Proceedings of the 2006 American Control Conference
Minneapolis, Minnesota, USA, June 14-16, 2006

Design and Analysis of a Novel $\mathcal{L}_1$ Adaptive Controller, Part II: Guaranteed Transient Performance

Chengyu Cao and Naira Hovakimyan

***Abstract*— In this paper, we present a novel adaptive control architecture that ensures that the input and output of an uncertain linear system track the input and output of a desired linear system during the transient phase, in addition to the asymptotic tracking. Design guidelines are presented to ensure that the desired transient specifications can be achieved for both system's input and output signals. The tools from this paper can be used to develop a theoretically justified verification and validation framework for adaptive systems. Simulation results illustrate the theoretical findings.**

I. INTRODUCTION

Model Reference Adaptive Control (MRAC) architecture was developed conventionally to control linear systems in the presence of parametric uncertainties [1], [2]. However, it offers no means for characterizing the system's input/output performance during the transient phase. Improvement of the transient performance of adaptive controllers has been addressed from various perspectives in numerous publications. One can find a detailed description of these results in Part I of this paper [3]. Therein, a novel $\mathcal{L}_1$ adaptive control design method is introduced, which guarantees that the control signal is in low-frequency range by definition. In this Part II, we give a slightly different design of the same $\mathcal{L}_1$ adaptive controller. To enable comparison with high-gain controllers, we replace the feedback module $\hat{K}(s)$, introduced in [3], by a linear constant gain feedback of the system states. For the sake of completeness, we give briefly the stability proof here for this design, which requires similar $\mathcal{L}_1$-gain minimization of a cascaded system as in [3]. The ideal (non-adaptive) version of this $\mathcal{L}_1$ adaptive controller is used along with the main system dynamics to define an extended closed-loop reference system, which gives an opportunity to estimate performance bounds in terms of $\mathcal{L}_\infty$ norms for both transient and steady state errors of both system's input and output signals as compared to the same signals of this reference system. These bounds immediately imply that the transient performance of the control signal in MRAC cannot be characterized. Design guidelines for selection of the low-pass filter ensure that the extended closed-loop reference system approximates the desired system response, despite the fact that it depends upon the unknown parameter.

The paper is organized as follows. Section II gives the problem formulation. In Section III, the new $\mathcal{L}_1$ adaptive controller is presented. Stability and tracking results of the $\mathcal{L}_1$ adaptive controller are presented in Section IV. Design guidelines for the $\mathcal{L}_1$ adaptive controller are presented in Section V. Comparison of the performance of $\mathcal{L}_1$ adaptive controller, MRAC and the high gain controller are discussed in section VI. In section VII, simulation results are presented, while Section VIII concludes the paper. Proofs are in Appendix.

Research is supported by AFOSR under Contract No. FA9550-05-1-0157 and partly by ADVANCE VT Institutional Transformation Research Seed Grant of NSF.

The authors are with AOE, Virginia Tech, Blacksburg, VA 24061-0203, e-mail: {chengyu, nhovakim}@vt.edu

II. PROBLEM FORMULATION

Consider the following single-input single-output system:

$$\begin{aligned} \dot{x}(t) &= A_m x(t) - b\theta^\top x(t) + bu(t), \ x(0) = x_0 \\ y(t) &= c^\top x(t), \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$ is the system state vector (measurable), $u \in \mathbb{R}$ is the control signal, $y \in \mathbb{R}$ is the regulated output, $b, c \in \mathbb{R}^n$ are known constant vectors, A_m is a known $n \times n$ Hurwitz matrix, $\theta \in \mathbb{R}^n$ is a vector of unknown parameters, which belongs to a given compact convex set Θ, i.e. $\theta \in \Theta$. The control objective is to design an adaptive controller to ensure that $y(t)$ tracks a given bounded continuous reference signal $r(t)$ *both in transient and steady state*, while all other error signals remain bounded. Rigorously, the control objective is to ensure $y(s) \approx D(s)r(s)$, where $y(s), r(s)$ are Laplace transformation of $y(t), r(t)$ respectively, and $D(s)$ is a strictly proper stable LTI system that specifies the desired transient and steady state performance.

III. $\mathcal{L}_1$ ADAPTIVE CONTROLLER

In this section, we develop a novel adaptive control architecture that permits complete transient characterization for both system input and output signals. The following control structure

$$u(t) = u_1(t) + u_2(t), \quad u_1(t) = -K^\top x(t), \tag{2}$$

where $u_2(t)$ is the adaptive controller to be determined later, while K is a nominal design gain and can be set to zero, leads to the following *partially* closed-loop dynamics:

$$\begin{aligned} \dot{x}(t) &= A_o x(t) - b\theta^\top x(t) + bu_2(t), \ x(0) = x_0 \\ y(t) &= c^\top x(t). \end{aligned} \tag{3}$$

The choice of K needs to ensure that $A_o = A_m - bK^\top$ is Hurwitz or, equivalently, that $H_o(s) = (s\mathbb{I} - A_o)^{-1}b$ is stable. One obvious choice is $K = 0$. For the linearly parameterized system in (3), we consider the following companion model

$$\begin{aligned} \dot{\hat{x}}(t) &= A_o \hat{x}(t) + b(u_2(t) - \hat{\theta}^\top(t) x(t)), \quad \hat{x}(0) = x_0 \\ \hat{y}(t) &= c^\top \hat{x}(t) \end{aligned} \tag{4}$$

