MA8203 - Algebraic geometry

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About

These lecture notes are based on my notes from the course MA8203 - Algebraic geometry taught by Peder Thompson at NTNU during the spring of 2021. The course followed the books *Algebraic geometry* by Perrin and *Algebraic geometry* by Hartshorne, hence these notes also have a similar progression. There are probably some mistakes in these notes, so take what is here with a pinch of salt, but most should hopefully be correct.

If you find mistakes you want corrected, or have some notes on my notes, then feel free to suggest changes and additions at the github repo. Feel free to use any figures or text in that repo for whatever you need.

Todo list

Contents

1 Lecture 1 - 12.01.21

Algebraic geometry is the study of algebraic varieties, which are roughly the zero loci of polynomials over fields. It is an old theory, spanning way back, but the focus of this course will be on the more modern theory developed by the likes of J.P. Serre and A. Grothendieck during the last half of the 20'th century.

Throughout the course k will be a field, usually algebraically closed, or at least infinite. The motivating examples will be $\mathbb{C}$ and $\mathbb{R}$. We will start by studying affine algebraic varieties, and later more general algebraic varieties and projective varieties, which are both locally modeled by the affine ones.

1.1 Introduction

Let $P \in k[X_1, \ldots X_n]$, i.e. a polynomial in n variables over a field k. Define its zero locus to be the set $V(P) = \{x \in k^n | P(x) = 0\} \subset k^n$. This set $V(P)$ is roughly what we mean by an algebraic variety. More generally we can use a set of polynomials instead of just one. Let $P = \{P_i\}$ be a collection of polynomials in $k[X_1, \ldots, X_n]$. We define the zero locus of the polynomials to be $V(P) = \{x \in k^n | P_i(x) = 0, \forall i\}$.

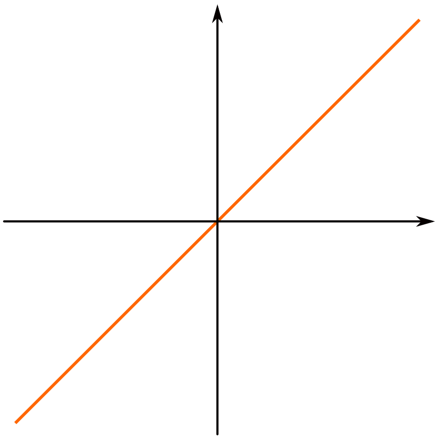
A subset $X \subset k^n$ is also an affine algebraic variety if $X = V(P)$ for some set of polynomials $P = \{P_i\}$ in $k[X_1, \ldots, X_n]$. Really this is what we call an affine algebraic set, but we will get back to this in lecture 2 when we take on some more proper definitions.

A really simple example is given by linear subspaces.

Example 1.1 If the P_i's all have degree 1, then $V(P)$ are affine linear subspaces of k^n, i.e. liner, planes and hyperplanes.

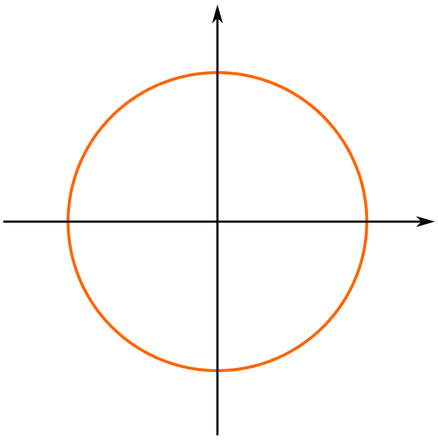
Another class of important intuition giving examples are planar curves, which often looks like the graph of a single polynomial in one variable. More specifically, if we let $n = 2$ and $k = \mathbb{R}$, then the zero locus of a single polynomial $P(X, Y)$ is a real plane curve. These will be important for this course, and can in many cases serve as nice intuition for these algebraic varieties. A concrete simple example is a line.

Example 1.2 An example of such a curve is $P(X, Y) = X - Y$, which has zero locus being the line $f(x) = x$ in the plane, i.e.



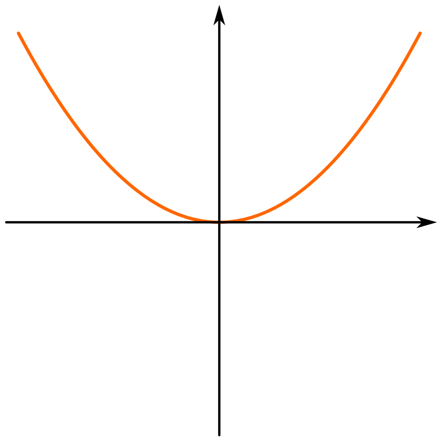
The above example uses a polynomial of degree one, and so must describe a line, but we can have polynomials with higher degrees - making more interesting algebraic varieties.

Example 1.3 If we require the polynomial to be of degree 2, then we can have $P(X,Y) = X^2 + Y^2 - 1$, which has zero locus equal to the unit circle in $\mathbb{R}^2$.



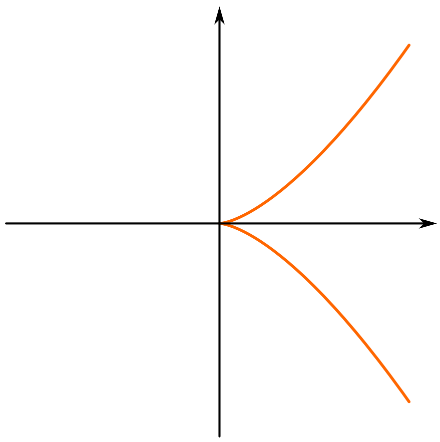
We see in the above example that a planar curve does not always have to be the graph of a single polynomial, as the circle requires more two one-variable polynomials to describe it as its curves.

Example 1.4 Another example in degree 2 is $P(X,Y) = y - x^2$, which has zero locus equal to the graph of $f(x) = x^2$.



We need not stop at polynomials of degree two, and the higher degree we get, the more complicated - and interesting - the planar curves get.

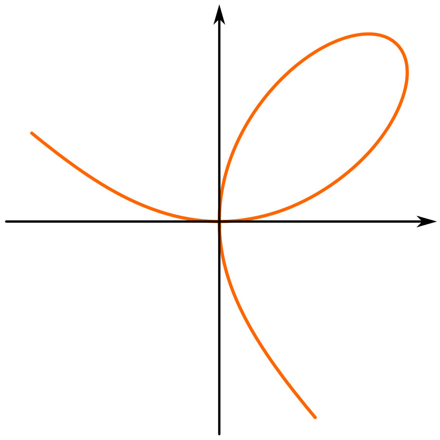
Example 1.5 If we have degree 3 polynomials we have for example $P(X,Y) = Y^2 - X^3$ which have zero locus being a cuspidal curve.



The above curve has a singularity, which we will come back to later in the course. Figuring out what a singularity is mathematically and not just intuitively by looking at the curve is a bit tricky, so it will take some rigorous exploration of dimension theory to do it. More on this in several weeks.

Such singularities need not be only of the form above, they can also come from self-intersecting curves, like the following example.

Example 1.6 Another degree 3 example is $P(X,Y) = X^3 + Y^3 - XY$, which has a loop shaped zero locus.



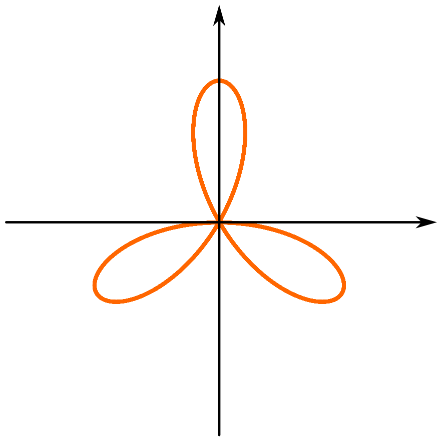
There exists a certain class of planar curves, which are all non-singular, i.e. they have none of these singularities or self intersections. They are famously connected to several different important mathematical theories including number theory and cryptography - namely the elliptic curves.

Example 1.7 The below curve is the elliptic curve given by [Some polynomial]

fix image of the elliptic curve

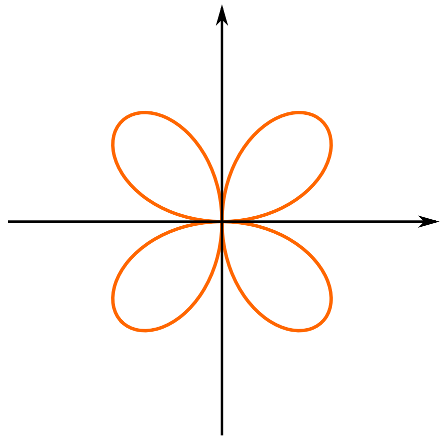
As we said, the higher the degree the more complicated the planar curve, but lets see some nice higher degree curves anyway.

Example 1.8 One example is the trefoil-curve, given by the polynomial $P(X, Y) = (X^2 + Y^2)^2 + 3X^2Y - Y^3$:



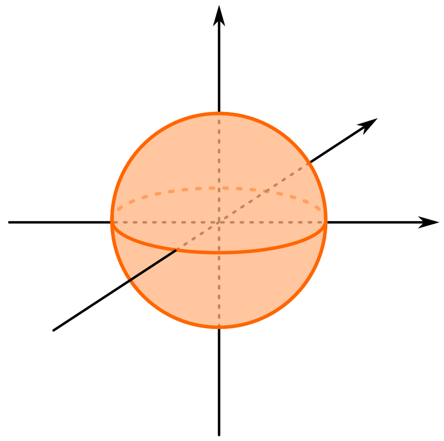
The polynomial P has degree four.

Example 1.9 Another degree four example is the quadrafoil curve, given by the polynomial $P(X, Y) = (X^2 + Y^2)^2 - 4X^2Y^2$.

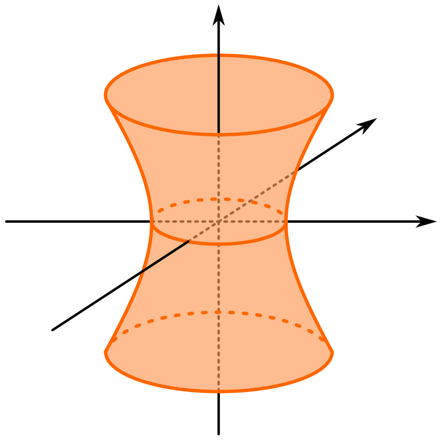


Ok, until now we have almost only seen planar curves, which is only a fraction of the possible algebraic varieties. One way to get other objects is to try to make "curves" in higher dimensions. If we for example let $n = 3$ and still have $k = \mathbb{R}$ then the zero loci of single polynomials are surfaces. If we first restrict ourselves to polynomials of degree 2, we get the quadratic surfaces we know and love from calculus.

Example 1.10 The most recognized quadratic surface is maybe the zero locus of $P(X, Y, Z) = X^2 + Y^2 + Z^2 - 1$, which is equal to the unit sphere in $\mathbb{R}^3$



Example 1.11 Another example of a surface is given by $P(X, Y, Z) = X^2 + Y^2 - Z^2 - 1$ which has zero locus equal to a hyperboloid.



So just looking at the zero-locus of a single polynomial already produces a rich and interesting theory, as we can study both elliptic curves, cubic surfaces and many many more.

To round off the examples, we consider an algebraic variety given by two polynomials. We still consider $n = 3$ and $k = \mathbb{R}$, but now the zero-loci of these two polynomialsgives us the space-curves.

Example 1.12 An example is the zero locus $V = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 - z^2 - 1 = 0, y = 0\}$, which is the part of the previous hyperboloid lying in the xz-plane

Make graphics for this example

Note: This study depends very much on which field we work in. We mentioned that we are going to mostly assume that our field is algebraically closed. This places an emphasis on the equations defining our varieties, rather than their actual points. We will hopefully understand a bit better what this means later in the course, when we discuss schemes.

1.2 Intersection of curves

We mentioned that the planar curves were very important for us, and we will now try to explain why. There is a famous important theorem, called Bézout's theorem, which this course will be centered around proving in detail. The theorem centeres around the following question.

Question: How can two plane curves intersect?

If we take a pause to think about it, the answer is quite simple. There are in general four ways two planar curves can intersect:

1. They can not intersect at all.
2. They can be tangent to each other, hence only intersect at a single point.
3. They can intersect in some finite number of points.

4. They can be overlapping, and hence have a continuum of intersections.

We shall mainly be interested in the third point - when two planar curves intersect at a finite number of points. As this number is finite, we would like to know which number it is. Is there a nice way to calculate this number? Is it related to other properties of the two planar curves? Bézout's theorem will give us a really satisfying answer to this. It arises from the following questions:

Questions: How does the number of intersections relate to the degrees of the curves? What are the maximum and minimum number of intersections? What are the special cases, and what is the general behavior?

Let C and C' be plane curves defined by polynomials of degrees d and d' respectively. If we take a little time to think about the above questions, we are quite naturally led to two more questions:

- Do C and C' always intersect in at most dd' points?
- When is the number of intersection points equal to dd'?

Intuitively there seems to be three obstructions to the number of intersections being dd'. These are

1. The curves have a common component, i.e. they have an infinite number of intersection points
2. The curves have no overlap, for example two parallel lines
3. The curves are tangent, and can therefore have less intersection points

So, if we remove all these obstructions, we should have a nice theorem that relates the number of intersection points to the product of the degrees of the planar curves, and this is exactly what Bézout's theorem is.

Theorem 1.13: Bézouts theorem

Let C and C' be two projective plane curves of degree d and d', defined over an algebraically closed field, with no common components. Then the number of intersections, counted with multiplicity, is dd'.

The fact that the three obstructions are naturally removed in the theorem is not an easy task to see just yet, as we have not properly discussed any of them. But, intuitively, no common components means we have a finite number of intersections, projective curves mean we don't have parallel lines and counting with multiplicity means tangential curves are counted right.

Defining these terms, formalizing the entire construction and proving this theorem will be the first - and biggest - goal for this course.

2 Lecture 2 - 18.01.21

Last time we intuitively introduced algebraic varieties, looked at several examples and provided a goal for the course - Bézout's theorem. To properly build up the theory from scratch we need to reintroduce what we only covered briefly in the introduction, namely algebraic varieties, or actually to start off, affine algebraic sets.

2.1 Affine algebraic sets

Affine algebraic sets are the building blocks of algebraic geometry. All the other objects are built out of these sets, are inspired by them, or act as generalizations of them. So, to understand algebraic geometry and its important objects like algebraic varieties, projective varieties, planar curves, projective planar curves, affine schemes and schemes, we need to have a firm proper understanding of these building blocks - the affine algebraic sets.

Let k be a field. We saw the definition of an affine algebraic set last time in the introduction, but for completeness and rigorousness we include it again.

Definition 2.1

Let $S \subset k[X_1, \ldots, X_n]$ be any subset. We define the affine algebraic set generated by S to be $V(S) = \{x \in k^n | P(x) = 0, \forall P \in S$.

Definition 2.2: Affine algebraic set

We say a subset $X \subset k^n$ is an affine algebraic set if we have $X = V(S)$ for some subset $S \subset k[X_1, \ldots, X_n]$.

The assignment of the generated affine algebraic set from a subset of polynomials is an order reversing assignment. This means simply that given $S \subset S' \subset k[X_1, \ldots, X_n]$, then we have $V(S') \subset V(S) \subset k^n$.

We saw many geometric examples of such sets last time, so lets try to be a bit more algebraic this time around.

Example 2.3 Let $1 \in k[X_1, \ldots, X_n]$ be the identity polynomial. Then we have $V(\{1\}) = \emptyset$.

Example 2.4 Let $0 \in k[X_1, \ldots, X_n]$ be the zero polynomial. Then we have $V(\{0\}) = k^n$.

These two examples seem trivial, but they are will be important soon. They are rarely used, but important to keep in mind.

Example 2.5 Let $n = 1$ and let S consist of a single polynomial P. Then $V(S)$ is a finite set. This is because it consists of the zeroes of the polynomial, which is finite because we are working over a field.

Example 2.6 Let $n = 2$, then the affine sets in k^2 are

- the empty set $\emptyset$
- the affine planes
- the curves
- finite sets of points

As k^n itself is an affine algebraic set, we call the sets in the above example the affine algebraic subsets of k^n. We can study affine algebraic subsets of any affine algebraic set in general, not just k^n.

The assignment $V(S)$ of some set of polynomials $S \subset k[X_1, \dots, X_n]$ consists as we know of its common zeroes. So, if $P(x) = 0$ for some polynomial P in S, then also $aP(x) = 0$ for some element $a \in k$. Also, for two polynomials P, Q in S then if $P(x) = 0 = Q(x)$, then also $(P + Q)(x) = 0$. So it seems like we can expand a bit this set S, and still generate the same affine algebraic set. The below definition and lemma makes this observation rigorous.

Definition 2.7

Let $S \subset R$ be any subset of a ring R. The ideal generated by S is the ideal $(S) = \{\sum_{i=0}^{r} a_i f_i | a_i \in k[X_1, \dots, X_n], f_i \in S\}$.

Lemma 2.8

The affine algebraic variety generated by a set $S \subset k[X_1, \dots, X_n]$ is the same as the affine algebraic variety generated by the ideal (S).

Proof. Since the assignment $V(-)$ is order reversing, and we have $S \subset (S)$ then we immediately have $V((S)) \subset V(S)$. For the other inclusion we let $x \in V(S)$ be any point. Then by definition we know that $P(x) = 0$ for all $P \in S$. Let now $Q \in (S)$. We have

$$\begin{aligned} Q(x) &= \sum_{i=0}^{r} a_i f_i(x) \\ &= 0 \end{aligned}$$

since we know that $f_i(x) = 0$ for all i. This holds for all elements $Q \in (S)$, hence we have $x \in V((S))$, which proves that $V(S) \subset V((S))$. $\square$

This fact allows us to reduce the size of the set of polynomials S, bue to the following result.

Proposition 2.9

Every affine algebraic set X is generated by a finite set of polynomials, i.e. $X = V(f_1, \dots f_r)$.

Proof. Since k is a field we have by the Hilbert basis theorem that $k[X_1, \dots X_2]$ is a Noetherian ring. In a Noetherian ring all ideals are finitely generated. Since X is an affine algebraic set we know that $X = V(S)$ for some subset $S \subset k[X_1, \dots, X_n]$. By the previous lemma we know that $V(S) = V((S))$, where now (S) is an ideal. But this we know is finitely generated, i.e. $(S) = (f_1, \dots, f_r)$ for some r, and hence we conclude that $X = V(f_1, \dots, f_r)$. □

We remark that we also have $V(f_1, \dots f_r) = V(f_1) \cap V(f_2) \cap \dots \cap V(f_r)$. This is relatively easy to convince ourselves of, as the left hand side consists of all points in k^n such that the polynomials vanish simultaneously, and the right hand side consists of the intersection of all points in k^n such that the polynomials vanish individually. We often refer to the affine algebraic set generated by a single polynomial, $V(f)$, as a hypersurface in k^n.

As ideals in a ring are its nice "substructures" and these substructures gets sent by the assignment $V(-)$ to some affine algebraic sets, it would be nice to have a correspondence between substructures of the ring and the substructures of the affine algebraic sets. This turns out to be the case, but we need to know what "nice substructures" of affine algebraic sets are.

Proposition 2.10

The affine algebraic subsets are the closed sets in a topology on k^n.

Proof. For the set of affine algebraic sets to define a topology on k^n we need three things:

1. $\emptyset$ and k^n are affine algebraic sets.
2. arbitrary intersections of affine algebraic sets is again an affine algebraic set.
3. finite union of affine algebraic sets is again an affine algebraic set.

We have already seen that $\emptyset$ and k^n are affine algebraic sets, because we have $\emptyset = V(\{1\})$ and $k^n = V(\{0\})$, so the first point is done.

For the second one we will show that $\bigcap_{i \in I} V(S_i) = V(\bigcup_{i \in I} V(S_i))$.

For the third point it is enough to show that the union of two affine algebraic sets generated by two ideals is again an affine algebraic set generated by an ideal.

Let I, J be two ideals in $k[X_1, \ldots, X_n]$. We know that the product of the ideals, IJ, is contained in both of the ideals, i.e. $IJ \subset I$ and $IJ \subset J$. By the order reversing property of $V(-)$ we have $V(J) \subset V(IJ)$ and $V(I) \subset V(IJ)$. Since both of them are contained, then also their union is, i.e. $V(I) \cup V(J) \subset V(IJ)$. This proves on of the inclusions. For the other one we let $x \in V(IJ)$ such that $x \notin V(I)$. Then by definition there exists a polynomial $P \in I$ such that $P(x) \neq 0$. Since we have $PQ \in IJ$ for all $Q \in J$ we know that x has to be an element of $V(J)$ because $(PQ)(x) = 0$, which implies $Q(x) = 0$ because $k[X_1, \ldots, X_n]$ is a domain. The same argument shows $x \in V(I)$ if $x \notin V(J)$. Hence we have $V(I) \cup V(J) = V(IJ)$. □

proove second point

This topology is called the Zariski topology on k^n and is hugely important for this course and for the field of algebraic geometry in general.

This topology is very different from the standard topology we are used to on k^n. To give an example we have that the closed sets in k^3 with the Zariski topology are the points, the curves and the planes. These are very "thin" and "small" sets compared to the closed balls that generate the closed sets in k^n with the standard topology. A nice intuition to have is that the closed sets in the Zariski topology are "sliced" out of the space we are in, and that slicing a closed set again yields a closed set. These slices have to be polynomially defined of course.

Definition 2.11: Standard open sets

Let $f \in k[X_1, \ldots, X_n]$. The standard open sets of k^n are the sets $D(f) = k^n \setminus V(f)$, i.e. the complements of the hypersurfaces.

The important thing about the standard open sets is that they form a basis for the Zariski topology on k^n.

Problem 2.12 Show that the intersection of any two open sets is non-empty.

Solution:

Since the standard open sets form a basis for the open sets, it is enough to show that $D(f) \cap D(g) \neq \emptyset$ for some arbitrary polynomials $f, g \in k[X_1, \ldots, X_n]$. This is proven using elementary set theory and the previous properties we have shown.

$$\begin{aligned} D(f) \cap D(g) &= k^n \setminus V(f) \cap k^n \setminus V(g) \\ &= k^n \setminus (V(f) \cup V(g)) \\ &= k^n \setminus (V((f)) \cup V((g))) \\ &= k^n \setminus V((f)(g)) \\ &\neq \emptyset \end{aligned}$$

where the last equality comes from the fact that $(f)(g) \neq (0)$ as $k[X_1, \ldots, X_n]$ is a domain.

2.2 The ideal of an affine algebraic set

We are now hopefully starting to sense and feel the duality between the geometry and the algebra. This duality comes through the connection between affine algebraic sets and ideals in rings. Until now we have only passed from the algebra to the geometry by assigning a geometric object to an algebraic one. In order to have a nice duality theory we want some way to also go the other way, i.e. some kind of dual or inverse to $V(-)$.

Definition 2.13: Ideal of affine subset

Let $V \subset k^n$ be some subset. The ideal of V is defined to be $I(V) = \{f \in k[X_1, \ldots, X_n] | f(x) = 0, \forall x \in V\}$.

It is maybe not obvious that this set is an ideal, so lets prove that it is. We define the morphism $r : k[X_1, \ldots, X_n] \longrightarrow \mathcal{F}(V, k)$ by sending a polynomial P to its restriction $P_{|V}$. Here $\mathcal{F}(V, k)$ is the ring of ring homomorphisms from V to k. We need to show that r is a ring homomorphism. It is because $r(P + Q) = (P + Q)_{|V} = P_{|V} + Q_{|V} = r(P) + r(Q)$ and because $r(PQ) = (PQ)_{|V} = P_{|V} Q_{|V} = r(P)r(Q)$.

Since the kernel of r is exactly the polynomials in $k[X_1, \ldots, X_n]$ that vanish when restricted to V we have $\mathrm{Ker}(r) = I(V)$. The kernels of ring homomorphisms are ideals, and thus we have shown that $I(V)$ is an ideal.

Definition 2.14: Coordinate ring

Let $V \subset k^n$. We define the coordinate ring of V to be the finite type k-algebra $\Gamma(V) = \mathrm{Im}(r)$, i.e. the polynomial functions on V.

By the first isomorphism theorem we have that $\Gamma(V) \cong k[X_1, \ldots, X_n]/I(V)$.

Similarily to $V(-)$, the assignment $I(-)$ is order reversing, i.e. if $V \subset V'$ then $I(V') \subset I(V)$.

Problem 2.15 What is the relationship between V and $V(I(V))$?

Solution:

By definition we have that $V(I(V))$ is the set of elements $y \in k^n$ such that $P(y) = 0$ for all polynomials $P \in I(V)$. These are exactly the polynomials $P \in k[X_1, \ldots, X_n]$ such that $P(z) = 0$ for all $z \in V$. So, if we have $x \in V$ then $P(x) = 0$ for all $P \in I(V)$. This also means that x is an element such that $P(x) = 0$ for all $P \in I(V)$ which means that $x \in V(I(V))$. Hence we have $V \subset V(I(V))$.

Now, take a point $x \in V(I(V))$. By definition we have $P(x) = 0$ for all polynomials $P \in I(V)$.

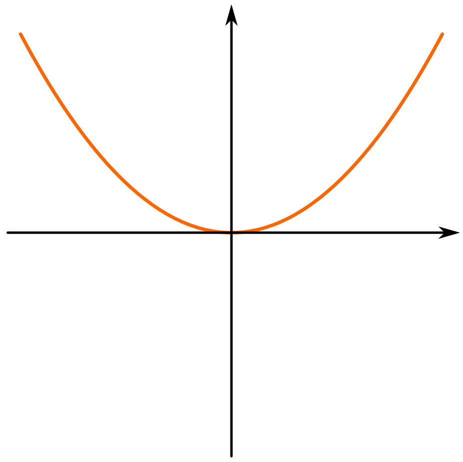
do rest of solution

Problem 2.16 What is the relationship between I and $I(V(I))$?

Solution:

Let $I \subset k[X_1, \ldots, X_n]$ be an ideal and $P \in I$. We have $V(I) = \{x \in k^n | f(x) = 0 \forall f \in I\}$. In particular we have $P(x) = 0$ for all $x \in V(I)$, and thus $P \in I(V(I))$. This shows $I \subset I(V(I))$.

As a counterexample to the equality we can for example take the ideal $I = (X^2 - Y)^2 \subset k[X, Y]$. Then $V(I)$ looks something like



We know that the polynomial $P = (X^2 - Y)^2$ is zero on this affine algebraic set, but so is $X^2 - Y$, which is not in I. Hence $I \neq I(V(I))$ in general.

Problem 2.17 Show that $I(\emptyset) = k[X_1, \ldots, X_n]$.

Solution:

This statement is vacuously true. The set $I(\emptyset)$ consists of all polynomials $P \in k[X_1, \ldots, X_n]$ such that $P(x) = 0$ for all $x \in \emptyset$, which is true because $\emptyset$ contains no elements.

Proposition 2.18

If k is an infinite field then $I(k^n) = (0)$.

Proof. We will use induction on n.

First let $n = 1$. Then $I(k) = \{f \in k[X_1] | f(x) = 0, \forall x \in k\}$. A non-zero polynomial

f has a finite set of roots, hence we can find a point $y \in k$ such that $f(y) \neq 0$ due to k being infinite. Hence if we have $f \in I(k)$ then f has to be zero outside of all of its roots, and hence it must be the zero polynomial.

Now let $n \geq 1$. Suppose that $f \in k[X_1, \ldots, X_n] \setminus \{0\}$ is non-constant. We can then express f as $f = a_r(X_1, \ldots, X_n)X_n^r + \ldots$ for some $r \geq 1$. By induction we can find $x_1, \ldots, x_{n-1}$ such that $a_r(x_1, \ldots, x_{n-1}) \neq 0$. Hence $f(x_1, \ldots, x_{n-1}, X_n)$ has at most r roots. We can then find a point y such that $f(y) \neq 0$ due to k being infinite. $\square$

details

Problem 2.19 Show that $I(\{(a_1, \ldots, a_n)\}) = (X_1 - a_1, \ldots X_n - a_n)$, i.e. points correspond to maximal ideals.

Problem 2.20 Compute $I(-)$ and $V(-)$ for some examples.

3 Lecture 3 - 19.01.21

To warm up our brains we start the lecture with a problem to solve.

Problem 3.1 Compute $I(V((X,Y^2)))$ in $k[X,Y,Z]$.

Solution:

The algebraic set $V((X,Y^2))$ consists of all points $(x,y,z) \in k^3$ such that $P(x,y,z) = 0$ for all polynomials $P \in (X,Y^2)$. Polynomials in (X,Y^2) are of the form $P = aX^n + bY^{2m}$, where $a, b \in k[X,Y,Z]$. For these to be equal to zero at (x,y,z) we need that $x = 0 = y$. We also have no restrictions on z. Hence $V((X,Y^2)) = \{(0,0,z) \in k^3\}$. If we look at $I(V((X,Y^2)))$, then this consists of all polynomials which are zero at all points in $V((X,Y^2)) = \{(0,0,z) \in k^3\}$, i.e. polynomials generated by X and Y. Hence we have $I(V((X,Y^2))) = (X,Y)$.

3.1 Irreducibility

Definition 3.2: Irreducible topological space

Let $X \neq \emptyset$ be a topological space (for us this means an affine algebraic set with the Zariski topology). We call X irreducible if whenever we can write $X = F \cup G$ with F, G closed subsets of X, then either $F = X$ or $G = X$. If this is not the case, then we call X reducible, or decomposable.

Theorem 3.3

Let $V \subset k^n$ be an affine algebraic set with the Zariski topology. Then V is irreducible if and only if $I(V)$ is a prime ideal in $k[X_1, \dots, X_n]$. Equivalently, V is irreducible if and only if $\Gamma(V)$ is a domain.

Proof. Assume that V is irreducible. We want to show that $I(V)$ is prime, i.e. that if we have two elements $f, g \in k[X_1, \dots, X_n]$ such that $fg \in I(V)$, then either $f \in I(V)$ or $g \in I(V)$ (or both of course).

Assume that we have two such elements f, g such that $fg \in I(V)$. Since $(fg) \subset I(V)$ we have by the order reversing property of $V(-)$ that $V(I(V)) \subset V((fg)) = V(fg)$. By the properties of the Zariski topology we know that $V(fg) = V(f) \cup V(g)$, and thus $V \subset V(f) \cup V(g)$. Then $V = (V \cap V(f)) \cup (V \cap V(g))$, which are two closed subsets of V. Since we assumed that V was irreducible we know that either $V = V \cap V(f)$ or $V = V \cap V(g)$. Assume without loss of generality that $V = V \cap V(f)$. Then we have that $V \subset V(f)$, which by the order reversing property of $I(-)$ means that $I(V(f)) \subset I(V)$. And since we know $f \in I(V(f))$ we have finally $f \in I(V)$.

Assume now that $I(V)$ is prime. We will show V irreducible by showing that a given decomposition of $V = V_1 \cup V_2$, where $V \neq V_1$, $V \neq V_2$, leads to a contradiction to $I(V)$ being prime.

