Epidemic models

Torbjørn Seland

Sep 15, 2014

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1 Introduction to epidemic models

Throughout the history large epidemic diseases has spread around the world. Often over large geographical areas. These diseases has done great harm on the population, and millions of people has died. Black Death and Cholera are epidemics that has moved over large distances, both into Europe. When Black Death came to Europe in 1347, it killed about a third of the population, which at that time was about 85 million. These diseases often gave physical symptoms, which has been important knowledge through history for preventing new outburst and to cure already infected humans. These diseases has various outbreaks, but often related to connection between humans and animals. Malaria is an example of a disease that transmits from mosquito to human. But there have been various explanations of the spread and cause of epidemics. AIDS(autoimmune deficiency syndrome) has been ascribed by many as a punishment sent by God.

The first major epidemic in the U.S.A was the Yellow Fever, discovered in 1793 in Philadelphia. About 5000 died out of a population of around 50 000. About 20 000 fled the city and the situation was quite chaotic. This had a major impact on the subsequent life and politics of the country. The power of a disease can do larger damage, with respect to death, than a war.

After the world war II, public health strategy has focused on elimination and control of organisms which cause disease. United Nations sat in 1978 a goal to eradication all disease by 2000. This was before AIDS was recognized, but a large job has been done and smallpox is an example on a disease that was last seen in Somalia in 1977.

Another important aspect in the current spread of diseases, is the displacement of human populations. About a million people cross international borders daily. The growth of human population, especially in underdeveloped countries, is also a factor that affects spread, specially microbes. These conditions played a key role in the spread of HIV(human immunodeficiency) in the 1980's. World Health Organization has estimated that around 30 million are at current time infected with the HIV virus.

Knowledge through history is important for the control of different epidemics, but also important in detecting new diseases. The plague of Athens had been studied in great detail by Thucydides in 430-438 BC. Similar had been done with the 'sweating sickness' in the late 15th and first of the 16th centuries in England. The symptoms of 'sweating sickness' was detected in 1993 in the Southwest U.S.A. Here the disease was called hanta virus. There is likely that this is the same disease, but that the 'sweating sickness' has been dormant for couple of hundred years.

There are four main microorganisms that can be disease-causing. These are; viruses, bacteria, parasites and fungi. This chapter will focus on the population dynamics. Spatiotemporal models will also be studied. These mathematical models has been important for the combating of the diseases. Both in describing the movement pattern but also in giving reasoned estimated numbers for the level of vaccination.

2 Simple Epidemic models

These models that will be shown in this chapter, will have a constant population. This may differ from the real world. Another assumption that has to be done, is how the population interact. This has to be similar for the whole area that is modeled. The population can be divided into three different groups. The first group is Susceptible,S, which consist of humans that are healthy and at risk of becoming infected. Infective,I,this group has the disease or only carriers of the disease. This group can infect the Susceptible. The last group is the removed, R. This group consist of either dead or recovered recovered humans, often people that already have had the disease. The natural order for a human is,

$$S \rightarrow I \rightarrow R. \tag{1}$$

This model is called SIR model, but the number of classes can be changed. SI only consist of the two first groups and $SEIR$ has added an extra class E, where the disease is latent. This can be used to model the incubation time.

A couple of important assumptions that has to be done for the model, are the transmission of the infection and incubation period. These are reflected in the terms of the equations. The amount of people in each class can be seen as a function of time, expressed as $S(t), I(t)$ and $R(t)$. The growth of I caused by Susceptible, can be viewed as a rate proportional to the number of Infective and Susceptible multiplied by a constant,rSI, where $r > 0$. This constant controls the efficiency of the transmission from S to I. This will appear as a reduction in the function $S(t)$. The rate of removal from Infective to Removed can be viewed as the number of Infective times a constant,aI, where $a > 0$ controls the time spent in the Infection state. The incubation time is here negligible. This could have been taken into account by the $SEIR$ model. The dynamic model will be,

$$\begin{aligned}\frac{dS}{dt} =& - rSI \\ \frac{dI}{dt} =& rSI - aI \\ \frac{dR}{dt} =& aI\end{aligned}$$

This model is called the Kermack-McKendrick(1927) model. Is is considered that the classes are uniformly mixed and that there is equal probability for contact for all individuals. This assumptions will not be correct for all diseases, especially sexually transmitted diseases.The total number for the population will stay constant. This can be seen on the total change in the classes,

$$\frac{dS}{dt} + \frac{dI}{dt} + \frac{dR}{dt} = 0 \tag{2}$$

Therefore the total size of the population, N, will be constant.

$$S(t) + I(t) + R(t) = N \tag{3}$$

2.1 Threshold phenomenon

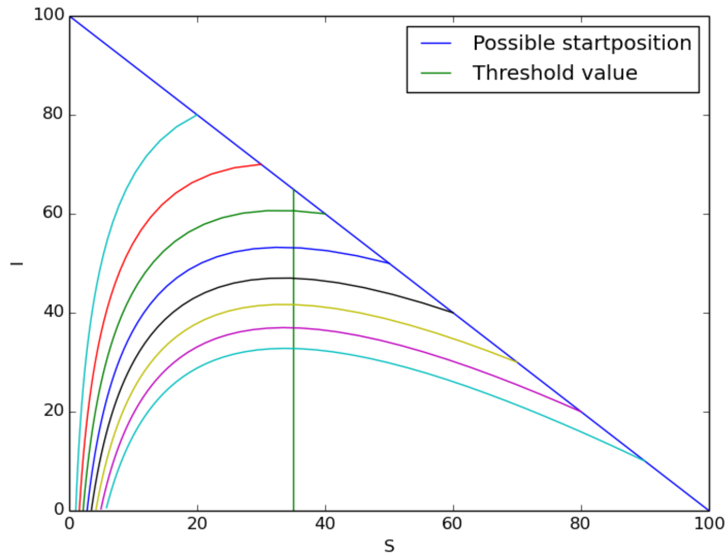
An important question to study when looking the at an epidemic model, is the threshold phenomenon. To cause an epidemic situation, the model need to fulfill $I(t) > I_0$ for some $t > 0$, where I_0 describe the initial condition for Infective. The initial conditions can be given as,