1-4244-0210-7/06/$20.00 ©2006 IEEE

along with the adaptive law for $\hat{\theta}(t)$:

$$\dot{\hat{\theta}}(t) = \Gamma \mathrm{Proj}(\hat{\theta}(t), x(t)\tilde{x}^\top(t)P_o b), \quad \hat{\theta}(0) = \hat{\theta}_0 \,, \tag{5}$$

where $\tilde{x}(t) = \hat{x}(t) - x(t)$ is the tracking error, $\Gamma \in \mathbb{R}^{n\times n} = \Gamma_c \mathbb{I}_{n\times n}, \Gamma_c > 0$ is a positive definite matrix of adaptation gains, and $P_o = P_o^\top > 0$ is the solution of the algebraic equation $A_o^\top P_o + P_o A_o = -Q_o$ for arbitrary $Q_o > 0$. Letting

$$\bar{r}(t) = \hat{\theta}^\top(t)x(t), \tag{6}$$

the companion model in (4) can be viewed as a low-pass system with $u(t)$ being the control signal, $\bar{r}(t)$ being a time-varying disturbance, which is not prevented from having high-frequency oscillations. Consider the following control design for (4):

$$u_2(s) = C(s)\big(\bar{r}(s) + k_g r(s)\big)\,, \tag{7}$$

where $\bar{r}(s), r(s)$ are the Laplace transformations of $\bar{r}(t), r(t)$, respectively, $C(s)$ is a stable and strictly proper system with low-pass gain $C(0) = 1$, and $k_g = 1/(c^\top H_o(0))$. Closed-loop companion model in (4) with the control signal in (7) can be viewed as an LTI system with two inputs $r(t)$ and $\bar{r}(t)$:

$$\hat{x}(s) = \bar{G}(s)\bar{r}(s) + G(s)r(s) \tag{8}$$

$$\bar{G}(s) = H_o(s)(C(s) - 1) \tag{9}$$

$$G(s) = k_g H_o(s)C(s)\,, \tag{10}$$

where $\hat{x}(s)$ is the Laplace transformation of $\hat{x}(t)$. We note that $\bar{r}(t)$ is related to $\hat{x}(t)$, $u(t)$ and $r(t)$ via nonlinear relationships. Let

$$\theta_{\max} = \max_{\theta\in\Theta} \sum_{i=1}^{n} |\theta_i|\,, \tag{11}$$

where θ_i is the i^{th} element of θ, Θ is the compact set, where the unknown parameter lies. We now give the $\mathcal{L}_1$ performance requirement for the design of K and the strictly proper stable system $C(s)$.

$\mathcal{L}_1$-gain requirement: Design K and $C(s)$ to satisfy

$$\lambda \triangleq \|\bar{G}(s)\|_{\mathcal{L}_1}\theta_{\max} < 1, \tag{12}$$

where $\theta_{\max}$ is defined in (11).

IV. Analysis of $\mathcal{L}_1$ Adaptive Controller

A. Stability and Asymptotic Convergence

Consider the following Lyapunov function candidate:

$$V(\tilde{x}(t), \tilde{\theta}(t)) = \tilde{x}^\top(t)P_o\tilde{x}(t) + \tilde{\theta}^\top(t)\Gamma^{-1}\tilde{\theta}(t)\,, \tag{13}$$

where P_o and Γ are introduced in (5). It follows from (3) and (4) that

$$\dot{\tilde{x}}(t) = A_o\tilde{x}(t) - b\tilde{\theta}^\top(t)x(t)\,, \quad \tilde{x}(0) = 0\,. \tag{14}$$

Hence, it is straightforward to verify from (5) that

$$\dot{V}(t) \le -\tilde{x}^\top(t)Q_o\tilde{x}(t) \le 0\,. \tag{15}$$

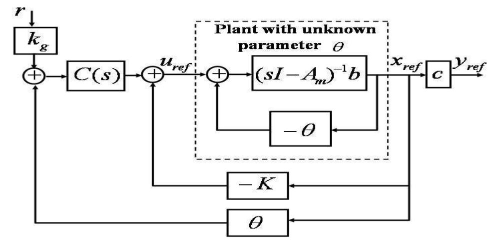


Fig. 1. Block diagram of the reference LTI system

Notice that the result in (15) is independent of $u_2(t)$, however one cannot deduce stability from it. One needs to prove in addition that with the $\mathcal{L}_1$ adaptive controller the state of the companion model will remain bounded. Boundedness of the system state then will follow.

Similar to Theorem 3 in Part I [3], we have:

Theorem 1: Given the system in (1) and the $\mathcal{L}_1$ adaptive controller defined via (2), (4), (5), (7) subject to (12), the tracking error $\tilde{x}(t)$ converges to zero asymptotically:

$$\lim_{t\to\infty} \tilde{x}(t) = 0. \tag{16}$$

B. Reference System

Recall that in the conventional MRAC architecture the proof on asymptotic stability was implying that the state of the system was tracking the state of the reference model. With the $\mathcal{L}_1$ adaptive controller one should question what is the reference system that the closed loop system with the $\mathcal{L}_1$ adaptive controller tracks. In this section we characterize the reference system, which is being tracked by both the system state and control input of the system (1) via the $\mathcal{L}_1$ adaptive controller in (2), (4), (5), (7) both in transient and steady state. Towards that end, consider the following *ideal* version of the adaptive controller in (2), (7):

$$u_r(s) = C(s)(k_g r(s) + \theta^\top x_r(s)) - K^\top x_r(s)\,, \tag{17}$$

where $x_r(s)$ denotes Laplace transformation of the state $x_r(t)$ of the closed-loop system. The block diagram of the system (1) with the controller (17) is shown in Fig. 1.

Remark 1: Notice that when $C(s) = 1$ and $K = 0$, one recovers the reference model of MRAC, and the *ideal* controller in (17) reduces to the conventional ideal controller $u(t) = \theta^\top x(t) + k_g r(t)$ of MRAC. If $C(s) \ne 1$ and $K \ne 0$, then the control law in (17) changes the bandwidth of it.

