11 Lecture 11 - 16.02.21

11.1 Sheaves of rings

The main type of sheaves we will use in this course is sheaves of rings. This is because they play an important role in defining schemes later, which will be our generalization of affine and projective algebraic sets in order to properly count multiplicity of intersection. We have already defined sheaves, and thereby sheaves of rings, i.e. *Ring*-valued sheaves, but to really hit the nail on the head we go through it again. Note that all rings mentioned will be commutative and have a multiplicative identity.

Definition 11.1: Sheaf of rings

A sheaf of rings is a sheaf $\mathscr{F}$ on a topological space X such that for each open set $U \subseteq X$ we have that $\mathscr{F}(U)$ is a ring, and that the restriction maps $\mathscr{F}(U) \longrightarrow \mathscr{F}(V)$ for any $V \subseteq U$ is a ring homomorphism.

If we in the above definition let $X = \{x\}$ be a singleton space, then we see that a sheaf of rings on X is really just a choice of a ring R. Hence the study of sheaves of rings vastly generalize the study of rings.

Definition 11.2: Ringed space

A ringed space, denoted $(X, \mathscr{O}_X)$, consists of a topological space X and a sheaf of rings $\mathscr{O}_X$, called the structure sheaf on X.

Recall that for a sheaf $\mathscr{F}$ on a space X we defined the stalk at a point $p \in X$ to be the set $\mathscr{F}_p = \lim_{p\in U} \mathscr{F}(U) = \{[U, f]\}/\sim = \{f_p\}$.

Problem 11.3 Let $\mathscr{F}$ be a sheaf of rings and p a point in X. Show that $\mathscr{F}_p$ is a ring.

Solution:

Let $[U, f], [V, g]$ be to elements in $\mathscr{F}_p$. Since U, V are open sets there exists an open set $W \subseteq U \cap V$ containing p. Notice that $[W, f_{|W}]$ is a representative for the same class as $[U, f]$, similarly for $[W, g_{|W}]$. But now both $f_{|W}$ and $g_{|W}$ are elements in $\mathscr{F}(W)$, which we know is a ring because $\mathscr{F}$ is a sheaf of rings. Hence $f_{|W} \cdot g_{|W}$ is well defined and satisfies all the ring axioms.

For a ring R we can study its modules. These are abelian groups M together with an action from R, i.e. a map $R \times M \longrightarrow M$. We just noted that sheaves of rings are a generalization of just rings, so for this generalization to be nice we really should be able to study some sort of modules on sheaves of rings. Luckily we can just generalize the module axioms into the world of sheaves to get what we want.

Definition 11.4: $\mathscr{O}_X$-modules

Let $(X, \mathscr{O}_X)$ be a ringed space. An $\mathscr{O}_X$-module $\mathscr{F}$ is a sheaf of abelian groups such that $\mathscr{F}(U)$ is an $\mathscr{O}_X(U)$-module with action α_U for all open $U \subseteq X$, such that

$$\begin{tikzcd} \mathscr{O}(U) \times \mathscr{F}(U) \arrow[r, "\alpha_U"] \arrow[d, "res^{\mathscr{O}_X}_{U,V} \times res^{\mathscr{F}}_{U,V}"'] & \mathscr{F}(U) \arrow[d, "res^{\mathscr{F}}_{U,V}"] \\ \mathscr{O}(V) \times \mathscr{F}(V) \arrow[r, "\alpha_V"] & \mathscr{F}(V) \end{tikzcd}$$

commutes for all open sets $V \subseteq U$.

If we again choose $X = \{x\}$ we see that the study of $\mathscr{O}_X$-modules vastly generalize the study of modules over rings.

Problem 11.5 Let $(X, \mathscr{O}_X)$ be a ringed space and $\mathscr{F}$ an $\mathscr{O}_X$-module. Show for every $p \in X$ that $\mathscr{F}_p$ is an $(\mathscr{O}_X)_p = \mathscr{O}_{X,p}$ module.

solution

11.2 Morphisms of sheaves

For the following discussion we let $\mathscr{C}$ be a nice, concrete category.

Definition 11.6

Let $\mathscr{F}, \mathscr{G}$ be $\mathscr{C}$-valued sheaves on a topological space X. A morphism $\phi\colon \mathscr{F} \longrightarrow \mathscr{G}$ consists of morphisms $\phi(U)\colon \mathscr{F}(U) \longrightarrow \mathscr{G}(U)$ in $\mathscr{C}$ for all open sets $U \subseteq X$, such that

$$\begin{tikzcd} \mathscr{F}(U) \arrow[r, "\phi_U"] \arrow[d, "res^{\mathscr{F}}_{U,V}"'] & \mathscr{G}(U) \arrow[d, "res^{\mathscr{G}}_{U,V}"] \\ \mathscr{F}(V) \arrow[r, "\phi_V"] & \mathscr{G}(V) \end{tikzcd}$$

commutes for all open sets $V \subseteq U$.

We define an isomorphism of sheaves to be a morphism with a two-sided inverse.

Note that a morphism $\phi\colon \mathscr{F} \longrightarrow \mathscr{G}$ of sheaves on a space X induces a morphism on stalks $\phi_p\colon \mathscr{F}_p \longrightarrow \mathscr{G}_p$ for all $p \in X$. This is because a morphism of sheaves gives us maps $\mathscr{F}(U) \longrightarrow \mathscr{F}(G)$ that respect restriction for all open sets containing p. Hence we have a morphism of directed systems

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] \\
\mathscr{F}(U) \arrow[r, "\phi(U)"] \arrow[d, "res^{\mathscr{F}}_{U,V}"'] & \mathscr{G}(U) \arrow[d, "res^{\mathscr{G}}_{U,V}"] \\
\mathscr{F}(V) \arrow[r, "\phi(V)"] \arrow[d, "res^{\mathscr{F}}_{V,W}"'] & \mathscr{G}(V) \arrow[d, "res^{\mathscr{G}}_{V,W}"] \\
\mathscr{F}(W) \arrow[r, "\phi(W)"] \arrow[d] & \mathscr{G}(W) \arrow[d] \\
\vdots & \vdots
\end{tikzcd}$$

A morphism of directed systems induces a morphism on its direct limit, which is the definition of the stalk at p, i.e. a map $\lim_{p\in U}\mathscr{F}(U) \longrightarrow \lim_{p\in U}\mathscr{G}(U)$.

Problem 11.7 Let $[U,f]\in\mathscr{F}_p$. Where does it go under the map ϕ_p? Why is it well defined?

