1 Lecture 1 - 12.01.21

1.1 Administrative information and setup

- Treat classes over zoom as a normal class.
- We will meet Mondays and Tuesdays from 10.15 to 12.00. Classes will not be recorded.
- The exam will be oral, possibly digital.
- It will be based on what is done in class, so if one wants to take the exam one should be present in most classes. The focus on the exam will be on the problems given throughout the course.
- The course will somewhat closely follow the book *Algebraic Geometry - An introduction* by Daniel Perrin.

Algebraic geometry is the study of algebraic varieties, which are roughly the zero loci of polynomials over fields. It is an old theory, spanning way back, but the focus of this course will be on the more modern theory developed by the likes of J.P. Serre and A. Grothendieck during the last half of the 20'th century.

Throughout the course k will be a field, usually algebraically closed, or at least infinite. The motivating examples will be $\mathbb{C}$ and $\mathbb{R}$. We will start by studying affine algebraic varieties, and later projective varieties, which are locally modeled by the affine ones.

1.2 Introduction

Let $P \in k[X_1, \dots X_n]$, i.e. a polynomial in n variables over a field k. Define its zero locus to be the set $V(P) = \{x \in k^n | P(x) = 0\} \subset k^n$. This set $V(P)$ is roughly what we mean by an algebraic variety. More generally we can use a set of polynomials instead of just one. Let $P = \{P_i\}$ be a collection of polynomials in $k[X_1, \dots, X_n]$. We define the zero locus of the polynomials to be $V(P) = \{x \in k^n | P_i(x) = 0, \forall i\}$.

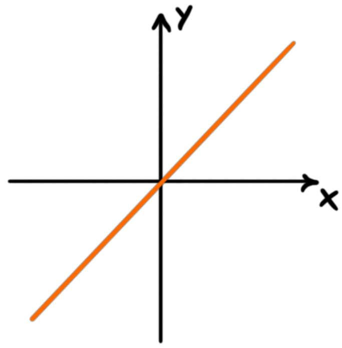
A subset $X \subset k^n$ is also an affine algebraic variety if $X = V(P)$ for some set of polynomials $P = \{P_i\}$ in $k[X_1, \dots, X_n]$. Really this is what we call an affine algebraic set, but we will get back to this in lecture 2.

Examples

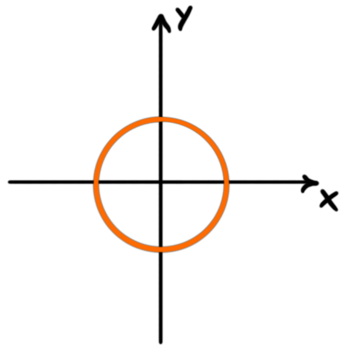
Example 1.1 If the P_i's all have degree 1, then $V(P)$ are affine linear subspaces of k^n, i.e. liner, planes and hyperplanes.

Let $n = 2$ and $k = \mathbb{R}$. Then the zero locus of a single polynomial $P(X, Y)$ is a real plane curve.

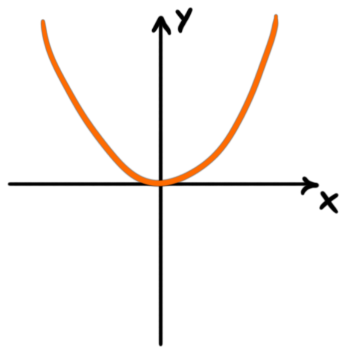
Example 1.2 An example of such a curve is $P(X, Y) = X - Y$, which has zero locus being the line $f(x) = x$ in the plane, i.e.



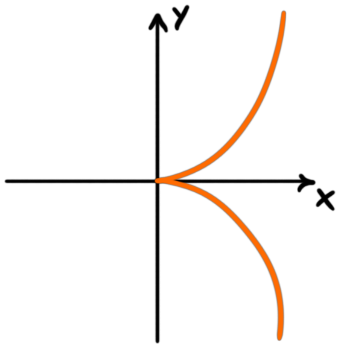
Example 1.3 If we require the polynomial to be of degree 2, then we can have $P(X, Y) = X^2 + Y^2 - 1$, which has zero locus equal to the unit circle in $\mathbb{R}^2$.



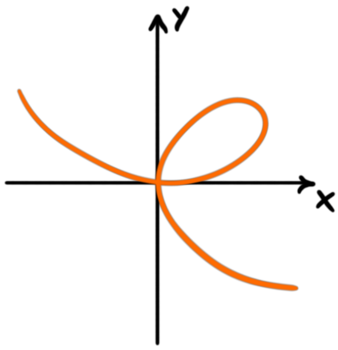
Example 1.4 Another example in degree 2 is $P(X, Y) = y - x^2$, which has zero locus equal to the graph of $f(x) = x^2$.



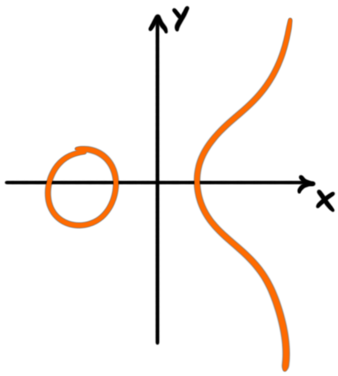
Example 1.5 If we have degree 3 polynomials we have for example $P(X,Y) = Y^2 - X^3$ which have zero locus being a cuspidal curve.



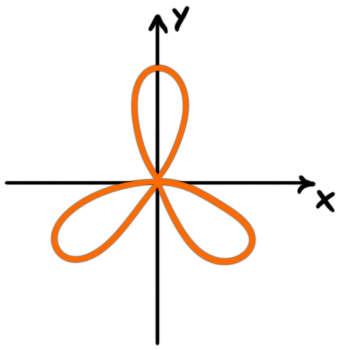
Example 1.6 Another degree 3 example is $P(X,Y) = X^3 + Y^3 - XY$, which has a loop shaped zero locus.



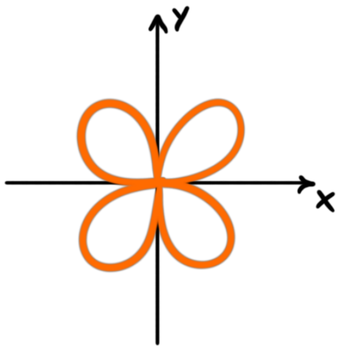
Example 1.7 A more famous degree 3 example are elliptic curves, for example the zero locus of $P(X,Y) = Y^2 - X(X-1)(X+1)$.



Example 1.8 Some higher degree examples include the trefoil curve, $P(X,Y) = (X^2+Y^2)^2 + 3X^2Y - Y^3$,

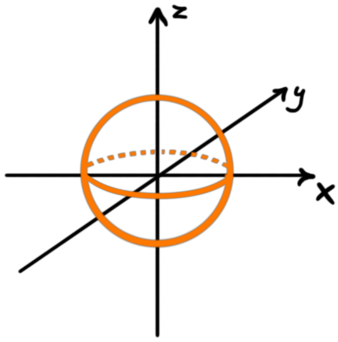


and the quadrafoil curve $P(X,Y) = (X^2+Y^2)^2 - 4X^2Y^2$.

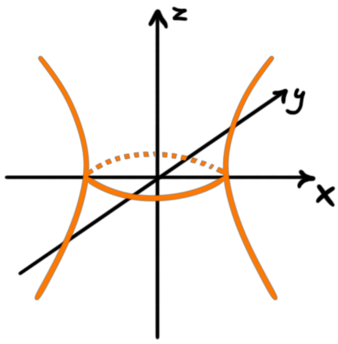


If we let $n = 3$ and still have $k = \mathbb{R}$ then the zero loci of single polynomials are surfaces. If we have degree 2 polynomials we get the quadratic surfaces we know from calculus.

Example 1.9 The most recognized quadratic surface is maybe the zero locus of $P(X,Y,Z) = X^2+Y^2+Z^2-1$, which is equal to the unit sphere in $\mathbb{R}^3$



Example 1.10 Another example of a surface is given by $P(X, Y, Z) = X^2 + Y^2 - Z^2 - 1$ which has zero locus equal to a hyperboloid.



Let us still consider $n = 3$ and $k = \mathbb{R}$. If we instead of just a single polynomial have two polynomials we get the space-curves as their zero loci.

Example 1.11 An example is the zero locus $V = \{(x, y, z) \in \mathbb{R}^3 | x^2+y^2-z^2-1 = 0, y = 0\}$, which is the part of the previous hyperboloid lying in the xz-plane

Note: This study depends very much on which field we work in. We mentioned that we are going to mostly assume that our field is algebraically closed. This places an emphasis on the equations defining our varieties, rather than their actual points.

1.3 Intersection of curves

Question: How can two plane curves intersect?

1. They can not intersect at all.
2. They can be tangent to each other, hence only intersect at a single point.
3. They can intersect in some finite number of points.

4. They can be overlapping, and hence have a continuum of intersections.

Discuss in groups: How does the number of intersections relate to the degrees of the curves? What are the maximum and minimum number of intersections? What are the special cases, and what is the general behavior?

Let C and C' be plane curves of degrees d and d' respectively. The natural questions that arise is the discussions are:

- Do C and C' always intersect in at most dd' points?
- When is the number of intersection points equal to dd'?

There are three obstructions to the number of intersections being dd'. These are

1. The curves have a common component, i.e. they have an infinite number of intersection points
2. The curves have no overlap, for example two parallel lines
3. The curves are tangent, and can therefore have less intersection points

Theorem 1.12: Bézouts theorem

Let C and C' be two projective plane curves of degree d and d', defined over an algebraically closed field, with no common components. Then the number of intersections, counted with multiplicity, is dd'.

To formalize and prove this theorem is one of the main goals of this course. Notice that the three obstructions are naturally removed in the theorem, because no common components mean we have a finite number of intersections, projective curves mean we don't have parallel lines and counting with multiplicity means tangential curves are counted right.

2 Lecture 2 - 18.01.21

2.1 Affine algebraic sets

Throughout we will still let k be a field.

Definition 2.1

Let $S \subset k[X_1, \dots, X_n]$ be any subset. We define the affine algebraic set generated by S to be $V(S) = \{x \in k^n | P(x) = 0, \forall P \in S$.

Lets see some easy examples.

Example 2.2

Let $1 \in k[X_1, \dots, X_n]$ be the identity polynomial. Then we have $V(\{1\}) = \emptyset$.

Example 2.3

Let $0 \in k[X_1, \dots, X_n]$ be the zero polynomial. Then we have $V(\{0\}) = k^n$.

Example 2.4

Let $n = 1$ and let S consist of a single polynomial P. Then $V(S)$ is a finite set. This is because it consists of the zeroes of the polynomial, which is finite because we are working over a field.

Example 2.5

Let $n = 2$, then the affine subsets of k^2 are

- the empty set $\emptyset$
- the affine planes
- the curves
- finite sets of points

Definition 2.6: Affine algebraic set

We say a subset $X \subset k^n$ is an affine algebraic set if we have $X = V(S)$ for some subset $S \subset k[X_1, \dots, X_n]$.

The assignment of the generated affine algebraic set from a subset of polynomials is an order reversing assignment. This means simply that given $S \subset S' \subset k[X_1, \ldots, X_n]$, then we have $V(S') \subset V(S) \subset k^n$.

Definition 2.7

Let $S \subset R$ be any subset of a ring R. The ideal generated by S is the ideal $(S) = \{\sum_{i=0}^{r} a_i f_i | a_i \in k[X_1, \ldots, X_n], f_i \in S\}$.

Lemma 2.8

The affine algebraic variety generated by a set $S \subset k[X_1, \ldots, X_n]$ is the same as the affine algebraic variety generated by the ideal (S).

Proof. Since the assignment $V(-)$ is order reversing, and we have $S \subset (S)$ then we immediately have $V((S)) \subset V(S)$. For the other inclusion we let $x \in V(S)$ be any point. Then by definition we know that $P(x) = 0$ for all $P \in S$. Let now $Q \in (S)$. We have

$$\begin{aligned} Q(x) &= \sum_{i=0}^{r} a_i f_i(x) \\ &= 0 \end{aligned}$$

since we know that $f_i(x) = 0$ for all i. This holds for all elements $Q \in (S)$, hence we have $x \in V((S))$, which proves that $V(S) \subset V((S))$. □

Proposition 2.9

Every affine algebraic set X is generated by a finite set of polynomials, i.e. $X = V(f_1, \ldots f_r)$.