Under the control action (17), $x_r(s)$ can be expressed as:

$$x_r(s) = (\mathbb{I} - \bar{G}(s)\theta^\top)^{-1}G(s)r(s)\,. \tag{18}$$

Lemma 1: If the condition in (12) holds, then

$$(i) \qquad (\mathbb{I} - \bar{G}(s)\theta^\top)^{-1} \quad \text{is stable;} \tag{19}$$

$$(ii) \quad (\mathbb{I} - \bar{G}(s)\theta^\top)^{-1}G(s) \quad \text{is stable.}$$

The proof follows from Theorem 1 in [3] easily.

C. System Response and the $\mathcal{L}_1$ Adaptive Control Signal

Letting $r_1(t) = \tilde{\theta}^\top(t)x(t)$, it follows from (6) that

$$\bar{r}(t) = \theta^\top(\hat{x}(t) - \tilde{x}(t)) + r_1(t), \qquad t \geq 0.$$

Hence, the companion model in (8) can be rewritten as

$$\begin{aligned}\hat{x}(s) \;&=\; (\mathbb{I} - \bar{G}(s)\theta^\top)^{-1}\Big(- \bar{G}(s)\theta^\top \tilde{x}(s) + \\ &\quad \bar{G}(s)r_1(s) + G(s)r(s)\Big)\,, \end{aligned} \tag{20}$$

where $r_1(s)$ is the Laplace transformation of $r_1(t)$. It follows from (14) that

$$\tilde{x}(s) = -H_o(s)r_1(s). \tag{21}$$

Substituting (9), (21) into (20) leads to $\hat{x}(s) = (\mathbb{I} - \bar{G}(s)\theta^\top)^{-1}G(s)r(s) + (\mathbb{I} - \bar{G}(s)\theta^\top)^{-1}(-\bar{G}(s)\theta^\top\tilde{x}(s) - (C(s)-1)\tilde{x}(s))$. Using $x_r(s)$ from (18) and recalling the definition of $\tilde{x}(s) = \hat{x}(s) - x(s)$, one arrives at

$$\begin{aligned}x(s) \;&=\; x_r(s) - \Big(\mathbb{I} + (\mathbb{I} - \bar{G}(s)\theta^\top)^{-1} \\ &\quad \big(\bar{G}(s)\theta^\top + (C(s)-1)\mathbb{I}\big)\Big)\tilde{x}(s). \end{aligned} \tag{22}$$

It follows from (2), (7) and (17) that

$$u(s) = u_r(s) + C(s)r_1(s) + (C(s)\theta^\top - K^\top)(x(s) - x_r(s))\,. \tag{23}$$

D. Asymptotic Performance and Steady State Error

Theorem 2: Given the system in (1) and the $\mathcal{L}_1$ adaptive controller defined via (2), (4), (5), (7) subject to (12), we have:

$$\lim_{t\to\infty} \|x(t) - x_r(t)\| \;=\; 0\,, \tag{24}$$

$$\lim_{t\to\infty} |u(t) - u_r(t)| \;=\; 0\,. \tag{25}$$

Lemma 2: Given the system in (1) and the $\mathcal{L}_1$ adaptive controller defined via (2), (4), (5), (7) subject to (12), if $r(t)$ is constant, we have: $\lim\limits_{t\to\infty} y(t) = r$.

The closed-loop system response with the $\mathcal{L}_1$ controller to a time varying input $r(t)$ is given in the next Section.

E. Transient Performance

We note that $(A_m - bK^\top, b)$ is the state space realization of $H_o(s)$. Since (A_m, b) is controllable, it can be proved easily that $(A_m - bK^\top, b)$ is also controllable. It follows from Lemma 4 in [3] that there exists $c_o \in \mathbb{R}^n$ such that

$$c_o^\top H_o(s) = N_n(s)/N_d(s)\,, \tag{26}$$

where the order of $N_d(s)$ is one more than the order of $N_n(s)$, and both $N_n(s)$ and $N_d(s)$ are stable polynomials.

Theorem 3: Given the system in (1) and the $\mathcal{L}_1$ adaptive controller defined via (2), (4), (5), (7) subject to (12), we have:

$$\|x - x_r\|_{\mathcal{L}_\infty} \;\leq\; \gamma_1/\sqrt{\Gamma_c} \tag{27}$$

$$\|y - y_r\|_{\mathcal{L}_\infty} \;\leq\; \gamma_1\|c^\top\|_{\mathcal{L}_1}/\sqrt{\Gamma_c} \tag{28}$$

$$\|u - u_r\|_{\mathcal{L}_\infty} \;\leq\; \gamma_2/\sqrt{\Gamma_c}, \tag{29}$$

where $\|c^\top\|_{\mathcal{L}_1}$ is $\mathcal{L}_1$ gain of $c^\top$, $H_2(s)$ is defined in (47),

$$\gamma_1 = \|H_2(s)\|_{\mathcal{L}_1}\sqrt{\bar{\theta}_{\max}/(\lambda_{\max}(P_o))}\,, \tag{30}$$

$$\begin{aligned}\gamma_2 = &\left\|C(s)\frac{1}{c_o^\top H_o(s)}c_o^\top\right\|_{\mathcal{L}_1}\sqrt{\frac{\bar{\theta}_{\max}}{\lambda_{\max}(P_o)}} + \\ &\|C(s)\theta^\top - K^\top\|_{\mathcal{L}_1}\gamma_1. \end{aligned} \tag{31}$$

Corollary 1: Given the system in (1) and the $\mathcal{L}_1$ adaptive controller defined via (2), (4), (5), (7) subject to (12), we have:

$$\lim_{\Gamma_c\to\infty}(x(t) - x_r(t)) \;=\; 0\,, \qquad \forall t \geq 0, \tag{32}$$

$$\lim_{\Gamma_c\to\infty}(y(t) - y_r(t)) \;=\; 0\,, \qquad \forall t \geq 0, \tag{33}$$

$$\lim_{\Gamma_c\to\infty}(u(t) - u_r(t)) \;=\; 0\,, \qquad \forall t \geq 0\,. \tag{34}$$

Corollary 1 states that $x(t)$, $y(t)$ and $u(t)$ follow $x_r(t)$, $y_r(t)$ and $u_r(t)$ not only asymptotically but also during the transient, provided that the adaptive gain is selected sufficiently large. Thus, the control objective is reduced to designing K and $C(s)$ to ensure that the reference LTI system has the desired response $D(s)$.