$$S(0) = S_0 > 0, \qquad I(0) = I_0 > 0, \qquad R(0) = 0. \tag{4}$$

The initial conditions for Susceptible and Infective combined with r and a will control the epidemic situation. These will affect the spread if the infection. From equation (**??**) the function for Infection at initial time can be written,

$$\left[\frac{dI}{dt}\right]_{t=0} = I_0(rS_0 - a) \tag{5}$$

The expression inside the brackets controls the change in I. The function will increase if $S_0 > \frac{a}{r}$, this will therefore be the threshold value for the function. This can be shown with a Phase trajectories of the Susceptible and the Infective.



The simulation shows that when the value of S is larger than $\frac{a}{r}$ the value of I is increasing. But when $S = \frac{a}{r}$, the value of I reaches a maximum value and then start to decrease. The reproduction rate can be described

$$R_0 = \frac{rS_0}{a} \tag{6}$$

If $R_0 > 1$ it will cause an epidemic reaction. This parameter is crucial in the understanding with the work with the disease. To be able to keep $R_0 < 1$ will prevent a dispersion. Global vaccination programs are an effective way to get control. Smallpox is an example on a disease that has been close to eradicating around the world. This is caused because of the reduction of Susceptible. But there is always a small chance for side effects when using vaccination. And therefore some people choose to skip it. This is quite critical for the fight of total eradication. Not only is it a big risk for the specific person, but it also increase the number of Susceptible. An important thing to remember is that an epidemic situation can quickly grow again if the reproduction rate reaches the threshold

Some analytical studies can be done on the simple model by using (**??**).

$$\frac{dI}{dS} = -\frac{(rS-a)I}{rSI} = -1 + \frac{\rho}{S}, \qquad \rho = \frac{a}{r}, (I \neq 0). \tag{7}$$

The singularities will all lie on the I=0 axis. This equation can be integrated and will then give phase plane trajectories in the (I,S) plane. This can be seen in the figure above.

$$I + S - \rho \ln S = constant = I_0 + S_0 - \rho \ln S_0 \tag{8}$$

An observation is that all initial values satisfy $I_0 + S_0 = N$ since $R(0) = 0$. This will change when $t > 0$. If a disease appear it would be important to know the severity of the disease, and also if it can develops to an epidemic. Therefore the maximum value of I,I_{max} which occurs when $S = \rho$. At this point, $\frac{dI}{dt} = 0$. This can be found by using (**??**)

$$
\begin{aligned}
I + S - \rho \ln S =& I_0 + S_0 - \rho \ln S_0 \\
I_{max} + \rho - \rho \ln \rho =& I_0 + S_0 - \rho \ln S_0 \\
I_{max} =& -\rho + \rho \ln \rho + I_0 + S_0 - \rho \ln S_0 \\
I_{max} =& N - \rho + \rho \ln \frac{\rho}{S_0}
\end{aligned}
$$

The trajectory in the figure above shows quite clear the difference between $S_0 > \rho$ and $S_0 < \rho$. In the cases where S_0 is higher, an increasing of the I will happend. While a decreasing will happend when S_0 is lower. An example can be shown. The ρ in the simulation above is sat to 35. While $N = 100$ for all trajectories. A calculation can be done on the lowest trajectory which has the initial conditions $S_0 = 90$ and $I_0 = 10$

$$
\begin{aligned}
I_max =& N - \rho + \rho \ln \frac{\rho}{S_0} \\
I_max =& 100 - 35 + 35 \ln \frac{35}{90} \\
I_max =& 31.94
\end{aligned}
$$

This situation causes an epidemic situation since I_{max} is quite much higher than the initial condition. The figure above shows that the trajectory of this function starts decreasing after this point. In the two upper trajectories $S_0 < \rho$, then I will decrease from initial condition. $I(\infty) \to 0$ will move towards zero as $t \to \infty$.

The Susceptible, S will always has a decreasing solution since $\frac{dS}{dt} < 0$ when $S \neq 0$ and $I \neq 0$. By using (**??**),

$$\frac{dS}{dR} = -\frac{S}{\rho} \tag{9}$$

By some integration the equation can be expressed

$$S = S_0 e^{-R/\rho} \geq S_0 e^{-R/\rho} > 0 \tag{10}$$

As $t \to \infty$ the total number of Susceptible will be in the range $0 < S(\infty) \leq N$. This range can be reduced even more by knowing that I will increase as long $S > \rho$. This will lead a decreasing for S. The number of Susceptible will be $0 < S(\infty) \leq \rho$. Since I will be zero when the time goes towards infinity, the Removed class can be described,$R(\infty) = N - S(\infty). Nowthiscanbeaddedinto(\mathbf{??}), whichgives, S(\infty) = S_0 \exp\left(-\frac{N-S(\infty)}{\rho}\right)$ (11)$S(\infty)$ be found as the positive root in the transcendental equation. This can be used to find the total number of people who catch the disease.