Solution

Proposition 11.8

Let $\phi\colon\mathscr{F}\longrightarrow\mathscr{G}$ be a morphism of sheaves of abelian groups on a space X. Then ϕ is an isomorphism if and only if each induced morphism on stalks ϕ_p is an isomorphism.

Proof. Assume that ϕ is an isomorphism of sheaves. This means that $\phi(U)\colon\mathscr{F}(U)\longrightarrow\mathscr{G}(U)$ is an isomorphism for all open sets $U\subseteq X$. Since each ϕ_p is a direct limit of a direct system of isomorphisms it is again an isomorphism.

Assume now that ϕ_p is an isomorphism of abelian groups for all p. If we had inverses $\phi^{-1}(U)\colon\mathscr{G}(U)\longrightarrow\mathscr{F}(U)$ for all U we could collect these together to form an inverse ϕ^{-1}. Hence it is enough to show that $\phi(U)$ is an isomorphism for all open sets $U\subseteq X$.

We start by showing that ϕ is injective. Let $f\in\mathscr{F}(U)$ and suppose $\phi(U)(f)=0$. This means that for all $p\in U$ we have $\phi(U)(f)_p=0$. We have $\phi(U)(f)_p=\phi_p(f_p)$ and since ϕ_p is assumed injective we must have $f_p=0$ for all p. This means that we for all $p\in U$ can find an open set W_p, containing p, such that $f_{|W_p}=0$. We can find a cover of U using these W_p, i.e. $U=\cup_{p\in U}W_p$. Since $\mathscr{F}$ is a sheaf it must satisfy the glueability axiom, which means that $f=0$. This is because the gluing is unique, and $f_{|W_p}=0_{|W_p}$ on all W_p's, hence $f=0$ as $0_{|W_p}$ glues back to 0. This means that the only element that gets sent to zero is the zero element, which means $\phi(U)$ is injective.

Surjectivity is a little trickier, but let's try our best. Suppose $g \in \mathscr{G}(U)$. For each $p \in U$ we let $g_p \in \mathscr{G}_p$ be its germ at p. Since ϕ_p is assumed surjective we can find $f_p \in \mathscr{F}_p$ such that $\phi_p(f_p) = g_p$. Since $\phi_p(f_p) = g_p$ we can find a small neighbourhood $V_p \subseteq U$ containing p such that $\phi(V_p)(f'_p) = g_{|V_p}$, where $f'_p \in \mathscr{F}(V_p)$ is a representative for f_p in V_p.

These sets V_p form a cover for U, i.e. $U = \cup_{p \in U} V_p$. We want to apply the glueability axiom for the sheaf $\mathscr{G}$, and to do that we need to have $f'_{p|V_p \cap V_q} = f'_{q|V_p \cap V_q}$. Both of these gets sent to $g_{|V_p \cap V_q}$ by the map $\phi(V_p \cap V_q)$, which we above proves is injective. Hence $f'_{p|V_p \cap V_q} = f'_{q|V_p \cap V_q}$. By glueability there exists a section $f \in \mathscr{F}(U)$ such that $f_{|V_p} = f'_p$.

Finally we need to check that this glued together section f actually maps to g under $\phi(U)$. We have $\phi(U)(f)_{|V_p} = g_{|V_p}$, and hence by the unique glueability in $\mathscr{G}$ we must have $\phi(U)(f) = g$. Hence every $g \in \mathscr{G}(U)$ gets hit by an $f \in \mathscr{F}(U)$ by $\phi(U)$, which means that it is surjective.

Since $\phi(U)$ is both injective and surjective it must be an isomorphism, which is what we wanted to show. □

11.3 Kernels, cokernels and images

Let $\phi \colon \mathscr{F} \longrightarrow \mathscr{G}$ be a morphism of sheaves of abelian groups on a topological space X.

Definition 11.9

The presheaf kernel of ϕ, denoted $\operatorname{Ker} \phi$, is the assignment of the abelian group $\operatorname{Ker} \phi(U)$ to every open set $U \subseteq X$.

Definition 11.10

The presheaf image of ϕ, denoted $\operatorname{Im} \phi$, is the assignment of the abelian group $\operatorname{Im} \phi(U)$ to every open set $U \subseteq X$.

Definition 11.11

The presheaf cokernel of ϕ, denoted $\operatorname{Cok} \phi$, is the assignment of the abelian group $\operatorname{Cok} \phi(U)$ to every open set $U \subseteq X$.

Problem 11.12 Show that these assignments are functors, i.e. that they are again presheaves.

Problem 11.13 Show that $\operatorname{Ker}\phi$ is a sheaf.

In general $\operatorname{Im}\phi$ and $\operatorname{Cok}\phi$ are not sheaves. To fix this we introduce the notion of sheafification of a presheaf. This will allow us to define the image and cokernel sheaf by sheafifying their respective presheaves.

Proposition 11.14

Let $\mathscr{F}$ be a presheaf on a topological space X. There is a sheaf $\mathscr{F}^+$, unique up to unique isomorphism, and a morphism $\theta\colon \mathscr{F} \longrightarrow \mathscr{F}^+$ such that for any other sheaf $\mathscr{G}$ we have

$$\begin{tikzcd} \mathscr{F} \arrow[r] \arrow[d] & \mathscr{F}^+ \arrow[dl, dotted, "\exists!"] \\ \mathscr{G} & \end{tikzcd}$$

i.e. that any other map into a sheaf factorizes through $\mathscr{F}^+$.

The proof of this will be given next time.

Definition 11.15: Sheafification

Let $\mathscr{F}$ be a presheaf on a topological space X. We define the sheafification of $\mathscr{F}$ to be the sheaf $\mathscr{F}^+$ as in the proposition above.

12 Lecture 12 - 22.02.21

As we began introducing last time we want to talk about the kernel, cokernel and image sheaves of morphisms of sheaves. We stated that these not necessarily were sheaves, so we needed a way to fix this, which is done by sheafifying the presheaves.

