Algebraic structures in monochromatic homotopy theory

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Abstract

This thesis consists of three papers—all focused on understanding certain features of localizing subcategories; all motivated by understanding features of chromatic homotopy theory.

Our first paper uses Patchkoria–Pstrągowski's version of Franke's algebraicity theorem to prove that monochromatic homotopy theory is completely algebraic when the prime p is large compared to the chromatic height n. In particular, we prove that the category of spectra, localized the Morava K-theory spectrum $K_p(n)$, is equivalent to the derived I_n-complete objects in Franke's category of periodic comodules over E_*E—the Hopf algebroid associated to height n Morava E-theory.

In the second paper we introduce contramodules over a cocommutative coalgebra in a presentably symmetric monoidal ∞-category. When the coalgebra C is coidempotent, we prove that there is a symmetric monoidal duality between comodules and contramodules over C, which we call Positselski duality. Furthermore, when the ambient category is stable and compactly generated by dualizable objects, this duality recovers local duality in the sense of Hovey–Palmieri–Strickland, allowing us to describe the category of $K_p(n)$-local spectra as contramodules over the monochromatization of the sphere spectrum.

The third paper studies how certain localizing subcategories compatible with a given t-structure on a stable ∞-category $\mathcal{C}$, can be classified by using the associated Grothendieck prestable ∞-category $\mathcal{C}_{\geqslant 0}$ and the associated Grothendieck abelian heart $\mathcal{C}^{\heartsuit}$. In particular, we prove that there is a one-to-one correspondence between t-structure compatible localizing subcategories in $\mathcal{C}$, and prestable localizing subcategories of $\mathcal{C}_{\geqslant 0}$. This allows us to extend a result of Lurie to the stable setting, and prove a classification result for these localizing subcategories.

Sammendrag

Denne avhandlingen består av tre artikler—alle fokusert på å forstå visse egenskaper av lokaliserende underkategorier; alle motivert av å forstå egenskaper iboende kromatisk homotopiteori.

Vår første artikkel bruker Patchkoria–Pstrągowskis versjon av Frankes algebraisitetsteorem til å bevise at monokromatisk homotopiteori er fullstendig algebraisk når primtallet p er mye større enn den kromatiske høyden n. Mer presist viser vi at kategorien av spectra lokalisert ved Morava K-teorispektrumet $K_p(n)$, er ekvivalent til I_n-komplette objekter i Frankes kategori av periodiske komodules over E_*E—Hopf algebroiden assosiert til Morava E-teori med høyde n.

I den andre artikkelen introduserer vi kontramoduler over en kokommutativ koalgebra i en presenterbar symmetrisk monoidal ∞-kategori. Når koalgebraen C er koidempotent viser vi at det er en symmetrisk monoidal dualitet mellom komoduler og kontramoduler over C, som vi kaller Positselski-dualitet. Videre viser vi at dersom bakgrunnskategorien er stabil og kompakt-generert av dualiserbare objekter, så gjennskaper dette Hovey–Palmieri–Stricklands teori om lokal dualitet, som lar oss beskrive kategorien av $K_p(n)$-lokale spektra som kontramoduler over monokromatiseringen av sfærespektumet.

Den tredje artikkelen studerer hvordan lokaliserende underkategorier som er kompatible med en gitt t-struktur på en stabil ∞-kategori $\mathcal{C}$, kan klassifiseres via den tilhørende Grothendieck-prestabile ∞-kategorien $\mathcal{C}_{\geqslant 0}$ og det Grothendieck-abelske hjertet $\mathcal{C}^{\heartsuit}$. Vi viser at det er en en-til-en korrespondanse mellom disse t-struktur-kompatible lokaliserende underkategoriene av $\mathcal{C}$, og prestabile lokaliserende underkategorier av $\mathcal{C}_{\geqslant 0}$. Dette lar oss utvide et resultat av Lurie til den stabile settingen, og bevise et klassifiserings resultat for disse lokaliserende underkategoriene.

Information

The contents of this thesis consist mainly of material from the papers [Aam24a], [Aam24c] and [Aam24b], where the candidate is the only author. In addition there are some added remarks, some further results not yet presented in any papers, some more historical background, as well as more in-depth introductions to the central ideas of the thesis.

This material is structured into four chapters. Chapter 0 consists of mathematical preliminaries and background, as well as a short summary of each paper. There is also an introduction for the layperson, mostly aimed at family and friends, for situating the topics of this thesis amidst the broad world of mathematics, and to give a vague sense of what its contents is about.

The three remaining chapters each consists of one of the above-mentioned papers. The first chapter presents the paper *Algebraicity in monochromatic homotopy theory* ([Aam24a]). The second chapter presents the paper *Positselski duality in stable ∞-categories* ([Aam24c]) as well as an addendum on contramodules over topological rings. Lastly, the third chapter presents the paper *Classification of localizing subcategories along t-structures* ([Aam24b]), together with an addendum on monochromatic synthetic spectra, linking the third paper back to the contents of the first one, attempting to create a certain sense of cohesion and circularity.

Before each of the papers there is a title-page, a poem and a drawing, each representing the contents of the paper. These are all made by the author. The poems are in the form of limericks, and each describe the main result from the associated paper. The drawings have two functions: enumerate the papers and give a visual feel for what the paper is about. A description of the drawing can be found on each subsequent page. The style of the drawings is inspired by the iconic album art of Joy Division's debut album *Unknown Pleasures*, [Div79], made by Peter Saville.

Acknowledgements

List of symbols

Standard notation

$\mathbb{N}$	The natural numbers
$\mathbb{Z}$	The integers
$\mathbb{Q}$	The rational numbers
$\mathbb{F}_p$	The finite field of characteristic p
$\mathbb{Z}_{(p)}$	The p-local integers
$\mathbb{Z}_p$	The p-adic integers

Chromatic homotopy theory

$\mathcal{S}$	The category of spaces
Sp	The category of spectra
$\mathbb{S}$	The sphere spectrum
$\mathbb{S}_{(p)}$	The p-local sphere spectrum
HG	The Eilenberg–Maclane spectrum of G
$E_{n,p}$	Morava E-theory
$E_p(n)$	Johnson–Wilson theory
$K_p(n)$	Morava K-theory
$\mathrm{Sp}_{n,p}$	$E_{n,p}$-local spectra
$\mathcal{M}_{n,p}$	Monochromatic spectra
$\mathrm{Sp}_{K_p(n)}$	$K_p(n)$-local spectra
$L_{n,p}$	$E_{n,p}$-localization functor
$M_{n,p}$	Monochromatization functor
$L_{K_p(n)}$	$K_p(n)$-localization functor
$L_{n,p}\mathbb{S}$	The $E_{n,p}$-local sphere spectrum
$L_{K_p(n)}\mathbb{S}$	The $K_p(n)$-local sphere spectrum
$M_{n,p}\mathbb{S}$	The monochromatic sphere spectrum
I_n	Landweber ideal $(p, v_1, \ldots, v_{n-1})$
LSyn_E	$E_{n,p}$-local synthetic spectra

General notation

$\mathbb{R}F$	Total right derived functor of F
$\mathbb{L}F$	Total left derived functor of F

Contents

Chapter 0

Introduction

0.1 The laypersons introduction

Mathematics is one of the longest, richest and best preserved traditions humanity has ever created. In all probability, mathematics started by humans (and animals) realizing that things can be counted—that collections of things can be said to have a certain numerical size. This developed to the simplest theory of numbers: *arithmetic*. For millennia counting and arithmetic was used to create new knowledge and new technology, everything from understanding how the seasons change, understanding lunar cycles, solar cycles and astronomy, to agriculture, crop cycles and animal populations. Another part of mathematics came about later, when humans needed to more precisely describe the shape of land they owned—forming the field of *geometry*.

These two fields of mathematics was essentially all there was for thousands of years, and in some very general way, these are still all there is to mathematics as a whole. Arithmetic and geometry continued in their own traditions, all the way through antiquity and the premodern eras. This is seen, for example, in the *seven liberal arts*, which the students at the first universities had to master. The words "arts" here has a different meaning than in modern society, where the original meaning is closer to that of a skill. These seven skills are what was deemed necessary in antiquity in order to be *free*—a person worthy of attending public debates, defending themselves in court, participating in juries and serving in the military. The first three, named the *trivium*, consisted of: grammar, rhetoric and logic. The remaining four, called the *quadrivium*, consisted of music, arithmetic, geometry and astronomy. Here we again see the appearance of our two fields of mathematics—arithmetic and geometry. These are still somewhat kept separate and studied in different ways, and by different tools.

In more modern times, knowledge, education and research has received several sorely needed revolutions. Research and knowledge is by now an incredibly rigorous process; education a well oiled machine accessible not only to the elites. The two fields of mathematics have expanded to immense sizes, and now contain hundreds upon hundreds of subfields. One of the most interesting development—in my humble unbiased opinion—was the development of bridges, connections and similarities between the two fields of mathematics. For example, numbers—now meaning not only the numbers used for counting, but also concepts like real and complex numbers—gave way to coordinate systems, functions and analysis. Things that locally looked like coordinate systems became manifolds, which are incredibly geometric, and shape-like in nature.

The study of the structure of systems of numbers turned into the mathematical field of *algebra*, the study of the structure of shapes formed the mathematical field of *topology*, and the interactions between these two became *algebraic topology*.

An example of such an interaction is the following. Imagine you have two circles and want to understand how many continuous functions there are between them. Up to a bit of rotation and scaling, there is essentially only one thing you can do: you can wind the first circle around the other. This can be done in either a clockwise or counter-clockwise fashion, and with as many turns as one could wish for. If we first loop five times around clockwise, and then two times around counter-clockwise, then we have effectively looped around only three times clockwise. This means that we can "add" or "subtract" these windings around the circle together. In particular, this has exactly the same structure as the whole numbers $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$. This study of all continuous maps between the circle and some space X is called the *fundamental group* of X, and is always a "number system" where one can add and subtract. This then, forms a bridge between shapes and number systems.

The act of building connections between the two fields of mathematics is also what this thesis is about. It is about continuing the long and deep tradition of understanding the interplay of these fields—about furthering the development and understanding in one of their modern subfields. These subfields are *homological* *algebra* and *stable homotopy theory*. The former is a subfield of algebra, where one tries to study the structure of the systems of all different systems of numbers; the latter is a subfield of topology, where one tries to study the structure of systems of all different systems of shapes. This is not very precise, but the precise concepts are technical in nature and will be introduced in the mathematical introduction of the thesis.