Assume V has such a decomposition, i.e. that V is reducible. Since $V_i \subset V$ we have by the order reversing property of $I(-)$ that $I(V) \subset I(V_i)$. Since $V_i \subsetneq V$ we also have $I(V) \subsetneq I(V_i)$, because $V(I(-)) = Id(-)$ implies that $I(-)$ is an injection. Hence there exists $f_1 \in I(V_1) \setminus I(V)$ and $f_2 \in I(V_2) \setminus I(V)$. But, notice that $f_1 f_2$ in fact vanishes on V since it vanishes on both V_1 and V_2 and hence on their union, which is V. Thus $f_1 f_2 \in I(V)$, but we explicitly chose f_1 and f_2 not in $I(V)$, and hence this is a contradiction to $I(V)$ being a prime ideal, meaning that our assumption about V being reducible must have been wrong. □

Corollary 3.4

If k is an infinite field, then k^n is irreducible.

Proof. When k is infinite we have previously shown that $I(k^n) = (0)$. Since (0) is a prime ideal in $k[X_1, \ldots, X_n]$, because it is a domain, we have by the previous theorem that k^n must be irreducible. □

Notice that this is not true in general when k is a finite field.

Theorem 3.5

Let $V \subset k^n$ be a non-empty affine algebraic set. Then there exists a (up to permutation) unique collection of irreducible affine algebraic sets $V_1, \ldots, V_r$ such that $V_i \nsubseteq V_j$ for $i \neq j$, and $V = V_1 \cup \cdots \cup V_r$.

Proof. There are two parts to this proof, showing such a decomposition exists, and showing it is unique. We start by showing existence.

Assume that we have a non-decomposable affine algebraic set V. This has a corresponding ideal $I(V)$. Since k is a field, then $k[X_1, \ldots, X_n]$ is Noetherian by the Hilbert basis theorem. This means we can choose V to be the affine algebraic set such that the ideal $I(V)$ is maximal. Here we don't necessarily mean that $I(V)$ is a maximal ideal, but that is the biggest with respect to the property that $I(V)$ is non-decomposable. Since V is assumed non-decomposable we must have a decomposition $V = F \cup G$ of V into closed subsets such that $F \neq V$, $G \neq V$.

We previously said that $I(-)$ is injective, and hence we have $I(V) \subsetneq I(F)$ and $I(V) \subsetneq I(G)$. Since $I(V)$ was maximal among the ideals of non-decomposable affine algebraic sets we must have that F and G are decomposable. Then we have $F = \bigcup_{i=1}^{s} V_i$ and $G = \bigcup_{i=s+1}^{r} V_i$, where V_i is irreducible. This of course gives us a decomposition $V = V_1 \cup \cdots \cup V_r$ into irreducible sets. By potentially removing some overlapping sets we also get $V_i \nsubseteq V_j$ for $i \neq j$. Hence all affine algebraic sets are decomposable.

Lets show that this decomposition is unique up to permutation.

Assume we have two decompositions $V = V_1 \cup \cdots \cup V_r$ and $V = W_1 \cup \cdots \cup W_s$. Then we have $V_1 = V \cap V_1 = (W_1 \cap V_1) \cup \cdots \cup (W_s \cap V_1)$. Since V_1 is irreducible by assumption we must have $V_1 = W_j \cap V_1$ for some j. This implies $V_1 \subseteq W_j$. Similarily we have $W_j = V \cap W_j = (V_1 \cap W_j) \cup \cdots \cup (V_r \cap W_j)$, which by the irreducibility of W_j must mean that $W_j = V_k \cap W_j$ for some k, and hence $V_j \subseteq V_k$.

But this means that we have $V_1 \subseteq W_j \subseteq V_k$, which cant be the case as $V_1 \not\subseteq V_i$ for all $i \neq 1$. This means that $k = 1$, and in turn $V_1 = V_k$. Since W_j is squeezed in the middle of two equal sets, it also mus be equal to them, i.e. $V_1 = W_j$. We now reorder the decomposition into $V = W_j \cup W_1 \cup \cdots W_{j-1} \cup W_{j+1} \cup \cdots \cup W_s$ and repeat the process by using V_2 instead. We can continue this process until we have associated every V_i with some W_j, which means that $r = s$ and that the two decompositions are the same after some reordering. □

3.2 Hilbert's nullstellensatz

For the rest of this lecture we will assume that k is an algebraically closed field. This is essential to the theorem. We will first prove the weak nullstellensatz and use that to prove the strong one.

Theorem 3.6: Hilbert's weak nullstellensatz

Let $I \subsetneq k[X_1, \ldots, X_n]$ be a proper ideal. Then $V(I) \neq \emptyset$.

In class we omitted the proof of this due to proving it in the course MA8202 - Commutative algebra, which this course has as a prerequisite. But, for completeness sake I have added a proof.

Proof. Every proper ideal is, or is contained in a maximal ideal. Hence, for some maximal ideal $M \subseteq k[X_1, \ldots, X_n]$ we have $I \subseteq M$. By the order reversing property of $V(-)$ we get $V(M) \subseteq V(I)$, so it is in fact enough to look at maximal ideals.

In the last lecture we saw that $I \subseteq I(V(I))$ for some ideal $I \subseteq k[X_1, \ldots, X_n]$. If we apply this to M we get $M \subseteq I(V(M))$, which implies either $I(V(M)) = M$ or $I(V(M)) = k[X_1, \ldots, X_n]$ as M is a maximal ideal. By section 3.1 this means that either $V(M)$ is an irreducible affine algebraic set, as $I(V(M)) = M$ is a prime ideal, or that we have $I(V(M)) = k[X_1, \ldots, X_n]$ which means $V(M) = \emptyset$. Hence we need to justify that there exists points in $V(M)$.

Notice that for some point $a = (a_1, \ldots, a_n) \in k^n$ we have that $M_a = (X_1 - a_1, \ldots, X_n - a_n)$ is a maximal ideal. If we somehow could prove that all ideals are of this form, then we would be done as $V(M_a)$ would contain $(a_1, \ldots, a_n)$ and hence be non-empty. So lets prove this.

We define the evaluation morphism as follows:

$$\begin{aligned} e_a : k[X_1, \ldots, X_n] &\longrightarrow k \\ f &\longmapsto f(a). \end{aligned}$$

Note that it is a surjective k-algebra homomorphism and since k is algebraically closed, it has kernel M_a.

Let M be some arbitrary maximal ideal in $k[X_1, \ldots, X_n]$. Then $k[X_1, \ldots, X_n]/M$ is a finitely generated field extension of k. By Zariski's lemma, $k[X_1, \ldots, X_n]/M$ is in fact a finite field extension, better known as a finite dimensional vector space. Since k is algebraically closed, there is an isomorphism of k-algebras

$$k[X_1, \ldots, X_n]/M \longrightarrow k.$$

Now, let a_i denote the image of X_i. Then we get that $M_a \subseteq M$, which implies $M_a = M$ since M_a is a maximal ideal.

Hence we know that any maximal ideal M will have $V(M) \neq$, and hence we are done. □

Notice here that we used Zariski's lemma in our proof. This states that if K is a finitely generated k-algebra, such that K is also a field, then K is a finite field extension of k, i.e. a finite dimensional k-vector space. The proof is omitted here, but a discussion about the geometry behind it can be found on my blog.

Also note that the affine algebraic set $V((X^2 + Y^2 + 1))$ is empty in $\mathbb{R}^2$, even though $(X^2 + Y^2 + 1)$ is prime in $\mathbb{R}[X, Y]$. Hence the weak nullstellensatz does not hold for non-algebraically closed fields.

Theorem 3.7: Hilbert's nullstellensatz

Let $I \subseteq k[X_1, \ldots, X_n]$ be an ideal. Then $I(V(I)) = \sqrt{I}$.

I didn't quite understand the proof presented in class. I will still put it below, but I have also put a (in my opinion) easier to understand proof in the appendix. It seems to be essentially the same, but some parts are wrapped up nicely in Zariski's lemma. The proof can be found at appendix A.1.

Proof. Since $k[X_1, \ldots, X_n]$ is noetherian we know that ideals are finitely generated. So, choose generators $I = (P_1, \ldots, P_r)$. We first prove the inclusion $\sqrt{I} \subseteq I(V(I))$.

Let $f \in \sqrt{I}$. By definition this means that $f^m \in I$ for some m. Since $f^m \in I$ we know that f^m vanishes on $V(I)$, i.e. $f^m \in I(V(I))$. But, if f^m vanishes on $V(I)$, then so does f, hence $f \in I(V(I))$.

For the other inclusion we let $f \in I(V(I))$, and we want to show that there is a m such that $f^m \in I$. We note that it is enough to show that $Ik[X_1, \ldots, X_n]_f = k[X_1, \ldots, X_n]_f$. This is because we would have $1_{k[X_1,\ldots,X_n]_f} = \sum P_i \frac{Q_i}{f^{n_i}}$, which would imply that $f^m = \sum P_i(Q_i f^{m-n_i})$, where $m = max\{n_i\}$. This means that we would have written f^m as a linear combination of parts with a P_i in each summand, which means $f^m \in (\{P_i\}) = I$.

We note that $k[X_1, \ldots, X_n]_f \cong k[X_1, \ldots, X_n, T]/(1 - Tf)$. This is like saying that inverting all powers of f, which is what localizing at f does, is the same as adding a new variable to the polynomial ring with the property that "it is the inverse of f".

Now we have $Ik[X_1, \ldots, X_n]_f = (P_1, \ldots, P_r, 1-Tf)/(1-Tf)$. Set $J = (P_1, \ldots, P_r, 1-Tf) \subseteq k[X_1, \ldots, X_n, T]$. We claim that $V(J) = \emptyset \subset k^{n+1}$. Suppose that this is not the case. This means that there is an element $(x_1, \ldots, x_n, t) \in V(J)$. We have $P_i(x_1, \ldots, x_n) = 0$, hence $(x_1, \ldots, x_n) \in V(I)$. This implies that $f(x_1, \ldots, x_n) = 0$ as we have chosen $f \in I(V(I))$. But then $(1 - Tf)(x_1, \ldots, x_n) \neq 0$ which means that $(x_1, \ldots, x_n, t) \notin V(J)$, which is a contradiction. Hence we must have $V(J) = \emptyset$.

By the weak nullstellensatz we then have $J = k[X_1, \ldots, X_n, T]$, which means $Ik[X_1, \ldots, X_n]_f = k[X_1, \ldots, X_n]_f$ and by the previous discussion that $f^m \in I$ which by definition means $f \in \sqrt{I}$. $\square$

One immediate application is that we now have a bijection between the set of affine algebraic sets in k^n and the radical ideals in $k[X_1, \ldots, X_n]$, given by $I(-)$ and $V(-)$.

4 Lecture 4 - 25.01.21

4.1 Applications of Hilbert's nullstellensatz

Last time we proved Hilbert's nullstellensatz, that tells us the precise duality between the algebra and the geometry. It states that $I(V(I)) = \sqrt{I}$. Since we already have $V(I(V)) = V$ for affine algebraic sets V we now have the following correspondence when we restrict ourselves to only radical ideals, i.e. ideals such that $\sqrt{I} = I$.

Proposition 4.1

The assignments $V(-)$ and $I(-)$ gives us a bijection between the affine algebraic sets in k^n and the radical ideals in $k[X_1, \ldots, X_n]$.

Moreover we get that

1. V is irreducible if and only if $I(V)$ is prime.
2. V is a point if and only if $I(V)$ is maximal.

The first point we have proven earlier, but lets prove the second one.

Proof. Assume $V = \{x\}$. We know that $I(V)$ is an ideal, and that it is contained in some maximal ideal $I(V')$. By the order reversing property of $V(-)$ we get $V(I(V')) \subset V(I(V))$. But, we knot that $V(I(V)) = V$, hence $V' \subset V = \{x\}$, so either V' is equal to V or V' is empty. But the latter cant be true by the weak nullstellensatz (section 3.2). Hence $V' = V$ which means that $I(V)$ is maximal.

For the converse we assume that $I(V)$ is maximal. We know again by the weak nullstellensatz that V is non-empty, so we can take $x \in V$. By the order reversing property of $I(-)$ we have that $I(V) \subset I(\{x\})$.. But $I(V)$ is assumed maximal, hence either $I(\{x\}) = k[X_1, \ldots, X_n]$ or $I(\{x\}) = I(V)$. But the former cant be true as the variety generating the whole polynomial ring is the empty set, which we know $\{x\}$ isn't. Hence $I(\{x\}) = I(V)$ which means $V = \{x\}$ is a point. $\square$

Proposition 4.2

Let $V \subset k^n$ be a affine algebraic set. Then V is finite if and only if $\Gamma(V)$ is a finite dimensional k-vector space.

Proof. Assume first that $V = \{u_1, \ldots, u_r\}$ is a finite set and consider the ring homomorphism

$$\begin{aligned} \phi\colon k[X_1, \ldots, X_n] &\longrightarrow k^r \\ F &\longmapsto (F(u_1), \ldots, F(u_r)) \end{aligned}$$

Notice that the kernel of ϕ are the maps F such that $(F(u_1), \ldots, F(u_r)) = (0, \ldots, 0)$, in other words they are the maps that vanish on all points in V. Hence $\operatorname{Ker} \phi = I(V)$.

This means that we have $\Gamma(V) = k[X_1, \dots, X_n]/\operatorname{Ker}\phi$, which by the first isomorphism theorem is isomorphic to $\Im\phi \subset k^r$, i.e. $\Gamma(V) \cong k^s$, where $s \leq r$. This shows that $\Gamma(V)$ is a finite dimensional vector space.

For the other direction we assume that $\Gamma(V)$ is a finite dimensional vector space. There exists $s \geq 1$ and $a_i \in k$ such that

$$P_j = a_s\bar{X_j}^s + a_{s-1}\bar{X_j}^{s-1} + \dots + a_1\bar{X_j} + a_0 1 = 0$$

where $\bar{X_j}^i$ are linearly dependent over k. If $u = (x_1, \dots, x_n) \in V$, then $P_j(u) = 0$ for each j. These are polynomials in just one variable, hence only have a finite set of roots, meaning V must be finite as it vanishes on them all. $\square$

Problem 4.3 Find a k-basis for $\Gamma(V) = k[X,Y]/I(V)$ where $V = V(Y^2, Y - X^2 + 1)$.

Solution:

Enter solution

Let now $W \subset V$ both be affine algebraic sets. Then we have $I(V) \subset I(W)$. In $\Gamma(V)$ we have an ideal $I(W)/I(V)$ as $I(W) \subset k[X_1, \dots, X_n]$. We denote this ideal by $I_V(W)$. Notice that the inclusion $I(V) \to I(W)$ induces a surjection $\Gamma(V) \to \Gamma(W)$ which has kernel $I_V(W)$. Hence we have by the first isomorphism theorem $\Gamma(V)/I_V(W) \cong \Gamma(W)$.

This fact generalizes the correspondence we had earlier as a consequence of the nullstellensatz.

Proposition 4.4

The assignments $V(-)$ and $I(-)$ gives us a bijection between the affine algebraic sets in V and the radical ideals in $\Gamma(V)$.

Similarly to last time we get that

1. W is irreducible if and only if $I_V(W)$ is prime in $\Gamma(V)$
2. W is a point in V if and only if $I_V(W)$ is maximal in $\Gamma(V)$
3. W is an irreducible component of V if and only if $I_V(W)$ is a minimal prime ideal of $\Gamma(V)$.

Corollary 4.5

The points in V are in a one-to-one correspondence with the maximal ideals in $\Gamma(V)$.

Problem 4.6 Find all the maximal ideals in $k[X,Y]/(XY)$.

Write down solution

4.2 First steps towards Bezout's theorem

We want to show that even stating Bezout's theorem makes sense. By this we mean that the intersection points of two plane curves actually can be counted. Before we state this as a theorem precisely we prove a lemma we will need in the proof of the theorem.

Lemma 4.7

Let $F, G \in K[X,Y]$ be two non-zero polynomials with no common factors. Then there exists a non-zero polynomial $d \in k[X]$ such that $d = AF + BG$ for some $A, B \in k[X,Y]$. In particular $d \in (F,G)$.

Proof. Let $k(X)$ denote the field of fractions of $k[X]$. As this is a field we have that adjoining a variable Y, i.e. $k(X)[Y]$ is a PID. This means that the ideal (F,G) generated by F and G is actually generated by a single element, which we denote by d. Now, d must divide both F and G, hence $F = d \cdot \frac{f}{p(x)}$ and $G = d \cdot \frac{g}{q(x)}$. By multiplying with the denominators we get $p(x)F = df$ and $q(x)G = dg$ in $k[X,Y]$. Since our polynomial ring is over a field we are in a UFD, hence we have unique factorization. This means that d divides $p(x)$ or d divides $q(x)$, meaning $d \in k[X]$. $\square$

Theorem 4.8

Let $F, G \in K[X,Y]$ be two non-zero polynomials with no common factors. Then $V(F) \cap V(G)$ is finite.

We will prove the theorem next time.

5 Lecture 5 - 26.01.21

We will first prove the theorem we left off at last time. We recall its statement.

Theorem 5.1

Let $F, G \in K[X, Y]$ be two non-zero polynomials with no common factors. Then $V(F) \cap V(G)$ is finite.

Proof. Let $(x, y) \in V(F) \cap V(G)$. This means in particular that $F(x, y) = 0 = G(x, y)$. By the lemma we proved last time there exists some $d \in k[X]$ such that $d = AF + BG$. Since both F and G vanish on (x, y) we have $d(x) = 0$. Since d is a polynomial in one variable over a field it has only a finite set of roots. Hence we only have finite choices for x.

Symmetrically we can find a polynomial $d' \in k[Y]$ with the same above properties. By the same argument er only have finite choices for y. Hence $V(F) \cap V(G)$ is a finite set. $\square$

Theorem 5.2

Let $F, G \in K[X, Y]$ be two non-zero polynomials with no common factors. Then $k[X, Y]/(F, G)$ is a finite dimensional vector space.

Problem 5.3 Compare this to section 4.1

Proof. By the same lemma used before we know there exists $d, d' \in (F, G)$ such that $d \in k[X]$ and $d' \in k[Y]$. Write

- $d(X) = X^n + a_{n-1}X^{n-1} + \ldots + a_0$
- $d'(Y) = Y^m + b_{m-1}Y^{m-1} + \ldots + b_0$

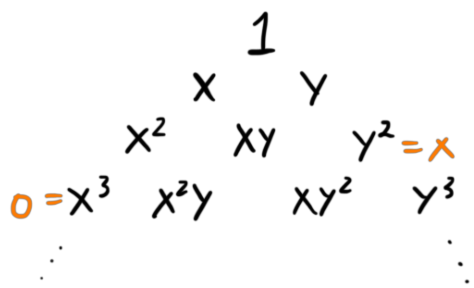
There are both zero in $k[X, Y]/(F, G)$ as they belong to (F, G). Hence, in $k[X, Y]/(F, G)$ we have $X^n \in (X^{n-1}, \ldots, X, 1)$ and $Y^m \in (Y^{m-1}, \ldots, Y, 1)$, which means

$$\{X^i Y^j\}_{0 \leq i \leq n-1, 0 \leq j \leq m-1}$$

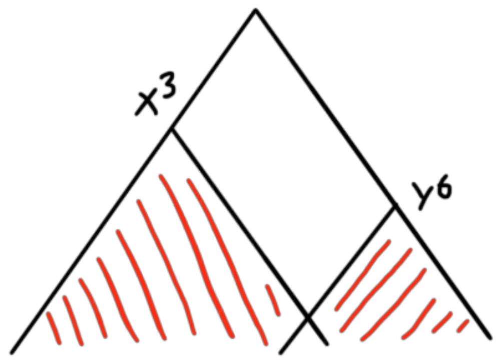
is a generating set for $k[X, Y]/(F, G)$. This set is finite, which means we are done we are done. $\square$

Example 5.4 Find a generating set for $k[X, Y]/(X^3, Y^2 - X)$.

We can think of the generators as the following system.



The relations makes some products of generators die. If we continue this we get something like:



And we can then count the ones that have not perished (the ones inside the non-red area).

5.1 Introduction to morphisms

For this part we let k be an infinite field.

Definition 5.5: Regular map

Let $V \subset k^n$, $W \subset k^m$ be affine algebraic sets. A morphism, called a regular map, $\phi : V \longrightarrow W$ is a collection of maps $\phi_i : k^n \longrightarrow k^m$ such that each ϕ_i is polynomial, i.e. $\phi_i \in \Gamma(V)$.

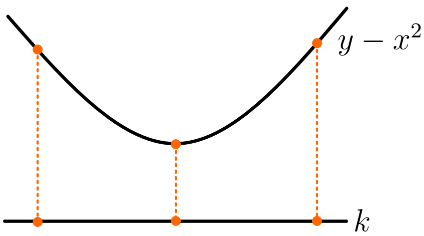
We denote the set of regular maps between V and W by $Reg(V, W)$. These maps make the collection of affine algebraic sets over k into a category, denoted $Aff(k)$.

Remark: The regular maps are continuous in the Zariski topology, but they are not all of the continuous maps.

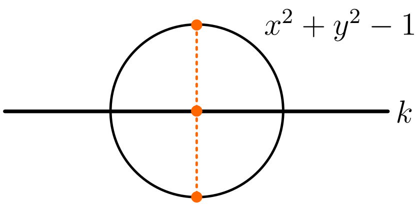
Example 5.6 Elements of $\Gamma(V)$ are the regular maps $V \longrightarrow k$.

Example 5.7 Any projection $V \subset k^n \longrightarrow k^p$, where $p \leq n$ is a regular map.

Example 5.8 $\phi : V(Y - X^2) \longrightarrow k$ defined by $(x, y) \mapsto x$ is actually a regular isomorphism. The inverse is given by $\phi^{-1}(x) = (x, x^2)$. Visually it looks like



Example 5.9 $\phi : V(X^2 + Y^2 - 1)$ sending $(x, y) \mapsto x$ is regular but not injective. It looks like



Let $\phi : V \longrightarrow W$ be regular map. For any $f \in \Gamma(W)$ we set $\phi^*(f) = f \circ \phi \in \Gamma(V)$, i.e. the pre-composition with ϕ. This makes Γ into a contravariant functor. It sends V to $\Gamma(V)$ and $\phi : V \longrightarrow W$ to $\phi^* : \Gamma(W) \longrightarrow \Gamma(V)$.

Proposition 5.10

The functor Γ is a fully faithful functor.

We can in fact show something even stronger.

Theorem 5.11

Let k be algebraically closed, and let $ftr - Alg_k$ denote the category of finite type reduced k-algebras. Then we have an equivalence of categories

$$\Gamma : Aff(k) \longrightarrow ftr - Alg_k$$

Before we prove this lets justify that Γ only hits these types of algebras. We know that $\Gamma(V) \cong k[X_1, \ldots, X_n]/I(V)$ is a k-algebra, and it is generated by $x_1, \ldots, x_n$ and is thus of finite type. Such a k-algebra is reduced if and only if $I(V)$ is a radical ideal. This we know is true by Hilbert's nullstellensatz, hence we know that algebras in the image of Γ are in $ftr - Alg_k$.

Proof. Lets first show that Γ is fully faithful. This means that it is an isomorphism on the sets of morphisms. We start with showing it is injective.

Let $\phi, \psi \colon V \longrightarrow W$ be regular maps such that $\phi^* = \psi^*$. Through the isomorphisms $\Gamma(W) \cong k[Y_1, \dots, Y_m]/I(W)$ and $\Gamma(V) \cong k[X_1, \dots, X_n]/I(V)$ we have that ϕ^* sends Y_i to ϕ_i. Since ψ^* also does this they must have the same images, i.e. $\phi_i = \psi_i$. Since all the components are the same the maps are the same.

We now show Γ is surjective on the morphisms.

Let $\theta : k[Y_1, \dots, Y_m]/I(W) \longrightarrow k[X_1, \dots, X_n]/I(V)$ be an algebra morphism. Set $\phi_i = \theta(Y_i)$ and $\phi = (\phi_1, \dots, \phi_m)$. We claim that $\phi : V \longrightarrow W$, i.e. $\operatorname{Im} \phi \subseteq W$ and hence that Γ is surjective on morphisms.

For Γ to be an equivalence of categories we also need it to be a dense functor, also called essentially surjective.

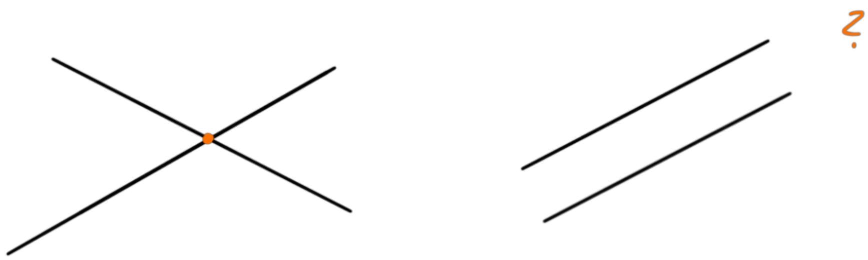
Let A be a finite type reduced k-algebra. Since A is of finite type we have some generators $X_1, \dots, X_n$ and some relations, that generate an ideal I, such that $A = k[X_1, \dots, X_n]/I$. Since A is reduced wee know that I is a radical ideal. By Hilbert's nullstellensatz we know that $I = I(V(I))$, and hence we have $A \cong \Gamma(V(I))$.

This shows that Γ is an equivalence of categories. $\square$

Show surjectivity

5.2 Projective algebraic sets

Affine algebraic sets are nice and easy, but they have some problem. Most notably for us now is the case of intersection points of curves. The general structure is that two lines always meet at a point, but this has an exception, namely parallel curves.



The reason we move to projective algebraic sets instead is that these exceptions disappear.

Definition 5.12

Let $n \geq 0$ be an integer and let E be a k-vector space of dimension $n+1$. For $x, y \in E \setminus \{0\}$ we define the relation $x \sim y$ if there exists a $\lambda \in k^\times$ such that $y = \lambda x$.

Problem 5.13 Show that this relation is an equivalence relation.

Note that the equivalence classes of this relations are the lines in E through the origin.

Definition 5.14: Projective space

The projective space of E, denoted $\mathbb{P}(E)$, is the set $(E \setminus \{0\})/\sim$, i.e. the set of lines in E.

If $E = k^n$ then $\mathbb{P}(E) = \mathbb{P}^n(k)$ is called the standard projective n-space .

Let $\pi : k^{n+1} \setminus \{0\} \longrightarrow \mathbb{P}^n(k)$ be the canonical projection, and let $x = (x_0, \ldots, x_n) \in k^n \setminus \{0\}$. Then $\bar{x} = \pi(x)$ is a point in $\mathbb{P}^n(k)$ with homogeneous coordinates $(x_0, \ldots, x_n)$ often written $[x_0 : \cdots : x_n]$.

Note that if $\lambda \neq 0$, then $[\lambda x_0 : \cdots : \lambda x_n]$ is another system of homogeneous coordinates for $\bar{x}$.

Example 5.15 Let $k = \mathbb{R}$. Then $\mathbb{P}^1(\mathbb{R})$ is the set of lines trough the origin in $\mathbb{R}^2$.

Problem 5.16 Read and learn about projective space.

6 Lecture 6 - 01.02.21

The lecture was focused on going through the exercises.

7 Lecture 7 - 02.02.21

7.1 Projective algebraic sets

We recall the definition of the projective space of a vector space E from last time.

Definition 7.1

Let E be a $n+1$-dimensional vector space over a field k. The projective space of E, denoted $\mathbb{P}(E)$, is the set $(E \setminus \{0\})/\sim$, where $x \sim y \iff \exists\lambda$ such that $x = \lambda y$.

If $E = k^{n+1}$ then we denote $\mathbb{P}(E)$ by $\mathbb{P}^n(k)$, which we call projective n-space. We have a map

$$\begin{aligned} k^{n+1} &\longrightarrow \mathbb{P}^n(k) \\ (x_0, \ldots, x_n) &\longmapsto [x_0 : \cdots : x_n] \end{aligned}$$

We call $(x_0, \ldots, x_n)$ the homogeneous coordinates.

Definition 7.2

Let E be a $n+1$-dimensional vector space over k, $0 \leq m \leq n$ be an integer and F a $m+1$-dimensional subspace of E. The image of $F \setminus \{0\}$ in $\mathbb{P}(E)$ is called the projective subspace of dimension m, denoted $\overline{F}$.

If $m = 0$ then $\overline{F}$ is a point. If $m = 1$ then $\overline{F}$ is a line. If $m = n-1$ then $\overline{F}$ is a hyperplane.

Proposition 7.3

Let E be a $n+1$-dimensional vector space over k.Let further V, W be two projective subspaces of $\mathbb{P}(E)$ with dimension r and s respectively such that $r + s - n \geq 0$. Then $V \cap W$ is a projective subspace of dimension greater than $r + s - n$. In particular, $V \cap W \neq \emptyset$.

Remark: This proposition is not true in the affine case. This is due to the existance of paralell lines, which are both affine subspaces, but has empty intersection.

Proof. We write $V = \overline{F_V}$ and $W = \overline{F_W}$ for F_V, F_W two subspaces of E with dimension $r+1$ and $s+1$ respectively. Note that the intersection of two vector subspaces is again a vector subspace, hence $\overline{F_V \cap F_W}$ is actually a projective subspace. Moreover we have:

$$\begin{aligned} dim(F_V \cap F_W) &= dim(F_V) + dim(F_W) - dim(F_V + F_W) \\ &\geq (r+1) + (s+1) - (n+1) \\ &= r + s - n + 1 \end{aligned}$$

This means that $dim(V \cap W) = dim(\overline{F_V} \cap \overline{F_W}) = dim(\overline{F_V \cap F_W}) \geq r + s - n$. □

Example 7.4 Two distinct lines in $P^2(k)$ meet at a unique point.

7.2 Homography

We will not cover this in detail, but we quickly mention what these are.

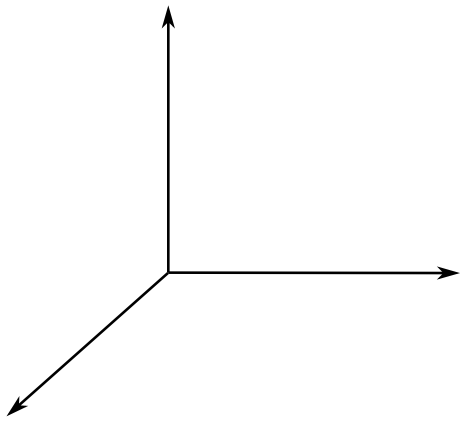
Definition 7.5: Homography

Let u be a vector space isomorphism on a vector space E, i.e. $u \in GL(E)$. The induced map $\overline{u} : \mathbb{P}(E) \longrightarrow \mathbb{P}(E)$ is called a homography. These are isomorphisms of projective spaces, and are as you can see the ones induced by vector space isomorphisms.

7.3 What does projective space look like?

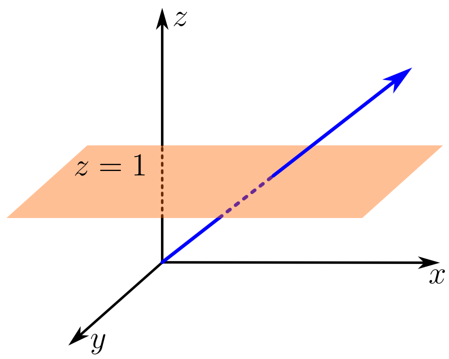
For easy visualization we let $k = \mathbb{R}$. Since we are on our goal to understand and prove Bezouts theorem, we are mostly interested in $\mathbb{P}^2(k)$, so lets try to visualize and understand $\mathbb{P}^2(\mathbb{R})$. I did this a bit in 5.2 but it never hurts to do it again.

We start by drawing $\mathbb{R}^3$, as this is where we get our points from.



