
Immersion and Invariance

[1] describe immersion and invariance as a new tool for stabilization and adaptive control of nonlinear systems. This work note have applied some the tools to a surface vessel which is affected by a external disturbance and uncertainties in the damping matrix and on the control input.

System

The 3 DOF dynamics of a surface vessel can be stated as:

$$\dot{\boldsymbol{\eta}} = \mathbf{R}(\psi)\boldsymbol{\nu} \quad (1)$$

$$\mathbf{M}\dot{\boldsymbol{\nu}} + \mathbf{C}(\boldsymbol{\nu})\boldsymbol{\nu} + \mathbf{D}(\boldsymbol{\nu})\boldsymbol{\nu} = \boldsymbol{\tau} + \mathbf{R}^\top(\psi)\mathbf{w}, \quad (2)$$

where

$$\mathbf{R}(\psi) = \begin{bmatrix} \cos(\psi) & -\sin(\psi) & 0 \\ \sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (3)$$

is a rotation matrix $\mathbf{R} \in SO(3)$, and where \mathbf{M} , $\mathbf{C}(\boldsymbol{\nu})$, $\mathbf{D}(\boldsymbol{\nu})$, $\boldsymbol{\tau}$ and \mathbf{w} represent the real inertia matrix, Coriolis and centripetal matrix, damping matrix, control input vector and disturbance vector, respectively. Here, the system matrices are assumed to satisfy the properties $\mathbf{M} = \mathbf{M}^\top > 0$, $\mathbf{C}(\boldsymbol{\nu}) = -\mathbf{C}(\boldsymbol{\nu})^\top$ and $\mathbf{D}(\boldsymbol{\nu}) > 0$.

I & I adaptive control with only uncertainties in the damping matrix

We consider the dynamic ship model (1) and (2) which can be rewritten as

$$\dot{\boldsymbol{\eta}} = \mathbf{R}(\psi)\boldsymbol{\nu} \quad (4)$$

$$\mathbf{M}\dot{\boldsymbol{\nu}} = \boldsymbol{\tau} - \mathbf{C}(\boldsymbol{\nu})\boldsymbol{\nu} + \mathbf{g}(\boldsymbol{\nu})\boldsymbol{\nu} + \boldsymbol{\Phi}(\boldsymbol{\nu})\boldsymbol{\varphi}. \quad (5)$$

Here, $\mathbf{g}(\boldsymbol{\nu})\boldsymbol{\nu}$ is the know part of $-\mathbf{D}(\boldsymbol{\nu})\boldsymbol{\nu}$ and

$$\boldsymbol{\varphi} = [Y_{|r|v}, Y_r, Y_{|v|r}, Y_{|r|r}, N_{|r|v}, N_r, N_{|v|r}, N_{|r|r}]^\top \quad (6)$$

$$\boldsymbol{\Phi}(\boldsymbol{\nu}) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ |r|v & r & |v|r & |r|r & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & |r|v & r & |v|r & |r|r \end{bmatrix} \quad (7)$$

Start by defining the error variables \mathbf{z}_1 and \mathbf{z}_2 :

$$\mathbf{z}_1 \triangleq \mathbf{R}^\top(\psi)(\boldsymbol{\eta} - \boldsymbol{\eta}_d) \quad (8)$$

$$\mathbf{z}_2 \triangleq \boldsymbol{\nu} - \boldsymbol{\alpha}, \quad (9)$$

where $\boldsymbol{\alpha} \in \mathbb{R}^3$ is a vector of stabilising functions to be designed.

The stabilising function can now be chosen as

$$\boldsymbol{\alpha} = \mathbf{R}^\top(\psi)\dot{\boldsymbol{\eta}}_d - \mathbf{K}_1\mathbf{z}_1, \quad (10)$$

where $\mathbf{K}_1 > 0$.

Let the estimation error \mathbf{s} be defined as

$$\mathbf{s} = \hat{\boldsymbol{\varphi}} - \boldsymbol{\varphi} + \boldsymbol{\beta}(\mathbf{z}) \quad (11)$$

After differentiating (11) with respect to time, the dynamics of \mathbf{s} is given by

$$\dot{\mathbf{s}} = \dot{\hat{\boldsymbol{\varphi}}} + \frac{\partial \boldsymbol{\beta}}{\partial \mathbf{z}_1^\top} \dot{\mathbf{z}}_1 + \frac{\partial \boldsymbol{\beta}}{\partial \mathbf{z}_2^\top} \dot{\mathbf{z}}_2 \quad (12)$$

$$= \dot{\hat{\boldsymbol{\varphi}}} + \frac{\partial \boldsymbol{\beta}}{\partial \mathbf{z}_1^\top} \dot{\mathbf{z}}_1 + \frac{\partial \boldsymbol{\beta}}{\partial \mathbf{z}_2^\top} (\boldsymbol{\tau} - \mathbf{C}(\boldsymbol{\nu})\boldsymbol{\nu} + \mathbf{g}(\boldsymbol{\nu})\boldsymbol{\nu} + \boldsymbol{\Phi}(\boldsymbol{\nu})(\hat{\boldsymbol{\varphi}} + \boldsymbol{\beta}(\mathbf{z}) - \mathbf{s}) - \mathbf{M}\dot{\boldsymbol{\alpha}}) \quad (13)$$