$$I_{total} = I_0 + S_0 - S(\infty) \tag{12}$$

This analysis is based on the important implication that the disease dies out because the Infective class goes towards zero, and not because of the lack of susceptibles. This is the case for all diseases and the important factor is the removal rate, ρ. This will affect the number of susceptible that can be infected. This removal rate will also varies with respect to different parameters as population density, incubation time and the length of the period of sickness. The two equations () and () gives an understanding of the maximum number of infective and the total number, but this demands the exact number of ρ, I_0, S_0 and S, which is quite hard to get exactly in a real situation. The challenging thing is often to know how many that are infected at each time. The easiest group to control, is the number of removed, R. This group is assisted with medical help. So to model a realistic situation, the number of removed as a function of time dR/dt is a realistic model. Here the equations from (**??**) ,(**??**) and (**??**) can be used.

$$\begin{aligned}\frac{dR}{dt} =& aI \\ =& a(N - R - S) \\ =& a(N - R - S_0 e^{-R/\rho})\end{aligned}$$

This solution also demands several parameters as a,r,S_0 and N to solve this numerically. Then it is normal to adjust the parameters after the epidemic to get the best result as possible. But it the epidemic is not to large, R/ρ will be quite small, at least under 1. Then another model from Kermack and McKendrick(1927) can be used.

$$\frac{dR}{dt} = a(N - S_0 + (\frac{S_0}{\rho} - 1)R - \frac{S_0 R^2}{2\rho^2}) \tag{13}$$

By factoring the right-hand side of this equation, the equation can be integrated. After some complicated steps the equation can be given by (**??**) is used in my equation

$$R(t) = \frac{\rho^2}{S_0}\left[\left(\frac{S_0}{\rho} - 1\right) + \alpha \tanh\left(\frac{\alpha a t}{2} - \phi\right)\right] \tag{14}$$

where

$$\alpha = \left[\left(\frac{S_0}{\rho} - 1\right) + \frac{2S_0(N - S_0)}{\rho^2}\right]^{1/2}, \phi = \tanh^{-1}\frac{1}{\alpha}\left(\frac{S_0}{\rho} - 1\right) \tag{15}$$

This will give sech

$$\frac{dR}{dt} = \frac{a\alpha^2 \rho^2}{2S_0}^2 \left(\frac{\alpha a t}{2} - \phi\right) \tag{16}$$

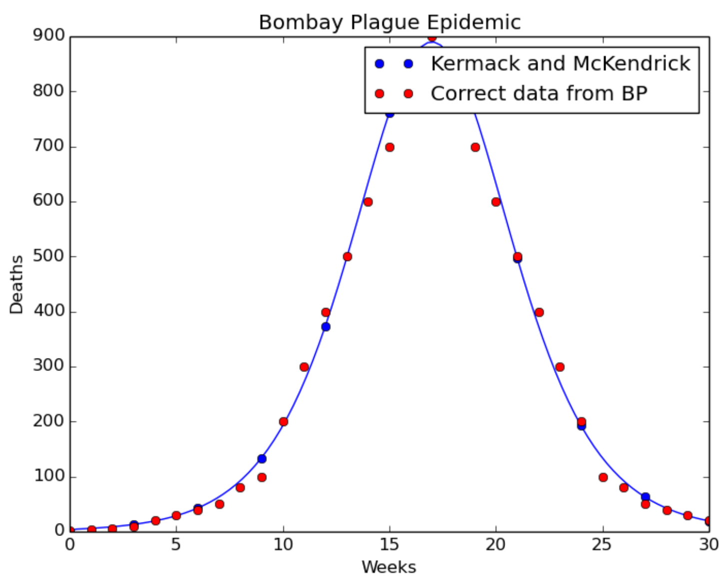
The advantage with () is that this equation only consist of three parameters. This is easier to adjust to a real situation. If the exact number of the removed class, R, is collected and the epidemic situation is quite small, the model (**??**) should be used. Both these two models demands that R/ρ is quite small. It not the model (**??**) has to be used.

2.2 Bombay Plague Epidemic

The model (**??**) was used by Kermack and McKendrick(1927) to model the epidemic in Bombay. The data that they were able to get was the number of deaths per week. By these result they were able to modify the equation to simulate the number of deaths per week. The three parameters were sech

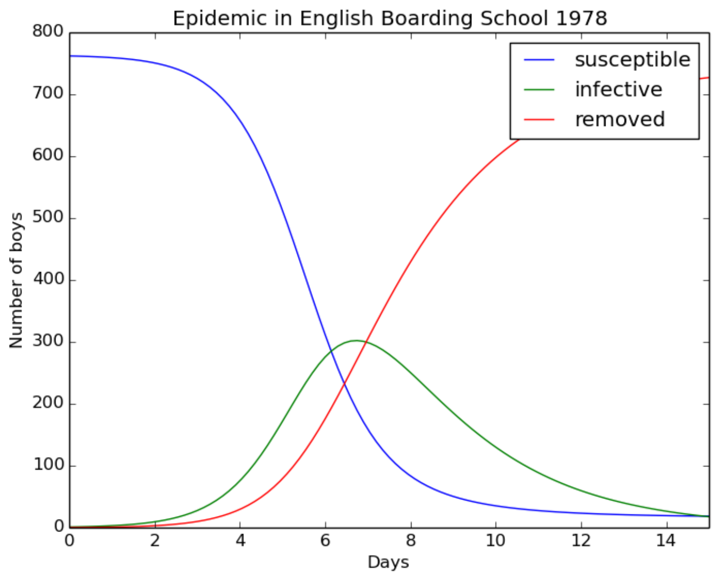
$$\frac{dR}{dt} = 890^2(0.2t - 3.4) \tag{17}$$

