}-u^n_{i-1,j})}{\Delta x^2}&\\
+\ \frac{q_{i,j+\frac{1}{2}}(u^n_{i,j+1}-u^n_{i,j})-q_{i,j-\frac{1}{2}}(u^n_{i,j}-u^n_{i,j-1})}{\Delta y^2}+f^n_{i,j}\Bigg)\Bigg]&
\end{aligned}
\tag{8}
$$

To make this equation cleaner and easier to read, we put in som constants for some of the equations.

$$
\begin{aligned}
Cx2&=\left(\frac{\Delta t}{\Delta x}\right)^2\\
Cy2&=\left(\frac{\Delta t}{\Delta y}\right)^2\\
R1&=\frac{1}{1+\frac{b\Delta t}{2}}\\
R2&=\frac{b\Delta t}{2}-1\\
dt2&=\Delta t^2
\end{aligned}
\tag{9}
$$

Then the equation will look like this:

$$\begin{aligned} u_{i,j}^{n+1} &= R1\left[2u_{i,j}^{n} + R2u_{i,j}^{n-1} + \right. \\ Cx2&\left(q_{i+\frac{1}{2},j}(u_{i+1,j}^{n} - u_{i,j}^{n}) - q_{i-\frac{1}{2},j}(u_{i,j}^{n} - u_{i-1,j}^{n})\right) \\ + Cy2&\left.\left(q_{i,j+\frac{1}{2}}(u_{i,j+1}^{n} - u_{i,j}^{n}) - q_{i,j-\frac{1}{2}}(u_{i,j}^{n} - u_{i,j-1}^{n})\right) + dt2f_{i,j}^{n}\right] \end{aligned} \tag{10}$$

But we are not able to use this on the edges

First step

The challenge her is that we only have the values for one step earlier. Since our equation demands values from two step earlier, we have to rewrite the equation.

We use the initial conditions

$$\begin{aligned} u(x,y,0) = I(x,y) &\simeq u_{i,j}^{0} = I_{i,j} \\ u_t(x,y,0) = V(x,y) \simeq \frac{u_{i,j}^{n+1} - u_{i,j}^{n-1}}{2\Delta x} &= V_{i,j}, n = 0 \\ u_{i,j}^{n-1} &= u^{n+1} - 2\Delta t V_{i,j} \\ n &= 0 \end{aligned} \tag{11}$$

Since we now have an expression for $u_{i,j}^{n-1}$, we can just put this in to equation (10)

$$\begin{aligned} u_{i,j}^{n+1} &= R1\left[2u_{i,j}^{n} + R2(u^{n+1} - 2\Delta t V_{i,j}) + \right. \\ Cx2&\left(q_{i+\frac{1}{2},j}(u_{i+1,j}^{n} - u_{i,j}^{n}) - q_{i-\frac{1}{2},j}(u_{i,j}^{n} - u_{i-1,j}^{n})\right) \\ + Cy2&\left.\left(q_{i,j+\frac{1}{2}}(u_{i,j+1}^{n} - u_{i,j}^{n}) - q_{i,j-\frac{1}{2}}(u_{i,j}^{n} - u_{i,j-1}^{n})\right) + dt2f_{i,j}^{n}\right] \end{aligned}$$

$$\begin{aligned} u_{i,j}^{n+1}(1 - R1R2) &= R1\left[2u_{i,j}^{n} - R2 * 2\Delta t V_{i,j} + \right. \\ Cx2&\left(q_{i+\frac{1}{2},j}(u_{i+1,j}^{n} - u_{i,j}^{n}) - q_{i-\frac{1}{2},j}(u_{i,j}^{n} - u_{i-1,j}^{n})\right) \\ + Cy2&\left.\left(q_{i,j+\frac{1}{2}}(u_{i,j+1}^{n} - u_{i,j}^{n}) - q_{i,j-\frac{1}{2}}(u_{i,j}^{n} - u_{i,j-1}^{n})\right) + dt2f_{i,j}^{n}\right] \end{aligned}$$

