Escaping the Zombie Threat by Mathematics

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January 16, 2013

1 Introduction

Mass extinction of man is a major threat in a world with zombies. It is therefore of fundamental importance for humans to figure out how to conquer zombies and the extremely serious disease known as zombification. The right strategy must be founded on a thorough understanding of how zombies and humans interact and how zombification spreads in a population. Although one gains some understanding of a world with zombies by watching relevant movies and reading books, this understanding is just as partial and incomplete as our understanding of the real world by just observing and experiencing it.

In the real world, it has been a tremendous success to understand the functioning of Nature by making mathematical models and studying these models. For example, Newton was the first to create mathematical models of how objects move under influence of forces. Such models laid the foundation of the industrial revolution and made it later possible to land a man on the moon. Maxwell and other physicists formulated models for electricity that gave us the understanding and the subsequent technology to create things like radio, TV, and wireless internet. It cannot be underestimated how important mathematical models have been for understanding the world and predicting the future.

So, in the world of movie making, it might well be the case that the realistic revolutionary ideas for how to conquer zombies in future movies arise from studying mathematical models of the human-zombie interaction. Note the world "realistic": how we conquer the zombies must be consistent with how a certain world with zombies actually functions. Mathematics can ensure this consistency in a simple way. Nothing can happen against the rules we define. Moreover, once the mathematical model with full consistency is in place, we can use it to learn about human-zombie interaction, change rules and assumptions, and finally predict how the future will be. There is always an uncertainty in the predictions, partly because we make many simplifications of how things are functioning, when we create a mathematical model, and partly because the model needs some data that can be hard to measure precisely.

How can we use mathematics to escape the zombie threat? As always, we look for existing applications of mathematics to similar problems, and then we try to migrate the thinking behind the model, in a well-understood problem, to a new problem. This particular ability of humans is the single most important reason for the dramatic technological development our specie has created. Hence, it seems fruitful to apply this strategy to prevent the extinction of man when the zombies attack.

Although there are no zombies (yet) in the real world, there are indeed processes going on that are similar to what we see in movies. The layman may think that a zombie reminds of a leper, and then we are almost there: according to the modern popular culture, humans get infected by zombies, and large-scale zombification is basically a question of how a disease spreads in a population. This latter topic has obviously received a lot of attention among scientists for centuries, and mathematics has in fact been central to the understanding for about 80 years. Recently, the spreading of the swine flu also made laymen interested in the science field known as epidemiology. We shall adopt mathematical ideas and techniques from this field to show how we can understand more about the dynamics of a population threatened by zombification.

The bottom line of the mathematics of zombification is that the number of zombies at time t, denoted by $Z(t)$, can be computed by a quite simple formula that is repeated a lot of times. That is, the formula would be tedious, and in fact impossible, to evaluate by pen and paper, so we need a computer to automate the job and do the calculations with high speed. The idea is that we compute Z at a set of discrete points in time, named $t_0 < t_1 < t_2 \cdots < t_{n-1} < t_n$. We have to know the initial number of zombies at time t_0, but how this number evolves in the future, can be computed. Let Z_i represent Z at time t_i, $i = 0, 1, \ldots, n$. The formula is of the form

$$Z_{i+1} = Z_i + \text{other known quantities at time } t_i$$

That is, we know some quantities at time t_i, and then we can evaluate a formula to compute a quantity like Z at some future time t_{i+1}. For such computations to be fairly accurate, the distance $t_{i+1} - t_i$ in time must be quite small[1]. One aspect that complicates the equation for Z is that that we must solve some similar equations for other quantities, because these quantities are needed in the formula for Z_{i+1}. We shall show in detail how we reason to construct and solve such equations. You hardly need high school math to understand it, all you need is some idea of what a function is. However, we may argue that the way we use mathematics in the present chapter represents an new, attractive way to introduce young people to the

[1]Mathematicians can give precise meaning to the adjectives “fairly” and “quite” in this context, and turn such vague qualitative statements into quantitative facts.

fundamental ideas and the craft of modern computation-based science and engineering.

2 Problem Definition

In a pioneering paper, Munz et. al [3] modeled zombie outbreaks mathematically and predicted extinction of mankind if zombies come to existence. Although doomsday was nearly inevitable, it could possibly be prevented by devastating and early counter attacks. This conclusion is supported by most modern zombie movies and stories. In the present paper, we refine this conclusion and use mathematics, computer simulations, and empirical data from movies to question the realism of the doomsday scenario, and also the realism of how zombie movies start.

The zombie archetype that forms the basis for this paper is based on the classical 1968 movie "The Night of The Living Dead" [5] by George A. Romero, which has inspired most modern zombie movies. Zombies are flesh-hungry, mindless, and almost dead people that spread the disease by biting humans. There is no magic involved, as is more common in the zombie variant that originates from Caribbean and African culture. In Romero's movie, zombies are subject to the nature of the laws of physics and biology, having similarity to animals infected by rabies, for instance. Zombies are not dead, they are simply a primitive and ruined form of life. Furthermore, zombies are hard to kill and may survive gunshots that humans would certainly die from. The only effective way of killing a zombie is to destroy its central nervous system. Outsiders of modern popular culture would probably suggest to conquer zombies by fiction, i.e., invetion of new magic effects. This is obviously an unethic approach: in all virtual worlds, from computer games to internet societies, the environment itself may be fiction, but the accepted behavior in that environment must be in accordance with the laws and rules of the environment.

Munz et. al [3] modeled zombie outbreaks by extending the commonly SIR-type differential equation models for the spreading of diseases such as flu and HIV. The conclusions in that paper are based on mathematical tools for analyzing the stability of nonlinear dynamical systems described by ordinary differential equations. It was shown theoretically that an equilibrium state corresponding to a zombie-free world was unstable. However, humans do conquer zombies in movies. Munz et al. [3] incorporated this effect by adding impulsive human attacks on zombies at some distinct points in time.

In the present work, we take a different approach. First, we phrase the mathematical model directly as a set of difference equations, incorporating all the effects suggested in [3]. However, since we have excluded magic, we argue that dead zombies cannot become functioning zombies again, a fact that has a fundamental impact on whether the doomsday scenario is likely or not. Second, we propose that the parameters in the model change with time, according to the phases of the human-zombie interaction observed in movies. Third, we put effort into estimating the parameters of the model, based on watching a single movie, The Night of The Living Dead. We fit the model to this movie, so that we can reproduce its scenarios, and thereafter we shall use the model to predict how a zombie outbreak will most likely behave in a bigger community with other initial conditions than in the film.

The paper is organized as follows. Section 2 lists the mathematical model and defines the input parameters required by the model. Section 3 is devoted to estimating parameters from one movie and demonstrating how the model predicts a zombie outbreak. We discuss the limitation of the study in Section 4 and make some concluding remarks about zombie movies. A series of appendices have been written with a two-fold goal: we want to document in detail how the model is derived, and we want to explain the derivation to a wide audience. To reach the latter goal, we avoid differential equations and work directly with difference equations. We have also chosen to start with explaining the well-established and simple SIR model, and then show how this model can be extended to include more complicated effects, ending up with the complete model for zombie outbreaks. Computer programs in Python for solving the equations arising in the models are inserted in the text to illustrate all the "nuts and bolts" necessary to bring the mathematics to action in real or virtual problems[2]. Knowledge of, and some interest in, basic high school mathematics should be sufficient to understand most of the mathematical details in the appendices. Hopefully, the exposition can help to show that mathematics can be useful far beyond the reader's imagination.

3 The Model for Human-Zombie Interaction

A detailed derivation of our mathematical model for human-zombie interaction can be found in the appendices. The model and its parameters are summarized below. We have four categories of individuals:

1. S: susceptible humans who can become zombies.
2. I: infected humans, being bitten by zombies.
3. Z: zombies.
4. R: removed individuals, either conquered zombies or dead humans.

[2] The web page `http://www.simula.no/Zombies` contains more programs, with additional graphical features.

Figure 1: The first zombie encountered in the movie The Night of The Living Dead [5].