And the function of deaths per week can be modeled,

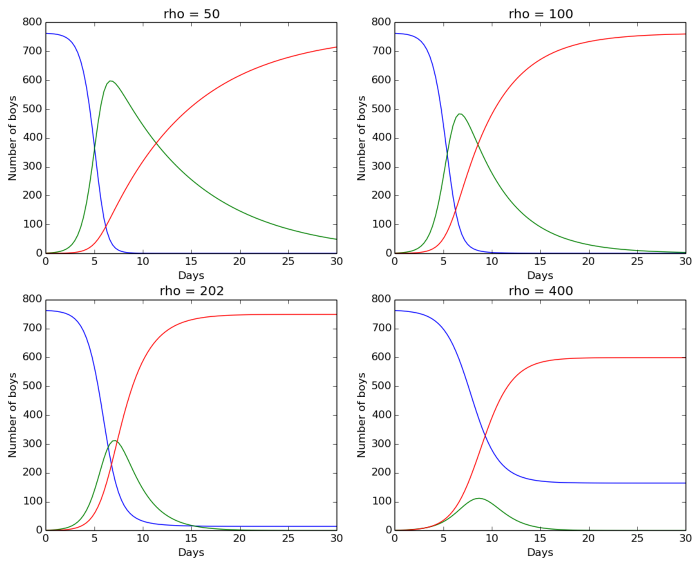


2.3 Epidemic in an English Boarding School 1978

The British medical journal had in 1978 a report from a boarding school in England. One of the boys had brought with him a disease back to the school. Since this was a boarding school, they were totally isolated from other and a closed system to model with the system. The following parameters has been used for the simulation, $N = 763$,$S_0 = 762$,$I_0 = 1$,$\rho = 202$ and $r = 2.18x10^{-3}$. Here $S_0 > \rho$ and an increasing of I can be expected.



By some variation in ρ, the results from the epidemic could turn out quite different. Here are some examples where ρ varies from 50 up to 400.



A small ρ gives an aggressive disease, since I will increase until $S = \rho$. The class of infective will increase until the number of susceptible falls down to 50. This results in in a total majority of infected students. A large ρ has the opposite effect. The total number of susceptible stays around 170 students, and will go towards steady number as $I(\infty) = 0$.

3 Zombification

One of the worst epidemics that can affect the human population is a zombie attack. This will have a huge impact on the way humans to day live. Several movies and series has illustrated this type of situation, but by using mathematical models this is possible to simulate. There have been written a couple of papers about this. Munz et al. used the SIR model to simulate an possible upcoming zombiefication, where the infected class, I, is switched with a zombie class, Z. The following model was used,

$$\frac{dS}{dt} = \Sigma - \beta SZ - \delta S$$
$$\frac{dZ}{dt} = \beta SZ + \zeta R - \alpha SZ$$
$$\frac{dR}{dt} = \delta S + \alpha SZ - \zeta R$$

This is a bit more complicated than the standard SIR model that was presented above. Here Σ describe the birthrate for new susceptibles. $\frac{dS}{dt}$ is now able to be positive. βSZ describe the numbers of susceptible that become zombies, by interactions between the two groups. Similar as the case for rSI. The last factor for the susceptible is δS. This describe the number of naturals deaths among the group. ζR controls the number of removed that arises as zombies. To avoid total eradication the humans will try to fight back, αSZ, describe the number of zombies killed by humans in the zombie attacks.

This model was challenged by the paper from Langtangen, Mardal and Røtnes, where they developed another model. They added another class, I, which here describe infected humans. This is a pre-zombie stadium, before becoming zombies. There is also added a function $\omega(t)$, which creates a massive attack from the humans. This is controlled by time and give the susceptibles a chance to fight back. The system:

$$\begin{aligned}
\frac{dS}{dt} =& \Sigma - \beta SZ - \delta_S S \\
\frac{dI}{dt} =& \beta SZ - \varrho I - \delta_I I \\
\frac{dZ}{dt} =& \varrho I - (\alpha + \omega(t))SZ + \zeta R \\
\frac{dR}{dt} =& \delta_S S + \delta_I I - \zeta R + (\alpha + \omega(t))SZ
\end{aligned}$$

The new elements are the different δ's, which describe the probability that the specific type is killed or dies. This can only harm the humans,S, or the infected group,I. The probability for a human to wake up as a zombie is here controlled by ϱ. This will have major impact on the amount of zombies. In the system from Munz et al. everyone ended up as zombies. But in this case the result can variy from everyone down to none, if $\varrho = 0$. The function:

$$\omega(t) = a \sum_{i=0}^{m} \exp\left(\frac{1}{2}\left(\frac{t - T_i}{\sigma}\right)^2\right) \tag{18}$$

controls the attacks from the susceptible will only be fired at choosen times. These are controlled by the parameters. a here works as a similar parameter as α, but will only active when the susceptible group is organized and ready to attack. T contains a list of numbers. This controls the time when the attacks will occur.

3.1 Parameters used in the model

An important factor while modelling a zombie attack is the parameters. In the ODE system from Langtangen, Mardal and Røtnes parameters from "The night of the living dead" was used. The TV series "Walking dead" has been study in detalj. The data will be based on the first five episodes and are constructed by watching the show carefully. The three phases in a zombie attack will be based on the form used in the paper from LMR.