Proof. Since k is a field we have by the Hilbert basis theorem that $k[X_1, \ldots X_2]$ is a Noetherian ring. In a Noetherian ring all ideals are finitely generated. Since X is an affine algebraic set we know that $X = V(S)$ for some subset $S \subset k[X_1, \ldots, X_n]$. By the previous lemma we know that $V(S) = V((S))$, where now (S) is an ideal. But this we know is finitely generated, i.e. $(S) = (f_1, \ldots, f_r)$ for some r, and hence we conclude that $X = V(f_1, \ldots, f_r)$. □

We remark that we also have $V(f_1, \ldots f_r) = V(f_1) \cap V(f_2) \cap \cdots \cap V(f_r)$. This is relatively easy to convince ourselves of, as the left hand side consists of all points in k^n such that the polynomials vanish simultaneously, and the right hand side consists of the intersection of all points in k^n such that the polynomials vanish individually. We often reffer to the affine algebraic set generated by a single polynomial, $V(f)$, as a hypersurface in k^n.

Proposition 2.10

The affine algebraic subsets are the closed sets in a topology on k^n.

Proof. For the set of affine algebraic sets to define a topology on k^n we need three things:

1. $\emptyset$ and k^n are affine algebraic sets.
2. arbitrary intersections of affine algebraic sets is again an affine algebraic set.
3. finite union of affine algebraic sets is again an affine algebraic set.

We have already seen that $\emptyset$ and k^n are affine algebraic sets, because we have $\emptyset = V(\{1\})$ and $k^n = V(\{0\})$, so the first point is done.

For the second one we will show that $\bigcap_{i\in I} V(S_i) = V(\bigcup_{i\in I} V(S_i))$.

For the third point it is enough to show that the union of two affine algebraic sets generated by two ideals is again an affine algebraic set generated by an ideal.

Let I, J be two ideals in $k[X_1, \dots, X_n]$. We know that the product of the ideals, IJ, is contained in both of the ideals, i.e. $IJ \subset I$ and $IJ \subset J$. By the order reversing property of $V(-)$ we have $V(J) \subset V(IJ)$ and $V(I) \subset V(IJ)$. Since both of them are contained, then also their union is, i.e. $V(I) \cup V(J) \subset V(IJ)$. This proves on of the inclusions. For the other one we let $x \in V(IJ)$ such that $x \notin V(I)$. Then by definition there exists a polynomial $P \in I$ such that $P(x) \neq 0$. Since we have $PQ \in IJ$ for all $Q \in J$ we know that x has to be an element of $V(J)$ because $(PQ)(x) = 0$, which implies $Q(x) = 0$ because $k[X_1, \dots, X_n]$ is a domain. The same argument shows $x \in V(I)$ if $x \notin V(J)$. Hence we have $V(I) \cup V(J) = V(IJ)$. $\square$

proove second point

This topology is called the Zariski topology on k^n and is hugely important for this class and for the field of algebraic geometry in general.

This topology is very different from the standard topology we are used to on k^n. To give an example we have that the closed sets in k^3 with the Zariski topology are the points, the curves and the planes. These are very "thin" and "small" sets compared to the closed balls that generate the closed sets in k^n with the standard topology.

Definition 2.11: Standard open sets

Let $f \in k[X_1, \dots, X_n]$. The standard open sets of k^n are the sets $D(f) = k^n \setminus V(f)$, i.e. the complements of the hypersurfaces.

The important thing about the standard open sets is that they form a basis for the Zariski topology on k^n.

Problem 2.12 Show that the intersection of any two open sets is non-empty.

Solution:

Since the standard open sets form a basis for the open sets, it is enough to show that $D(f) \cap D(g) \neq \emptyset$ for some arbitrary polynomials $f, g \in k[X_1, \ldots, X_n]$. This is proven using elementary set theory and the previous properties we have shown.

$$\begin{aligned} D(f) \cap D(g) &= k^n \setminus V(f) \cap k^n \setminus V(g) \\ &= k^n \setminus (V(f) \cup V(g)) \\ &= k^n \setminus (V((f)) \cup V((g))) \\ &= k^n \setminus V((f)(g)) \\ &\neq \emptyset \end{aligned}$$

where the last equality comes from the fact that $(f)(g) \neq (0)$ as $k[X_1, \ldots, X_n]$ is a domain.

2.2 The ideal of an affine algebraic set

We are starting to see and feel the duality between the geometry and the algebra, through the connection between affine algebraic sets and ideals. Until now we have passe from the algebra to the geometry by assigning a geometric object to an algebraic one. We want some way to also go the other way, i.e. som kind of dual or inverse to $V(-)$.

Definition 2.13

Let $V \subset k^n$ be some subset. The ideal of V is defined to be $I(V) = \{f \in k[X_1, \ldots, X_n] | f(x) = 0, \forall x \in V\}$.

It is maybe not obvious that this set is an ideal, so lets prove that it is. We define the morphism $r : k[X_1, \ldots, X_n] \longrightarrow \mathcal{F}(V, k)$ by sending a polynomial P to its restriction $P_{|V}$. Here $\mathcal{F}(V, k)$ is the ring of ring homomorphisms from V to k. We need to show that r is a ring homomorphism. It is because $r(P + Q) = (P + Q)_{|V} = P_{|V} + Q_{|V} = r(P) + r(Q)$ and because $r(PQ) = (PQ)_{|V} = P_{|V}Q_{|V} = r(P)r(Q)$.

Since the kernel of r is exactly the polynomials in $k[X_1, \ldots, X_n]$ that vanish when restricted to V we have $ker(r) = I(V)$. The kernels of ring homomorphisms are ideals, and thus we have shown that $I(V)$ is an ideal.

Definition 2.14

Let $V \subset k^n$. We define the affine algebra of V to be the finite type k-algebra $\Gamma(V) = Im(r)$, i.e. the polynomial functions on V.

By the first isomorphism theorem we have that $\Gamma(V) \cong k[X_1, \ldots, X_n]/I(V)$.

Similarily to $V(-)$, the assignment $I(-)$ is order reversing, i.e. if $V \subset V'$ then $I(V') \subset I(V)$.

Problem 2.15 What is the relationship between V and $V(I(V))$?

Solution:

By definition we have that $V(I(V))$ is the set of elements $y \in k^n$ such that $P(y) = 0$ for all polynomials $P \in I(V)$. These are exactly the polynomials $P \in k[X_1, \ldots, X_n]$ such that $P(z) = 0$ for all $z \in V$. So, if we have $x \in V$ then $P(x) = 0$ for all $P \in I(V)$. This also means that x is an element such that $P(x) = 0$ for all $P \in I(V)$ which means that $x \in V(I(V))$. Hence we have $V \subset V(I(V))$.

Now, take a point $x \in V(I(V))$. By definition we have $P(x) = 0$ for all polynomials $P \in I(V)$.

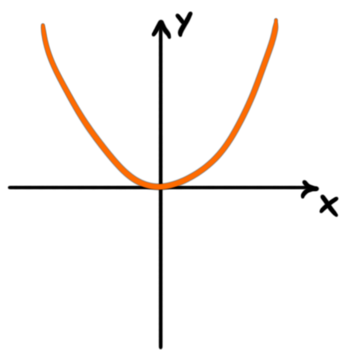
do rest of solution

Problem 2.16 What is the relationship between I and $I(V(I))$?

Solution:

Let $I \subset k[X_1, \ldots, X_n]$ be an ideal and $P \in I$. We have $V(I) = \{x \in k^n | f(x) = 0 \forall f \in I\}$. In particular we have $P(x) = 0$ for all $x \in V(I)$, and thus $P \in I(V(I))$. This shows $I \subset I(V(I))$.

As a counterexample to the equality we can for example take the ideal $I = (X^2 - Y)^2 \subset k[X, Y]$. Then $V(I)$ looks something like



We know that the polynomial $P = (X^2 - Y)^2$ is zero on this affine algebraic set, but so is $X^2 - Y$, which is not in I. Hence $I \neq I(V(I))$ in general.

Problem 2.17 Show that $I(\emptyset) = k[X_1, \ldots, X_n]$.

Solution:

This statement is vacuously true. The set $I(\emptyset)$ consists of all polynomials $P \in k[X_1, \ldots, X_n]$ such that $P(x) = 0$ for all $x \in \emptyset$, which is true because $\emptyset$ contains no elements.

Proposition 2.18

If k is an infinite field then $I(k^n) = (0)$.

Proof. We will use induction on n.

First let $n = 1$. Then $I(k) = \{f \in k[X_1] | f(x) = 0, \forall x \in k\}$. A non-zero polynomial f has a finite set of roots, hence we can find a point $y \in k$ such that $f(y) \neq 0$ due to k being infinite. Hence if we have $f \in I(k)$ then f has to be zero outside of all of its roots, and hence it must be the zero polynomial.

Now let $n \geq 1$. Suppose that $f \in k[X_1, \ldots, X_n] \setminus \{0\}$ is non-constant. We can then express f as $f = a_r(X_1, \ldots, X_n)X_n^r + \ldots$ for some $r \geq 1$. By induction we can find $x_1, \ldots, x_{n-1}$ such that $a_r(x_1, \ldots, x_{n-1}) \neq 0$. Hence $f(x_1, \ldots, x_{n-1}, X_n)$ has at most r roots. We can then find a point y such that $f(y) \neq 0$ due to k being infinite. $\square$

details

Problem 2.19 Show that $I(\{(a_1, \ldots, a_n)\}) = (X_1 - a_1, \ldots X_n - a_n)$, i.e. points correspond to maximal ideals.

Problem 2.20 Compute $I(-)$ and $V(-)$ for some examples.

3 Lecture 3 - 19.01.21

To warm up our brains we start the lecture with a problem to solve.

Problem 3.1 Compute $I(V((X, Y^2)))$ in $k[X, Y, Z]$.

Solution:

The algebraic set $V((X, Y^2))$ consists of all points $(x, y, z) \in k^3$ such that $P(x, y, z) = 0$ for all polynomials $P \in (X, Y^2)$. Polynomials in (X, Y^2) are of the form $P = aX^n + bY^{2m}$, where $a, b \in k[X, Y, Z]$. For these to be equal to zero at (x, y, z) we need that $x = 0 = y$. We also have no restrictions on z. Hence $V((X, Y^2)) = \{(0, 0, z) \in k^3\}$. If we look at $I(V((X, Y^2)))$, then this consists of all polynomials which are zero at all points in $V((X, Y^2)) = \{(0, 0, z) \in k^3\}$, i.e. polynomials generated by X and Y. Hence we have $I(V((X, Y^2))) = (X, Y)$.

3.1 Irreducibility

Definition 3.2: Irreducible

Let $X \neq \emptyset$ be a topological space (for us this means an affine algebraic set with the Zariski topology). We call X irreducible if whenever we can write $X = F \cup G$ with F, G closed subsets of X, then either $F = X$ or $G = X$. If this is not the case, then we call X reducible, or decomposable.

Theorem 3.3

Let $V \subset k^n$ be an affine algebraic set with the Zariski topology. Then V is irreducible if and only if $I(V)$ is a prime ideal in $k[X_1, \dots, X_n]$. Equivalently, V is irreducible if and only if $\Gamma(V)$ is a domain.

Proof. Assume that V is irreducible. We want to show that $I(V)$ is prime, i.e. that if we have two elements $f, g \in k[X_1, \dots, X_n]$ such that $fg \in I(V)$, then either $f \in I(V)$ or $g \in I(V)$ (or both of course).

Assume that we have two such elements f, g such that $fg \in I(V)$. Since $(fg) \subset I(V)$ we have by the order reversing property of $V(-)$ that $V(I(V)) \subset V((fg)) = V(fg)$. By the properties of the Zariski topology we know that $V(fg) = V(f) \cup V(g)$, and thus $V \subset V(f) \cup V(g)$. Then $V = (V \cap V(f)) \cup (V \cap V(g))$, which are two closed subsets of V. Since we assumed that V was irreducible we know that either $V = V \cap V(f)$ or $V = V \cap V(g)$. Assume without loss of generality that $V = V \cap V(f)$. Then we have that $V \subset V(f)$, which by the order reversing property of $I(-)$ means that $I(V(f)) \subset I(V)$. And since we know $f \in I(V(f))$ we have finally $f \in I(V)$.

Assume now that $I(V)$ is prime. We will show V irreducible by showing that a given decomposition of $V = V_1 \cup V_2$, where $V \neq V_1$, $V \neq V_2$, leads to a contradiction to $I(V)$ being prime.

