L_1-Adaptive Control: Stability, Robustness, and Interpretations

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***Abstract*—An adaptive control design approach that involves the insertion of a strictly proper stable filter at the input of standard Model Reference Adaptive Control (MRAC) schemes has been proposed in the recent years. This approach was given the name L_1-Adaptive Control (L_1-AC) due to the L_1 bounds obtained for various signals. As part of the approach it is recommended to use very high adaptive gains for fast and robust adaptation. The purpose of this note is to analyze whether L_1-AC provides any improvements to existing MRAC schemes by focusing on a simple plant whose states are available for measurement presented in [1]. Our analysis shows that the insertion of the proposed filter deteriorates the performance and robust stability margin bounds compared to standard MRAC, i.e., when the filter is removed. The use of high adaptive gains recommended in the L_1-AC approach may cause two major problems. First, it makes the differential equation of the adaptive law very stiff leading to possible numerical instabilities. Second, it makes the adaptive scheme less robust with respect to unmodeled dynamics.**

***Index Terms*— Model reference adaptive control (MRAC) .**

I. Introduction

Many attempts were made since the early 90s to design adaptive control schemes for aerospace applications. Among these efforts, a research approach emerged that involves the insertion of a stable strictly proper filter, usually first order, at the input of a standard Model Reference Adaptive Control (MRAC) scheme. As part of the approach, it is recommended to use very high adaptive gains for fast and robust adaptation [2]–[5]. This approach is given the name L_1 Adaptive Control (L_1-AC) due to the L_1 bounds obtained for several signals and led to a body of work where the same approach is applied to different classes of plants [1]–[5]. Subsequently, the name L_1-AC that was motivated from this filtering approach was also given to nonadaptive schemes that involved fixed controllers with integral control action which initially created some confusion which has been recently clarified by other authors in [6] and [7].

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The purpose of this note is to discuss the possible benefits that the L_1-AC approach was intended to achieve. We focus on a simple plant whose states are available for measurement presented in [1]. The conclusions derived are applicable to all other results on L_1-AC that involve insertion of strictly proper stable filters at the input of MRAC schemes for different plant models and plant assumptions. The reason is that the insertion of any strictly proper stable filter at the input of MRAC changes the relative degree of the plant and violates the necessary condition for tracking the output of the reference model at all frequencies [8]–[19]. The methodology used in this note in evaluating the L_1-AC approach involves the following steps: Identify the control objective; examine whether it can be met and how closely; and evaluate whether there are any improvements, i.e., what is gained by the filter so that when it is removed we can see a deterioration.

It is evident in [1] and other L_1-AC publications [2]–[5] that the control objective is not clear and that there is no adequate discussion as to how it is achieved despite the numerous bounds. For example, we clarify in this note that in [1] and other publications [2]–[5], there are two control objectives; one that aims to track the reference input of the reference model and one that aims to track the output of the reference model. In addition to the above observation, the note derives the following results: The L_1-AC approach applied to a simple plant in [1] cannot meet the control objective of tracking the output of the reference model as the filter acts as a perturbation (by violating the order and relative degree matching between the plant and reference model), in contrast to MRAC which can always achieve this objective. Also, the L_1-AC approach cannot meet the objective of the tracking of a reference input. Since L_1-AC for the class of plants presented in [1] is equivalent to MRAC with a filter and that MRAC is designed to track the output of the reference model, there is nothing about the proposed filter that makes the modified scheme track both the input and output of the reference model at all frequencies even approximately. The objectives of tracking the input and output of the reference model coincide when the reference input is constant and the dc gain of the reference model is equal to unity.

In summary, the first point that this note makes is this: The insertion of the filter deteriorates the tracking performance of the L_1-AC both for transient and steady state, and leads to higher bounds than in the absence of the filter which is the standard MRAC scheme. The second point that this note makes is that due to the phase lag introduced by the input filter in the L_1-AC, the stability margins deteriorate. This in turn imposes a bound on the allowable parametric uncertainty.

The third and final point that this note makes is that the design recommendation of using very high adaptive gains, an important hallmark of L_1-AC in [1] and other papers [2]–[5], brings up two problems that have not been addressed in the literature of L_1-AC. The first problem is that the use of high adaptive gains has a negative effect on robustness as documented in the literature of robust adaptive control [8]–[19]. Second, very high adaptive gains make the differential equation of the adaptive law stiff and difficult to solve numerically. This numerical instability often leads to unbounded signals or when parameter projection is used leads to high frequency oscillations in the parameter estimates. It is likely that the filtering of these numerical oscillations from entering the plant may have motivated the use of the filter in the L_1-AC approach. We should note that such numerical instabilities are present in all control schemes that involve very high gains and/or discontinuities that lead to very high derivatives of signals. The limitations of such techniques are well known in the control literature.