12.1 Sheafification

Proposition 12.1

Let $\mathscr{F}$ be a presheaf on a topological space X. There is a sheaf $\mathscr{F}^+$ on X, unique up to unique isomorphism, and a map $\Theta\colon \mathscr{F} \longrightarrow \mathscr{F}^+$, such that for any sheaf $\mathscr{G}$ with morphism $\mathscr{F} \longrightarrow \mathscr{G}$ we have a commutative diagram

$$\begin{tikzcd} \mathscr{F} \arrow[r, "\Theta"] \arrow[d] & \mathscr{F}^+ \arrow[dl, dotted, "\exists!"] \\ \mathscr{G} & \end{tikzcd}$$

Proof. For any open set $U \subseteq X$ we define

$$\mathscr{F}^+(U) = \{s : U \longrightarrow \bigcup_{p\in U} \mathscr{F}_p \,|\, 1.\text{ and } 2.\text{ holds}\},$$

where

1. $\forall p$ we have $s(p) \in \mathscr{F}_p$
2. $\forall p$ there exists a neighborhood of p, $V \subseteq U$ and $t \in \mathscr{F}(V)$ such that $\forall q \in V$ we have $t_q = s(q)$

Our claim is that $\mathscr{F}^+$ defined this way is a sheaf that satisfies the above proposition.

For $V \subseteq U$ we have that $\mathscr{F}(U) \longrightarrow \mathscr{F}(V)$ is the actual restruction maps, as we defined $\mathscr{F}^+$ using functions. This means that $\mathscr{F}^+$ is at least a presheaf.

Assume we have a set with an open cover, i.e. $U = \bigcup_i U_i$ and sections $s_i \in \mathscr{F}^+(U_i)$ such that $s_{i|U_i\cap U_j} = s_{j|U_i\cap U_j}$. We define $s \in \mathscr{F}^+(U)$ to be the map that sends $x \in U$ to $s_i(x)$ when $x \in U_i$. This gives the existence of a glued section.

Assume that we have two sections $s, s' \in \mathscr{F}^+(U)$ such that $s_{|U_i} = s_i = s'_{|U_i}$. Since $\{U_i\}$ is a cover we know that all $x \in U$ lie at least in one of the U_i's. We assume $x \in U_i$. Then

$$s(x) = s_{|U_i}(x) = s_i(x) = s'_{|U_i}(x) = s'(x)$$

holds for all points $x \in U$, hence $s = s'$. This shows that the map s we defined above is unique, and hence that $\mathscr{F}^+$ has unique gluing, and is thus a sheaf.

The map Θ is defined as follows. For every open set $U \subseteq X$ define $\Theta(U)$ as

$$\begin{aligned} \mathscr{F}(U) &\longrightarrow \mathscr{F}^+(U) \\ s &\longmapsto [U \to \cup\mathscr{F}_p, x \mapsto s_x] \end{aligned}$$

$\square$

Check that sheaves factor through theta

As mentioned last time we define $\mathscr{F}^+$ to be the sheafification of a presheaf $\mathscr{F}$.

Problem 12.2 Prove that for all $p \in X$ that $\mathscr{F}_p^+ = \mathscr{F}_p$.

This means that at the level of stalks, working with the sheafification is relatively easy.

Let $\phi_{\mathscr{F}} \longrightarrow \mathscr{G}$ be a morphism of sheaves (of abelian groups). Recall that $\operatorname{Ker}\phi$, $\operatorname{Im}_{pre}\phi$ and $\operatorname{Cok}_{pre}\phi$ are presheaves. The kernel sheaf $\operatorname{Ker}\phi$ is in fact also a sheaf.

Definition 12.3

A subsheaf of a sheaf $\mathscr{F}$ on X is a sheaf $\mathscr{F}'$ such that for all open sets $U \subseteq X$ we have that $\mathscr{F}'(U)$ is a subobject of $\mathscr{F}(U)$, and that the restriction maps in $\mathscr{F}'$ are induced by the ones in $\mathscr{F}$.

Note that this makes $\mathscr{F}_p'$ a subobject of $\mathscr{F}_p$.

Definition 12.4

We say a morphism $\phi\colon \mathscr{F} \longrightarrow \mathscr{G}$ is injective if $\operatorname{Ker}\phi = 0$.

Note that ϕ is injective if and only if $\phi(U)$ is injective for all open sets $U \subseteq X$.

Definition 12.5

We define the image sheaf of a morphism $\phi\colon \mathscr{F} \to \mathscr{G}$ to be $\operatorname{Im}\phi = \operatorname{Im}_{pre}\phi^+$, i.e. the sheafification of the image presheaf.

Calling it the image is justified as we have an injective morphism $\operatorname{Im}\phi \longrightarrow \mathscr{G}$. This map exists because of universal property of the sheafification

$$\begin{tikzcd}
\mathscr{F} \arrow[rr, "\phi"] \arrow[dr] & & \mathscr{G} \\
& \operatorname{Im}_{pre}\phi \arrow[ur, "\psi'"] \arrow[d] & \\
& \operatorname{Im}\phi \arrow[uur, dotted, "\exists\psi"'] &
\end{tikzcd}$$

It is injective because $\psi'(U)$ is injective for all U, hence ψ' injective. This implies $\psi_p'\colon (\operatorname{Im}_{pre}\phi)_p \longrightarrow \mathscr{G}_p$ is injective, which by the problem above, i.e. $(\operatorname{Im}_{pre}\phi)p =$

$\operatorname{Im}\phi_p$, means that ψ_p is injective as well. Being injective on all germs is sufficien to be injective as morphism of sheaves, hence ψ is injective.

Definition 12.6

We say a morphism $\phi\colon \mathscr{F} \longrightarrow \mathscr{G}$ is surjective if $\operatorname{Im}\phi = \mathscr{G}$.

Definition 12.7

Let $\mathscr{F}'$ be a subsheaf of a sheaf $\mathscr{F}$ on a topological space X. We define the quotient sheaf $\mathscr{F}/\mathscr{F}'$ by sending an open set $U \subseteq X$ to $\mathscr{F}(U)/\mathscr{F}'(U)$.

Problem 12.8 Show that this is in fact a sheaf.

Definition 12.9

Let $\phi\colon \mathscr{F} \longrightarrow \mathscr{G}$ be a morphism of sheaves. We define its cokernel sheaf to be the sheaf $\operatorname{Cok}\phi = \operatorname{Cok}_{pre}\phi^{+}$, i.e. the sheafification of the cokernel presheaf.

Definition 12.10

A sequence of sheaves on a topological space X,

$$\dots \xrightarrow{\phi^{i-2}} \mathscr{F}^{i-1} \xrightarrow{\phi^{i-1}} \mathscr{F}^{i} \xrightarrow{\phi^{i}} \mathscr{F}^{i+1} \xrightarrow{\phi^{i+1}} \dots$$

is called exact at degree i if $\operatorname{Ker}\phi^i = \Im\phi^{i-1}$. The sequence is called exact if it is exact at all i.