We can then make a very general description of what the contents of this thesis is about: it is about studying three different bridges between "shapes" and "numbers". These three bridges are each located in their own research paper, which form the main content of this thesis. Let us very briefly try to explain what each of these bridges are:

The first paper is the most direct bridge. The system of shapes at hand is in some sense a very "fundamental" system, as it arises as the smallest pieces—the atoms, if you will—of perhaps the most important system we have in stable homotopy theory. We directly compare this atomic system of shapes to a specific system of numbers, and prove that they are in fact equivalent—they have exactly the same structure.

The second paper has a bit more of an indirect bridge, where we take a concept from algebra and try to study an analogous concept in topology. Doing this we are able to recover and generalize some already known results in topology, now seen from a completely new angle. For example, we get a new description of the atomic system we studied in the first paper.

The third paper is again more direct, where we have a direct comparison between certain substructures of shapes, to certain substructures of numbers. We prove that there is a one-to-one correspondence between these collections of substructures, providing new insight into the substructures, and illuminating previously known such relationships.

0.2 Central ideas

As a backdrop for this entire thesis lies the ubiquitous concept of ∞-*categories*, as developed by Joyal, Lurie and others—the canonical references being [Joy02], [Lur09] and [Lur17]. We will assume familiarity with ∞-categories and their associated standard constructions, and use them all willy-nilly throughout the rest of the thesis. We will sometimes drop the prefix, and hope that the distinction between ∞-categories and classical categories will be clear from context.

Most ∞-categories considered will be *presentable*, in the sense of [Lur09, Chapter 5].

Example 0.2.1. The most important example is perhaps the ∞-category of *spaces*, denoted $\mathcal{S}$. This is an ∞-categorical version of the classical category of topological spaces.

The $(\infty, 2)$-category of presentable ∞-categories, where the morphisms are the colimit-preserving functors, is denoted Pr^L. It has a symmetric monoidal structure via the Lurie tensor product $\otimes^L$, and we will say a presentable ∞-category is *presentably symmetric monoidal* if it is a commutative monoid in Pr^L. Any such category $\mathcal{C}$ has a symmetric monoidal structure, with the property that the tensor product in $\mathcal{C}$ preserves colimits separately in each variable. The symmetric monoidal product will usually be denoted $- \otimes_{\mathcal{C}} -$, and its associated unit by $\mathbb{1}_{\mathcal{C}}$.

We will also assume knowledge about *stable* ∞-categories, which are ∞-categorical enhancements of triangulated categories. One can, for example, construct presentable stable ∞-categories from presentable ∞-categories via *stabilization*, defined by formally inverting the desuspension functor Ω. The $(\infty, 2)$-category of presentable stable ∞-categories and exact colimit preserving functors, denoted Pr^L_{st}, inherits a symmetric monoidal structure from Pr^L. An ∞-category $\mathcal{C}$ is a *presentably symmetric monoidal stable ∞-category* if it is a commutative monoid in Pr^L_{st}. This means that it is presentably symmetric monoidal, and the tensor product preserves the stable structure.

Example 0.2.2. The stabilization of the ∞-category of spaces is the ∞-category of *spectra*, denoted Sp. It is the unit for the Lurie tensor product on Pr^L_{st}, and it is the initial presentably symmetric monoidal stable ∞-category. The unit of the symmetric monoidal structure is the *sphere spectrum* $\mathbb{S}$, which is the suspension spectrum of S^0.

Example 0.2.3. Another example is the derived ∞-category of abelian groups, $\mathrm{D}(\mathbb{Z})$. This is an ∞-categorical version of the classical triangulated derived category of $\mathbb{Z}$. It is a presentably symmetric monoidal stable ∞-category, and the unit is the integers $\mathbb{Z}$, treated as a chain complex concentrated in degree 0.

The derived ∞-category of $\mathbb{Z}$, as well as the ∞-categories of spaces, spectra, and many other interesting ∞-categories, satisfy some even nicer conditions than merely being presentable: they have an explicit collection of generators, which satisfy some "smallness" condition.

Definition 0.2.4. An object $x \in \mathcal{C}$ is said to be *compact* if the functor $\mathrm{Hom}_{\mathcal{C}}(x, -)$ commutes with filtered colimits. The full subcategory of compact objects in $\mathcal{C}$ will be denoted $\mathcal{C}^{\omega}$.

Example 0.2.5. The compact objects in $\mathcal{S}$ are the *finite spaces*, which correspond to the classical finite CW-complexes. The compact generators in Sp are the *finite spectra*, where a spectrum is finite if it is the desuspension of a suspension spectrum $\Sigma^{-n}\Sigma^{\infty}K$ for some number n, where K is a finite space. The compact objects in $\mathrm{D}(\mathbb{Z})$ are the *perfect complexes*, which are the bounded complexes of finitely generated projective abelian groups.

Definition 0.2.6. A presentable ∞-category $\mathcal{C}$ is *compactly generated* if $\mathcal{C}^{\omega}$ generates $\mathcal{C}$ under filtered colimits.

All three of the categories $\mathcal{S}$, Sp and $\mathrm{D}(\mathbb{Z})$ are compactly generated by their respective collection of compact objects. For the latter two, even more is true: they are compactly generated by their monoidal unit, being $\mathbb{S}$ and $\mathbb{Z}$ respectively. Such categories are sometimes called *monogenic*, but we will not focus on these in this thesis.

In the presence of symmetric monoidal structures we have another "smallness" condition, slightly different from being compact. As the symmetric monoidal structure is assumed to preserve colimits separately in each variable, the functor $X \otimes (-)$ has a right adjoint, denoted $\underline{\mathrm{Hom}}_{\mathcal{C}}(X, -)$, equipping $\mathcal{C}$ with an *internal hom* $\underline{\mathrm{Hom}}_{\mathcal{C}}(-,-)\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \longrightarrow \mathcal{C}$. This gives, in particular, a duality functor $(-)^\vee := \underline{\mathrm{Hom}}_{\mathcal{C}}(-, \mathbb{1}_{\mathcal{C}})\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathcal{C}$, sometimes referred to as the *linear dual*. The unit map on $X^\vee$ induces by adjunction a map $X \otimes X^\vee \longrightarrow \mathbb{1}_{\mathcal{C}}$, sometimes called the evaluation map. For any $Y \in \mathcal{C}$ this gives a map $X \otimes X^\vee \otimes X \longrightarrow Y$, given as $ev \otimes Y$.

Definition 0.2.7. An object $X \in \mathcal{C}$ is *dualizable* if for any other object Y, the map $X^\vee \otimes Y \longrightarrow \underline{\mathrm{Hom}}(X, Y)$, adjoint to the map $ev \otimes Y$, is an equivalence. The full subcategory of dualizable objects in $\mathcal{C}$ will be denoted $\mathcal{C}^{\mathrm{dual}}$.

In a certain sense, being compact is about being small with respect to colimits, while being dualizable is about being small with respect to the monoidal structure. For presentably symmetric monoidal ∞-categories we always have $\mathcal{C}^\omega \subseteq \mathcal{C}^{\mathrm{dual}}$, but in very well-behaved categories, these two notions of smallness do in fact coincide.

Definition 0.2.8. A presentably symmetric monoidal stable ∞-category $\mathcal{C}$ is *rigidly compactly generated* if it is compactly generated and $\mathcal{C}^\omega \simeq \mathcal{C}^{\mathrm{dual}}$.

Example 0.2.9. An example is again our favorite stable ∞-category Sp. Every compact object—being the finite spectra—is dualizable, and conversely, every dualizable object is compact. These also generate Sp, hence it is a rigidly compactly generated symmetric monoidal stable ∞-category. Another example is the derived category $\mathrm{D}(\mathbb{Z})$, which is is rigidly compactly generated by the perfect complexes.

Remark 0.2.10. As shown in [HPS97, 2.1.3], a presentably symmetric monoidal ∞-category $\mathcal{C}$ is rigidly compactly generated if $\mathcal{C}$ is compactly generated by dualizable objects, and the unit $\mathbb{1}_{\mathcal{C}}$ is compact.

0.2.1 Localizing subcategories and ideals

If we were to assign this thesis a single protagonist, it would be the idea of a localizing subcategory. It will heavily feature in all the different parts of the thesis:

1. In paper I we study how a specific localizing subcategory, appearing in chromatic homotopy theory, interacts with a specific homological functor.
2. In paper II we study how, in certain situations, the category of comodules over a coalgebra in a stable ∞-category forms a localizing subcategory.
3. In paper III we prove a classification result for certain localizing subcategories along nicely behaved t-structures on stable ∞-categories.

Given a presentable stable ∞-category $\mathcal{C}$, one should think about a localizing subcategory as being a collection of objects in $\mathcal{C}$, that themselves form a nice presentable stable ∞-category, compatible with $\mathcal{C}$. In other words, they are the "structure preserving subcategories", in a certain precise way.

Definition 0.2.11. If $\mathcal{C}$ is a presentable stable ∞-category, then a full subcategory $\mathcal{L} \subseteq \mathcal{C}$ is *localizing* if it is closed under retracts, desuspensions and colimits.

This means that $\mathcal{L}$ is itself a presentable stable ∞-category, and that computing colimits in $\mathcal{L}$ is equivalent to computing colimits in $\mathcal{C}$.

Definition 0.2.12. Let $\mathcal{C}$ be a presentable stable ∞-category. Given a collection of objects $\mathcal{K} \subseteq \mathcal{C}$ we denote by $\mathrm{Loc}_{\mathcal{C}}(\mathcal{K})$ the smallest localizing subcategory of $\mathcal{C}$ containing $\mathcal{K}$. We will often call it the localizing subcategory *generated* by $\mathcal{K}$. If the category $\mathcal{C}$ is understood, we will sometimes just write $\mathrm{Loc}(\mathcal{K})$ for simplicity.

Localizing subcategories are often very interconnected with the idea of "primeness". We give some examples for $\mathrm{D}(\mathbb{Z})$.

Example 0.2.13. Given a chain complex of abelian groups A one can form its p-localization $A_{(p)}$ by inverting the map $q\colon A \longrightarrow A$ for all other primes $q \neq p$. The derived category of all p-local abelian groups, denoted $\mathrm{D}(\mathbb{Z})^{p-\mathrm{loc}}$, is a localizing subcategory of $\mathrm{D}(\mathbb{Z})$. In fact, it is equivalent to $\mathrm{D}(\mathbb{Z}_{(p)})$, the derived category of the p-local integers, hence also generated by $\mathbb{Z}_{(p)}$, treated as a complex in degree 0.

Example 0.2.14. Inverting all primes p gives the *rationalization* of the complex A, often denoted $A_{\mathbb{Q}}$. The derived category of rational abelian groups is equivalent to the derived category of $\mathbb{Q}$, and is also a localizing subcategory of $\mathrm{D}(\mathbb{Z})$. This is also generated by the object $\mathbb{Q}$, treated as a chain complex in degree 0.