The three points of the note mentioned above are discussed in Sections II, III, and IV, respectively.

II. What is L_1-Adaptive Control?

We consider the class of plants considered in several L_1-AC publications (see for example [1] and [20])

$$\dot{x} = A_m x + b\theta^{*T} x + bu, \quad y = c^T x \tag{1}$$

where the state vector $x \in \mathbb{R}^n$ is available for measurement, $x(0) = x_0$, A_m is a known matrix with all its eigenvalues in $Re\{s\} < 0$, b, c are known vectors, the pair (A_m, b) is controllable, and $\theta^* \in \mathbb{R}^n$ is an unknown parameter vector. The control objective according to L_1-AC design copied from reference [1] is "*to design an adaptive controller to ensure that the system output $y(t)$ follows a given reference signal $r(t)$ with quantifiable transient and steady-state performance bounds*." The L_1-AC design proposed in [1] to meet the above control objective involves the following equations:

$$\dot{\hat{x}} = A_m \hat{x} + b\theta^T x + bu, \qquad \hat{x}(0) = x_0 \tag{2}$$

$$\dot{\theta} = \Gamma x \tilde{x}^T P b, \qquad \theta(0) = \theta_0 \tag{3}$$

where $\tilde{x} = x - \hat{x}$, $P = P^T > 0$ is the solution of the Lyapunov equation $A_m^T P + P A_m = -Q$, $Q = Q^T > 0$, and $\Gamma > 0$ is the adaptive gain assumed to be a scalar[1] as in [6]. Equation (3) is a standard parameter estimator scheme that can be designed and analyzed using the existing adaptive control literature (see [10] and [11]) and it is considered as one of the simplest adaptive schemes as the vector b is known and x is available for measurement. Using a simple Lyapunov analysis i.e., $V = \tilde{x}^T P \tilde{x} + \tilde{\theta}^T \Gamma^{-1} \tilde{\theta}$, where $\tilde{\theta}(t) = \theta(t) - \theta^*$, it follows that $\dot{V} = -\tilde{x}^T Q \tilde{x} \leq 0$ which implies that $\tilde{\theta}$, $\tilde{x}$ are bounded. Using well-known results from the early years of adaptive control literature, we can also establish that if u is bounded and sufficiently rich, then both $\hat{x}$ and θ converge exponentially fast to x and θ^*, respectively. Therefore, (2) and (3) correspond to a standard on-line parameter estimator for θ^*. The combination of this parameter estimator with a control law leads to adaptive control methods (see for example [10]–[16]) derived over the past two decades. The L_1-AC proposes the control law

$$u = C(s)[-\theta^T x + k_0 r] \tag{4}$$

where $C(s)$ is a stable strictly proper transfer function with $C(0) = 1$ and $k_0 = -1/(c^T A_m^{-1} b)$. As pointed out in [1], the above control law without the filter $C(s)$ (i.e., $C(s) = 1$) is simply the MRAC law

$$u = -\theta^T x + k_0 r. \tag{5}$$

The MRAC law (5) together with (3) guarantees that x follows the state $\hat{x}$ of the reference model:

$$\dot{\hat{x}} = A_m \hat{x} + b_m r, \qquad b_m = k_0 b \tag{6}$$

for any reference input r. Needless to say that (6) is identical to (2) and is obtained by substituting (5) into (2) which makes the direct and indirect schemes to be identical in this case [10]–[16] as no indirect calculations of the controller parameters are required for implementation.

It is clear that the L_1-AC and MRAC have the following differences. The MRAC objective is for the plant state/output to track the state/output of the reference model for any reference input signal r [10], whereas the control objective of L_1-AC as stated in [1] and elsewhere is for the plant output to track the reference input r. What is surprising is that the only difference between the two schemes is just the insertion of the input filter $C(s)$ in the MRAC law (5) to obtain (4). The important question is whether by simply inserting the filter $C(s)$, the filtered MRAC law referred to as L_1-AC, can meet the modified control objective of [1] given that MRAC is designed to track the output of the reference model which is a completely different objective than tracking its input. In order to answer this question, we recall the certainty equivalence principle that is used in the design of MRAC. According to this principle, for the adaptive control to meet the control objective, it has to be shown that the solution to the problem exists. That is there exists a controller parameter vector θ^* that meets the control objective. If such a θ^* does not exist, then it is unlikely that an adaptive law that is designed to reduce the parameter error between θ^* and its estimate can achieve the control objective. The following lemma shows that such a θ^* does not exist to meet the control objective of L_1-AC.