3.1 The System of Difference Equations

The mathematical model expresses the temporal transfer of individuals between the four categories: S, I, Z, and R. We introduce the number of individuals in each category as the functions of time t: $S(t)$, $I(t)$, $Z(t)$, and $R(t)$. These four functions are computed at discrete points in time: $t_i = i\Delta t$, $i = 1, 2, \ldots, n$. The values at time $t_0 = 0$ must be known. Introducing $S^i = S(t_i)$ and a similar short notation for the other three functions, we can write the equations governing the temporal evolution of S, I, Z, and R as follows:

$$
\begin{aligned}
S^{i+1} &= S^i + \Delta t(\Sigma - \beta S^i Z^i - \delta_S S^i), && (1)\\
I^{i+1} &= I^i + \Delta t(\beta S^i Z^i - \rho I^i - \delta_I I^i), && (2)\\
Z^{i+1} &= Z^i + \Delta t(\rho I^i - (\alpha + \omega(t)) S^i Z^i + \zeta R^i), && (3)\\
& \omega(t) = a \sum_{i=0}^{m} \exp\left(\frac{1}{2}\left(\frac{t - T_i}{\sigma}\right)^2\right), && (4)\\
R^{i+1} &= R^i + \Delta t(\delta_S S^i + \delta_I I^i - \zeta R^i + (\alpha + \omega(t)) S^i Z^i)\,. && (5)
\end{aligned}
$$

The parameters Σ, β, δ_S, δ_I, ρ, ζ, α, a, σ, and $T_0, \ldots, T_m$ must given. We must also know the distribution of individuals initially, i.e., S^0, I^0, Z^0, and R^0.

The interpretations of the parameters are as follows:

- Σ: the number of new humans brought into the zombified area per unit time.

- β: the probability that a theoretically possible human-zombie pair actually meets physically, during a unit time interval, with the result that the human is infected.

- δ_S: the probability that a susceptible human is killed or dies, in a unit time interval.

- δ_I: the probability that an infected human is killed or dies, in a unit time interval.

- ρ: the probability that an infected human is turned into a zombie, during a unit time interval.

- ζ: the probability that a removed individual turns into a zombie, during a unit time interval.

- α: the probability that, during a unit time interval, a theoretically possible human-zombie pair fights and the human kills the zombie.

- a: as α, but the probability relates to killing a zombie in an organized and effective war on zombies.

- $T_0, \ldots, T_m$: points in time with strong attacks (war) on zombies.

- σ: length of attacks in the war on zombies (typically, 4σ measures the length and should be much smaller than the time interval $T_i - T_{i-1}$ between attacks).

Note that probabilities per unit time do not necessarily lie in the interval $[0, 1]$. The real probability, lying between 0 and 1, arises after multiplication by the time interval of interest.

3.2 Computing the Solution to the Equations (1)–(5)

Below is a small computer program, written in the Python language, for calculating the evolution of the S^i, I^i, Z^i, and R^i quantities, for the time levels corresponding to $i = 0, 1, 2, \ldots, n$.

```
Sigma = 0
beta = 0.03125
delta_S = 0
delta_I = 0
rho = 1
zeta = 0
alpha = 0.2*beta

a = 10*beta
sigma = 0.5
attacks = [5, 10, 18]
from math import exp
def omega(t, a, sigma, T):
```

```
    return a*sum(exp(-0.5*(t-T[i])**2/sigma) for i in range(len(T)))

dt = 0.1          # time step measured in hours
D = 0.7           # simulation lasts for D days
n = int(D*24/dt)  # corresponding total no of hours

from numpy import zeros
S = zeros(n+1)
I = zeros(n+1)
Z = zeros(n+1)
R = zeros(n+1)

# initial conditions:
S[0] = 50
I[0] = 0
Z[0] = 3
R[0] = 0

# step equations forward in time:
for i in range(n):
    t = i*dt
    omega_t = omega(t, a, sigma, attacks)
    S[i+1] = S[i] + dt*(Sigma - beta*S[i]*Z[i] - delta*S[i])
    I[i+1] = I[i] + dt*(beta*S[i]*Z[i] - rho*I[i] - delta*I[i])
    Z[i+1] = Z[i] + dt*(rho*I[i] - (alpha + omega_t)*S[i]*Z[i] + \
                        zeta*R[i])
    R[i+1] = R[i] + dt*(delta*S[i] - zeta*R[i] + delta*I[i] + \
                        (alpha + omega_t)*S[i]*Z[i])
```

3.3 A Corresponding System of Differential Equations

Mathematical modeling in epidemiology, which is the scientific discipline laying the foundation for the system of difference equations (1)–(5) modeling zombification, is very much about systems of *differential equations* (ODEs), not systems of difference equations. However, in the limit $\Delta t \to 0$, the difference equations (1)–(5) approach a system of ODEs[3]. Since some readers may be well educated in differential equations, and find them easier to interpret than difference equations, we list the ODE system corresponding to (1)–(5):

$$
\begin{aligned}
S' &= \Sigma - \beta SZ - \delta_S S, && (6)\\
I' &= \beta SZ - \rho I - \delta_I I, && (7)\\
Z' &= \rho I - (\alpha + \omega(t))SZ + \zeta R, && (8)\\
& \omega(t) = a \sum_{i=0}^{m} \exp\left(\frac{1}{2}\left(\frac{t - T_i}{\sigma}\right)^2\right), && (9)\\
R' &= \delta_S S + \delta_I I - \zeta R + (\alpha + \omega(t))SZ\,. && (10)
\end{aligned}
$$

The prime denotes the derivative in time. We shall not, however, deal more with the equations (6)–(10) in this paper.

[3] Just move the first term on the right-hand sides to the left-hand side, divide by Δt, and observe that the left-hand sides are difference approximations to derivatives, which become derivatives as $\Delta t \to 0$.

4 Estimation of Parameters

To be able to estimate realistic parameters for the model we have chosen to divide zombie outbreaks in three different phases:

1. the initial phase,
2. the hysterical phase,
3. the counter attack.

The initial phase is characterized by the fact that humans do not know that zombies are devastating man-eating monsters. In this phase, humans typically try to bring infected and zombies to hospitals for treatment. The consequence is that the disease spreads fast during this phase and that few zombies are killed. The initial phase is usually short. Humans soon realize that helping zombies will likely cause their own death. Needless to say, zombies often have a frightening look and appearance that prevent them from gaining sympathy and rescue attempts.

The second phase is usually characterized by hysteria. Humans, facing a pressed situation, try to hide and only fight back with tools at their disposals. Often humans barricade themselves in a house and try to communicate with others or gather information through telephones, radio, or TV. Zombification does not spread much during this phase.

In the final phase, man has fully realized the threat. Humans gathere weapons and fight back against zombies in an intelligent and strategic manner. However, in this phase zombies often outnumber humans by orders of magnitude, caused by a rapid increase of zombification during the first phase.

Example 4.1. *The initial phase in The Night of the Living Dead.*

The initial phase in The Night of The Living Dead movie seems to last for a few hours, taken as 4 hours in our calculations. During this time, two humans meet one zombie, see Figure 1, and one of the humans get infected or eaten when trying to help the zombie (the zombie appears to be a human stumbling around without any sense). It is, of course, difficult to estimate parameters from such an isolated incidence, but let us try. According to the derivation of (1), as given in Appendix B, the term $\Delta t \beta S Z$ models the increase in the number of infected individuals during a time interval Δt, which is assumed small. When applying this formula for a long time interval of 4 hours, we choose to use the value SZ at the beginning of the time interval, when $S = 2$ and $Z = 1$. It is unclear whether the human gets killed or infected so let us say that there is a 50% chance for survival, resulting in 1/2 infected, which is turned into a zombie. This means that $\beta \cdot 4 \cdot 2 \cdot 1 = 0.5$, implying $\beta = 0.0625$. Furthermore, no zombies are killed in

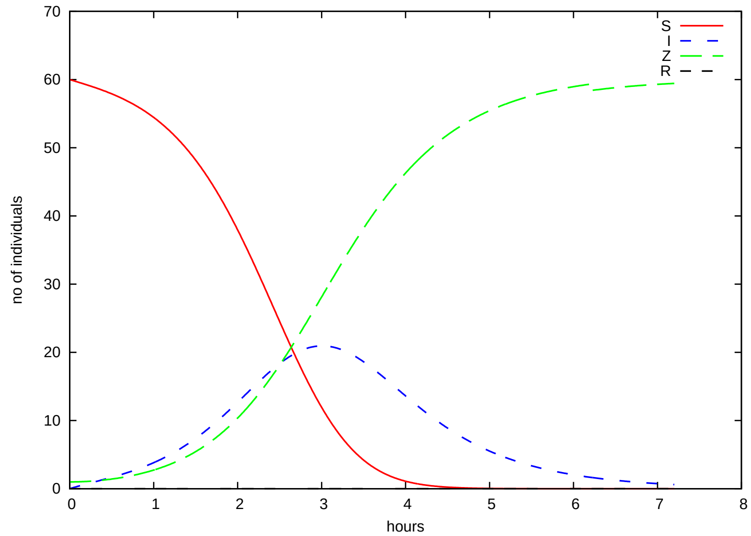

Figure 2: The initial phase of The Night of The Living Dead, with $\beta = 0.0625$. Humans are eradicated before the main part of the movie starts.

this initial phase so $\alpha = a = 0$. We assume that infected humans become zombies within one hour with probability 1, giving $\rho = 1$.