Remark 2: Notice that if we set $C(s) = 1$, then the $\mathcal{L}_1$ adaptive controller is equivalent to MRAC. In that case $\left\|C(s)\frac{1}{c_o^\top H_o(s)}c_o^\top\right\|_{\mathcal{L}_1}$ cannot be finite, since $H_o(s)$ is strictly proper. Therefore, from (31) it follows that $\gamma_2 \to \infty$, and hence for the control signal in MRAC one can not reduce the bound in (29) by increasing the adaptive gain.

V. Design of the $\mathcal{L}_1$ Adaptive Controller

We proved that the error between the state and the control signal of the closed-loop system with $\mathcal{L}_1$ adaptive controller in (1), (2), (4), (5), (7) and the state and the control signal of the closed-loop reference system in (17), (18) can be rendered arbitrarily small by choosing large adaptive gain. Therefore, the control objective is reduced to determining K and $C(s)$ to ensure that the reference system in (17), (18) (Fig. 1) has the desired response $D(s)$ from $r(t)$ to $y_r(t)$. Notice that the reference system in Fig. 1 depends upon the unknown parameter θ.

Consider the following signals:

$$y_d(s) = c^\top G(s)r(s) = C(s)k_g c^\top H_o(s)r(s) \tag{35}$$

$$u_d(s) = k_g C(s)(1 + C(s)\theta^\top H_o(s) - K^\top H_o(s))r(s)\,. \tag{36}$$

We note that $u_d(t)$ depends on the unknown parameter θ, while $y_d(t)$ does not.

Lemma 3: For the LTI system in Fig. 1, subject to (12), the following upper bounds hold:

$$\|y_r - y_d\|_{\mathcal{L}_\infty} \leq \frac{\lambda}{1-\lambda}\|c^\top\|_{\mathcal{L}_1}\|G(s)\|_{\mathcal{L}_1}\|r\|_{\mathcal{L}_\infty} \tag{37}$$

$$\|y_r - y_d\|_{\mathcal{L}_\infty} \leq (\|c^\top\|_{\mathcal{L}_1}\|h_3\|_{\mathcal{L}_\infty})/(1-\lambda) \tag{38}$$

$$\begin{aligned}\|u_r - u_d\|_{\mathcal{L}_\infty} \leq &(\lambda\|C(s)\theta^\top - K^\top\|_{\mathcal{L}_1} \\ &\|G(s)\|_{\mathcal{L}_1}\|r\|_{\mathcal{L}_\infty})/(1-\lambda) \end{aligned} \tag{39}$$

$$\begin{aligned}\|u_r - u_d\|_{\mathcal{L}_\infty} \leq &(\|C(s)\theta^\top - K^\top\|_{\mathcal{L}_1} \\ &\|h_3\|_{\mathcal{L}_\infty})/(1-\lambda)\,, \end{aligned} \tag{40}$$

where $h_3(t)$ is the inverse Laplace transformation of

$$H_3(s) = (C(s)-1)C(s)r(s)k_g H_o(s)\theta^\top H_o(s). \quad (41)$$

Thus, we need to determine K and $C(s)$ such that

$$(i) \qquad \lambda \text{ or } \|h_3\|_{\mathcal{L}_\infty} \text{ are sufficiently small}, \quad (42)$$

$$(ii) \qquad y_d(s) \approx D(s)r(s)\,, \quad (43)$$

where $D(s)$ is the desired LTI system. It consequently implies that the output $y(t)$ of the system in (1) and its $\mathcal{L}_1$ adaptive control signal $u(t)$ will follow $y_d(t)$ and $u_d(t)$ both in transient and steady state with quantifiable bounds given in (25), (29) and (38)-(40). Thus, for the given desired specifications, one needs to ensure that (42) and (43) are satisfied, which in turn imply that the $\mathcal{L}_1$ adaptive controller controls a partially known system with satisfactory performance. We note that the minimization of λ in (42) is consistent with the stability requirement in (12), while the other requirements in (42) and (43) can be achieved via two different design methods: i) fix $C(s)$ and minimize $\|H_o(s)\|_{\mathcal{L}_1}$, ii) fix $H_o(s)$ and minimize the $\mathcal{L}_1$-gain of one of the cascaded systems $\|H_o(s)(C(s)-1)\|_{\mathcal{L}_1}$, $\|(C(s)-1)r(s)\|_{\mathcal{L}_1}$ or $\|C(s)(C(s)-1)\|_{\mathcal{L}_1}$ via the choice of $C(s)$. The important point to emphasize is that the requirements in (42) and (43) are *not* in conflict with each other.