Assume V has such a decomposition, i.e. that V is reducible. Since $V_i \subset V$ we have by the order reversing property of $I(-)$ that $I(V) \subset I(V_i)$. Since $V_i \subsetneq V$ we also have $I(V) \subsetneq I(V_i)$, because $V(I(-)) = Id(-)$ implies that $I(-)$ is an injection. Hence there exists $f_1 \in I(V_1) \setminus I(V)$ and $f_2 \in I(V_2) \setminus I(V)$. But, notice that $f_1 f_2$ in fact vanishes on V since it vanishes on both V_1 and V_2 and hence on their union, which is V. Thus $f_1 f_2 \in I(V)$, but we explicitly chose f_1 and f_2 not in $I(V)$, and hence this is a contradiction to $I(V)$ being a prime ideal, meaning that our assumption about V being reducible must have been wrong. □

Corollary 3.4

If k is an infinite field, then k^n is irreducible.

Proof. When k is infinite we have previously shown that $I(k^n) = (0)$. Since (0) is a prime ideal in $k[X_1, \dots, X_n]$, because it is a domain, we have by the previous theorem that k^n must be irreducible. □

Notice that this is not true in general when k is a finite field.

Theorem 3.5

Let $V \subset k^n$ be a non-empty affine algebraic set. Then there exists a (up to permutation) unique collection of irreducible affine algebraic sets $V_1, \dots, V_r$ such that $V_i \not\subseteq V_j$ for $i \neq j$, and $V = V_1 \cup \dots \cup V_r$.

Proof. There are two parts to this proof, showing such a decomposition exists, and showing it is unique. We start by showing existence.

Assume that we have a non-decomposable affine algebraic set V. This has a corresponding ideal $I(V)$. Since k is a field, then $k[X_1, \dots, X_n]$ is Noetherian by the Hilbert basis theorem. This means we can choose V to be the affine algebraic set such that the ideal $I(V)$ is maximal. Here we don't necessarily mean that $I(V)$ is a maximal ideal, but that is the biggest with respect to the property that $I(V)$ is non-decomposable. Since V is assumed non-decomposable we must have a decomposition $V = F \cup G$ of V into closed subsets such that $F \neq V$, $G \neq V$.

We previously said that $I(-)$ is injective, and hence we have $I(V) \subsetneq I(F)$ and $I(V) \subsetneq I(G)$. Since $I(V)$ was maximal among the ideals of non-decomposable affine algebraic sets we must have that F and G are decomposable. Then we have $F = \bigcup_{i=1}^{s} V_i$ and $G = \bigcup_{i=s+1}^{r} V_i$, where V_i is irreducible. This of course gives us a decomposition $V = V_1 \cup \dots \cup V_r$ into irreducible sets. By potentially removing some overlapping sets we also get $V_i \not\subseteq V_j$ for $i \neq j$. Hence all affine algebraic sets are decomposable.

Lets show that this decomposition is unique up to permutation.

Assume we have two decompositions $V = V_1 \cup \cdots \cup V_r$ and $V = W_1 \cup \cdots \cup W_s$. Then we have $V_1 = V \cap V_1 = (W_1 \cap V_1) \cup \cdots \cup (W_s \cap V_1)$. Since V_1 is irreducible by assumption we must have $V_1 = W_j \cap V_1$ for some j. This implies $V_1 \subseteq W_j$. Similarily we have $W_j = V \cap W_j = (V_1 \cap W_j) \cup \cdots \cup (V_r \cap W_j)$, which by the irreducibility of W_j must mean that $W_j = V_k \cap W_j$ for some k, and hence $V_j \subseteq V_k$.

But this means that we have $V_1 \subseteq W_j \subseteq V_k$, which cant be the case as $V_1 \nsubseteq V_i$ for all $i \neq 1$. This means that $k = 1$, and in turn $V_1 = V_k$. Since W_j is squeezed in the middle of two equal sets, it also mus be equal to them, i.e. $V_1 = W_j$. We now reorder the decomposition into $V = W_j \cup W_1 \cup \cdots W_{j-1} \cup W_{j+1} \cup \cdots \cup W_s$ and repeat the process by using V_2 instead. We can continue this process until we have associated every V_i with some W_j, which means that $r = s$ and that the two decompositions are the same after some reordering. □

3.2 Hilbert's nullstellensatz

For the rest of this lecture we will assume that k is an algebraically closed field. This is essential to the theorem. We will first prove the weak nullstellensatz and use that to prove the strong one.

Theorem 3.6: Hilbert's weak nullstellensatz

Let $I \subsetneq k[X_1, \ldots, X_n]$ be a proper ideal. Then $V(I) \neq \emptyset$.

In class we omitted the proof of this due to proving it in the course MA8202 - Commutative algebra, which this course has as a prerequisite. But, for completeness sake I have added a proof.

Proof. Every proper ideal is, or is contained in a maximal ideal. Hence, for some maximal ideal $M \subseteq k[X_1, \ldots, X_n]$ we have $I \subseteq M$. By the order reversing property of $V(-)$ we get $V(M) \subseteq V(I)$, so it is in fact enough to look at maximal ideals.

In the last lecture we saw that $I \subseteq I(V(I))$ for some ideal $I \subseteq k[X_1, \ldots, X_n]$. If we apply this to M we get $M \subseteq I(V(M))$, which implies either $I(V(M)) = M$ or $I(V(M)) = k[X_1, \ldots, X_n]$ as M is a maximal ideal. By section 3.1 this means that either $V(M)$ is an irreducible affine algebraic set, as $I(V(M)) = M$ is a prime ideal, or that we have $I(V(M)) = k[X_1, \ldots, X_n]$ which means $V(M) = \emptyset$. Hence we need to justify that there exists points in $V(M)$.

Notice that for some point $a = (a_1, \ldots, a_n) \in k^n$ we have that $M_a = (X_1 - a_1, \ldots, X_n - a_n)$ is a maximal ideal. If we somehow could prove that all ideals are of this form, then we would be done as $V(M_a)$ would contain $(a_1, \ldots, a_n)$ and hence be non-empty. So lets prove this.

We define the evaluation morphism as follows:

$$\begin{aligned} e_a : k[X_1, \ldots, X_n] &\longrightarrow k \\ f &\longmapsto f(a). \end{aligned}$$

Note that it is a surjective k-algebra homomorphism and since k is algebraically closed, it has kernel M_a.

Let M be some arbitrary maximal ideal in $k[X_1, \ldots, X_n]$. Then $k[X_1, \ldots, X_n]/M$ is a finitely generated field extension of k. By Zariski's lemma, $k[X_1, \ldots, X_n]/M$ is in fact a finite field extension, better known as a finite dimensional vector space. Since k is algebraically closed, there is an isomorphism of k-algebras

$$k[X_1, \ldots, X_n]/M \longrightarrow k.$$

Now, let a_i denote the image of X_i. Then we get that $M_a \subseteq M$, which implies $M_a = M$ since M_a is a maximal ideal.

Hence we know that any maximal ideal M will have $V(M) \neq$, and hence we are done. □

Notice here that we used Zariski's lemma in our proof. This states that if K is a finitely generated k-algebra, such that K is also a field, then K is a finite field extension of k, i.e. a finite dimensional k-vector space. The proof is omitted here, but a discussion about the geometry behind it can be found on my blog.

Also note that the affine algebraic set $V((X^2 + Y^2 + 1))$ is empty in $\mathbb{R}^2$, even though $(X^2 + Y^2 + 1)$ is prime in $\mathbb{R}[X, Y]$. Hence the weak nullstellensatz does not hold for non-algebraically closed fields.

Theorem 3.7: Hilbert's nullstellensatz

Let $I \subseteq k[X_1, \ldots, X_n]$ be an ideal. Then $I(V(I)) = \sqrt{I}$.

I didn't quite understand the proof presented in class. I will still put it below, but I have also put a (in my opinion) easier to understand proof in the appendix. It seems to be essentially the same, but some parts are wrapped up nicely in Zariski's lemma. The proof can be found at **??**.

Proof. Since $k[X_1, \ldots, X_n]$ is noetherian we know that ideals are finitely generated. So, choose generators $I = (P_1, \ldots, P_r)$. We first prove the inclusion $\sqrt{I} \subseteq I(V(I))$.

Let $f \in \sqrt{I}$. By definition this means that $f^m \in I$ for some m. Since $f^m \in I$ we know that f^m vanishes on $V(I)$, i.e. $f^m \in I(V(I))$. But, if f^m vanishes on $V(I)$, then so does f, hence $f \in I(V(I))$.

For the other inclusion we let $f \in I(V(I))$, and we want to show that there is a m such that $f^m \in I$. We note that it is enough to show that $Ik[X_1, \ldots, X_n]_f = k[X_1, \ldots, X_n]_f$. This is because we would have $1_{k[X_1,\ldots,X_n]_f} = \sum P_i \frac{Q_i}{f^{n_i}}$, which would imply that $f^m = \sum P_i(Q_i f^{m-n_i})$, where $m = max\{n_i\}$. This means that we would have written f^m as a linear combination of parts with a P_i in each summand, which means $f^m \in (\{P_i\}) = I$.

We note that $k[X_1, \dots, X_n]_f \cong k[X_1, \dots, X_n, T]/(1 - Tf)$. This is like saying that inverting all powers of f, which is what localizing at f does, is the same as adding a new variable to the polynomial ring with the property that "it is the inverse of f".

Now we have $Ik[X_1, \dots, X_n]_f = (P_1, \dots, P_r, 1-Tf)/(1-Tf)$. Set $J = (P_1, \dots, P_r, 1-Tf) \subseteq k[X_1, \dots, X_n, T]$. We claim that $V(J) = \emptyset \subset k^{n+1}$. Suppose that this is not the case. This means that there is an element $(x_1, \dots, x_n, t) \in V(J)$. We have $P_i(x_1, \dots, x_n) = 0$, hence $(x_1, \dots, x_n) \in V(I)$. This implies that $f(x_1, \dots, x_n) = 0$ as we have chosen $f \in I(V(I))$. But then $(1 - Tf)(x_1, \dots, x_n) \neq 0$ which means that $(x_1, \dots, x_n, t) \notin V(J)$, which is a contradiction. Hence we must have $V(J) = \emptyset$.

By the weak nullstellensatz we then have $J = k[X_1, \dots, X_n, T]$, which means $Ik[X_1, \dots, X_n]_f = k[X_1, \dots, X_n]_f$ and by the previous discussion that $f^m \in I$ which by definition means $f \in \sqrt{I}$. □

One immediate application is that we now have a bijection between the set of affine algebraic sets in k^n and the radical ideals in $k[X_1, \dots, X_n]$, given by $I(-)$ and $V(-)$.

4 Lecture 4 - 25.01.21

4.1 Applications of Hilbert's nullstellensatz

Last time we proved Hilbert's nullstellensatz, that tells us the precise duality between the algebra and the geometry. It states that $I(V(I)) = \sqrt{I}$. Since we already have $V(I(V)) = V$ for affine algebraic sets V we now have the following correspondence when we restrict ourselves to only radical ideals, i.e. ideals such that $\sqrt{I} = I$.

Proposition 4.1

The assignments $V(-)$ and $I(-)$ gives us a bijection between the affine algebraic sets in k^n and the radical ideals in $k[X_1, \ldots, X_n]$.

Moreover we get that

1. V is irreducible if and only if $I(V)$ is prime.
2. V is a point if and only if $I(V)$ is maximal.

The first point we have proven earlier, but lets prove the second one.

Proof. Assume $V = \{x\}$. We know that $I(V)$ is an ideal, and that it is contained in some maximal ideal $I(V')$. By the order reversing property of $V(-)$ we get $V(I(V')) \subset V(I(V))$. But, we knot that $V(I(V)) = V$, hence $V' \subset V = \{x\}$, so either V' is equal to V or V' is empty. But the latter cant be true by the weak nullstellensatz (section 3.2). Hence $V' = V$ which means that $I(V)$ is maximal.

For the converse we assume that $I(V)$ is maximal. We know again by the weak nullstellensatz that V is non-empty, so we can take $x \in V$. By the order reversing property of $I(-)$ we have that $I(V) \subset I(\{x\})$.. But $I(V)$ is assumed maximal, hence either $I(\{x\}) = k[X_1, \ldots, X_n]$ or $I(\{x\}) = I(V)$. But the former cant be true as the variety generating the whole polynomial ring is the empty set, which we know $\{x\}$ isn't. Hence $I(\{x\}) = I(V)$ which means $V = \{x\}$ is a point. $\square$

Do we know that every maximal ideal comes from a variety?