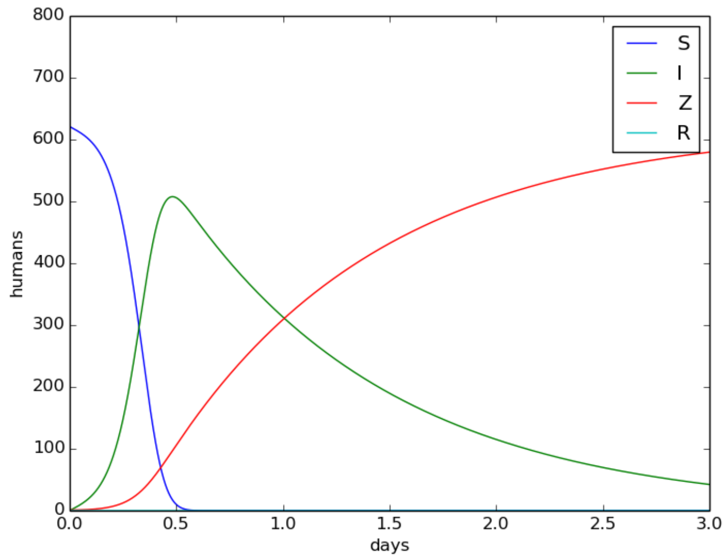
The initial phase. In this phase the disease is yet not known and humans often tries to save the sick ones by taking them to hospitals or get some kind of treatment. Because of this ignorance related to the disease, the number of infected humans is high. This phase is often quite short and humans soon starts realise that the risk of saving other is really high. Walking Dead never shows anything from this phase, but the viewer only sees the damages when the main character sheriff Rick Grimes wakes up at the local hospital after a coma caused by shooting accident. What he wakes up to is the major damage caused in the initial phase and the society is in the hysterical phase.

To denote the question about the time, the length of Ricks coma is essential. There are several factors that gives an indication of the time aspect. When Rick wakes up at the hospital he has grown a smooth beard of about 1 cm. This would correspond to 1 month in average for an European origin. He also has some flowers which has dried out, these also give the impression that there has been some weeks. The hospital is running on its emergency generator. This would probably not last for many days with a fully operational hospital. The hospital is as well as shut down when Rick wakes up. This will be a factor that can give the emergency generator a longer lifetime. Dr. Edwin Jenner gives the viewer in episode 5. Here he records a videolog describing his result.He tells the viewer that it was 63 days since the epidemic started spreading. By studying the first five episodes in detail, the series gives a sense that the time aspect has not been in focus. Therefore the different phases are constructed from the information that has been given. Rick Grimes has probably been in coma for a month and the number of zombies he meets the first days will give the indication of the numbers that will be used in the model. The total amount of people in the model will be based on the number of humans, dead and zombies seen in the first five episodes. The number of humans has been estimated to 62. 20 living in the camp with Rick, 40 humans in the old nursing home and the father and son in episode 1. The number of dead is estimated to 200. This is based on the amount of dead outside the hospital where Rick wakes up. The number of zombies are assume to be 360. These are based on the 30 outside the house of Morgan Jones and his son Duane, 300 zombies in the city Atlanta and 30 zombies attacking the camp. The total number will then be 622. Since these numbers are calculated for about a month after the outbreak of the disease and the initial phase probably are over in three days. After this the hysterical phase starts.Over the three first days when Rick is awake about 20 zombies are killed by humans and 1 human is killed by zombies. This can be used to calculate the final number in the initial phase. To do this calculation easier, a month is assumed to be 30 days and 27 days earlier. The numbers are set to:

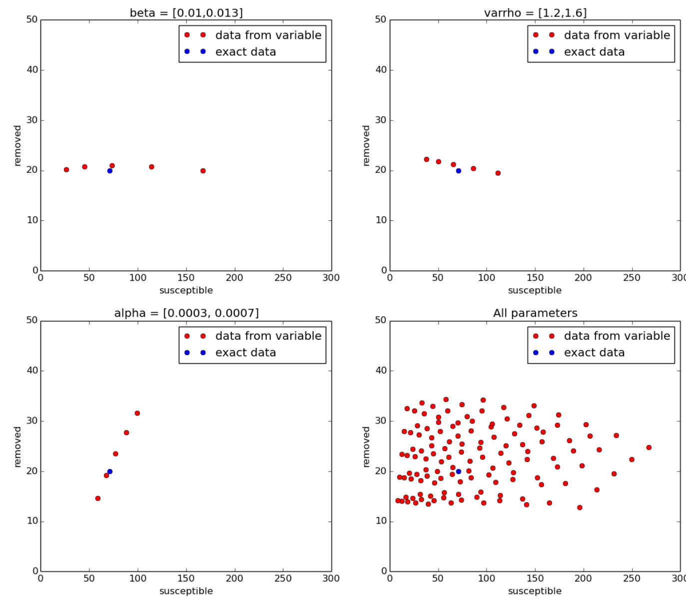
$$\frac{27}{3} * 20 = 180$$
$$\frac{27}{3} * 1 = 9$$

180 zombies and 9 humans are killed in this period. The final number for the initial phase can then be set to 71 humans, 540 zombies/infected and 20 dead. This is the same number as for the initial values for the hysterical phase. Another question to discuss is the incubation time. Here there are two examples that can be used. The first transformation from human to zombie happens for Amy, who was bit in the arm by a zombie. The transformation happens in about 12 hours. The other character Jim has a slower transformation. This last for about two days before the rest of the group leave him alongside the road on their way to CDC(Center for Disease Control). An average of the incubation time can then be set to 24 hours based on these two transformations.