Design Method 1. Set $C(s) = D(s)$. Then minimization of $\|H_o(s)\|_{\mathcal{L}_1}$ can be achieved via high-gain feedback by choosing K sufficiently large. However, minimized $\|H_o(s)\|_{\mathcal{L}_1}$ via large K leads to large poles of $H_o(s)$, which is typical for high-gain design methods. Since $C(s)$ is a strictly proper system containing the dominant poles of the closed-loop system in $k_g c^\top H_o(s)C(s)$ and $k_g c^\top H_o(0) = 1$, we have $k_g c^\top H_o(s)C(s) \approx C(s) = D(s)$. Hence, the system response $y_r(s) \approx D(s)r(s)$. We note that with large feedback K, $\mathcal{L}_1$ adaptive controller degenerates into a high-gain robust one. The shortcoming of this design is that the high gain feedback K leads to a reduced phase margin and affects robustness.

Design Method 2. As in MRAC, assume that we can select A_m to ensure that $k_g c^\top (s\mathbb{I}-A_m)^{-1}b \approx D(s)$. Then we can set $K = 0$. Or one can alternatively choose K to ensure $k_g c^\top H_o(s) \approx D(s)$. Let $C(s) = \omega/(s+\omega)$.

Lemma 4: For any single input n-output strictly proper stable system $H(s)$ the following is true: $\lim_{\omega\to\infty} \|(C(s)-1)H(s)\|_{\mathcal{L}_1} = 0$.

The proof is straightforward, and is therefore omitted. Lemma 4 states that if one chooses $k_g c^\top H_o(s)r(s) \approx D(s)$, then by increasing the bandwidth of the low-pass system $C(s)$, it is possible to render $\|\bar{G}(s)\|_{\mathcal{L}_1}$ arbitrarily small. With large ω, the pole $-\omega$ due to $C(s)$ is omitted, and $H_o(s)$ is the dominant reference system leading to $y_r(s) \approx k_g c^\top H_o(s)r(s) \approx D(s)r(s)$. We note that $k_g c^\top H_o(s)$ is exactly the reference model of the MRAC design. Therefore this approach is equivalent to mimicking MRAC, and, hence, high-gain feedback can be completely avoided.

However, increasing the bandwidth of $C(s)$ is not the only choice for minimizing $\|\bar{G}(s)\|_{\mathcal{L}_1}$. Since $C(s)$ is a low-pass filter, its complementary $1-C(s)$ is a high-pass filter with its cutoff frequency approximating the bandwidth of $C(s)$. Since both $H_o(s)$ and $C(s)$ are strictly proper systems, $\bar{G}(s) = H_o(s)(C(s)-1)$ is equivalent to cascading a low-pass system $H_o(s)$ with a high-pass system $C(s)-1$. If one chooses the cut-off frequency of $C(s)-1$ larger than the bandwidth of $H_o(s)$, it ensures that $\bar{G}(s)$ is a "no-pass" system, and hence its $\mathcal{L}_1$ gain can be rendered suitably small. This can be done via higher order filter design methods.

Next, consider the minimization of $\|h_3\|_{\mathcal{L}_\infty}$. We note that $\|h_3\|_{\mathcal{L}_\infty}$ can be upperbounded in two ways:

$$(i) \qquad \|h_3\|_{\mathcal{L}_\infty} \le \|(C(s)-1)r(s)\|_{\mathcal{L}_1}\|h_4\|_{\mathcal{L}_\infty}\,,$$

where $h_4(t)$ is the inverse Laplace transformation of $H_4(s) = C(s)k_g H_o(s)\theta^\top H_o(s)$, and

$$(ii) \qquad \|h_3\|_{\mathcal{L}_\infty} \le \|(C(s)-1)C(s)\|_{\mathcal{L}_1}\|h_5\|_{\mathcal{L}_\infty}\,,$$

where $h_5(t)$ is the inverse Laplace transformation of $H_5(s) = r(s)k_g H_o(s)\theta^\top H_o(s)$.

We note that since $r(t)$ is a bounded signal and $C(s), H_o(s)$ are stable proper systems, $\|h_4\|_{\mathcal{L}_\infty}$ and $\|h_5\|_{\mathcal{L}_\infty}$ are finite. Therefore, $\|h_3\|_{\mathcal{L}_\infty}$ can be minimized by minimizing $\|(C(s)-1)r(s)\|_{\mathcal{L}_1}$ or $\|(C(s)-1)C(s)\|_{\mathcal{L}_1}$.

First, consider minimizing $\|(C(s)-1)r(s)\|_{\mathcal{L}_1}$. Since $r(t)$ is usually in low-frequency range, one can choose the cut-off frequency of $C(s)-1$ to be larger than the bandwidth of the reference signal $r(t)$ to minimize $\|(C(s)-1)r(s)\|_{\mathcal{L}_1}$.

Second, consider minimizing $\|C(s)(C(s)-1)\|_{\mathcal{L}_1}$. If $C(s)$ is an ideal low-pass filter, it can be checked easily that $C(s)(C(s)-1) = 0$ and hence $\|h_3\|_{\mathcal{L}_\infty} = 0$. Although an ideal low-pass filter is not physically implementable, one can still minimize $\|C(s)(C(s)-1)\|_{\mathcal{L}_1}$ via the choice of the low-pass filter $C(s)$.

The above presented approaches ensure that $C(s) \approx 1$ in the bandwidth of $r(s)$ and $H_o(s)$. Therefore it follows from (35) that $y_d(s) = C(s)k_g c^\top H_o(s)r(s) \approx k_g c^\top H_o(s)r(s)$, which conseuquently implies that $y_d(s) \approx D(s)r(s)$.