Proposition 4.2

Let $V \subset k^n$ be a affine algebraic set. Then V is finite if and only if $\Gamma(V)$ is a finite dimensional k-vector space.

Proof. Assume first that $V = \{u_1, \ldots, u_r\}$ is a finite set and consider the ring homomorphism

$$\begin{aligned}\phi\colon k[X_1, \ldots, X_n] &\longrightarrow k^r\\ F &\longmapsto (F(u_1), \ldots, F(u_r))\end{aligned}$$

Notice that the kernel of ϕ are the maps F such that $(F(u_1), \ldots, F(u_r)) = (0, \ldots, 0)$, in other words they are the maps that vanish on all points in V. Hence $\operatorname{Ker}\phi = I(V)$. This means that we have $\Gamma(V) = k[X_1, \ldots, X_n]/\operatorname{Ker}\phi$, which by the first isomorphism theorem is isomorphic to $\Im\phi \subset k^r$, i.e. $\Gamma(V) \cong k^s$, where $s \leq r$. This shows that $\Gamma(V)$ is a finite dimensional vector space.

For the other direction we assume that $\Gamma(V)$ is a finite dimensional vector space. There exists $s \geq 1$ and $a_i \in k$ such that

$$P_j = a_s\bar{X}_j^{\,s} + a_{s-1}\bar{X}_j^{\,s-1} + \ldots + a_1\bar{X}_j + a_0 1 = 0$$

where $\bar{X}_j^{\,i}$ are linearly dependent over k. If $u = (x_1, \ldots, x_n) \in V$, then $P_j(u) = 0$ for each j. These are polynomials in just one variable, hence only have a finite set of roots, meaning V must be finite as it vanishes on them all. □

Problem 4.3 Find a k-basis for $\Gamma(V) = k[X, Y]/I(V)$ where $V = V(Y^2, Y - X^2 + 1)$.

Solution:

Enter solution

Let now $W \subset V$ both be affine algebraic sets. Then we have $I(V) \subset I(W)$. In $\Gamma(V)$ we have an ideal $I(W)/I(V)$ as $I(W) \subset k[X_1, \ldots, X_n]$. We denote this ideal by $I_V(W)$. Notice that the inclusion $I(V) \to I(W)$ induces a surjection $\Gamma(V) \to \Gamma(W)$ which has kernel $I_V(W)$. Hence we have by the first isomorphism theorem $\Gamma(V)/I_V(W) \cong \Gamma(W)$.

This fact generalizes the correspondence we had earlier as a consequence of the nullstellensatz.

Proposition 4.4

The assignments $V(-)$ and $I(-)$ gives us a bijection between the affine algebraic sets in V and the radical ideals in $\Gamma(V)$.

Similarly to last time we get that

1. W is irreducible if and only if $I_V(W)$ is prime in $\Gamma(V)$
2. W is a point in V if and only if $I_V(W)$ is maximal in $\Gamma(V)$
3. W is an irreducible component of V if and only if $I_V(W)$ is a minimal prime ideal of $\Gamma(V)$.

Corollary 4.5

The points in V are in a one-to-one correspondence with the maximal ideals in $\Gamma(V)$.

Problem 4.6 Find all the maximal ideals in $k[X,Y]/(XY)$.

Write down solution

4.2 First steps towards Bezout's theorem

We want to show that even stating Bezout's theorem makes sense. By this we mean that the intersection points of two plane curves actually can be counted. Before we state this as a theorem precisely we prove a lemma we will need in the proof of the theorem.

Lemma 4.7

Let $F, G \in K[X,Y]$ be two non-zero polynomials with no common factors. Then there exists a non-zero polynomial $d \in k[X]$ such that $d = AF + BG$ for some $A, B \in k[X,Y]$. In particular $d \in (F,G)$.

Proof. Let $k(X)$ denote the field of fractions of $k[X]$. As this is a field we have that adjoining a variable Y, i.e. $k(X)[Y]$ is a PID. This means that the ideal (F,G) generated by F and G is actually generated by a single element, which we denote by d. Now, d must divide both F and G, hence $F = d \cdot \frac{f}{p(x)}$ and $G = d \cdot \frac{g}{q(x)}$. By multiplying with the denominators we get $p(x)F = df$ and $q(x)G = dg$ in $k[X,Y]$. Since our polynomial ring is over a field we are in a UFD, hence we have unique factorization. This means that d divides $p(x)$ or d divides $q(x)$, meaning $d \in k[X]$. $\square$

Theorem 4.8: 5.1

Let $F, G \in K[X,Y]$ be two non-zero polynomials with no common factors. Then $V(F) \cap V(G)$ is finite.

We will prove the theorem next time.

5 Lecture 5 - 26.01.21

We will first prove the theorem we left off at last time. We recall its statement.

Theorem 5.1: 5.1

Let $F, G \in K[X, Y]$ be two non-zero polynomials with no common factors. Then $V(F) \cap V(G)$ is finite.

Proof. Let $(x, y) \in V(F) \cap V(G)$. This means in particular that $F(x, y) = 0 = G(x, y)$. By the lemma we proved last time there exists some $d \in k[X]$ such that $d = AF + BG$. Since both F and G vanish on (x, y) we have $d(x) = 0$. Since d is a polynomial in one variable over a field it has only a finite set of roots. Hence we only have finite choices for x.

Symmetrically we can find a polynomial $d' \in k[Y]$ with the same above properties. By the same argument er only have finite choices for y. Hence $V(F) \cap V(G)$ is a finite set. ☐

Theorem 5.2: 5.2

Let $F, G \in K[X, Y]$ be two non-zero polynomials with no common factors. Then $k[X, Y]/(F, G)$ is a finite dimensional vector space.

Problem 5.3 Compare this to section 4.1

Proof. By the same lemma used before we know there exists $d, d' \in (F, G)$ such that $d \in k[X]$ and $d' \in k[Y]$. Write

- $d(X) = X^n + a_{n-1}X^{n-1} + \ldots + a_0$
- $d'(Y) = Y^m + b_{m-1}Y^{m-1} + \ldots + b_0$

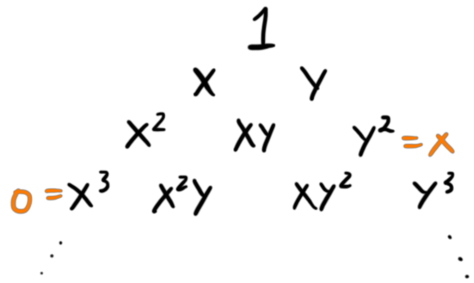
There are both zero in $k[X, Y]/(F, G)$ as they belong to (F, G). Hence, in $k[X, Y]/(F, G)$ we have $X^n \in (X^{n-1}, \ldots, X, 1)$ and $Y^m \in (Y^{m-1}, \ldots, Y, 1)$, which means

$$\{X^i Y^j\}_{0 \leq i \leq n-1, 0 \leq j \leq m-1}$$

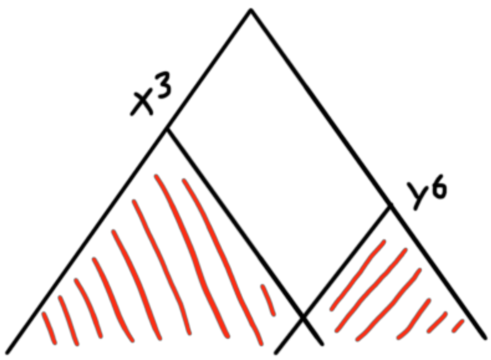
is a generating set for $k[X, Y]/(F, G)$. This set is finite, which means we are done we are done. ☐

Example 5.4 Find a generating set for $k[X, Y]/(X^3, Y^2 - X)$.

We can think of the generators as the following system.



The relations makes some products of generators die. If we continue this we get something like:



And we can then count the ones that have not perished (the ones inside the non-red area).

5.1 Introduction to morphisms

For this part we let k be an infinite field.

Definition 5.5

Let $V \subset k^n$, $W \subset k^m$ be affine algebraic sets. A morphism, called a regular map, $\phi : V \longrightarrow W$ is a collection of maps $\phi_i : k^n \longrightarrow k^m$ such that each ϕ_i is polynomial, i.e. $\phi_i \in \Gamma(V)$.

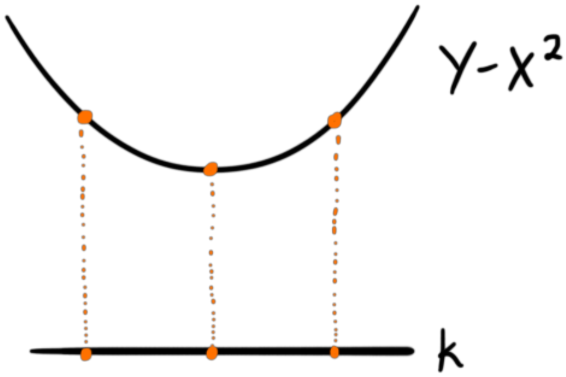
We denote the set of regular maps between V and W by $Reg(V, W)$. These maps make the collection of affine algebraic sets over k into a category, denoted $Aff(k)$.

Remark: The regular maps are continuous in the Zariski topology, but they are not all continuous maps.

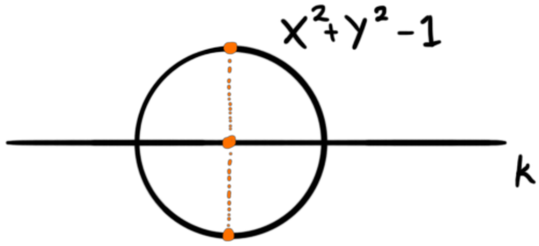
Example 5.6 Elements of $\Gamma(V)$ are the regular maps $V \longrightarrow k$.

Example 5.7 Any projection $V \subset k^n \longrightarrow k^p$, where $p \leq n$ is a regular map.

Example 5.8 $\phi : V(Y - X^2) \longrightarrow k$ defined by $(x, y) \mapsto x$ is actually a regular isomorphism. The inverse is given by $\phi^{-1}(x) = (x, x^2)$. Visually it looks like



Example 5.9 $\phi : V(X^2 + Y^2 - 1)$ sending $(x, y) \mapsto x$ is regular but not injective. It looks like



Let $\phi : V \longrightarrow W$ be regular map. For any $f \in \Gamma(W)$ we set $\phi^*(f) = f \circ \phi \in \Gamma(V)$, i.e. the pre-composition with ϕ. This makes Γ into a contravariant functor. It sends V to $\Gamma(V)$ and $\phi : V \longrightarrow W$ to $\phi^* : \Gamma(W) \longrightarrow \Gamma(V)$.

Proposition 5.10

The functor Γ is a fully faithful functor.

We can in fact show something even stronger.

Theorem 5.11

Let k be algebraically closed, and let $ftr - Alg_k$ denote the category of finite type reduced k-algebras. Then we have an equivalence of categories

$$\Gamma : Aff(k) \longrightarrow ftr - Alg_k$$

Before we prove this lets justify that Γ only hits these types of algebras. We know that $\Gamma(V) \cong k[X_1, \ldots, X_n]/I(V)$ is a k-algebra, and it is generated by $x_1, \ldots, x_n$ and is thus of finite type. Such a k-algebra is reduced if and only if $I(V)$ is a radical ideal. This we know is true by Hilbert's nullstellensatz, hence we know that algebras in the image of Γ are in $ftr - Alg_k$.

Proof. Lets first show that Γ is fully faithful. This means that it is an isomorphism on the sets of morphisms. We start with showing it is injective.

Let $\phi, \psi_V \longrightarrow W$ be regular maps such that $\phi^* = \psi^*$. Through the isomorphisms $\Gamma(W) \cong k[Y_1, \ldots, Y_m]/I(W)$ and $\Gamma(V) \cong k[X_1, \ldots, X_n]/I(V)$ we have that ϕ^* sends Y_i to ϕ_i. Since ψ^* also does this they must have the same images, i.e. $\phi_i = \psi_i$. Since all the components are the same the maps are the same.

We now show Γ is surjective on the morphisms.

Let $\theta : k[Y_1, \ldots, Y_m]/I(W) \longrightarrow k[X_1, \ldots, X_n]/I(V)$ be an algebra morphism. Set $\phi_i = \theta(Y_i)$ and $\phi = (\phi_1, \ldots, \phi_m)$. We claim that $\phi : V \longrightarrow W$, i.e. $\operatorname{Im} \phi \subseteq W$ and hence that Γ is surjective on morphisms.

For Γ to be an equivalence of categories we also need it to be a dense functor, also called essentially surjective.