As we have $x = y$ in $\mathbb{P}(E)$ whenever there exists a λ such that $\lambda x = y$ we get in our coordinates that $[x : y : z] = [\frac{x}{z} : \frac{y}{z} : 1]$ which means that we can just look a the plane $z = 1$ in $\mathbb{R}^3$.

Every point on that plane uniquely determines a line through the origin, as visualized below.



This means that much of $\mathbb{P}^2(\mathbb{R})$ acts like $\mathbb{R}^2$, and we get the intuitive equality $\mathbb{P}^2(\mathbb{R})$" $=''$ $\mathbb{R}^2 \cup \{\text{points at infinity}\}$. Lets make this a bit more rigorous.

We define $H = V(z) \subseteq \mathbb{R}^3$, and $\overline{H} \subset \mathbb{P}^2(\mathbb{R})$ to be its projective space. We then set $U = \mathbb{P}^2(\mathbb{R}) \setminus \overline{H}$. There is a bijection

$$\begin{aligned} \phi : U &\longrightarrow \mathbb{R}^2 \\ [x : y : z] &\longmapsto (\frac{x}{z}, \frac{y}{z}) \\ [x : y : 1] &\longleftarrow\!\shortmid (x, y) \end{aligned}$$

Furthermore, the inclusion $\overline{H} \hookrightarrow \mathbb{P}^2(\mathbb{R}), [x : y : 0] \mapsto [x : y : 0]$ splits by the map

$$\begin{aligned} \mathbb{P}^2(\mathbb{R}) &\longrightarrow \overline{H} \\ [x : y : z] &\longmapsto [x : y : z] \end{aligned}$$

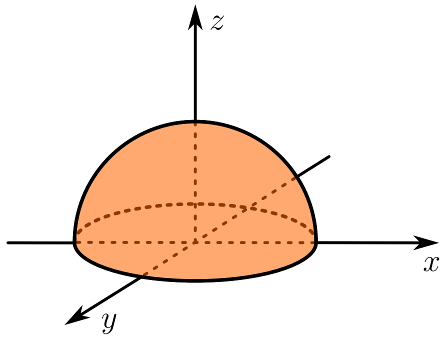
meaning that we have $\mathbb{P}^2(\mathbb{R}) = \overline{H} \coprod U = \mathbb{P}^1(\mathbb{R}) \coprod \mathbb{R}^2$. These we think about as "points at infinity" and "points at a finite distance" respectively.

Problem 7.6 What does $\mathbb{P}^1(\mathbb{R})$ look like?

Solution:

By the same argument as earlier we can look at the $y = 1$ line in $\mathbb{R}^2$ to determine how $\mathbb{P}^1(\mathbb{R})$ looks. Every point on the line determines uniquely a line with a unique slope m. All possible slopes $m \in \mathbb{R}$ are hit by one such line, except $m = 0$. This line never intersects the $y = 1$ line, and is hence designated as the "point at infinity". As we have a line isomorphic to $\mathbb{R}$, with the same "point at infinity" in both ends, this shape is homeomorphic to the circle S^1. We think of this as folding the real line into a circle and gluing it together at this "point at infinity".

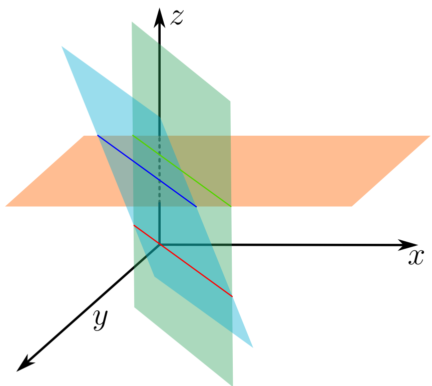
Another way to visualize projective space is to use spheres. Since the space $\mathbb{R}^3 \setminus \{0\}$ is homeomorphic to S^2 we can study that instead. A line through the origin intersects the sphere at two points. These points are antipodal. We can then reduce to thinking about the half-sphere instead.



We still have identified the antipodal points at the equator, but all other points correspond to a unique line, i.e. a point in projective space. The points on the equator are then the "points at infinity" while the non-equatorial points are the "points at finite distance". As with $\mathbb{P}^1(\mathbb{R})$ we can think of this as gluing each line in the plane $z = 1$ to its endpoints at infinity.

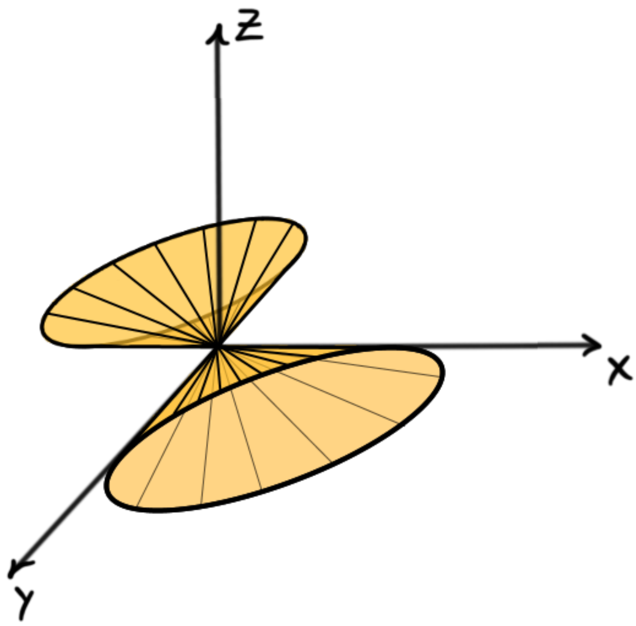
Example 7.7 We can also study projective lines in $\mathbb{P}^2(\mathbb{R})$. Lets see what happens with "parallel lines", as these are the ones we are trying to get rid of using projective space instead of affine.

A 1-dim projective line is the image of a 2-dim subspace of $\mathbb{R}^3$, i.e. a plane. This means that the points at finite distance of the projective line is the intersection of that plane and the plane $z = 1$. We draw this for two projective lines.

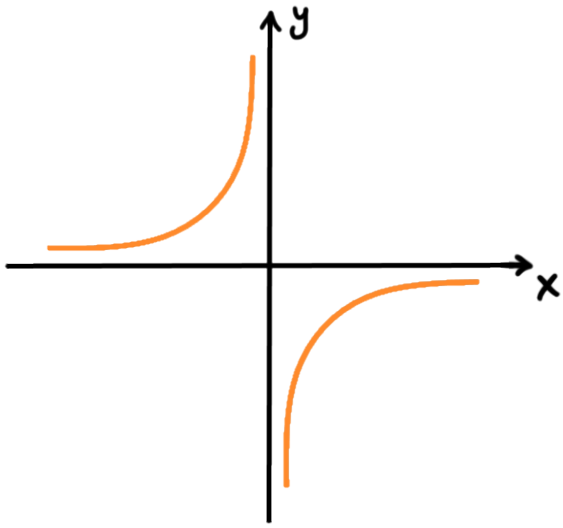


We see that the two planes cut out two "parallel lines" on the plane. But, the planes also intersect elsewhere, namely at the red line in the xy-plane. That line determines a point at infinity in $\mathbb{P}^2(\mathbb{R})$, hence the two projective lines do indeed meet at a point.

Another example is the affine algebraic set given by $C = V(XY - Z^2) \subseteq \mathbb{R}^3$.



We denote $\overline{C}$ its image in $\mathbb{P}^2(\mathbb{R})$. It intersects the plane $z = 1$ at the graph of $xy = 1$.



$\overline{C}$ also intersects $V(Z)$ at two points in the $z = 0$ plane, namely $[1 : 0 : 0]$ and $[0 : 1 : 0]$, i.e. the two axes.

We see that the two "points at infinity" correspond to the asymptotes of the graph that intersects $z = 1$, i.e. the asymptotes of $xy = 1$.

Problem 7.8 Where does $\overline{C}$ intersect the projective line $x - z = 0$?

Solution:

Solution

We see by the previous problem that the conics (circles, ellipses, hyperbola, parabola) are all "affine constructs". In projective space, these properties could be characterized by how the conic cuts the line at infinity. It can cut it into 0, 1 or 2 parts, which correspond to the three types of conics.

8 Lecture 8 - 08.02.21

8.1 Projective algebraic sets

A problem we run into while trying to define projective algebraic sets if we were to do it similarly to affine algebraic sets is that polynomials are not always functions on $\mathbb{P}^n(k)$.

Definition 8.1: Projective zero of a polynomial

Let $\overline{x} \in \mathbb{P}^n(k)$ and $F \in k[X_0, \ldots, X_n]$. We say $\overline{x}$ is a zero of F, sometimes called a projective zero, if $F(\lambda x) = 0$ for all $\lambda \in k^\times$. We write $F(\overline{x}) = 0$ even though F is not necessarily a function.

Proposition 8.2

If $F \in k[X_0, \ldots, X_n]$ is a homogeneous polynomial and $F(x) = 0$ for some x, then $F(\overline{x}) = 0$, i.e. $\overline{x}$ is a zero of F.

Proposition 8.3

Decompose the polynomial F into its homogeneous components, i.e. $F = F_0 + F_1 + \ldots + F_r$, where $r = deg(F)$ and F_i is the degree i homogeneous component. Then $F(\overline{x}) = 0 \iff F_i(\overline{x}) = 0$ for all i.

Problem 8.4 Proove this.

Definition 8.5: Projective algebraic set

Let $S \subseteq k[X_0, \ldots, X_n]$ be a subset. We call

$$V_{\text{proj}}(S) = \{\overline{x} \in \mathbb{P}^n(k) | F(\overline{x}) = 0, \forall F \in S\}$$

the projective algebraic set of S.

We note that $V_{\text{proj}}(S) = V_{\text{proj}}((S))$, where (S) if the ideal generated by S. This is exactly the same as for the affine case. As $k[X_0, \ldots, X_n]$ is noetherian, we can by Hilbert's basis theorem assume that S is a finite set.

Example 8.6 $V_{\text{proj}}((0)) = \mathbb{P}^n(k)$

Before we see the next example we need one more definition.

Definition 8.7: The irrelevant ideal

The ideal $R^+ = (X_0, \ldots, X_n) \subset k[X_0, \ldots, X_n]$, i.e. the ideal generated by the indeterminates, is called the irrelevant ideal.

Example 8.8 $V_{\text{proj}}(R^+) = \emptyset$. This is because $0 \notin \mathbb{P}^n(k)$.

Example 8.9 Let $\overline{x} = [x_0 : \cdots : x_n] \in \mathbb{P}^n(k)$. As $0 \notin \mathbb{P}^n(k)$ we can without loss of generality assume that $x_0 \neq 0$. This means that $\overline{x} = [1 : x_1/x_0 : \cdots : x_n/x_0]$. We can relabel to get $\overline{x} = [1 : x_1 : \cdots : x_n] \in \mathbb{P}^n(k)$.

Then $\{\overline{x}\} = V_{\text{proj}}(X_1 - x_1X_0, \ldots, X_n - x_nX_0)$.

In the affine case we had $\{x\} = V(X_0 - x_0, \ldots, X_n - x_n)$, but when we want projective algebraic sets we need homogeneuity. This we can get y multiplying by X_0, which acts as 1.

As in the affine case we have some standard properties these projective algebraic sets satisfy:

1. If $S \subseteq S'$ then $V_{\text{proj}}(S') \subseteq V_{\text{proj}}(S)$, i.e. $V_{\text{proj}}(-)$ has the order reversing property.
2. $\cap_{i \in I} V_{\text{proj}}(S_i) = V_{\text{proj}}(\cup_{i \in I} S_i)$ for an arbitrary index set I.
3. $\cup_{i=1}^n V_{\text{proj}}(S_i) = V_{\text{proj}}(\prod_{i=1}^n) S_i$.

Together with the previous remark that $\mathbb{P}^n(k)$ and $\emptyset$ lie in the image of $V_{\text{proj}}(-)$ we still have the same Zariski topology on $\mathbb{P}^n(k)$, where the projective algebraic sets are the closed sets.

8.2 Ideal of a projective algebraic set

Definition 8.10: Ideal of a projective set

Let $V \subset \mathbb{P}^n(k)$ be a subset. We call

$$I_{\text{proj}}(V) = \{F \in k[X_0, \ldots, X_n] | F(\overline{x}) = 0, \forall \overline{x} \in V\}$$

the ideal of V.

Definition 8.11: Homogeneous ideal

We call an ideal a homogeneous ideal if it is generated by homogeneous elements.

1. If $V \subset V'$ then $I_{\text{proj}}(V') \subseteq I_{\text{proj}}(V)$.

2. $I_{\text{proj}}(V)$ is a homogeneous and radical ideal.
3. If V is a projective algebraic set, then $V_{\text{proj}}(I_{\text{proj}}(V)) = V$.
4. If I is an ideal, then $I \subseteq I_{\text{proj}}(V_{\text{proj}}(I))$.
5. $I_{\text{proj}}(\mathbb{P}^n(k)) = (0)$.
6. $I_{\text{proj}}(\emptyset) = k[X_0, \ldots, X_n]$.
7. Irreducibly makes sense.

8.3 The cone construction

The cone construction is a technique for reducing the case of projective algebraic sets to affine algebraic sets. If we let $V \subseteq \mathbb{P}^n(k)$ be a projective algebraic set, and π the canonical projection

$$\pi\colon k^{n+1} \setminus \{0\} \longrightarrow \mathbb{P}^n(k)$$

then the cone of V is defined to be the set $\text{Cone}(V) = \pi^{-1}(V) \cup \{0\}$.

The cone of a projective algebraic set has the following two properties.

1. If $I \subsetneq k[X_0, \ldots, X_n]$ is a proper homogeneous ideal, then $\text{Cone}(V_{\text{proj}}(I)) = V(I) \subseteq k^{n+1}$.

   Proof. □

   proof

2. If $I = k[X_0, \ldots, X_n]$, then $\text{Cone}(V_{\text{proj}}(I)) = \text{Cone}(\emptyset) = \{0\}$.

   Proof. We know that $V_{\text{proj}}(k[X_0, \ldots, X_n]) = \emptyset$, which means that

   $$\text{Cone}(V_{\text{proj}}(k[X_0, \ldots, X_n])) = \text{Cone}(\emptyset) = \pi^{\emptyset} \cup \{0\}.$$

   The inverse image of the empty set under the canonical projection is again the empty set, thus we have $\text{Cone}(V_{\text{proj}}(k[X_0, \ldots, X_n])) = \{0\}$. □

Theorem 8.12: Projective nullstellensatz

Let k be an algebraically closed field and $I \subseteq k[X_0, \ldots, X_n]$ a homogeneous ideal. Then $V_{\text{proj}}(I) = \emptyset$ if and only if $R^+ \subseteq \sqrt{I}$. If $V_{\text{proj}}(I) \neq \emptyset$, then $I_{\text{proj}}(V_{\text{proj}}(I)) = \sqrt{I}$.

Proof. Notice that if $I = k[X_0, \ldots, X_n]$, then $R^+ \subseteq \sqrt{k[X_0, \ldots, X_n]} = k[X_0, \ldots, X_n]$, i.e. the first statement holds. Hence we can assume that I is a proper ideal. Also notice that $V_{\text{proj}}(I) = \emptyset$ if and only if $\text{Cone}(V_{\text{proj}}(I)) = \{0\} = V(I)$.

Let's prove the first statement. Assume that $V_{\text{proj}}(I) = \emptyset$. Then

$$R^+ = (X_0, \ldots, X_n) \subseteq I(0) = I(V(I)),$$

where the last equality is by the above remark. By the nullstellensatz for affine algebraic sets we know that $I(V(I)) = \sqrt{I}$, hence $R^+ \subseteq \sqrt{I}$.

Assume now that $R^+ \subseteq \sqrt{I}$. By the affine nullstellensatz we have $\sqrt{I} = I(V(I))$, hence we have $R^+ \subseteq I(V(I))$. Then by the order reversing property of $V(-)$ we have

$$\emptyset \neq V(I) = V(I(V(I))) \subseteq V(R^+) = \{0\}.$$

This means that $V(I) = \{0\}$ which means that $V_{\text{proj}}(I) = \emptyset$ by the above remark.

We now prove the second part, i.e. that if the projective algebraic set of an ideal is non-empty, then $I_{\text{proj}}(V_{\text{proj}}(I)) = \sqrt{I}$. Assume $V_{\text{proj}}(I) \neq \emptyset$. This gives us that $I_{\text{proj}}(V_{\text{proj}}(I)) \subseteq k[X_0, \ldots, X_n]$ is a proper ideal. We claim that $I_{\text{proj}}(V_{\text{proj}}(I)) = I(\text{Cone}(V_{\text{proj}}(I)))$. We show both containments.

Let $F \in I_{\text{proj}}(V_{\text{proj}}(I))$. This means that $F(\overline{x}) = 0$ for all $\overline{x} \in V_{\text{proj}}(I)$, i.e. that $F(\lambda x) = 0$ for all $\lambda \in k^\times$. By letting $\lambda = 1$ we see that $F(x) = 0$, which means that $F \in I(\pi^{-1}(V_{\text{proj}}(I)))$. As $I_{\text{proj}}(V_{\text{proj}}(I))$ is a proper homogeneous ideal we also see that $F(0) = 0$, as F can't be a constant, i.e. a homogeneous polynomial of degree 0. This means that $F \in I(\text{Cone}(V_{\text{proj}}(I)))$. Hence we have $I_{\text{proj}}(V_{\text{proj}}(I)) \subseteq I(\text{Cone}(V_{\text{proj}}(I)))$.

Let $F \in I(\text{Cone}(V_{\text{proj}}(I))$. This means that $F(x) = 0$ for all $x \in \text{Cone}(V_{\text{proj}}(I)) = V(I)$, as I is a proper homogeneous ideal. We can decompose F into its homogeneous components, i.e. $F = F_0 + \cdots + F_r$, where $r = deg(F)$. As $F(0) = 0$ we know that $F_0 = 0$. Thus $F(\lambda x) = \lambda F_1(x) + \cdots \lambda^r F_r(x)$. All these $F_i(x)$ must vanish since $F(x) = 0$, hence $F_i(x) = 0$. By 8.3 this means that $F(\overline{x}) = 0$ and thus $F \in I_{\text{proj}}(V_{\text{proj}}(I))$. Hence $I(\text{Cone}(V_{\text{proj}}(I))) \subseteq I_{\text{proj}}(V_{\text{proj}}(I))$.

We then finally have

$$I_{\text{proj}}(V_{\text{proj}}(I)) = I(\text{Cone}(V_{\text{proj}}(I))) = I(V(I)) = \sqrt{I},$$

where the last equality is by the affine nullstellensatz and the middle equality is due to the properties we described above. □

Proposition 8.13

There is a bijection between non-empty projective algebraic sets $V \subseteq \mathbb{P}^n(k)$ and homogeneous radical ideals in $k[X_0, \ldots, X_n]$ not containing R^+.

As in the affine case we have that irreducible projective algebraic sets correspond to the prime ideals. Also as in the affine case we can compare to certain k-algebras. If I is a homogeneous radical ideal not containing R^+ corresponding to an affine algebraic set V, then $k[X_0, \ldots, X_n]/I$ is a k-algebra, which we denote by $\Gamma_{homog}(V)$. These are often called the homogeneous coordinate ring of V. For the affine case we had that points on an affine algebraic set corresponded to maximal ideals in $\Gamma(V)$, but for the projective case this is not the case.

Definition 8.14

Let $V \subseteq \mathbb{P}^n(k)$ be a projective algebraic set and $f \in \Gamma_{homog}(V)$ be homogeneous with positive degree. We define $D^+(f) = \{\overline{x} \in V | f(\overline{x}) \neq 0\}$.

These are open sets in the Zariski topology on V.

Example 8.15 $\mathbb{P}^n(k) = D^+(X_0) \cup D^+(X_1) \cup \cdots D^+(X_n)$.

Each one of these $D^+(X_i)$ are isomorphic to k, which means that projective space is locally affine! These are far from being disjoint unions, but they do form a covering.

Example 8.16 $\mathbb{P}^2(\mathbb{R}) = D^+(X) \cup D^+(Y) \cup D^+(Z)$. Here $D^+(X)$ consists of all points $[x : y : z] \in \mathbb{P}^2(\mathbb{R})$ such that $x \neq 0$, and the other ones are the same for the other coordinates. As these are projective coordinates we can look at $[1 : y/x : z/x]$ instead, which means that $D^+(X)$ is isomorphic to $\mathbb{R}^2$ through $\phi([x : y : z]) = (y/x, z/x)$, and $\phi^{-1}(a, b) = [1 : a : b]$.

Problem 8.17 Lets take the affine algebraic set $V(Y - X^3) \subseteq \mathbb{R}^2$. We can make it projective by instead studying $V_{\text{proj}}(X^3 - YZ^2) \subseteq \mathbb{P}^2(\mathbb{R})$. Call this projective algebraic set V. What does V look like?

Solution:

We try to understand how it looks like on each of the open sets $D^+(X)$, $D^+(Y)$ and $D^+(Z)$. On $D^+(X)$ it looks like $V(YZ^2 - 1)$, on $D^+(Y)$ like $V(Z^2 - X^3)$ and on $D^+(Z)$ like $V(Y - X^3)$. We can visualize them as:

We see that all three of them contain the point $[1 : 1 : 1]$.

Visualization

Problem 8.18 Find some other points and try to understand how these three open sets glue together globally.

9 Lecture 9 - 09.02.21

9.1 Motivation on sheaves

We start by some motivation as to why we would need something more that just affine and projective algebraic sets.

The first clue is that the correspondence between affine algebraic sets and reduced finite type k-algebras does not include a lot of rings that we would like to study using the same techniques. For example:

- $\mathbb{Z}$ and $\mathbb{Z}[i]$ (They are not k-algebras)
- $k(x)$ and $k[x]_{(x)}$, being all rational functions and rational functions defined at 0 respectively (they are not finite type k-algebras)

We would also like to study multiplicity of points. For example $V = V(Y-X^2) \cap V(Y)$. We can study this by looking at its coordinate ring $\Gamma(V)$. We find the ideal first:

$$\begin{aligned} I(V(Y - X^2) \cap V(Y)) &= I(V(Y - X^2, Y)) \\ &= I(V(X^2, Y)) \\ &= \sqrt{(X^2, Y)} \\ &= (X, Y), \end{aligned}$$

where the second to last equality is due to the affine nullstellensatz. We then get $\Gamma(V) \cong k[X, Y]/(X, Y) \cong k$, which is a 1-dimensional k-algebra. This means that we just counted the intersection point once, even though the zero has multiplicity 2. So the coordinate ring forgets multiplicity.

What if we instead didn't take the radical? Then we would get $\Gamma \cong k[X, Y]/(X^2, Y) \cong k[X]/(X^2)$ which is a 2-dimensional k-algebra. This means we have counted multiplicity right! But, this algebra is not reduced, as the ideal is not radical.

What we want is a correspondence between all commutative rings, and something geometric, more general than affine algebraic sets. In this way we can count multiplicity right, we can study rings geometrically and more interesting stuff we will get to later. These geometric objects are called affine schemes.

To make the definition of an affine scheme we need one more tool, namely sheaves. Let's see some motivation on these to get a feeling for what they are and why they are needed.

For affine algebraic sets V we had a correspondence with $\Gamma(V)$ which were the ring of functions on V. In the projective case, i.e. for a projective algebraic set V, this $\Gamma_{homog}(V)$ did not have elements that were functions on V. To solve this we recall that $\mathbb{P}^n(k) = D^+(X_0) \cup \cdots \cup D^+(X_n)$, i.e. projective space is locally affine. So maybe we could define functions on these affine spaces and glue them together to form functions on the projective space?

Sadly this does not work.. Lets see an example of why.

Example 9.1 We look at $\mathbb{P}^1(k)$, which is covered by $D^+(X)$ and $D^+(Y)$. Here $D^+(X)$ are the points $[a : b]$ where a is non-zero, meaning we can look at just $[a : b] = [1 : \frac{b}{a}]$. Now $D^+(X)$ is isomorphic to k through the isomorphism $\phi([a : b]) = \frac{b}{a}$, with inverse $\phi^{-1}(c) = [1 : c]$. Similarily $D^+(Y)$ consists on points $[a : b] \in \mathbb{P}^1(k)$ such that b is non-zero. We again write $[a : b] = [\frac{a}{b} : 1]$ and an isomorphism $\psi : D^+(Y) \longrightarrow k$ defined by $\psi([a : b]) = \frac{a}{b}$ with inverse $\psi^{-1}(c) = [c : 1]$.

A function $f : D^+(X) \longrightarrow k$ is "good" if the induced function $\overline{f} : k \longrightarrow k$, defined such that $f([a : b]) = \overline{f}(\frac{b}{a})$, is a polynomial function. Similarly $g : D^+(Y) \longrightarrow k$ is "good" if the induced function $\overline{g} : k \longrightarrow k$, defined such that $g([a : b]) = \overline{g}(\frac{a}{b})$ is polynomial.

To get a polynomial function on $\mathbb{P}^1(k)$ these two functions f and g would need to agree on the intersection $D^+(X) \cap D^+(Y)$. This means that for $[a : b] \in D^+(X) \cap D^+(Y)$ we need $\overline{f}(\frac{b}{a}) = \overline{g}(\frac{a}{b})$. Since these are polynomial we get

$$a_n(\frac{b}{a})^n + \ldots + a_1(\frac{b}{a}) + a_0 = b_m(\frac{a}{b})^m + \ldots + b_1(\frac{a}{b}) + b_0.$$

By clearing the denominators we get

$$a_nb^{n+m} + a_{n-1}b^{n+m-1}a + \ldots + a_0b^ma^n - b_ma^{n+m} - b_{m-1}a^{n+m-1}b - \ldots - b_0b^ma^n = 0$$

And since k is infinite this means that $a_i = 0 = b_i$ for all $i > 0$. The resulting equation is then $a_0 - b_0 = 0$, which means that f and g were the same constant functions.

Hence all global functions on $\mathbb{P}^1(k)$ are constant.

We want a more encompassing notion of function than just constant functions, so wee need to expand to looking at locally defined functions instead of globally defined ones. This leads us directly to the notion of a sheaf.

9.2 Definition and examples

We first study the sheaf of k-valued functions on a topological space. Let X be a topological space and let U, V be open sets in X. Let further $\mathscr{F}(U)$ and $\mathscr{F}(V)$ denote the set of functions on U and V respectively. Note that if we have $V \subseteq U$, then a function $f \in \mathscr{F}(U)$ defines a function on V by restriction, i.e. $f_{|V} \in \mathscr{F}(V)$. Suppose we have an open set U that decomposes into two open sets, $U = U_0 \cup U_1$. If we have functions $f_0 \in \mathscr{F}(U_0)$ and $f_1 \in \mathscr{F}(U_1)$ such that $f_{0|U_0 \cap U_1} = f_{1|U_0 \cap U_1}$, then there exists an unique function $f \in \mathscr{F}(U)$ with $f_{|U_i} = f_i$. This means that functions on the decomposition glue together to a function on the union.

These are all the things we want a sheaf to be. An assignment some set to all open sets on a topological space, such that we have restriction and nice gluing. We remarked earlier that functions are too restrictive, so we don't want to restrict $\mathscr{F}(U)$

to being functions in our proper definition. Before we define a sheaf properly we need some other definitions.

Definition 9.2: Category of open sets

Let X be a topological space. We define $Open(X)$ to be the category consisting of

- Objects: Open sets in X
- Morphisms:

$$Hom(V, U) = \begin{cases} V \subseteq U, & \text{if } V \subseteq U \\ \emptyset, & \text{if } V \nsubseteq U \end{cases}$$

Definition 9.3: Presheaf

Let X be a topological space and $\mathscr{C}$ be some concrete category [a]. A $\mathscr{C}$-valued presheaf on X is a contravariant functor $\mathscr{F} : Open(X) \longrightarrow \mathscr{C}$.

[a] Most often either Grp, $Ring$, $Mod\Lambda$ or Set, but could in theory be any.

This means in particular that

- For each $U \in Open(X)$ we have an object $\mathscr{F}(U) \in \mathscr{C}$.
- If $V \subseteq U$ then we have a map $\mathscr{F}(V \subseteq U) = res_{U,V} : \mathscr{F}(U) \longrightarrow \mathscr{F}(V)$ which we call restriction.
- If $W \subseteq V \subseteq U$, then $res_{U,W} = res_{V,W} \circ res_{U,V}$.
- $res_{U,U} = Id_{\mathscr{F}(U)}$.

Since our intuition comes from the image of $\mathscr{F}$ geing a set of functions, we denote $res_{U,V}(f) = f_{|V}$ for $f \in \mathscr{F}(U)$.

Definition 9.4: Sheaf

A $\mathscr{C}$-valued presheaf $\mathscr{F}$ on a topological space X is called a sheaf if it satisfies the following property, which we call the gluability or gluing property.

If $U \in Open(X)$ is covered by $\{U_i\}_{i \in I}$, where $U_i \in Open(X)$, then for any choice of $f_i \in \mathscr{F}(U_i)$ such that $f_{i|U_i \cap U_j} = f_{j|U_i \cap U_j}$, there exists a unique $f \in \mathscr{F}(U)$ with $f_{|U_i} = f_i$ for all $i \in I$.

We often denote $\mathscr{F}(U) = \Gamma(U, \mathscr{F})$ and call elements sections of $\mathscr{F}$ over U.

Example 9.5 (The sheaf of continuous real-valued functions).

Let X be a topological space. Define a contravariant functor $\mathscr{O} : Open(X) \longrightarrow$

Ring by $\mathscr{O}(U) = \{f : U \to R | f \text{ continuous}\}$. This is a ring by defining addition and multiplication pointwise.

Claim: This is a sheaf.

Proof. We first show that it is a presheaf, and then show it satisfies the gluability condition.

- For each $U \in Open(X)$ we have that $\mathscr{O}(U)$ is a ring, as defined above.
- If $V \subseteq U$, then $\mathscr{O}(U) \longrightarrow \mathscr{O}(V)$ defined by $f \longmapsto f_{|V}$ is continuous. This is because for any open set $W \subset \mathbb{R}$ we have $f^{-1}(W) \subset U$ is open, meaning that $f_{|V}^{-1}(W) = f^{-1}(W) \cap V$ is also open, and hence $f_{|V}$ continuous.
- If we have $W \subseteq V \subseteq U$, then for any $f \in \mathscr{O}(U)$ we have $(f_{|V})_{|W} = f_{|W}$.
- If $f \in \mathscr{O}(U)$ for some $U \in Open(X)$, then $f_{|U} = f$, meaning that $res_{U,U} = Id_{\mathscr{O}(U)}$.

The above points show that $\mathscr{O}$ is a presheaf. Assume now that $U = \cup_{i \in I} U_i$ is an open set in X covered by a family of open sets $\{U_i\}_{i \in I}$. Suppose that $f_i \in \mathscr{O}(U_i)$ for all $i \in I$ such that $f_{i|U_i \cap U_j} = f_{j|U_i \cap U_j}$. We need to show there exists a unique map $f : U \longrightarrow \mathbb{R}$ such that $f_{|U_i} = f_i$.

We define f by $f(x) = f_i(x)$ if $x \in U_i$. This is well defined because of the fact that the maps f_i agree on the overlaps of sets in the open cover. For some open set $W \subseteq \mathbb{R}$ we have that $f^{-1}(W) = \cup_{i \in I} f_i^{-1}(W)$. This set is open because $f_i^{-1}(W)$ open since f_i continuous, and arbitrary union of open sets is again open. Hence f is continuous.