These parameters are believed to be representative for the population in this small area of the outbreak. In order to produce as many zombies as appear later in the movie, we must assume that more humans are present initially, say 60. Figure 2 shows the consequence of starting with $S^0 = 60$, $I^0 = 3$ and using the estimated β, ρ, and α values. The other parameters, δ_S, δ_I, a, and ζ are not considered relevant and hence set to zero. The outbreak is seen to be very fast: the humans are eradicated after four hours, that is, during the initial phase of the movie. This is obviously not consistent with what we watch later. The problem relates to the large value of β. Therefore, we look more into the sensitivity to β before proceeding with the other two phases.

Example 4.2. *Sensitivity to parameters in the initial phase.*

The evolution of humans and zombies in the first phase is very sensitive to the value of β. To investigate this sensitivity, we have run series of simulations where β is varied and where we have measured the number of hours it takes to reduce the human population with 95% compared to the initial value (S^0). Figure 4 shows the sensitivity to β for some choices of initial conditions S^0. Other key parameters for these runs are $\Sigma = \alpha = \delta_S = \delta_I = a = 0$, and $Z^0 = 1$. We have also computed the sensitivity to β for various values of Z^0. The curves are very similar to the ones in Figure 4. Since we have a logarithmic scale on the axis in in Figure 4,

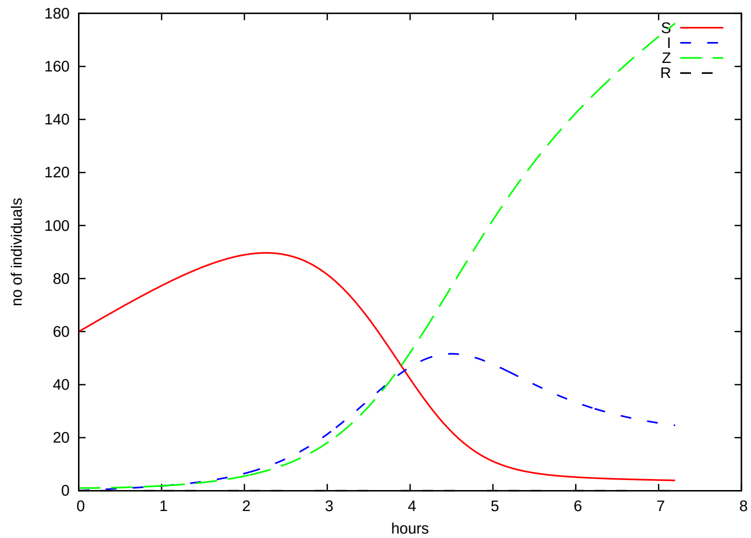

Figure 3: The initial phase of The Night of The Living Dead with $\Sigma = 20$, $\beta = 0.03$, $S^0 = 60$, and $Z^0 = 1$.

the time to eradcate humans behaves as $C_1\beta^{-C_2}$ for positive constants C_1 and C_2. We notice that decreasing the initial amount of humans (S^0), or the initial amount of zombies (Z^0), may help to make the initial phase last longer, but a decrease in S^0 also leaves few humans available for the next phase.

A different strategy to arrive at a more realistic model for the initial phase, is to reduce β to 0.03 and steadily bring in more humans from the outside into the zombified area. Introducing $\Sigma = 20$, so that 20 humans arrive in the zombified area every hour, shifts the $S(t)$ to the right, as depicted in Figure 3. These graphs result in about 10 humans and 100 zombies after five hours, a state that may be compatible with what we see in the next phase of the movie. Another strategy is to reduce β further, e.g., to 0.02 as used later in Figure 10.

It might be of interest to see the sensitivity to β for various Σ values, and Figure 5 provides one example (with S^0 and Z^0 fixed at 60 and 1, respectively).

A conclusion is that the estimated $\beta = 0.0625$ is too large to give meaningful results. We either have to decrease β, say to less than 0.02 according to Figure 4, or we have to feed in new humans, say $\Sigma = 20$, and halve the β value.

Example 4.3. *The hysterical phase in The Night of The Living Dead.*

Most of the movie concerns the hysterical phase, which seems to last about 24 hours. During this time the main characters arrive at an isolated

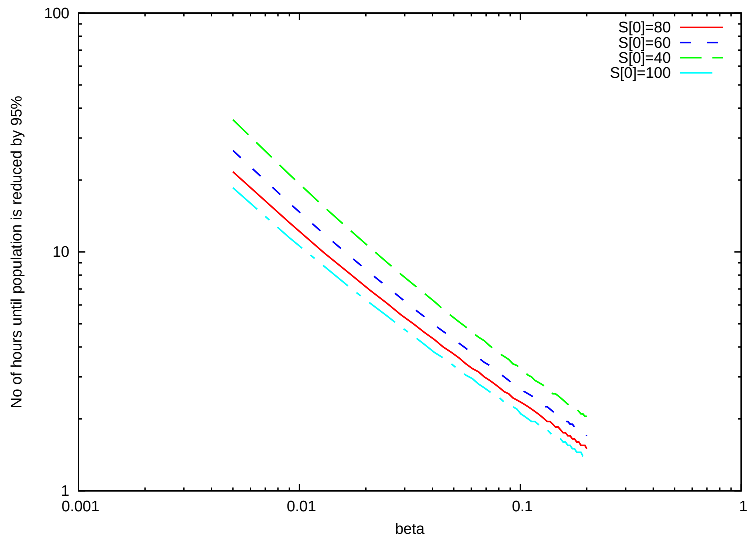

Figure 4: Time to 95% reduction in human population in the initial phase as function of β. Different initial conditions S^0 are varied.

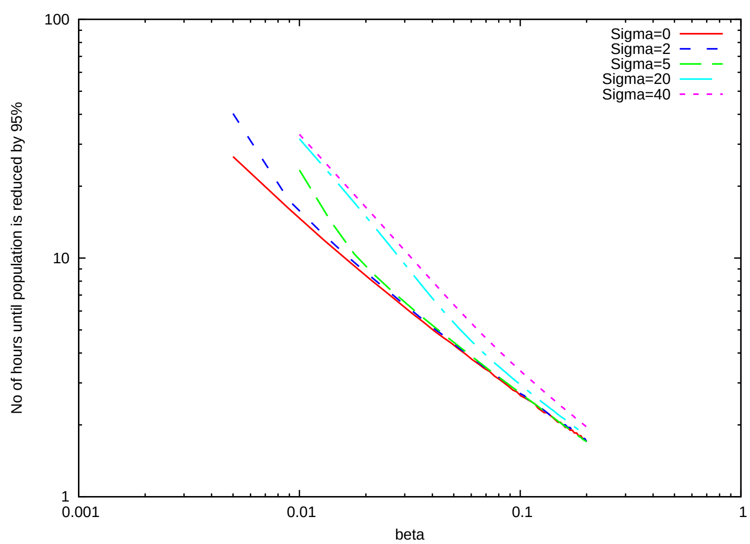

Figure 5: Time 95% reduction in human population in the initial phase as function of β. Different values of Σ (Sigma).

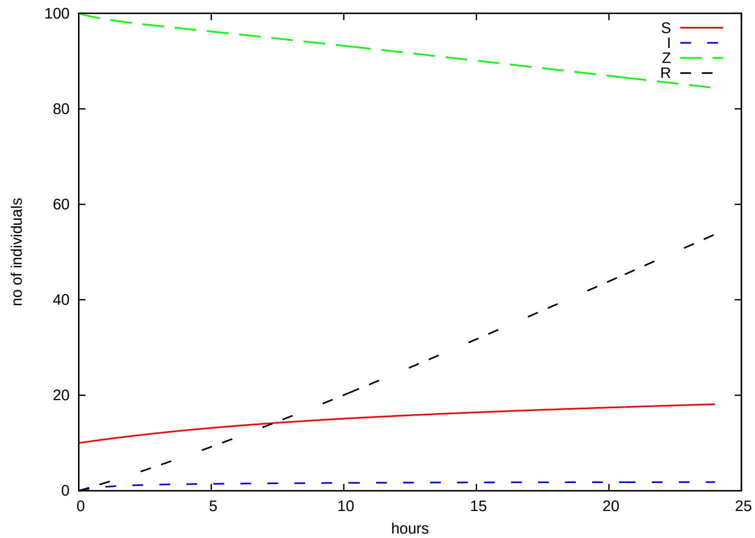

Figure 6: The hysterical phase of The Night of The Living Dead ($\beta = 0.0012$, $\alpha = 0.0016$, $\delta_I = 0.014$, $\Sigma = 2$, with all other parameters set to zero).