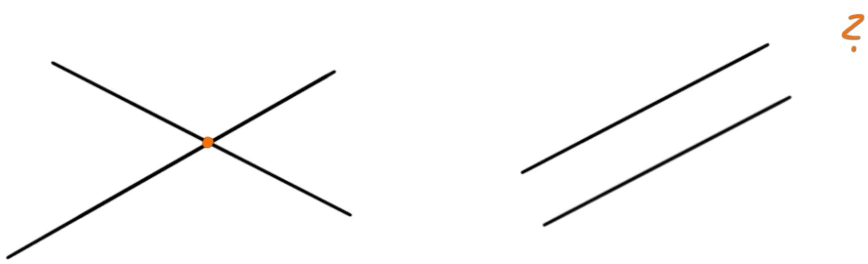
Let A be a finite type reduced k-algebra. Since A is of finite type we have some generators $X_1, \ldots, X_n$ and some relations, that generate an ideal I, such that $A = k[X_1, \ldots, X_n]/I$. Since A is reduced wee know that I is a radical ideal. By Hilbert's nullstellensatz we know that $I = I(V(I))$, and hence we have $A \cong \Gamma(V(I))$.

This shows that Γ is an equivalence of categories. □

Show surjectivity

5.2 Projective algebraic sets

Affine algebraic sets are nice and easy, but they have some problem. Most notably for us now is the case of intersection points of curves. The general structure is that two lines always meet at a point, but this has an exception, namely parallel curves.



The reason we move to projective algebraic sets instead is that these exceptions disappear.

Definition 5.12

Let $n \geq 0$ be an integer and let E be a k-vector space of dimension $n+1$. For $x, y \in E \setminus \{0\}$ we define the relation $x \sim y$ if there exists a $\lambda \in k^\times$ such that $y = \lambda x$.

Problem 5.13 Show that this relation is an equivalence relation.

Note that the equivalence classes of this relations are the lines in E through the origin.

Definition 5.14

The projective space of E, denoted $\mathbb{P}(E)$, is the set $(E \setminus \{0\})/\sim$, i.e. the set of lines in E.

If $E = k^n$ then $\mathbb{P}(E) = \mathbb{P}^n(k)$ is called the standard projective n-space.

Let $\pi : k^{n+1} \setminus \{0\} \longrightarrow \mathbb{P}^n(k)$ be the canonical projection, and let $x = (x_0, \dots, x_n) \in k^n \setminus \{0\}$. Then $\bar{x} = \pi(x)$ is a point in $\mathbb{P}^n(k)$ with homogeneous coordinates $(x_0, \dots, x_n)$ often written $[x_0 : \cdots : x_n]$.

Note that if $\lambda \neq 0$, then $[\lambda x_0 : \cdots : \lambda x_n]$ is another system of homogeneous coordinates for $\bar{x}$.

Example 5.15 Let $k = \mathbb{R}$. Then $\mathbb{P}^1(\mathbb{R})$ is the set of lines trough the origin in $\mathbb{R}^2$.

Problem 5.16 Read and learn about projective space.

6 Lecture 6 - 01.02.21

The lecture was focused on going through the exercises. Solutions and notes to these can be found at **??**.

7 Lecture 7 - 02.02.21

7.1 Projective algebraic sets

We recall the definition of the projective space of a vector space E from last time.

Definition 7.1

Let E be a $n{+}1$-dimensional vector space over a field k. The projective space of E, denoted $\mathbb{P}(E)$, is the set $(E \setminus \{0\})/\sim$, where $x \sim y \iff \exists \lambda$ such that $x = \lambda y$.

If $E = k^{n+1}$ then we denote $\mathbb{P}(E)$ by $\mathbb{P}^n(k)$, which we call projective n-space. We have a map

$$\begin{aligned} k^{n+1} &\longrightarrow \mathbb{P}^n(k) \\ (x_0, \dots, x_n) &\longmapsto [x_0 : \dots : x_n] \end{aligned}$$

We call $(x_0, \dots, x_n)$ the homogeneous coordinates.

Definition 7.2

Let E be a $n+1$-dimensional vector space over k, $0 \leq m \leq n$ be an integer and F a $m+1$-dimensional subspace of E. The image of $F \setminus \{0\}$ in $\mathbb{P}(E)$ is called the projective subspace of dimension m, denoted $\overline{F}$.

If $m = 0$ then $\overline{F}$ is a point. If $m = 1$ then $\overline{F}$ is a line. If $m = n - 1$ then $\overline{F}$ is a hyperplane.

Proposition 7.3

Let E be a $n + 1$-dimensional vector space over k.Let further V, W be two projective subspaces of $\mathbb{P}(E)$ with dimension r and s respectively such that $r + s - n \geq 0$. Then $V \cap W$ is a projective subspace of dimension greater than $r + s - n$. In particular, $V \cap W \neq \emptyset$.

Remark: This proposition is not true in the affine case. This is due to the existance of paralell lines, which are both affine subspaces, but has empty intersection.

Proof. We write $V = \overline{F_V}$ and $W = \overline{F_W}$ for F_V, F_W two subspaces of E with dimension $r + 1$ and $s + 1$ respectively. Note that the intersection of two vector subspaces is again a vector subspace, hence $\overline{F_V \cap F_W}$ is actually a projective subspace. Moreover we have:

$$\begin{aligned} dim(F_V \cap F_W) &= dim(F_V) + dim(F_W) - dim(F_V + F_W) \\ &\geq (r + 1) + (s + 1) - (n + 1) \\ &= r + s - n + 1 \end{aligned}$$

This means that $dim(V \cap W) = dim(\overline{F_V} \cap \overline{F_W}) = dim(\overline{F_V \cap F_W}) \geq r + s - n$. $\square$

Example 7.4 Two distinct lines in $P^2(k)$ meet at a unique point.

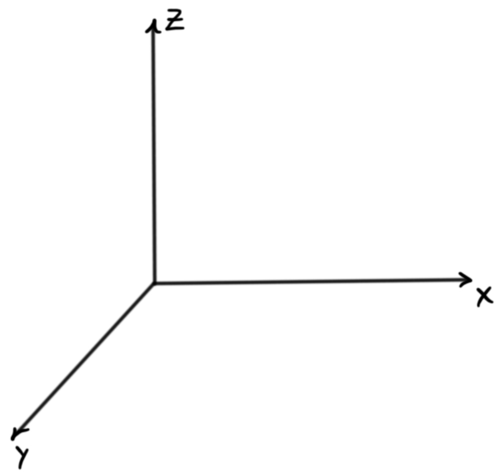
7.2 Homography

We will not cover this in detail, but we quickly mention what these are. Let u be a vector space isomorphism on a vector space E, i.e. $u \in GL(E)$. The induced map $\overline{u} : \mathbb{P}(E) \longrightarrow \mathbb{P}(E)$ is called a homography. These are isomorphisms of projective spaces, and are as you can see the ones induced by vector space isomorphisms.

7.3 What does projective space look like?

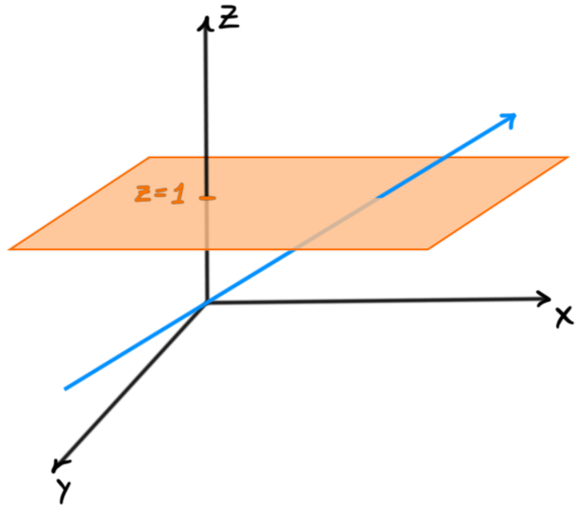
For easy visualization we let $k = \mathbb{R}$. Since we are on our goal to understand and prove Bezouts theorem, we are mostly interested in $\mathbb{P}^2(k)$, so lets try to visualize and understand $\mathbb{P}^2(\mathbb{R})$. I did this a bit in 5.2 but it never hurts to do it again.

We start by drawing $\mathbb{R}^3$, as this is where we get our points from.



As we have $x = y$ in $\mathbb{P}(E)$ whenever there exists a λ such that $\lambda x = y$ we get in our coordinates that $[x : y : z] = [\frac{x}{z} : \frac{y}{z} : 1]$ which means that we can just look a the plane $z = 1$ in $\mathbb{R}^3$.

Every point on that plane uniquely determines a line through the origin, as visualized below.



This means that much of $\mathbb{P}^2(\mathbb{R})$ acts like $\mathbb{R}^2$, and we get the intuitive equality $\mathbb{P}^2(\mathbb{R})$" $=$" $\mathbb{R}^2 \cup \{\text{points at infinity}\}$. Lets make this a bit more rigorous.

We define $H = V(z) \subseteq \mathbb{R}^3$, and $\overline{H} \subset \mathbb{P}^2(\mathbb{R})$ to be its projective space. We then set $U = \mathbb{P}^2(\mathbb{R}) \setminus \overline{H}$. There is a bijection

$$\begin{aligned} \phi : U &\longrightarrow \mathbb{R}^2 \\ [x : y : z] &\longmapsto (\frac{x}{z}, \frac{y}{z}) \\ [x : y : 1] &\longleftarrow\!\shortmid (x, y) \end{aligned}$$

Furthermore, the inclusion $\overline{H} \hookrightarrow \mathbb{P}^2(\mathbb{R}), [x : y : 0] \mapsto [x : y : 0]$ splits by the map

$$\begin{aligned} \mathbb{P}^2(\mathbb{R}) &\longrightarrow \overline{H} \\ [x : y : z] &\longmapsto [x : y : z] \end{aligned}$$

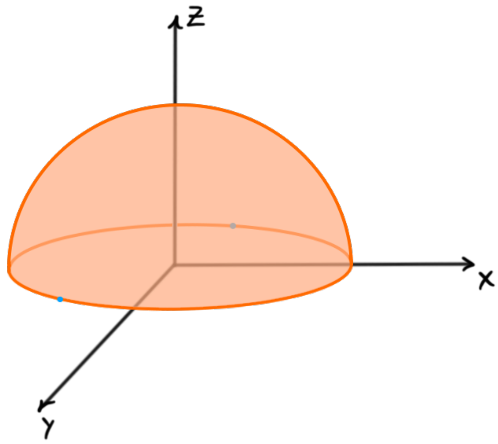
meaning that we have $\mathbb{P}^2(\mathbb{R}) = \overline{H} \coprod U = \mathbb{P}^1(\mathbb{R}) \coprod \mathbb{R}^2$. These we think about as "points at infinity" and "points at a finite distance" respectively.

Problem 7.5 What does $\mathbb{P}^1(\mathbb{R})$ look like?

Solution:

By the same argument as earlier we can look at the $y = 1$ line in $\mathbb{R}^2$ to determine how $\mathbb{P}^1(\mathbb{R})$ looks. Every point on the line determines uniquely a line with a unique slope m. All possible slopes $m \in \mathbb{R}$ are hit by one such line, except $m = 0$. This line never intersects the $y = 1$ line, and is hence designated as the "point at infinity". As we have a line isomorphic to $\mathbb{R}$, with the same "point at infinity" in both ends, this shape is homeomorphic to the circle S^1. We think of this as folding the real line into a circle and gluing it together at this "point at infinity".

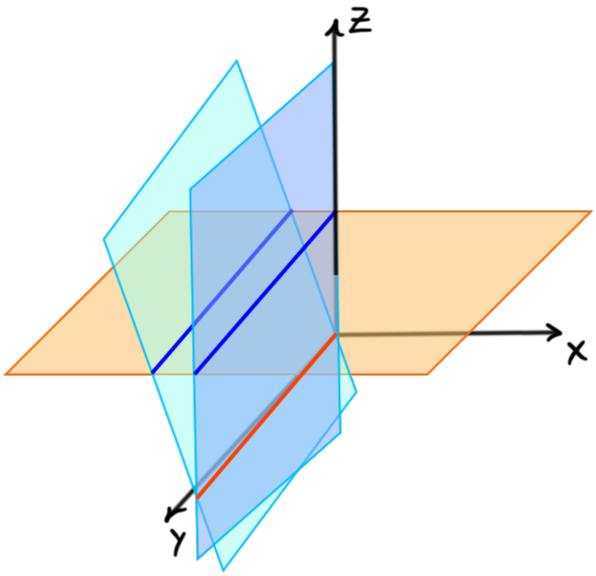
Another way to visualize projective space is to use spheres. Since the space $\mathbb{R}^3 \setminus \{0\}$ is homeomorphic to S^2 we can study that instead. A line through the origin intersects the sphere at two points. These points are antipodal. We can then reduce to thinking about the half-sphere instead.