Assume that we have two maps $f, g \in \mathscr{O}(U)$ such that $f_{|U_i} = f_i = g_{|U_i}$. Since $\{U_i\}$ is a cover we know that all $x \in U$ lie at least in one of the U_i's. We assume $x \in U_i$. Then

$$f(x) = f_{|U_i}(x) = f_i(x) = g_{|U_i}(x) = g(x)$$

holds for all points $x \in U$, hence $f = g$. This shows that the map f we found above is unique, and hence that $\mathscr{O}$ is a sheaf. □

We also want to see a non-example, to get a feeling on where things might go wrong.

Example 9.6 (Non-example).

Let $X = \mathbb{R}$ be our topological space and define $\mathscr{P} : Open(X) \longrightarrow Ring$ by $\mathscr{P}(U) = \{f : U \to \mathbb{R} | f \text{ is bounded}\}$. We claim that $\mathscr{P}$ is a presheaf, but not a sheaf.

Proof. The restriction of a bounded function is again bounded, hence we in fact have a presheaf. It is not a sheaf as we can define $f_i = Id_{U_i}$, where $U_i = (i-1, i+1)$. We have $\cup U_i = \mathbb{R}$, but the f_i's glue to $f = Id_{\mathbb{R}}$, which is not a bounded function. □

10 Lecture 10 - 15.02.21

We continue studying sheaves. Let's recall the definition. A $\mathscr{C}$-valued presheaf on a topological space X is a contravariant functor $\mathscr{F} : Open(X) \longrightarrow \mathscr{C}$. A sheaf $\mathscr{F}$ is a presheaf that satisfies the glueability axiom.

- If $U \subseteq X$ is an open set, covered by other open sets $\{U_i\}_{i\in I}$, then for any choice of sections $f_i \in \mathscr{F}(U_i)$ such that $f_{i|U_1\cap U_j} = f_{j|U_i\cap U_j}$, there exists a unique section $f \in U$ such that $f_{|U_1} = f_i$.

Notice that this requires the category $\mathscr{C}$ to be concrete, i.e. have objects that consists of elements. If we have an abelian category instead, we can define the glueability axiom in an equivalent way without using elements. This equivalent axiom is given by the exacness of the following sequence:

$$0 \longrightarrow \mathscr{F}(U) \xrightarrow{\phi} \prod_{i\in I} \mathscr{F}(U_i) \xrightarrow{\alpha-\beta} \prod_{i\in I} \mathscr{F}(U_i \cap U_j)$$

where α is the map given by projecting to $\mathscr{F}(U_i)$, then restricting to $\mathscr{F}(U_i \cap U_j)$ and including into the product. The map β is given the same way, but by first projecting to $\mathscr{F}(U_j)$ instead. The uniqueness condition comes from exactness at the left part of the sequence, and the existence comes from the exactness in the middle.

There is even an even more general axiom, that does not require $\mathscr{C}$ to be abelian. This axiom is given by the sequence

$$\mathscr{F}(U) \longrightarrow \prod_{i\in I} \mathscr{F}(U_i) \rightrightarrows \prod_{i\in I} \mathscr{F}(U_i \cap U_j)$$

being an equalizer sequence.

Problem 10.1 Let $U = U_1 \cup U_2$ and $\mathscr{F}$ a sheaf of abelian groups. Show that the definition using abelian categories and concrete categories are the same in this setting.

Proof. The sequence in this setting becomes

$$\mathscr{F}(U) \longrightarrow \mathscr{F}(U_1) \times \mathscr{F}(U_2) \longrightarrow \mathscr{F}(U_1 \cap U_2)$$

where α is the map $\mathscr{F}(U_1) \times \mathscr{F}(U_2) \xrightarrow{p_1} \mathscr{F}(U_1) \xrightarrow{res_{U_1,U_1\cap U_2}} \mathscr{F}(U_1 \cap U_2)$, β the map $\mathscr{F}(U_1) \times \mathscr{F}(U_2) \xrightarrow{p_2} \mathscr{F}(U_2) \xrightarrow{res_{U_2,U_1\cap U_2}} \mathscr{F}(U_1 \cap U_2)$ and ϕ the map

If (f_1, f_2) lies in the kernel of $\alpha - \beta$, this means that $f_{1|U_1\cap U_2} - f_{2|U_1\cap U_2}$, i.e. $f_{1|U_1\cap U_2} = f_{2|U_1\cap U_2}$. Since this is a complex there exists a section $f \in \mathscr{F}(U)$ that gets mapped to (f_1, f_2) by ϕ. Since ϕ is injective, this section is unique. This shows the two definitions are the same in this small setting. $\square$

10.1 Stalks and germs

We want to examine what a sheaf does really locally, i.e. on a single point. We most often won't have that a singleton is an open set in the topological space, so we have to kind of zoom in using open neighbourhoods of the point. This is done formally by using limits.

Definition 10.2

Let $\mathscr{F}$ be a (pre)sheaf on a topological space X and fix a point $p \in X$. The stalk of $\mathscr{F}$ at p is defined as $\mathscr{F}_p = \lim_{p \in U} \mathscr{F}(U)$.

Alternatively we can define the stalk $\mathscr{F}_p$ by

$$\mathscr{F}_p = \{(U, f) | p \in U, f \in \mathscr{F}(U)\} / \sim$$

where $(U, f) \sim (V, g)$ is there exists an open set W, containing p, such that $f_{|W} = g_{|W}$.

The equivalence class $[U, f]$, which we denote f_p, is called the germ of $f \in \mathscr{F}(U)$ at p.

Here goes agricultural image

Example 10.3 Let X be a topological space, and $\mathscr{O} : Open(X) \longrightarrow Ring$ the sheaf of continuous functions, i.e. $\mathscr{F}(U) = \{f : U \longrightarrow \mathbb{R} | f \text{ continuous}\}$. We showed earlier that this is in fact a sheaf. Let $p \in X$ and define $\phi \mathscr{O}_p \longrightarrow \mathbb{R}$ by $\mathscr{O}(f_p) = f(p) \in \mathbb{R}$.

We have that ϕ is surjective, as every real number is hit by its corresponding constant function. We also have that the kernel, $\operatorname{Ker} \phi = \{[U, f] \in \mathscr{O}_p | f(p) = 0\}$ is an ideal in $\mathscr{O}_p$.

Notice that $\mathscr{O}_p / \operatorname{Ker} \phi \cong \mathbb{R}$, hence $\operatorname{Ker} \phi$ is maximal. Also, if $[U, f] \in \mathscr{O}_p \setminus \operatorname{Ker} \phi$, then $f(p) \neq 0$. Since f is a continuous function, we know that there exists a neighbourhood V around p such that $f_{|V} \neq 0$. This means that $[U, f]$ is invertible in $\mathscr{O}_p$ and hence that $(\mathscr{O}_p, \operatorname{Ker} \phi)$ is a local ring!

This is the construction that justifies the name “local”.

Fill in details, show Op a ring etc

Problem 10.4 Given a sheaf $\mathscr{F}$ on a topological space X and some point $p \in X$. Is $\mathscr{F}_p$ a local ring?

It is always possible to consider a sheaf (of sets) as a sheaf of functions. We show this as follows.

Let $\mathscr{F}$ be a sheaf on X and let $K = \coprod_{p \in X} \mathscr{F}_p$. Define $i_U : \mathscr{F}(U) \longrightarrow Map(U, K)$ by $f \mapsto [p \mapsto f_p]$.

Problem 10.5 Show that i_U is an injection and that it is compatible with restriction maps.

This means that any sheaf (of sets) can be considered as a sub sheaf of this above construction.

Example 10.6 (Restriction sheaf)**.**

Let $\mathscr{F}$ be a sheaf on X and $U \subseteq X$ be an open set. The restriction of $\mathscr{F}$ to U, defined by sending open sets $V \subseteq U$ to $\mathscr{F}_{|U}(V) = \mathscr{F}(V)$ is again a sheaf, called the restriction sheaf.

Problem 10.7 Show that this is in fact a sheaf.

Example 10.8 (The constant presheaf)**.**

Let S be a set. The constant presheaf at S is defined by $\underline{S}_{pre}(U) = S$ for all open sets $U \subseteq X$. This is not a sheaf in general as it fails the glueability axiom.

Example 10.9 (The constant sheaf)**.**

Let S be a set. The constant sheaf at S is given by

$$\underline{S}(U) = \{f : U \longrightarrow S | f \text{ locally constant}\}$$

Equivalently we can define it by letting $\mathscr{F}_p = S$ for every $p \in X$.

Example 10.10 (The pushforward sheaf)**.**

Let $\mathscr{F}$ be a presheaf on X and let $\pi \colon X \longrightarrow Y$ be a continuous map. The pushforward presheaf of $\mathscr{F}$ along π is defined by $\pi_*\mathscr{F}(V) = \mathscr{F}(\pi^{-1}(U))$ for open sets $V \subseteq Y$.

Problem 10.11 If F is a sheaf, show that $\pi_*\mathscr{F}$ is again a sheaf.

Example 10.12 (The skyscraper sheaf)**.**

Let $p \in X$ be a point and S a set. We endow $\{p\}$ with the discrete topology and define $i_p \colon \{p\} \hookrightarrow X$ to be continuous. The skyscraper sheaf is defined as $(i_p)_*\underline{S}$.

Problem 10.13 What does this sheaf look like? Hint: Look at the stalks.

11 Lecture 11 - 16.02.21

11.1 Sheaves of rings

The main type of sheaves we will use in this course is sheaves of rings. This is because they play an important role in defining schemes later, which will be our generalization of affine and projective algebraic sets in order to properly count multiplicity of intersection. We have already defined sheaves, and thereby sheaves of rings, i.e. *Ring*-valued sheaves, but to really hit the nail on the head we go through it again. Note that all rings mentioned will be commutative and have a multiplicative identity.

Definition 11.1: Sheaf of rings

A sheaf of rings is a sheaf $\mathscr{F}$ on a topological space X such that for each open set $U \subseteq X$ we have that $\mathscr{F}(U)$ is a ring, and that the restriction maps $\mathscr{F}(U) \longrightarrow \mathscr{F}(V)$ for any $V \subseteq U$ is a ring homomorphism.

If we in the above definition let $X = \{x\}$ be a singleton space, then we see that a sheaf of rings on X is really just a choice of a ring R. Hence the study of sheaves of rings vastly generalize the study of rings.

Definition 11.2: Ringed space

A ringed space, denoted $(X, \mathscr{O}_X)$, consists of a topological space X and a sheaf of rings $\mathscr{O}_X$, called the structure sheaf on X.

Recall that for a sheaf $\mathscr{F}$ on a space X we defined the stalk at a point $p \in X$ to be the set $\mathscr{F}_p = \lim_{p \in U} \mathscr{F}(U) = \{[U, f]\}/\sim = \{f_p\}$.

Problem 11.3 Let $\mathscr{F}$ be a sheaf of rings and p a point in X. Show that $\mathscr{F}_p$ is a ring.

Solution:

Let $[U, f], [V, g]$ be to elements in $\mathscr{F}_p$. Since U, V are open sets there exists an open set $W \subseteq U \cap V$ containing p. Notice that $[W, f_{|W}]$ is a representative for the same class as $[U, f]$, similarly for $[W, g_{|W}]$. But now both $f_{|W}$ and $g_{|W}$ are elements in $\mathscr{F}(W)$, which we know is a ring because $\mathscr{F}$ is a sheaf of rings. Hence $f_{|W} \cdot g_{|W}$ is well defined and satisfies all the ring axioms.

For a ring R we can study its modules. These are abelian groups M together with an action from R, i.e. a map $R \times M \longrightarrow M$. We just noted that sheaves of rings are a generalization of just rings, so for this generalization to be nice we really should be able to study some sort of modules on sheaves of rings. Luckily we can just generalize the module axioms into the world of sheaves to get what we want.

Definition 11.4: $\mathscr{O}_X$-modules

Let $(X, \mathscr{O}_X)$ be a ringed space. An $\mathscr{O}_X$-module $\mathscr{F}$ is a sheaf of abelian groups such that $\mathscr{F}(U)$ is an $\mathscr{O}_X(U)$-module with action α_U for all open $U \subseteq X$, such that

$$\begin{tikzcd} \mathscr{O}(U) \times \mathscr{F}(U) \arrow[r, "\alpha_U"] \arrow[d, "res^{\mathscr{O}_X}_{U,V} \times res^{\mathscr{F}}_{U,V}"'] & \mathscr{F}(U) \arrow[d, "res^{\mathscr{F}}_{U,V}"] \\ \mathscr{O}(V) \times \mathscr{F}(V) \arrow[r, "\alpha_V"] & \mathscr{F}(V) \end{tikzcd}$$

commutes for all open sets $V \subseteq U$.

If we again choose $X = \{x\}$ we see that the study of $\mathscr{O}_X$-modules vastly generalize the study of modules over rings.

Problem 11.5 Let $(X, \mathscr{O}_X)$ be a ringed space and $\mathscr{F}$ an $\mathscr{O}_X$-module. Show for every $p \in X$ that $\mathscr{F}_p$ is an $(\mathscr{O}_X)_p = \mathscr{O}_{X,p}$ module.

solution

11.2 Morphisms of sheaves

For the following discussion we let $\mathscr{C}$ be a nice, concrete category.

Definition 11.6

Let $\mathscr{F}, \mathscr{G}$ be $\mathscr{C}$-valued sheaves on a topological space X. A morphism $\phi\colon \mathscr{F} \longrightarrow \mathscr{G}$ consists of morphisms $\phi(U)\colon \mathscr{F}(U) \longrightarrow \mathscr{G}(U)$ in $\mathscr{C}$ for all open sets $U \subseteq X$, such that

$$\begin{tikzcd} \mathscr{F}(U) \arrow[r, "\phi_U"] \arrow[d, "res^{\mathscr{F}}_{U,V}"'] & \mathscr{G}(U) \arrow[d, "res^{\mathscr{G}}_{U,V}"] \\ \mathscr{F}(V) \arrow[r, "\phi_V"] & \mathscr{G}(V) \end{tikzcd}$$

commutes for all open sets $V \subseteq U$.

We define an isomorphism of sheaves to be a morphism with a two-sided inverse.

Note that a morphism $\phi\colon \mathscr{F} \longrightarrow \mathscr{G}$ of sheaves on a space X induces a morphism on stalks $\phi_p\colon \mathscr{F}_p \longrightarrow \mathscr{G}_p$ for all $p \in X$. This is because a morphism of sheaves gives us maps $\mathscr{F}(U) \longrightarrow \mathscr{F}(G)$ that respect restriction for all open sets containing p. Hence we have a morphism of directed systems

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] \\
\mathscr{F}(U) \arrow[r, "\phi(U)"] \arrow[d, "res^{\mathscr{F}}_{U,V}"'] & \mathscr{G}(U) \arrow[d, "res^{\mathscr{G}}_{U,V}"] \\
\mathscr{F}(V) \arrow[r, "\phi(V)"] \arrow[d, "res^{\mathscr{F}}_{V,W}"'] & \mathscr{G}(V) \arrow[d, "res^{\mathscr{G}}_{V,W}"] \\
\mathscr{F}(W) \arrow[r, "\phi(W)"] \arrow[d] & \mathscr{G}(W) \arrow[d] \\
\vdots & \vdots
\end{tikzcd}$$

A morphism of directed systems induces a morphism on its direct limit, which is the definition of the stalk at p, i.e. a map $\lim_{p\in U} \mathscr{F}(U) \longrightarrow \lim_{p\in U} \mathscr{G}(U)$.

Problem 11.7 Let $[U, f] \in \mathscr{F}_p$. Where does it go under the map ϕ_p? Why is it well defined?

Solution

Proposition 11.8

Let $\phi\colon \mathscr{F} \longrightarrow \mathscr{G}$ be a morphism of sheaves of abelian groups on a space X. Then ϕ is an isomorphism if and only if each induced morphism on stalks ϕ_p is an isomorphism.

Proof. Assume that ϕ is an isomorphism of sheaves. This means that $\phi(U)\colon \mathscr{F}(U) \longrightarrow \mathscr{G}(U)$ is an isomorphism for all open sets $U \subseteq X$. Since each ϕ_p is a direct limit of a direct system of isomorphisms it is again an isomorphism.

Assume now that ϕ_p is an isomorphism of abelian groups for all p. If we had inverses $\phi^{-1}(U)\colon \mathscr{G}(U) \longrightarrow \mathscr{F}(U)$ for all U we could collect these together to form an inverse ϕ^{-1}. Hence it is enough to show that $\phi(U)$ is an isomorphism for all open sets $U \subseteq X$.

We start by showing that ϕ is injective. Let $f \in \mathscr{F}(U)$ and suppose $\phi(U)(f) = 0$. This means that for all $p \in U$ we have $\phi(U)(f)_p = 0$. We have $\phi(U)(f)_p = \phi_p(f_p)$ and since ϕ_p is assumed injective we must have $f_p = 0$ for all p. This means that we for all $p \in U$ can find an open set W_p, containing p, such that $f_{|W_p} = 0$. We can find a cover of U using these W_p, i.e. $U = \cup_{p\in U} W_p$. Since $\mathscr{F}$ is a sheaf it must satisfy the glueability axiom, which means that $f = 0$. This is because the gluing is unique, and $f_{|W_p} = 0_{|W_p}$ on all W_p's, hence $f = 0$ as $0_{|W_p}$ glues back to 0. This means that the only element that gets sent to zero is the zero element, which means $\phi(U)$ is injective.

Surjectivity is a little trickier, but let's try our best. Suppose $g \in \mathscr{G}(U)$. For each $p \in U$ we let $g_p \in \mathscr{G}_p$ be its germ at p. Since ϕ_p is assumed surjective we can find $f_p \in \mathscr{F}_p$ such that $\phi_p(f_p) = g_p$. Since $\phi_p(f_p) = g_p$ we can find a small neighbourhood $V_p \subseteq U$ containing p such that $\phi(V_p)(f'_p) = g_{|V_p}$, where $f'_p \in \mathscr{F}(V_p)$ is a representative for f_p in V_p.

These sets V_p form a cover for U, i.e. $U = \cup_{p\in U} V_p$. We want to apply the glueability axiom for the sheaf $\mathscr{G}$, and to do that we need to have $f'_{p|V_p\cap V_q} = f'_{q|V_p\cap V_q}$. Both of these gets sent to $g_{|V_p\cap V_q}$ by the map $\phi(V_p \cap V_q)$, which we above proves is injective. Hence $f'_{p|V_p\cap V_q} = f'_{q|V_p\cap V_q}$. By glueability there exists a section $f \in \mathscr{F}(U)$ such that $f_{|V_p} = f'_p$.

Finally we need to check that this glued together section f actually maps to g under $\phi(U)$. We have $\phi(U)(f)_{|V_p} = g_{|V_p}$, and hence by the unique glueability in $\mathscr{G}$ we must have $\phi(U)(f) = g$. Hence every $g \in \mathscr{G}(U)$ gets hit by an $f \in \mathscr{F}(U)$ by $\phi(U)$, which means that it is surjective.

Since $\phi(U)$ is both injective and surjective it must be an isomorphism, which is what we wanted to show. □

11.3 Kernels, cokernels and images

Let $\phi\colon \mathscr{F} \longrightarrow \mathscr{G}$ be a morphism of sheaves of abelian groups on a topological space X.

Definition 11.9

The presheaf kernel of ϕ, denoted $\operatorname{Ker}\phi$, is the assignment of the abelian group $\operatorname{Ker}\phi(U)$ to every open set $U \subseteq X$.

Definition 11.10

The presheaf image of ϕ, denoted $\operatorname{Im}\phi$, is the assignment of the abelian group $\operatorname{Im}\phi(U)$ to every open set $U \subseteq X$.

Definition 11.11

The presheaf cokernel of ϕ, denoted $\operatorname{Cok}\phi$, is the assignment of the abelian group $\operatorname{Cok}\phi(U)$ to every open set $U \subseteq X$.

Problem 11.12 Show that these assignments are functors, i.e. that they are again presheaves.

Problem 11.13 Show that $\operatorname{Ker}\phi$ is a sheaf.

In general $\operatorname{Im}\phi$ and $\operatorname{Cok}\phi$ are not sheaves. To fix this we introduce the notion of sheafification of a presheaf. This will allow us to define the image and cokernel sheaf by sheafifying their respective presheaves.

Proposition 11.14

Let $\mathscr{F}$ be a presheaf on a topological space X. There is a sheaf $\mathscr{F}^+$, unique up to unique isomorphism, and a morphism $\theta\colon \mathscr{F} \longrightarrow \mathscr{F}^+$ such that for any other sheaf $\mathscr{G}$ we have

$$\begin{tikzcd} \mathscr{F} \arrow[r] \arrow[d] & \mathscr{F}^+ \arrow[dl, dotted, "\exists!"] \\ \mathscr{G} & \end{tikzcd}$$

i.e. that any other map into a sheaf factorizes through $\mathscr{F}^+$.

The proof of this will be given next time.

Definition 11.15: Sheafification

Let $\mathscr{F}$ be a presheaf on a topological space X. We define the sheafification of $\mathscr{F}$ to be the sheaf $\mathscr{F}^+$ as in the proposition above.

12 Lecture 12 - 22.02.21

As we began introducing last time we want to talk about the kernel, cokernel and image sheaves of morphisms of sheaves. We stated that these not necessarily were sheaves, so we needed a way to fix this, which is done by sheafifying the presheaves.

12.1 Sheafification

Proposition 12.1

Let $\mathscr{F}$ be a presheaf on a topological space X. There is a sheaf $\mathscr{F}^+$ on X, unique up to unique isomorphism, and a map $\Theta\colon \mathscr{F} \longrightarrow \mathscr{F}^+$, such that for any sheaf $\mathscr{G}$ with morphism $\mathscr{F} \longrightarrow \mathscr{G}$ we have a commutative diagram

$$\begin{tikzcd} \mathscr{F} \arrow[r, "\Theta"] \arrow[d] & \mathscr{F}^+ \arrow[dl, dotted, "\exists!"] \\ \mathscr{G} & \end{tikzcd}$$

Proof. For any open set $U \subseteq X$ we define

$$\mathscr{F}^+(U) = \{s : U \longrightarrow \bigcup_{p\in U} \mathscr{F}_p \,|\, 1. \text{ and } 2. \text{ holds}\},$$

where

1. $\forall p$ we have $s(p) \in \mathscr{F}_p$
2. $\forall p$ there exists a neighborhood of p, $V \subseteq U$ and $t \in \mathscr{F}(V)$ such that $\forall q \in V$ we have $t_q = s(q)$

Our claim is that $\mathscr{F}^+$ defined this way is a sheaf that satisfies the above proposition.

For $V \subseteq U$ we have that $\mathscr{F}(U) \longrightarrow \mathscr{F}(V)$ is the actual restruction maps, as we defined $\mathscr{F}^+$ using functions. This means that $\mathscr{F}^+$ is at least a presheaf.

Assume we have a set with an open cover, i.e. $U = \bigcup_i U_i$ and sections $s_i \in \mathscr{F}^+(U_i)$ such that $s_{i|U_i\cap U_j} = s_{j|U_i\cap U_j}$. We define $s \in \mathscr{F}^+(U)$ to be the map that sends $x \in U$ to $s_i(x)$ when $x \in U_i$. This gives the existence of a glued section.

Assume that we have two sections $s, s' \in \mathscr{F}^+(U)$ such that $s_{|U_i} = s_i = s'_{|U_i}$. Since $\{U_i\}$ is a cover we know that all $x \in U$ lie at least in one of the U_i's. We assume $x \in U_i$. Then

$$s(x) = s_{|U_i}(x) = s_i(x) = s'_{|U_i}(x) = s'(x)$$

holds for all points $x \in U$, hence $s = s'$. This shows that the map s we defined above is unique, and hence that $\mathscr{F}^+$ has unique gluing, and is thus a sheaf.

The map Θ is defined as follows. For every open set $U \subseteq X$ define $\Theta(U)$ as

$$\begin{aligned} \mathscr{F}(U) &\longrightarrow \mathscr{F}^+(U) \\ s &\longmapsto [U \to \cup\mathscr{F}_p, x \mapsto s_x] \end{aligned}$$

□

Check that sheaves factor through theta

As mentioned last time we define $\mathscr{F}^+$ to be the sheafification of a presheaf $\mathscr{F}$.

Problem 12.2 Prove that for all $p \in X$ that $\mathscr{F}^+_p = \mathscr{F}_p$.

This means that at the level of stalks, working with the sheafification is relatively easy.

Let $\phi_{\mathscr{F}} \longrightarrow \mathscr{G}$ be a morphism of sheaves (of abelian groups). Recall that $\operatorname{Ker}\phi$, $\operatorname{Im}_{pre}\phi$ and $\operatorname{Cok}_{pre}\phi$ are presheaves. The kernel sheaf $\operatorname{Ker}\phi$ is in fact also a sheaf.

Definition 12.3

A subsheaf of a sheaf $\mathscr{F}$ on X is a sheaf $\mathscr{F}'$ such that for all open sets $U \subseteq X$ we have that $\mathscr{F}'(U)$ is a subobject of $\mathscr{F}(U)$, and that the restriction maps in $\mathscr{F}'$ are induced by the ones in $\mathscr{F}$.

Note that this makes $\mathscr{F}'_p$ a subobject of $\mathscr{F}_p$.

Definition 12.4

We say a morphism $\phi\colon \mathscr{F} \longrightarrow \mathscr{G}$ is injective if $\operatorname{Ker}\phi = 0$.

Note that ϕ is injective if and only if $\phi(U)$ is injective for all open sets $U \subseteq X$.

Definition 12.5

We define the image sheaf of a morphism $\phi\colon \mathscr{F} \to \mathscr{G}$ to be $\operatorname{Im}\phi = \operatorname{Im}_{pre}\phi^+$, i.e. the sheafification of the image presheaf.

Calling it the image is justified as we have an injective morphism $\operatorname{Im}\phi \longrightarrow \mathscr{G}$. This map exists because of universal property of the sheafification

$$\begin{tikzcd}
\mathscr{F} \arrow[rr, "\phi"] \arrow[dr] & & \mathscr{G} \\
& \operatorname{Im}_{pre}\phi \arrow[ur, "\psi'"] \arrow[d] & \\
& \operatorname{Im}\phi \arrow[uur, dotted, "\exists\psi"'] &
\end{tikzcd}$$

It is injective because $\psi'(U)$ is injective for all U, hence ψ' injective. This implies $\psi'_p\colon (\operatorname{Im}_{pre}\phi)_p \longrightarrow \mathscr{G}_p$ is injective, which by the problem above, i.e. $(\operatorname{Im}_{pre}\phi)p =$

$\operatorname{Im}\phi_p$, means that ψ_p is injective as well. Being injective on all germs is sufficien to be injective as morphism of sheaves, hence ψ is injective.

Definition 12.6

We say a morphism $\phi\colon \mathscr{F} \longrightarrow \mathscr{G}$ is surjective if $\operatorname{Im}\phi = \mathscr{G}$.

Definition 12.7

Let $\mathscr{F}'$ be a subsheaf of a sheaf $\mathscr{F}$ on a topological space X. We define the quotient sheaf $\mathscr{F}/\mathscr{F}'$ by sending an open set $U \subseteq X$ to $\mathscr{F}(U)/\mathscr{F}'(U)$.

Problem 12.8 Show that this is in fact a sheaf.

Definition 12.9

Let $\phi\colon \mathscr{F} \longrightarrow \mathscr{G}$ be a morphism of sheaves. We define its cokernel sheaf to be the sheaf $\operatorname{Cok}\phi = \operatorname{Cok}_{pre}\phi^+$, i.e. the sheafification of the cokernel presheaf.

Definition 12.10

A sequence of sheaves on a topological space X,

$$\cdots \xrightarrow{\phi^{i-2}} \mathscr{F}^{i-1} \xrightarrow{\phi^{i-1}} \mathscr{F}^{i} \xrightarrow{\phi^{i}} \mathscr{F}^{i+1} \xrightarrow{\phi^{i+1}} \cdots$$

is called exact at degree i if $\operatorname{Ker}\phi^i = \Im\phi^{i-1}$. The sequence is called exact if it is exact at all i.

Note that the sequence

$$\cdots \xrightarrow{\phi^{i-2}} \mathscr{F}^{i-1} \xrightarrow{\phi^{i-1}} \mathscr{F}^{i} \xrightarrow{\phi^{i}} \mathscr{F}^{i+1} \xrightarrow{\phi^{i+1}} \cdots$$

is exact if and only if the sequence

$$\cdots \xrightarrow{\phi^{i-2}(U)} \mathscr{F}^{i-1}(U) \xrightarrow{\phi^{i-1}(U)} \mathscr{F}^{i}(U) \xrightarrow{\phi^{i}(U)} \mathscr{F}^{i+1}(U) \xrightarrow{\phi^{i+1}(U)} \cdots$$

is exact for all open sets $U \subseteq X$.

Definition 12.11

Let $\mathscr{F}$ and $\mathscr{G}$ be sheaves on a topological space X. For an open set $U \subseteq X$ we define $\mathscr{H}om(\mathscr{F},\mathscr{G})(U) = Mor(\mathscr{F}_{|U}, \mathscr{G}_{|U})$.

Proposition 12.12

The assignment $\mathscr{H}om(\mathscr{F},\mathscr{G})$ defines a sheaf on X.

Proof. We first see that it is a presheaf. Take a subset $V \subseteq U$, we need to define a map $Mor(\mathscr{F}_{|U},\mathscr{G}_{|U}) \longrightarrow Mor(\mathscr{F}_{|V},\mathscr{G}_{|V})$. For any subset $W \subseteq V$ we have that $\mathscr{F}_{|V}(W) = \mathscr{F}_{|U}(W)$, as the extra restriction of the sheaf itself does not do anything since we are looking at function on an even smaller set. Hence we get a map $\mathscr{F}_{|V}(W) \longrightarrow \mathscr{G}_{|V}(W)$ from the map we already have from $\mathscr{F}_{|U}(W) \longrightarrow \mathscr{G}_{|U}(W)$.

Let now $U = \bigcup U_i$ be an open cover and suppose $s_i \in \mathscr{H}om(\mathscr{F},\mathscr{G})(U_i)$ such that $s_{i|U_i\cap U_j} = s_{j|U_i\cap U_j}$. For $V \subseteq U$ set $V_i = V \cap U_i$, which means that $V = \bigcup V_i$ is an open cover.

For $f \in \mathscr{F}(V)$ we can restrict it to V_i to get $f_{|V_i} \in \mathscr{F}(V_i)$. We can map this to $\mathscr{G}(V_i)$ by using $s_i(V_i)$ to get some $g_i \in \mathscr{G}(V_i)$. Since $\mathscr{G}$ is a sheaf we can glue these g_i to get an unique section $g \in \mathscr{G}(V)$. We then simply define $s \in \mathscr{H}om(\mathscr{F},\mathscr{G})(U)$ by $s(V)(f) = g$.

This defines $s\colon \mathscr{F}_{|U} \longrightarrow \mathscr{G}_{|U}$ such that $s_{|U_i} = s_i$, and thus we have existence of a glued section.