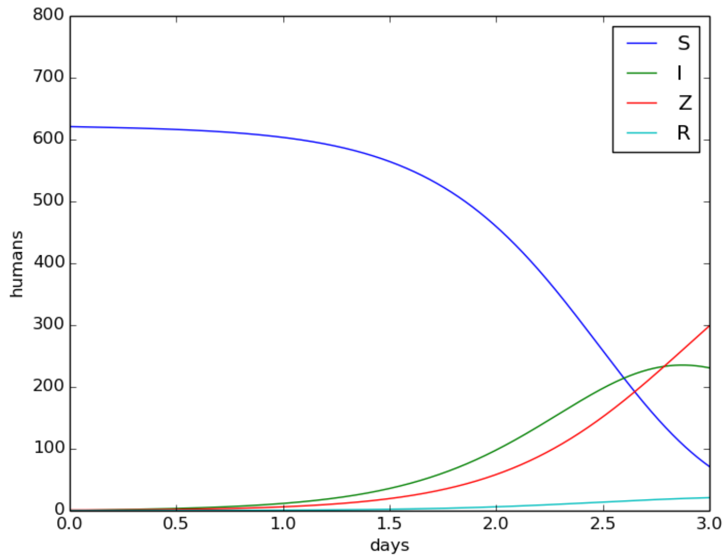
Now the ODEmodel can be used to model the initial phase. $S_0 = 621$ and $Z_0 = 1$ while the two other groups are set to zero. β can be found with the expression $\beta \Delta t S Z$ from the first ODE equation. After three days about 90 percent of the humans are killed. The probability for a human being infected will be sat to $\beta = 0.3$. The natural death rate is sat to $\delta_S = 2.2 * 10^{-5}$ based on [**?**]. It is quite hard to find similar realistic data for infected humans, so $\delta_I = \delta_S$. The number of births is sat to $\Sigma = 3.45 * 10^{-5}$. This is based on data from CDC from 2012 [**?**]. Since this is data for the initial phase, zombies are seen as infeteced humans that can be saved. Therefore $\alpha = 0$. And the two last parameters are also zero, $a = \zeta = 0$. This model will be:



This result shows that the human population is eradicated in about a half day. This is not the case, and some adjustments need to be done. There are three parameters that are interesting to study. The first is β, which describes how many humans that get infected in a human-zombie collision. Second one is ϱ. This parameter controls the incubation time. The last parameter that can affect the number in each group is α. This describe the number of zombies that gets killed in a human-zombie collision. These variables are plotted separately and combined in the last subplot. The idea here is to produce results that fulfill the final number for the susceptible class and the removed class, which is 71 and 20. The blue dot in each plot is describe this value. A rough estimate has been done for each parameter before using it. This is why they all lie in different regions than the values for the plot above



These plots give an important knowledge in the effect of varying the parameters. β and ϱ mainly affect the number of susceptible while α affect them both. By choosing $\beta = 0.01155$, $\varrho = 1.37$ and $\alpha = 0.00044$, the following plot can be produced:



Here the final values are $S_n = 71.3, I_n = 230.8, Z_n = 298.9$ and $R_n = 21$, which is quite close to the result from the movie. By looking at these changes, it is possible to argue for them. By increasing ϱ to 1.37 reduces the incubation time. Now the average time will be about 17.5 hours which is quite realistic. The probability β is quite sensitive and has a major effect only from small variations. This is due to the term that it is a part of. $\Delta t SZ\beta$. A couple of examples demonstrate this. $\Delta t = 1/24$, this is equivalent to one hour. When using the initial values for the classes S and Z and $\beta = 0.01155$ from the second plot, the number of infected in the first hour will be $(1/24) * 721 * 1 * 0.01155 = 0.34$. $1/3$ of a human in the first hour seems as a slow and not very aggressive disease. But when the number of zombies slowly increase, this will affect the numbers of infected. By looking at the hour when the number is equal between humans and zombies, about 200. The number of infected this hour will be 19.25 per hour. This result in about 10 percent. By changing β to the value from the first plot, the number of infected will be 500 per hour and it is quite easy to see that this will lead to eradication in a short amount of time. The last parameter α controls the number of zombies that dies in an interaction between zombies and humans. While humans still think that this is a disease that the infected can be saved from, it is still a chance that the result of a collision can end with a murder. By some adjustments these results can be seen as realistic values.

The hysterical phase. Now the humans start to avoid the infected and some tries to fight them. The humans often gather in groups and try to find safe spots away from the zombies. Important emergency as weapons and food are their main priority. Barricades are build and the guarding is strict. When Rick Grimes wakes up, the hospital is abandoned and the halls are filled up with dead people. Quite fast he understand that he needs to reach safety and after a certain time he ends up in a camp outside Atlanta city, where the zombies are. A couple of elementary changes in the interaction between humans and infected/zombies. In the initial phase, the humans tried to help the infected humans. This result in a high percent of infected. While they now understand this risk and keep distance to those who are infected. This will affect β which will have lower value. Another important assumption that the humans have done is the morality for a zombie kill. While this was seen as no opinion, this is now totally okay. The humans have started treat zombies and infected as a enemy instead as a sick allied. This result in a higher dead rate among the zombies, so α will be higher. These are the main changes.