Example 0.2.15. The p-power torsion of a complex A is defined as $T_pA := \{a \in A \mid p^k a = 0 \text{ for some } k > 0\}$. There is a natural map $T_pA \longrightarrow A$, and the complex A is said to be p-power torsion if this map is an equivalence. The full subcategory of p-power torsion objects in $\mathrm{D}(Z)$, denoted $\mathrm{D}(\mathbb{Z})^{p-\mathrm{tors}}$, is a localizing subcategory.

Remark 0.2.16. If the collection $\mathcal{K} \subseteq \mathcal{C}$ consists of only compact objects, in the sense of Definition 0.2.4, then the localizing subcategory $\mathrm{Loc}_{\mathcal{C}}(\mathcal{K})$ is said to be a *compactly generated* localizing subcategory.

Example 0.2.17. Even though $\mathbb{Z}_{(p)}$ and $\mathbb{Q}$ are not perfect complexes in $\mathrm{D}(\mathbb{Z})$, their associated localizing subcategories are in fact compactly generated. The same is true for $\mathrm{D}(\mathbb{Z})^{p-\mathrm{tors}}$. We will come back to this in the next section.

Remark 0.2.18. A more rigorous way to state that a presentable stable ∞-category $\mathcal{C}$ is compactly generated—as defined in Definition 0.2.6—is to say that it is so if and only if the smallest localizing subcategory containing the collection of all compact objects $\mathcal{C}^\omega$ is the entire ∞-category $\mathcal{C}$. In other words, there is an equivalence $\mathcal{C} \simeq \mathrm{Loc}_{\mathcal{C}}(\mathcal{C}^\omega)$ of presentable stable ∞-categories.

If our presentable stable ∞-category is also symmetric monoidal, then we we want a version of localizing subcategories that preserve the monoidal structure. If one thinks of a presentably symmetric monoidal stable ∞-category as a categorified version of a ring, then the natural such sub-structure should model that of an ideal in a ring.

Definition 0.2.19. If $\mathcal{C}$ is a presentably symmetric monoidal stable ∞-category, then a full subcategory $\mathcal{L} \subseteq \mathcal{C}$ is a *localizing $\otimes$-ideal* if it is a localizing subcategory, and for any $L \in \mathcal{L}$ and $X \in \mathcal{C}$, we have $L \otimes X \in \mathcal{L}$.

The definition of an ideal here is completely analogous to the classical setting of discrete rings.

Definition 0.2.20. Let $\mathcal{C}$ be a presentably symmetric monoidal stable ∞-category. Given a collection of objects $\mathcal{K} \subseteq \mathcal{C}$ we denote by $\mathrm{Loc}^{\otimes}_{\mathcal{C}}(\mathcal{K})$ the smallest localizing $\otimes$-ideal of $\mathcal{C}$ containing $\mathcal{K}$. We will, as before, often refer to this as the localizing $\otimes$-ideal *generated* by $\mathcal{K}$.

Any ideal I in a discrete ring R is a non-unital subring of R. This is also the case for a localizing $\otimes$-ideal $\mathcal{L} \subseteq \mathcal{C}$, which becomes a non-unital presentably symmetric monoidal stable ∞-category. However, in some good cases $\mathcal{L}$ is actually unital, but the unit $\mathbb{1}_{\mathcal{L}}$ will naturally have to be different than the unit $\mathbb{1}_{\mathcal{C}}$, otherwise we would have $\mathcal{L} = \mathcal{C}$. The localizing ideals we study in Chapter 1, as well as some of the ones in Chapter 2, will have this property. In particular, as we will see in the next section, any localizing $\otimes$-ideal which is compactly generated in the sense of Remark 0.2.16 will have this property.

Example 0.2.21. The three examples we saw earlier, Example 0.2.13, Example 0.2.14 and Example 0.2.15, are all localizing $\otimes$-ideals. We will see in the next section that they, or some slight variation of these categories, are compactly generated localizing $\otimes$-ideals, which by the above comment mean they are all presentably symmetric monoidal stable ∞-categories themselves.

0.2.2 Local duality

The theory of abstract local duality, proved in [HPS97] and generalized to the ∞-categorical setting in [BHV18], is one of the central ideas of this thesis that will show up several times.

0.2.2.1 Localizations

To understand local duality, and also the use of localizing subcategories, we look at certain functors, called localizations. In spirit, these are functors that invert a certain class of maps.

Definition 0.2.22. Let $\mathcal{C}$ and $\mathcal{D}$ be two presentable stable ∞-categories. A *localization* is a colimit preserving functor $f\colon \mathcal{C} \longrightarrow \mathcal{D}$ such that the right adjoint i is fully faithful.

There are several ways to construct localizations, but one method particularly important for us will be via localizing subcategories.

Definition 0.2.23. Let $\mathcal{L} \subseteq \mathcal{C}$ be a full subcategory. The *right orthogonal complement* of $\mathcal{L}$, is the full subcategory $\mathcal{L}^{\perp}$ consisting of objects $X \in \mathcal{C}$ such that $\mathrm{Hom}_{\mathcal{C}}(L, X) \simeq 0$ for all $L \in \mathcal{L}$.

Example 0.2.24. Let $\mathcal{L} \subseteq \mathcal{C}$ be a localizing subcategory. The inclusion of the complement $\mathcal{L}^{\perp} \hookrightarrow \mathcal{C}$ is fully faithful and has a left adjoint $L\colon \mathcal{C} \longrightarrow \mathcal{L}^{\perp}$. Hence, L is a localization, and its kernel is precisely $\mathcal{L}$.

Example 0.2.25. Let $\mathcal{C} = \mathrm{D}(\mathbb{Z})$ and $\mathcal{L} = \mathrm{D}(\mathbb{Z}_{(p)})$ the derived category of p-local abelian groups. The right orthogonal complement of $\mathrm{D}(\mathbb{Z}_{(p)})$ is the category of derived p-complete abelian groups, $\mathrm{D}(\mathbb{Z})^{p-\mathrm{comp}}$. The associated localization Λ_p is the total left derived functor of the p-adic completion functor C^p, defined by sending an abelian group A to the colimit $A_p^{\wedge} = \mathrm{colim}_k\, A/p^k$. In other words, we have a localization

$$\Lambda_p \simeq \mathbb{L}C^p\colon \mathrm{D}(\mathbb{Z}) \longrightarrow \mathrm{D}(\mathbb{Z})^{p-\mathrm{comp}}.$$

We will mostly focus on localizations of presentably symmetric monoidal stable ∞-categories, hence we also want to make sure that the localizations of interest preserve the monoidal structure. This is done as follows.

Definition 0.2.26. Let $\mathcal{C}$ and $\mathcal{D}$ be two presentably symmetric monoidal stable ∞-categories and $f\colon \mathcal{C} \longrightarrow \mathcal{D}$ a functor. A map φ in $\mathcal{C}$ is called an *f-equivalence* if $f(\varphi)$ is an equivalence. The functor f is said to be *tensor-compatible* if being an f-equivalence is stable under tensor product: in the sense that for any f-equivalence $X \longrightarrow Y$ and object $Z \in \mathcal{C}$, the induced map $X \otimes Z \longrightarrow Y \otimes Z$ is again an f-equivalence.

Definition 0.2.27. Let $\mathcal{C}$ and $\mathcal{D}$ be two presentably symmetric monoidal stable ∞-categories. A *monoidal localization* is a tensor-compatible functor $f\colon \mathcal{C} \longrightarrow \mathcal{D}$ with a fully faithful right adjoint i.

Remark 0.2.28. For the rest of this thesis we will assume that, whenever $\mathcal{C}$ is a presentably symmetric monoidal, that any localization of $\mathcal{C}$ is tensor-compatible. We will usually omit the prefix "monoidal" from localizations. It is really only in paper III that we will see non-monoidal localizations, hence the distinction between them should hopefully be clear from the context.

Remark 0.2.29. Let $f\colon \mathcal{C} \longrightarrow \mathcal{D}$ be a localization. The composition of f with the fully faithful right adjoint i is denoted L. The functor i gives an equivalence between $\mathcal{D}$ and a full subcategory of $\mathcal{C}$, denoted $\mathcal{C}_L$. By [Lur09, 5.2.7.4] there is an equivalence between localizations $f\colon \mathcal{C} \longrightarrow \mathcal{D}$ and functors $L\colon \mathcal{C} \longrightarrow \mathcal{C}_L$ (L viewed as a functor to its essential image) that are left adjoint to the inclusion. We will usually use this perspective, using the functor L rather than f.

Definition 0.2.30. Given a localization $L\colon \mathcal{C} \longrightarrow \mathcal{C}_L$, any object $X \in \mathcal{C}$ admits a map $X \longrightarrow LX$ coming from the unit of the adjunction, called its *L-localization*. The object X is said to be *L-local* if this is an L-equivalence. By definition the category of L-local objects is $\mathcal{C}_L$.

Proposition 0.2.31 ([Lur17, 1.3.4.3])**.** *If $L\colon \mathcal{C} \longrightarrow \mathcal{C}_L$ is a localization, then the category of local objects $\mathcal{C}_L$ is equivalent to* *the full subcategory of $\mathcal{C}$ obtained by inverting the collection of L-equivalences W_L. In other words, there is an equivalence of symmetric monoidal stable ∞-categories $\mathcal{C}_L \simeq \mathcal{C}[W_L^{-1}]$.*

Remark 0.2.32. Let $L\colon \mathcal{C} \longrightarrow \mathcal{C}_L$ be a localization on a presentably symmetric monoidal stable ∞-category $\mathcal{C}$. The symmetric monoidal structure on $\mathcal{C}$ induces a symmetric monoidal structure on $\mathcal{C}_L$, defined by $L(- \otimes_{\mathcal{C}} -)$, making L into a symmetric monoidal functor. This follows from [Lur17, 2.2.1.9] by our standing assumption that all localizations on symmetric monoidal categories are tensor-compatible, see Remark 0.2.28.

Remark 0.2.33. If $L\colon \mathcal{C} \longrightarrow \mathcal{C}_L$ is a monoidal localization, then the kernel $\operatorname{Ker} L$ is a localizing $\otimes$-ideal of $\mathcal{C}$.