For uniqueness we assume there exists $s, t \in \mathscr{H}om(\mathscr{F},\mathscr{G})(U)$ such that $s_{|U_i} = t|U_i$. For $V \subseteq U$ we have a diagram

$$\begin{tikzcd}
\mathscr{F}_{|U}(V) \arrow[r, bend left, "s(V)"] \arrow[r, bend right, "t(V)"'] \arrow[d] & \mathscr{G}_{|U}(V) \arrow[d] \\
\mathscr{F}(V_i) \arrow[r, "s(V_i)=t(V_i)"'] & \mathscr{F}(V_i)
\end{tikzcd}$$

where the vertical arrows are restriction maps. Let $f \in \mathscr{F}(V)$. We want to compare $s(V)(f)$ and $t(V)(f)$. For all i we have

$$s(V)(f)_{|V_i} = s(V_i)(f_{|V_i}) = t(v_i)(f_{|V_i}) = t(V)(f)_{|V_i}$$

which by $\mathscr{G}$ being a sheaf means that $s(V)(f) = t(V)(f)$. So $s(V)$ and $t(V)$ are pointwise equal, i.e. the same map, hence $s(V) = t(V)$. This holds for all open sets V, hence also s and t are pointwise equal, making them again equal, i.e. $s = t$. Hence the gluing is unique and we are done. $\square$

Definition 12.13

Let now $\mathscr{F}$ and $\mathscr{G}$ be two $\mathscr{O}_X$ modules. We can define their tensor product $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G}$ to be the sheaf associted to the presheaf $U \longrightarrow \mathscr{F}(U)\otimes_{\mathscr{O}_X(U)\mathscr{G}(U)}$.

12.2 Sheaves and varieties

Let V be an affine algebraic set. We want to study some special ringed spaces, $(V, \mathscr{O}_V)$, which will be the affine algebraic varieties. In more generality we will have algebraic varieties which will be ringed spaces $(X, \mathscr{O}_X)$ that are locally affine. If $V \subseteq \mathbb{P}^n(k)$ is projective we will get projective algebraic varieties, which will be examples of these more general algebraic varieties.

In even more generality, these will all be examples of schemes, which also are locally ringed spaces.

13 Lecture 13 - 23.02.21

Last time we ended on a little note explaining where we are headed today, i.e. looking at algebraic varieties.

13.1 The structural sheaf

Let $V \subseteq k^n$ be an affine algebraic set and recall that we have a basis for its open sets by the distinguished open sets $D(f) = V \setminus V(f)$.

Lemma 13.1

To define a sheaf on a topological space X, it is enough to define it on the basis elements for the topology on X.

Problem 13.2 Prove this lemma, at least for sheaves of functions.

We define a sheaf, called the structural sheaf, or the structure sheaf, $\mathscr{O}_V$ on V by $\mathscr{O}_V(D(f)) = \Gamma(V)_f$.

Proposition 13.3

$\mathscr{O}_V$ in fact defines a sheaf on V.

Proof. For $D(f) \subseteq D(g)$ we have $V(g) \subseteq V(f)$, hence by the nullstellensatz we have $f \in \sqrt{g}$. This means that we can find a h and a natural number n such that $f^n = gh$. We have

$$\begin{aligned} \Gamma(V)_g &\longrightarrow \Gamma(V)_f \\ \frac{u}{g^i} &\longmapsto \frac{uh^i}{g^ih^i} = \frac{uh^i}{f^{ni}} \end{aligned}$$

Thus $\mathscr{O}_V$ is a presheaf of rings on V.

Assume now $D(f) = \bigcup D(f_i)$ and $s_i \in \mathscr{O}_V(D(f_i)$ such that $s_i i|D(f_i) \cap D(f_j) = s_{j|D(f_i)\cap D(f_j)}$. For $\mathscr{O}_V$ to be a sheaf we need a unique section $s \in \mathscr{O}_V(D(f))$ that restricts to the s_i's.

Lets first look at a special case where V is an irreducible affine algebraic set. In this case recall that we have $I(V)$ a prime ideal. Assume also that $D(f) = D(f_1) \cup D(f_2)$. Then we have a sequence

$$0 \longrightarrow \mathscr{O}_V(D(f)) \longrightarrow \mathscr{O}_V(D(f_1)) \oplus \mathscr{O}_V(D(f_2)) \longrightarrow \mathscr{O}_V(D(f_1f_2))$$

where exactness at $\mathscr{O}_V(D(f_1)) \oplus \mathscr{O}_V(D(f_2))$ yields existence of a section, and $\mathscr{O}_V(D(f))$ yields uniqueness.

This we get from the totalization of the following commutative square

$$\begin{array}{ccc} \Gamma(V)_f & \longrightarrow & \Gamma(V)_{f_1} \\ \downarrow & & \downarrow \\ \Gamma(V)_{f_2} & \longrightarrow & \Gamma(V)_{f_1 f_2} \end{array}$$

Ok, back to the general case. Write $s_i = \frac{a'_i}{f_i^{n_i}}$. We can choose $n = maxn_i$ to get $s_i = \frac{a_i}{f_i^n}$ instead.

We have that $s_{i|D(f_i)\cap D(f_j)} = s_{j|D(f_i)\cap D(f_j)}$ if and only if $\frac{a_i}{f_i^n} = \frac{a_j}{f_j^n}$ in $\Gamma(V)_{f_1 f_2}$, which again hold if and only if $f_i^N f_j^N (a_i f_j^n - a_j f_i^n) = 0$ for some N. As f vanishes on $V(f_1^{n+N}, \ldots, f_r^{n+N})$ we have by k being algebraically closed that $f \in \sqrt{(f_1^{n+N}, \ldots, f_r^{n+N})}$. Hence there exists $m \geq 1$ and $b_j \in \Gamma(V)$ such that $f^m = \sum_{j=1}^r b_j f_j^{n+N}$. Set $a = \sum_{j=1}^r a_j b_j f_j^N$ and then $s = \frac{a}{f^m}$. We claim that this is our glued section. To confirm this we need to show that it restricts to the s_i's, i.e. that $\frac{a}{f^m} = \frac{a_i}{f_i^n}$ in $\Gamma(V)_{f_i}$. We have

$$\begin{aligned} f_i^N (a_i f^m - a f_i^n) &= f_i^N a_i f^m - a f_i^{n+N} \\ &= f_i^N a_i \sum_{j=1}^r b_j f_j^{n+N} - a f_i^{n+N} \\ &= \sum_{j=1}^r a_i b_j f_i^N f_j^{n+N} - a f_i^{n+N} \\ &= \sum_{j=1}^r a_j b_b f_i^{n+N} f_j^N - a f_i^{n+N} \\ &= a f_i n + N - a f_i^{n+N} \\ &= 0 \end{aligned}$$

where the fourth equality comes from the previous equation $f_i^N f_j^N (a_i f_j^n - a_j f_i^n) = 0$. Hence s restricts to s_i and we are done.

□

13.2 Algebraic varieties

Definition 13.4

An affine algebraic variety is a ringed space isomorphic (as ringed spaces) to $(V, \mathscr{O}_V)$ for some affine algebraic set V, where $\mathscr{O}_V$ is defined as above.

Such an isomorphism is a homeomorphism of the topological spaces, and an isomorphism of sheaves.

Proposition 13.5

Let V be an affine algebraic set and $f \in \Gamma(V)$. Then $(D(f), \mathscr{O}_{V|D(f)})$ is an affine algebraic variety.

Proof. Assume $V \subseteq k^n$ and let F be a polynomial corresponding to f. Define $\phi\colon D(f) \longrightarrow k^{n+1}$ by sending

$$(x_1, \dots, x_n) \longmapsto (x_1, \dots, x_n, \frac{1}{f(x_1, \dots, x_n)})$$

□

Problem 13.6 Check that $\operatorname{Im}\phi = V(J)$, where $J = I(V) + (X_{n+1}F - 1)$, and that ϕ is a homeomorphism.

Definition 13.7

An algebraic variety, often just called a variety, is a quasi-compact ringed space $(X, \mathscr{O}_X)$ such that for any $x \in X$ there is an open neighborhood $U \subseteq X$, containing x, such that $(U, \mathscr{O}_{X|U}$ is an affine algebraic variety.

This is what we mean when we say a general algebraic variety is locally affine. By quasi-compact we mean that any open cover has a finite subcover. It is the same as being compact minus the Hausdorff property.

Proposition 13.8

Let $(X, \mathscr{O}_X)$ be an algebraic variety. Any open set $U \subseteq X$ is a finite union of affine open sets, i.e. sets U_i such that $(U_i, \mathscr{O}_{X|U_i})$ is an affine algebraic variety.

Proof. Write $X = \bigcup_{i=1}^r U_i$, U_i open affine. This decomposition exists as X is quasi-compact and locally affine. Let $U \subseteq X$ be open and write $U = \bigcup_{i=1}^r U \cap U_i$. The $U \cap U_i$'s are affine open. □

Problem 13.9 Fill in details.

Problem 13.10 Is every open subvariety of an affine algebraic variety again affine?

Problem 13.11 Let $(X, \mathscr{O}_X)$ be an algebraic variety and let $x \in X$. Prove that $\mathscr{O}_{X,x}$ is a local ring with maximal ideal $M = \{f \in \mathscr{O}_{X,x} \,|\, f(x) = 0\}$

Hint: Define $\phi\colon \mathscr{O}_{X,x} \longrightarrow k$ by $\phi(U, f) = f(x)$.

Proposition 13.12

Let $(X, \mathscr{O}_X)$ be an algebraic variety, $x \in X$ and $U \subseteq X$ be an open set containing x. Set $A = \mathscr{O}_X(U)$ and let M be the maximal ideal that corresponds to x. Then $\mathscr{O}_{X,x} \cong A_m$.

13.3 Projective algebraic varieties

Let $V \subseteq \mathbb{P}^n(k)$ be a projective algebraic set. We need to define a sheaf, and as before we do so on the basis $D^+(f)$ where $f \in \Gamma_{homog}(V)$.

Definition 13.13

Let R be a graded ring and let $f \in R$ be a homogeneous element of degree d. Then R_f is also graded, and $deg(\frac{g}{f^r} = deg(g) - r \cdot d$. Define $R_{(f)}$ to be the set of degree 0 in R_f.

Definition 13.14

Let V be a projective algebraic set. Define $\mathscr{O}_V(D^+(f)) = \Gamma_{homog}(V)_{(f)}$.

This is in fact a sheaf on V.

Proposition 13.15

Let V be a projective algebraic set. Then $(V, \mathscr{O}_V)$ is an algebraic variety, called a projective algebraic variety.

14 Lecture 14 - 01.03.21

The lecture was focused on going through the exercises. Solutions and notes to these can be found at **??**.

15 Lecture 15 - 02.03.21

15.1 Sheaves of modules on varieties

Let $(V, \mathscr{O}_V)$ be an affine algebraic variety. Recall that $\mathscr{O}_V(D(f)) = \Gamma(V)_f$. Set $A = \mathscr{O}(V) = \Gamma(V, \mathscr{O}_V)$, which we call the global sections, and let $\mathscr{F}$ be an $\mathscr{O}_V$-module. In particular $\mathscr{F}(V)$ is an A-module.

A question we want to answer is weather we can create $\mathscr{O}_V$-modules from an A-module.

Definition 15.1

Let M be an A-module. Define an $\mathscr{O}_V$-module $\widetilde{M}$ by $\widetilde{M}(D(f)) = M_f$ for $f \in A$.

Notice that $M_f = M \otimes_A A_f$, so this could also be used as an alternative definition. Notice also that we have $\widetilde{M}(V) = M$.

Problem 15.2 What is $\widetilde{A}$?

Solution:

$\widetilde{A} = \mathscr{O}_V$.

Problem 15.3 Check that $\widetilde{M}$ is a sheaf. This should be a similar proof as for $\mathscr{O}_V$ being a sheaf, but carrying the tensor around.

Proposition 15.4

The assignment

$$\begin{aligned} A - \text{modules} &\longrightarrow \mathscr{O}_V - \text{modules} \\ M &\longmapsto \widetilde{M} \end{aligned}$$

is functorial, exact and preserves direct sums and tensor products.

Proof. This is true because the localization functor has these properties. □

Definition 15.5

Let $\mathscr{F}$ be an $\mathscr{O}_V$-module. We say $\mathscr{F}$ is quasi-coherent if $\mathscr{F} \cong \widetilde{M}$ for some A-module M and coherent if this M is finitely generated.

Hence we have an equivalence of categories between the category of A-modules and the category of quasi-coherent sheaves on V, $QCoh(V)$.

Problem 15.6 Check that $QCoh(V)$ is a category, and that the above described functor indeed gives an equivalence of categories.

Definition 15.7

Let $(X, \mathscr{O}_X)$ be an algebraic variety and $\mathscr{F}$ an $\mathscr{O}_X$-module. We say $\mathscr{F}$ is quasi-coherent if $\exists$ an open affine cover $X = \bigcup U_i$ such that $\mathscr{F}_{|U_i} \cong \widetilde{M_i}$ for some $\mathscr{O}_X(U_i)$-module M_i.

Equivalently we could define it to be quasi-coherent if for any open set $U \subset X$ we have $\mathscr{F}_{|U} \cong \widetilde{M}$ for some $\mathscr{O}_X(U)$-module M.

We say $\mathscr{F}$ is coherent if in either of the above definitions either all the M_i's or all the M's are finitely generated modules.

Proposition 15.8

Let $(X, \mathscr{O}_X)$ be an algebraic variety and let $\mathscr{F}$ and $\mathscr{G}$ be quasi-coherent sheaves on X. Then $\mathscr{F} \otimes \mathscr{G}$ is again quasi-coherent.

Proof. For any open set $U \subseteq X$ consider

$$(\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G})_{|U} \cong \mathscr{F}_{|U} \otimes_{\mathscr{O}_U} \mathscr{G}_{|U}$$

These are both sheaves with the same underlying presheaf, namely the assignment $W \longmapsto \mathscr{F}(W) \otimes_{\mathscr{O}_X(W)} \mathscr{G}(W)$. Hence they are isomorphic.

Let $U \subseteq X$ be open affine. As $\mathscr{F}$ and $\mathscr{G}$ are quasi-coherent, we can find A-modules M and N such that $\mathscr{F}_{|U} \cong \widetilde{M}$ and $\mathscr{G}_{|U} \cong \widetilde{N}$. Thus

$$(\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G})_{|U} \cong \mathscr{F}_{|U} \otimes_{\mathscr{O}_U} \mathscr{G}_{|U} \cong \widetilde{M} \otimes_{\mathscr{O}_U} \widetilde{N} \cong M \widetilde{\otimes_{\mathscr{O}_U(U)}} N$$

Where the last isomorphism comes from $M \otimes_A N \otimes A_f \cong (M \otimes_A A_f) \otimes_{A_f} (N \otimes_A A_f)$. $\square$

Example 15.9 Let $(X, \mathscr{O}_X)$ be an algebraic variety, then $\mathscr{O}_X$ is coherent. This holds because for any affine open set $U \subseteq X$ we have that $\mathscr{O}_{X|U} = \widetilde{\mathscr{O}_{X|U}(U)}$ is a finitely generated module over itself.

15.2 Projective varieties

Let k be an algebraically closed field and $V \subseteq \mathbb{P}^n(k)$ be a projective algebraic set. Our goal is to look at $\mathscr{O}_V$ and to show that $(V, \mathscr{O}_V)$ is an algebraic variety.

Definition 15.10

Let R be a graded ring and f a degree d homogeneous element. Then R_f is graded, and $deg(\frac{g}{f^r}) = deg(g) - r \cdot d$. The degree zero component is denoted by $(R_f)_0$.

Definition 15.11

The structure sheaf $\mathscr{O}_V$ for V is defined as

$$\mathscr{O}_V(D^+(f)) = (\Gamma_{\text{homog}}(V)_f)_0$$

where f is homogeneous of positive degree.

Problem 15.12 Show that $\mathscr{O}_V$ is a sheaf.

Solution:

It is a presheaf because for $D^+(f) \subseteq D^+(g)$ we have $V \setminus V_{\text{proj}}(f) \subseteq V \setminus V_{\text{proj}}(g)$ which again means that $V_{\text{proj}}(g) \subseteq V_{\text{proj}}(f)$. By the projective nullstellensatz we then have $(f) \subseteq \sqrt{(g)}$, which means there is an h such that $f^r = g \cdot h$.

The restriction maps are then

$$\begin{aligned} (\Gamma_{\text{homog}}(V)_g)_0 &\longrightarrow (\Gamma_{\text{homog}}(V)_f)_0 \\ \frac{u}{g^i} &\longmapsto \frac{uh^i}{f^{ni}} \end{aligned}$$

Then sheaf condition is the same as for the affine case.

To prove that this sheaf makes our projective algebraic set into an algebraic variety we need to show it is locally isomorphic to an affine algebraic variety. This isomorphism is as ringed spaces, so we need to know what such a map is.

Definition 15.13

A morphism of ringed spaces

$$(\phi, \phi^{\#})\colon (X, \mathscr{O}_X) \longrightarrow (Y, \mathscr{O}_Y)$$

is a continuous map $\phi\colon X \longrightarrow Y$ of topological spaces, and a map $\phi^{\#}: \mathscr{O}_Y \longrightarrow \phi_*\mathscr{O}_X$ of schemes. Here the map of schemes is often thought of as a pullback.

We also need the concept of homogenization and dehomogenization. These processes are morphisms between $k[X_0, \dots, X_n]$ and $k[X_1, \dots, X_n]$. Dehomogenization, denoted by ${}^b(-)$ is defined by ${}^b(F(X_0, \dots, X_n)) = F(1, X_1, \dots, X_n)$. Homogenization is a bit more difficult, but we describe it by an example. The homogenization, denoted $^h(-)$, of the element $X_1 + X_2^3 + X_3^4 \in k[X_1, X_2, X_3]$ is $X_0^3X_1 + X_0X_2^3 + X_3^4$. It finds the greatest degree and multiplies the other components by X_0 until it gets to that degree.

Proposition 15.14

Let $V \subseteq \mathbb{P}^n(k)$ be a projective algebraic set. Then $(V, \mathscr{O}_V)$ is an algebraic variety.

Proof. We reduce to only proving t for $V = \mathbb{P}^n(k)$.

Cover $\mathbb{P}^n(k)$ by $D^+(X_i)$. We will show that $D^+(X_0)$ is an affine algebraic variety. By transferring the same argument over a homography, this is also sufficient.

Set $U_0 = D^+(X_0) = \{[x_0 : \cdots : x_n] \in \mathbb{P}^n(k) \,|x_0 \neq 0\}$. We have earlier seen that we have a bijection $j\colon k^n \longrightarrow U_0$, given by sending $(a_1, \ldots, a_n)$ to $[1 : a_1 : \cdots : x_n]$.

We claim that $(k^n, \mathscr{O}_{k^n}) \overset{(j,j^\#)}{\longrightarrow} (U_0, \mathscr{O}_{\mathbb{P}^n(k)|U_0}$ is an isomorphism of ringed spaces.

If $D^+(F) \subseteq U_0$ where $F \in k[X_0, \ldots, X_n]$ is homogeneous, then $j^{-1}(D^+(F)) = j^{-1}(D^+(F)) \cap U_0 = D(^bF)$, which means j is continuous. We also have $j(D(f) = D^+(^hf) \cap U_0$, hence j^{-1} is also continuous, meaning that j is a homeomorphism.

For $W \subseteq V$ open in $\mathbb{P}^n(k)$ we have

$$\begin{array}{ccc} \mathscr{O}_{\mathbb{P}^n(k)|U_0}(V \cap U_0) & \xrightarrow{\cong} & \mathscr{O}_{k^n}(j^{-1}(V \cap U_0)) \\ \downarrow & & \downarrow \\ \mathscr{O}_{\mathbb{P}^n(k)|U_0}(W \cap U_0) & \xrightarrow{\cong} & \mathscr{O}_{k^n}(j^{-1}(W \cap U_0)) \end{array}$$

We need to show there is an isomorphism $\mathscr{O}_{\mathbb{P}^n(k)}(D^+(F) \cap U_0) \cong \mathscr{O}_{k^n}(D(^bF))$. Notice here that the latter is just $k[X_1, \ldots, X_n]_{^bF}$, while the former is $(k[X_0, \ldots, X_n]_{FX_0})0$. Hence we define, for a homogeneous element p with $deg(p) = r(deg(F) + 1)$, the map

$$\begin{aligned} \phi\colon (k[X_0, \ldots, X_n]_{FX_0})_0 &\longrightarrow k[X_1, \ldots, X_n]_{^bF} \\ \frac{p}{F^rX_0^r} &\longmapsto \frac{^bp}{^b(F^rX_0^r)} = \frac{^bp}{^bF^r} \end{aligned}$$

which is an isomorphism. Hence $j^\#$ is an isomorphism of sheaves, and we are done. □

We say an algebraic variety is a projective algebraic variety, or sometimes a projective variety, if it is of the form from the above proposition.

16 Lecture 16 - 08.03.21

16.1 Dimension

Since we are working both with algebra and topology we have two separate notions of dimension, one algebraic and one topological.

Definition 16.1

Let X be a topological space. The dimension of X is defined to be

$$\dim X = \sup_{n \in \mathbb{Z}} \{X_0 \subsetneq X_1 \subsetneq \cdots \subsetneq X_n\}$$

where X_i is a closed irreducible subset of X.

Note that this definition is really only useful when working with topologies similar or equal to the Zariski topology. In most standard topologies there is usually not very many such sets, so the dimension is often just zero or one.

Proposition 16.2

Let $Y \subseteq X$ be a subspace. Then $\dim X \geq \dim Y$. If X is irreducible and finite dimensional, and Y is closed in X, then $\dim X > \dim Y$.

Proof. Let $F_0 \subsetneq \cdots \subsetneq F_n$ be a chain of closed irreducible subsets of Y, then $\overline{F_0} \subseteq \cdots \subseteq \overline{F_n}$ is a chain of closed subsets of X. As we have $F_i = \overline{F_i} \cap Y$ we must have that the $\overline{F_i}$'s are distinct, because if $\overline{F_i} = \overline{F_{i+1}}$ then $F_i = F_{i+1}$, which we have assumed is not so. Hence we have a chain $\overline{F_0} \subsetneq \cdots \subsetneq \overline{F_n}$. Suppose that $\overline{F_i} = U \cup V$, where U, V closed and non-empty. Then $F_i = \overline{F_i} \cap Y = (U \cap Y) \cup (V \cap Y)$, which contradicts the assumption that F_i is irreducible. Hence $\overline{F_i}$ is irreducible, and we have at least a chain of length n in X for all chains of length n in Y, hence $\dim Y \leq \dim X$.

Now, assume that X is irreducible, $\dim X < \infty$ and Y closed in X. By the above argument we also have $\dim Y < \infty$, so let $\dim Y = n$. Then a maximal length chain in Y looks like $F_0 \subsetneq \cdots \subsetneq F_n$ for some closed irreducible subsets F_i. Then we have that $\overline{F_0} \subsetneq \cdots \subsetneq \overline{F_n} \subsetneq X$ is a chain in X of length $n+1$, meaning $\dim Y < \dim X$. $\square$

Proposition 16.3

If $X = \bigcup_{i=1}^n X_i$ where X_i is closed in X, then we have $\dim X = \sup \dim X_i$.

Proof. We know from the previous proposition that $\dim X \geq \dim X_i$ for all i, hence we also have $\sim X \geq \sup \dim X_i$. Notice that we are done if $\sup \dim X_i = \infty$, hence we assume $\sup \dim X_i = p$. Suppose now that $p < \dim X$. That means that we can find a chain $F_0 \subsetneq \cdots \subsetneq F_p \subsetneq F_{p+1}$ of closed irreducible subsets of X. We then have $F_{p+1} = \bigcup_{i=1}^n F_{p+1} \cap X_i$, but F_{p+1} is irreducible, hence $F_{p+1} = F_{p+1} \cap X_i$ for some

i. Hence we have $F_{p+1} \subseteq X_i$, and hence X_i now has a chain of length $p+1$, i.e. $\dim X_i > \sup \sim X_i$, which of course is absurd. Hence $\dim X = p$. $\square$

Recall that if X is an algebraic variety, then $X = \bigcup_{i=1}^{n} F_i$, where F_i is irreducible closed and does not contain each other. Thus we usually reduce to studying dimension of irreducible algebraic varieties.

16.2 Relation to Krull dimension

For an ring R we define its Krull dimension to be $\operatorname{krull.dim} R = \sup\{n \in \mathbb{Z} | p_0 \subsetneq \cdots \subsetneq p_n\}$ where the p_i's are prime ideals of R. In other words, the Krull dimension is the maximal length of a chain of prime ideals.

Proposition 16.4

Let V be an affine algebraic variety. Then $\dim V = \operatorname{krull.dim} \Gamma(V)$.

Proof. Using the Nullstellensatz, there is a correspondence between closed irreducible subsets of V and prime ideals in $\Gamma(V)$. A chain on one side of this correspondence immediately gives a chain of the same length on the other side. $\square$

Example 16.5 Let $V = k^n$, hence $\Gamma(V) = k[X_1, \ldots, X_n]$. Then

$$V(X_1, \ldots, X_n) \subsetneq V(X_1, X_{n-1}) \subsetneq \cdots \subsetneq V(X_1, X_2) \subsetneq V(X_1)$$

is a chain of length n in V. This corresponds to the chain prime ideals

$$(X_1) \subsetneq (X_1, X_2) \subsetneq \cdots \subsetneq (X_1, \ldots, X_{n-1} \subsetneq (X_1, \ldots, X_n))$$

in $\Gamma(V)$. Hence $\dim V = \operatorname{krull.dim} \Gamma(V) \geq n$, as we don't yet know that these are maximal length chains.

Our goal is then to show what our intuition tells us, i.e. that $\dim k^n = n = \operatorname{krull.dim} k[X_0, \ldots, X_n]$.

Lemma 16.6: Noethers normalization lemma

Let A be a finitely generated k-algebra. There exists algebraically independent elements $y_1, \ldots, y_r \in A$ such that A is integral over $k[y_1, \ldots, y_r]$.

Example 16.7 Let x denote the image of X in $k[X, Y]/(XY)$. Then $k[X, Y]/(XY)$ is integral over $k[x]$.

Note that the number of these algebraically independent elements used in Noethers normalization lemma is going to be the dimension.

Theorem 16.8: The dimension theorem

Let A be a domain that is also a finite type k-algebra. Then $\operatorname{krull.dim} A = \operatorname{tr.deg}_k \operatorname{Fr}(A)$, where $\operatorname{Fr}(A)$ is the fraction field of A.

Proof. Noether normalization gives us an injection

$$k[y_1, \ldots, y_r] \hookrightarrow A$$

The extension $k \to k(y_1, \ldots, y_r) = \operatorname{Fr}(k[y_1, \ldots, y_r])$ has transcendence degree r.

Since A is a finitely generated domain over $k[y_1, \ldots, y_r]$ we have by the Going up theorem that

$$\operatorname{krull.dim} A = \operatorname{krull.dim} k[y_1, \ldots, y_r]$$

Hence it is enough to consider the proof for $k[y_1, \ldots, y_r]$.

Our strategy is to show that $k[y_1, \ldots, y_r]$ has no chain of prime ideals with length greater than r. We do this by induction on r. For $r = 0$ this is ok.

Assume $r > 0$ and that there exists a chain $p_0 \subsetneq \ldots \subsetneq p_m$ for some $m > r$. Choose an element $a_1 \in p_1 \setminus p_0$. Since $a_1 \in k[y_1, \ldots, y_r]$ is not a constant polynomial there is "a lemma" that says there exists $a_2, \ldots, a_r \in k[y_1, \ldots, y_r]$ such that $k[y_1, \ldots, y_r]$ is finitely generated over $k[a_1, \ldots, a_r]$. We then have

$$\begin{tikzcd}
k[Z_1, \ldots, Z_r] \arrow[r, "f.g."] \arrow[d, two heads] & k[y_1, \ldots, y_r] \arrow[d, two heads] \\
k[Z_1, \ldots Z_r]/(Z_1) \arrow[r, dotted, "f.g."] & k[y_1, \ldots, y_r]/p_1
\end{tikzcd}$$

where the top map sends Z_i to a_i.

Notice also that $k[Z_1, \ldots Z_r]/(Z_1) \cong k[Z_2, \ldots Z_r]$. Now, say we have a chain $q_1 \subsetneq \cdots \subsetneq q_m$ in $k[Z_2, \ldots Z_r]$. Then the going down theorem makes sure we have a chain $\overline{p_1} \subsetneq \cdots \subsetneq \overline{p_m}$. By induction we must have that $m - 1 \leq r - 1$ i.e. that $m \leq r$, which contradicts our assumption that $m > r$. $\square$

Corollary 16.9

We have $\operatorname{krull.dim} k[X_1, \ldots, X_n] = n$.

Corollary 16.10

If V is an irreducible affine algebraic variety, then $\dim V < \infty$.

Corollary 16.11

We have $\dim k^n = n$.

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We continue looking at dimension theory.

Proposition 17.1

Let X be an irreducible algebraic variety and $U \subseteq X$ be a non-empty open subset. Then $\dim X = \dim U$.

Proof. We first prove the statement for affine algebraic varieties, so let's assume X is affine. Since $U \neq \emptyset$ there exists a distinguished open set $D(f) \subseteq U$, where $f \in \Gamma(X)$.

We $\dim D(f) \leq \dim U \leq \dim X$ by last lecture, and also that

$$\dim D(f) = \operatorname{krull.dim} \Gamma(X)_f = \operatorname{tr.deg}_k \operatorname{Fr}(\Gamma(X)_f)$$

and

$$\dim X = \operatorname{krull.dim} \Gamma(X) = \operatorname{tr.deg}_k \operatorname{Fr}(\Gamma(X))$$

The only difference between $\operatorname{Fr}(\Gamma(X)_f)$ and $\operatorname{Fr}(\Gamma(X))$ is that f gets inverted earlier, which does not matter in the end, as everything is invertible in the fraction field. Hence $\operatorname{Fr}(\Gamma(X)_f) = \operatorname{Fr}(\Gamma(X))$ which means they have the same transcendence degree. Hence $\dim D(f) = \dim X$, which means $\dim U = \dim X$ when X is affine.

In the general case, i.e. X not affine, the above proof shows that all non-empty irreducible open affine subsets of X have the same dimension, say r. As the distinguished opens form a basis for these affine subsets, we have also that any two sets with non-empty intersection must have the same dimension.

Assume now that $\dim X > r$. This means that we have a chain $F_0 \subsetneq \cdots \subsetneq F_n$ of closed irreducible subsets, where $n > r$. Let $x \in F_0$, contained in some open affine set U. Then $U \cap F_0 \subsetneq \cdots \subsetneq U \cap F_n$ is a chain of closed irreducible distinct subsets in U, meaning that $n = r = \dim U$, and hence that $\dim X = r$.

Now, if instead of being affine U is any open set, then there exists an open affine set $U' \subseteq U$. Hence $\dim U' \leq \dim U \leq \dim X$, but $\dim U' = \dim X$, hence also $\dim U = \dim X$. □

Example 17.2 ($\mathbb{P}^n(k)$).

By exercise 4 in chapter 1 we know that the union of overlapping irreducible subsets is again irreducible, hence that $\mathbb{P}^n(k)$ is irreducible. Hence we have

$$\dim \mathbb{P}^n(k) = \dim D^+(X_0) = \dim k^n = n$$

17.1 Dimension and counting equations

Let V be a d-dimensional affine algebraic variety and let $f \in \Gamma(V)$. We want to understand a bit better the dimension of $V(f)$.

Example 17.3 Let $V = k^3$. Then $\dim V(X) = 2 = 3 - 1$, $\dim V(X,Y) = 1 = 3 - 2$ and $\dim V(X,Y,Z) = 0 = 3 - 3$.