The the hysterical model can be constructed based on the data found in the initial phase. First the initial values will be based on the final data from the initial phase. These are $S_0 = 71.3, I_0 = 230.8, Z_0 = 298.9$ and $R_0 = 21$. As the final result will be $S_n = 62, I_n Z_n = 360$ and $R_n = 200$. Here infected and zombies are added together. The time aspect will be modeled for 30 days, which result in ten times longer simulation. Since the final results are known here, a similar adjustment of the parameters can be done. Here some test simulations are done to get the range for the parameters.

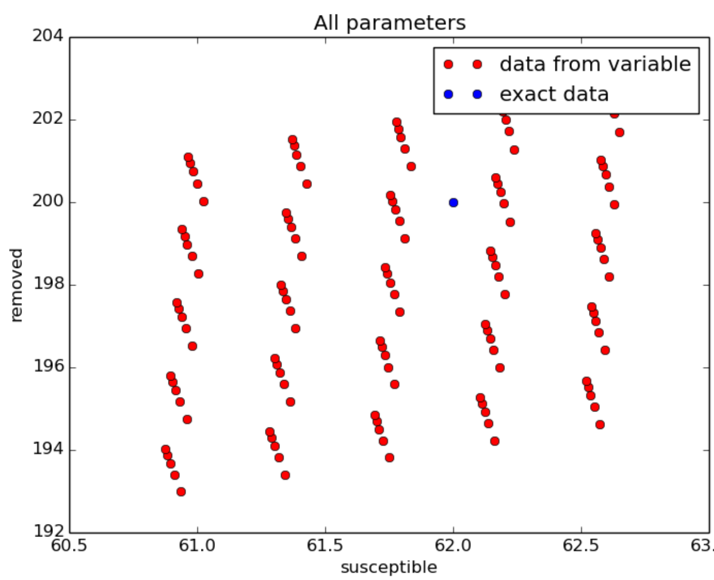


Figure 1: The final result for each combination can be shown as a grid. $\beta = [1 * 10^{-5}, 1.2 * 10^{-5}]$,$\varrho = [1, 2]$ and $\alpha[2 * 10^{-4}, 2.2 * 10^{-4}]$

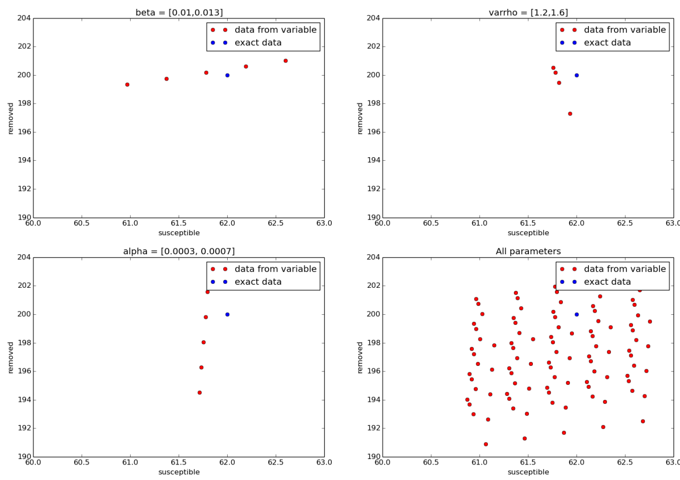
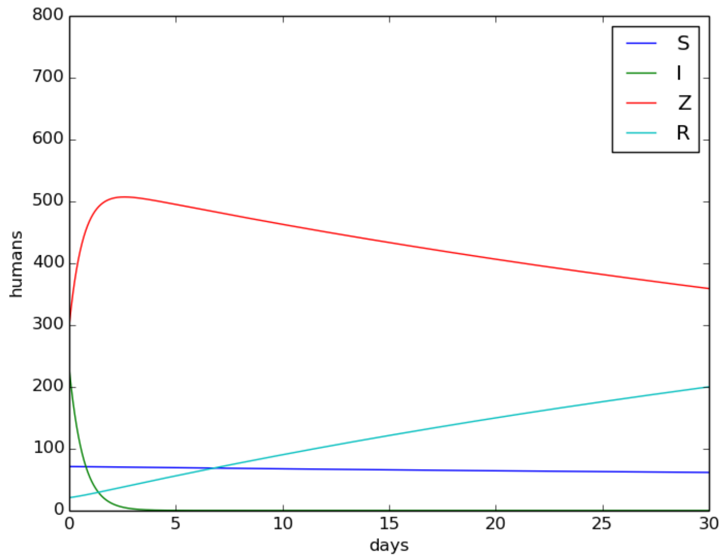


Figure 2: The final result for each combination can be shown as a grid. $\beta = [1 * 10^{-5}, 1.2 * 10^{-5}]$,$\varrho = [1, 2]$ and $\alpha[2 * 10^{-4}, 2.2 * 10^{-4}]$

This plot gives insight in how the parameters affects the final result. By decreasing β, it will essentially increase the number of humans that survive. But it will also increase the number of dead. This may at first glance seems quite strange. Should not the number of deaths decrease when the number of humans increase? This can be explained with the idea that was shown for β in the initial phase. Since βSZ gets smaller when β get smaller, the combination of SZ will stay higher for a longer time. This again affects αSZ, which regulates the number of zombies that dies. The larger this combination is, the more zombies will die. Then by looking at α, this mainly affect the number of removed, here the dead ones.

By increasing α, the number of removed also increase. But similar to the increasing of the removed it also has a slight increase on the number of susceptible. Here the argument for β above can be reversed. Since a higher α leads to a higher death rate among the zombies, the combination SZ will be smaller and the number infected humans caused by βSZ is lower.