Remark 0.2.34. Similarly to localizations, we can define *colocalizations* as functors $g\colon \mathcal{C} \longrightarrow \mathcal{D}$ admitting a fully faithful left adjoint i. The composition $i \circ g$ is denoted Γ. The adjoint gives an equivalence between $\mathcal{D}$ and a full subcategory $\mathcal{C}^{\Gamma}$ of $\mathcal{C}$, and the datum of a colocalization is equivalent to the datum of a functor $\Gamma\colon \mathcal{C} \longrightarrow \mathcal{C}^{\Gamma}$ that is right adjoint to the inclusion. Dually to localizations, we get for any $X \in \mathcal{C}$ a colocalization map $\Gamma X \longrightarrow X$, and we say X is Γ-*colocal* if this is an equivalence.

Let $\mathcal{C}$ be a presentably symmetric monoidal stable ∞-category. For any localization $L\colon \mathcal{C} \longrightarrow \mathcal{C}_L$, the image of the unit $L\mathbb{1}_{\mathcal{C}}$ is a ring object, and any L-local object X admits the structure of an $L\mathbb{1}_{\mathcal{C}}$ module via the map of functors $L\mathbb{1}_{\mathcal{C}} \otimes L(-) \longrightarrow L(-)$.

Remark 0.2.35. By the tensor-internal hom adjunction, the $L\mathbb{1}_{\mathcal{C}}$-module structure on an L-local object X is equivalent to a map $L(-) \longrightarrow \underline{\operatorname{Hom}}(L\mathbb{1}_{\mathcal{C}}, -)$.

Definition 0.2.36. We say a localization L is *smashing* if the $L\mathbb{1}_{\mathcal{C}}$-module map above is an equivalence. This is equivalent to the dual map $L(-) \longrightarrow \underline{\operatorname{Hom}}_{\mathcal{C}}(L\mathbb{1}, -)$ being an equivalence.

Example 0.2.37. There is a p-localization functor $L\colon \mathrm{D}(\mathbb{Z}) \longrightarrow \mathrm{D}(\mathbb{Z}_{(p)})$ given by inverting all primes $q \neq p$. This is a smashing localization, and is then given by $L \simeq \mathbb{Z}_{(p)} \otimes_{\mathbb{Z}} (-)$.

Remark 0.2.38. For a smashing localization L we have by definition that being L-local is equivalent to being a module over $L\mathbb{1}_{\mathcal{C}}$. This gives a symmetric monoidal equivalence $\mathcal{C}_L \simeq \mathrm{Mod}_{L\mathbb{1}_{\mathcal{C}}}(\mathcal{C})$.

Remark 0.2.39. Similarly, for a colocalization Γ there are maps $\Gamma\mathbb{1}_{\mathcal{C}} \otimes \Gamma(-) \longrightarrow \Gamma(-)$ and $\Gamma(-) \longrightarrow \underline{\mathrm{Hom}}(\Gamma\mathbb{1}_{\mathcal{C}}, -)$. The colocalization Γ is said to be *smashing* if the former is an equivalence. In Remark 0.2.35 we noted that for a localization the module structure was equivalent to a map into $\underline{\mathrm{Hom}}(L\mathbb{1}_{\mathcal{C}}, -)$. For colocalizations this is no longer true, leading to two different notions of "modules" in this setting. This setup is studied in detail in paper II.

Remark 0.2.40. Any monoidal localization L equips $\mathcal{C}_L$ with a symmetric monoidal structure, as seen in Remark 0.2.32. If L is a smashing localization, then the induced tensor product is the same as in the category $\mathcal{C}$. The same applies to smashing colocalizations.

0.2.2.2 The local duality theorem

We are now ready to present the setup for local duality, which is a natural duality theory for compactly generated localizing $\otimes$-ideals.

Definition 0.2.41. A pair $(\mathcal{C}, \mathcal{K})$, consisting of a presentably symmetric monoidal stable ∞-category $\mathcal{C}$, that is compactly generated by dualizable objects, and a subset $\mathcal{K} \subseteq \mathcal{C}^{\omega}$, is called a *local duality context*.

Any choice of local duality context allows us to assign to it three new categories, which together decomposes the category $\mathcal{C}$.

Construction 0.2.42. Let $(\mathcal{C}, \mathcal{K})$ be a local duality context. We define the category of *$\mathcal{K}$-torsion objects* in $\mathcal{C}$ to be the localizing $\otimes$-ideal generated by $\mathcal{K}$, and denote it by $\mathcal{C}^{\mathcal{K}-\mathrm{tors}} := \mathrm{Loc}^{\otimes}_{\mathcal{C}}(\mathcal{K})$. Further we define the category of *$\mathcal{K}$-local objects* in $\mathcal{C}$ to be the right orthogonal complement—see Definition 0.2.23—of $\mathcal{C}^{\mathcal{K}-\mathrm{tors}}$. In other words $\mathcal{C}^{\mathcal{K}-\mathrm{loc}} := (\mathcal{C}^{\mathcal{K}-\mathrm{tors}})^{\perp}$. Finally we define the category of $\mathcal{K}$-*complete objects* in $\mathcal{C}$ to be the right orthogonal complement of $\mathcal{C}^{\mathcal{K}-\mathrm{loc}}$, i.e., $\mathcal{C}^{\mathcal{K}-\mathrm{comp}} = (\mathcal{C}^{\mathcal{K}-\mathrm{loc}})^{\perp}$.

These three categories have fully faithful inclusions into $\mathcal{C}$, denoted $i_{\mathcal{K}-\mathrm{tors}}$, $i_{\mathcal{K}-\mathrm{loc}}$ and $i_{\mathcal{K}-\mathrm{comp}}$ respectively. By the adjoint functor theorem, [Lur09, 5.5.2.9], the inclusions $i_{\mathcal{K}-\mathrm{loc}}$ and $i_{\mathcal{K}-\mathrm{comp}}$ have left adjoints $L_{\mathcal{K}}$ and $\Lambda_{\mathcal{K}}$ respectively, while $i_{\mathcal{K}-\mathrm{tors}}$ and $i_{\mathcal{K}-\mathrm{loc}}$ have right adjoints $\Gamma_{\mathcal{K}}$ and $V_{\mathcal{K}}$ respectively. These are then, by definition, localizations and colocalizations.

The torsion, local and complete objects all form $\otimes$-ideals, meaning that the localizations and colocalizations above are compatible with the symmetric monoidal structure of $\mathcal{C}$, in the sense of Definition 0.2.26. In particular, by Remark 0.2.32 the categories inherit unique induced symmetric monoidal structures such that $L_{\mathcal{K}}$, $\Lambda_{\mathcal{K}}$, $\Gamma_{\mathcal{K}}$ and $V_{\mathcal{K}}$ are symmetric monoidal functors.

For any $X \in \mathcal{C}$, these functors assemble into two cofiber sequences:

$$\Gamma_{\mathcal{K}}X \longrightarrow X \longrightarrow L_{\mathcal{K}}X \quad \text{and} \quad V_{\mathcal{K}}X \longrightarrow X \longrightarrow \Lambda_{\mathcal{K}}X.$$

Note also that these functors only depend on the localizing subcategory $\mathcal{C}^{\mathcal{K}-\mathrm{tors}}$, not on the particular choice of generators $\mathcal{K}$. Thus, when the set $\mathcal{K}$ is clear from the context, we sometimes omit it as a subscript when writing the functors.

Remark 0.2.43. By definition $\mathcal{C}^{\mathcal{K}-\mathrm{tors}}$ is compactly generated, and by [BHV18, 2.17] both $\mathcal{C}^{\mathcal{K}-\mathrm{loc}}$ and $\mathcal{C}^{\mathcal{K}-\mathrm{comp}}$ are as well.

The following theorem is a slightly simplified version of the abstract local duality theorem of [HPS97, 3.3.5] and [BHV18, 2.21].

Theorem 0.2.44. *If $(\mathcal{C}, \mathcal{K})$ is a local duality context, then*

1. *Γ is a smashing colocalization and L is a smashing localization,*
2. *there are equivalences of functors $\Lambda \simeq \underline{\mathrm{Hom}}(\Gamma\mathbb{1}, -)$ and $V \simeq \underline{\mathrm{Hom}}(L\mathbb{1}, -)$, and*

3. the functors

$$\Gamma\colon \mathcal{C}^{\mathcal{K}-\mathrm{comp}} \longrightarrow \mathcal{C}^{\mathcal{K}-\mathrm{tors}} \text{ and } \Lambda\colon \mathcal{C}^{\mathcal{K}-\mathrm{tors}} \longrightarrow \mathcal{C}^{\mathcal{K}-\mathrm{comp}}$$

are mutually inverse equivalences of symmetric monoidal stable ∞-categories.

This can be summarized by the following diagram of adjoints

$$\begin{tikzcd}
 & \mathcal{C}^{\mathcal{K}-\mathrm{loc}} \arrow[d, shift right=2] \arrow[d, shift left=2] \arrow[ddr, dashed, bend left] & \\
 & \mathcal{C} \arrow[u, "L"] \arrow[u, shift left=4, "V"'] \arrow[dl, shift left, "\Gamma"] \arrow[dr, shift left, "\Lambda"] & \\
\mathcal{C}^{\mathcal{K}-\mathrm{tors}} \arrow[ur, shift left] \arrow[rr, "\simeq"] \arrow[uur, dashed, bend left] & & \mathcal{C}^{\mathcal{K}-\mathrm{comp}} \arrow[ul, shift left]
\end{tikzcd}$$

Remark 0.2.45. Theorem 0.2.44 implies, in particular, that the symmetric monoidal structure induced by the localization L and the colocalization Γ is just the symmetric monoidal structure on $\mathcal{C}$ restricted to the full subcategories. This is not the case for $\mathcal{C}^{\mathcal{K}-\mathrm{comp}}$, where the symmetric monoidal structure is given by $\Lambda(- \otimes_{\mathcal{C}} -)$. The functor V also induces a symmetric monoidal structure on $\mathcal{C}^{\mathcal{K}-\mathrm{loc}}$, but this coincides with the one induced by L, due to their associated endofunctors on $\mathcal{C}$ defining an adjoint symmetric monoidal monad-comonad pair. Note that we will not need or focus on the functor V, hence it will usually be omitted from the local duality diagrams for the rest of the thesis.