Based on [2], the adaptive state feedback update law $\hat{\boldsymbol{\varphi}}$ and control input $\boldsymbol{\tau}$ are designed as follows:

$$\dot{\hat{\boldsymbol{\varphi}}} = -\frac{\partial \boldsymbol{\beta}}{\partial \mathbf{z}_1^\top} \dot{\mathbf{z}}_1 - \frac{\partial \boldsymbol{\beta}}{\partial \mathbf{z}_2^\top} (\boldsymbol{\tau} - \mathbf{C}(\boldsymbol{\nu})\boldsymbol{\nu} + \mathbf{g}(\boldsymbol{\nu})\boldsymbol{\nu} + \boldsymbol{\Phi}(\boldsymbol{\nu})(\hat{\boldsymbol{\varphi}} + \boldsymbol{\beta}(\mathbf{z})) - \mathbf{M}\dot{\boldsymbol{\alpha}}) \quad (14)$$

$$\boldsymbol{\tau} = -\mathbf{K}_2 \mathbf{z}_2 - \mathbf{z}_1 + \mathbf{C}(\boldsymbol{\nu})\boldsymbol{\nu} - \mathbf{g}(\boldsymbol{\nu})\boldsymbol{\nu} - \boldsymbol{\Phi}(\boldsymbol{\nu})(\hat{\boldsymbol{\varphi}} + \boldsymbol{\beta}(\mathbf{z})) + \mathbf{M}\dot{\boldsymbol{\alpha}} \quad (15)$$

Based on [3] we select

$$\frac{\partial \boldsymbol{\beta}}{\partial \mathbf{z}_1^\top} = 0 \quad (16)$$

$$\frac{\partial \boldsymbol{\beta}}{\partial \mathbf{z}_2^\top} = \boldsymbol{\Gamma} \boldsymbol{\Phi}(\boldsymbol{\nu})^\top \quad (17)$$

According to this the adaptive state feedback update law $\hat{\boldsymbol{\varphi}}$ and control input $\boldsymbol{\tau}$ for the system are

$$\dot{\hat{\boldsymbol{\varphi}}} = -\boldsymbol{\Gamma} \boldsymbol{\Phi}(\boldsymbol{\nu})^\top \left(\boldsymbol{\tau} - \mathbf{C}(\boldsymbol{\nu})\boldsymbol{\nu} + \mathbf{g}(\boldsymbol{\nu})\boldsymbol{\nu} + \boldsymbol{\Phi}(\boldsymbol{\nu})(\hat{\boldsymbol{\varphi}} + \boldsymbol{\Gamma} \boldsymbol{\Phi}(\boldsymbol{\nu})^\top \mathbf{z}_2) - \mathbf{M}\dot{\boldsymbol{\alpha}} \right) \quad (18)$$

$$\boldsymbol{\tau} = -\mathbf{K}_2 \mathbf{z}_2 - \mathbf{z}_1 + \mathbf{C}(\boldsymbol{\nu})\boldsymbol{\nu} - \mathbf{g}(\boldsymbol{\nu})\boldsymbol{\nu} - \boldsymbol{\Phi}(\boldsymbol{\nu})(\hat{\boldsymbol{\varphi}} + \boldsymbol{\Gamma} \boldsymbol{\Phi}(\boldsymbol{\nu})^\top \mathbf{z}_2) + \mathbf{M}\dot{\boldsymbol{\alpha}}. \quad (19)$$

Substituting (19) into (18) and (13), they become

$$\dot{\hat{\boldsymbol{\varphi}}} = -\boldsymbol{\Gamma} \boldsymbol{\Phi}(\boldsymbol{\nu})^\top (-\mathbf{K}_2 \mathbf{z}_2 - \mathbf{z}_1) \quad (20)$$

$$\begin{aligned} \dot{\mathbf{s}} &= \dot{\hat{\boldsymbol{\varphi}}} + \boldsymbol{\Gamma} \boldsymbol{\Phi}(\boldsymbol{\nu})^\top (-\mathbf{K}_2 \mathbf{z}_2 - \mathbf{z}_1 - \boldsymbol{\Phi}(\boldsymbol{\nu})\mathbf{s}) \\ &= -\boldsymbol{\Gamma} \boldsymbol{\Phi}(\boldsymbol{\nu})^\top \boldsymbol{\Phi}(\boldsymbol{\nu})\mathbf{s} \end{aligned} \quad (21)$$