Note that the sequence

$$\dots \xrightarrow{\phi^{i-2}} \mathscr{F}^{i-1} \xrightarrow{\phi^{i-1}} \mathscr{F}^{i} \xrightarrow{\phi^{i}} \mathscr{F}^{i+1} \xrightarrow{\phi^{i+1}} \dots$$

is exact if and only if the sequence

$$\dots \xrightarrow{\phi^{i-2}(U)} \mathscr{F}^{i-1}(U) \xrightarrow{\phi^{i-1}(U)} \mathscr{F}^{i}(U) \xrightarrow{\phi^{i}(U)} \mathscr{F}^{i+1}(U) \xrightarrow{\phi^{i+1}(U)} \dots$$

is exact for all open sets $U \subseteq X$.

Definition 12.11

Let $\mathscr{F}$ and $\mathscr{G}$ be sheaves on a topological space X. For an open set $U \subseteq X$ we define $\mathscr{H}om(\mathscr{F},\mathscr{G})(U) = Mor(\mathscr{F}_{|U},\mathscr{G}_{|U})$.

Proposition 12.12

The assignment $\mathscr{H}om(\mathscr{F},\mathscr{G})$ defines a sheaf on X.

Proof. We first see that it is a presheaf. Take a subset $V \subseteq U$, we need to define a map $Mor(\mathscr{F}_{|U},\mathscr{G}_{|U}) \longrightarrow Mor(\mathscr{F}_{|V},\mathscr{G}_{|V})$. For any subset $W \subseteq V$ we have that $\mathscr{F}_{|V}(W) = \mathscr{F}_{|U}(W)$, as the extra restriction of the sheaf itself does not do anything since we are looking at function on an even smaller set. Hence we get a map $\mathscr{F}_{|V}(W) \longrightarrow \mathscr{G}_{|V}(W)$ from the map we already have from $\mathscr{F}_{|U}(W) \longrightarrow \mathscr{G}_{|U}(W)$.

Let now $U = \bigcup U_i$ be an open cover and suppose $s_i \in \mathscr{H}om(\mathscr{F},\mathscr{G})(U_i)$ such that $s_{i|U_i\cap U_j} = s_{j|U_i\cap U_j}$. For $V \subseteq U$ set $V_i = V \cap U_i$, which means that $V = \bigcup V_i$ is an open cover.

For $f \in \mathscr{F}(V)$ we can restrict it to V_i to get $f_{|V_i} \in \mathscr{F}(V_i)$. We can map this to $\mathscr{G}(V_i)$ by using $s_i(V_i)$ to get some $g_i \in \mathscr{G}(V_i)$. Since $\mathscr{G}$ is a sheaf we can glue these g_i to get an unique section $g \in \mathscr{G}(V)$. We then simply define $s \in \mathscr{H}om(\mathscr{F},\mathscr{G})(U)$ by $s(V)(f) = g$.

This defines $s\colon \mathscr{F}_{|U} \longrightarrow \mathscr{G}_{|U}$ such that $s_{|U_i} = s_i$, and thus we have existence of a glued section.

For uniqueness we assume there exists $s, t \in \mathscr{H}om(\mathscr{F},\mathscr{G})(U)$ such that $s_{|U_i} = t|U_i$. For $V \subseteq U$ we have a diagram

$$\begin{tikzcd}
\mathscr{F}_{|U}(V) \arrow[r, bend left, "s(V)"] \arrow[r, bend right, "t(V)"'] \arrow[d] & \mathscr{G}_{|U}(V) \arrow[d] \\
\mathscr{F}(V_i) \arrow[r, "s(V_i)=t(V_i)"'] & \mathscr{F}(V_i)
\end{tikzcd}$$

where the vertical arrows are restriction maps. Let $f \in \mathscr{F}(V)$. We want to compare $s(V)(f)$ and $t(V)(f)$. For all i we have

$$s(V)(f)_{|V_i} = s(V_i)(f_{|V_i}) = t(v_i)(f_{|V_i}) = t(V)(f)_{|V_i}$$

which by $\mathscr{G}$ being a sheaf means that $s(V)(f) = t(V)(f)$. So $s(V)$ and $t(V)$ are pointwise equal, i.e. the same map, hence $s(V) = t(V)$. This holds for all open sets V, hence also s and t are pointwise equal, making them again equal, i.e. $s = t$. Hence the gluing is unique and we are done. □

Definition 12.13

Let now $\mathscr{F}$ and $\mathscr{G}$ be two $\mathscr{O}_X$ modules. We can define their tensor product $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G}$ to be the sheaf associted to the presheaf $U \longrightarrow \mathscr{F}(U)\otimes_{\mathscr{O}_X(U)\mathscr{G}(U)}$.

12.2 Sheaves and varieties

Let V be an affine algebraic set. We want to study some special ringed spaces, $(V, \mathscr{O}_V)$, which will be the affine algebraic varieties. In more generality we will have algebraic varieties which will be ringed spaces $(X, \mathscr{O}_X)$ that are locally affine. If $V \subseteq \mathbb{P}^n(k)$ is projective we will get projective algebraic varieties, which will be examples of these more general algebraic varieties.

In even more generality, these will all be examples of schemes, which also are locally ringed spaces.

13 Lecture 13 - 23.02.21

Last time we ended on a little note explaining where we are headed today, i.e. looking at algebraic varieties.

13.1 The structural sheaf

Let $V \subseteq k^n$ be an affine algebraic set and recall that we have a basis for its open sets by the distinguished open sets $D(f) = V \setminus V(f)$.

Lemma 13.1

To define a sheaf on a topological space X, it is enough to define it on the basis elements for the topology on X.

Problem 13.2 Prove this lemma, at least for sheaves of functions.

We define a sheaf, called the structural sheaf, or the structure sheaf, $\mathscr{O}_V$ on V by $\mathscr{O}_V(D(f)) = \Gamma(V)_f$.

Proposition 13.3

$\mathscr{O}_V$ in fact defines a sheaf on V.

Proof. For $D(f) \subseteq D(g)$ we have $V(g) \subseteq V(f)$, hence by the nullstellensatz we have $f \in \sqrt{g}$. This means that we can find a h and a natural number n such that $f^n = gh$. We have

$$\begin{aligned} \Gamma(V)_g &\longrightarrow \Gamma(V)_f \\ \frac{u}{g^i} &\longmapsto \frac{uh^i}{g^ih^i} = \frac{uh^i}{f^{ni}} \end{aligned}$$

Thus $\mathscr{O}_V$ is a presheaf of rings on V.