house which they barricade with simple means. During this phase, six humans get to the house and all but one get infected or killed. Hence we take five to be infected, two of these are eaten (killed) by zombies, while the other three turn into zombies. At the same time seven zombies are killed by the involved characters. It may appear that there are around 30 zombies that interact with the humans in this phase of the movie. To estimate β and α, we use the terms $\Delta t \beta SZ$ ($= 3+5$) and $\Delta t \alpha SZ$ ($= 7$) from the equations, with $\Delta t = 24$ hours, $S = 6$, and $Z = 30$. Hence, $\beta = 3/(6 \cdot 30 \cdot 24) \approx 0.0012$ and $\alpha = 7/(6 \cdot 30 \cdot 24) = 0.0016$. We also take the two infected humans that are killed into account in the I equation, giving a contribution $\delta_I \Delta t S$ equal to 2 in that equation, i.e., $\delta_I = 2/(6 \cdot 24) = 0.014$. With these parameters, there will be some reduction in the human population, but not much. However, starting with 10 humans and 100 zombies, which is the state after five hours in the initial phase, the number of humans is further reduced, leaving too few for the final phase. We may either assume more humans at the beginning of the hysterical phase, or we may bring in new ones ($\Sigma \neq 0$). Going for the latter strategy and $\Sigma = 2$, Figure 6 shows the evolution of zombies during the course of 24 hours. In this situation, we see that human and zombies may co-exist for a long time.

Example 4.4. *The counter attack in The Night of The Living Dead.* Finally, in the end of the movie, the humans start their counter-attack. In this phase, about 30 zombies are killed by about 30 humans in a matter of hours. Here, the humans act strategically and effectively with weapons to completely de-

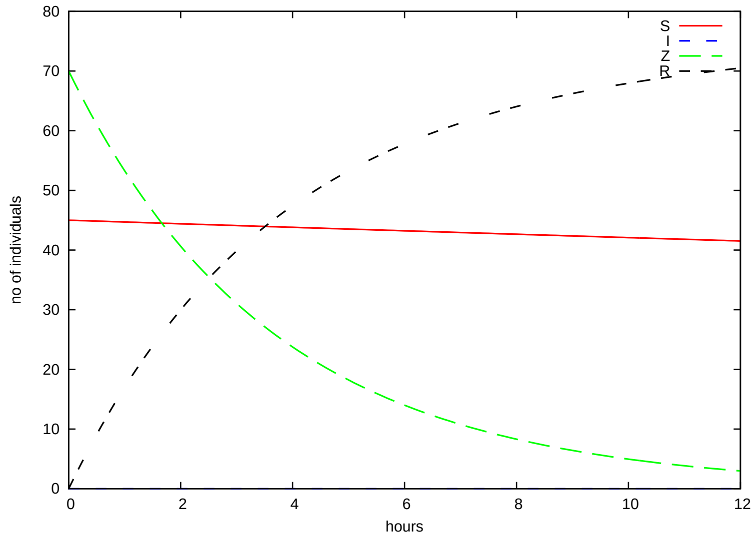

Figure 7: The counter attack in The Night of The Living Dead.

stroy zombies. One sheltered human accidently gets killed because he got mistaken for a zombie, but otherwise human destroy the zombies without risking their own lives.

Assuming a time frame of five hours, we can make some estimates: $\alpha \cdot 5 \cdot 30 \cdot 30 = 30$, giving $\alpha = 0.006$; $\beta = 0$ (no humans get infected); $\delta_S \cdot 5 \cdot 30 = 1$, implying $\delta_S = 0.0067$. Figure 7 shows the evolution of humans and zombies during this final phase. We may also put all the three phases together to show the evolution of zombies and humans through the complete movie. Figure 8 combines the phases and show the evolution over three days. Using the originally estimated $\beta = 0.0625$ and without introducing new humans into the zombified area, we get the evolution as depicted in Figure 9, where doomsday appears before the core of the movie starts. This is obviously unrealistic. A smaller β value, 0.02, gives a much more realistic scenario, see Figure 10, also without any transport of humans into the zombified area.

As we have seen in the previous examples, the relation between α and β determines whether mankind faces extinction or not. The parameters are of course very uncertain, even when we base them on specific movies. Furthermore, the $\alpha - \beta$ relation varies a lot within a single movie. In The Night of The Living Dead, the zombies are mindless, clumsy, and slow, while in Zombieland [1] and Død snø [2], they may even outrun at least an untrained person. However, in all movies it is natural to assume that $\alpha \geq \beta$ in phase 2 and 3 because zombies always demonstrate a complete lack of intelligence. Furthermore, in phase 3, $\alpha \gg \beta$ in all movies, since attacks

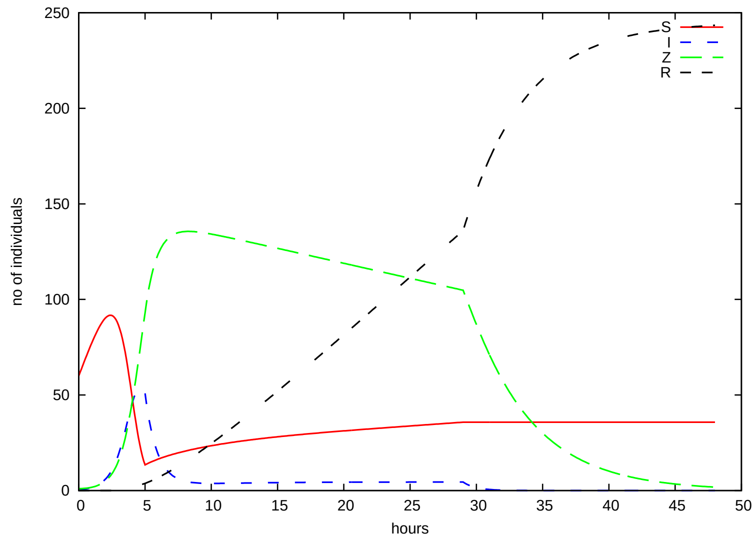

Figure 8: All three phases in The Night of the Living Dead.

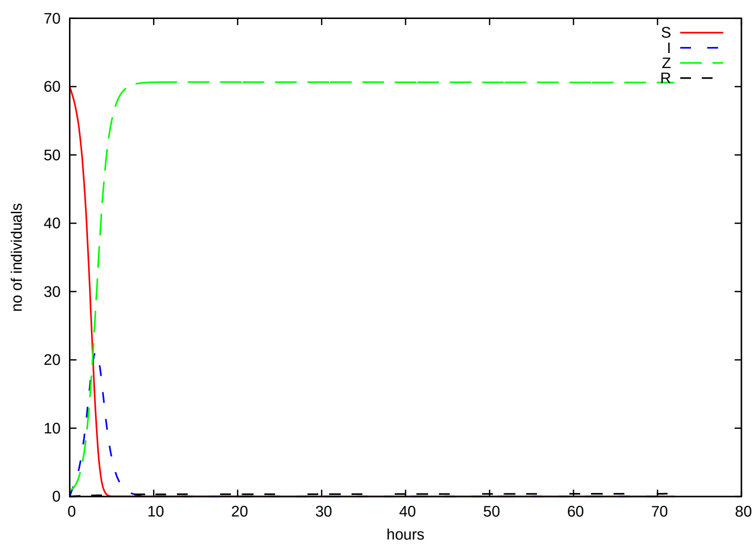

Figure 9: All three phases in The Night of the Living Dead, using $\beta = 0.0625$ for the first phase and $\Sigma = 0$ in all phases.

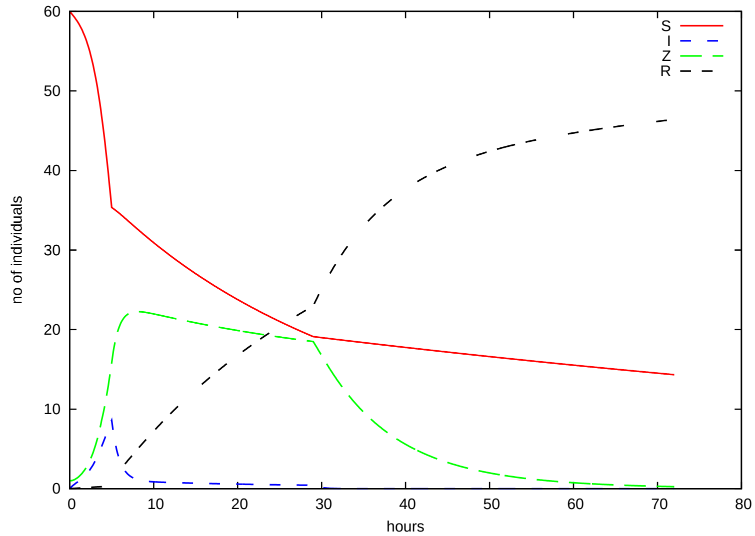

Figure 10: All three phases in The Night of the Living Dead, using $\beta = 0.02$ for the first phase and $\Sigma = 0$ in all phases.

are planned and effective. Hence, the crucial parameters that determine mankind's survival under a zombie outbreak is the duration of the initial phase and the value of the corresponding β. These two parameters are also the most uncertain parameters in typical zombie films since the initial phase is often short, and mostly used to introduce the main characters in the movie. Our sensitivity analysis shows that the time to human extinction has a power-law dependence on β.