Remark 3: From Corollary 1 and Lemma 3 it follows that the $\mathcal{L}_1$ adaptive controller can generate a system response to track (35) and (36) both in transient and steady state, if we set the adaptive gain large and minimize λ or $\|h_3\|_{\mathcal{L}_\infty}$. Notice that $u_d(t)$ in (36) depends upon the unknown parameter θ, while $y_d(t)$ in (35) does not. This implies that for different values of θ, the $\mathcal{L}_1$ adaptive controller will generate different control signals (dependent on θ) to ensure uniform system response (independent of θ). This is natural, since different unknown parameters imply different systems, and to have similar response for different systems the control signals have to be different. Here is the obvious advantage of the $\mathcal{L}_1$ adaptive controller in a sense that it controls a partially known system as an LTI feedback controller would have done if the unknown parameters were known. Finally, we note that if the term

$k_g C(s)C(s)\theta^\top H_o(s)$ is dominated by $k_g C(s)K^\top H_o(s)$, then the controller in (36) turns into a robust one, and $\mathcal{L}_1$ adaptive controller degenerates into robust design.

VI. Discussion

We use a scalar system to compare the performance of $\mathcal{L}_1$ and high-gain controllers. Towards that end, consider $\dot{x}(t) = \theta x(t) + u(t)$, where $x \in \mathbb{R}$ is the measurable system state, $u \in \mathbb{R}$ is the control signal and $\theta \in \mathbb{R}$ is unknown, which belongs to a given compact set $[\theta_{\min}, \theta_{\max}]$. Let $u(t) = -kx(t)$, leading to the following closed-loop system: $\dot{x}(t) = (\theta - k)x(t) + kr(t)$. We need to choose $k > \theta_{\max}$ to guarantee stability. We note that both the steady state error and transient performance depend on the unknown parameter value θ. By further introducing a proportional-integral controller, we can achieve zero steady error. When we choose $k \gg \max\{\theta_{\max}, \theta_{\min}\}$, we have $x(s) = \frac{k}{s-(\theta-k)} r(s) \approx \frac{k}{s+k} r(s)$, which leads to high-gain system. To apply the $\mathcal{L}_1$ adaptive controller, consider the following desired reference system: $D(s) = \frac{2}{s+2}$. Let $u_1 = -2x$, $k_g = 2$, leading to $H_o(s) = \frac{1}{s+2}$. Choose $C(s) = \frac{\omega_n}{s+\omega_n}$ with large ω_n, and set adaptive gain Γ_c large. Then it follows from Theorem 3 that

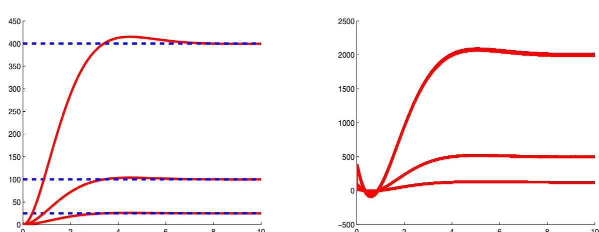
$$x(s) \approx x_r(s) \approx 2/(s+2)\,, \tag{44}$$

$$u(s) \approx u_r(s) = (-2+\theta)x_r(s) + 2r(s)\,. \tag{45}$$

The relationship in (44) implies that the control objective is met, while the relationship in (45) states that the $\mathcal{L}_1$ adaptive controller approximates $u_r(t)$, that cancels the unknown θ.

VII. Simulations

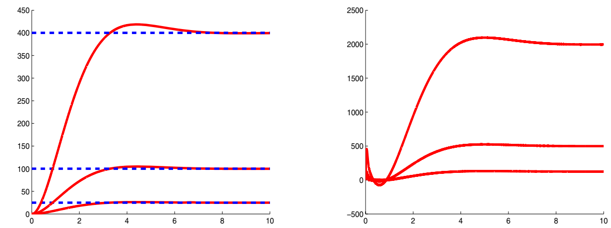
Consider the following system parameters: $A_m = \begin{bmatrix} 0 & 1 \\ -1 & -1.4 \end{bmatrix}$, $b = [0 \quad 1]^\top$, $c = [1 \quad 0]^\top$, where the unknown parameter $\theta = [4 \quad -4.5]^\top$ belongs to a known compact set: $\theta_i \in [-10, \ 10]$, $i = 1, 2$. We set $K = 0$ to avoid the linear feedback completely and present two different simulation scenarios for different choices of $C(s)$. Simulation results are in Figs. 2-3.



(a) $y(t)$ (solid) and $r(t)$ (dashed) (b) Control signal

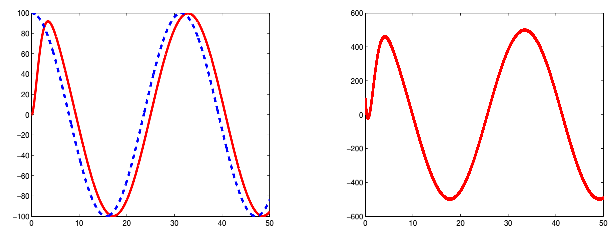
Fig. 2. Performance of $\mathcal{L}_1$ adaptive controller with $C(s) = \frac{160}{s+160}$, $\Gamma = 40000$ for $r = 25, \ 100, \ 400$ with uniform bound $\|y_{ref} - y_{des}\|_{\mathcal{L}_\infty} \leq 0.0946\|r\|_{\mathcal{L}_\infty}$, while $\lambda = 0.1725$.

It can be seen that the $\mathcal{L}_1$ adaptive controller leads to scaled control signal and scaled system response for



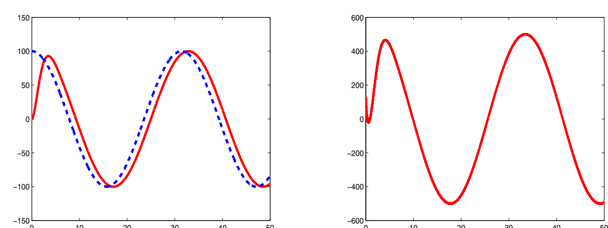
(a) $y(t)$ (solid) and $r(t)$ (dashed) (b) Control signal

Fig. 3. Performance of $\mathcal{L}_1$ adaptive controller with $C(s) = \frac{3\omega^2 s+\omega^3}{(s+\omega)^3}$, $\omega = 50, \Gamma = 400$ for $r = 25, \ 100, \ 400$ with uniform bound $\|y_{ref} - y_{des}\|_{\mathcal{L}_\infty} \leq 0.0721\|r\|_{\mathcal{L}_\infty}$, while $\lambda = 0.3984$.