Example 0.2.46. The object $\mathbb{Z}_{(p)}/p \cong \mathbb{F}_p$ is compact in the derived category of p-local abelian groups, $\mathrm{D}(\mathbb{Z}_{(p)})$. This means that $(\mathrm{D}(\mathbb{Z}_{(p)}), \mathcal{K})$ forms a local duality context, where $\mathcal{K}$ is the singleton set $\{\mathbb{F}_p\}$. The category of local objects, $\mathrm{D}(\mathbb{Z}_{(p)})^{\mathcal{K}-\mathrm{loc}}$, has objects in which p is invertible. But, as all other primes are already invertible, all of these are necessarily rational, giving $\mathrm{D}(\mathbb{Z}_{(p)})^{\mathcal{K}-\mathrm{loc}} \simeq \mathrm{D}(\mathbb{Q})$. The category $\mathrm{D}(\mathbb{Z}_{(p)})^{\mathcal{K}-\mathrm{tors}}$ is equivalent to the category of derived p-torsion objects in $\mathrm{D}(\mathbb{Z}_{(p)})$. Dually, the category $\mathrm{D}(\mathbb{Z}_{(p)})^{\mathcal{K}-\mathrm{comp}}$ is equivalent to the derived p-complete objects in $\mathrm{D}(\mathbb{Z})_{(p)}$, which gives a local duality diagram

$$\begin{tikzcd}
 & \mathrm{D}(\mathbb{Q}) \arrow[dr, dashed, bend left] \arrow[d, shift left] & \\
 & \mathrm{D}(\mathbb{Z}_{(p)}) \arrow[u, shift left, "L_p"] \arrow[dl, shift left, "\Gamma^p"] \arrow[dr, shift left, "\Delta_p"] & \\
\mathrm{D}(\mathbb{Z}_{(p)})^{p-\mathrm{tors}} \arrow[uur, dashed, bend left] \arrow[ur, shift left] \arrow[rr, "\simeq"] & & \mathrm{D}(\mathbb{Z}_{(p)})^{p-\mathrm{comp}} \arrow[ul, shift left]
\end{tikzcd}$$

0.2.3 Chromatic homotopy theory

As mentioned in the abstract, all of the three papers in this thesis are motivated by understanding certain aspects of chromatic homotopy theory. Even though good introductions to this topic abound, we give here an overview, trying in particular to tie it to the ideas already introduced. Our approach is inspired by [BB19].

Chromatic homotopy theory, in very crude words, is really a perspective—or maybe a toolbox—to study the ∞-category of spectra, Sp, in which one decomposes it to its smallest fundamental pieces. An often repeated analogy is that of a prism. If Sp consists of pure white light, then shining it through the "chromatic lens" decomposes it to distinct colors, labeled by an integer n called the *chromatic height*.

$$\begin{tikzcd}
 & & \mathcal{C}_\infty \\
 & & \vdots \\
\mathrm{Sp} \arrow[r] & \text{Chromatic lens} \arrow[uur] \arrow[ur] \arrow[r] \arrow[dr] \arrow[ddr] & \mathcal{C}_n \\
 & & \vdots \\
 & & \mathcal{C}_0
\end{tikzcd}$$

The key to this perspective is that the individual pieces of information can be reassembled back to give information about Sp. This happens in the form of a filtration on the sphere spectrum $\mathbb{S}$, called the *chromatic filtration*. The colimit of this filtration recovers the sphere, hence we can think of the main idea of chromatic homotopy theory as the following: in order to understand Sp, it should be enough to understand the "chromatic pieces" individually.

Remark 0.2.47. Historically these chromatic pieces came about from the relationship between spectra and the algebraic geometry of formal groups, as studied by Quillen in his seminal paper [Qui69]. To any complex oriented ring spectrum E, one can associate to it a formal group, see for example [Rav86, Appendix 2], and Quillen proved that the formal group associated to the complex cobordism spectrum MU is the universal formal group over the Lazard ring. The moduli stack of formal groups has a filtration by the *height* of a formal group, and pulling back this filtration to spectra gives precisely the chromatic filtration hinted to above.

0.2.3.1 Fracture squares and field objects

In light of Waldhausen's viewpoint of stable homotopy theory as an enhancement of algebra, usually called *brave new algebra*, one should view the category of spectra Sp as a homotopical enrichment of the derived category of abelian groups $\mathrm{D}(\mathbb{Z})$. We have seen earlier that abelian groups can be studied one prime at the time, which corresponds to studying $\mathrm{D}(\mathbb{Z}_{(p)})$, the p-local derived category.

In [Bou79b], Bousfield developed a general machinery for studying localizations on Sp, by inverting maps that are equivalences after tensoring with some spectrum F, now called *Bousfield localizations*. The corresponding localization functor is denoted L_F.

Example 0.2.48. We can create a version of p-localization on Sp, by Bousfield localizing at the p-local Moore spectrum $M\mathbb{Z}_{(p)}$, giving a functor usually written

$$L_{(p)}\colon \mathrm{Sp} \longrightarrow \mathrm{Sp}_{(p)}.$$

On homotopy groups this has the effect of p-localizing, in the sense that

$$\pi_* L_{(p)} X \simeq \pi_* X[S^{-1}] \simeq \pi_* X \otimes \mathbb{Z}_{(p)}$$

where S is the set of all primes $q \neq p$. The category of p-local spectra, denoted $\mathrm{Sp}_{(p)}$, should then be thought of as a homotopical enrichment of $\mathrm{D}(\mathbb{Z}_{(p)})$. Just as in Example 0.2.37, the p-localization functor $L_{(p)}$ is a smashing localization.

By using the classical arithmetic fracture square,

$$\begin{tikzcd}
\mathbb{Z}_{(p)} \arrow[r] \arrow[d] & \mathbb{Z}_p \arrow[d] \\
\mathbb{Q} \arrow[r] & \mathbb{Q} \otimes \mathbb{Z}_p
\end{tikzcd}$$

we see that we can decompose the p-local integers into a rational part and a p-complete part. This also extends to a general chain complex $A \in \mathrm{D}(\mathbb{Z})_{(p)}$, where we have a homotopy pullback square

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & A_p^\wedge \arrow[d] \\
\mathbb{Q} \otimes A \arrow[r] & \mathbb{Q} \otimes_{\mathbb{Z}} A_p^\wedge
\end{tikzcd}$$

where $(A)_p^\wedge$ denotes derived p-completion of A, as in Example 0.2.25. We want to use this to decompose $\mathrm{D}(\mathbb{Z}_{(p)})$ even further; reduce it to its "atomic pieces". These is done via its minimal localizing subcategories.

Definition 0.2.49. A localizing subcategory $\mathcal{L} \subseteq \mathcal{C}$ is said to be *minimal* if any proper localizing subcategory $\mathcal{L}' \subset \mathcal{L}$ is (0).

Remark 0.2.50. If $\mathcal{L}$ is a minimal localizing subcategory, then any non-zero object $K \in \mathcal{L}$ generates $\mathcal{L}$ as a localizing subcategory: $\mathrm{Loc}_{\mathcal{C}}(K) \simeq \mathcal{L}$.

The study of minimal localizing subcategories is tightly connected to local duality, as in Section 0.2.2. By [BHV18, 2.26], we get from any local duality diagram a fracture square, which for the local duality context $(\mathrm{D}(\mathbb{Z}_{(p)}), \mathbb{F}_p)$ gives precisely the classical arithmetic fracture square above.

Proposition 0.2.51. *Let $\mathcal{L}$ be a minimal localizing subcategory of $\mathrm{D}(\mathbb{Z}_{(p)})$. Then either $\mathcal{L} \simeq \mathrm{D}(\mathbb{Q})$ or $\mathcal{L}$ is equivalent to the category of derived p-complete objects, $\mathcal{L} \simeq \mathrm{D}(\mathbb{Z}_{(p)})^{p-\mathrm{comp}}$.*

Now, if $\mathrm{Sp}_{(p)}$ is supposed to be a homotopical enrichment of $\mathrm{D}(\mathbb{Z}_{(p)})$, we should expect there to be an analogy of this decomposition for p-local spectra, which is indeed the case. The first to study such squares in topology was Sullivan in his 1970 MIT notes, where he constructed the analogous square for nilpotent spaces, see [Sul05, 3.20]. This was later lifted up to spectra by Bousfield in [Bou79b, 2.9], and takes the following form.

If $\mathbb{S}_{(p)}$ denotes the p-local sphere spectrum, we have a spectral arithmetic fracture square

$$\begin{tikzcd}
\mathbb{S}_{(p)} \arrow[r] \arrow[d] & \mathbb{S}_p^{\wedge} \arrow[d] \\
H\mathbb{Q} \arrow[r] & H\mathbb{Q} \otimes \mathbb{S}_p^{\wedge}
\end{tikzcd}$$

where $\mathbb{S}_p^{\wedge}$ denotes the p-complete sphere. This also extends to any object $X \in \mathrm{Sp}_{(p)}$, just like for $A \in \mathrm{D}(\mathbb{Z}_{(p)})$.

We can then ask the natural question: do these give all the minimal localizing subcategories of $\mathrm{Sp}_{(p)}$? This was indeed the case for $\mathrm{D}(\mathbb{Z}_{(p)})$, but now, in the more complicated world of spectra, this is no longer true. We now have an infinite sequence of minimal localizing subcategories, indexed by a natural number n, that "interpolates" between the two minimal localizing subcategories of $\mathrm{D}(\mathbb{Z}_{(p)})$.

Remark 0.2.52. In fact, even more is true: By the failure of the telescope conjecture, see [Bur+23], there are at least two such infinite sequences. We can make sure that there is a single such sequence if we translate over to compactly generated $\otimes$-ideals, but for the above exposition, we have chosen to push these details under a big old telescope-shaped rug.

We can identify these "intermediary" subcategories by an analysis of field objects. For $\mathrm{D}(\mathbb{Z}_{(p)})$ there are exactly two field objects associated to the unit $\mathbb{Z}_{(p)}$, namely $\mathbb{Q}$ and $\mathbb{F}_p$, each obtained by either killing or inverting p.

For $\mathrm{Sp}_{(p)}$ we have a field object for any number $n \in \mathbb{N} \cup \{\infty\}$, usually denoted $K(n)$, or $K_p(n)$ if we want to remember the prime. We have $K_p(0) = H\mathbb{Q}$ and $K_p(\infty) = H\mathbb{F}_p$, which shows that this sequence of field objects really forms an interpolation between the two field objects coming from algebra.

Notation 0.2.53. The object $K_p(n)$ is called the *Morava K-theory of height n*. The associated category of $K_p(n)$-local spectra—meaning the category obtained by Bousfield localization at $K_p(n)$—is denoted $\mathrm{Sp}_{K_p(n)}$.

These field objects $K_p(n)$ were constructed by Morava in the early 70's, by topologically interpreting the unique geometric point in the moduli stack of formal groups, determined by the height n Honda formal group. The categories of $K_p(n)$-local spectra have been under intense study ever since. We do not cover precise constructions here and instead refer the interested reader to [HS99]. The Morava K-theory spectrum $K_p(n)$ is, however, uniquely determined by the following properties.