Assume now $D(f) = \bigcup D(f_i)$ and $s_i \in \mathscr{O}_V(D(f_i)$ such that $s_i i | D(f_i) \cap D(f_j) = s_{j|D(f_i) \cap D(f_j)}$. For $\mathscr{O}_V$ to be a sheaf we need a unique section $s \in \mathscr{O}_V(D(f))$ that restricts to the s_i's.

Lets first look at a special case where V is an irreducible affine algebraic set. In this case recall that we have $I(V)$ a prime ideal. Assume also that $D(f) = D(f_1) \cup D(f_2)$. Then we have a sequence

$$0 \longrightarrow \mathscr{O}_V(D(f)) \longrightarrow \mathscr{O}_V(D(f_1)) \oplus \mathscr{O}_V(D(f_2)) \longrightarrow \mathscr{O}_V(D(f_1f_2))$$

where exactness at $\mathscr{O}_V(D(f_1)) \oplus \mathscr{O}_V(D(f_2))$ yields existence of a section, and $\mathscr{O}_V(D(f))$ yields uniqueness.

This we get from the totalization of the following commutative square

$$\begin{array}{ccc} \Gamma(V)_f & \longrightarrow & \Gamma(V)_{f_1} \\ \downarrow & & \downarrow \\ \Gamma(V)_{f_2} & \longrightarrow & \Gamma(V)_{f_1 f_2} \end{array}$$

Ok, back to the general case. Write $s_i = \frac{a_i'}{f_i^{n_i}}$. We can choose $n = maxn_i$ to get $s_i = \frac{a_i}{f_i^n}$ instead.

We have that $s_{i|D(f_i)\cap D(f_j)} = s_{j|D(f_i)\cap D(f_j)}$ if and only if $\frac{a_i}{f_i^n} = \frac{a_j}{f_j^n}$ in $\Gamma(V)_{f_1 f_2}$, which again hold if and only if $f_i^N f_j^N(a_i f_j^n - a_j f_i^n) = 0$ for some N. As f vanishes on $V(f_1^{n+N}, \ldots, f_r^{n+N})$ we have by k being algebraically closed that $f \in \sqrt{(f_1^{n+N}, \ldots, f_r^{n+N})}$. Hence there exists $m \geq 1$ and $b_j \in \Gamma(V)$ such that $f^m = \sum_{j=1}^r b_j f_j^{n+N}$. Set $a = \sum_{j=1}^r a_j b_j f_j^N$ and then $s = \frac{a}{f^m}$. We claim that this is our glued section. To confirm this we need to show that it restricts to the s_i's, i.e. that $\frac{a}{f^m} = \frac{a_i}{f_i^n}$ in $\Gamma(V)_{f_i}$. We have

$$\begin{aligned} f_i^N(a_i f^m - a f_i^n) &= f_i^N a_i f^m - a f_i^{n+N} \\ &= f_i^N a_i \sum_{j=1}^r b_j f_j^{n+N} - a f_i^{n+N} \\ &= \sum_{j=1}^r a_i b_j f_i^N f_j^{n+N} - a f_i^{n+N} \\ &= \sum_{j=1}^r a_j b_b f_i^{n+N} f_j^N - a f_i^{n+N} \\ &= a f_i n + N - a f_i^{n+N} \\ &= 0 \end{aligned}$$

where the fourth equality comes from the previous equation $f_i^N f_j^N(a_i f_j^n - a_j f_i^n) = 0$. Hence s restricts to s_i and we are done.

□

13.2 Algebraic varieties

Definition 13.4

An affine algebraic variety is a ringed space isomorphic (as ringed spaces) to $(V, \mathscr{O}_V)$ for some affine algebraic set V, where $\mathscr{O}_V$ is defined as above.

Such an isomorphism is a homeomorphism of the topological spaces, and an isomorphism of sheaves.

Proposition 13.5

Let V be an affine algebraic set and $f \in \Gamma(V)$. Then $(D(f), \mathscr{O}_{V|D(f)})$ is an affine algebraic variety.

Proof. Assume $V \subseteq k^n$ and let F be a polynomial corresponding to f. Define $\phi\colon D(f) \longrightarrow k^{n+1}$ by sending

$$(x_1, \ldots, x_n) \longmapsto (x_1, \ldots, x_n, \frac{1}{f(x_1, \ldots, x_n)})$$

□

Problem 13.6 Check that $\operatorname{Im}\phi = V(J)$, where $J = I(V) + (X_{n+1}F - 1)$, and that ϕ is a homeomorphism.

Definition 13.7

An algebraic variety, often just called a variety, is a quasi-compact ringed space $(X, \mathscr{O}_X)$ such that for any $x \in X$ there is an open neighborhood $U \subseteq X$, containing x, such that $(U, \mathscr{O}_{X|U}$ is an affine algebraic variety.

This is what we mean when we say a general algebraic variety is locally affine. By quasi-compact we mean that any open cover has a finite subcover. It is the same as being compact minus the Hausdorff property.

Proposition 13.8

Let $(X, \mathscr{O}_X)$ be an algebraic variety. Any open set $U \subseteq X$ is a finite union of affine open sets, i.e. sets U_i such that $(U_i, \mathscr{O}_{X|U_i})$ is an affine algebraic variety.

Proof. Write $X = \bigcup_{i=1}^r U_i$, U_i open affine. This decomposition exists as X is quasi-compact and locally affine. Let $U \subseteq X$ be open and write $U = \bigcup_{i=1}^r U \cap U_i$. The $U \cap U_i$'s are affine open. □

Problem 13.9 Fill in details.

Problem 13.10 Is every open subvariety of an affine algebraic variety again affine?

Problem 13.11 Let $(X, \mathscr{O}_X)$ be an algebraic variety and let $x \in X$. Prove that $\mathscr{O}_{X,x}$ is a local ring with maximal ideal $M = \{f \in \mathscr{O}_{X,x} \,|\, f(x) = 0\}$

Hint: Define $\phi\colon \mathscr{O}_{X,x} \longrightarrow k$ by $\phi(U, f) = f(x)$.

Proposition 13.12

Let $(X, \mathscr{O}_X)$ be an algebraic variety, $x \in X$ and $U \subseteq X$ be an open set containing x. Set $A = \mathscr{O}_X(U)$ and let M be the maximal ideal that corresponds to x. Then $\mathscr{O}_{X,x} \cong A_m$.

13.3 Projective algebraic varieties

Let $V \subseteq \mathbb{P}^n(k)$ be a projective algebraic set. We need to define a sheaf, and as before we do so on the basis $D^+(f)$ where $f \in \Gamma_{homog}(V)$.