$$\begin{aligned} u_{i,j}^{n+1} &= \left(R1\left[2u_{i,j}^{n} - R2 * 2\Delta t V_{i,j} + \right.\right. \\ Cx2&\left(q_{i+\frac{1}{2},j}(u_{i+1,j}^{n} - u_{i,j}^{n}) - q_{i-\frac{1}{2},j}(u_{i,j}^{n} - u_{i-1,j}^{n})\right) \\ + Cy2&\left.\left.\left(q_{i,j+\frac{1}{2}}(u_{i,j+1}^{n} - u_{i,j}^{n}) - q_{i,j-\frac{1}{2}}(u_{i,j}^{n} - u_{i,j-1}^{n})\right) + dt2f_{i,j}^{n}\right]\right) / (1 - R1R2) \end{aligned} \tag{12}$$

And this we can use for the first step.

Boundary poins

Here we have to use the Boundary conditions from (4). Then we will get

$$\begin{aligned} W &\Rightarrow u^n_{i+1,j} = u^n_{i-1,j}, i = 0 \\ E &\Rightarrow u^n_{i+1,j} = u^n_{i-1,j}, i = N_x \\ N &\Rightarrow u^n_{i,j+1} = u^n_{i,j-1}, i = 0 \\ S &\Rightarrow u^n_{i,j+1} = u^n_{i,j-1}, i = N_y \end{aligned} \tag{13}$$

And we use eq(10) and put in these values

West

$$q_{i+\frac{1}{2},j} + q_{i-\frac{1}{2},j} = 2q_{i,j}$$

$$\begin{aligned} u^{n+1}_{i,j} = R1 \Big[& 2u^n_{i,j} + R2u^{n-1}_{i,j} + Cx2 \left(2q_{i,j}(u^n_{i+1,j} - u^n_{i,j})\right) \\ & + Cy2 \left(q_{i,j+\frac{1}{2}}(u^n_{i,j+1} - u^n_{i,j}) - q_{i,j-\frac{1}{2}}(u^n_{i,j} - u^n_{i,j-1}) \right) + dt2f^n_{i,j} \Big] \end{aligned} \tag{14}$$

East

$$\begin{aligned} u^{n+1}_{i,j} = R1 \Big[& 2u^n_{i,j} + R2u^{n-1}_{i,j} + Cx2 \left(2q_{i,j}(u^n_{i-1,j} - u^n_{i,j})\right) \\ & + Cy2 \left(q_{i,j+\frac{1}{2}}(u^n_{i,j+1} - u^n_{i,j}) - q_{i,j-\frac{1}{2}}(u^n_{i,j} - u^n_{i,j-1}) \right) + dt2f^n_{i,j} \Big] \end{aligned} \tag{15}$$

North

$$\begin{aligned} u^{n+1}_{i,j} = R1 \Big[& 2u^n_{i,j} + R2u^{n-1}_{i,j} + \\ & Cx2 \left(q_{i+\frac{1}{2},j}(u^n_{i+1,j} - u^n_{i,j}) - q_{i-\frac{1}{2},j}(u^n_{i,j} - u^n_{i-1,j}) \right) \\ & + Cy2 \left(2q_{i,j}(u^n_{i,j+1} - u^n_{i,j})\right) + dt2f^n_{i,j} \Big] \end{aligned} \tag{16}$$

South

$$\begin{aligned} u^{n+1}_{i,j} = R1 \Big[& 2u^n_{i,j} + R2u^{n-1}_{i,j} + \\ & Cx2 \left(q_{i+\frac{1}{2},j}(u^n_{i+1,j} - u^n_{i,j}) - q_{i-\frac{1}{2},j}(u^n_{i,j} - u^n_{i-1,j}) \right) \\ & + Cy2 \left(2q_{i,j}(u^n_{i,j-1} - u^n_{i,j})\right) + dt2f^n_{i,j} \Big] \end{aligned} \tag{17}$$

All directions in the first step

West

Here we use equation(12) with the same method as over.