Proposition 0.2.54. *Let p be a prime and n a natural number. The height n Morava K-theory spectrum $K_p(n)$ is a complex oriented $\mathbb{E}_1$-ring spectrum with coefficients*

$$K_p(n)_* := \pi_* K_p(n) \simeq \mathbb{F}_p[v_n^{\pm}],$$

with $|v_n| = 2p^n - 2$, whose associated formal group is the height n Honda formal group. Furthermore, for any two spectra X and Y, there is a Künneth isomorphism

$$K_p(n)_*(X \times Y) \simeq K_p(n)_* X \otimes_{K_p(n)_*} K_p(n)_* Y.$$

Remark 0.2.55. By [HS99, 7.5] the categories of $K_p(n)$-local spectra are minimal. One can show that for finite heights n, the categories $\mathrm{Sp}_{K_p(n)}$ are all the minimal localizing subcategories of $\mathrm{Sp}_{(p)}$. But, there are some issues with height ∞, not yet allowing us to create a full classification.

Remark 0.2.56. While the $\mathbb{E}_1$-ring structure on $K_p(n)$ can be shown to be essentially unique, it does admit uncountably many $\mathbb{E}_1$-MU-algebra structures—see [Ang11].

Remark 0.2.57. The category $\mathrm{Sp}_{K_p(n)}$ is compactly generated by dualizable objects, but it is *not* a rigidly compactly generated category, in the sense of Definition 0.2.8, as the unit $L_{K_p(n)}\mathbb{S}$—the $K_p(n)$-local sphere—is not compact.

It now remains to understand how these field objects $K_p(n)$ are related to the spectral arithmetic fracture square above. If the $\mathrm{Sp}_{K_p(n)}$'s all form minimal localizing subcategories, and they in some sense interpolate between rational information at height 0, and p-local information at height ∞, then we should perhaps expect there to be an infinite sequence of fracture squares—starting from $L_{\mathbb{Q}}\mathbb{S} \simeq H\mathbb{Q}$, and converging to $\mathbb{S}_{(p)}$. This is indeed the case.

Construction 0.2.58. Let $L_{n,p} := L_{K_p(0)\vee\cdots\vee K_p(n)}$. By Ravenel's smash product theorem, see [Rav92, 7.5.6], the functor $L_{n,p}\colon \mathrm{Sp}_{(p)} \longrightarrow \mathrm{Sp}_{(p)}$ is a smashing localization. The *chromatic fracture square*, generally attributed to Hopkins, is then of the form

$$\begin{tikzcd} L_{n,p}\mathbb{S} \arrow[r] \arrow[d] & L_{K_p(n)}\mathbb{S} \arrow[d] \\ L_{n-1,p}\mathbb{S} \arrow[r] & L_{n-1,p}\mathbb{S} \otimes L_{K_p(n)}\mathbb{S} \end{tikzcd}$$

By definition we ave $L_{0,p}\mathbb{S} = L_{\mathbb{Q}}\mathbb{S} \simeq H\mathbb{Q}$, hence the starting point is exactly what we wanted. As alluded to earlier, the spectra $L_{n,p}\mathbb{S}$ assemble into a a filtration,

$$\cdots \longrightarrow L_{3,p}\mathbb{S} \longrightarrow L_{2,p}\mathbb{S} \longrightarrow L_{1,p}\mathbb{S} \longrightarrow L_{0,p}\mathbb{S} = L_{\mathbb{Q}}\mathbb{S}$$

called the chromatic filtration. By the chromatic convergence theorem of Hopkins-Ravenel, see [Rav92, 7.5.7], we can recover $\mathbb{S}_{(p)}$

as the colimit of this diagram. This is the more precise meaning of the statement that the Morava K-theory spectra $K_p(n)$ allow us to interpolate between rational and p-local information.

Remark 0.2.59. The arithmetic fracture square for $\mathrm{D}(\mathbb{Z}_{(p)})$ was "categorified" into the local duality diagram for the local duality context $(\mathrm{D}(\mathbb{Z}_{(p)}), \mathbb{F}_p)$, in the sense that the associated fracture square to the diagram was exactly the arithmetic one. We would like to have a similar property for the chromatic fracture square. In order to do this, we first need to understand the localization functor $L_{n,p}$ that showed up above.

0.2.3.2 Morava E-theories

In the previous section, we obtained a localization functor $L_{n,p}$, which collected the information coming from height 0 up to and including height n. This localization is good for many purposes, but when we later want to tie the homotopy theory to algebra, we need another approach. In particular, we want a spectrum E such that the Bousfield localization L_E is the same as $L_{n,p}$. There are several approaches to obtaining such a spectrum E, and the goal of this section is to give a brief overview of the ones we will need later. We will assume general knowledge about formal groups—all needed background can be found in [Rav86, Appendix 2]. Our overview is inspired by lecture notes made by Rognes, [Rog23].

The first construction of a spectrum E satisfying the above is due to Morava, and is based on the aforementioned connection between complex oriented cohomology theories and formal groups. In honor of this one usually refers to any conveniently nice spectrum with the property that $L_E \simeq L_{n,p}$ as *Morava E-theory*.

Construction 0.2.60. Let p be a prime and k a perfect field of characteristic p. Lubin and Tate proved in [LT66] that for any formal group law F of height n over k, there is a universal deformation $\bar{F}$ over the ring $E(k, F) = \mathbb{W}(k)[\![u_1, \ldots, u_{n-1}]\!]$ of formal power series over the Witt vectors of k. This ring is now usually called the *Lubin-Tate ring* of F. Using the algebraic geometry of formal groups, Morava interpreted this universal deformation as a formal neighborhood of the height n Honda formal group law, and via a topological realization process obtained a spectrum $E_{n,p}^{Mor}$.

We will not explain in detail how Morava obtained such a spectrum, and instead cover a more simple approach, yielding a slightly different spectrum. This spectrum was originally constructed by Johnson and Wilson in [JW75], by using the theory of manifolds with singularities developed by Baas-Sullivan (see [Baa73a] and [Baa73b]). We take a more modern approach, utilizing Brown representability.

Construction 0.2.61. Let p be a prime, n a natural number and $E_p(n)_* := \mathbb{Z}_{(p)}[v_1, \dots, v_{n-1}, v_n^{\pm}]$. This is obtained from the coefficient ring of the Brown–Peterson spectrum BP—essentially a p-local version of the complex cobordism spectrum MU—by killing all the generators v_k for $k > n$ and inverting v_n. This ring is a BP_*-module, and satisfies a certain flatness condition called *Landweber flatness*, see [Lan76]. Tensoring with $E_p(n)_*$ is not exact as a $\mathrm{Mod}_{\mathrm{BP}_*} \longrightarrow \mathrm{Mod}_{E_p(n)_*}$, but it *is* exact as a functor on $\mathrm{Comod}_{\mathrm{BP}_*\mathrm{BP}}$. Hence, as BP-homology lands in this Grothendieck abelian category—BP is Adams type—we can define a functor

$$\mathrm{Sp} \xrightarrow{\mathrm{BP}_*} \mathrm{Comod}_{\mathrm{BP}_*\mathrm{BP}} \xrightarrow{E_p(n)_* \otimes_{\mathrm{BP}_*} -} \mathrm{grAb},$$

which by the properties mentioned is a homology theory. Via Brown's representability theorem, see [Bro62, Theorem 1], this homology theory is governed by a spectrum $E_p(n)$ such that $\pi_* E_p(n) \simeq E_p(n)_*$. This spectrum is called the height n *Johnson-Wilson theory*.

Remark 0.2.62. The ring $E_p(n)_*$ is local, and has a unique maximal ideal $I_n = (p, v_1, \dots, v_{n-1})$ called the Landweber ideal. The quotient of $E_p(n)_*$ by this maximal ideal gives $E_p(n)_*/I_n \cong \mathbb{F}_p[v_n^{\pm}] = K_p(n)_*$. By utilizing some results from [EK20], this can also be suitably interpreted as a quotient of spectra.

Definition 0.2.63. An $\mathbb{E}_1$-ring spectrum R is said to be concentrated in degrees divisible by q if $\pi_k R \cong 0$ for all $k \neq 0 \bmod q$.

Proposition 0.2.64. *Let p be a prime and n a natural number. Height n Johnson–Wilson theory $E_p(n)$ is a complex oriented, Landweber exact, $\mathbb{E}_1$-ring spectrum concentrated in degrees divisible by $2p-2$.*

Remark 0.2.65. The Johnson–Wilson spectrum $E_p(n)$ is also periodic, with period $2p^n - 2$. This is the same periodicity that the Morava K-theory spectrum has.

Later, using a 2-periodic analogue of the universal deformation theory of Lubin and Tate, Hopkins and Miller constructed a 2-periodic $\mathbb{E}_1$-version of Morava's spectrum, which was later enhanced to an $\mathbb{E}_\infty$-ring spectrum $E_{n,p}$ via Goerss–Hopkins theory, see [GH04], or [PV22] for a modern treatment. In essence, Hopkins and Miller constructed a functor from pairs (k, F) of a perfect field k of characteristic p, together with a choice of height n formal group law F, to even periodic ring spectra. For a specific choice of (k, F), we can summarize the properties as follows.

Proposition 0.2.66. *Let p be a prime, k a perfect field of characteristic p, and F a formal group law of height n over k. The spectrum $E(k, F)$ is a 2-periodic, complex oriented, Landweber exact, $\mathbb{E}_\infty$-ring spectrum, such that $\pi_0 E(k, F) = \mathbb{W}(k)[\![u_1, \ldots, u_{n-1}]\!]$ and the associated formal group law is the universal deformation of F.*

Definition 0.2.67. For the specific choice $(k, F) = (\mathbb{F}_{p^n}, H_n)$ we simply write $E(\mathbb{F}_{p^n}, H_n) = E_{n,p}$, and call it the height n *Lubin–Tate theory* at the prime p.

Remark 0.2.68. One can also study maps of ring spectra

$$E_{n,p} \longrightarrow K_{n,p}$$

such that the induced map on homotopy groups is given by taking the quotient by the maximal ideal, just as in Remark 0.2.62. Such spectra $K_{n,p}$ are 2-periodic versions of Morava K-theory and have been studied, for example, in [HL17] and [BP23].