Choosing a positive definite control Lyapunov functions (CLF)

$$V_1 = \frac{1}{2} \mathbf{z}_1^\top \mathbf{z}_1 + \frac{1}{2} \mathbf{z}_2^\top \mathbf{M} \mathbf{z}_2 + \frac{1}{2} \mathbf{s}^\top \mathbf{s} \quad (22)$$

the derivative of V_1 with respect to (w.r.t) time along the dynamics gives

$$\begin{aligned} \dot{V}_1 &= \mathbf{z}_1^\top \dot{\mathbf{z}}_1 + \mathbf{z}_2^\top \mathbf{M} \dot{\mathbf{z}}_2 + \mathbf{s}^\top \dot{\mathbf{s}} \\ &= -\mathbf{z}_1^\top \mathbf{K}_1 \mathbf{z}_1 - \mathbf{z}_2^\top (\mathbf{K}_2 \mathbf{z}_2 + \boldsymbol{\Phi}(\boldsymbol{\nu})\mathbf{s}) - \mathbf{s}^\top \boldsymbol{\Gamma} \boldsymbol{\Phi}(\boldsymbol{\nu})^\top \boldsymbol{\Phi}(\boldsymbol{\nu})\mathbf{s} \end{aligned} \quad (23)$$

$$\dot{V}_1 \leq -\mathbf{z}_1^\top \mathbf{K}_1 \mathbf{z}_1 - \frac{1}{2} \mathbf{z}_2^\top \mathbf{K}_2 \mathbf{z}_2 - \frac{1}{2} \mathbf{s}^\top \boldsymbol{\Gamma} \boldsymbol{\Phi}(\boldsymbol{\nu})^\top \boldsymbol{\Phi}(\boldsymbol{\nu})\mathbf{s} < 0 \quad \forall \mathbf{z}_1 \wedge \mathbf{z}_2 \wedge \mathbf{s} \neq 0. \quad (24)$$

Comparing the I & I stability result with the stability result of an adaptive backstepping controller [4] which is

$$V_2 = \frac{1}{2} \mathbf{z}_1^\top \mathbf{z}_1 + \frac{1}{2} \mathbf{z}_2^\top \mathbf{M} \mathbf{z}_2 + \frac{1}{2} \tilde{\boldsymbol{\varphi}}^\top \tilde{\boldsymbol{\varphi}}. \quad (25)$$

And the derivative of V_2 with respect to (w.r.t) time along the dynamics gives

$$\begin{aligned} \dot{V}_2 &= \mathbf{z}_1^\top \dot{\mathbf{z}}_1 + \mathbf{z}_2^\top \mathbf{M} \dot{\mathbf{z}}_2 + \tilde{\boldsymbol{\varphi}}^\top \dot{\tilde{\boldsymbol{\varphi}}} \\ &= -\mathbf{z}_1^\top \mathbf{K}_1 \mathbf{z}_1 - \mathbf{z}_2^\top (\mathbf{K}_2 \mathbf{z}_2 + \boldsymbol{\Phi}(\boldsymbol{\nu}) \tilde{\boldsymbol{\varphi}}) + \tilde{\boldsymbol{\varphi}}^\top \boldsymbol{\Gamma} \boldsymbol{\Phi}(\boldsymbol{\nu})^\top \mathbf{z}_2 \end{aligned} \quad (26)$$

$$= -\mathbf{z}_1^\top \mathbf{K}_1 \mathbf{z}_1 - \mathbf{z}_2^\top \mathbf{K}_2 \mathbf{z}_2. \quad (27)$$

Consult [5] for more details about the control algorithm and stability analysis.

Comparing Performance between I & I and AB

The control and adaptation gain are chosen as $\mathbf{K}_1 = \text{diag}([0.05, 0.05, 0.22])$, $\mathbf{K}_2 = \text{diag}([5, 7, 15])$ and $\boldsymbol{\Gamma} = \text{diag}([8, 4, 8, 8, 8, 4, 8, 8])$.

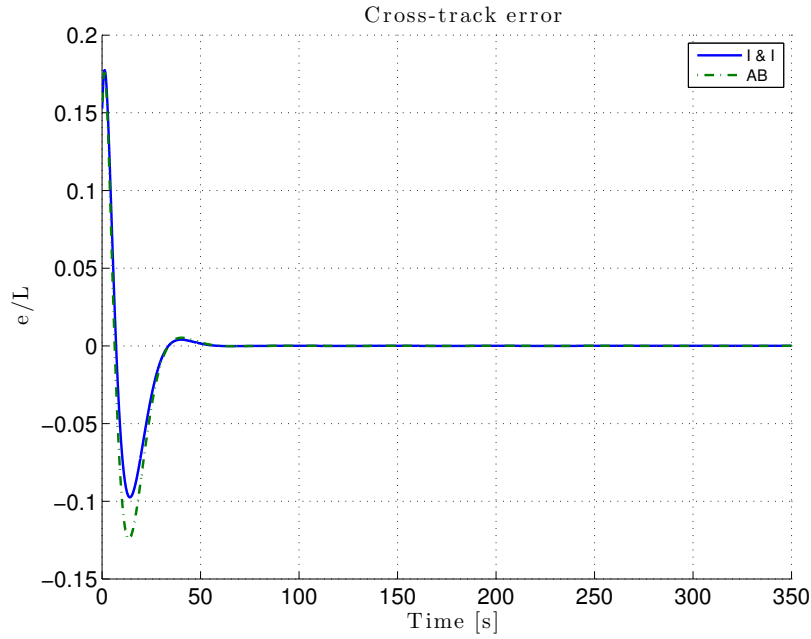


Figure 1: The cross-track error scaled by the vessel length, in the straight-line motion scenario