Definition 13.13

Let R be a graded ring and let $f \in R$ be a homogeneous element of degree d. Then R_f is also graded, and $deg(\frac{g}{f^r} = deg(g) - r \cdot d$. Define $R_{(f)}$ to be the set of degree 0 in R_f.

Definition 13.14

Let V be a projective algebraic set. Define $\mathscr{O}_V(D^+(f)) = \Gamma_{homog}(V)_{(f)}$.

This is in fact a sheaf on V.

Proposition 13.15

Let V be a projective algebraic set. Then $(V, \mathscr{O}_V)$ is an algebraic variety, called a projective algebraic variety.

14 Lecture 14 - 01.03.21

The lecture was focused on going through the exercises. Solutions and notes to these can be found at **??**.

15 Lecture 15 - 02.03.21

15.1 Sheaves of modules on varieties

Let $(V, \mathscr{O}_V)$ be an affine algebraic variety. Recall that $\mathscr{O}_V(D(f)) = \Gamma(V)_f$. Set $A = \mathscr{O}(V) = \Gamma(V, \mathscr{O}_V)$, which we call the global sections, and let $\mathscr{F}$ be an $\mathscr{O}_V$-module. In particular $\mathscr{F}(V)$ is an A-module.

A question we want to answer is weather we can create $\mathscr{O}_V$-modules from an A-module.

Definition 15.1

Let M be an A-module. Define an $\mathscr{O}_V$-module $\widetilde{M}$ by $\widetilde{M}(D(f)) = M_f$ for $f \in A$.

Notice that $M_f = M \otimes_A A_f$, so this could also be used as an alternative definition. Notice also that we have $\widetilde{M}(V) = M$.

Problem 15.2 What is $\widetilde{A}$?

Solution:

$\widetilde{A} = \mathscr{O}_V$.

Problem 15.3 Check that $\widetilde{M}$ is a sheaf. This should be a similar proof as for $\mathscr{O}_V$ being a sheaf, but carrying the tensor around.

Proposition 15.4

The assignment

$$\begin{aligned} A - \text{modules} &\longrightarrow \mathscr{O}_V - \text{modules} \\ M &\longmapsto \widetilde{M} \end{aligned}$$

is functorial, exact and preserves direct sums and tensor products.

Proof. This is true because the localization functor has these properties. □

Definition 15.5

Let $\mathscr{F}$ be an $\mathscr{O}_V$-module. We say $\mathscr{F}$ is quasi-coherent if $\mathscr{F} \cong \widetilde{M}$ for some A-module M and coherent if this M is finitely generated.

Hence we have an equivalence of categories between the category of A-modules and the category of quasi-coherent sheaves on V, $QCoh(V)$.

Problem 15.6 Check that $QCoh(V)$ is a category, and that the above described functor indeed gives an equivalence of categories.

Definition 15.7

Let $(X, \mathscr{O}_X)$ be an algebraic variety and $\mathscr{F}$ an $\mathscr{O}_X$-module. We say $\mathscr{F}$ is quasi-coherent if $\exists$ an open affine cover $X = \bigcup U_i$ such that $\mathscr{F}_{|U_i} \cong \widetilde{M_i}$ for some $\mathscr{O}_X(U_i)$-module M_i.

Equivalently we could define it to be quasi-coherent if for any open set $U \subset X$ we have $\mathscr{F}_{|U} \cong \widetilde{M}$ for some $\mathscr{O}_X(U)$-module M.

We say $\mathscr{F}$ is coherent if in either of the above definitions either all the M_i's or all the M's are finitely generated modules.

Proposition 15.8

Let $(X, \mathscr{O}_X)$ be an algebraic variety and let $\mathscr{F}$ and $\mathscr{G}$ be quasi-coherent sheaves on X. Then $\mathscr{F} \otimes \mathscr{G}$ is again quasi-coherent.

Proof. For any open set $U \subseteq X$ consider

$$(\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G})_{|U} \cong \mathscr{F}_{|U} \otimes_{\mathscr{O}_U} \mathscr{G}_{|U}$$

These are both sheaves with the same underlying presheaf, namely the assignment $W \longmapsto \mathscr{F}(W) \otimes_{\mathscr{O}_X(W)} \mathscr{G}(W)$. Hence they are isomorphic.

Let $U \subseteq X$ be open affine. As $\mathscr{F}$ and $\mathscr{G}$ are quasi-coherent, we can find A-modules M and N such that $\mathscr{F}_{|U} \cong \widetilde{M}$ and $\mathscr{G}_{|U} \cong \widetilde{N}$. Thus

$$(\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G})_{|U} \cong \mathscr{F}_{|U} \otimes_{\mathscr{O}_U} \mathscr{G}_{|U} \cong \widetilde{M} \otimes_{\mathscr{O}_U} \widetilde{N} \cong M \widetilde{\otimes_{\mathscr{O}_U(U)}} N$$

Where the last isomorphism comes from $M \otimes_A N \otimes A_f \cong (M \otimes_A A_f) \otimes_{A_f} (N \otimes_A A_f)$. $\square$

Example 15.9 Let $(X, \mathscr{O}_X)$ be an algebraic variety, then $\mathscr{O}_X$ is coherent. This holds because for any affine open set $U \subseteq X$ we have that $\mathscr{O}_{X|U} = \widetilde{\mathscr{O}_{X|U}(U)}$ is a finitely generated module over itself.

15.2 Projective varieties

Let k be an algebraically closed field and $V \subseteq \mathbb{P}^n(k)$ be a projective algebraic set. Our goal is to look at $\mathscr{O}_V$ and to show that $(V, \mathscr{O}_V)$ is an algebraic variety.

Definition 15.10

Let R be a graded ring and f a degree d homogeneous element. Then R_f is graded, and $deg(\frac{g}{f^r}) = deg(g) - r \cdot d$. The degree zero component is denoted by $(R_f)_0$.

Definition 15.11

The structure sheaf $\mathscr{O}_V$ for V is defined as

$$\mathscr{O}_V(D^+(f)) = (\Gamma_{\text{homog}}(V)_f)_0$$

where f is homogeneous of positive degree.

Problem 15.12 Show that $\mathscr{O}_V$ is a sheaf.

Solution:

It is a presheaf because for $D^+(f) \subseteq D^+(g)$ we have $V \setminus V_{\text{proj}}(f) \subseteq V \setminus V_{\text{proj}}(g)$ which again means that $V_{\text{proj}}(g) \subseteq V_{\text{proj}}(f)$. By the projective nullstellensatz we then have $(f) \subseteq \sqrt{(g)}$, which means there is an h such that $f^r = g \cdot h$.