We still have identified the antipodal points at the equator, but all other points correspond to a unique line, i.e. a point in projective space. The points on the equator are then the "points at infinity" while the non-equatorial points are the "points at finite distance". As with $\mathbb{P}^1(\mathbb{R})$ we can think of this as gluing each line in the plane $z = 1$ to its endpoints at infinity.

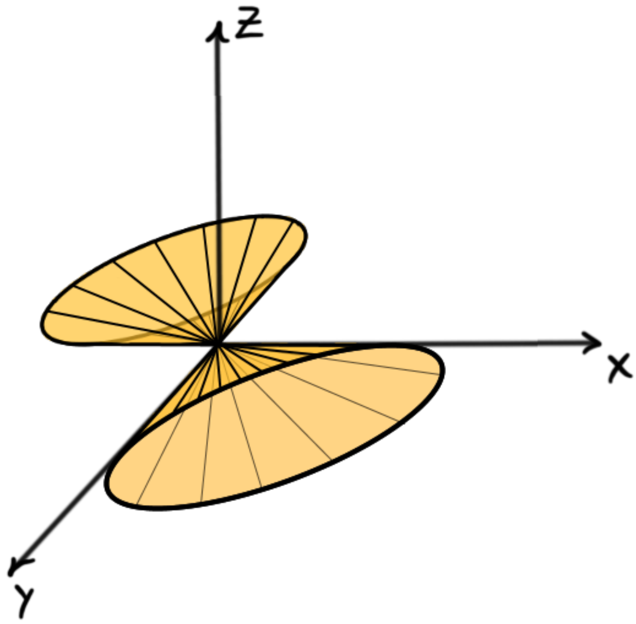
Example 7.6 We can also study projective lines in $\mathbb{P}^2(\mathbb{R})$. Lets see what happens with "parallel lines", as these are the ones we are trying to get rid of using projective space instead of affine.

A 1-dim projective line is the image of a 2-dim subspace of $\mathbb{R}^3$, i.e. a plane. This means that the points at finite distance of the projective line is the intersection of that plane and the plane $z = 1$. We draw this for two projective lines.

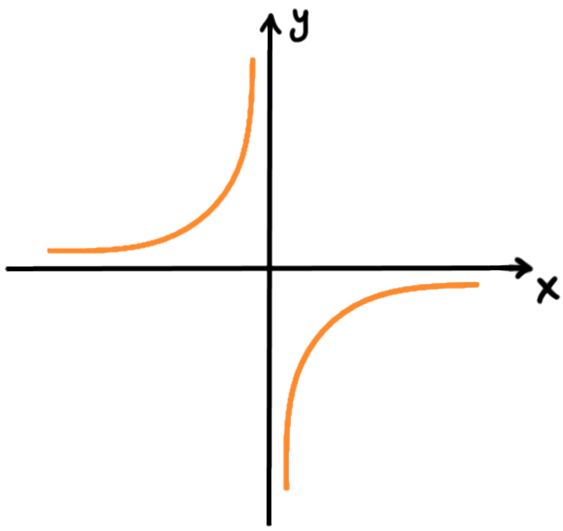


We see that the two planes cut out two "parallel lines" on the plane. But, the planes also intersect elsewhere, namely at the red line in the xy-plane. That line determines a point at infinity in $\mathbb{P}^2(\mathbb{R})$, hence the two projective lines do indeed meet at a point.

Another example is the affine algebraic set given by $C = V(XY - Z^2) \subseteq \mathbb{R}^3$.



We denote $\overline{C}$ its image in $\mathbb{P}^2(\mathbb{R})$. It intersects the plane $z = 1$ at the graph of $xy = 1$.



$\overline{C}$ also intersects $V(Z)$ at two points in the $z = 0$ plane, namely $[1 : 0 : 0]$ and $[0 : 1 : 0]$, i.e. the two axes.

We see that the two "points at infinity" correspond to the asymptotes of the graph that intersects $z = 1$, i.e. the asymptotes of $xy = 1$.

Problem 7.7 Where does $\overline{C}$ intersect the projective line $x - z = 0$?

Solution:

Solution

We see by the previous problem that the conics (circles, ellipses, hyperbola, parabola) are all "affine constructs". In projective space, these properties could be characterized by how the conic cuts the line at infinity. It can cut it into 0, 1 or 2 parts, which correspond to the three types of conics.

8 Lecture 8 - 08.02.21

8.1 Projective algebraic sets

A problem we run into while trying to define projective algebraic sets if we were to do it similarly to affine algebraic sets is that polynomials are not always functions on $\mathbb{P}^n(k)$.

Definition 8.1

Let $\overline{x} \in \mathbb{P}^n(k)$ and $F \in k[X_0, \ldots, X_n]$. We say $\overline{x}$ is a zero of F if $F(\lambda x) = 0$ for all $\lambda \in k^\times$. We write $F(\overline{x}) = 0$ even though F is not necessarily a function.

Proposition 8.2

If $F \in k[X_0, \ldots, X_n]$ is a homogeneous polynomial and $F(x) = 0$ for some x, then $F(\overline{x}) = 0$, i.e. $\overline{x}$ is a zero of F.

Proposition 8.3

Decompose the polynomial F into its homogeneous components, i.e. $F = F_0 + F_1 + \ldots + F_r$, where $r = deg(F)$ and F_i is the degree i homogeneous component. Then $F(\overline{x}) = 0 \iff F_i(\overline{x}) = 0$ for all i.

Problem 8.4 Proove this.

Definition 8.5

Let $S \subseteq k[X_0, \ldots, X_n]$ be a subset. We call

$$V_{\text{proj}}(S) = \{\overline{x} \in \mathbb{P}^n(k) | F(\overline{x}) = 0, \forall F \in S\}$$

the projective algebraic set of S.

We note that $V_{\text{proj}}(S) = V_{\text{proj}}((S))$, where (S) if the ideal generated by S. This is exactly the same as for the affine case. As $k[X_0, \ldots, X_n]$ is noetherian, we can by Hilbert's basis theorem assume that S is a finite set.

Example 8.6 $V_{\text{proj}}((0)) = \mathbb{P}^n(k)$

Before we see the next example we need one more definition.

Definition 8.7

The ideal $R^+ = (X_0, \dots, X_n) \subset k[X_0, \dots, X_n]$, i.e. the ideal generated by the indeterminates, is called the irrelevant ideal.

Example 8.8 $V_{\text{proj}}(R^+) = \emptyset$. This is because $0 \notin \mathbb{P}^n(k)$.

Example 8.9 Let $\overline{x} = [x_0 : \dots : x_n] \in \mathbb{P}^n(k)$. As $0 \notin \mathbb{P}^n(k)$ we can without loss of generality assume that $x_0 \neq 0$. This means that $\overline{x} = [1 : x_1/x_0 : \dots : x_n/x_0]$. We can relabel to get $\overline{x} = [1 : x_1 : \dots : x_n] \in \mathbb{P}^n(k)$.

Then $\{\overline{x}\} = V_{\text{proj}}(X_1 - x_1X_0, \dots, X_n - x_nX_0)$.

In the affine case we had $\{x\} = V(X_0 - x_0, \dots, X_n - x_n)$, but when we want projective algebraic sets we need homogeneuity. This we can get y multiplying by X_0, which acts as 1.

As in the affine case we have some standard properties these projective algebraic sets satisfy:

1. If $S \subseteq S'$ then $V_{\text{proj}}(S') \subseteq V_{\text{proj}}(S)$, i.e. $V_{\text{proj}}(-)$ has the order reversing property.
2. $\cap_{i \in I} V_{\text{proj}}(S_i) = V_{\text{proj}}(\cup_{i \in I} S_i)$ for an arbitrary index set I.
3. $\cup_{i=1}^n V_{\text{proj}}(S_i) = V_{\text{proj}}(\prod_{i=1}^n) S_i$.

Together with the previous remark that $\mathbb{P}^n(k)$ and $\emptyset$ lie in the image of $V_{\text{proj}}(-)$ we still have the same Zariski topology on $\mathbb{P}^n(k)$, where the projective algebraic sets are the closed sets.

8.2 Ideal of a projective algebraic set

Definition 8.10

Let $V \subset \mathbb{P}^n(k)$ be a subset. We call

$$I_{\text{proj}}(V) = \{F \in k[X_0, \dots, X_n] | F(\overline{x}) = 0, \forall \overline{x} \in V\}$$

the ideal of V.

Definition 8.11

We call an ideal a homogeneous ideal if it is generated by homogeneous elements.

1. If $V \subset V'$ then $I_{\text{proj}}(V') \subseteq I_{\text{proj}}(V)$.

2. $I_{\text{proj}}(V)$ is a homogeneous and radical ideal.
3. If V is a projective algebraic set, then $V_{\text{proj}}(I_{\text{proj}}(V)) = V$.
4. If I is an ideal, then $I \subseteq I_{\text{proj}}(V_{\text{proj}}(I))$.
5. $I_{\text{proj}}(\mathbb{P}^n(k)) = (0)$.
6. $I_{\text{proj}}(\emptyset) = k[X_0, \dots, X_n]$.
7. Irreducibly makes sense.

8.3 The cone construction

The cone construction is a technique for reducing the case of projective algebraic sets to affine algebraic sets. If we let $V \subseteq \mathbb{P}^n(k)$ be a projective algebraic set, and π the canonical projection

$$\pi\colon k^{n+1} \setminus \{0\} \longrightarrow \mathbb{P}^n(k)$$

then the cone of V is defined to be the set $\text{Cone}(V) = \pi^{-1}(V) \cup \{0\}$.

The cone of a projective algebraic set has the following two properties.

1. If $I \subsetneq k[X_0, \dots, X_n]$ is a proper homogeneous ideal, then $\text{Cone}(V_{\text{proj}}(I)) = V(I) \subseteq k^{n+1}$.

   Proof. □

   proof

2. If $I = k[X_0, \dots, X_n]$, then $\text{Cone}(V_{\text{proj}}(I)) = \text{Cone}(\emptyset) = \{0\}$.

   Proof. We know that $V_{\text{proj}}(k[X_0, \dots, X_n]) = \emptyset$, which means that

   $$\text{Cone}(V_{\text{proj}}(k[X_0, \dots, X_n])) = \text{Cone}(\emptyset) = \pi^{\emptyset} \cup \{0\}.$$

   The inverse image of the empty set under the canonical projection is again the empty set, thus we have $\text{Cone}(V_{\text{proj}}(k[X_0, \dots, X_n])) = \{0\}$. □

Theorem 8.12: Projective nullstellensatz

Let k be an algebraically closed field and $I \subseteq k[X_0, \dots, X_n]$ a homogeneous ideal. Then $V_{\text{proj}}(I) = \emptyset$ if and only if $R^+ \subseteq \sqrt{I}$. If $V_{\text{proj}}(I) \neq \emptyset$, then $I_{\text{proj}}(V_{\text{proj}}(I)) = \sqrt{I}$.

Proof. Notice that if $I = k[X_0, \dots, X_n]$, then $R^+ \subseteq \sqrt{k[X_0, \dots, X_n]} = k[X_0, \dots, X_n]$, i.e. the first statement holds. Hence we can assume that I is a proper ideal. Also notice that $V_{\text{proj}}(I) = \emptyset$ if and only if $\text{Cone}(V_{\text{proj}}(I)) = \{0\} = V(I)$.

Let's prove the first statement. Assume that $V_{\text{proj}}(I) = \emptyset$. Then

$$R^+ = (X_0, \dots, X_n) \subseteq I(0) = I(V(I)),$$

where the last equality is by the above remark. By the nullstellensatz for affine algebraic sets we know that $I(V(I)) = \sqrt{I}$, hence $R^+ \subseteq \sqrt{I}$.

Assume now that $R^+ \subseteq \sqrt{I}$. By the affine nullstellensatz we have $\sqrt{I} = I(V(I))$, hence we have $R^+ \subseteq I(V(I))$. Then by the order reversing property of $V(-)$ we have

$$\emptyset \neq V(I) = V(I(V(I))) \subseteq V(R^+) = \{0\}.$$

This means that $V(I) = \{0\}$ which means that $V_{\text{proj}}(I) = \emptyset$ by the above remark.