There seems to be a correspondence between the number of variables and the dimension in the above example. If we define the codimension of $V(f)$ in V by

$$\operatorname{codim} V(f) = \dim V - \dim V(f)$$

then we can rewrite the above example as

- $\operatorname{codim} V(X) = 1$
- $\operatorname{codim} V(X,Y) = 2$
- $\operatorname{codim} V(X,Y,Z) = 3$

This looks very much like an equality between the number of equation and the codimension of the generated variety.

Question: When does this correspondence hold true in general?

Let's try to solve this problem. Our goals are

1. Show that $\operatorname{codim} V(f)$ "should" be 1.
2. Give some relation between $\operatorname{codim} W$ and the number of equations defining W.

Proposition 17.4: Two extremal cases

1. The set $V(f)$ is empty if and only if f is invertible in $\Gamma(V)$.
2. The set $V(f)$ contains an irreducible component if and only if f is a zero devisor in $\Gamma(V)$.

Proof. 1. Assume $V(f) = \emptyset$. Recall that $\Gamma(V) = k[X_1, /ldots, X_n]/I(V) = k[X_1, \ldots, X_n]/\sqrt{I}$. Denote the image of f in this algebra by F. We know $V(f) \subseteq V(I)$, hence $\sqrt{I} \subseteq \sqrt{(F)}$. The weak nullstellensatz gives us that if $\sqrt{(F)} \subsetneq k[X_1, \ldots, X_n]$, then $V(f) \neq \emptyset$, hence we must have

$$\sqrt{(F)} = k[X_1, \ldots, X_n]$$

This means that there exists a unit u such that $u^m = F \cdot G$ for some m. This is still a unit, which means that F is invertible, which again means that f is invertible.

Assume now that f is invertible in $\Gamma(V)$. This means that the ideal generated by it is the whole ring, i.e. $(f) = \Gamma(V)$. Hence we have that $I(V(f)) = \sqrt{(f)} = \Gamma(V)$, which is only the case for $V(f) = \emptyset$.

2. Assume $V(f)$ contains an irreducible component. Then we can find a g that vanishes on another component. This gives us that $f \cdot g = 0$, and hence that f is a zero devisor.

   Assume now that f is a zero devisor. Hence there exists a $g \neq 0$ such that $f \cdot g = 0$.Thus $V = V(f) \cup V(g)$, where $V(g) \neq V$ as $g \neq 0$.

   If V_i is an irreducible component of V, then

   $$V_i = (V(f) \cap V_i) \cup (V(g) \cap V_i).$$

   Since V_i is irreducible on of these must be empty, meaning that either $V_i \subset V(f)$ or $V(i) \subset V(g)$. But not all irreducible components V_i can be a subset of $V(g)$ as we know that $V(g) \neq V$. Hence there must exist some i such that $V_i \subseteq V(f)$.

□

Details on part 2 forward direction

Definition 17.5

An algebraic variety X is called equidimensional if all of its irreducible components have the same dimension.

Theorem 17.6: Geometric Krull's principal ideal theorem

Let V be an equidimensional affine algebraic variety of dimension n. Let further $f \in \Gamma(V)$ be non-invertible and not a zero devisor. Then $V(f)$ is an equidimensional affine algebraic variety with $\operatorname{codim} V(f) = 1$.

Theorem 17.7: Algebraic Krull's principal ideal theorem

Let A be a commutative noetherian ring and $f \in A$ be non-invertible and not a zero devisor. Then a minimal prime P over (f) has height 1.

Here being a minimal prime over (f) means that $(f) \subseteq P$ but there are no other prime ideals between them. Having height 1 means that P only contains one other prime ideal.

Corollary 17.8

If (A, M) is a local ring and $f \in M$ is not a zero devisor, then

$$\operatorname{krull.dim} A/(f) = \operatorname{krull.dim} A - 1$$

Corollary 17.9

Let V be an equidimensional affine algebraic variety of dimension n and let $f_1, \ldots, f_r \in \Gamma(V)$. If W is an irreducible component of $V(f_1, \ldots, f_r)$, then $\operatorname{codim} W \leq r$.

Proof. Induct on the codimension r. □

Example 17.10 Let $V = V(X, Y)$ and $f = X(X + Y + 1)$. Then f is a zero devisor and $V(f)$ is a non-equidimensional variety consisting of a line and a point.

Proposition 17.11

Let X and Y be irreducible algebraic varieties in k^n of dimension r and s respectively. Then every irreducible component of $X \cap Y$ has dimension greater than or equal to $r + s - n$.

Question: If $W \subseteq V$ has codimension r, can we define W by r equations?

Proposition 17.12

Let V be an irreducible affine algebraic variety such that $\Gamma(V)$ is a unique factorization domain (UFD). Let $W \subseteq V$ be a closed irreducible subset of codimension 1. Then there exists an element $f \in \Gamma(V)$ such that $W = V(f)$.

Proof. Since W is irreducible we know that $I(W)$ is a prime ideal in $\Gamma(V)$. Irreducible subsets of W are in one-to-one correspondence with prime ideals in $\Gamma(V)$ containing $I(W)$, which by codimension 1 gives us that the height of $I(W) \leq 1$.

We claim that $I(W)$ is a principal ideal.

Let $0 \neq g \in I(W)$. Since we are in a UFD we can factorize g into $g = u \cdot f_1 \cdot \ldots \cdot f_t$, where f_i are irreducible. Since $I(W)$ is prime then $f_i \in I(W)$. Since we are in a UFD then (f_i) is a prime ideal. Hence we have $(0) \subsetneq (f_i) \subseteq I(W)$. But, this can't happen as we have height 0 or 1. Hence there must be some i such that $(f_i) = I(W)$. This then gives us that

$$W = V(I(W)) = V(f_i)$$

which is what we wanted. □

Proposition 17.13

Let V be an irreducible affine agebraic variety, and W an irreducible affine subvariety with $\operatorname{codim} W = r > 1$. Then there exists $f_1, \ldots, f_r \in \Gamma(V)$ such that W is an irreducible component of $V(f_1, \ldots, f_r)$.

Problem 17.14 The elements $f_1, \ldots, f_r$ in the above proposition are called a system of parameters. Are these related in any meaningful way to the more standard notion of a system of parameters used in algebra?

18 Lecture 18 - 16.03.21

18.1 Morphisms and dimension

Let $\phi\colon X \longrightarrow Y$ be a morphism of irreducible algebraic varieties. In the more general case of non-irreducible varieties we can always decompose into its irreducible components, so choosing these looses no generality. Recall that a morphism of algebraic varieties is given by

$$\begin{aligned} \phi\colon X &\longrightarrow Y \\ \phi^{\#}\colon \mathscr{O}_X &\longrightarrow \phi_* \mathscr{O}_Y \end{aligned}$$

where ϕ is a continuous map of topological spaces and $\phi^{\#}$ is a morphism of sheaves of rings.

Let $y \in Y$. We call the set $\phi^{-1}(y) = \{x \in X | \phi(x) = y\}$ the fiber of y.

Our goal is to compare $\dim X$ and $\dim Y$. These will relate through $\dim \phi^{-1}(y)$.

Example 18.1 Let $X = k^{n+d}$, $Y = k^n$ and $\pi\colon X \longrightarrow Y$ be the projection onto the first n coordinates, i.e. $(x_1, \ldots, x_{n+d}) \mapsto (x_1, \ldots, x_n)$. Then $\dim X = n + d$, $\dim Y = n$ and

$$\begin{aligned} \dim \pi^{-1}(a_1, \ldots, a_n) &= \dim\{(a_1, \ldots, a_n, x_{n+1}, \ldots, x_{n+d})\} \\ &= \dim k^d \\ &= d \end{aligned}$$

This gives us that $\dim X = \dim Y + \dim \pi^{-1}(a_1, \ldots, a_n)$.

Question: Does this always work, i.e. do we always have for $y \in Y$ that

$$\dim X = \dim Y + \dim \phi^{-1}(y) \tag{1}$$

Example 18.2 (Counterexample)**.**

Consider algebraic varieties X and Y and fix some $y \in Y$. Define $\phi(x) = b$ for all $x \in X$. Then

$$\phi^{-1}(y) = \begin{cases} \emptyset \text{ if } y \neq b \\ X \text{ if } y = b \end{cases}$$

Hence we have that $\dim X = \dim \phi^{-1}(b)$. If we want to fulfill eq. (1), then we must have $\dim Y = 0$, but we can in this case choose Y to be any dimension we want.

In hindsight the obvious reason that the above counter example does not work while the projection example works is due to ϕ not being surjective. So a possible fix is to add a surjectivity criteria. This turns out to actually be a bit too strong of a requirement.

Example 18.3 Let $V = V(XY - 1) \subseteq k^2$ and $\phi\colon V \longrightarrow k$ the projection $\phi(x, y) = x$. What is $\dim \phi^{-1}(x)$?

If $x = 0$ then $\phi^{-1}(x) = \emptyset$, while if $x \neq 0$ then there exists a unique $y \in k$ that is an inverse to x, i.e. $xy = 1$. Hence there is a unique preimage, meaning that $\dim \phi^{-1}(x) = 0$.

Hence eq. (1) holds for all $x \neq 0$.

Notice that we have $\overline{\operatorname{Im} \phi} = k$.

Example 18.4 Let $V = V(XZ - Y) \subseteq k^3$ and $\psi\colon V \longrightarrow k^2$ be the projection $\psi()x, y, z) = (x, y)$. What is $\dim \psi^{-1}(x, y)$?

If $x = 0$ then we must also have $y = 0$ to have a non-empty inverse image. But we can let z vary as we want, thus the inverse image of $(0, 0)$ is one-dimensional. If $x \neq 0$ and $y = 0$, then we must have $z = 0$, meaning that the inverse image of $(x, 0)$ is zero-dimensional. For both x and y non-zero, then we can find a unique z in the preimage, given by $z = x^{-1}y$.

Hence eq. (1) holds for all $x \neq 0$.

Notice that we also here have $\overline{\operatorname{Im} \psi} = k^2$. This prompts the following definition.

Definition 18.5: Dominant morphism

A morphism $\phi\colon X \longrightarrow Y$ between two algebraic varieties X, Y is called dominant if $\overline{\operatorname{Im} \phi} = Y$.

The moral of the above two examples and the definition is that if we have a dominant morphism, then the "general" or "typical" fiber will have the expected dimension, but fibers over "special" points might be different.

Lemma 18.6

Assume $\phi\colon X \longrightarrow Y$ is a dominant morphism of irreducible algebraic varieties and let $y \in \operatorname{Im} \phi$ and Z be an irreducible component of $\phi^{-1}(y)$. Then there are non-empty affine algebraic sets $U \subseteq X$ and $V \subseteq Y$ such that:

1. $\phi(U) \subseteq V$
2. $\phi_{|U}\colon U \longrightarrow V$ is dominant
3. $y \in V$

4. $Z \cap U \neq \emptyset$.

Proof. Let $V \subseteq Y$ be an open affine set with $y \in V$. Then $\phi^{-1}(V) \subseteq X$ is open, and $Z \subseteq \phi^{-1}(y) \subseteq \phi^{-1}(V)$. For any point $z \in Z$ there is an open affine set $U \subseteq \phi^{-1}(V)$ such that $z \in U$, meaning that $Z \cap U \neq \emptyset$.

We claim that $\phi_{|U}$ is dominant. Let $W \subseteq V$ be a non-empty open set (not necessarily affine). If we can show that $W \cap \phi_{|U}(U) \neq \emptyset$ then we have shown that it is dense in V, which means it is dominant. As ϕ itself is dominant then we know that $W \cap \phi(X) \neq \emptyset$ and hence that $\phi^{-1}(W) \subseteq X$ is non-empty and open. Since X is irreducible we know that $\phi^{-1}(W) \cap U \neq \emptyset$, which means that there exists an element $w \in W$ such that $\phi(x) \in W \cap \phi_{|U}(U)$. This means that $W \cap \phi_{|U}(U) \neq \emptyset$ and hence that $\overline{\phi_{|U}(U)} = V$, which is the definition of being dominant. By that we are done. □

A consequence of this lemma is that we can in most cases reduce to affine algebraic varieties when proving stuff about dimension. This is because $\dim X = \dim U$, $\dim Y = \dim V$ and $\dim Z = \dim(Z \cap U)$, where then $Z \cap U$ is an ireducible component of $\phi_{|U}^{-1}(y)$.

Another one is that if $\phi \colon X \longrightarrow Y$ is a dominant morphism, then $\dim Y \leq \dim X$. This is because ϕ being dominant gives us that $\phi_* \colon \Gamma(Y) \longrightarrow \Gamma(X)$ is injective, which again implies that

$$\dim Y = \operatorname{tr.deg}_k \operatorname{Fr}(Gamma(Y)) \leq \operatorname{tr.deg}_k \operatorname{Fr}(\Gamma(X)) = \dim X$$

Theorem 18.7: The dimension theorem

Let $\phi \colon X \longrightarrow Y$ be a dominant morphism of irreducible algebraic varieties.

1. Let $y \in Y$. Every irreducible component of $\phi^{-1}(y)$ has dimension at least $\dim X - \dim Y$.
2. There exists a non-empty open set $U \subseteq \phi(X)$ such that for all $y \in U$ we have $\dim \phi^{-1}(y) = \dim X - \dim Y$.

In fact, every irreducible component of $\phi^{-1}(y)$ is of dimension $\dim X - \dim Y$.

Proof. We only sketch the proof of part 2:

By the previous lemma we can assume that X and Y are affine algebraic varieties. When ϕ is dominant we have that $\phi_* \colon \Gamma(Y) \longrightarrow \Gamma(X)$ is injective. The algebra $\Gamma(X)$ is of finite type over k and by ϕ_* injective it is also of finite type over $\Gamma(Y)$. Hence

$$\Gamma(X) = \Gamma(Y)[b_1, \ldots, b_r, \ldots, b_n]$$

where $b_1, \ldots, b_r$ is a trancendence basis of $\Gamma(X)_{(0)}$ over $\Gamma(Y)_{(0)}$.

We have

$$\begin{aligned} \Gamma(Y)[b_1, \ldots, b_r] &\cong \Gamma(Y) \otimes_k k[T_1, \ldots, T_r] \\ &= \Gamma(Y) \otimes_k \Gamma(K^r) \end{aligned}$$

Which means that $\Gamma(Y)[b_1, \ldots, b_r] = \Gamma(Z)$ for the algebraic variety $Z = Y \times k^r$. Hence we have

$$\begin{tikzcd} X \arrow[rr, "\phi"] \arrow[dr, "\psi"'] & & Y \\ & Z \arrow[ur, two heads, "\pi"'] & \end{tikzcd}$$

where π is the projection onto Y. Since ϕ is dominant, then ψ has to be as well.

1. First show that there exists a non-empty open set $U \subseteq \phi(X)$. It suffices to show that there exists a non-empty open set $\Omega \subseteq \psi(X)$, as we have that $\pi(\Omega) \subseteq \phi(X)$.
2. Using the algebraically dependent elements $b_{r+1}, \ldots, b_n$ we get equations

$$c_{n_i,i} b_i^{n_i} + \cdots + c_0 = 0$$

for all $i = r+1, \ldots, n$ where the coefficients lie in $\Gamma(Y)[b_1, \ldots, b_r]$ and the first coefficient is non-zero. Set $0 \neq f = \Pi_{i=r+1}^{n} c_{n_i,i} \in \Gamma(Y)[b_1, \ldots, b_r]$ and let $\Omega = D_Z(f) \subseteq Z$.

Then $\psi^{-1}(D_Z(f)) = D_X(\psi^*(f) \subseteq X$.

The map $\Gamma(Y)[b_1, \ldots, b_r] \longrightarrow \Gamma(X)$ induces an integral extension

$$\Gamma(Y)[b_1, \ldots, b_r]_f \longrightarrow \Gamma(X)_{\psi^* f}$$

hence we get a map

$$\psi : D_X(\psi^* f) \longrightarrow D_Z(f)$$

which is surjective by the going up theorem.

This shows an outline of the proof of existence of almost the set we wanted, but we need to find another set with the correct dimension. Then we intersect these to get the set U.

$\square$

19 Lecture 19 - 22.03.21

Add drawing of tangent space for intuition

For most points on a algebraic variety V we expect that $\dim T_a W = \dim V$, i.e. that the dimension of the tangent space at that point is the same as the dimension of the variety. However it can also happen that $\dim T_b W \geq \dim W$. Such points b will be called a singular point, while a will be non-singular, or regular.

We can also study these properties through $\mathscr{O}_{V,a}$ and $\mathscr{O}_{W,b}$. Both these will be local rings, but $\mathscr{O}_{V,a}$ will be regular and $\mathscr{O}_{W,b}$ will not be. This is also the justification for the name regular.

19.1 Motivation for tangent spaces

Let $f(x_1, \ldots, x_n) \in C^\infty$ i.e. a real valued infinitely differentiable function. Let also $S = V(f) \subseteq \mathbb{R}^n$ and $a \in S$.

Question: What is the tangent space of S at a?

We can use Taylors formula at $x = a + h$ to get

$$f(x) = f(a) + \sum_{i=1}^{n} h_i \frac{\partial f}{\partial x_i}(a) + \frac{1}{2} \sum_{i,j} h_i h_j \frac{\partial^2 f}{\partial x_i \partial x_j}(a) + \ldots$$

Since $a \in S$ we know that $f(a) = 0$. This gives us an approximation of f by a "tangent hyperplane". We ignore the higher order terms and look only at

$$\sum_{i=1}^{n} h_i \frac{\partial f}{\partial x_i}(a_1, \ldots, a_n) = \sum_{i=1}^{n} (x_i - a_i) \frac{\partial f}{\partial x_i}(a_1, \ldots, a_n) = 0$$

Example 19.1 Let $V = V(Y - X^2) \subseteq \mathbb{R}^2$ and $a = (0, 0)$.

drawing of V

Then we have

$$(x - 0)\frac{\partial f}{\partial x}(0) + (y - 0)\frac{\partial f}{\partial y}(0) = 0.$$

This gives us

$$x(2x)(0) + y(1)(0) = y = 0$$

meaning that we must have $y = 0$, as would be the expected tangent space.

In the more standard setting of Euclidean space we can use wording like h is small, or converges to zero. In algebraic geometry we replace "small/close" by something called infinitesimal deformations in order to have greater generality and applicability to more fields than just $\mathbb{R}$. The idea is to replace $a + h$ by $a + b\epsilon$ where $b \in k^n$ and $\epsilon \neq 0$ such that $\epsilon^2 = 0$. Hence we work in $k[\epsilon] = k[x]/x^2$.

Note that $k[\epsilon]$ is not a reduced algebra, hence it does not actually correspond to an algebraic variety itself.

19.2 Tangent spaces

Let V be an affine algebraic variety with $x \in V$. Recall that points in algebraic varieties correspond to maximal ideals in $\Gamma(V)$. These maximal ideals are kernels of morphisms of algebras $\Gamma(V) \longrightarrow k$, more precisely of the characters, i.e. maps $\chi_x(f) = f(x)$.

So an inclusion $x \to V$ correspond to projection $\chi_x \colon \Gamma(V) \longrightarrow k$.

Intuitively we want to replace "point" P by "fat point" P_ϵ in order to gain a notion of "closeness".

Here "fat points" P_ϵ corresponds to the algebra $\Gamma(P_\epsilon) = k[\epsilon] = k[x]/x^2$.

Definition 19.2: Deformation

A deformation of an affine algebraic variety V at a point x is a morphism of ringed spaces

$$t \colon P_\epsilon \longrightarrow V$$

given by $t(P_\epsilon) = x$. The set of deformations of V at x is denoted by $\mathrm{Def}(V, x)$.

Equivalently we can define a deformation in terms of $\Gamma(V)$. It is then defined as a k-algebra morphism

$$t^* \colon \Gamma(V) \longrightarrow k[\epsilon]$$

such that

$$\begin{tikzcd} \Gamma(V) \arrow[rr, "t^*"] \arrow[dr, "\chi_x"'] & & k[\epsilon] \arrow[dl, "\pi"] \\ & k & \end{tikzcd}$$

commutes. The set of deformations is then denoted by $\mathrm{Def}(\Gamma(V), x)$.

Since the above diagram commutes, and π is a projection, we can write

$$t^*(f) = f(x) + V_t(f)\epsilon$$

where $V_t(f) \in k$, and hence that $t^* = \chi_x + V_t\epsilon$.

Note that

$$\begin{aligned} t^*(fg) &= t^*(f)t^*(g) \\ &= (f(x) + V_t(f)\epsilon)(g(x) + V_t(g)\epsilon) \\ &= f(x)g(x) + f(x)V_t(g)\epsilon + g(x)V_t(f)\epsilon + V_t(f)V_t(g)\epsilon^2 \\ &= f(x)g(x) + f(x)V_t(g)\epsilon + g(x)V_t(f)\epsilon \end{aligned}$$

and hence that

$$V_t(fg) = f(x)V_t(f) + g(x)V_t(f)$$

which is a certain kind of nice function called a derivation.

Definition 19.3: Derivation

Let A be a k-algebra and M an A-module. A map $D\colon A \longrightarrow M$ is a derivation if

- D is k-linear
- We have $D(ab) = aD(b) + bD(a)$

We denote the set of derivations between A and M by $\mathrm{Der}(A, M)$.

Example 19.4 The map

$$\begin{aligned} k[x_1, \dots, x_n] &\longrightarrow k[x_1, \dots, x_n] \\ F &\longmapsto \frac{\partial F}{\partial X_i} \end{aligned}$$

is a derivation. This is also the intuition for the name.

Definition 19.5: Tangent space

Let V be an affine algebraic variety and $x \in V$. The tangent space of V at x is defined by

$$T_x V = \mathrm{Der}(\Gamma(V), k)$$

where the module action of $\Gamma(V)$ on k is defined by $f \cdot \lambda = f(x)\lambda$.

Proposition 19.6

The tangent space of an algebraic variety V at a point x is also given by $\mathrm{Def}(V, x)$.

Proof.

$$\mathrm{Def}(V, x) \cong Def(\Gamma(V), x) \cong \mathrm{Der}_k(\Gamma(V), k) = T_x V$$

where the maps are given by $t \longmapsto t^* \longmapsto V_t$. □

Note that a morphism $\phi\colon V \longrightarrow W$ of affine algebraic varieties induces morphisms

$$\begin{aligned} \mathrm{Def}(V, x) &\longrightarrow \mathrm{Def}(W\phi(x)) \\ t &\longmapsto t\phi \end{aligned}$$

and

$$\begin{aligned} T_xV &\longrightarrow T_{\phi(x)}W \\ V_t &\longmapsto V_t\phi^* \end{aligned}$$

Example 19.7 Let $V = V(Y - X^2) \subseteq \mathbb{R}^2$ and $a = (0,0)$. Then

$$\begin{aligned} T_{(0,0)}V(Y - X^2) &= \mathrm{Der}(\mathbb{R}[X,Y](Y - X^3), \mathbb{R}) \\ &\cong \mathrm{Def}(\mathbb{R}[X,Y](Y - X^3), (0,0)) \end{aligned}$$

where $\mathrm{Def}(\mathbb{R}[X,Y](Y - X^3), (0,0))$ is the set of k-algebra morphisms such that $X \longmapsto b\epsilon$ and $Y \longmapsto c\epsilon$. But notice that $Y - X^2$ projects to $c\epsilon - b^2\epsilon^2 = c\epsilon$ in k but gets sent to 0 in $k[\epsilon]$, hence we must have $c = 0$.

This gives us that $T_{(0,0)}V = \{(b,0)\} = V(Y)$, which is what we expected, and what we got in the motivation as well.

Example 19.8 Let now $V = k^n$ and $a = (a_1, \dots, a_n)$. Then

$$\begin{aligned} T_ak^n &\cong \mathrm{Def}(k[X_1, \dots X_n], k[\epsilon]) \\ &= \{t^* \colon k[X_1, \dots, X_n] \to k[\epsilon], X_i \mapsto a_i + b_i\epsilon\} \\ &\cong \{(b_1, \dots, b_n) \in k^n\} \qquad = k^n \end{aligned}$$

Example 19.9 Let $V \subseteq k^n$ be an algebraic variety with $I(V) = (F_1, \dots, F_r)$ and $a = (a_1, \dots, a_n) \in V$. We want to find the tangent space T_aV.

Consider the deformation

$$\begin{tikzcd} k[X_1, \dots, X_n]/(F_1, \dots, F_r) \arrow[rr, "t^*"] \arrow[dr, "\chi_x"'] & & k[\epsilon] \arrow[dl, "\pi"] \\ & k & \end{tikzcd}$$

with $t^*(x_i) = a_i + b_i\epsilon$ and such that $F_j(a + b\epsilon) = 0$. Then

$$\sum_{i=1}^{n} b_i \frac{\partial F_j}{\partial X_j}(a_1, \dots, a_n) = 0$$

for all $j = 1, \dots, r$. But notice that this is exactly the condition that the Jacobian of $(F_1, \dots F_r)$ is zero in a. Recall that the Jacobian is the matrix

$$J_x(F_1, \dots, F_r) = \begin{bmatrix} \frac{\partial F_1}{\partial X_1}(x) & \cdots & \frac{\partial F_1}{\partial X_n}(x) \\ \vdots & & \vdots \\ \frac{\partial F_r}{\partial X_1})(x) & \cdots & \frac{\partial F_r}{\partial X_n}(x) \end{bmatrix}$$

which is a map $k^n \longrightarrow k^r$. Hence we have

$$T_aV = \text{Ker } J_a(F_1, \dots F_r)$$

This is an important observation, and it makes it much simpler to calculate tangent spaces of varieties. Lets try this method on the earlier example of $V = V(Y - X^2)$. We get

$$\begin{aligned} T_(0,0)V(Y - X^2) &= \text{Ker } J_{(0,0)}(Y - X^2) \\ &= \text{Ker}[-2x, 1] \\ &= [0, 1] \\ &= V(Y) \end{aligned}$$

which is exactly the same as we have gotten before, but this time the calculation was much simpler.

Problem 19.10 Try repeating this for another couple curves.

Proposition 19.11

Let V be an affine algebraic variety and $x \in V$ with corresponding maximal ideal m_x. Then there is a isomorphism of vector spaces

$$T_xV \cong Hom_k(m_x/m_x^2, k).$$

Proof. Let $v\colon \Gamma(V) \longrightarrow k$ be an element of T_xV. Let $v_{|m_x} : m_x \longrightarrow k$ be its restriction to the maximal ideal corresponding to x.

Note that for $f, g \in m_x$ we have $v(fg) = f(x)v(g) + g(x)v(f) = 0$ because $f(x) = 0 = g(x)$. hence there exists a map $\overline{v}\colon m_x/m_x^2 \longrightarrow k$.

On the other hand, if $\theta \in Hom_k(m_x/m_x^2, k)$ then define $v(f) = \theta(\overline{f - f(x)})$. It can be checked that this is a derivation and thus an element in the tangent space. □

Corollary 19.12

If $(\mathscr{O}_{V,x}, m_{V,x})$ is the local ring of V at x, then $T_xV = Hom_k(m_{V,x}/m_{V,x}^2, k)$.

Problem 19.13 Note that $m_{V,x} = m_x\mathscr{O}_{V,x}$ so there is actually something to prove in the above corollary. Do this proof.

This now means that the tangent space only depends on the local ring of V at x, which maybe should not be surprising as it should intuitively model local behaviour of the variety. This also means that we can define tangent spaces for arbitrary algebraic varieties, not just affine ones.

20 Lecture 20 - 23.03.21

Definition 20.1

Let V be an irreducible algebraic variety and $x \in V$. We say x is smooth or regular, or that V is non-singular at x if

$$\dim V = \dim_k T_x V.$$

A point that is not non-singular is called singular.

Note: We always have $\dim V \leq \dim_k T_x V$ and $\dim_x V \leq \dim_k T_x V$, where

$$\dim_x V = \sup\{\dim V_i\}$$

where V_i is an irreducible component or V containing x.

Example 20.2 Let $V = V(Y^2 - X^3)$. We have $\operatorname{krull.dim}(k[X,Y]/(Y^2 - X^3)) = 1$, and hence that $\dim V = 1$. Lets see how the tangent spaces relate to this dimension. The tangent space at the origin is given by

$$\begin{aligned} T_{(0,0)}V &= \operatorname{Ker} J_{(0,0)}(Y^2 - X^3) \\ &= \operatorname{Ker}[-3x^2(0), 2y(0)] \\ &= \operatorname{Ker}[0, 0] \\ &= k^2 \end{aligned}$$

while the tangent space at some non zero point $(a, b) \neq (0, 0)$ is given by

$$\begin{aligned} T_{(a,b)}V &= \operatorname{Ker} J_{(a,b)}(Y^2 - X^3) \\ &= \operatorname{Ker}[-3a^2, 2b] \\ &\cong k \end{aligned}$$

We see that $(0, 0)$ is the only singular point.

Theorem 20.3

Let $V \subseteq k^n$ be an irreducible affine algebraic variety. Assume $I(V) = (F_1, \ldots, F_r)$. Then V is non-singular at a point x if and only if $\operatorname{rank} J_x(F_1, \ldots, F_r) = n - \dim V$.

Proof. We know that V is non-singular at x if $\dim V = \dim_k T_x V$, which is given by

$$\begin{aligned} \dim_k T_x V &= \dim_k \operatorname{Ker} J_x(F_1, \ldots, F_r) \\ &= n - \operatorname{rank} J_x(F_1, \ldots, F_r) \end{aligned}$$

where the last equality is due to the rank-nullity theorem from linear algebra. $\square$

Corollary 20.4

If $F(X,Y)$ is a polynomial without any common factors, i.e. $I(V(F)) = (F)$, then a point (a,b) is singular in $V(F)$ if and only if

$$\frac{\partial F}{\partial X}(a,b) = 0 = \frac{\partial F}{\partial Y}(a,b)$$

Proof. The proof is an exercise. $\square$

Do the proof

Example 20.5 Let $F(X,Y) = X^3 + Y^3 - XY$ and $V = V(F)$. Then both

$$\frac{\partial F}{\partial X} = 3X^2 - Y$$
$$\frac{\partial F}{\partial Y} = 3^2 - X$$

are zero at the point $(0,0)$. Thus V is singular at $(0,0)$.

Proposition 20.6

Let $V \subseteq \mathbb{P}^n(k)$ be an irreducible projective algebraic variety and $x = [x_0 : \dots : x_n] \in V$ with $x_0 \neq 0$. Let further $I(V) = (F_1, \dots, F_r)$ with F_i homogeneous.

Define

$$A(x) = \begin{bmatrix} \frac{\partial F_1}{\partial X_0}(x) & \cdots & \frac{\partial F_1}{\partial X_n}(x) \\ \vdots & & \vdots \\ \frac{\partial F_r}{\partial X_0}(x) & \cdots & \frac{\partial F_r}{\partial X_n}(x) \end{bmatrix}$$

Then V is non-singular at x if and only if $\operatorname{rank} A(x) = n - \dim V$.