Example 4.5. *Allowing dead zombies to re-enter the action.* So far, we have assumed that dead zombies are out of the play, i.e., $\zeta = 0$, which is in accordance with the non-magic zombie character from The Night of The Living Dead. Since Munz et al. [3] allow $\zeta \neq 0$, it is of interest to introduce the magic that dead zombies can turn into live zombies again. Obviously, this effect changes the picture dramatically. Even a small value, $\zeta = 0.05$, leads to a doomsday scenario over eight days, as shown in Figure 11.

Example 4.6. *A theoretical initial counter attack.*

We have seen that zombification is rapid in the first phase. A better strategy than depicted in typical zombie movies is to follow the recommendation of Munz et al. [3] and as soon as possible start with a war on zombies, containing impulsive attacks. Our $\omega(t)$ function in the mathematical model is exactly designed for this purpose. As a demonstration, we start out with 50 humans and 3 zombies, and $\beta = 0.0625$ as estimated from The Night of The Living Dead movie. These values leads to a rapid zombification.

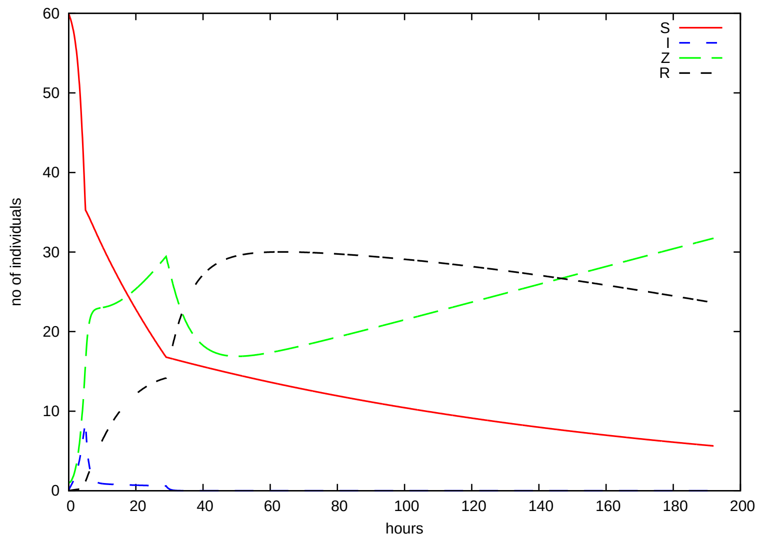

Figure 11: All three phases in The Night of the Living Dead, using $\beta = 0.02$ for the first phase, $\Sigma = 0$ in all phases, but $\zeta = 0.05$ to allow dead zombies to re-enter the zombie category.

We assume there are some small resistances against zombies from the humans: $\alpha = 0.2\beta$. However, the humans implement three strong attacks, $a = 50\alpha$, at 5, 10, and 18 hours after zombification starts. The attacks last for about 2 hours ($\sigma = 0.5$). It appears from Figure 12 that such attacks are sufficient to save mankind in this particular case.

5 Conclusion

Zombies are un-intelligent, flesh-hungry, and clumsy beings. Consequently, once people realize the threat they will need to protect themselves fiercely. In some more recent movies like Død Snø [2] and Zombieland [1], the zombies are fast and may even outrun an untrained individual, but they are always un-intelligent. Because of this lack of intelligence, the main characters in the movies always kill huge amounts of zombies. Hence, if the main characters are representative humans, then an average human could kill more zombies than an average zombie could infect. Humans can therefore quickly eradicate zombies as soon as they realize the threat. The key point is that the time it takes to realize this fact must be smaller than the time it takes to infect nearly all humans in the initial phase.

Our estimation of the speed of zombification in the initial phase gave a too high value, as no disease spreads that fast. The estimation procedure

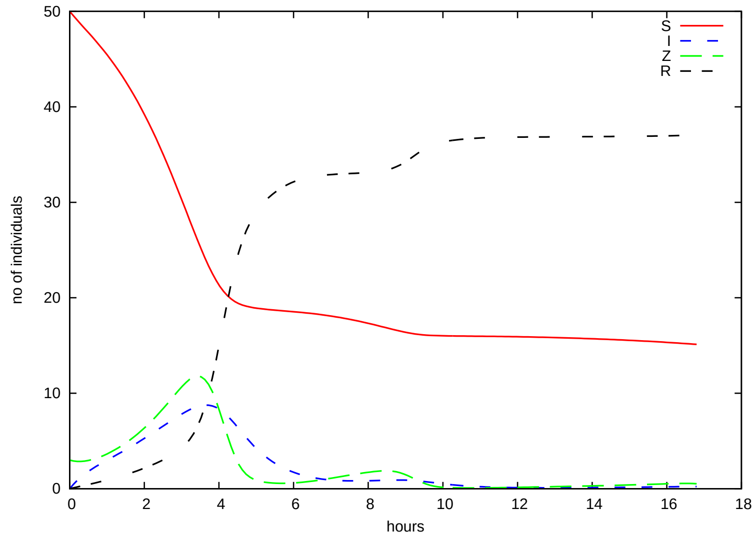

Figure 12: Simulation of a zombie outbreak with β as in the initial phase of The Night of The Living Dead, but with an effective war on zombies with strong attacks after 5, 10, and 18 hours ($\delta_S = \delta_I = \zeta = \Sigma = 0$, $\rho = 1$, $\alpha = 0.2\beta$, $a = 50\alpha$, $\sigma = 0.5$).

used terms in the difference equations, with time scales much larger than assumed when introducing these terms in the modeling. The data was also based on watching the movie, i.e., what we see is what there is. Claiming that the initial phase of The Night of The Living Dead is unrealistic, and that zombification spreads at a significantly lower speed, we could obtain a model that fits the observed evolution of humans and zombies in that movie.

Zombies will never outnumber people if humans realize the danger, and realizing that zombies are monsters is almost inevitable. Zombies will therefore never be a threat to mankind, unless magic ($\zeta > 0$) is involved.

A Basic Mathematical Modeling Concepts

Our aim with this section is to show in detail how one can apply mathematics to investigate spreading of diseases. The next section will then demonstrate how we can transfer this knowledge to a world where humans fight zombies.

A.1 Spreading of a Flu

Imagine a boarding school out in the country side. This school is a small and closed society. Suddenly, one or more of the pupils get a flu. We expect that the flu may spread quite effectively, but how many of the pupils and the school's staff will be affected? Some quite simple mathematics can help us to achieve insight into the dynamics of how the disease spreads.

Let $S(t)$ count how many individuals, at time t, that have the possibility to get infected. Here, t may count hours or days, for instance. These individuals make up a category called susceptibles, labeled as S. Another category, I, consists of the individuals that are infected. Let $I(t)$ count how many there are in category I at time t. An individual having recovered from the disease is assumed to gain immunity. There is also a small possibility that an infected will die. In either case, the individual is moved from the I category to a category we call the removed category, labeled with R. We let $R(t)$ count the number of individuals in the R category at time t. Those who enter the R category, cannot leave this category.

To summarize, the spreading of this disease is essentially the dynamics of moving individuals from the S to the I and then to the R category. We can use mathematics to more precisely describe the exchange between the categories. The fundamental idea is to describe the changes that take place during a small time interval, denoted by Δt.

The problem setting is that we assume we know S at a particular time t_i and want to compute S at some future time $t_{i+1} > t_i$. Here, i is an integer counter, $i = 0, 1, 2, \ldots, n$, used to label some discrete points in time: $t_0 < t_1 < t_2 < \cdots < t_n$. Our aim is to compute S at these discrete points only, not for any value of t. We still consider S as a continuous function of t, but our computations will be performed by a computer and only produce S values at the time points $t_0, t_1, \ldots, t_n$. Aiming for S at a finite number of discrete points instead of finding a general mathematical formula for the function $S(t)$ is very much simpler, more general, and more powerful.