We now prove the second part, i.e. that if the projective algebraic set of an ideal is non-empty, then $I_{\text{proj}}(V_{\text{proj}}(I)) = \sqrt{I}$. Assume $V_{\text{proj}}(I) \neq \emptyset$. This gives us that $I_{\text{proj}}(V_{\text{proj}}(I)) \subseteq k[X_0, \dots, X_n]$ is a proper ideal. We claim that $I_{\text{proj}}(V_{\text{proj}}(I)) = I(\text{Cone}(V_{\text{proj}}(I)))$. We show both containments.

Let $F \in I_{\text{proj}}(V_{\text{proj}}(I))$. This means that $F(\overline{x}) = 0$ for all $\overline{x} \in V_{\text{proj}}(I)$, i.e. that $F(\lambda x) = 0$ for all $\lambda \in k^\times$. By letting $\lambda = 1$ we see that $F(x) = 0$, which means that $F \in I(\pi^{-1}(V_{\text{proj}}(I)))$. As $I_{\text{proj}}(V_{\text{proj}}(I))$ is a proper homogeneous ideal we also see that $F(0) = 0$, as F can't be a constant, i.e. a homogeneous polynomial of degree 0. This means that $F \in I(\text{Cone}(V_{\text{proj}}(I)))$. Hence we have $I_{\text{proj}}(V_{\text{proj}}(I)) \subseteq I(\text{Cone}(V_{\text{proj}}(I)))$.

Let $F \in I(\text{Cone}(V_{\text{proj}}(I))$. This means that $F(x) = 0$ for all $x \in \text{Cone}(V_{\text{proj}}(I)) = V(I)$, as I is a proper homogeneous ideal. We can decompose F into its homogeneous components, i.e. $F = F_0 + \cdots + F_r$, where $r = deg(F)$. As $F(0) = 0$ we know that $F_0 = 0$. Thus $F(\lambda x) = \lambda F_1(x) + \cdots \lambda^r F_r(x)$. All these $F_i(x)$ must vanish since $F(x) = 0$, hence $F_i(x) = 0$. By 8.3 this means that $F(\overline{x}) = 0$ and thus $F \in I_{\text{proj}}(V_{\text{proj}}(I))$. Hence $I(\text{Cone}(V_{\text{proj}}(I))) \subseteq I_{\text{proj}}(V_{\text{proj}}(I))$.

We then finally have

$$I_{\text{proj}}(V_{\text{proj}}(I)) = I(\text{Cone}(V_{\text{proj}}(I))) = I(V(I)) = \sqrt{I},$$

where the last equality is by the affine nullstellensatz and the middle equality is due to the properties we described above. □

Proposition 8.13

There is a bijection between non-empty projective algebraic sets $V \subseteq \mathbb{P}^n(k)$ and homogeneous radical ideals in $k[X_0, \dots, X_n]$ not containing R^+.

As in the affine case we have that irreducible projective algebraic sets correspond to the prime ideals. Also as in the affine case we can compare to certain k-algebras. If I is a homogeneous radical ideal not containing R^+ corresponding to an affine algebraic set V, then $k[X_0, \dots, X_n]/I$ is a k-algebra, which we denote by $\Gamma_{homog}(V)$. These are often called the homogeneous coordinate ring of V. For the affine case we had that points on an affine algebraic set corresponded to maximal ideals in $\Gamma(V)$, but for the projective case this is not the case.

Definition 8.14

Let $V \subseteq \mathbb{P}^n(k)$ be a projective algebraic set and $f \in \Gamma_{homog}(V)$ be homogeneous with positive degree. We define $D^+(f) = \{\overline{x} \in V | f(\overline{x}) \neq 0\}$.

These are open sets in the Zariski topology on V.

Example 8.15 $\mathbb{P}^n(k) = D^+(X_0) \cup D^+(X_1) \cup \cdots D^+(X_n)$.

Each one of these $D^+(X_i)$ are isomorphic to k, which means that projective space is locally affine! These are far from being disjoint unions, but they do form a covering.

Example 8.16 $\mathbb{P}^2(\mathbb{R}) = D^+(X) \cup D^+(Y) \cup D^+(Z)$. Here $D^+(X)$ consists of all points $[x : y : z] \in \mathbb{P}^2(\mathbb{R})$ such that $x \neq 0$, and the other ones are the same for the other coordinates. As these are projective coordinates we can look at $[1 : y/x : z/x]$ instead, which means that $D^+(X)$ is isomorphic to $\mathbb{R}^2$ through $\phi([x : y : z]) = (y/x, z/x)$, and $\phi^{-1}(a, b) = [1 : a : b]$.

Problem 8.17 Lets take the affine algebraic set $V(Y - X^3) \subseteq \mathbb{R}^2$. We can make it projective by instead studying $V_{\text{proj}}(X^3 - YZ^2) \subseteq \mathbb{P}^2(\mathbb{R})$. Call this projective algebraic set V. What does V look like?

Solution:

We try to understand how it looks like on each of the open sets $D^+(X)$, $D^+(Y)$ and $D^+(Z)$. On $D^+(X)$ it looks like $V(YZ^2 - 1)$, on $D^+(Y)$ like $V(Z^2 - X^3)$ and on $D^+(Z)$ like $V(Y - X^3)$. We can visualize them as:

We see that all three of them contain the point $[1 : 1 : 1]$.

Visualization

Problem 8.18 Find some other points and try to understand how these three open sets glue together globally.

9 Lecture 9 - 09.02.21

9.1 Motivation on sheaves

We start by some motivation as to why we would need something more that just affine and projective algebraic sets.

The first clue is that the correspondence between affine algebraic sets and reduced finite type k-algebras does not include a lot of rings that we would like to study using the same techniques. For example:

- $\mathbb{Z}$ and $\mathbb{Z}[i]$ (They are not k-algebras)
- $k(x)$ and $k[x]_{(x)}$, being all rational functions and rational functions defined at 0 respectively (they are not finite type k-algebras)

We would also like to study multiplicity of points. For example $V = V(Y - X^2) \cap V(Y)$. We can study this by looking at its coordinate ring $\Gamma(V)$. We find the ideal first:

$$\begin{aligned} I(V(Y - X^2) \cap V(Y)) &= I(V(Y - X^2, Y)) \\ &= I(V(X^2, Y)) \\ &= \sqrt{(X^2, Y)} \\ &= (X, Y), \end{aligned}$$

where the second to last equality is due to the affine nullstellensatz. We then get $\Gamma(V) \cong k[X, Y]/(X, Y) \cong k$, which is a 1-dimensional k-algebra. This means that we just counted the intersection point once, even though the zero has multiplicity 2. So the coordinate ring forgets multiplicity.

What if we instead didn't take the radical? Then we would get $\Gamma \cong k[X, Y]/(X^2, Y) \cong k[X]/(X^2)$ which is a 2-dimensional k-algebra. This means we have counted multiplicity right! But, this algebra is not reduced, as the ideal is not radical.

What we want is a correspondence between all commutative rings, and something geometric, more general than affine algebraic sets. In this way we can count multiplicity right, we can study rings geometrically and more interesting stuff we will get to later. These geometric objects are called affine schemes.

To make the definition of an affine scheme we need one more tool, namely sheaves. Let's see some motivation on these to get a feeling for what they are and why they are needed.

For affine algebraic sets V we had a correspondence with $\Gamma(V)$ which were the ring of functions on V. In the projective case, i.e. for a projective algebraic set V, this $\Gamma_{homog}(V)$ did not have elements that were functions on V. To solve this we recall that $\mathbb{P}^n(k) = D^+(X_0) \cup \cdots \cup D^+(X_n)$, i.e. projective space is locally affine. So maybe we could define functions on these affine spaces and glue them together to form functions on the projective space?

Sadly this does not work.. Lets see an example of why.

Example 9.1 We look at $\mathbb{P}^1(k)$, which is covered by $D^+(X)$ and $D^+(Y)$. Here $D^+(X)$ are the points $[a:b]$ where a is non-zero, meaning we can look at just $[a:b] = [1:\frac{b}{a}]$. Now $D^+(X)$ is isomorphic to k through the isomorphism $\phi([a:b]) = \frac{b}{a}$, with inverse $\phi^{-1}(c) = [1:c]$. Similarily $D^+(Y)$ consists on points $[a:b] \in \mathbb{P}^1(k)$ such that b is non-zero. We again write $[a:b] = [\frac{a}{b}:1]$ and an isomorphism $\psi : D^+(Y) \longrightarrow k$ defined by $\psi([a:b]) = \frac{a}{b}$ with inverse $\psi^{-1}(c) = [c:1]$.

A function $f : D^+(X) \longrightarrow k$ is "good" if the induced function $\overline{f} : k \longrightarrow k$, defined such that $f([a:b]) = \overline{f}(\frac{b}{a})$, is a polynomial function. Similarly $g : D^+(Y) \longrightarrow k$ is "good" if the induced function $\overline{g} : k \longrightarrow k$, defined such that $g([a:b]) = \overline{g}(\frac{a}{b})$ is polynomial.

To get a polynomial function on $\mathbb{P}^1(k)$ these two functions f and g would need to agree on the intersection $D^+(X) \cap D^+(Y)$. This means that for $[a:b] \in D^+(X) \cap D^+(Y)$ we need $\overline{f}(\frac{b}{a}) = \overline{g}(\frac{a}{b})$. Since these are polynomial we get

$$a_n(\frac{b}{a})^n + \ldots + a_1(\frac{b}{a}) + a_0 = b_m(\frac{a}{b})^m + \ldots + b_1(\frac{a}{b}) + b_0.$$

By clearing the denominators we get

$$a_n b^{n+m} + a_{n-1} b^{n+m-1} a + \ldots + a_0 b^m a^n - b_m a^{n+m} - b_{m-1} a^{n+m-1} b - \ldots - b_0 b^m a^n = 0$$

And since k is infinite this means that $a_i = 0 = b_i$ for all $i > 0$. The resulting equation is then $a_0 - b_0 = 0$, which means that f and g were the same constant functions.

Hence all global functions on $\mathbb{P}^1(k)$ are constant.

We want a more encompassing notion of function than just constant functions, so wee need to expand to looking at locally defined functions instead of globally defined ones. This leads us directly to the notion of a sheaf.

9.2 Definition and examples

We first study the sheaf of k-valued functions on a topological space. Let X be a topological space and let U, V be open sets in X. Let further $\mathscr{F}(U)$ and $\mathscr{F}(V)$ denote the set of functions on U and V respectively. Note that if we have $V \subseteq U$, then a function $f \in \mathscr{F}(U)$ defines a function on V by restriction, i.e. $f_{|V} \in \mathscr{F}(V)$. Suppose we have an open set U that decomposes into two open sets, $U = U_0 \cup U_1$. If we have functions $f_0 \in \mathscr{F}(U_0)$ and $f_1 \in \mathscr{F}(U_1)$ such that $f_{0|U_0 \cap U_1} = f_{1|U_0 \cap U_1}$, then there exists an unique function $f \in \mathscr{F}(U)$ with $f_{|U_i} = f_i$. This means that functions on the decomposition glue together to a function on the union.

These are all the things we want a sheaf to be. An assignment some set to all open sets on a topological space, such that we have restriction and nice gluing. We remarked earlier that functions are too restrictive, so we don't want to restrict $\mathscr{F}(U)$

to being functions in our proper definition. Before we define a sheaf properly we need some other definitions.

Definition 9.2: Category of open sets

Let X be a topological space. We define $Open(X)$ to be the category consisting of

- Objects: Open sets in X
- Morphisms:
$$Hom(V,U) = \begin{cases} V \subseteq U, & \text{if } V \subseteq U \\ \emptyset, & \text{if } V \nsubseteq U \end{cases}$$

Definition 9.3: Presheaf

Let X be a topological space and $\mathscr{C}$ be some concrete category [a]. A $\mathscr{C}$-valued presheaf on X is a contravariant functor $\mathscr{F} : Open(X) \longrightarrow \mathscr{C}$.

[a] Most often either Grp, $Ring$, $Mod\Lambda$ or Set, but could in theory be any.

This means in particular that

- For each $U \in Open(X)$ we have an object $\mathscr{F}(U) \in \mathscr{C}$.
- If $V \subseteq U$ then we have a map $\mathscr{F}(V \subseteq U) = res_{U,V} : \mathscr{F}(U) \longrightarrow \mathscr{F}(V)$ which we call restriction.
- If $W \subseteq V \subseteq U$, then $res_{U,W} = res_{V,W} \circ res_{U,V}$.
- $res_{U,U} = Id_{\mathscr{F}(U)}$.