Remark 0.2.69. One nice benefit of $E_{n,p}$ compared to $E_p(n)$—other than it being fully coherently commutative rather than just

$\mathbb{E}_1$—is that the former is $K_p(n)$-local, making its chromatic behavior even more interesting. In fact, the unit map $L_{K_p(n)}\mathbb{S} \longrightarrow E_{n,p}$ is a pro-Galois extension in the sense of [Rog08], where the Galois group is the extended Morava stabilizer group $\mathbb{G}_n$, see [DH04]. We can, however, also make the latter $K_p(n)$-local, by instead using a completed version $\widehat{E}_p(n)$, often called *completed Johnson-Wilson theory*. It has most of the same properties as that of $E_p(n)$, except that it is $K_p(n)$-local and its coefficients are p-adic and I_n-complete:

$$\widehat{E}_p(n)_* \simeq \mathbb{Z}_p[v_1, \cdots, v_{n-1}, v_n^{\pm}]^{\wedge}_{I_n}.$$

Remark 0.2.70. An $\mathbb{E}_\infty$-version of Morava's original spectrum $E_{n,p}^{Mor}$ can be recovered from $E_{n,p}$ by taking the homotopy fixed points with respect to the Galois action from $\mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p) \cong \mathbb{Z}/n$. Another alternative is to use $E_{n,p}^{h\mathbb{F}_p^\times}$. This spectrum is concentrated in degrees divisible by $2p-2$, hence serves as a nice $\mathbb{E}_\infty$-version of the $\mathbb{E}_1$-ring spectrum $E_p(n)$. This is the model of E used, for example, in Barkan's monoidal algebraicity theory, see [Bar23].

We have now introduced several versions of Morava E-theories, all in light of trying to understand the localization functor $L_{n,p}$. Hence, we round off this section by stating that the Bousfield localizations at any of the above E-theories are equivalent.

Proposition 0.2.71 ([Hov95, 1.12])**.** *If p is a prime and n a natural number, then there are symmetric monoidal equivalences of stable ∞-categories*

$$\mathrm{Sp}_{L_{n,p}} \simeq \mathrm{Sp}_{E_p(n)} \simeq \mathrm{Sp}_{E(k,F)} \simeq \mathrm{Sp}_{E_{n,p}} \simeq \mathrm{Sp}_{\widehat{E}_p(n)} \simeq \mathrm{Sp}_{E_{n,p}^{h\mathbb{F}_p^\times}}.$$

In fact, all of these categories are equivalent as subcategories of Sp*. Furthermore, if E is any Landweber exact v_n-periodic* BP*-algebra, then* Sp_E *is equivalent to the above categories.*

Notation 0.2.72. We will use the common notation $\mathrm{Sp}_{n,p}$ for any of the above categories, and call it the category of $E_{n,p}$-local spectra, or sometimes the category of height n spectra, or even just the category of E-local spectra, when the height and prime is understood.

Remark 0.2.73. The category $\mathrm{Sp}_{n,p}$ is compactly generated by the collection of dualizable objects $\{L_{n,p}F\}$, where F is a finite spectrum. In fact, the monoidal unit $L_{n,p}\mathbb{S}$ *is* a compact object, hence $\mathrm{Sp}_{n,p}$ is rigidly compactly generated, in contrast to $\mathrm{Sp}_{K_p(n)}$—see Remark 0.2.57.

Remark 0.2.74. Note that even though the different models for $\mathrm{Sp}_{n,p}$ are equivalent, some of them have non-equivalent associated module categories. For example, $\mathrm{Mod}_{E_{n,p}} \not\simeq \mathrm{Mod}_{E_p(n)}$, as the ring spectra $E_{n,p}$ and $E_p(n)$ have different periodicity—the former is 2-periodic while the latter is $(2p^n-2)$-periodic. Whenever such a distinction is relevant, we will make this explicit.

0.2.3.3 Monochromatic spectra and local duality

Recall from Section 0.2.3.1 that our goal is to understand the $K_p(n)$-local pieces of the category of p-local spectra, $\mathrm{Sp}_{(p)}$. By Remark 0.2.59, we are looking for a local duality theory that categorifies the chromatic fracture square. In this section, we construct precisely such a local duality theory, both for $\mathrm{Sp}_{n,p}$ and for modules over E for some choice of Morava E-theory.

Definition 0.2.75. A spectrum X is called *n-monochromatic* if it is $E_{n,p}$-local and $E_{n-1,p}$-acyclic. The full subcategory of n-monochromatic spectra will be denoted $\mathcal{M}_{n,p}$ and referred to as the height n monochromatic category.

If the height is understood, we will sometimes drop the n from the notation. We have a convenient way to produce monochromatic spectra from $E_{n,p}$-local ones.

Definition 0.2.76. Let $X \in \mathrm{Sp}_{n,p}$. The fiber of the localization $X \longrightarrow L_{n-1,p}X$, which we denote $M_{n,p}X$ is called the n'th *monochromatic layer* of X at the prime p.

Remark 0.2.77. If X is a monochromatic spectrum, then it is $L_{n-1,p}$-acyclic by definition, i.e., $L_{n-1,p}X \simeq 0$. Hence the fiber sequence

$$M_{n,p}X \longrightarrow X \longrightarrow L_{n-1,p}X$$

gives an equivalence $X \simeq M_{n,p}X$. The fully faithful inclusion $\mathcal{M}_{n,p} \longrightarrow \mathrm{Sp}_{n,p}$ has a right adjoint, given by $X \longmapsto M_{n,p}X$, which we call the *monochromatization.*

Proposition 0.2.78. *The monochromatization functor*

$$M_{n,p}\colon \mathrm{Sp}_{n,p} \longrightarrow \mathcal{M}_{n,p}$$

is a smashing colocalization, in the sense of Remark 0.2.39.

Proof. As far we are aware, this proposition was first proved in [Bou96, Sec 6.3] in the case of finite monochromatization, i.e., the fiber functor of the finite localization $L^f_{n,p}$. The proof, however, uses the arguments from [Bou79a, 2.10], which also work for the non-finite case. A simplified argument uses Ravenel's smash product theorem, see [Rav92, 7.5.6], stating that the localization $L_{n-1,p} = L_{E_{n-1,p}}$ is smashing. Hence, we can simply compare the two fiber sequences

$$M_{n,p}\mathbb{S} \otimes X \longrightarrow X \longrightarrow L_{n-1,p}\mathbb{S} \otimes X$$

and

$$M_{n,p}X \longrightarrow X \longrightarrow L_{n-1,p}X,$$

which immediately identifies the fibers. $\square$

We are now almost ready to construct local duality for chromatic homotopy theory. The last thing we need is is a good set of compact objects to form our local duality context.

Definition 0.2.79. A finite p-local spectrum X is said to be of *type* n if $K_p(n)_*X \ncong 0$ and $K_p(m)_*X \cong 0$ for all $m < n$.

As a consequence of the thick subcategory theorem of Hopkins–Smith, [HS98, Theorem 7], such spectra exist for all primes p and natural numbers n. For example, if $n = 1$, we can choose the mod p Moore spectrum $\mathbb{S}/p$.

Construction 0.2.80. Let n be a non-negative integer and p a prime. For a finite type n spectrum $F(n)$, its $L_{n,p}$-localization $L_{n,p}F(n)$ is a compact object in $\mathrm{Sp}_{n,p}$ and hence generates a localizing tensor ideal $\mathrm{Sp}_{n,p}^{\mathcal{K}-\mathrm{tors}}$ in $\mathrm{Sp}_{n,p}$, where $\mathcal{K}$ denotes the singleton set $\{L_{n,p}F(n)\}$. By Theorem 0.2.44, we have a corresponding local duality diagram for the local duality context $(\mathrm{Sp}_{n,p}, \mathcal{K})$:

$$\begin{tikzcd}
 & \mathrm{Sp}_{n,p}^{\mathcal{K}-\mathrm{loc}} \arrow[d, shift left] \arrow[ddr, dashed, bend left] & \\
 & \mathrm{Sp}_{n,p} \arrow[u, shift left, "L"] \arrow[dl, shift left, "\Gamma"] \arrow[dr, shift left, "\Lambda"] & \\
\mathrm{Sp}_{n,p}^{\mathcal{K}-\mathrm{tors}} \arrow[ur, shift left] \arrow[uur, dashed, bend left] \arrow[rr, "\simeq"] & & \mathrm{Sp}_{n,p}^{\mathcal{K}-\mathrm{comp}} \arrow[ul, shift left]
\end{tikzcd}$$

We wanted to construct a local duality diagram that categorified the chromatic fracture square, and we claim that the above diagram does the trick.

Proposition 0.2.81. *There are equivalences*

1. $\mathrm{Sp}_{n,p}^{\mathcal{K}-\mathrm{tors}} \simeq \mathcal{M}_{n,p}$
2. $\mathrm{Sp}_{n,p}^{\mathcal{K}-\mathrm{loc}} \simeq \mathrm{Sp}_{n-1,p}$
3. $\mathrm{Sp}_{n,p}^{\mathcal{K}-\mathrm{comp}} \simeq \mathrm{Sp}_{K_p(n)}$

of symmetric monoidal stable ∞-categories.

Remark 0.2.82. These equivalences are by now classical, but we recall an arguments for the reader's convenience and for building intuition.

Proof. By definition the category $\mathcal{M}_{n,p}$ is the full subcategory of $L_{n-1,p}$-acyclic objects in $\mathrm{Sp}_{n,p}$ and $M_{n,p}$ coincides with the $L_{n-1,p}$-acyclification. By [HS99, 6.10] $L_{n-1,p}$-localization is the finite localization away from $L_{n,p}F(n)$, which proves equivalence (2). This also means that the $L_{n-1,p}$-acyclic objects are precisely the objects in

$$\mathrm{Loc}^{\otimes}_{\mathrm{Sp}_{n,p}}(L_{n,p}F(n)) =: \mathrm{Sp}_{n,p}^{\mathcal{K}-\mathrm{tors}},$$

giving the equivalences $\mathcal{M}_{n,p} \simeq \mathrm{Sp}_{n,p}^{\mathcal{K}-\mathrm{tors}}$ and $\Gamma \simeq M_{n,p}$, proving (1). One can also see this by the fact that $M_{n,p}$ preserves compact objects, as it is smashing by Proposition 0.2.78, which also implies that $\mathcal{M}_{n,p}$ is closed under colimits. The compact objects $L_{n,p}X \in \mathrm{Sp}_{n,p}$ for X any finite spectrum of type n are also monochromatic, as

$$E_{n-1,p*}L_{n,p}X \cong E_{n-1,p*}X \cong 0,$$

and they do in fact generate $\mathcal{M}_{n,p}$ under colimits.