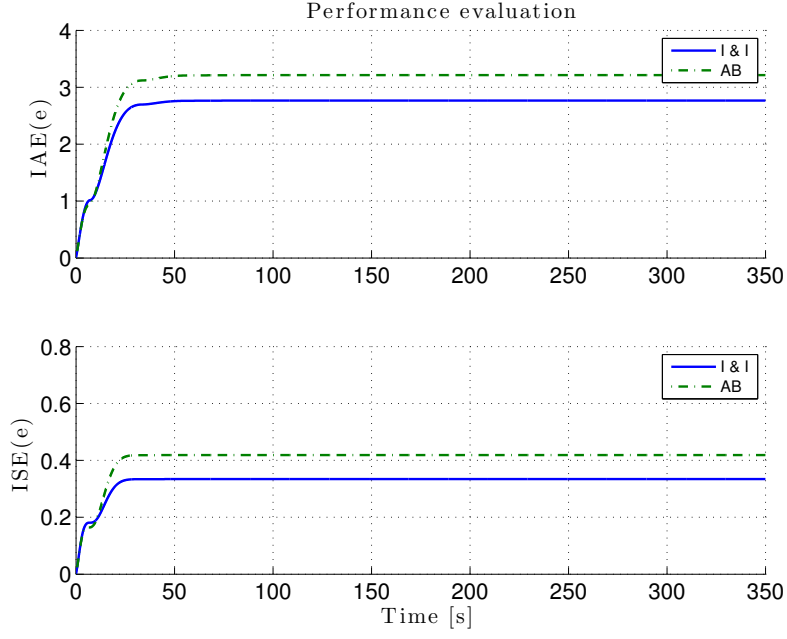


Figure 2: The IAE and ISE of the cross-track error in the straight-line motion scenario

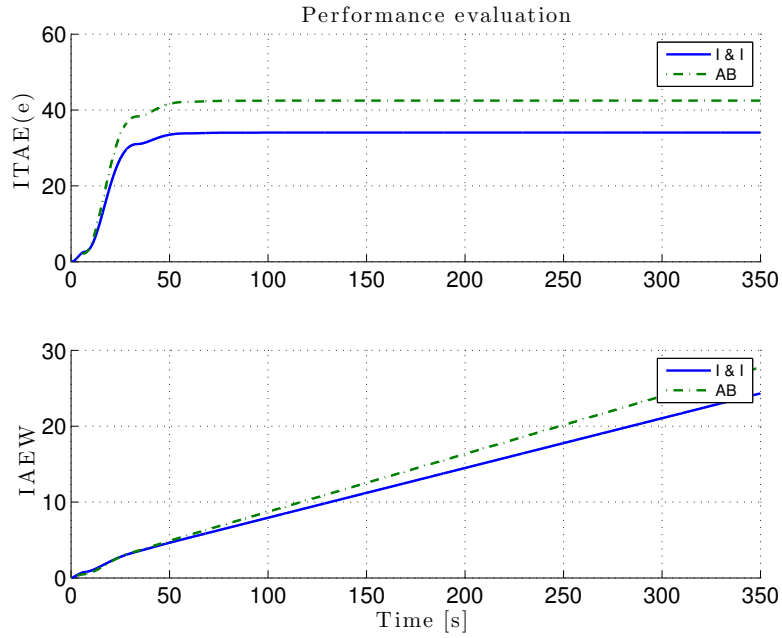


Figure 3: The ITAE and IAEW of the cross-track error in the straight-line motion scenario

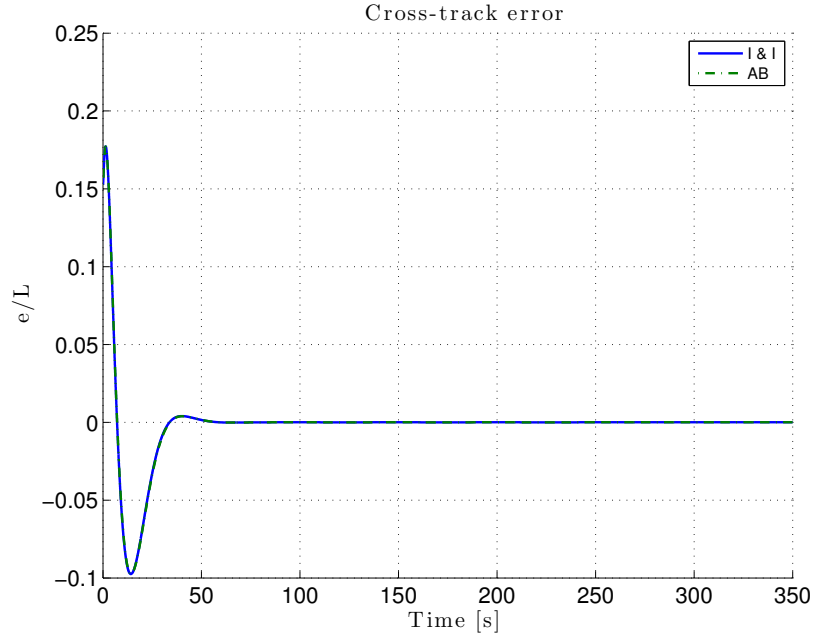


Figure 4: The cross-track error scaled by the vessel length, in the straight-line motion scenario