The restriction maps are then

$$\begin{aligned} (\Gamma_{\text{homog}}(V)_g)_0 &\longrightarrow (\Gamma_{\text{homog}}(V)_f)_0 \\ \frac{u}{g^i} &\longmapsto \frac{uh^i}{f^{ni}} \end{aligned}$$

Then sheaf condition is the same as for the affine case.

To prove that this sheaf makes our projective algebraic set into an algebraic variety we need to show it is locally isomorphic to an affine algebraic variety. This isomorphism is as ringed spaces, so we need to know what such a map is.

Definition 15.13

A morphism of ringed spaces

$$(\phi, \phi^\#) \colon (X, \mathscr{O}_X) \longrightarrow (Y, \mathscr{O}_Y)$$

is a continuous map $\phi \colon X \longrightarrow Y$ of topological spaces, and a map $\phi^\# \colon \mathscr{O}_Y \longrightarrow \phi_* \mathscr{O}_X$ of schemes. Here the map of schemes is often thought of as a pullback.

We also need the concept of homogenization and dehomogenization. These processes are morphisms between $k[X_0, \ldots, X_n]$ and $k[X_1, \ldots, X_n]$. Dehomogenization, denoted by ${}^b(-)$ is defined by ${}^b(F(X_0, \ldots, X_n)) = F(1, X_1, \ldots, X_n)$. Homogenization is a bit more difficult, but we describe it by an example. The homogenization, denoted ${}^h(-)$, of the element $X_1 + X_2^3 + X_3^4 \in k[X_1, X_2, X_3]$ is $X_0^3X_1 + X_0X_2^3 + X_3^4$. It finds the greatest degree and multiplies the other components by X_0 until it gets to that degree.

Proposition 15.14

Let $V \subseteq \mathbb{P}^n(k)$ be a projective algebraic set. Then $(V, \mathscr{O}_V)$ is an algebraic variety.

Proof. We reduce to only proving t for $V = \mathbb{P}^n(k)$.

Cover $\mathbb{P}^n(k)$ by $D^+(X_i)$. We will show that $D^+(X_0)$ is an affine algebraic variety. By transferring the same argument over a homography, this is also sufficient.

Set $U_0 = D^+(X_0) = \{[x_0 : \cdots : x_n] \in \mathbb{P}^n(k) \,|x_0 \neq 0\}$. We have earlier seen that we have a bijection $j\colon k^n \longrightarrow U_0$, given by sending $(a_1, \ldots, a_n)$ to $[1 : a_1 : \cdots : x_n]$.

We claim that $(k^n, \mathscr{O}_{k^n}) \xrightarrow{(j,j\#)} (U_0, \mathscr{O}_{\mathbb{P}^n(k)|U_0}$ is an isomorphism of ringed spaces.

If $D^+(F) \subseteq U_0$ where $F \in k[X_0, \ldots, X_n]$ is homogeneous, then $j^{-1}(D^+(F)) = j^{-1}(D^+(F)) \cap U_0 = D({}^bF)$, which means j is continuous. We also have $j(D(f) = D^+({}^hf) \cap U_0$, hence j^{-1} is also continuous, meaning that j is a homeomorphism.

For $W \subseteq V$ open in $\mathbb{P}^n(k)$ we have

$$\begin{array}{ccc} \mathscr{O}_{\mathbb{P}^n(k)|U_0}(V \cap U_0) & \xrightarrow{\cong} & \mathscr{O}_{k^n}(j^{-1}(V \cap U_0)) \\ \downarrow & & \downarrow \\ \mathscr{O}_{\mathbb{P}^n(k)|U_0}(W \cap U_0) & \xrightarrow{\cong} & \mathscr{O}_{k^n}(j^{-1}(W \cap U_0)) \end{array}$$

We need to show there is an isomorphism $\mathscr{O}_{\mathbb{P}^n(k)}(D^+(F) \cap U_0) \cong \mathscr{O}_{k^n}(D({}^bF))$. Notice here that the latter is just $k[X_1, \ldots, X_n]_{{}^bF}$, while the former is $(k[X_0, \ldots, X_n]_{FX_0})0$. Hence we define, for a homogeneous element p with $deg(p) = r(deg(F) + 1)$, the map

$$\begin{aligned} \phi\colon (k[X_0, \ldots, X_n]_{FX_0})_0 &\longrightarrow k[X_1, \ldots, X_n]_{{}^bF} \\ \frac{p}{F^r X_0^r} &\longmapsto \frac{{}^bp}{{}^b(F^r X_0^r)} = \frac{{}^bp}{{}^bF^r} \end{aligned}$$

which is an isomorphism. Hence $j^{\#}$ is an isomorphism of sheaves, and we are done. $\square$

We say an algebraic variety is a projective algebraic variety, or sometimes a projective variety, if it is of the form from the above proposition.

16 Lecture 16 - 08.03.21

16.1 Dimension

Since we are working both with algebra and topology we have two separate notions of dimension, one algebraic and one topological.

Definition 16.1

Let X be a topological space. The dimension of X is defined to be

$$dimX = \sup_{n\in\mathbb{Z}}\{X_0 \subsetneq X_1 \subsetneq \cdots \subsetneq X_n\}$$

where X_i is a closed irreducible subset of X.

Note that this definition is really only useful when working with topologies similar or equal to the Zariski topology. In most standard topologies there is usually not very many such sets, so the dimension is often just zero or one.

Proposition 16.2

Let $Y \subseteq X$ be a subspace. Then $\dim X \geq \dim Y$. If X is irreducible and finite dimensional, and Y is closed in X, then $\dim X > \dim Y$.

Proof. Let $F_0 \subsetneq \cdots \subsetneq F_n$ be a chain of closed irreducible subsets of Y, then $\overline{F_0} \subseteq \cdots \subseteq \overline{F_n}$ is a chain of closed subsets of X. As we have $F_i = \overline{F_i} \cap Y$ we must have that the $\overline{F_i}$'s are distinct, because if $\overline{F_i} = \overline{F_{i+1}}$ then $F_i = F_{i+1}$, which we have assumed is not so. Hence we have a chain $\overline{F_0} \subsetneq \cdots \subsetneq \overline{F_n}$. Suppose that $\overline{F_i} = U \cup V$, where U, V closed and non-empty. Then $F_i = \overline{F_i} \cap Y = (U \cap Y) \cup (V \cap Y)$, which contradicts the assumption that F_i is irreducible. Hence $\overline{F_i}$ is irreducible, and we have at least a chain of length n in X for all chains of length n in Y, hence $\dim Y \leq \dim X$.