The last parameter, ϱ, has nearly no effect. The red dots are combined with a pull down to the right when increasing ϱ. This parameter varies most, but has the least impact. This can be explained with the long time aspect and the number of infected compared to the number of zombies. Since the number of zombies are that much higher, the transformation length from infected to zombie is almost negligible.



This plot fulfill the result that was predicted based on the series. These numbers correspond to the number in each group when Rick woke up in the hospital. The parameters are set to $\beta = 0.000011$,$\varrho = 1.5$ and $\alpha = 0.000208$. The plot shows that the number of zombies increase quickly and reach its maximum value after a couple of days in this phase. After this, the number of infected dramatically decrease which again affect the number of zombies. Here the humans has been able to stabilize. Since the clashes between humans and zombies are dramatically decreased, there are nearly no one that gets infected. And in the cases where humans has to face zombies, the kill rate has increased. The increase of deaths is close to proportional to the decrease of zombies.

The counter attack. This counter attack is more complicated to model since this phase appear simultaneously as the hysterical phase in Walking Dead. The group of human are trying to avoid the zombies, but when they get to close the humans need to fight back. These situations are often caused by a high density of zombies. This force the zombies to spread to new areas. In Walking dead the counter attack appears when a group of 30 zombies reach the camp. This trigger a fight where all the zombies are killed and 4 of the humans are bitten. This shows that a counter attack from the humans causes a lot of damage. The time aspect is sat to 6 hours.

Now the function $\omega(t)$ will be used. This can be shown:

To get some startvalues, $SZ\omega(t) = 30$ can be used. Where $\omega(t)$ is the area under the function. By inserting the values for S and Z before the counter

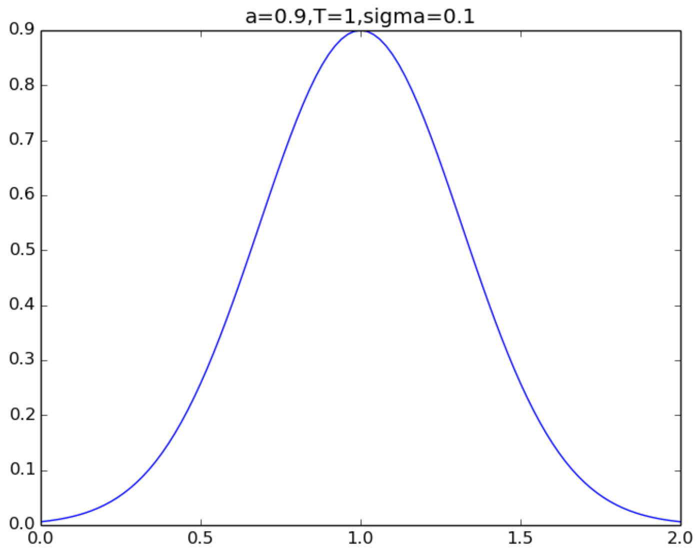
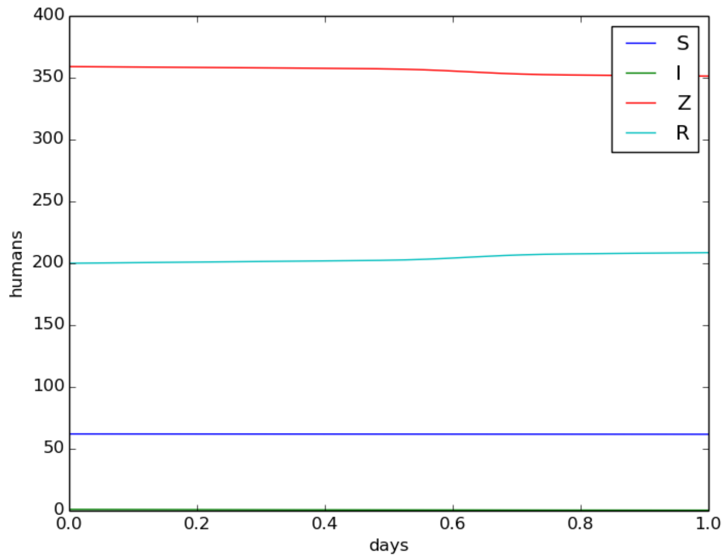


Figure 3: $\omega(t)$ is a Gaussian function where a controls the maximum value, T controls when the maximum value strikes and σ controls the width of the function.

attach, the area shall be $\omega(t) = 1.34 * 10^{-3}$. By using $a = 0.00103$ and $\sigma = 0.005$ this result can be reproduced. Then counter attack is sat to appear during the last part of the day [0.75,1]. the value of T is then sat to $T = [0.875]$.



This simulation result in about 8-9 dead zombies. The total number should be higher. Another problem is that no human died under this battle. The model from [?] is based on The Night of the Living Dead, where the amount of humans who are killed are close to zero. This is not the case in Walking dead. Therefore the risk is higher for humans in a counter attack. This solved by adding $\mu\omega(t)SZ$, where μ is the risk for human to get infected during this attack. The model (**??**) can then be expand,

$$\begin{aligned}
\frac{dS}{dt} &= \Sigma - (\beta + \mu\omega(t))SZ - \delta_S S \\
\frac{dI}{dt} &= (\beta + \mu\omega(t))SZ - \varrho I - \delta_I I \\
\frac{dZ}{dt} &= \varrho I - (\alpha + \omega(t))SZ + \zeta R \\
\frac{dR}{dt} &= \delta_S S + \delta_I I - \zeta R + (\alpha + \omega(t))SZ
\end{aligned}$$