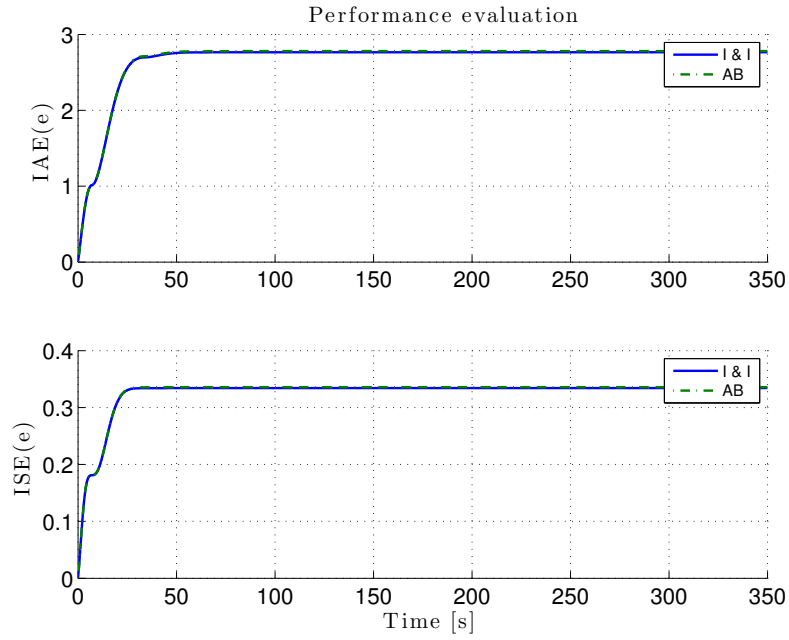


Figure 5: The IAE and ISE of the cross-track error in the straight-line motion scenario

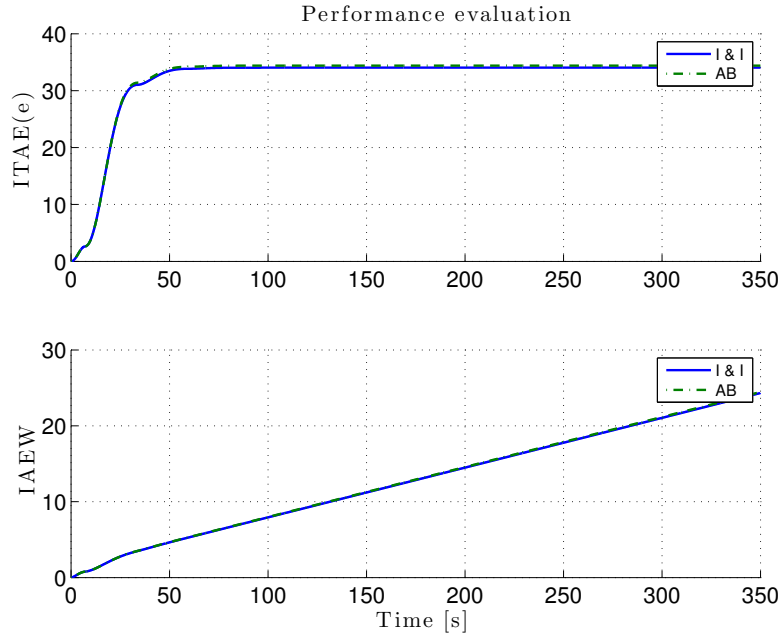


Figure 6: The ITAE and IAEW of the cross-track error in the straight-line motion scenario

Comparing Performance between I & I and AB with disturbance

$$\boldsymbol{\varphi} = [Y_{|r|v}, Y_r, Y_{|v|r}, Y_{|r|r}, N_{|r|v}, N_r, N_{|v|r}, N_{|r|r}, w_x^*, w_y^*, w_\psi^*]^\top \quad (28)$$

$$\boldsymbol{\Phi}(\boldsymbol{\nu}, \boldsymbol{\eta}) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cos(\psi) & \sin(\psi) & 0 \\ |r|v & r & |v|r & |r|r & 0 & 0 & 0 & 0 & -\sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 0 & 0 & |r|v & r & |v|r & |r|r & 0 & 0 & 1 \end{bmatrix} \quad (29)$$

The control and adaptation gain are chosen as $\mathbf{K}_1 = \text{diag}([0.05, 0.05, 0.22])$, $\mathbf{K}_2 = \text{diag}([5, 7, 15])$ and $\boldsymbol{\Gamma} = \text{diag}([8, 4, 8, 8, 8, 8, 4, 8, 8, 0.4, 0.4, 0.4])$.