We now let the discrete points in time be uniformly distributed such that the time between t_i and t_{i+1} is a constant Δt, for any i. This means that $t_i = i\Delta t$ if $t_0 = 0$ (which is a common choice). Given $S(t_i)$, we aim at predicting the future value $S(t_{i+1})$ under the assumption that Δt is small. That is, we want to look slightly into the future. What "small" means is vague at this stage, but for spreading a disease one may think of an hour or five minutes, not three days because during three days one may have moved from the S to R category. The idea is that Δt should be small enough so that changes in S are small.

In the time interval Δt we know that some pepole will be infected, so S will decrease. We shall soon argue by mathematics that there will be $\beta \Delta t S I$ new infected individuals in this time interval, where β is a parameter reflecting how easy people get infected during a time interval of unit length.

If the loss in S $\beta\Delta tSI$, we have that

$$S(t_{i+1}) - S(t_i) = -\beta\Delta tS(t_i)I(t_i),$$

which gives a formula for the future value:

$$S(t_{i+1}) = S(t_i) - \beta\Delta tS(t_i)I(t_i)\,. \tag{11}$$

We have simply evaluated the product SI at time t_i such that we can use known values $S(t_i)$ and $I(t_i)$. Evaluating this formula at t_{i+1}, or any time point in $[t_i, t_{i+1}]$ is also possible, but it will not matter much if Δt is small[4].

Let us step aside and explain the formula $\beta\Delta tSI$. We have S susceptibles and I infected people. These can make up SI pairs. Now, suppose that during a time interval T we measure that m actual pairwise meetings do occur among n theoretically possible pairings of people from the S and I categories. The probability that people meet in pairs during a time T is (by the empricial frequency definition of probability) equal to m/n, i.e., the number of successes divided by the number of possible outcomes. From such statistics we normally derive quantities expressed per unit time, i.e., here we want the probability per unit time, μ, which is found from dividing by T: $\mu = m/(nT)$. Given the probability μ, the expected number of meetings per time interval of SI possible pairs of people is (from basic statistics) μSI. During a time interval Δt, there will be $\mu SI\Delta t$ expected number of meetings between susceptibles and infected people such that the virus may spread. Only a fraction of the $\mu\Delta tSI$ meetings are effective in the sense that the susceptible actually becomes infected. Counting that m people get infected in n such pairwise meetings (say 5 are infected from 1000 meetings), we can estimate the probability of being infected as $p = m/n$. The expected number of individuals in the S category that in a time interval Δt catch the virus and get infected is then $p\mu\Delta tSI$. Introducing a new constant $\beta = p\mu$ to save some writing, we arrive at the formula $\beta\Delta tSI$.

Estimating the value of β is important before we can use (11) to predict the future. One possibility is to estimate p and μ from their meanings in the derivation above. Alternatively, we can observe an “experiment” where there are S_0 susceptibles and I_0 infected at some point in time. During a time interval T we count that N susceptibles have become infected. Using (11) as a rough approximation of how S has developed during time T (and now T is not necessarily small, but we use (11) anyway), we get

$$N = \beta TS_0I_0 \quad \Rightarrow \quad \beta = \frac{N}{TS_0I_0}\,. \tag{12}$$

Using (11) to compute $S(t_{i+1})$ is straightforward if we know β, $S(t_i)$, and $I(t_i)$. However, using (11) again to compute $S(t_{i+2})$ from

$$S(t_{i+2}) = S(t_{i+1}) - \beta\Delta tS(t_{i+1})I(t_{i+1})$$

[4]Viewing (11) as a differential equation in the limit $\Delta t \to 0$, the evaluation point for SI is determined by the numerical solution method applied to solve this differential equation.

is difficult since we do not know $I(t_{i+1})$ after having computed (11) only. We therefore need an additional equation to update I as well. Such an equation is easy to establish by noting that the loss in the S category is a corresponding gain in the I category. That is,

$$I(t_{i+1}) = I(t_i) + \beta\Delta t S(t_i)I(t_i)\,. \tag{13}$$

However, there is also a loss in the I category because people recover from the disease. Suppose that we can measure that m out of n individuals recover in a time period T (say 10 of 40 sick people recover during a day: $m = 10$, $n = 40$, $T = 24$ h). Now, $\gamma = m/(nT)$ is the probability that one individual recover in a unit time interval. Then (on average) $\gamma\Delta t I$ infected will recover in a time interval Δt. This quantity represents a loss in the I category and a gain in the R category. We can therefore write

$$I(t_{i+1}) = I(t_i) + \beta\Delta t S(t_i)I(t_i) - \gamma\Delta t I(t_i), \tag{14}$$

and

$$R(t_{i+1}) = R(t_i) + \gamma\Delta t I(t_i)\,. \tag{15}$$

Since there is no loss in the R category (people are either recovered and immune, or dead), we are done with the modeling of this category. In fact, we do not strictly need the equation (15) for R, but in later modeling examples we do.

To summarize, the three quations (11), (14), and (15) discribe the complete[5] dynamics of the spreading of the flu. The computational procedure goes as follows:

- Specify values for Δt, β, γ, $S(t_0)$, $I(t_0)$, and $R(t_0)$.
- For $i = 0, 1, 2, \ldots, n$:
  - Use (11) to compute $S(t_{i+1})$.
  - Use (14) to compute $I(t_{i+1})$.
  - Use (15) to compute $R(t_{i+1})$.

The nice thing is that the formulas (11), (14), and (15) are recursive, i.e., they can be used repeatedly. If the interest is in some future values of S and I, say $S(t_n)$ and $I(t_n)$, one could think of taking one single large jump and set $\Delta t = t_n - t_0$. However, the equations were derived to express small changes in small time intervals, so the equations cannot be used for large jumps into the future. When we create mathematical models of real or

[5]The model we have derived is only an approximation to reality, and to use the model, we need to have good estimates of β and γ. However, we may say that the model is complete in a mathematical sense: we have established enough formulas to enable computation of S, I, and R as far into the future as we wish.

virtual worlds, we are usually only able to find mathematical expressions for small changes in small time intervals. Seeing far into the future can be done by adding a large number of small changes, and that is what we do in the computational recipe above. The computational procedure is, of course, very tedious since we need to repeatedly calculate the same formulas over and over again, but all this work can be done by a computer.

For further work, we introduce some notation that can save writing. Let S^i, I^i, and R^i be a short notation for $S(t_i)$, $I(t_i)$, and $I(t_i)$, respectively. We can then write the equations (11), (14), and (15) more compactly, and better suited for a computer, as

$$S^{i+1} = S^i - \Delta t \beta S^i I^i, \tag{16}$$
$$I^{i+1} = I^i + \Delta t(\beta S^i I^i - \gamma I^i), \tag{17}$$
$$R^{i+1} = R^i + \Delta t \gamma I^i\,. \tag{18}$$

The computation of (16)–(18) can be readily made in a computer program. Below we present a program written in the Python language:

```
# time unit: 1 h
beta = 10./(40*8*24)
gamma = 3./(15*24)
dt = 0.1          # 6 min
D = 30            # simulate for D days
n = int(D*24/dt)  # corresponding no of hours

from numpy import zeros
S = zeros(n+1)
I = zeros(n+1)
R = zeros(n+1)

# initial condition:
S[0] = 50
I[0] = 1
R[0] = 0

# step equations forward in time:
for i in range(n):
    S[i+1] = S[i] - dt*beta*S[i]*I[i]
    I[i+1] = I[i] + dt*beta*S[i]*I[i] - dt*gamma*I[i]
    R[i+1] = R[i] + dt*gamma*I[i]
```

This program was written to investigate the spreading of a flu at the mentioned boarding school, and the reasoning for the specific choices β and γ goes as follows. At some other school where the disease has already spread, it was observed that in the beginning of a day there were 40 susceptibles and 8 infected, while the numbers were 30 and 18, respectively, 24 hours later. Using 1 h as time unit, we then have from (12) that $\beta = 10/(40 \cdot 8 \cdot 24)$. Among 15 infected, it was observed that 3 recoverd during a day, giving $\gamma = 3/(15 \cdot 24)$. Applying these parameters to a new case where there is one infected initially and 50 susceptibles, gives the graphs in Figure 13. These graphs are just straight lines between the values at times

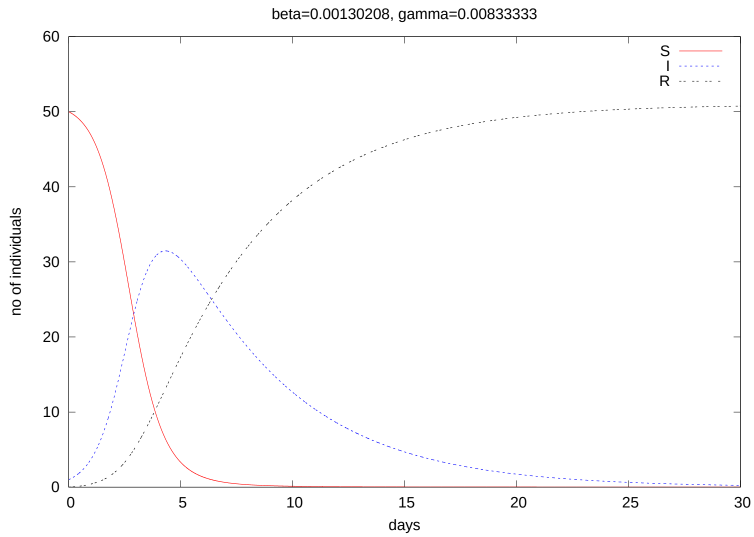

Figure 13: Natural evolution of a flu at a boarding school.