Now, assume that X is irreducible, $\dim X < \infty$ and Y closed in X. By the above argument we also have $\dim Y < \infty$, so let $\dim Y = n$. Then a maximal length chain in Y looks like $F_0 \subsetneq \cdots \subsetneq F_n$ for some closed irreducible subsets F_i. Then we have that $\overline{F_0} \subsetneq \cdots \subsetneq \overline{F_n} \subsetneq X$ is a chain in X of length $n+1$, meaning $\dim Y < \dim X$. □

Proposition 16.3

If $X = \bigcup_{i=1}^n X_i$ where X_i is closed in X, then we have $\dim X = \sup \dim X_i$.

Proof. We know from the previous proposition that $\dim X \geq \dim X_i$ for all i, hence we also have $\sim X \geq \sup \dim X_i$. Notice that we are done if $\sup \dim X_i = \infty$, hence we assume $\sup \dim X_i = p$. Suppose now that $p < \dim X$. That means that we can find a chain $F_0 \subsetneq \cdots \subsetneq F_p \subsetneq F_{p+1}$ of closed irreducible subsets of X. We then have $F_{p+1} = \bigcup_{i=1}^n F_{p+1} \cap X_i$, but F_{p+1} is irreducible, hence $F_{p+1} = F_{p+1} \cap X_i$ for some

i. Hence we have $F_{p+1} \subseteq X_i$, and hence X_i now has a chain of length $p+1$, i.e. $\dim X_i > \sup \sim X_i$, which of course is absurd. Hence $\dim X = p$. $\square$

Recall that if X is an algebraic variety, then $X = \bigcup_{i=1}^n F_i$, where F_i is irreducible closed and does not contain each other. Thus we usually reduce to studying dimension of irreducible algebraic varieties.

16.2 Relation to Krull dimension

For an ring R we define its Krull dimension to be $\operatorname{krull.dim} R = \sup\{n \in \mathbb{Z} | p_0 \subsetneq \cdots \subsetneq p_n\}$ where the p_i's are prime ideals of R. In other words, the Krull dimension is the maximal length of a chain of prime ideals.

Proposition 16.4

Let V be an affine algebraic variety. Then $\dim V = \operatorname{krull.dim} \Gamma(V)$.

Proof. Using the Nullstellensatz, there is a correspondence between closed irreducible subsets of V and prime ideals in $\Gamma(V)$. A chain on one side of this correspondence immediately gives a chain of the same length on the other side. $\square$

Example 16.5 Let $V = k^n$, hence $\Gamma(V) = k[X_1, \ldots, X_n]$. Then

$$V(X_1, \ldots, X_n) \subsetneq V(X_1, X_{n-1}) \subsetneq \cdots \subsetneq V(X_1, X_2) \subsetneq V(X_1)$$

is a chain of length n in V. This corresponds to the chain prime ideals

$$(X_1) \subsetneq (X_1, X_2) \subsetneq \cdots \subsetneq (X_1, \ldots, X_{n-1} \subsetneq (X_1, \ldots, X_n))$$

in $\Gamma(V)$. Hence $\dim V = \operatorname{krull.dim} \Gamma(V) \geq n$, as we don't yet know that these are maximal length chains.

Our goal is then to show what our intuition tells us, i.e. that $\dim k^n = n = \operatorname{krull.dim} k[X_0, \ldots, X_n]$.

Lemma 16.6: Noethers normalization lemma

Let A be a finitely generated k-algebra. There exists algebraically independent elements $y_1, \ldots, y_r \in A$ such that A is integral over $k[y_1, \ldots, y_r]$.

Example 16.7 Let x denote the image of X in $k[X,Y]/(XY)$. Then $k[X,Y]/(XY)$ is integral over $k[x]$.

Note that the number of these algebraically independent elements used in Noethers normalization lemma is going to be the dimension.

Theorem 16.8: The dimension theorem

Let A be a domain that is also a finite type k-algebra. Then $\operatorname{krull.dim} A = \operatorname{tr.deg}_k \operatorname{Fr}(A)$, where $\operatorname{Fr}(A)$ is the fraction field of A.

Proof. Noether normalization gives us an injection

$$k[y_1, \dots, y_r] \hookrightarrow A$$

The extension $k \to k(y_1, \dots, y_r) = \operatorname{Fr}(k[y_1, \dots, y_r])$ has transcendence degree r.

Since A is a finitely generated domain over $k[y_1, \dots, y_r]$ we have by the Going up theorem that

$$\operatorname{krull.dim} A = \operatorname{krull.dim} k[y_1, \dots, y_r]$$

Hence it is enough to consider the proof for $k[y_1, \dots, y_r]$.

Our strategy is to show that $k[y_1, \dots, y_r]$ has no chain of prime ideals with length greater than r. We do this by induction on r. For $r = 0$ this is ok.

Assume $r > 0$ and that there exists a chain $p_0 \subsetneq \dots \subsetneq p_m$ for some $m > r$. Choose an element $a_1 \in p_1 \setminus p_0$. Since $a_1 \in k[y_1, \dots, y_r]$ is not a constant polynomial there is "a lemma" that says there exists $a_2, \dots, a_r \in k[y_1, \dots, y_r]$ such that $k[y_1, \dots, y_r]$ is finitely generated over $k[a_1, \dots, a_r]$. We then have

$$\begin{tikzcd}
k[Z_1, \dots, Z_r] \arrow[r, "f.g."] \arrow[d, two heads] & k[y_1, \dots, y_r] \arrow[d, two heads] \\
k[Z_1, \dots Z_r]/(Z_1) \arrow[r, dotted, "f.g."] & k[y_1, \dots, y_r]/p_1
\end{tikzcd}$$

where the top map sends Z_i to a_i.

Notice also that $k[Z_1, \dots Z_r]/(Z_1) \cong k[Z_2, \dots Z_r]$. Now, say we have a chain $q_1 \subsetneq \dots \subsetneq q_m$ in $k[Z_2, \dots Z_r]$. Then the going down theorem makes sure we have a chain $\overline{p_1} \subsetneq \dots \subsetneq \overline{p_m}$. By induction we must have that $m - 1 \leq r - 1$ i.e. that $m \leq r$, which contradicts our assumption that $m > r$. □

Corollary 16.9

We have $\operatorname{krull.dim} k[X_1, \dots, X_n] = n$.

Corollary 16.10

If V is an irreducible affine algebraiv variety, then $\dim V < \infty$.

Corollary 16.11

We have $\dim k^n = n$.