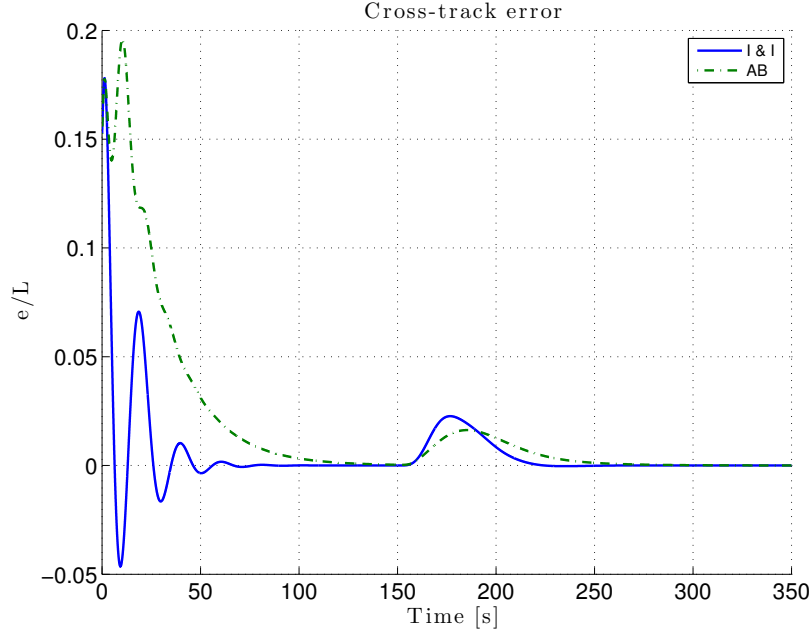


Figure 7: The cross-track error scaled by the vessel length, in the straight-line motion scenario

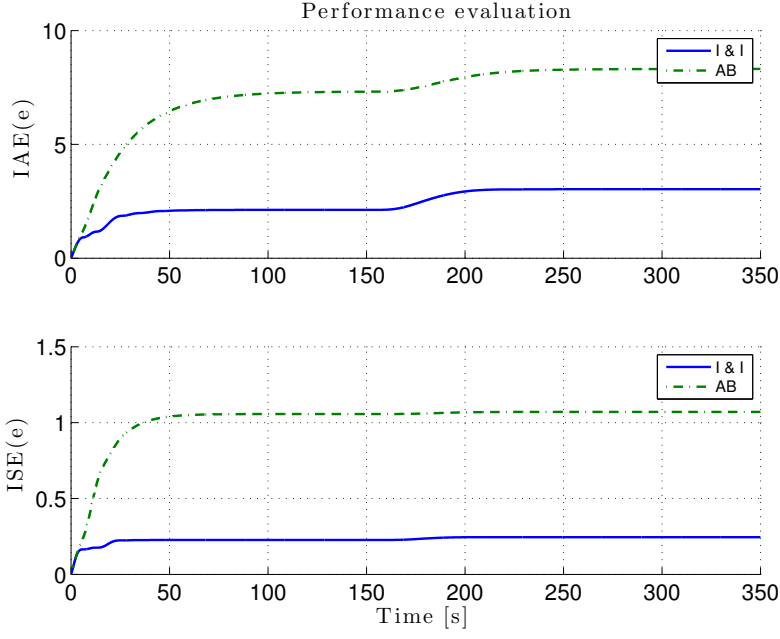


Figure 8: The IAE and ISE of the cross-track error in the straight-line motion scenario

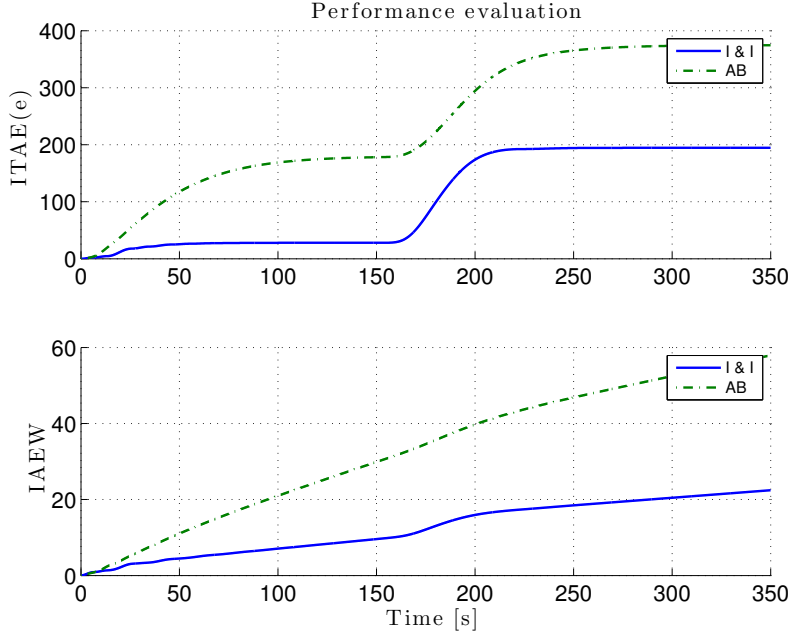


Figure 9: The ITAE and IAEW of the cross-track error in the straight-line motion scenario

Unknown Control Gain

Now, we consider the system

$$\dot{\eta} = \mathbf{R}(\psi)\boldsymbol{\nu} \quad (30)$$

$$M\dot{\boldsymbol{\nu}} = \theta\boldsymbol{\tau} - \mathbf{C}(\boldsymbol{\nu})\boldsymbol{\nu} + \mathbf{g}(\boldsymbol{\nu})\boldsymbol{\nu} + \boldsymbol{\Phi}(\boldsymbol{\nu}, \eta)\boldsymbol{\varphi}. \quad (31)$$

where $\theta \in \mathbb{R}$ and suppose that the sign of θ is known. Without loss of generality suppose that $\theta > 0$. The adaptation law and control from [5] and [6] are given as