$$u_{i,j}^{n+1} = \Big(R1\left[2u_{i,j}^{n} - R2 * 2\Delta t V_{i,j} + Cx2\left(2q_{i,j}(u_{i+1,j}^{n} - u_{i,j}^{n})\right)\right.$$
$$\left. + \ Cy2\left(q_{i,j+\frac{1}{2}}(u_{i,j+1}^{n} - u_{i,j}^{n}) - q_{i,j-\frac{1}{2}}(u_{i,j}^{n} - u_{i,j-1}^{n})\right) + dt2 f_{i,j}^{n}\right]\Big) / (1 - R1R2) \tag{18}$$

East

$$u_{i,j}^{n+1} = \Big(R1\left[2u_{i,j}^{n} - R2 * 2\Delta t V_{i,j} + Cx2\left(2q_{i,j}(u_{i-1,j}^{n} - u_{i,j}^{n})\right)\right.$$
$$\left. + \ Cy2\left(q_{i,j+\frac{1}{2}}(u_{i,j+1}^{n} - u_{i,j}^{n}) - q_{i,j-\frac{1}{2}}(u_{i,j}^{n} - u_{i,j-1}^{n})\right) + dt2 f_{i,j}^{n}\right]\Big) / (1 - R1R2) \tag{19}$$

North

$$\begin{aligned} u_{i,j}^{n+1} &= \Big(R1\Big[2u_{i,j}^{n} - R2 * 2\Delta t V_{i,j} + \\ Cx2 &\left(q_{i+\frac{1}{2},j}(u_{i+1,j}^{n} - u_{i,j}^{n}) - q_{i-\frac{1}{2},j}(u_{i,j}^{n} - u_{i-1,j}^{n})\right) \\ &+ \ Cy2\left(2q_{i,j}(u_{i,j+1}^{n} - u_{i,j}^{n})\right) + dt2 f_{i,j}^{n}\Big]\Big) / (1 - R1R2) \end{aligned} \tag{20}$$

South

$$\begin{aligned} u_{i,j}^{n+1} &= \Big(R1\Big[2u_{i,j}^{n} - R2 * 2\Delta t V_{i,j} + \\ Cx2 &\left(q_{i+\frac{1}{2},j}(u_{i+1,j}^{n} - u_{i,j}^{n}) - q_{i-\frac{1}{2},j}(u_{i,j}^{n} - u_{i-1,j}^{n})\right) \\ &+ \ Cy2\left(2q_{i,j}(u_{i,j-1}^{n} - u_{i,j}^{n})\right) + dt2 f_{i,j}^{n}\Big]\Big) / (1 - R1R2) \end{aligned} \tag{21}$$

Truncation error

$$\frac{\partial^2 u}{\partial t^2} + b\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(q(x,y)\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(q(x,y)\frac{\partial u}{\partial y}\right) + f(x,y,t) \tag{22}$$

In this case will we set q to be a constant, and we are able to rewrite it

$$\frac{\partial^2 u}{\partial t^2} + b\frac{\partial u}{\partial t} = q\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{23}$$

Then we can derive the truncation error

$$
\begin{aligned}
D_tD_tu + bD_{2t}u = q(d_xD_xu + D_yD_yu) + R \\
R^n = D_tD_tu + bD_{2t}u - q(d_xD_xu + D_yD_yu) \\
D_tD_tu = \frac{1}{12}u_{tttt}(t_n)\Delta t^2 \\
bD_{2t}u = \frac{b}{6}u_{ttt}(t_n)\Delta t^2 \\
qD_xD_xu = \frac{q}{12}u_{xxxx}(t_n)\Delta x^2 \\
qD_yD_yu = \frac{q}{12}u_{yyyy}(t_n)\Delta y^2
\end{aligned} \tag{24}
$$

$$
\begin{aligned}
R^n_{i,j} = \frac{1}{12}u_{tttt}(x_i, y_j, t_n)\Delta t^2 + \frac{b}{6}u_{ttt}(x_i, y_j, t_n)\Delta t^2 - \\
\frac{q}{12}\left(u_{xxxx}(t_n)\Delta x^2 + u_{yyyy}(t_n)\Delta y^2\right)
\end{aligned} \tag{25}
$$

This shows that we should get a second order error