(a) $y(t)$ (solid) and $r(t)$*(dashed)* (b) Time-history of $u(t)$

Fig. 4. $C(s) = \frac{160}{s+160}$, $\Gamma = 40000$ for $r = 100\cos(0.2t)$



(a) $y(t)$ (solid) and $r(t)$*(dashed)* (b) Time-history of $u(t)$

Fig. 5. $C(s) = \frac{350^2 s+50^3}{(s+50)^3}$, $\Gamma = 400$ for $r = 100\cos(0.2t)$

scaled reference inputs. We notice that in the second case higher order $C(s)$ leads to an improved bound with smaller bandwidth and smaller adaptive gain. While a rigorous relationship between the choice of adaptive gain and the bandwidth of the low-pass filter has not been derived at this stage, an insight into this can be gained from the following analysis. It follows from (3), (2) and (7) that

$$x(s) = k_g H_o(s)C(s)r(s) - H_o(s)\theta^\top x(s) + H_o(s)C(s)\bar{r}(s)\,,$$

while the companion model in (4) can be rewritten as

$$\hat{x}(s) = k_g H_o(s)C(s)r(s) + H_o(s)(C(s)-1)\bar{r}(s)\,.$$

We note that $\bar{r}(t)$ is divided into two parts. Its low-frequency component $C(s)\bar{r}(s)$ is what the system in (3) gets, while the complementary high-frequency component $(C(s)-1)\bar{r}(s)$ goes into the companion model. We recall that higher frequencies appear in $\bar{r}(t)$ in the presence of large adaptive gain. Therefore a first order $C(s)$ with large bandwidth achieves the desired performance with large adaptive gain. A higher order filter with smaller bandwidth and *reduced tailing effects* obtains similar performance with a smaller adaptive gain. Figs 4(a)-4(b), 5(a)-5(b) show the system response and control signal for reference input $r(t) = 100\cos(0.2t)$, without any retuning of the controller.

VIII. CONCLUSION

A novel $\mathcal{L}_1$ adaptive controller is developed that has guaranteed transient response in addition to stable tracking. The new low-pass control architecture tolerates high adaptation gains without generating high-frequency oscillations in the control signal and guarantees desired transient performance for both system's input and output signals. In [4], [5], the methodology is extended to systems with unknown time-varying parameters and bounded disturbances in the presence of unknown high-frequency gain, and stability margins are derived. These arguments enable development of theoretically justified tools for verification and validation of adaptive controllers.

REFERENCES

[1] K. Narendra, A. Annaswamy. *Stable Adaptive Systems*. Prentice Hall, 1989.

[2] J.-J. Slotine, W. Li. *Applied Nonlinear Control*. Prentice Hall, 1991.

[3] C. Cao and N. Hovakimyan. Design and analysis of a novel $\mathcal{L}_1$ adaptive controller, Part I: Control signal and asymptotic stability. *Americal Control Conference*, 2006.

[4] C. Cao and N. Hovakimyan. Guaranteed transient performance with $\mathcal{L}_1$ adaptive controller for systems with unknown time-varying parameters: Part I. *Conf. on Decision and Contr.*, Submitted 2006.

[5] C. Cao and N. Hovakimyan. Stability margins of $\mathcal{L}_1$ adaptive controller: Part II. *Conf. on Decision and Contr.*, Submitted 2006.

APPENDIX

Proof of Theorem 2: Let

$$r_2(s) = (\mathbb{I} + (\mathbb{I} - \bar{G}(s)\theta^\top)^{-1} (\bar{G}(s)\theta^\top + (C(s)-1)\mathbb{I}))\tilde{x}(s). \qquad (46)$$

It follows from (22) that $r_2(t) = x_r(t) - x(t)$. The signal $r_2(t)$ can be viewed as the response of the LTI system

$$H_2(s) = \mathbb{I} + (\mathbb{I} - \bar{G}(s)\theta^\top)^{-1}(\bar{G}(s)\theta^\top + (C(s)-1)\mathbb{I}) \qquad (47)$$

to the bounded error signal $\tilde{x}(t)$. It follows from (19) that $(\mathbb{I} - \bar{G}(s)\theta^\top)^{-1}$, $\bar{G}(s)$, $C(s)$ are stable and, therefore, $H_2(s)$ is stable. Hence, from (16) we have $\lim_{t\to\infty} r_2(t) = 0$, which confirms (24).

Let

$$r_3(s) = C(s)r_1(s) + (C(s)\theta^\top - K^\top)(x(s) - x_r(s)). \qquad (48)$$

It follows from (23) that $r_3(t) = u(t) - u_r(t)$. Since $\tilde{\theta}(t)$ is bounded, it follows from (14) and (16) that

$$\lim_{t\to\infty} r_1(t) = 0. \qquad (49)$$

Since $C(s)$ is a stable proper system, it follows from (24), (48) and (49) that $\lim_{t\to\infty} r_3(t) = 0$, which confirms (25). □

Proof of Lemma 2: Since $y_r(t) = c^\top x_r(t)$, it follows from (24) that $\lim_{t\to\infty}(y(t) - y_r(t)) = 0$. It follows from (18) that $y_r(s) = c^\top(\mathbb{I} - \bar{G}(s)\theta^\top)^{-1}G(s)r(s)$, and hence for a constant r, the end value theorem ensures $\lim_{t\to\infty} y_r(t) = \lim_{s\to 0} c^\top(\mathbb{I} - \bar{G}(s)\theta^\top)^{-1}G(s)r = c^\top H_o(0)C(0)k_g r$. The definition of k_g and the fact that $C(0) = 1$ lead to $\lim_{t\to\infty} y(t) = r$. □