$$\begin{aligned}\dot{\hat{\theta}} &= \left(\mathbf{I} + \frac{\partial \beta}{\partial \hat{\theta}} \right)^{-1} \left(\frac{\partial \beta}{\partial \mathbf{z}_1} \dot{\mathbf{z}}_1 + \frac{\partial \beta}{\partial \mathbf{z}_2} \dot{\mathbf{z}}_2 \right) \\ \tau &= -(\hat{\theta} + \beta_2(\mathbf{z}, \hat{\varphi}))(\mathbf{K}_2 \mathbf{z}_2 + \mathbf{z}_1 - \mathbf{C}(\nu)\nu + \mathbf{g}(\nu)\nu + \Phi(\nu, \eta)(\hat{\varphi} + \beta_1(\mathbf{z})) - M\dot{\alpha})\end{aligned}$$

where $\hat{\theta} = [\hat{\varphi}^\top, \hat{\theta}]^\top$ and $\beta(\cdot) = [\beta_1(\cdot)^\top, \beta_2(\cdot)^\top]^\top$, with

$$\beta_1(\mathbf{z}) = \Gamma_1 \int_0^{\mathbf{z}_2} \Phi(\mathbf{z}_1, \xi) d\xi \quad (32)$$

$$\beta_2(\mathbf{z}, \hat{\varphi}) = \gamma_2 \left(\frac{1}{2} \mathbf{z}_2^\top \mathbf{K}_2 \mathbf{z}_2 + \mathbf{z}_1^\top \mathbf{z}_2 \right) + \gamma_2 \int_0^{\mathbf{z}_2} \Phi(\mathbf{z}_1, \xi)^\top (\hat{\varphi} + \beta_1(\mathbf{z}_1, \xi)) d\xi, \quad (33)$$

where $\mathbf{K}_2 > 0$, $\Gamma_1 > 0$ and $\gamma_2 > 0$ and constant. Selecting

$$\beta_1(\mathbf{z}) = \Gamma_1 \Phi(\nu)^\top \mathbf{z}_2 \quad (34)$$

$\beta_2(\mathbf{z}, \hat{\varphi})$ becomes

$$\beta_2(\mathbf{z}, \hat{\varphi}) = \gamma_2 \left(\frac{1}{2} \mathbf{z}_2^\top \mathbf{K}_2 \mathbf{z}_2 + \mathbf{z}_1^\top \mathbf{z}_2 + \hat{\varphi}^\top \Phi(\nu, \eta)^\top \mathbf{z}_2 + \frac{1}{2} \mathbf{z}_2^\top \Phi(\nu, \eta)^\top \Gamma_1 \Phi(\nu) \mathbf{z}_2 \right) \quad (35)$$

From this the adaptation law become

$$\dot{\hat{\theta}} = \left(\mathbf{I} + \begin{bmatrix} \mathbf{0}_{11 \times 11} & \mathbf{0}_{11 \times 1} \\ \mathbf{z}_2^\top \Phi(\nu, \eta) & 0 \end{bmatrix} \right)^{-1} \begin{bmatrix} \Gamma_1 \Phi(\nu)^\top (-\mathbf{K}_2 \mathbf{z}_2 - \mathbf{z}_1) \\ \gamma_2 (\Delta(\cdot)) \end{bmatrix}, \quad (36)$$

where

$$\Delta(\cdot) = \mathbf{z}_2^\top \dot{\mathbf{z}}_1 + \mathbf{z}_2^\top \mathbf{K}_2 \dot{\mathbf{z}}_2 + \mathbf{z}_1^\top \dot{\mathbf{z}}_2 + \hat{\varphi}^\top \Phi(\nu, \eta)^\top \dot{\mathbf{z}}_2 + \mathbf{z}_2^\top \Phi(\nu, \eta)^\top \Gamma_1 \Phi(\nu) \dot{\mathbf{z}}_2 \quad (37)$$

Comparing Performance between I & I and AB with disturbance and Unknown Control Gain

$$\varphi = [Y_{|r|v}, Y_r, Y_{|v|r}, Y_{|r|r}, N_{|r|v}, N_r, N_{|v|r}, N_{|r|r}, w_x^*, w_y^*, w_\psi^*]^\top \quad (38)$$

$$\Phi(\nu, \eta) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cos(\psi) & \sin(\psi) & 0 \\ |r|v & r & |v|r & |r|r & 0 & 0 & 0 & 0 & -\sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 0 & 0 & |r|v & r & |v|r & |r|r & 0 & 0 & 1 \end{bmatrix} \quad (39)$$

The control and adaptation gain are chosen as $\mathbf{K}_1 = \text{diag}([0.05, 0.05, 0.22])$, $\mathbf{K}_2 = \text{diag}([5, 7, 15])$, $\Gamma_1 = \text{diag}([8, 4, 8, 8, 8, 4, 8, 8, 0.4, 0.4, 0.4])$, $\theta = 0.4$ and $\gamma_2 = 2$.