Lemma 1: The control law (4) with $\theta = \theta^*$ cannot force the output y to track a given reference signal $r(t)$ unless $r(t)$ is constant. Furthermore, for stability the size of θ^* has to be restricted to satisfy the inequality [1]

$$\left\| (sI - A_m)^{-1} b \left(C(s) - 1 \right) \right\|_1 \|\theta^*\|_1 < 1. \tag{7}$$

Proof: The closed-loop transfer function obtained by substituting (4) into (1) is given by $y = c^T (sI - A_m + b(C(s) - 1)\theta^{*T})^{-1} b C(s)[k_0 r]$. It follows from the small-gain theorem that if (7) is satisfied, the closed-loop system is stable. According to the certainty equivalence principle for the control objective of $y(t) \to r(t)$, as $t \to \infty$ for any given bounded r, to be possible we need to show the existence of a θ^* that satisfies (7) and guarantees that

$$c^T \left(sI - A_m + b \left(C(s) - 1 \right) \theta^{*T} \right)^{-1} b C(s) k_0 = 1, \qquad \forall s. \tag{8}$$

Since the left side of (8) is a strictly proper transfer function no θ^* exists to satisfy (8). Equation (8) however is satisfied when $s = 0$ i.e., when r is constant due to $k_0 = -1/(c^T A_m^{-1} b)$ and $C(0) = 1$. ■

Lemma 1 shows that tracking the reference input r at all frequencies using the MRAC structure is not possible for the simple reason that MRAC is designed to track the output of the reference model and not its input. The control objective of tracking a reference signal can be achieved using a different class of adaptive control schemes referred to as adaptive placement controllers [10]. The reason that the scheme is able to track a constant reference input is simply because the choice of $C(0) = 1$ and $k_0 = -1/(c^T A_m^{-1} b)$ guarantees that at steady state the reference input is equal to the output of the reference model. Equation (7) which is also presented in [1] clearly indicates the limitation on the size of the parametric uncertainty in order to guarantee stability. This limitation is clearly removed by removing the filter, i.e., with $C(s) = 1$. Lemma 1 clearly shows that the MRAC control structure cannot be used to track the reference signal given that it is designed to track the output of the reference model and the insertion of the filter does not change that.

In [1] and subsequent publications [2]–[5], we were not able to identify the reason why the MRAC control objective was changed to that of tracking the reference input that cannot be met by the chosen control structure. Instead, the following theorem and subsequent discussions

[1] A projection is usually added to the adaptive law in (3) to ensure boundedness in the parameter estimates in the presence of bounded disturbances. For ease of exposition, we have omitted it in this section where the focus is on the ideal case of no disturbances.

in [1] suggest shifting the control objective from tracking the reference input to that of tracking the reference model output.

Theorem 1—(Theorem 2.1 in [1]): Consider the feedback system referred to as the 'closed-loop reference system' in [1] given by

$$\begin{aligned}\dot{x}_{\rm ref} &= A_m x_{\rm ref} + b\theta^{*T} x_{\rm ref} + b u_{\rm ref}, \quad y_{\rm ref} = c^T x_{\rm ref}\\ u_{\rm ref} &= C(s)\left[-\theta^{*T} x_{\rm ref} + k_0 r\right]. \end{aligned} \tag{9}$$

where $x_{\rm ref}(0) = x(0)$, obtained by replacing $\theta(t)$ with θ^* in (1)–(4). If condition (7) is satisfied, then for the closed-loop system (1)–(4) we have $\lim_{t\to\infty} \|x(t) - x_{\rm ref}(t)\| = 0$, $\lim_{t\to\infty} |u(t) - u_{\rm ref}(t)| = 0$, and

$$\|x - x_{\rm ref}\|_\infty \le \frac{\gamma_1}{\sqrt{\Gamma}}, \quad \|u - u_{\rm ref}\|_\infty \le \frac{\gamma_2}{\sqrt{\Gamma}} \tag{10}$$

where γ_1 and γ_2 are positive constants independent of Γ.

Since (9) cannot be designed to represent the desired properties of the plant as θ^* is unknown, [1] introduced the following system as a desired reference model:

$$\dot{x}_{\rm des} = A_m x_{\rm des} + k_0 b r_0, \quad y_{\rm des} = c^T x_{\rm des}, \quad r_0 = C(s)[r]. \tag{11}$$

Lemma 7 in [1] is then derived to compare the fictitious reference system in (9) and the desired reference model in (11) as

$$\|y_{\rm ref} - y_{\rm des}\|_\infty \le \frac{\lambda}{1-\lambda}\|c\|_1 \, \|k_0 W_b(s) C(s)\|_1 \, \|r\|_\infty \tag{12}$$

where $W_b(s) = (sI - A_m)^{-1} b$, $\lambda = \|W_b(s)(C(s) - 1)\|_1 \theta_{\max}$, and $\theta_{\max}$ is an upper bound for the norm of θ^*. It is clear that the value of $C(s)$ that minimizes the bound in (12) is $C(s) = 1$. In other words, the bound itself suggests removing the filter and using the corresponding standard MRAC scheme. In fact, the MRAC law (3), (5) with (6) as the reference model (which coincides with (11) if $C(s) = 1$), guarantees a very precise result. This is summarized in Theorem 2 below.