$t_i = i\Delta t$ as computed by the program. We observe that S reduces as I and R grows. After about 30 days everyone has become ill and recovered again.

We can experiment with β and γ to see whether we get an outbreak of the disease or not. Imagine that a "wash your hands" campaign was successful and that the other school in this case experienced only 2 new susceptibles during 36 hours. This gives a reduced β. Assuming that the "wash your hands" campaign is equally effective in the society where we apply the model, we get the graphs as in Figure 14. This time the disease spreads very slowly.

Looking at the equation for I, it is clear that we must have $\beta S^i I^i - \gamma I^i > 0$ for I to increase. When we start the simulation it means that

$$\beta S(0)I(0) - \gamma I(0) > 0,$$

or simpler

$$\frac{\beta S(0)}{\gamma} > 1 \tag{19}$$

to increase the number of infected people and accelerate the spreading of the disease.

A.2 Defeating a More Serious Disease

Flu is seldom a really serious problem at a boarding school. What is more serious for pupils in the age group 16–19 is sexually transmitted diseases.

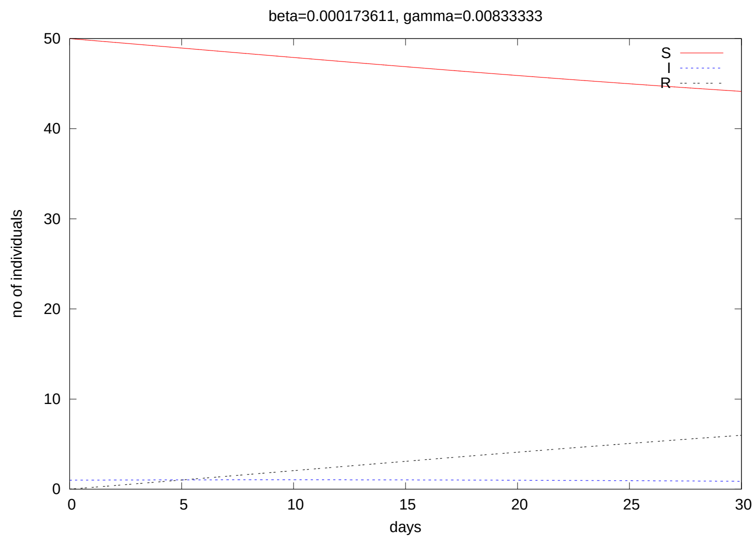

Figure 14: Small outbreak of a flu at a boarding school (β is much smaller than in Figure 13).

Because diseases like ghonorrea is a constant threat at the school, it becomes important to understand the dynamics of how the disease spreads, and what the effect of different actions will be. Insight can be gained from a small extension of the described SIR model.

Individuals now recover after being treated by antibiotics. It is natural to introduce the S, I, and R categories again. However, individuals in the R category are not immune to the infection. Some may be sufficiently scared so that they protect themselves appropriately, while others rely on luck to avoid being infected again. There is hence a leakage from the R category to the S category. As usual we collect some statistics: It appears that among n individuals who have recovered from the disease, m individuals have in a time period T fallen back to old habits of pairing up with infected people in a way that exposes them efficiently to the infection. This gives automatically a contribution to the S category. We can then define the probability of one recovered individual going from the R to the S category per unit time as $\delta = m/(nT)$. In a time interval Δt, $\delta \Delta t R$ individuals will leak from the R to the S category.

The modified system of equations, incorporating the transfer of individuals from the R to the S category, now looks like

$$S^{i+1} = S^i + \Delta t(\delta R^i - \beta S^i I^i), \tag{20}$$
$$I^{i+1} = I^i + \Delta t(\beta S^i I^i - \gamma I^i), \tag{21}$$
$$R^{i+1} = R^i + \Delta t(\gamma I^i - \delta R^i)\,. \tag{22}$$

Say that from previous experience it is recorded that out of 50 susceptibles

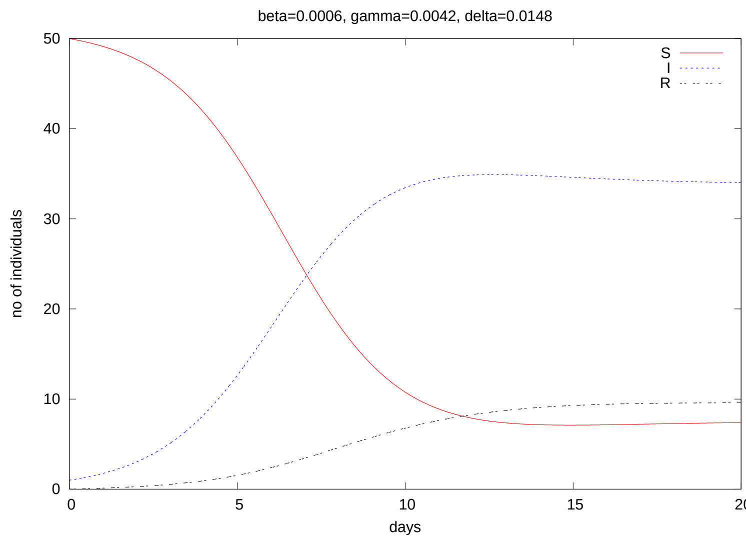

Figure 15: Explosion of a sexually transmitted disease with a significant leakage from the R to the S (and then to the I) category.

and 4 infected, 8 new individuals were infected during a 3-day period. The number of infected can be roughly estimated by (12), giving $\beta = 8/(3 \cdot 24 \cdot 50 \cdot 4)$, when the time unit is hours. Among 40 infected, 4 were free of symptoms during one day. This gives $\gamma = 4/(1 \cdot 24 \cdot 40)$. The bad thing is that out of 24 recovered, 17 ignored to protect themselves properly, over a 2-day period, and thus enter the S category again. From this observation, a rough estimate of δ, based on the same reasoning as for γ, becomes $17/(2 \cdot 24 \cdot 24)$. Suppose then that the number 17 can be reduced to 1 by active information. The corresponding two δ values lead to two different scenarios, as depicted in Figures 15 and 16: the large value implies an explosion of the disease and a constant, high number of infectives in the long run, while the low value cures the disease. An even more attractive scenario, where the sexual activity is also reduced, giving a direct reduction of β, appears in Figure 17.

B Modeling Zombie Infection

We start this section with a quick overview of how zombies behave and quote T. V. Wilson's article "How Zombies Work" [6]: *Many people credit George A. Romero with setting the standard for modern zombies. In the classic movie "Night of the Living Dead," Romero portrayed zombies as slow-moving, flesh-eating corpses, reanimated by radiation from a satellite returning from Venus. The radiation affected the recent, unburied dead,*

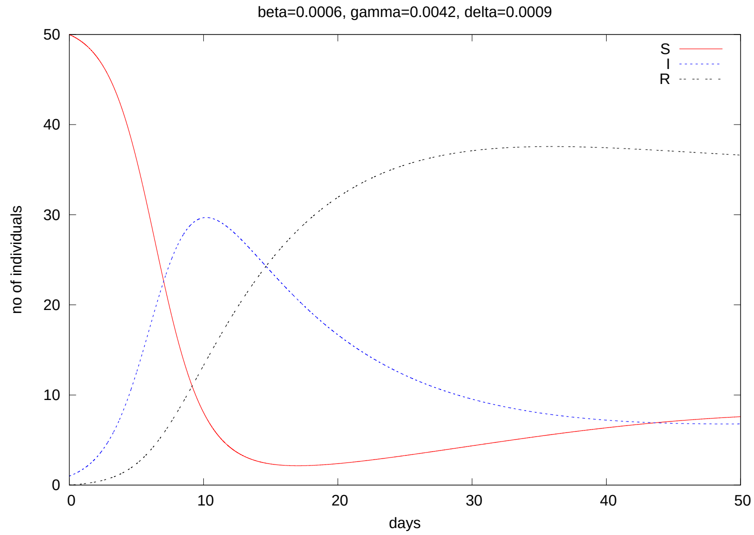

Figure 16: Stabilization of the disease through a small leakage from the R to the S category.