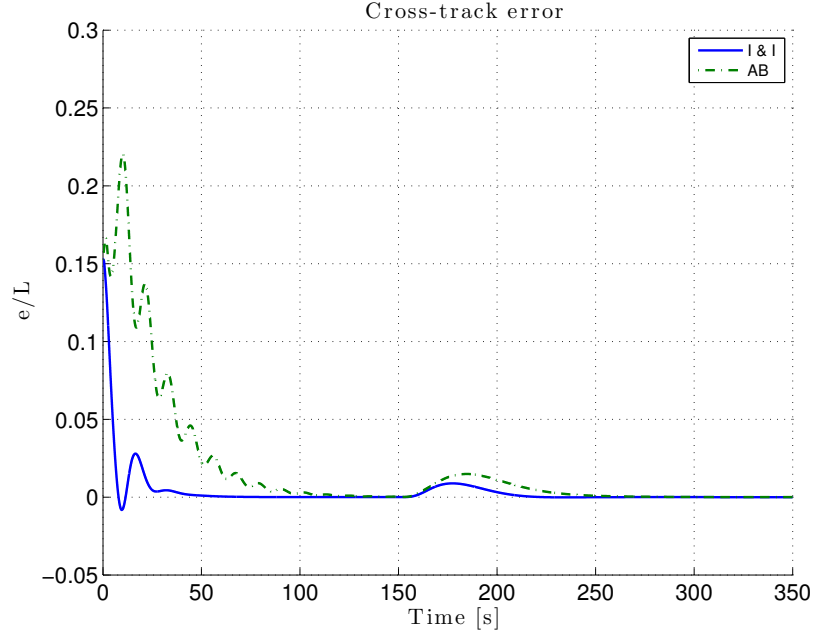


Figure 10: The cross-track error scaled by the vessel length, in the straight-line motion scenario

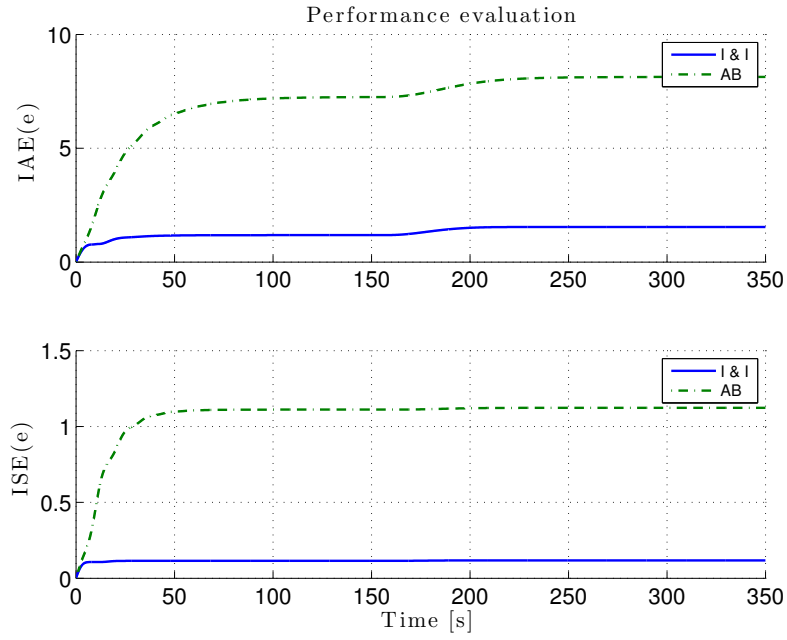


Figure 11: The IAE and ISE of the cross-track error in the straight-line motion scenario

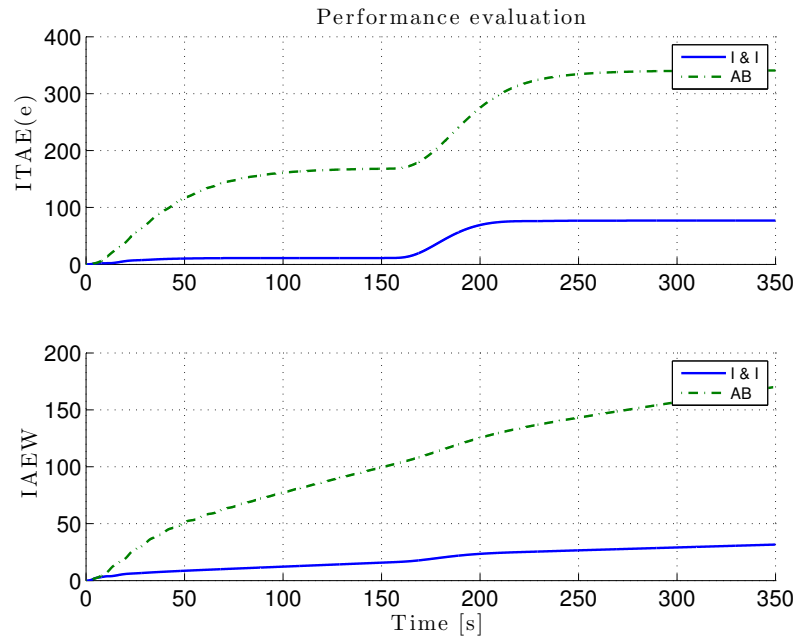


Figure 12: The ITAE and IAEW of the cross-track error in the straight-line motion scenario

References

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- [6] D. Karagiannis and A. Astolfi, “Adaptive state feedback design via immersion and invariance,” *In Proceeding of the 9th European Control Conference, Kos, Greece*, 2007.