Proof. As $x_0 \neq 0$ we can without loss of generality assume that $x = [1 : x_1 : \dots : x_n]$. Observe that x is regular if and only if $(x_1, \dots, x_n)$ is a regular point of $V^b = V \cap D^+(X)$ and $I(V^b) = (F_1^b, \dots, F_r^b)$. So $(x_1, \dots, x_n)$ is regular in V^b if and only if

$$\operatorname{rank} J_{(x_1,\dots,x_n)}(F_1^b, \dots, F_r^b) = n - \dim V$$

Note that if F is a homogeneous polynomial, then

$$\frac{\partial F}{\partial X_j}(x) = \frac{\partial F^b}{\partial X_i}(x_1, \dots, x_n)$$

and so we get that

$$A(x) = \left[\frac{\partial F_i}{\partial X_0} \quad | \quad J_{(x_1,\dots,x_n)}(F_1^b, \dots, F_r^b)\right]$$

Hence we must have $\operatorname{rank} A(x) \geq \operatorname{rank} J_{(x_1,\ldots,x_n)}(F_1^b,\ldots,F_r^b)$. Then Euler's formula gives us that

$$d \cdot F = \sum_{j=0}^{n} x_j \frac{\partial F}{\partial X_j}.$$

Now, if $\deg F_i = d_i$, then

$$d_i F_i = \sum_{j=1}^{r} x_j \frac{\partial F_i}{\partial X_j}(x)$$

which gives us that

$$\frac{\partial F_i}{\partial X_0}(x) = d_i F_i(x) - \sum_{j=1}^{n} x_j \frac{\partial F_i}{\partial X_j}(x)$$

Which again must mean that the first column of $A(x)$ is given by

$$\begin{bmatrix} A_1 \end{bmatrix} = -\sum x_j \begin{bmatrix} \frac{\partial F_1}{\partial X_j}(x) \\ \vdots \\ \frac{\partial F_r}{\partial X_j}(x) \end{bmatrix}$$

which means that $\operatorname{rank} A(x) = \operatorname{rank} J_{(x_1,\ldots,x_n)}(F_1^b,\ldots,F_r^b) = n - \dim V$. $\square$

Problem 20.7 Let $V = V(Y^2T - X(X-T)(X-\lambda T) \subseteq \mathbb{P}^2(k)$. For what values of λ is V smooth?

Solution:

We look at $T = 1$ in affine space. We have

$$J_x(Y^2T - X(X-1)(X-\lambda) = [-3X^2 + 2(\lambda-1)X + \lambda, 2Y](x)$$

For $x = (0,0)$ we have

$$\begin{aligned} J_{(0,0)}(Y^2T - X(X-1)(X-\lambda) &= [-3X^2 + 2(\lambda-1)X + \lambda, 2Y](0,0) \\ &= [\lambda, 0] \end{aligned}$$

So if we set $\lambda = 0$, then $\operatorname{Ker} J_{(0,0)} = k^2$ which means that V is singular at that point.

For $x = (1,0)$ we have

$$\begin{aligned} J_{(0,0)}(Y^2T - X(X-1)(X-\lambda) &= [-3X^2 + 2(\lambda-1)X + \lambda, 2Y](1,0) \\ &= [-3 + 2(\lambda-1) + \lambda, 0] \\ &= ?? \end{aligned}$$

Which means if we set $\lambda = 1$ then V is again singular at $(1,0)$.

This means that V is smooth, i.e. consists of only regular points when $\lambda \neq 0$ and $\lambda \neq 1$.

20.1 Regular local rings

Let (A, m, k) be a local ring. Then

1. $m/m^2 (\cong m \otimes_A k)$ is a vector space over k.
2. $\dim_k m/m^2 \geq \operatorname{krull.dim} A$.

Definition 20.8

We say A is a regular ring if we have equality in point 2 above.

Example 20.9 The ring $A = k[[X, Y]]$ with $m = (X, Y)$ is a regular ring.

Note that all regular rings are domains. Also, the number $\dim_k m/m^2$ also equals the minimal number of generators og m. Thus A is regular if and only if m is generated by $\operatorname{krull.dim} A$ number of elements.

Proposition 20.10

Let V be an algebraic variety and $x \in V$. Then V is regular at x if and only if $\mathscr{O}_{V,x}$ is a regular ring.

Proof. This proof is an exercise. □

Do the proof

Proposition 20.11

A point at the intersection of two irreducible components must be singular.

Proof. This proof is an exercise. □

Do the proof

20.2 Curves

Recall that a curve is an equidimensional algebraic variety of dimension 1.

Proposition 20.12

Let C be a curve and $x \in C$. Then x is non-singular if and only if $\mathscr{O}_{C,x}$ is a local pricipal ideal domain (PID).

Theorem 20.13

Let C be an irreducible affine curve. Then C is smooth if and only if $\Gamma(C)$ is integrally closed.

Example 20.14 Let $V = V(Y^2 - X^3)$. We have that $\frac{Y}{X} \in \mathrm{Fr}(\Gamma(V))$ but $(\frac{Y}{X})^2 - X = 0$, hence not an integral element. This means that $\Gamma(V)$ is not integrally closed, and by the last theorem there must exist some point x such that V is singular at x. We already know from an earlier example that this point is $x = (0, 0)$.

21 Lecture 21 - 12.04.21

Recall from the beginning of the course that if C and D are plane curves of degree s and t respectively, then they hopefully intersect in $s \cdot t$ points. We found some obstructions for this being the case, so we have to be a bit careful. These obstructions were:

1. No common components
2. The field k must be algebraically closed
3. The curves must be projective curves
4. We need to count intersection with multiplicity

So far in this course we have covered points 1, 2 and 3, so in order to make Bézout's theorem precise we now turn our heads to point 4.

21.1 Intersection multiplicity

Lets start by examining an example. Let our two plane curves be defined by $C = V(Y - X^2$ and $D_\lambda = V(Y - \lambda$ for some $\lambda \in k$. We then have

$$C \cap D_\lambda = V(Y - X^2, Y - \lambda)$$

Let $I_\lambda = (Y - \lambda, X^2 - \lambda)$ and $A_\lambda = k[X,Y]/I_\lambda$. Note that $A_\lambda \cong k[X]/(X^2 - \lambda)$.

We have two cases, $\lambda \neq 0$ or $\lambda = 0$. If $\lambda \neq 0$ then there is some $a \in k$ such that $\lambda = a^2$, which means

$$A_\lambda = k[X]/(X^2 - a^2) = k[X]/(X - a)(X + a) \cong k \times k$$

The last isomorphism is given by

$$\begin{aligned} k[X]/(X-a)(X+a) &\longrightarrow k \times k \\ 1 &\longmapsto (1,1) \\ x &\longmapsto (a, -a) \end{aligned}$$

Check that this is a ring iso

Hence $C \cap D_\lambda$ has the structure of an affine algebraic variety. It consists of two points, as A_λ is a 2-dimensional vector space.

If $\lambda = 0$ then

$$A_0 = k[X,Y]/(X^2 - Y, Y) \cong k[X]/(x^2) = k[\epsilon]$$

which gives us that $I(C \cap D_0) = \sqrt{I_0} = (X, Y)$. This means that $C \cap D_0$ only has one point, as its coordinate ring is just k.

So what is the solution to this problem? We must define $C \cap D_0$ as a finite scheme, rather that a variety. This will allow us to have coordinate ring A_0, which is 2-dimensional, which means we have the correct number of points.

Definition 21.1: Finite scheme

A finite scheme $(Z, \mathscr{O}_Z)$ is a ringed space such that Z is a finite discrete set, and $\mathscr{O}_Z(P)$ is a local k-algebra that is finite dimansional as a vector space for each point $P \in Z$.

Definition 21.2

Let $(Z, \mathscr{O}_Z)$ be a finite scheme. The multiplicity of Z at a point $P \in Z$ is

$$\mu_P(Z) = \dim_k \mathscr{O}_Z(P).$$

Problem 21.3 Show that a finite algebraic variety is a finite scheme where all multiplicities are 1.

Proposition 21.4

Let $(Z, \mathscr{O}_Z)$ be a finite scheme. Then for every subset $V \subseteq Z$ we have

$$\Gamma(V, \mathscr{O}_Z) = \prod_{p \in V} \mathscr{O}_Z(P).$$

Moreover, given a finite set Z, if we assign to each point in it a local finite k-algebra, then the formula above defines a finite scheme structure on Z.

Note that for a point $\{P\} \subseteq \{P, Q\}$ we get a projection $\mathscr{O}_Z(P) \times \mathscr{O}_Z(Q) \longrightarrow \mathscr{O}_Z(P)$.

Problem 21.5 Do the above proof. It should follow from generalizing the note above.

Proof

Definition 21.6

Let $(Z, \mathscr{O}_Z)$ be a finite scheme and $A = \mathscr{O}_Z(Z) = \Gamma(Z, \mathscr{O}_Z)$. Then $Z = \operatorname{Spec} A$, so

$$(Z, \mathscr{O}_Z) = (\operatorname{Spec} A, \mathscr{O}_{\operatorname{Spec} A}).$$

Note that a finite scheme $(\operatorname{Spec} A, \mathscr{O}_{\operatorname{Spec} A})$ is an algebraic variety if and only if A is a reduced k-algebra.

Problem 21.7 Find a finite scheme structure on a single point $\{P\} \in k^2$. As a finite scheme structure is an assignement of a finite dimensional local k-algebra to each point, we only need one to have a finite scheme structure.

Solution:

- $(\{P\}, k[X,Y]/(X,Y))$
- $(\{P\}, k[X,Y]/(X^2,Y)$
- $(\{P\}, k[X,Y]/(X^4,Y^7)$

As we wanted to use this new gadget to study the intersection points of curves we need to be able to define an finite scheme structure on the intersection of projective plane curves. We will do this by first looking at defining such a structure on the intersection of affine plane curves.

Let $F, G \in k[X,Y]$ be non-zero and without any common factors. We showed earlier that $Z = V(F,G) = V(F) \cap V(G)$ is finite set, and that $k[X,Y](F,G)$ is a finite k-algebra.

Proposition 21.8

Let $(X, \mathscr{O}_X)$ be an irreducible addine algebraic variety. Set $R = \Gamma(X)$ and let $I \subseteq R$ be an ideal such that $Z = V(I)$ is finite. Set $\mathscr{F} = \widetilde{R/I}$. For a standard open set $D(f) \subseteq X$ we have that

1. $D(f) \cap Z = \emptyset \implies \mathscr{F}(D(f)) = 0$
2. $D(f) \cap Z = \{x\} \implies \mathscr{F}(D(f)) = \mathscr{O}_X(x)/I\mathscr{O}_X(x)$
3. $D(f) \cap Z = \{x_1, \ldots, x_n\} \implies \mathscr{F}(D(f)) = \prod_{i=1}^n \mathscr{O}_X(x_i)/I\mathscr{O}_X(x_i)$.

We can use this to define a sheaf of rings $\mathscr{O}_Z$ on Z by $\Gamma(U, \mathscr{O}_Z) = \prod_{x \in U} \mathscr{O}_X(x)/I\mathscr{O}_X(x)$. In this case $\mathscr{F} = i_* \mathscr{O}_Z$, where $i \colon Z \hookrightarrow X$ is the inclusion. Then we have that $(Z, \mathscr{O}_Z)$ is a finite scheme, which we denote by $\operatorname{Spec} R/I$.

Problem 21.9 Prove the above.

Proof

Let now $F, G \in k[X,Y]$ be non-zero with no common factors and set $Z = V(F,G)$, $I(F,G)$ and $R = k[X,Y]$. Then $(Z, \mathscr{O}_Z)$ is a finite scheme as above. For $P \in k^2$ we have $\mathscr{O}_Z(P) = \mathscr{O}_{k^2}(P)/(F,G)$ and

$$k[X,Y]/(X,Y) = \prod_{P \in Z} \mathscr{O}_{k^2}(P)/(F,G) = \mathscr{O}_Z(Z).$$

Definition 21.10: Intersection multiplicity

The intersection multiplicity of the plane curves F and G is

$$\mu_P(F,G) = \mu_P(Z) =\sim_k (\mathscr{O}_Z(P) = \dim \mathscr{O}_{k^2}(O)/(F,G) = \dim k[X,Y]_P/(F,G)$$

Corollary 21.11

$$\sum_{P \in V(F,G)} \mu_P(F,G) = \dim_k k[X,Y]/(F,G)$$

Note that this does not give us Bézout's theorem, as this may still miss points at infinity. Take for example $F = X$ and $G = X - 1$.

Problem 21.12 What is the intersection multiplicity of $F = X^3 - X^2 - Y$ and $G = Y$ at the points $P = (0,0)$ and $Q = (1,0)$?

Solution:

We have $V(F,G) = V(X^3 - X^2 - Y, Y)$, so

$$k[X,Y]/(X^3 - X^2 - Y, Y) \cong k[X]/(X^3 - X^2) = k[X]/(X^2(X-1)).$$

Now, localizing at the point P is the same as localizing at the ideal (X). This means that we invert everything not in (X), hence $(X-1)$ becomes invertible. We then have

$$k[X]_{(X)}/(X^2(X-1)) \cong k[X]/(X^2)$$

which is 2-dimensional.

If we instead localize at Q, then this is the same as localizing at the ideal $(X-1)$. This means inverting everything not in the ideal, hence X^2 becomes invertible. We then get

$$k[X]_{(X-1)}/(X^2(X-1)) \cong k[X]/(X-1)$$

which is 1-dimensional.

This means that the intersection multiplicity at P is 2, and at Q it is 1.

But, for Bézouts theorem to work we need that the curves are projective, so we need to have a definition for this as well.

Definition 21.13: Projective intersection multiplicity

Let $F, G \in k[X,Y,T]$ be homogeneous, non-zero elements of degree s and t respectively with no common factors. Let $P = (X,Y,1) \in \mathbb{P}^2$ and define the intersection multiplicity of F and G at P by

$$\mu_P(F,G) = \mu_{(X,Y)(F_b,G_b)}.$$

22 Lecture 22 - 13.04.21

22.1 Bézout's theorem

Last time we defined the intersection multiplicity of curves in $\mathbb{P}^2(k)$. This was the last thing we needed to define before proving Bézout's theorem, but there is still a couple small things to check before we do that.

For the rest of the lecture we let

- k be an algebraically closed field
- $F, G \in k[X, Y, T]$ be non-zero with no common components
- $\deg F = s$ and $\deg G = t$.

Lemma 22.1

The projective intersection multiplicity does not depend on the choice of line at ∞. In fact

$$\mathscr{O}_{\mathbb{P}^2}(P)/(F, G)_P \cong \mathscr{O}_{k^2}(P)/(F_b, G_b)$$

which requires no such choice of ∞.

Proof. The morphism $(-)_b\colon k[X, Y, T] \longrightarrow k[X, Y]$ induced an morphism

$$(-)_b\colon k[X, Y, T]_{I(P)} \longrightarrow k[X, Y]_{m_p},$$

which restricts to an isomorphism

$$(-)_b\colon \mathscr{O}_{\mathbb{P}^2}(P) \longrightarrow k[X, Y]_{m_p}.$$

□

Details

We need some more facts about the $(-)_b$ and $(-)^\#$ operations.

Lemma 22.2

1. $(pq)^\# = p^\# q^\#$
2. $(p^\#)_b = p$
3. If p is homogeneous, then $p = T^r(P_b)^\#$
4. If p is homogeneous and $p_b = 0$, then $p = 0$.

Problem 22.3 Prove the above statements.

proof

Theorem 22.4: Bézout's theorem

Let $F, G \in k[X, Y, T]$ be non-zero, homogeneous of respective degree s and t such that they have no common components. Then

$$\sum_{P \in V(F,G)} \mu_P(F, G) = s \cdot t$$

.

We will proove the theorem using a sequence of lemmas and smaller results.

In order for Bézout's theorem to work we need to know that the intersection of to curves give only a finite number of points. We had this for affine curves, but we have not yet proved it for projective ones.

Lemma 22.5

$V(F) \cap V(G)$ is finite.

Proof. Take $T = 0$ to be the line at ∞ and denote it by D_∞.

We start by looking at the points at infinity.

There are two possibilities; $V(F) \cap D_\infty$ and $V(G) \cap D_\infty$ are either finite, or equal to D_∞. If $V(F) \cap D_\infty = D_\infty$, then I divides F. Similarly, if $V(G) \cap D_\infty = D_\infty$, then I divides G. We know that F and G have no common components, so both of these cant be true at the same time. Hence at least one of them is finite, which means that $V(F) \cap V(G) \cap D_\infty$ is finite.

For the affine points we identify $\mathbb{P}^2 \setminus D_\infty$ with k^2 as usual. It can be checked that $V(F) \cap k^2 = V(F_b)$, and similarly $V(G) \cap k^2 = V(G_b)$. By point 1) in the previous lemma we know that F_b and G_b still have no common components. Thus

$$V(F) \cap V(G) \cap k^2 = V(F_b) \cap V(G_b)$$

is finite.

Putting these two together we get that $V(F) \cap V(G)$ is finite. □

Lemma 22.6

There exists a line D which misses $V(F) \cap V(G)$.

Proof. Let $Z \subseteq \mathbb{P}^2(k)$ be finite and take some point $a \in \mathbb{P}^2$. There a infinitely many lines passing through a, but only finitely many of them meet Z.

Hence for $Z = V(F) \cap V(G)$ we can choose one of these lines not meeting it. □

Let D be a projective line not meeting Z. Up to homography we know that $D = V(T)$. We also know that

$$\sum_{P\in\mathbb{P}^2} \mu_P(F,G) = \sum_{P\in V(F_b,G_b)} \mu_P(F_b,G_b) = \dim_k \mathscr{O}_Z(Z)$$

where $(Z, \mathscr{O}_Z)$ is the finite scheme on the intersection of $V(F)$ and $V(G)$.

To prove Bézout's theorem it is enough to show that

$$\dim \mathscr{O}_Z(Z) =\sim_k k[X,Y]/(F_b,G_b) = s\cdot t$$

Note that this is absolutely not the same as just reducing to affine space, as we really need the whole system to be induced from projective space in the right way!

Set

- $S = k[X,Y,T]$
- $R = k[X,Y]$
- $J = (F,G)$
- $I = (F_b,G_b)$
- $i\colon Z \hookrightarrow k^2$
- $j\colon Z \hookrightarrow \mathbb{P}^2$.

Proposition 22.7

We have that $i_*\mathscr{O}_Z \cong \widetilde{R/I}$ and $j_*\mathscr{O}_Z \cong \widetilde{S/J}$.

In particular we have $\Gamma(Z,\mathscr{O}_Z) = \Gamma(\mathbb{P}^2, \widetilde{S/J})$. This is sort of similar to one of the isomorphism theorems from abstract algebra.

It Bézout's theorem it is then enough to show that

$$\dim_k \Gamma(\mathbb{P}^2, \widetilde{S/J}) = s\cdot t$$

Lemma 22.8

There is an exact sequence of graded S-modules

$$0 \longrightarrow S[-s-t] \xrightarrow{\begin{bmatrix}-G\\ F\end{bmatrix}} S[-s]\oplus S[-t] \xrightarrow{\begin{bmatrix}F & G\end{bmatrix}} S \xrightarrow{\pi} S/J \longrightarrow 0$$

Problem 22.9 Check this.

The above complex is in fact a Koszul complex.

There is another exact sequence of sheaves

$$0 \longrightarrow \mathscr{O}_{\mathbb{P}^2}[-s-t] \longrightarrow \mathscr{O}_{\mathbb{P}^2}[-s] \oplus \mathscr{O}_{\mathbb{P}^2}[-t] \longrightarrow \mathscr{O}_{\mathbb{P}^2} \longrightarrow \widetilde{S/J} \longrightarrow 0$$

This there is also an exact sequence of rings

$$0 \longrightarrow \Gamma(\mathbb{P}^2, \widetilde{J}) \longrightarrow \Gamma(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}) \longrightarrow \Gamma(\mathbb{P}^2, \widetilde{S/J})$$

in which the rightmost map is not surjective. We can see this clearly as the middle ring has dimension 1, but we hope that the rightmost ring has dimension $s \cdot t$. The failure of this to be right exact is given by sheaf cohomology, which we will get back to.

The solutiion is then to work with the shifted sheaves instead, $\mathscr{O}_{\mathbb{P}^2} = \widetilde{S(d)}$.

Recall that

$$\Gamma(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(d)) = \begin{cases} 0, & d < 0 \\ S_d, & d \geq 0 \end{cases}.$$

Hence we need to compare $\mathscr{O}_Z = \widetilde{S/J}$ with $\widetilde{S/J(d)} = \mathscr{O}_Z(d)$. The goal is to show that these are isomorphic.

Proposition 22.10

$$\alpha \colon S/J(-1) \xrightarrow{\cdot T} S/J$$

is an injection, and for $n \geq s + t - 1$,

$$\alpha_n \colon (S/J)_{n-1} \longrightarrow (S/J)_n$$

is a surjection.

Proof. We will leave out the proof for injectivity, but it can be read in Perrin. Surjectivity will follow from the next lemma. □

Lemma 22.11

For $d \geq s + t - 2$ we have $\dim_k (S/J)_d = s \cdot 2$.

Proof. Look at the exact sequence earlier in degree d.

$$0 \longrightarrow S[-s-t] \xrightarrow{\begin{bmatrix}-G \\ F\end{bmatrix}} S[-s] \oplus S[-t] \xrightarrow{\begin{bmatrix}F & G\end{bmatrix}} S \xrightarrow{\pi} S/J \longrightarrow 0$$

We then have the result by recalling that for an exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

we have $\dim B = \dim A + \dim C$. □

Example 22.12 Let $F = X^2 - Y$ and $G = Y$ and $d = 2$. In $k[X, Y, T]$ we have six generators in degree $d = 2$, namely $X^2, Y^2, T^2, XY, XT, YT$. In $k[X, Y, T]/(YT - X^2, Y)$ all of these except T^2 and XT vanish. Hence it has dimension $2 = \deg F \cdots \deg G = 2 \cdots 1$.

Corollary 22.13

$$\widetilde{S/J}(-1) \cong \widetilde{S/J} \implies \mathscr{O}_Z \cong \mathscr{O}_Z(d)$$

for any $d \in \mathbb{Z}$.

Now, we then need to show that

$$s \cdot t = \dim_k \Gamma(\mathbb{P}^2, \widetilde{S/J}) = \dim_k \Gamma(\mathbb{P}^2, \widetilde{S/J(d)})$$

Thus it is enough to show that there exists an isomorphism

$$(S/J)_d \longrightarrow k[X, Y]/(F_b, G_b)$$

as we know that $(S/J)_d$ has dimension $s \cdot t$.

Consider the ring homomorphism $(-)_b \colon k[X, Y, T] \longrightarrow k[X, Y]$. It induces a homomorphism $(-)_b \colon S/J \longrightarrow R/I$ which restricts to a morphism

$$(-)_b \colon (S/J)_d \longrightarrow R/I$$

which is an isomorphism for $d \geq s + t - 2$. This all together proves Bézout's theorem!

Example 22.14 Let $V(F)$ be the trefoil curve, and $V(G)$ the quadrafoil curve. They have degree 4 and 6 respectively. From Bézout's theorem we know that these have 24 intersection points.

Problem 22.15 It would be interesting to have a full write up of this problem with a complete solution.

Example 22.16 Let $F = X^2 - Y$ and $G = X^3 - Y$. Let the line at infinity be given by $T = 0$. We then get $k[X,Y]/(Y - X^2, Y_X^3)$ as our ring. If we localize at $(0,0)$ it is the same as localizing at the ideal (X,Y) and we get

$$\frac{k[X,Y]_{(X,Y)}}{(Y - X^2, Y - X^3)} \cong \frac{k[X]_{(X)}}{(X^3 - X^2)} = \frac{k[X]_{(X)}}{(X^2(X-1))} \cong \frac{k[X]}{(X^2)}$$

as $(X - 1)$ becomes a unit. This algebra has dimension 2, hence the point $(0,0)$ has multiplicity 2.

Localizing at $(1,1)$ gives

$$k[X]_{(X-1)}/(X^2(X-1)) \cong k[X]/(X-1) \cong k$$

This shows that the point $(1,1)$ has multiplicity 1.

We really should have 6 points here, which means they intersect at infinity in multiplicity 3. This means that the choice of line at infinity to be $T = 0$ was a bad choice...

Problem 22.17 Try the above example choosing $0 = X - Y - T$ as the line at infinity. We should then get three affine points with total multiplicity 6.

23 Lecture 23 - 19.04.21

The lecture was focused on going through the exercises. Solutions and notes to these can be found at

24 Lecture 24 - 20.04.21

24.1 Sheaf cohomology

Let $(X, \mathscr{O}_X)$ be an algebraic variety. Recall that if we have an exact sequence of $\mathscr{O}_X$-modules

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{G} \longrightarrow \mathscr{H} \longrightarrow 0$$

Then we get an exact sequence

$$0 \longrightarrow \Gamma(X, \mathscr{F}) \longrightarrow \Gamma(X, \mathscr{G}) \xrightarrow{\pi} \Gamma(X, \mathscr{H})$$

where π need not be surjective. This means that the functor $\Gamma(X, -)\colon \mathscr{O}_X - mod \longrightarrow Ab$ may not be a right exact functor.

24.1.1 Cech cohomology

Let X be some topological space, $\mathscr{F}$ a sheaf of abelian groups on X and $U = \{U_i\}_{i=0}^n$ an open cover of X. Define a complex of abelian groups $C^*(U, \mathscr{F})$ for $0 \leq p \leq n$ by

$$C^*(U, \mathscr{F}) = \prod_{i_0 < \cdots < i_p} \mathscr{F}(U_{i_0} \cap \cdots \cap \mathscr{F}(U_{i_p})$$

where the differetial is defined by

$$\begin{aligned} C^p(U, \mathscr{F}) &\longrightarrow C^{p+1}(U, \mathscr{F}) \\ s_{i_0 \cdots i_p} &\longmapsto \sum_{k=0}^{p+1} (-1)^k s_{i_0 \cdots \hat{i_k} \cdots i_{p+1} | U_{i_0} \cap \cdots \cap U i_{p+1}} \end{aligned}$$

where $\hat{i_k}$ means that we omit that index. This gives us the complex

$$0 \longrightarrow \prod_{i=0}^{n} \mathscr{F}(U_i) \longrightarrow \prod_{i_0 < i_1} \mathscr{F}(U_{i_0} \cap U_{i_1}) \longrightarrow \cdots \longrightarrow \mathscr{F}(U_0 \cap \cdots \cap U_n) \longrightarrow 0$$

For $n = 1$ we get the following complex:

$$0 \to \mathscr{F}(U_0) \times \mathscr{F}(U_1) \to \mathscr{F}(U_0 \cap U_1) \to 0$$

and the differential d^0 is given by

$$\begin{bmatrix} s_0 \\ s_1 \end{bmatrix} \longmapsto \left[(s_{\hat{1}0} - s_{1\hat{0}})_{|U_0 \cap U_1} \right] = \left[(s_1 - s_0)_{|U_0 \cap U_1} \right]$$

It is trivially a complex, as any way to compose to differentials must include one copy of the trivial map.

For $n = 2$ we get:

$$0 \to \mathscr{F}(U_0) \times \mathscr{F}(U_1) \times \mathscr{F}(U_2) \to \mathscr{F}(U_0 \cap U_1) \times \mathscr{F}(U_0 \cap U_2) \times \mathscr{F}(U_1 \cap U_2) \to \mathscr{F}(U_0 \cap U_1 \cap U_2) \to 0$$

We can explicitly describe d^0 by

$$\begin{bmatrix} s_0 \\ s_1 \\ s_2 \end{bmatrix} \longmapsto \begin{bmatrix} (s_{\hat{1}0} - s_{1\hat{0}})_{|U_0 \cap U_1} \\ (s_{\hat{0}2} - s_{0\hat{2}})_{|U_0 \cap U_2} \\ (s_{\hat{1}2} - s_{1\hat{2}})_{|U_1 \cap U_2} \end{bmatrix} = \begin{bmatrix} (s_1 - s_0)_{|U_0 \cap U_1} \\ (s_2 - s_0)_{|U_0 \cap U_2} \\ (s_2 - s_1)_{|U_1 \cap U_2} \end{bmatrix}$$

Problem 24.1 Check that this is in fact a complex.

Definition 24.2: Čech cohomology

We call this complex the Čech complex of the space X. Its cohomology groups

$$\check{H}^p(U, \mathscr{F}) = \operatorname{Ker} d^p / \operatorname{Im} d^{p+1}$$

are called the Cech cohomology groups of X.

Note that when we are working in algebraic geometry we need to choose the covering to be an open affine covering.

Proposition 24.3

$\check{H}^0(U, \mathscr{F}) \cong \Gamma(X, \mathscr{F})$.

Proof.

$$\begin{aligned}
\check{H}^0(U, \mathscr{F}) &= H^0(C^*(U, \mathscr{F})) \\
&= \operatorname{Ker}(d^0 \colon C^0(U, \mathscr{F}) \to C^1(U, \mathscr{F})) \\
&= \{(s_i)_{i_0}^n \in \prod_{i=0}^n \mathscr{F}(U_i) \mid (s_i)_{U_i \cap U_j} = (s_j)_{U_i \cap U_j} \, \forall i, j\}
\end{aligned}$$

As $\mathscr{F}$ is a sheaf, these local sections $s_i \in \mathscr{F}(U_i)$ can be glued uniquely to a global section $s \in \mathscr{F}(X)$. As we have $\mathscr{F}(X) = \Gamma(X, \mathscr{F})$ we indeed have $\check{H}^0(U, \mathscr{F}) \cong \Gamma(X, \mathscr{F})$. □

Theorem 24.4

Let X be an affine algebraic variety. Furthermore we let $A = \Gamma(X)$, M be an A-module, $\mathscr{F} = \widetilde{M}$ and $U = \{U_i\}_{i=0}^m$ an open cover of standard open affine sets. Then for $p > 0$ we have $\check{H}^p(U, \mathscr{F}) = 0$.

Corollary 24.5

If X is an affine algebraic variety, and

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{G} \longrightarrow \mathscr{H} \longrightarrow 0$$

an exact sequence of quasi-coherent sheaves, then there is an exact sequence

$$0 \longrightarrow \Gamma(X, \mathscr{F}) \longrightarrow \Gamma(X, \mathscr{G}) \longrightarrow \Gamma(X, \mathscr{H}) \longrightarrow 0.$$

Definition 24.6: Separated algebraic variety

Let X be an algebraic variety. We say X is separated if the diagonal Δ is closed in $X \times X$.

Notice that this is not the same as being Hausdorff, as the topology on $X \times X$ is not the product topology.

If X is a separated algebraic variety and $U, V \subseteq X$ are open affine, then their intersection $U \cap V$ is again open affine!

Example 24.7 All affine algebraic varieties, and all projective algbraic varieties are separated.

Theorem 24.8

Let X be a separated algebraic variety, U a finite open cover and

$$0 \longrightarrow \Gamma(X, \mathscr{F}) \longrightarrow \Gamma(X, \mathscr{G}) \longrightarrow \Gamma(X, \mathscr{H}) \longrightarrow 0$$

an exact sequence of quasi-coherent sheaves on X. Then

$$0 \to \check{H}^0(U, \mathscr{F}) \to \check{H}^0(U, \mathscr{G}) \to \check{H}^0(U, \mathscr{H}) \to \check{H}^1(U, \mathscr{F}) \to \cdots$$

is a long exact sequence.

Proof. There exists an exact sequence of complexes

$$0 \longrightarrow C^*(X, \mathscr{F}) \longrightarrow C^*(X, \mathscr{G}) \longrightarrow C^*(X, \mathscr{H}) \longrightarrow 0$$

which we can apply general techniques from homological algebra to to get the correct long exact sequence.

The above sequence is exact, because it is exact in each degree. □

We know that $\check{H}^0(U, \mathscr{F}) \cong \Gamma(X, \mathscr{F})$, and putting this together with the long exact sequence above we get that Čech cohomology actually measures the failure of the global sections functor $\Gamma(X, -)$ to be right exact.