Theorem 2:

(a) The MRAC scheme (3), (5), (6) guarantees that all signals are uniformly bounded and the tracking error $e \stackrel{\Delta}{=} x - x_m$ converges to zero for all bounded reference inputs r, where $x_m = \hat{x}$ is the state of the reference model (6). In addition, the tracking error satisfies the bound

$$\|e\|_\infty \le \frac{\nu_0}{\sqrt{\Gamma}}, \quad if\ e(0) = 0$$

where $\nu_0 = \|\tilde{\theta}(0)\|_2 / \sqrt{\lambda_{\min}\{P\}}$ is a constant independent of Γ. Furthermore, if r is sufficiently rich, θ and therefore x converge 'exponentially fast' to θ^* and x_m, respectively.

(b) The L_1-AC scheme (2), (3), (4) guarantees the following: Restrict θ^* to satisfy (7). Assume that the input filter is of the form $C(s) = 1/(\tau s + 1)$, where $\tau > 0$ is a design parameter to be suitably chosen. There exists a $\tau_{\max} > 0$ so that for any $\tau \in [0, \tau_{\max})$ all signals are uniformly bounded and the tracking error satisfies

$$\|e\|_\infty \le \tau\nu_1 + \frac{\nu_2}{\sqrt{\Gamma}}, \quad if\ e(0) = 0$$

where $\nu_1 = \|sW_b(s)\|_1(\theta_{\max}\bar{\mathrm{x}}_m + \bar{\mathrm{r}})/(1 - \tau/\tau_{\max})$ and $\nu_2 = \nu_0/(1 - \tau/\tau_{\max})$ are constants independent of Γ, $W_b(s) = (sI - A_m)^{-1} b$, $\tau_{\max} = 1/(\theta_{\max}\|sW_b(s)\|_1)$, and $\theta_{\max}$, $\bar{\mathrm{x}}_m$, and $\bar{\mathrm{r}}$ are upper bounds for the norm of $\theta(t)$, $x_m(t)$, and $k_0 r(t)$, respectively.

Proof: The proof is given in the Appendix. ■

According to Theorem 2, the standard MRAC meets its control objective exactly. It achieves its control objective without the limitations on θ^* imposed in L_1-AC indicated by (7). The L_1-AC scheme, in contrast, cannot guarantee a zero steady-state tracking error performance for non-constant reference inputs and in addition, the computed transient bound for the tracking error is larger than that with MRAC. If the reference input is constant and the reference model is designed to have a dc gain of 1 then it can be shown that for both schemes the plant output tracks both the reference input and the output of the reference model exactly [1], [10]. Given that in such a case, the performance of the MRAC is the same as that of L_1-AC, it is clear that the additional complexity of a filter offers no benefit given that the standard MRAC can meet the control objective exactly and with better bounds. In [1] and other L_1-AC publications, it is argued that $C(s)$ is introduced to filter plant uncertainties using bandwidth considerations, which are often translated to mean that the frequencies in the plant input can be controlled (see [20, p. 26]). The only uncertainty in the plant equation (1) is the unknown constant parameter vector θ^* which is not affected by filtering. In the following section we examine whether the use of the input filter as presented in [1] has any benefit on robustness with respect to unmodeled dynamics.

III. L_1-AC: Robustness

The effect of unmodeled dynamics can be analyzed by inserting a multiplicative uncertainty term $\Delta_m(s)$ in the plant dynamics (1) as

$$\dot{x} = A_m x + b\theta^{*T} x + b\left(1 + \Delta_m(s)\right)[u]. \tag{13}$$

The multiplicative uncertainty $\Delta_m(s)$ may represent unmodeled input delay, fast actuator dynamics and other modeling errors [10]. Its parameters and structure are assumed to be unknown. The question asked in robust adaptive control [10], [11] as well as in robust control [21] is whether a control scheme designed for $\Delta_m(s) = 0$ can tolerate the presence of a nonzero unknown $\Delta_m(s)$ and if so what is the size of the allowable uncertainty. The question is whether the use of the input filter in (4) can improve robustness with respect to the unknown $\Delta_m(s)$. As before, let us use the certainty equivalence principle to examine what stability properties can be obtained when we have perfect parameter information, i.e., $\theta = \theta^*$ in the control law. The control law in such a case is given by

$$u = C(s)\left[-\theta^{*T} x + k_0 r\right] \tag{14}$$

with $C(s) = 1/(\tau s + 1)$, $\tau > 0$ as proposed in [1].