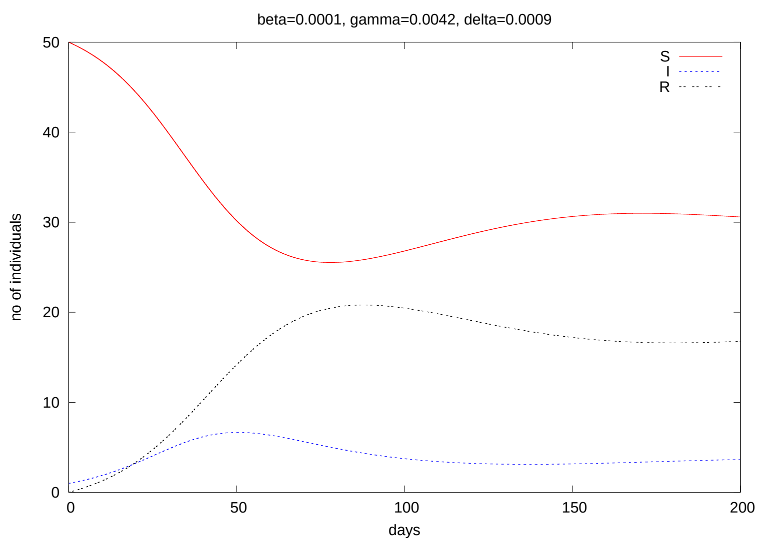

Figure 17: Evolution of a sexually transmitted disease with small infection probability and small leakage from the R to the S category. Note the long-term stabilization of the number of infected.

and the resulting zombies were invulnerable until someone destroyed their brains or separated their heads from their bodies. In "Night of the Living Dead," zombies were neither intelligent nor self-aware. They had a very limited use of tools, mostly confined to using blunt objects as cudgels. ... Many movies and video games have used Romero's concept of zombies. For the most part, zombies are: newly dead corpses reanimated by radiation, chemicals, viruses, sorcery or acts of God; human, although some depictions include zombie animals; very strong, but not very fast or agile; impervious to pain and able to function after sustaining extreme physical damage; invulnerable to injury, except for decapitation or destruction of the brain; relentlessly driven to kill and eat; afraid of fire and bright lights. Some modern movies introduce intelligent and fast-moving zombies, but these are often claimed to be contradictory to the mythology established by George A. Romero [4].

Only a small extension of the previous SIR model is necessary to model the effect of human-zombie interaction mathematically. The basic starting point is that zombification acts like a disease in the SIR model. That is, human susceptibles are getting infected by zombies. A fraction of the infected are then turned into zombies. On the other hand, humans can conquer zombies.

We introduce four categories: susceptibles (S), infected (I), zombies (Z), and removed (R). The corresponding functions counting how many individuals we have in each category are named $S(t)$, $I(t)$, $Z(t)$, and $R(t)$, respectively.

Now we shall precisely set up all the dynamic features of the human-zombie populations we aim to model.

- Changes in the S category are due to three effects.

  1. Susceptibles are infected by zombies, modeled by a term $-\Delta t\beta SZ$, similar to the S-I interaction in the SIR model.
  2. Susceptibles enter the removed category because they die naturally or get killed in the violent actions. If the probability that one susceptible dies during a unit time interval is δ_S, the total expected number of deaths in a time interval Δt becomes $\Delta t\delta_S S$.
  3. We also allow new humans to enter the area with zombies, as this effect may be necessary to successfully run a war on zombies (the other two effects alone will solely reduce the number susceptibles). The number of new individuals in the S category arriving per time unit is denoted by Σ, giving an increase in $S(t)$ by $\Delta t\Sigma$ during a time Δt.

  We could also add newborns to the S category, but we simply skip this effect since it will only be significant over time scales of a couple of decades. Our characteristic time in this study will be days rather than decades.

The balance equation of susceptibles, incorporating the three mentioned effects, becomes

$$S^{i+1} = S^i + \Delta t(\Sigma - \beta S^i Z^i - \delta_S S^i)\,. \tag{23}$$

- The infected category gets a contribution $\Delta t\beta SZ$ from the S category, but loses individuals to the Z and R cateogry. That is, some infected are turned into zombies, while others die. Movies reveal that infected may commit suicide or that others (suceptibles) may kill them. Let δ_I be the probability of being killed in a unit time interval. During time Δt, a total of $\delta_I \Delta t I$ will die and hence be transferred to the removed category. The probability that a single infected is turned into a zombie during a unit time interval is denoted by ρ, so that a total of $\Delta t\rho I$ individuals are lost from the I to the Z category in time Δt. The accounting in the I category becomes

$$I^{i+1} = I^i + \Delta t(\beta S^i Z^i - \rho I^i - \delta_I I^i)\,. \tag{24}$$

- The zombie category gains $-\Delta t\rho I$ individuals from the I category. Following Munz et al. [3], we also allow a fraction ζ per time unit of the removed category to turn into zombies. During time Δt a total number of $\Delta t\zeta R$ are moved from the R to the Z category. However, we question the relevance of this term, because magic is needed to turn dead people or zombies into alive zombies again. A quote from [6] supports this view: "In some portrayals, zombism is contagious, and people bitten by zombies become zombies themselves. In others, people die from the bite and are reanimated by the same force that created the other zombies." The latter case is modeled by β and ρ – it does not matter if the human or infected dies and turns into a zombie after a few minutes or if the death does not occur. The important effect with a $\Delta t\zeta R$ term is that conquered zombies can be zombies, i.e., there is feedback in the system, and obviously zombies overtake us all.

  A fundamental feature in zombie movies is that humans can conquer zombies. We introduce two flavors of this feature. First, zombies can be killed in a "man-to-man" human-zombie fight. This interaction resembles the nature of zombification (or the susceptible-infective interaction in the SIR model) and can be modeled by a loss $-\alpha SZ$ for some parameter α with an interpretation similar to that of β. Second, a war on zombies can be implemented with large-scale effective attacks. A possible model is to increase α by some amount $\omega(t)$, where $\omega(t)$ varies in time to model strong attacks at some distinct points of time $T_1 < T_2 < \cdots < T_m$. Around these t values we want ω to have a large value, while in between the attacks ω is small. One possible mathematical function with this behavior is a sum of bell functions:

$$\omega(t) = a \sum_{i=0}^{m} \exp\left(\frac{1}{2}\left(\frac{t - T_i}{\sigma}\right)^2\right), \tag{25}$$

where a measures the strength of the attacks (the maxium value of $\omega(t)$) and σ measures the length of the attacks, which should be much less than the time between the points of attack: typically, 4σ measures the length of an attack, and we must have $4\sigma \ll T_i - T_{i-1}$ to make the length of an attack much smaller than the time between two attacks. We should choose a significantly larger than α to make the attacks in the war on zombies much stronger than the "man-to-man" killing of zombies. We remark that (25) is our continuous way of modeling the discrete impulsive attacks that play a fundamental role in defeating zombies in the work by Munz et al. [3].

Summarizing the loss and gain in the zombie category leads to the following equation for Z:

$$Z^{i+1} = Z^i + \Delta t(\rho I^i - (\alpha + \omega(t))S^i Z^i + \zeta R^i)\,. \tag{26}$$

- The accounting in the R category consists of a gain δS of natural deaths from the S category, a corresponding loss ζR to the Z category (though claimed unrealistic), a gain δI from the I category, and a gain $(\alpha + \omega)SZ$ from defeated zombies:

$$R^{i+1} = R^i + \Delta t(\delta S^i - \zeta R^i + \delta I^i + (\alpha + \omega(t))S^i Z^i) \tag{27}$$

The classical SIR model for spreading of a disease has now been extended to model human-zombie interaction in terms of four key quantities $S(t)$, $I(t)$, $Z(t)$, and $R(t)$, which can be computed from the equations (23)–(27). The parameters Σ, β, δ, ρ, ζ, α, a, σ, and $T_0, \ldots, T_m$ must given. We must also know the distribution of individuals initially, i.e., $S(0)$, $I(0)$, $Z(0)$, and $R(0)$ must be given.

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