As this makes Čech cohomology the derived functor of the global sections functor, we get that the construction is in fact independent of the cover we chose! Hence we can denote the cohomology by $\check{H}(X, \mathscr{F})$. Note that there may be some niceness conditions on X and F in order to get this to work.

24.2 Vanishing theorems

Theorem 24.9

Let V be a separated algebraic variety of dimension n and let $\mathscr{F}$ be a quasi-coherent sheaf on V. Then $\check{H}(V, \mathscr{F}) = 0$ for $i > n$.

Theorem 24.10

Let $S = k[X_0, \cdots, X_n]$. Then

1. $\check{H}^0(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(d)) = S_d$
2. $\check{H}^i(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(d)) = 0$ for $0 < i < n$
3. $\check{H}^n(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(d)) = \check{H}^0(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(-d-n-1))$

Proof. Look at the the sequence

$$0 \to \prod S_{x_i} \to \prod_{i<j} S_{x_i x_j} \to \cdots \to S_{x_0 \cdots x_n} \to 0$$

$\square$

Problem 24.11 Look at and prove the above theorem for $n = 1$ in detail. Use it to compute $\check{H}^1(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}$.

25 Lecture 25 - 26.04.21

Last lecture we created a cocomple $\check{C}(X, \mathscr{F})$ called the Čech complex of a topological space X. Its homology coincided with the derived global sections.

Lemma 25.1

If $V \subseteq \mathbb{P}^N$ is an n-dimensional projective variety, then there exists a linear subvariety $W \subseteq \mathbb{P}^N$ of codimension $n+1$ such that $V \cap W = \emptyset$.

The proof can be read in Perrin.

Theorem 25.2

Let V be an n-dimensional separated algebraic variety and $\mathscr{F}$ a quasi-coherent sheaf on V. Then

$$\check{H}^i(V, \mathscr{F}) = 0$$

for all $i > n$.

We prove this in the case of V being a projective variety in $\mathbb{P}^N$.

Proof. We know from the above lemma that thee exists s linear subvariety W. Up to homography we can assume that $W = V(X_0, \dots, X_n)$. As $W \cap V$ is empty we have that $V \subseteq D^+(X_0) \cup \cdots D^+(X_n)$, which gives an open affine covering $\{V \cap D^+(X_i)\}$ of V. The Čech complex for this covering looks like

$$C^0 \longrightarrow \cdots \longrightarrow C^n \longrightarrow 0$$

which automatically makes $H^i(V, \mathscr{F}) = 0$ for all $i > n$. □

Theorem 25.3

Let X be a projective algebraic variety and $\mathscr{F}$ a quasi-coherent sheaf on X. Then $\dim_k \check{H}^i(X, \mathscr{F}) < \infty$ for all $i \geq 0$. Furthermore, there is an integer n_0, dependent on $\mathscr{F}$, such that for all $d \geq n_0$ and $i > 0$ we have $\check{H}(X, \mathscr{F}(d)) = 0$.

25.1 Riemann-Roch theorem

Let $\mathscr{F}$ be a coherent sheaf on a projective algebraic variety X. We know that $\dim_k \check{H}^i(X, \mathscr{F}) < \infty$, so every such number is an actual integer. These turn out to be highly interesting.

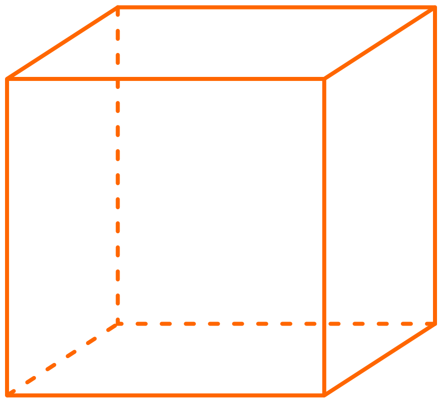
Definition 25.4

Let X be a projective algebraic variety and $\mathscr{F}$ a coherent sheaf on X. Then we define the Euler-Poincaré characteristic of X with respect to $\mathscr{F}$ by

$$\chi(X,\mathscr{F}) = \sum_{i=0}^{n}(-1)^i \dim_k \check{H}^i(X,\mathscr{F})$$

This should remind us of Eulers formula for polyhedra, which is given as follows. Consider a convex 3-dimensional polyhedra with V vertices, E edges and F faces. Then $V - E + F = 2$.

Example 25.5 Let P be the cube, i.e. the following convex polyhedra:



It has 8 vertices, 12, edges and 6 faces, meaning that $V - E + F = 8 - 12 + 6 = 2$.

Lemma 25.6

Let

$$0 \longrightarrow A_n \longrightarrow A_n - 1 \longrightarrow \cdots \longrightarrow A_1 \longrightarrow A_0 \longrightarrow 0$$

be an exact sequence of finite dimensional vector spaces. Then

$$\sum_{i=0}^{n}(-1)^i \dim_k A_i = 0$$

Proof. For $n = 1$ the statement is true because $A_1 \cong A_2$ which means they have the same dimension. For $n = 2$ we have the exact sequence

$$0 \longrightarrow A_2 \longrightarrow A_1 \longrightarrow A_0 \longrightarrow 0$$

which by the Rank-nullity theorem gives us $\dim A_1 = \dim A_2 + \dim A_0$, which means the formula holds.

Take now the full sequence

$$0 \longrightarrow A_n \longrightarrow A_n - 1 \longrightarrow \cdots \longrightarrow A_1 \longrightarrow A_0 \longrightarrow 0$$

and notice that for any three objects in the sequence we have a diagram

$$
\begin{tikzcd}
 & 0 & 0 & 0 & \\
 & \operatorname{Im} d^{i-1} \arrow[u] \arrow[r, "="] & \operatorname{Ker} d^i \arrow[u, leftarrow] \arrow[d] & \operatorname{Im} d^{i+1} \arrow[u] & \\
\cdots \arrow[r] & A_{i-1} \arrow[u] \arrow[r, "d^{i-1}"] & A_i \arrow[r, "d^i"] \arrow[d] & A_{i+1} \arrow[u] \arrow[r] & \cdots \\
 & \operatorname{Ker} d^{i-1} \arrow[u] & \operatorname{Im} d^i \arrow[r, "="] \arrow[d] & \operatorname{Ker} d^{i+1} \arrow[u] & \\
 & 0 \arrow[u] & 0 & 0 \arrow[u] &
\end{tikzcd}
$$

where the columns are exact. By the $n = 2$ case we know that $\dim A_i = \dim \operatorname{Ker} d^i + \dim \operatorname{Im} d^i$ for all i. Since the original sequence is exact we know that $\operatorname{Ker} d^i = \operatorname{Im} d^{i-1}$, which together with the fact that $\operatorname{Ker} d^n = 0$ and $\operatorname{Im} d^0 = 0$ allows us to construct the following sum:

$$
\begin{aligned}
\dim A_0 &= \dim \operatorname{Ker} d^0 \\
&= \dim \operatorname{Im} d^1 \\
&= \dim A_1 - \dim \operatorname{Ker} d^1 \\
&= \dim A_1 - \dim \operatorname{Im} d^1 \\
&= \dim A_1 - \dim A_2 + \dim \operatorname{Ker} d^2 \\
&\qquad\vdots \\
&= -\sum_{i=1}^{n-1} (-1)^i \dim A_i + (-1)^n \dim \operatorname{Ker} d^{n-1} \\
&= -\sum_{i=1}^{n-1} (-1)^i \dim A_i + (-1)^n \dim \operatorname{Im} d^n \\
&= -\sum_{i=1}^{n-1} (-1)^i \dim A_i + (-1)^n \dim A_n + (-1)^n \dim \operatorname{Ker} d^n \\
&= -\sum_{i=1}^{n-1} (-1)^i \dim A_i + (-1)^n \dim A_n
\end{aligned}
$$

Moving the sum to the left hand side we get finally

$$\sum_{i=0}^{n} (-1)^i \dim A_i = 0$$

which concludes the proof. $\square$

Proposition 25.7

Let

$$0 \longrightarrow \mathscr{F}' \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}'' \longrightarrow 0$$

be an exact sequence of coherent sheaves on an n-dimensional projective algebraic variety X. Then

$$\chi(X, \mathscr{F}) = \chi(X, \mathscr{F}') + \chi(X, \mathscr{F}'')$$

Proof. As the sequence is exact we get a long exact sequence in Čech cohomology

$$\begin{tikzcd}
0 \arrow[r] & \check{H}^0(X, \mathscr{F}') \arrow[r] & \check{H}^0(X, \mathscr{F}) \arrow[r] & \check{H}^0(X, \mathscr{F}'') \arrow[dll] & \\
& \check{H}^1(X, \mathscr{F}') \arrow[r] & \check{H}^1(X, \mathscr{F}) \arrow[r] & \check{H}^1(X, \mathscr{F}'') \arrow[dll] & \\
& \check{H}^2(X, \mathscr{F}') \arrow[r] & \cdots \arrow[r] & \check{H}^n(X, \mathscr{F}'') \arrow[r] & 0
\end{tikzcd}$$

where we know it ends at n because of the earlier results. This is a long exact sequence of vector spaces, meaning that we can apply the previous lemma. We then get

$$\begin{aligned}
0 &= \sum_{i=0}^{n} (-1)^i (\dim \check{H}^i(X, \mathscr{F}') - \dim \check{H}^i(X, \mathscr{F}) + \dim \check{H}^i(X, \mathscr{F}'')) \\
&= \sum_{i=0}^{n} (-1)^i \dim \check{H}^i(X, \mathscr{F}') - \sum_{i=0}^{n} (-1)^i \dim \check{H}^i(X, \mathscr{F}) + \sum_{i=0}^{n} (-1)^i \dim \check{H}^i(X, \mathscr{F}'') \\
&= \chi(X, \mathscr{F}') - \chi(X, \mathscr{F}) + \chi(X, \mathscr{F}'').
\end{aligned}$$

Rearranging we get the result we wanted:

$$\chi(X, \mathscr{F}) = \chi(X, \mathscr{F}') + \chi(X, \mathscr{F}'').$$

□

Let now $C \subseteq \mathbb{P}^N$ be an irreducible projective curve and denote

- $S = k[X_0, \dots, X_n]$
- $A = \Gamma_{\text{homog}}(C) = S/I(C)$, which is a graded domain
- $\widetilde{A} = \mathscr{O}_C$
- $n \in \mathbb{Z}$

Our goal is to calculate $\chi(\mathscr{O}_C(n)$. Notice that

$$\chi(\mathscr{O}_C(n)) = \dim_k \check{H}^0(X, \mathscr{O}_C(n)) - \dim_k \check{H}^1(X, \mathscr{O}_C(n))$$

as the curve C is of dimension 1.

Idea: Look at the intersection of C with a general enough hypersurface.

Proposition 25.8

Let $H = V(h)$ be a hypersurface not containing C. Then multiplication by $\overline{h} \in A = S/I(C)$ induces an exact sequence

$$0 \longrightarrow A(-1) \xrightarrow{\cdot \overline{h}} A \longrightarrow A/(\overline{h}) \longrightarrow 0$$

Proof. It is enough to show that $\cdot\overline{h}$ is an injection as the last map is a surjection by definition. As A is a domain it is enough to show that $\overline{h} \neq 0$, or equivalently, $\overline{h} \notin I(C)$.

Note that if we would have $h \in I(C)$, then $V(h) \supseteq V(I(C)) \supseteq C$, and we have $V(h) = H$ which is assumed to not contain C which is a contradiction. Hence $h \notin I(C)$. □

Let now $Z = C \cap H$, and notice that it is finite, as $H \supsetneq C$. We endow Z with a finite scheme structure by letting $\mathscr{O}_Z$ be the sheaf associated to $A/(\overline{h})$.

The exact sequence we got above by multiplying with $\overline{h}$ gives us an exact sequence

$$0 \longrightarrow \mathscr{O}_C(-1) \longrightarrow \mathscr{O}_C \longrightarrow \mathscr{O}_Z \longrightarrow 0,$$

which we can shift by n to get an exact sequence

$$0 \longrightarrow \mathscr{O}_C(n-1) \longrightarrow \mathscr{O}_C(n) \longrightarrow \mathscr{O}_Z(n) \longrightarrow 0$$

By the earlier proposition we know that if we take the Euler-Poincaré characteristics, we get

$$\chi(C, \mathscr{O}_C(n)) = \chi(C, \mathscr{O}_C(n-1) + \chi(Z, \mathscr{O}_Z(n)).$$

We have $\dim \check{H}^0(Z, \mathscr{O}_Z(n)) = \dim \check{H}^0(Z, \mathscr{O}_Z) = d$, as $\mathscr{O}_Z(n) \cong \mathscr{O}_Z$. As $Z \neq \emptyset$ we know that $d \geq 1$. This integer d is the number of intersection points, counted with multiplicity, between C and H.

We get

$$\begin{aligned}
\chi(\mathscr{O}_C(n)) &= d + \chi(\mathscr{O}_C(n-1)) \\
&= 2d + \chi(\mathscr{O}_C(n-2)) \\
&\;\;\vdots \\
&= nd + \chi(\mathscr{O}_C)
\end{aligned}$$

The part $nd + \chi(\mathscr{O}_C)$ can be though of as a polynomial of degree n in the variable n, and it shows up sometimes in dimension theory.

Note that the above calculation shows that the number d does not depend on the hyperplane H.

Definition 25.9: The dregree of a curve

The number d is called the degree of C.

Lemma 25.10

Let X be an irreducible projective algebraic variety. Then $\check{H}^0(X, \mathscr{O}_X) \cong k$.

Proof. We have that $\check{H}^0(X, \mathscr{O}_X) \cong \Gamma(X, \mathscr{O}_X)$ is a domain, that is finite dimensional over k. It is in fact a field, and is algebraic over k, which means it is isomorphic to k as k is algebraically closed. $\square$

Definition 25.11: The genus of a curve

We call the integer $g = \dim \check{H}(C, \mathscr{O}_C)$ the genus of C.

We have during this subsection actually proven the following result.

Theorem 25.12: The Riemann-Rock theorem

Let C be an irreducible projective algebraic curve of degree d and genus g. Then for all $n \in \mathbb{Z}$ we have

$$\chi(\mathscr{O}_C(n)) = nd + 1 - g$$

.

Example 25.13 Let $F \in k[X, Y, T]$ be irreducible, homogeneous and of polynomial degree $d > 0$. Let $C = V_{\text{proj}}(F) \subseteq \mathbb{P}^2$. Then the degree of C is d, and the genus of C is $\frac{(d-1)(d-2)}{2}$.

26 Lecture 26 - 27.04.21

26.1 Schemes

Let X be a topological space. Recall that a presheaf of abelian groups on X is a functor

$$\mathscr{F} : Open(X)^{op} \longrightarrow Ab$$

and that a sheaf is a presheaf such that if an open subset $U \subseteq X$ is covered by open set $\{U_i\}$, then for any $f_i \in \mathscr{F}(U_i)$ such that $f_{i|U_i \cap U_j} = f_{j|U_i \cap U_j}$ there exists a unique $f \in \mathscr{F}(U)$ such that $f_{|U_i} = f_i$ for all i.

Problem 26.1 Hartshorne uses a slightly different definition of a sheaf. Prove that these are equivalent.

Example 26.2 Let V be an affine alebraic set. For $U \subseteq V$, let $\mathscr{O}(U)$ be the ring of regular functions $U \longrightarrow k$. If $U' \subseteq U$, then $\mathscr{O}(U) \longrightarrow \mathscr{O}(U')$ is given by restriction.

This $\mathscr{O}$ is the sheaf of regular functions on V, and is the sheaf we defined for affine algebraic varieties earlier.

Recall that morphisms of sheaves $\phi\colon \mathscr{F} \longrightarrow \mathscr{G}$ are maps such that the diagram

$$\begin{array}{ccc} \mathscr{F}(U) & \longrightarrow & \mathscr{G}(U) \\ \downarrow & & \downarrow \\ \mathscr{F}(U') & \longrightarrow & \mathscr{G}(U') \end{array}$$

commutes for all open set $U' \subseteq U$.

Proposition 26.3

A morphism of sheaves $\phi\colon \mathscr{F} \longrightarrow \mathscr{G}$ is an isomorphism if and only if the induced map on stalks $\phi_p\colon \mathscr{F}_p \longrightarrow \mathscr{G}_p$ is an isomorphism for all points p.

Let A be a commutative ring with identity. Recall that $SpecA = \{p \subseteq A | p$ a prime ideal. For any ideal $a \subseteq A$ set

$$V(a) = \{p \in SpecA | a \subseteq p\}$$

Lemma 26.4

We have

- $V(ab) = V(a) \cup V(b)$
- $V(\sum_{i \in I} a_i) = \cap_{i \in I} V(a_i)$

- $V(a) \subseteq V(b)$ if and only if $\sqrt{b} \subseteq \sqrt{a}$.

Problem 26.5 Prove the above lemma.

This defines the Zariski topology on $SpecA$, where the sets $V(a)$ are the closed sets in the topology.

Now, define a sheaf of rings $\mathscr{O}_{SpecA}$ on $SpecA$ as follows:

For an open set $U \subseteq SpecA$ set

$$\mathscr{O}(U) = \{\text{functions } s \colon U \to \coprod_{p \in U} A_p | 1. \text{ and } 2.\}$$

where

1. $s(p) \in A_p$
2. For each $p \in U$ there exists a neighborhood $V \subset U$ around p, and elements $a, f \in A$, such that for all $q \in V$ with $f \notin q$ then $s(q) = q/f \in A_q$.

The last point can be thought of as s being locally a quotient of elements in A.

Example 26.6 ($Spec\mathbb{Z}$)**.**

The points correspond to the prime ideals in $\mathbb{Z}$, i.e. the ideal generated by prime numbers.

We have $(p) = V((p))$ and $V((0)) = Spec\mathbb{Z}$. This means that open sets are sets that miss a finite set of points, for example $U = Spec\mathbb{Z} \setminus (3)$.

What is then $\mathscr{O}(U) = \mathscr{O}(D((3)))$?

$$\{\frac{a}{3^n} | a \in \mathbb{Z}, n \geq 0\} = \mathbb{Z}[3^{-1}] \to \mathscr{O}(D((3)))$$

by sending $a/3^n$ to $[s \colon D((3)) \to \coprod_{p \in D((3))} \mathbb{Z}_p]$, defined by $s(p) = a/3^n \in \mathbb{Z}_p$.

Visuals of open sets etc

Proposition 26.7

Let A be a ring and let $\mathscr{O}$ be the sheaf defined above. Then

1. for $p \in SpecA$ we have $\mathscr{O}_p \cong A_p$
2. for $f \in A$ we have $\mathscr{O}(D(f)) \cong A_f$
3. $\Gamma(SpecA, \mathscr{O}) \cong A$

Proof. Notice that point 2 implies point 3 by letting $f = 1$.

We start by proving 1. For $p \in SpecA$ define

$$\begin{aligned} \phi\colon \mathscr{O}_p &\longrightarrow A_p \\ s_p &\longrightarrow s(p) \end{aligned}$$

Injectivity is left as an exercise. For surjectivity we notice that $A_p = \{a/f | f \notin p\}$. If we take a distinguished open set $D(f)$ which is a neigborhood of p, then $a/f \in \mathscr{O}(D(f))$, which means that we can let $s = a/f$. Then $s_p = [D(f), a/f] \mapsto a/f$.

We now prove 2. Define

$$\begin{aligned} A_f &\longrightarrow \mathscr{O}(D(f)) \\ \frac{a}{f^n} &\longmapsto s \in \mathscr{O}(D(f)) \end{aligned}$$

where $s(p) = a/f^n \in A_p$. This is in fact an isomorphism, but the proof is long and technical. It can be seen in Hartshorne. □

Note that we could have taken point 2. to be the definition of the sheaf, as it is enough to define it on a basis, which the distinguished open set $D(f)$ do form.

Definition 26.8: Ringed space

A ringed space is a pair $(X, \mathscr{O}_X)$, where X is a topological space and $\mathscr{O}_X$ is a sheaf on X.

Definition 26.9: Morphism of ringed spaces

A morphism $(f, f^\#)\colon (X, \mathscr{O}_X) \longrightarrow (Y, \mathscr{O}_Y)$ of ringed spaces consists of a continuous map

$$f\colon X \longrightarrow Y$$

and a morphism of sheaves

$$f^\#\colon \mathscr{O}_Y \longrightarrow f_*\mathscr{O}_X$$

Definition 26.10: Locally ringed space

A ringed space $(X, \mathscr{O}_X)$ is called a locally ringed space if for all $p \in X$ we have that $\mathscr{O}_{X,p}$ are local rings.

Proposition 26.11

If A is a ring, then $(SpecA, \mathscr{O}_{SpecA}$ is a locally ringed space.

Proposition 26.12

If $\phi\colon A \longrightarrow B$ is a ring homomorphism, then there is an induced morphism of ringed spaces

$$(\overline{\phi}, \phi^{\#})\colon (SpecB, \mathscr{O}_{SpecB}) \longrightarrow (SpecA, \mathscr{O}_{SpecA}).$$

If we are given such a map of ringed spaces, the this also induces a morphism of rings.

Definition 26.13: Affine scheme

An affine scheme is a locally ringed space $(X, \mathscr{O}_X)$ that is isomorphic (as locally ringed spaces) to $(SpecA, \mathscr{O}_{SpecA})$ for some ring A.

Definition 26.14

A scheme is a locally ringed space $(X, \mathscr{O}_X)$ where every point $x \in X$ has a neighborhood U such that $(U, \mathscr{O}_{X|U}$ is an affine scheme.

Problem 26.15 Compare to affine algebraic varieties.

27 Lecture 27 - 03.05.21

We return to the structure sheaf of $SpecA$.

Recall that we defined for an open set $U \subseteq SpecA$

$$\mathscr{O}(U) = \{s \colon U \to \coprod_{p \in U} A_p | 1. \text{ and } 2.\}$$

where

1. $s(p) \in A_p$
2. for each $p \in U$ there exists a neighborhood $V \subset U \cap D(f)$ around p, and elements $a, f \in A$, such that for all $q \in V$ with $f \notin q$ then $s(q) = q/f \in A_q$.

Note that this coproduct happens in the category of sets, so it is just disjoint union.

add picture

We need to check that $\mathscr{O}(U)$ is a ring, as we want it to be a sheaf of rings. Let $s, t \in \mathscr{O}(U)$. Define

- $(s + t)(p) = s(p) + t(p) \in A_p$
- $(s \cdot t)(p) = s(p) \cdot t(p) \in A_p$

Notice that point 1. is immediate, so we show point 2.

We know that for all p there exists elements $a', f' \in A$ and neighborhood $V' \subseteq U \cap D(f')$ of p such that $s_{|V'} = a'/f'$. Take again another such elements a'', f''.

For $s + t$ we set $V = V' \cap V'' \subseteq U \cap D(f') \cap D(f'') = U \cap D(f'f'')$. For all $q \in V$, $(s + t)(q) = \frac{a'}{f'} + \frac{a''}{f''} = \frac{a'f'' + a''f'}{f'f''} \in A_q$.

If we set $a = a'f'' + a''f'$ and $f = f'f''$ then all is well defined.

For $s \cdot t$ we again let $V = V' \cap V''$ and set $a = a'a''$ and $f = f'f''$. Thus we get $(s \cdot t)(q) = \frac{a'}{f'} \cdot \frac{a''}{f''} = \frac{a'a''}{f'f''}$, which shows that it is well defined.

The unit $1 \colon U \longrightarrow \coprod_{p \in U} A_p$ is defined by $1(p) = 1_{A_p} \in A_p$. Thus we can now check that $\mathscr{O}(U)$ is a commutative ring with identity, with the operations defined above.

We also need to check that $\mathscr{O}$ is a sheaf. So let $V \subseteq U$ be open sets. There is a map $\mathscr{O}(U) \to \mathscr{O}(V)$ given by sending s to $s_{|V}$, i.e. restriction. It is a ring homomorphism as

$$(s + t)_{|V} = s_{|V} + t_{|V}$$
$$(s \cdot t)_{|V} = s_{|V} \cdot t_{|V}$$

Together with the fact that point 1. and 2. are local conditions, this shows that $\mathscr{O}$ is a presheaf. Let then $U = \bigcup_{i \in I} U_i$ be an open cover and suppose $s_i \in \mathscr{O}(U_i)$ such that $s_{i|U_u \cap U_j} = s_{j|U_i \cap U_j}$.

Define $s\colon U \longrightarrow \coprod_{p\in U} A_p$ by $s(p) = s_{|U_i}(p)$, where U_i is some set in $\{U_i\}$ that contains p. We can check that $s \in \mathscr{O}(U)$. Also, s is uniquem hence we actually have a sheaf of rings.

Definition 27.1

The spectrum of A is defined as $(SpecA, \mathscr{O})$ where $\mathscr{O}$ is the above sheaf of rings.

Example 27.2 Let $A = k[x]_{(x)}$. Then $SpecA = \{(0), (x)\}$.

The closed sets are $\{V((x)) = \{(x)\}, V((0)) = SpecA, V(A) = \emptyset\}$, and the open sets are $\{D(x) = \{(0)\}, D(0) = \emptyset, D(1) = S\mathring{a}ecA\}$, hence we notice that all open sets are distinguished.

We have

$$\begin{aligned}\mathscr{O}(D(x)) &= \{s\colon \{(0)\} \to A_{(0)} | 1. \text{ and } 2.\} \\ &= A_{(0)} \\ &= k[x]_{(0)} \\ &= k(x) = \{\frac{p(x)}{q(x)} | p, q \in k[x], q \neq 0\} \\ &= (k[x]_{(x)})_x\end{aligned}$$

and

$$\begin{aligned}\mathscr{O}(D(0)) &= 0 \\ &= A_0\end{aligned}$$

and

$$\mathscr{O}(D(1)) = \{s\colon SpecA \to A_{(0)} \cup A | 1. \text{ and } 2.\}$$

where this last ring is isomorphic to just $A = A_1$ by the morphism $\psi\colon A \to \mathscr{O}(D(1)), a \mapsto s_a$, where $s_a(p) = a/1$. It is injective because if $s_a = 0$ then

$$0 = s_a((x)) = a \in A_{(x)}$$

but here $A_{(x)} = A$, hence $a = 0$.

For surjectivity we assume $s \in \mathscr{O}(D(1))$. The point 2. says there exists elements $a, f \in A$ and an open set $V \subseteq D(1)$ around (x) such that for all $q \in V$ we have $s(q) = a/f$. But $SpecA = V \subseteq SpecA \cap D(f)$, so if f is a unit in A then $a/f = f^{-1}a/1$. Hence $\psi(s_{f^{-1}a} = s$.

Proposition 27.3

For any $f \in A$ we have that $\mathscr{O}(D(f)) \cong A_f$.

Example 27.4 *Speck*$[x]$

Do example

Example 27.5 *Speck*$[x, y]$

Do example

Proposition 27.6

There is a fully faithful functor

$$Var(k) \longrightarrow Sch(Speck)$$

Problem 27.7 Look at some k-algebras and compare how they look as varieties vs schemes.

The analogue to projective varieties is the proj construction.

Let S be a graded ring $\bigoplus_{d \geq 0} S_d$. Define

$$ProjS = \{h \subseteq S | h \text{ homogeneous }, \bigoplus_{d>0} S_d \nsubseteq h\}$$

We can then define sets $V(a) = \{p \in ProjS | a \subseteq p\}$ for homogeneous ideals a in S. These sets again form the Zariski closed sets on a topology on $ProjS$.

We can define a sheaf of rings $\mathscr{O}$ on $ProjS$ by letting $D^+(f) = \{p \in ProjS | f \notin p\}$ where $f \in \bigoplus_{d>0} S_d$. These form a basis for the topology on $ProjS$. So defining

$$\mathscr{O}(D^+(f) = (S_f)_0$$

where $(-)_0$ means the degree zero component, gives a sheaf on $ProjS$.

This makes $(ProjS, \mathscr{O})$ into a scheme.

The scheme $Projk[X_0, \dots, X_n]$ is a scheme, whose subspace of closed points is homeomorphic to $\mathbb{P}^n(k)$.

28 Lecture 28 - 04.05.21

The lecture was focused on going through the exercises and last questions.

A Appendix A

A.1 Alternative proof of Hilbert's nullstellensatz

First we show that if B is a finitely generated k-algebra and $\mathfrak{b}$ be an ideal in B. Then we have

$$\sqrt{\mathfrak{b}} = \bigcap_{\mathfrak{b} \subseteq \mathfrak{m}} \mathfrak{m}.$$

where $\mathfrak{m}$ are the maximal ideals in B.

First, we note that the projection $\pi : B \to B/\mathfrak{b}$ induces bijections between the sets

- prime ideals in $B/\mathfrak{b}$ and prime ideals in B that contain $\mathfrak{b}$,
- maximal ideals in $B/\mathfrak{b}$ and maximal ideals in B that contain $\mathfrak{b}$,
- radical ideals in $B/\mathfrak{b}$ and radical ideals in B that contain $\mathfrak{b}$.

Hence we only need to prove the statement for $\mathfrak{b} = (0)$, and since it is clear that $\sqrt{(0)}$ is contained in every maximal ideal because $\sqrt{(0)}$ consists of all nilpotent elements, we only need to show that every element not contained in $\sqrt{(0)}$ is not contained in some maximal ideal.

Let $f \in B$ be non-nilpotent, i.e. $f \in \sqrt{(0)}$. This implies that

$$B_f \cong B[t]/(ft - 1)$$

is a non-trivial k-algebra, hence it has a maximal ideal $\mathfrak{m}$. Consider the morphism $\phi : B \longrightarrow B_f$. This is a morphism of finitely generated k-algebras, and by Zariski's lemma, $k \subseteq B/\phi^{-1}(\mathfrak{m}) \subseteq B_f/\mathfrak{m}$ is a finite extension, and hence $k \subseteq B/\phi^{-1}(\mathfrak{m})$ is an integral extension. Since k is a field, it is a field itself. This gives us that the inverse image of a maximal ideal is again a maximal ideal, i.e. $\phi^{-1}(\mathfrak{m})$ is a maximal ideal of B. But this ideal can't contain f. Hence we have shown that every non-nilpotent element is not contained in all maximal ideals.

Now we use this to deduce the result of the nullstellensatz.

Let $a \in k^n$. First, note that $a \in Z(\mathfrak{a})$ if and only if $\mathfrak{a} \subseteq \mathfrak{m}_a$. Hence, the maximal ideals containing $\mathfrak{a}$ is just the maximal ideals $\mathfrak{m}_a$ such that $a \in V(\mathfrak{a})$. Above we showed that the radical of an ideal was equal to the intersection of all maximal ideals containing it, hence we have

$$\sqrt{\mathfrak{a}} = \bigcap_{a \in V(\mathfrak{a})} \mathfrak{m}_a.$$

For the final part, we have for $f \in k[t_1, \ldots, t_n]$ and $a \in k^n$ that $f(x) = 0$ if and only if $f \in \mathfrak{m}_a$. Hence we have for subsets $V \subseteq k^n$ that $I(V) = \bigcap_{a \in V} \mathfrak{m}_a$. And since $a \in Z(\mathfrak{a})$ if and only if $\mathfrak{a} \subseteq \mathfrak{m}_a$, we finally have

$$I(V(\mathfrak{a})) = \bigcap_{a \in V(\mathfrak{a})} \mathfrak{m}_a = \sqrt{\mathfrak{a}}.$$

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