Theorem 3: The closed-loop system (13), (14) guarantees the following. If

$$\|W_b \Delta_m\|_1 \le \frac{1}{\|\theta^*\|_1} - \tau\|sW_b\|_1 \tag{15}$$

then all signals in the closed-loop plant are bounded and the tracking error $e \stackrel{\Delta}{=} x - x_m$ satisfies

$$\begin{aligned}\|e_t\|_\infty \le\ & \frac{1}{1 - \tau\|sW_b\|_1\|\theta^*\|_1 - \|W_b\Delta_m\|_1\|\theta^*\|_1}\\ &\times \left(\tau\|sW_b\|_1 \left\|\left(\theta^{*T} x_m - k_0 r\right)_t\right\|_\infty\right.\\ &\left. + \|W_b\Delta_m\|_1 \left\|\left(\theta^{*T} x_m - k_0 r\right)_t\right\|_\infty\right), \qquad \forall t \ge 0\end{aligned}$$

where $W_b(s) = (sI - A_m)^{-1} b$ and $x_m = \hat{x}$ is the state of the reference model (6).

Proof: The proof is based on the use of the small-gain theorem and input-output stability results [10], [22] and it is omitted due to lack of space. ■

It is clear from Theorem 3 that the presence of the input filter, i.e., $\tau \ne 0$, reduces the stability margin bound and increases the bound for the norm of the tracking error. Similar to the arguments presented in Section II, the bound in (15) suggests removing the filter i.e., use of the corresponding MRAC. We clarify that if the bound in (15) is not satisfied, the stability of the corresponding adaptive scheme where $\theta(t)$, the estimate of θ^*, is used instead of θ^*, cannot be established either. In fact for $\theta \neq \theta^*$, the stability margin bounds (15) are reduced further and a more elaborate analysis is required to establish them (see [10, chap. 6] and [11, chap. 8]).

A careful analysis of the results of L_1-AC leads us to the following: The L_1-AC results did not address the same robustness problem that has been addressed in the robust adaptive control literature (see [10]–[16] for example), which is the stability and performance of adaptive control schemes for plant (13) designed for $\Delta_m(s) = 0$ but analyzed for $\Delta_m(s) \neq 0$ with no knowledge of $\Delta_m(s)$. Rather, the effect of the multiplicative uncertainty was parameterized and expressed as $\theta^T x(t)$, where $x(t)$ is a known signal vector and θ is unknown (see [20, chap. 2]). This essentially converts the unmodeled dynamics, which is a dynamic uncertainty into a parametric uncertainty; in such a case, all standard adaptive methods without any modification can be directly used [10], [11]. In reference [20, p. 68, eq. 2.130], the knowledge of $\Delta_m(s)$ is required in order to design the filter $C(s)$. This implies that $C(s)$ is no longer *ad hoc* but depends directly on the knowledge of $\Delta_m(s)$ which makes the latter no longer an uncertainty.

IV. Effect of High Adaptive Gains

In addition to the input filter, the L_1-AC approach suggests the use of very high adaptive gains motivated from the bounds in (10) in order to achieve fast convergence of x close to x_{des}. Such a use contradicts the results of robust adaptive control that go as far back as 1980 (see [10] and [11] for example) which establish that robustness is often obtained with much smaller adaptive gains. In addition, very high adaptive gains make the differential equation of the adaptive law very stiff and could easily lead to numerical instability, even in a high speed computer, as explained below.

As the adaptive gain goes to infinity, the derivative $\dot{\theta}(t)$ goes to infinity, which at one point will cause any differential equation solver to either fail or generate erroneous results. Therefore, from a numerical point of view, there is an upper limit to the adaptive gain that not only depends on the CPU of the computer but also on the differential equation solver and the sampling rate used. The design suggestion often made in the L_1-AC publications, that the adaptive gains could be as high as the CPU of the computer allows it to be, should therefore be clarified by bringing up numerical instability considerations. Such considerations are important as numerical instability does not always lead to unbounded signals and therefore may go unnoticed. For example, when modifications such as parameter projection are used, numerical instability may lead to fast oscillations of the parameter estimates between projection bounds with learning totally lost and with no parameter convergence even in the presence of persistence of excitation. We analyze this phenomenon below:

The L_1-AC approach often involves the use of projection to force the estimate $\theta(t)$ to satisfy the bound $\|\theta(t)\| \leq \theta_{\max}$. The use of projection was developed in the robust adaptive control literature back in the late 70s and early 80s as one of the modifications to counteract parameter drift [8], [10], [11], [19]. The general version of the adaptive law with projection is given by $\dot{\theta}(t) = \Gamma \mathrm{Proj}(\theta(t), \tilde{x}^T P b x)$, where

$$\mathrm{Proj}(\theta, y) = \begin{cases} y - \frac{\nabla f(\theta)(\nabla f(\theta))^T}{\|\nabla f(\theta)\|^2} y f(\theta) & \text{if } f(\theta) > 0 \wedge \\ & \quad y^T \nabla f(\theta) > 0 \\ y & \text{otherwise} \end{cases}$$

and $f(\cdot)$ is a convex function, one choice for which is given by $f(\theta) = (\|\theta\|^2 - \vartheta^2)/(\varepsilon^2 + 2\varepsilon\vartheta)$, where ϑ and ε are arbitrary positive constants. Let Ω_0 and Ω_1 be defined as $\Omega_0 = \{\theta \in \mathbb{R}^n | f(\theta) \leq 0\}$ and $\Omega_1 = \{\theta \in \mathbb{R}^n | f(\theta) \leq 1\}$. It is easy to show that the boundedness of θ follows for any choice of y [10], [23] due to the fact that the derivative of $f(\theta)$ is zero on Ω_1 and negative inside Ω_1. A direct consequence of the projection algorithm is therefore that the normal component of $\dot{\theta}$ is proportional to Γ on Ω_0 and zero on the boundary of Ω_1. This implies that as $\Gamma \to \infty$, this normal component of $\dot{\theta}$ is discontinuous, which can lead to chatter and numerical instability. This analysis clearly explains the high frequency oscillations in the estimated parameters due to the presence of high adaptive gains in combination with projection.

As discussed in [1] and [20], the bounds in (10) are significant, as the state of the plant and the control input can be forced to be arbitrarily close to that of an LTI system (9) by significantly increasing the adaptive gains. Referred to as a 'scaling' property in [20], this guarantees that the closed-loop adaptive control system behaves like an LTI system. As discussed in this section, the adaptive gains cannot be made arbitrarily large as they lead to numerical instability. This makes the bounds in (10) rather academic. The bound for $\|u - u_{\mathrm{ref}}\|_\infty$ in (10) is based on the assumption that $C(s)$ is strictly proper; otherwise, for $C(s) = 1$, which is the standard MRAC case, the L_1-norm in the constant γ_2 in (10) does not exist due to the underlying transfer function becoming improper. In the case of MRAC, i.e., $C(s) = 1$, the following bounds can be derived

$$\|x - x_{\mathrm{ref}}\|_\infty \leq \frac{\kappa_0}{\sqrt{\Gamma}}, \quad \|u - u_{\mathrm{ref}}\|_\infty \leq \kappa_1 \left\|\tilde{\theta}(0)\right\|_2 + \frac{\kappa_2}{\sqrt{\Gamma}}$$

where κ_0, κ_1, and κ_2 are positive constants independent of Γ which confirms the uniform boundedness of the control input u. The strong result of Theorem 2 for MRAC cannot be traded off with this 'scaling' property of L_1-AC since it does not appear to have any practical significance as it cannot be realized in practice due to restrictions on the size of the adaptive gain.

As an illustration of the analytical results we consider the plant $\dot{x} = -x + \theta^* x + u$, $x(0) = 1$, where $\theta^* \in [-5, 5]$ is an unknown parameter. The reference model is $\dot{x}_m = -x_m + r$ and $r(t) = 100\sin(0.5t)$. We assume that the input filter is of the form $C(s) = 1/(\tau s + 1)$ for which the stability condition (7) is satisfied for any $\theta^* \in [-5, 5]$ if $\tau < 0.13$. We choose $\tau = 0.05$ and assume the true value of the unknown parameter $\theta^* = 2$. Fig. 1 shows the simulations generated using Matlab demonstrating the results that are consistent with what we have shown analytically. The sampling period is 10^{-4} sec and a forth order Runge-Kutta method is used. Fig. 1(a) shows the lack of tracking performance in the case of L_1-AC whereas (b) shows that removal of the filter, i.e., with MRAC, the tracking performance is recovered. Fig. 1(c) illustrates the effect of a high adaptive gain for the same example, which shows high frequency oscillations in θ demonstrating the inherent numerical instability. It should be noted that a projection algorithm was used in 1(c), with $\vartheta = 5$ and $\varepsilon = 0.1$. In part (c), the tracking error response does not reveal any suspicion about numerical instability by looking at $e = x - x_m$, but it is obvious in the parameter estimate response θ. By removing projection, this numerical instability leads to unbounded signals as illustrated in Fig. 1(d).

In addition to all of the above, another negative effect of the use of high adaptive gains is the 'freezing' of the adaptive law as the adaptive gain becomes large, as discussed in [24] and [25].