Since our intuition comes from the image of $\mathscr{F}$ geing a set of functions, we denote $res_{U,V}(f) = f_{|V}$ for $f \in \mathscr{F}(U)$.

Definition 9.4: Sheaf

A $\mathscr{C}$-valued presheaf $\mathscr{F}$ on a topological space X is called a sheaf if it satisfies the following property, which we call the gluability or gluing property.

If $U \in Open(X)$ is covered by $\{U_i\}_{i \in I}$, where $U_i \in Open(X)$, then for any choice of $f_i \in \mathscr{F}(U_i)$ such that $f_{i|U_i \cap U_j} = f_{j|U_i \cap U_j}$, there exists a unique $f \in \mathscr{F}(U)$ with $f_{|U_i} = f_i$ for all $i \in I$.

We often denote $\mathscr{F}(U) = \Gamma(U, \mathscr{F})$ and call elements sections of $\mathscr{F}$ over U.

Example 9.5 (The sheaf of continuous real-valued functions)**.**

Let X be a topological space. Define a contravariant functor $\mathscr{O} : Open(X) \longrightarrow$

Ring by $\mathscr{O}(U) = \{f : U \to R | f \text{ continuous}\}$. This is a ring by defining addition and multiplication pointwise.

Claim: This is a sheaf.

Proof. We first show that it is a presheaf, and then show it satisfies the gluability condition.

- For each $U \in Open(X)$ we have that $\mathscr{O}(U)$ is a ring, as defined above.
- If $V \subseteq U$, then $\mathscr{O}(U) \longrightarrow \mathscr{O}(V)$ defined by $f \longmapsto f_{|V}$ is continuous. This is because for any open set $W \subset \mathbb{R}$ we have $f^{-1}(W) \subset U$ is open, meaning that $f_{|V}^{-1}(W) = f^{-1}(W) \cap V$ is also open, and hence $f_{|V}$ continuous.
- If we have $W \subseteq V \subseteq U$, then for any $f \in \mathscr{O}(U)$ we have $(f_{|V})_{|W} = f_{|W}$.
- If $f \in \mathscr{O}(U)$ for some $U \in Open(X)$, then $f_{|U} = f$, meaning that $res_{U,U} = Id_{\mathscr{O}(U)}$.

The above points show that $\mathscr{O}$ is a presheaf. Assume now that $U = \cup_{i \in I} U_i$ is an open set in X covered by a family of open sets $\{U_i\}_{i \in I}$. Suppose that $f_i \in \mathscr{O}(U_i)$ for all $i \in I$ such that $f_{i|U_i \cap U_j} = f_{j|U_i \cap U_j}$. We need to show there exists a unique map $f : U \longrightarrow \mathbb{R}$ such that $f_{|U_i} = f_i$.

We define f by $f(x) = f_i(x)$ if $x \in U_i$. This is well defined because of the fact that the maps f_i agree on the overlaps of sets in the open cover. For some open set $W \subseteq \mathbb{R}$ we have that $f^{-1}(W) = \cup_{i \in I} f_i^{-1}(W)$. This set is open because $f_i^{-1}(W)$ open since f_i continuous, and arbitrary union of open sets is again open. Hence f is continuous.

Assume that we have two maps $f, g \in \mathscr{O}(U)$ such that $f_{|U_i} = f_i = g_{|U_i}$. Since $\{U_i\}$ is a cover we know that all $x \in U$ lie at least in one of the U_i's. We assume $x \in U_i$. Then

$$f(x) = f_{|U_i}(x) = f_i(x) = g_{|U_i}(x) = g(x)$$

holds for all points $x \in U$, hence $f = g$. This shows that the map f we found above is unique, and hence that $\mathscr{O}$ is a sheaf. □

We also want to see a non-example, to get a feeling on where things might go wrong.

Example 9.6 (Non-example)**.**

Let $X = \mathbb{R}$ be our topological space and define $\mathscr{P} : Open(X) \longrightarrow Ring$ by $\mathscr{P}(U) = \{f : U \to \mathbb{R} | f \text{ is bounded}\}$. We claim that $\mathscr{P}$ is a presheaf, but not a sheaf.

Proof. The restriction of a bounded function is again bounded, hence we in fact have a presheaf. It is not a sheaf as we can define $f_i = Id_{U_i}$, where $U_i = (i-1, i+1)$. We have $\cup U_i = \mathbb{R}$, but the f_i's glue to $f = Id_{\mathbb{R}}$, which is not a bounded function. □

10 Lecture 10 - 15.02.21

We continue studying sheaves. Let's recall the definition. A $\mathscr{C}$-valued presheaf on a topological space X is a contravariant functor $\mathscr{F} : Open(X) \longrightarrow \mathscr{C}$. A sheaf $\mathscr{F}$ is a presheaf that satisfies the glueability axiom.

- If $U \subseteq X$ is an open set, covered by other open sets $\{U_i\}_{i\in I}$, then for any choice of sections $f_i \in \mathscr{F}(U_i)$ such that $f_{i|U_1\cap U_j} = f_{j|U_i\cap U_j}$, there exists a unique section $f \in U$ such that $f_{|U_1} = f_i$.

Notice that this requires the category $\mathscr{C}$ to be concrete, i.e. have objects that consists of elements. If we have an abelian category instead, we can define the glueability axiom in an equivalent way without using elements. This equivalent axiom is given by the exacness of the following sequence:

$$0 \longrightarrow \mathscr{F}(U) \xrightarrow{\phi} \prod_{i\in I} \mathscr{F}(U_i) \xrightarrow{\alpha-\beta} \prod_{i\in I} \mathscr{F}(U_i \cap U_j)$$

where α is the map given by projecting to $\mathscr{F}(U_i)$, then restricting to $\mathscr{F}(U_i \cap U_j)$ and including into the product. The map β is given the same way, but by first projecting to $\mathscr{F}(U_j)$ instead. The uniqueness condition comes from exactness at the left part of the sequence, and the existence comes from the exactness in the middle.

There is even an even more general axiom, that does not require $\mathscr{C}$ to be abelian. This axiom is given by the sequence

$$\mathscr{F}(U) \longrightarrow \prod_{i\in I} \mathscr{F}(U_i) \rightrightarrows \prod_{i\in I} \mathscr{F}(U_i \cap U_j)$$

being an equalizer sequence.

Problem 10.1 Let $U = U_1 \cup U_2$ and $\mathscr{F}$ a sheaf of abelian groups. Show that the definition using abelian categories and concrete categories are the same in this setting.

Proof. The sequence in this setting becomes

$$\mathscr{F}(U) \longrightarrow \mathscr{F}(U_1) \times \mathscr{F}(U_2) \longrightarrow \mathscr{F}(U_1 \cap U_2)$$

where α is the map $\mathscr{F}(U_1) \times \mathscr{F}(U_2) \xrightarrow{p_1} \mathscr{F}(U_1) \xrightarrow{res_{U_1,U_1\cap U_2}} \mathscr{F}(U_1 \cap U_2)$, β the map $\mathscr{F}(U_1) \times \mathscr{F}(U_2) \xrightarrow{p_2} \mathscr{F}(U_2) \xrightarrow{res_{U_2,U_1\cap U_2}} \mathscr{F}(U_1 \cap U_2)$ and ϕ the map

If (f_1, f_2) lies in the kernel of $\alpha - \beta$, this means that $f_{1|U_1\cap U_2} - f_{2|U_1\cap U_2}$, i.e. $f_{1|U_1\cap U_2} = f_{2|U_1\cap U_2}$. Since this is a complex there exists a section $f \in \mathscr{F}(U)$ that gets mapped to (f_1, f_2) by ϕ. Since ϕ is injective, this section is unique. This shows the two definitions are the same in this small setting. $\square$

10.1 Stalks and germs

We want to examine what a sheaf does really locally, i.e. on a single point. We most often won't have that a singleton is an open set in the topological space, so we have to kind of zoom in using open neighbourhoods of the point. This is done formally by using limits.

Definition 10.2

Let $\mathscr{F}$ be a (pre)sheaf on a topological space X and fix a point $p \in X$. The stalk of $\mathscr{F}$ at p is defined as $\mathscr{F}_p = \lim_{p \in U} \mathscr{F}(U)$.

Alternatively we can define the stalk $\mathscr{F}_p$ by

$$\mathscr{F}_p = \{(U, f) | p \in U, f \in \mathscr{F}(U)\} / \sim$$

where $(U, f) \sim (V, g)$ is there exists an open set W, containing p, such that $f_{|W} = g_{|W}$.

The equivalence class $[U, f]$, which we denote f_p, is called the germ of $f \in \mathscr{F}(U)$ at p.

Here goes agricultural image

Example 10.3 Let X be a topological space, and $\mathscr{O} : Open(X) \longrightarrow Ring$ the sheaf of continuous functions, i.e. $\mathscr{F}(U) = \{f \colon U \longrightarrow \mathbb{R} | f \text{ continuous}\}$. We showed earlier that this is in fact a sheaf. Let $p \in X$ and define $\phi \mathscr{O}_p \longrightarrow \mathbb{R}$ by $\mathscr{O}(f_p) = f(p) \in \mathbb{R}$.

We have that ϕ is surjective, as every real number is hit by its corresponding constant function. We also have that the kernel, $\operatorname{Ker} \phi = \{[U, f] \in \mathscr{O}_p | f(p) = 0\}$ is an ideal in $\mathscr{O}_p$.

Notice that $\mathscr{O}_p / \operatorname{Ker} \phi \cong \mathbb{R}$, hence $\operatorname{Ker} \phi$ is maximal. Also, if $[U, f] \in \mathscr{O}_p \setminus \operatorname{Ker} \phi$, then $f(p) \neq 0$. Since f is a continuous function, we know that there exists a neighbourhood V around p such that $f_{|V} \neq 0$. This means that $[U, f]$ is invertible in $\mathscr{O}_p$ and hence that $(\mathscr{O}_p, \operatorname{Ker} \phi)$ is a local ring!

This is the construction that justifies the name "local".

Fill in details, show Op a ring etc

Problem 10.4 Given a sheaf $\mathscr{F}$ on a topological space X and some point $p \in X$. Is $\mathscr{F}_p$ a local ring?

It is always possible to consider a sheaf (of sets) as a sheaf of functions. We show this as follows.

Let $\mathscr{F}$ be a sheaf on X and let $K = \coprod_{p \in X} \mathscr{F}_p$. Define $i_U \colon \mathscr{F}(U) \longrightarrow Map(U, K)$ by $f \mapsto [p \mapsto f_p]$.

Problem 10.5 Show that i_U is an injection and that it is compatible with restriction maps.

This means that any sheaf (of sets) can be considered as a sub sheaf of this above construction.

Example 10.6 (Restriction sheaf)**.**

Let $\mathscr{F}$ be a sheaf on X and $U \subseteq X$ be an open set. The restriction of $\mathscr{F}$ to U, defined by sending open sets $V \subseteq U$ to $\mathscr{F}_{|U}(V) = \mathscr{F}(V)$ is again a sheaf, called the restriction sheaf.

Problem 10.7 Show that this is in fact a sheaf.

Example 10.8 (The constant presheaf)**.**

Let S be a set. The constant presheaf at S is defined by $\underline{S}_{pre}(U) = S$ for all open sets $U \subseteq X$. This is not a sheaf in general as it fails the glueability axiom.

Example 10.9 (The constant sheaf)**.**

Let S be a set. The constant sheaf at S is given by

$$\underline{S}(U) = \{f : U \longrightarrow S | f \text{ locally constant}\}$$

Equivalently we can define it by letting $\mathscr{F}_p = S$ for every $p \in X$.

Example 10.10 (The pushforward sheaf)**.**

Let $\mathscr{F}$ be a presheaf on X and let $\pi \colon X \longrightarrow Y$ be a continuous map. The pushforward presheaf of $\mathscr{F}$ along π is defined by $\pi_*\mathscr{F}(V) = \mathscr{F}(\pi^{-1}(U))$ for open sets $V \subseteq Y$.

Problem 10.11 If F is a sheaf, show that $\pi_*\mathscr{F}$ is again a sheaf.

Example 10.12 (The skyscraper sheaf)**.**

Let $p \in X$ be a point and S a set. We endow $\{p\}$ with the discrete topology and define $i_p \colon \{p\} \hookrightarrow X$ to be continuous. The skyscraper sheaf is defined as $(i_p)_*\underline{S}$.

Problem 10.13 What does this sheaf look like? Hint: Look at the stalks.