Proof of Theorem 3: It follows from (46) and Corollary 1 in [3] that $\|r_2\|_{\mathcal{L}_\infty} \leq \|H_2(s)\|_{\mathcal{L}_1}\|\tilde{x}\|_{\mathcal{L}_\infty}$. It follows from (13), (14), (15) that

$$\|\tilde{x}\|_{\mathcal{L}_\infty} \leq \sqrt{(\bar{\theta}_{\max})/(\lambda_{\max}(P_o)\Gamma_c)}\,. \qquad (50)$$

Therefore, $\|r_2\|_{\mathcal{L}_\infty} \leq \|H_2(s)\|_{\mathcal{L}_1}\sqrt{\frac{\bar{\theta}_{\max}}{\lambda_{\max}(P_o)\Gamma_c}}$, which along with (30) leads to (27). The upper bound in (28) follows from (27) and Lemma 2 in [3] directly. From (21), we have $r_3(s) = C(s)\frac{1}{c_o^\top H_o(s)}c_o^\top H_o(s)r_1(s) + (C(s)\theta^\top - K^\top)(x(s) - x_r(s)) = -C(s)\frac{1}{c_o^\top H_o(s)}c_o^\top\tilde{x}(s) + (C(s)\theta^\top - K^\top)(x(s) - x_r(s))$, where c_o is introduced in (26). It follows from (26) that $C(s)\frac{1}{c_o^\top H_o(s)} = C(s)\frac{N_d(s)}{N_n(s)}$, where $N_d(s)$, $N_n(s)$ are stable polynomials and the order of $N_n(s)$ is one less than the order of $N_d(s)$. Since $C(s)$ is stable and strictly proper, the complete system $C(s)\dfrac{1}{c_o^\top H_o(s)}$ is proper and stable, which implies that its $\mathcal{L}_1$ gain exists and is finite. Thus, $\|r_3\|_{\mathcal{L}_\infty} \leq \left\|C(s)\frac{1}{c_o^\top H_o(s)}c_o^\top\right\|_{\mathcal{L}_1}\|\tilde{x}\|_{\mathcal{L}_\infty} + \|C(s)\theta^\top - K^\top\|_{\mathcal{L}_1}\|x - x_r\|_{\mathcal{L}_\infty}$, which with (27), (50) leads to (29). □

Proof of Lemma 3: It follows from (18) that $y_r(s) = c^\top(\mathbb{I} - \bar{G}(s)\theta^\top)^{-1}G(s)r(s)$. Following Lemma 1, the condition in (12) ensures stability of the reference system. Since $(\mathbb{I} - \bar{G}(s)\theta^\top)^{-1}$ is stable, then one can expand it into convergent series:

$$\begin{aligned} y_r(s) &= c^\top(\mathbb{I} + \sum_{i=1}^{\infty}(\bar{G}(s)\theta^\top)^i)G(s)r(s) = y_d(s) \\ &\quad + c^\top(\sum_{i=1}^{\infty}(\bar{G}(s)\theta^\top)^i)G(s)r(s). \end{aligned} \qquad (51)$$

Let $r_4(s) = c^\top(\sum_{i=1}^{\infty}(\bar{G}(s)\theta^\top)^i)G(s)r(s)$. Then $r_4(t) = y_r(t) - y_d(t)$. It follows from Lemma 2 in [3] that

$$\begin{aligned} \|r_4\|_{\mathcal{L}_\infty} &\leq (\sum_{i=1}^{\infty}\lambda^i)\|c^\top\|_{\mathcal{L}_1}\|G(s)\|_{\mathcal{L}_1}\|r\|_{\mathcal{L}_\infty} \\ &= \frac{\lambda}{1-\lambda}\|c^\top\|_{\mathcal{L}_1}\|G(s)\|_{\mathcal{L}_1}\|r\|_{\mathcal{L}_\infty}. \end{aligned} \qquad (52)$$

Using (9), (10) and (41), from (51) one can derive

$$y_r(s) = y_d(s) + c^\top(\sum_{i=1}^{\infty}(\bar{G}(s)\theta^\top)^{i-1})H_3(s),$$

upon which Corollary 1 in [3] leads to (38). Comparing $u_d(s)$ in (36) to $u_r(s)$ in (17) it follows that $u_d(s)$ can be written as $u_d(s) = k_g C(s)r(s) + (C(s)\theta^\top - K^\top)x_d(s)$, where $x_d(s) = C(s)k_g H_o(s)r(s)$. Therefore $u_r(s) - u_d(s) = (C(s)\theta^\top - K^\top)(x_r(s) - x_d(s))$. Hence, it follows from Lemma 1 in [3]

$$\|u_r - u_d\|_{\mathcal{L}_\infty} = \|C(s)\theta^\top - K^\top\|_{\mathcal{L}_1}\|x_r - x_d\|_{\mathcal{L}_\infty}.$$

Using the same steps as for $\|y_r - y_d\|_{\mathcal{L}_\infty}$, we have

$$\begin{aligned} \|x_r - x_d\|_{\mathcal{L}_\infty} &\leq \lambda\|G(s)\|_{\mathcal{L}_1}\|r\|_{\mathcal{L}_\infty}/(1-\lambda), \\ \|x_r - x_d\|_{\mathcal{L}_\infty} &\leq \|h_3\|_{\mathcal{L}_\infty}/(1-\lambda)\,, \end{aligned}$$

and hence (39) and (40) are proved. □