V. Conclusion

In this note, we analyze properties of an adaptive control design approach that involves the insertion of a stable strictly proper filter

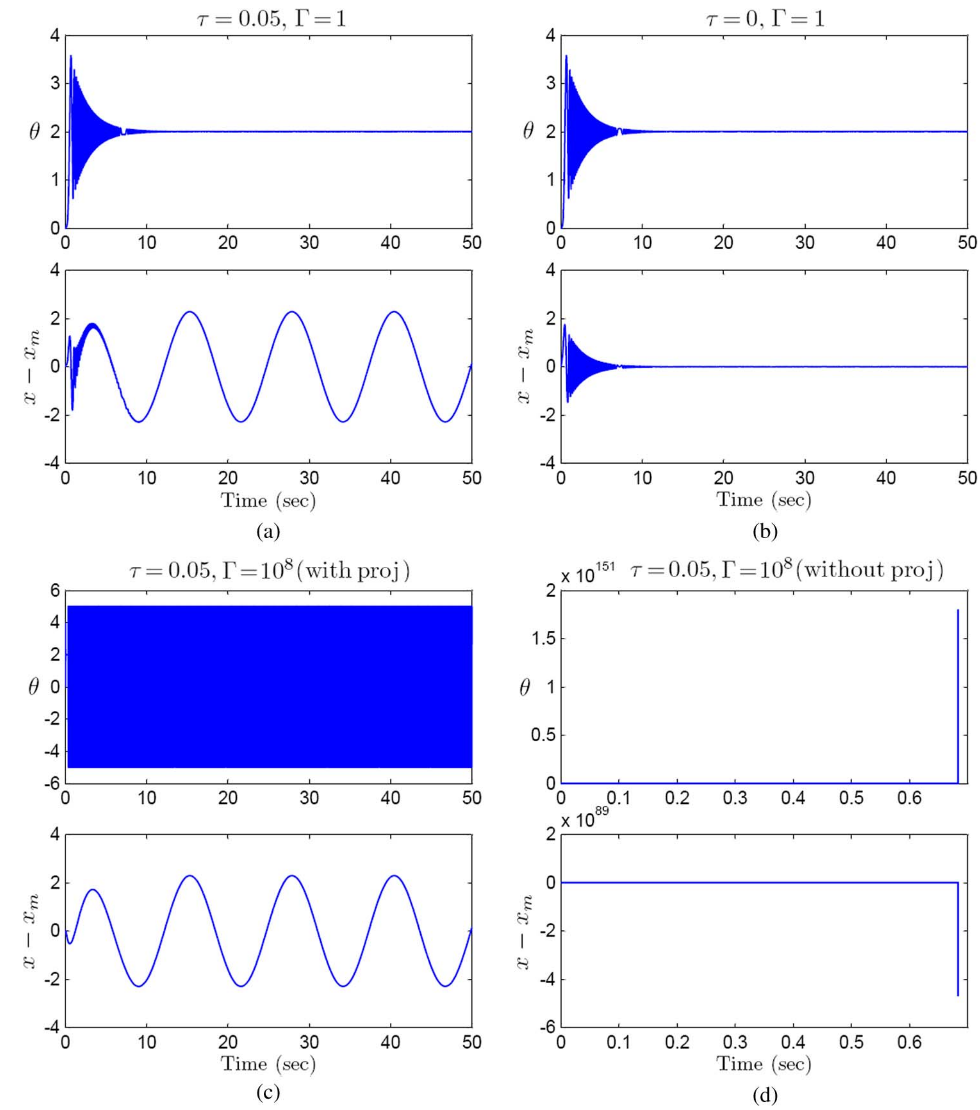

Fig. 1. The L_1-AC performance: (a) For a non-constant r, the tracking error $e = x - x_m$ does not go to zero. (b) For $C(s) = 1$ (which is equivalent to the standard MRAC), e converges to zero. (c) The use of very high Γ leads to numerical instability and to the loss of learning even in the presence of persistently exciting signals as $\theta(t)$ oscillates between the projection bounds. (d) Without projection all signals in the closed-loop system go unbounded due to numerical instability.

at the input of standard MRAC and the use of very high adaptive gains. The approach was given the name L_1-AC and led to a body of work involving different classes of plants. We have shown analytically that the L_1-AC approach for a class of simple LTI plants with states accessible presented in [1] and [20] offers no benefits in terms of performance, robustness or bounds that suggest useful trade offs. On the contrary, the approach deteriorates the very properties of the MRAC that it is purportedly trying to improve.

APPENDIX

Proof of Theorem 2:

(a) Using the Lyapunov candidate $V = e^T Pe + \tilde{\theta}^T \Gamma^{-1} \tilde{\theta}$, where P is the solution of $A_m^T P + PA_m = -I$, it can be shown that $\theta, e, \dot{e} \in \mathcal{L}_\infty$ and $e \in \mathcal{L}_2$ which imply all signals are uniformly bounded and that $e \to 0$ as $t \to \infty$, by first showing that $\dot{V} = -e^T e \leq 0$. It follows that if $e(0) = 0$:

$$e^T(t)Pe(t) \leq V(t) \leq V(0) = \frac{\tilde{\theta}^T(0)\tilde{\theta}(0)}{\Gamma}, \qquad \forall t.$$

Then, we have

$$\lambda_{\min}\{P\}\|e_t\|_\infty^2 \leq \lambda_{\min}\{P\}\|e_t\|_2^2 \leq V(t) \leq \frac{1}{\Gamma}\tilde{\theta}^T(0)\tilde{\theta}(0)$$

where $\lambda_{\min}\{P\} > 0$ denotes the minimum eigenvalue of P. Therefore, $\|e\|_\infty \leq \nu_0/\sqrt{\Gamma}$, where $\nu_0 = \|\tilde{\theta}(0)\|_2/\sqrt{\lambda_{\min}\{P\}}$.

(b) Using the same V with e replaced by $\tilde{x}$, it can be shown that $\theta, \tilde{x} \in \mathcal{L}_\infty$, $\tilde{x} \in \mathcal{L}_2$ and that $\|\tilde{x}\|_\infty \leq \nu_0/\sqrt{\Gamma}$.

Equation (2) can be written as

$$\hat{x} = W_b(1-C)[\theta^T \hat{x}] + W_b(1-C)[\theta^T \tilde{x}] + W_b C[k_0 r] + w$$

where $W_b(s) = (sI - A_m)^{-1}b$ and $w(t) = \mathcal{L}^{-1}\{(sI - A_m)^{-1}x_0\}$ is a bounded decaying function. Then

$$\|\hat{x}_t\|_\infty \leq \|W_b(1-C)\|_1 \left\|(\theta^T \hat{x})_t\right\|_\infty + \|W_b(1-C)\|_1 \left\|(\theta^T \tilde{x})_t\right\|_\infty + \|W_b C\|_1 \|(k_0 r)_t\|_\infty + \|w_t\|_\infty.$$

Then, with filter $C(s)=1/(\tau s+1)$ we have

$$\begin{aligned}\|\hat{x}_t\|_\infty \leq& \tau\theta_{\max}\|sW_b\|_1 \left\|\frac{1}{\tau s+1}\right\|_1 \|\hat{x}_t\|_\infty\\ &+\tau\frac{\nu\theta_{\max}}{\sqrt{\Gamma}}\|sW_b\|_1\left\|\frac{1}{\tau s+1}\right\|_1+\nu\|W_b\|_1\left\|\frac{1}{\tau s+1}\right\|_1+\nu\\ =& \tau\theta_{\max}\|sW_b\|_1\|\hat{x}_t\|_\infty+\tau\frac{\nu\theta_{\max}}{\sqrt{\Gamma}}\|sW_b\|_1+\nu\|W_b\|_1+\nu\end{aligned}$$

where $\theta_{\max}$ is an upper bound for $\|\theta\|_1$ and $\nu>0$ is a constant independent of τ, Γ.

We choose τ small enough so that $\tau\theta_{\max}\|sW_b\|_1<1$, then for any $\tau\in[0,\tau_{\max})$, where $\tau_{\max}=1/(\theta_{\max}\|sW_b\|_1)$, $\hat{x}$ and therefore all signals are uniformly bounded. In addition, since $\dot{\tilde{x}}\in\mathcal{L}_\infty$ and $\tilde{x}\in\mathcal{L}_2\cap\mathcal{L}_\infty$, $\tilde{x}(t)$ goes to zero as $t\to\infty$.

The tracking error can be expressed as $e=W_b(1-C)[\theta^T e]+W_b(1-C)[\theta^T x_m-k_0 r]-W_b[\tilde{\theta}^T x]$ and also we have $\tilde{x}=-W_b[\tilde{\theta}^T x]$, then

$$e=W_b(1-C)[\theta^T e]+W_b(1-C)[\theta^T x_m-k_0 r]+\tilde{x}.$$

Then, we obtain

$$\begin{aligned}\|e_t\|_\infty \leq& \tau\theta_{\max}\|sW_b\|_1\|e_t\|_\infty\\ &+\tau\|sW_b\|_1\left\|(\theta^T x_m-k_0 r)_t\right\|_\infty+\|\tilde{x}_t\|_\infty.\end{aligned}$$

Then, since $\|\tilde{x}\|_\infty\leq\nu_0/\sqrt{\Gamma}$, then $\forall\tau\in[0,\tau_{\max})$ we have

$$\|e_t\|_\infty\leq\tau\nu_1+\frac{\nu_2}{\sqrt{\Gamma}},\qquad\forall t\geq 0$$

where $\nu_1,\nu_2>0$ are independent of Γ. ∎

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