The equivalence in (3) follows from [BHV18, 2.34], which shows that $\Lambda_{\mathcal{K}}$ can be identified with the Bousfield localization L_K whenever the set of compact objects in a local duality context $(\mathcal{C}, \mathcal{K})$ consists of a single element $\mathcal{K} = \{K\}$. Note that the localization L_K is not the same as the functor $L_{\mathcal{K}}$. This, together with the fact that the Bousfield localizations $L_{K_p(n)}$ and $L_{L_{n,p}F(n)}$ agree, see [HS99, 7.1], proves (3). □

Remark 0.2.83. The equivalence $\mathrm{Sp}_{n,p}^{\mathcal{K}-\mathrm{tors}} \xrightarrow{\simeq} \mathrm{Sp}_{n,p}^{\mathcal{K}-\mathrm{comp}}$ is then given by the adjoint pair $(L_{K_p(n)} \dashv M_{n,p})$, which recovers the symmetric monoidal equivalence $\mathcal{M}_{n,p} \simeq \mathrm{Sp}_{K_p(n)}$ of [HS99, 6.19].

Remark 0.2.84. The local duality diagram we obtained in Construction 0.2.80 gives via [BHV18, 2.26] precisely the chromatic fracture square

$$\begin{tikzcd}
L_{n,p}\mathbb{S} \arrow[r] \arrow[d] & L_{K_p(n)}\mathbb{S} \arrow[d] \\
L_{n-1,p}\mathbb{S} \arrow[r] & L_{n-1,p}\mathbb{S} \otimes L_{K_p(n)}\mathbb{S}
\end{tikzcd}$$

just as we wanted in Remark 0.2.59.

Remark 0.2.85. By Remark 0.2.43, all the categories in the local duality diagram are compactly generated. But, the unit $L_{K_p(n)}\mathbb{S}$ in $\mathrm{Sp}_{K_p(n)}$ is not compact, so by Remark 0.2.10 the compact objects and the dualizables might differ. The same is then necessarily true for $\mathcal{M}_{n,p}$.

We have a similar construction in the case of modules over any of the models for Morava E-theory we presented in Section 0.2.3.2.

Construction 0.2.86. Let n be a non-negative integer, p a prime, and E some height n Morava E theory (eg. Johnson–Wilson theory or Lubin–Tate theory). The object E/I_n is compact in Mod_E and generates a localizing tensor ideal

$$\mathrm{Mod}_E^{I_n-\mathrm{tors}} := \mathrm{Loc}^{\otimes}_{\mathrm{Mod}_E}(E/I_n),$$

where we use the superscript I_n for simplicity. By Theorem 0.2.44, we have a corresponding local duality diagram for the local duality context $(\mathrm{Mod}_E, \{E/I_n\})$:

$$\begin{tikzcd}
 & \mathrm{Mod}_E^{I_n-\mathrm{loc}} \arrow[d, shift left] \arrow[ddr, dashed, bend left] & \\
 & \mathrm{Mod}_E \arrow[u, shift left, "L"] \arrow[dl, shift left, "\Gamma"] \arrow[dr, shift left, "\Lambda"] & \\
\mathrm{Mod}_E^{I_n-\mathrm{tors}} \arrow[ur, shift left] \arrow[uur, dashed, bend left] \arrow[rr, "\simeq"] & & \mathrm{Mod}_E^{I_n-\mathrm{comp}} \arrow[ul, shift left]
\end{tikzcd}$$

Just as in Construction 0.2.80 there are equivalences

$$\mathrm{Mod}_E^{I_n-\mathrm{tors}} \simeq \mathcal{M}_{n,p}\mathrm{Mod}_E \tag{1}$$
$$\mathrm{Mod}_E^{I_n-\mathrm{loc}} \simeq L_{n-1,p}\mathrm{Mod}_E \tag{2}$$
$$\mathrm{Mod}_E^{I_n-\mathrm{comp}} \simeq L_{K_p(n)}\mathrm{Mod}_E \tag{3}$$

where the first category is the full subcategory of monochromatic E-modules, the second is the full subcategory of $E_{n-1,p}$-local E-modules and the third is the full subcategory of $K_p(n)$-local E-modules.

0.2.4 Hopf algebroids and their comodules

As mentioned before, this thesis is about understanding certain relationship between stable homotopy theory and homological algebra. One of these relationships—perhaps the most well known one—comes from the notion of *homology*.

Definition 0.2.87. Let R be a ring spectrum. Associated to R we have an *R-homology* functor

$$R_*\colon \mathrm{Sp} \longrightarrow \mathrm{Ab}$$

defined by $R_*(X) := \pi_*(R \otimes X)$.

As $R \otimes X$ is an R-module for any spectrum X, the R-homology functor actually lands in the category of R_*-modules, where

$$R_* := R_*(\mathbb{S}) = \pi_* R.$$

In fact, even more is true: the image of a spectrum X under $R_*(-)$ has certain cooperations, coming from the relationship between R_* and $R_*R := R_*(R)$, making it more structured than just an R_*-module. This section is precisely about understanding this extra structure.

Definition 0.2.88. A ring spectrum R is called *flat* if R_*R is a flat module over R_*. We say R is of *Adams type* if it can be written as a filtered colimit $R \simeq \operatorname{colim}_\alpha R_\alpha$, where each R_α is a finite spectrum such that R_*R_α is a finitely generated projective R_*-module and the natural map

$$R^*R_\alpha \longrightarrow \operatorname{Hom}_{R_*}(R_*R_\alpha, R_*)$$

is an isomorphism.

In particular, all Adams type ring spectra are flat, as the filtered colimit $R \simeq \operatorname{colim}_\alpha R_\alpha$ gives a filtered colimit

$$R_*R \cong \operatorname{colim}_\alpha R_*R_\alpha$$

of projective objects.

Most of the following examples were given by Adams in [Ada74, III.13.4]—except for the Morava theories.

Example 0.2.89 ([Hov04, 1.4.7, 1.4.9])**.** The ring spectra MU, MSp, KO, $H\mathbb{F}_p$, $K_p(n)$, $E_p(n)$, $E_{n,p}$ are all of Adams type. We also have the following class of examples: if R is Adams type, then any Landweber exact R-algebra is also Adams type.

Given a flat ring spectrum R, we can now make the following constructions.

Construction 0.2.90. From the unit map $\mathbb{S} \longrightarrow R$, the multiplication map $\mu\colon R \otimes R \longrightarrow R$ and the twist map $\tau\colon R \otimes R \longrightarrow R \otimes R$, we get maps on R_*-homology

1. $\eta_L\colon R_* \longrightarrow R_*R$, from the identification $R \otimes \mathbb{S} \simeq R$
2. $\eta_R\colon R_* \longrightarrow R_*R$, from the identification $\mathbb{S} \otimes R \simeq R$
3. $\varepsilon\colon R_*R \longrightarrow R$, from μ
4. $c\colon R_*R \longrightarrow R_*R$, from τ
5. $R_*(R \otimes R) \longrightarrow R_*R$, from μ

There is a comparison map

$$R_*R \otimes_{R_*} R_*R \longrightarrow R_*(R \otimes R) \simeq \pi_*(R \otimes R \otimes R),$$

which is an isomorphism precisely when R is flat. There is also a map

$$R \otimes R \longrightarrow R \otimes \mathbb{S} \otimes R \longrightarrow R \otimes R \otimes R$$

induced by the unit map, which on homotopy groups gives

$$\nabla\colon R_*R \longrightarrow \pi_*(R \otimes R \otimes R) \xleftarrow{\cong} R_*R \otimes_{R_*} R_*R.$$

The left and right unit maps η_L and η_R make R_*R a bimodule over R_*, hence this map is a *comultiplication* on R_*R. As $R \otimes R$ is again a ring spectrum, we also get an induced multiplication map

$$\Delta\colon R_*R \otimes_{R_*} R_*R \longrightarrow R_*R$$

The relations on ring spectra also induce relations on the pair (R_*, R_*R), like coassociativity, counitality, and the antipode relation.

Remark 0.2.91. If R is a field object, for example, $K_p(n)$ or $H\mathbb{F}_p$, then the operations described above, together with the associated relations, make (R_*, R_*R) into a Hopf algebra. In particular, the left and right unit maps are equal: $\eta_L = \eta_R$. In the case of $H\mathbb{F}_p$, the Hopf algebra $H\mathbb{F}_{p*}H\mathbb{F}_p$ is the dual Steenrod algebra $\mathcal{A}_p^*$, see [Mil58].

Not all flat ring spectra are field objects, so we cannot expect to obtain a Hopf algebra in general. However, we do obtain a similar, more general structure, encapsulated by the concept of Hopf algebroids.

Definition 0.2.92. A (graded) *Hopf algebroid* is a cogroupoid object (A, Ψ) in the category of graded commutative rings.

By now the use of Hopf algebroids in situations related to homotopy theory has a long tradition. Good introductions can be found in [Rav86, A.1] and [Hov04].

We can also mimic the notion of being Adams type in the setting of Hopf algebroids, and then naturally relate these to ring spectra of Adams type.

Definition 0.2.93. We say a Hopf algebroid (A, Ψ) is of *Adams type* if Ψ is a filtered colimit of dualizable comodules Ψ_j.

The key feature, which by now hopefully is expected, is that the maps in Construction 0.2.90 make the pair (R_*, R_*R) into a Hopf algebroid.

Proposition 0.2.94 ([Hov04, 1.4.6])**.** *If R is a flat ring spectrum, then the pair (R_*, R_*R) is a Hopf algebroid. If R is Adams type, then (R_*, R_*R) is an Adams Hopf algebroid.*

Given a Hopf algebroid (A, Ψ) we can always talk about modules over the ring A. The added extra structure of cooperations, as mentioned earlier, now comes from adding in the relationship to Ψ.

Definition 0.2.95. Let (A, Ψ) be a Hopf algebroid. A *comodule* over (A, Ψ), sometimes referred to as a Ψ-comodule, is an A-module M together with a coassociative and counital map $\psi\colon M \longrightarrow M \otimes_A \Psi$. The category of comodules over (A, Ψ) is denoted Comod_Ψ.

Example 0.2.96. For any commutative graded ring A, the pair (A, A) is a called a *discrete Hopf algebroid*. The category of comodules over this Hopf algebroid is the normal Grothendieck abelian category Mod_A of modules over A.

Construction 0.2.97. Given an Adams Hopf algebroid (A, Ψ), we can define a discretization map $\varepsilon\colon (A, \Psi) \longrightarrow (A, A)$, which is given by the identity on A and the counit on Ψ. By [Rav86, A1.2.1] and [BHV18, 4.6] it induces a faithful exact forgetful functor $\varepsilon_*\colon \mathrm{Comod}_\Psi \longrightarrow \mathrm{Mod}_A$ with a right adjoint ε^* given by $\varepsilon^*(M) \simeq \Psi \otimes_A M$. A comodule in the essential image of ε^* is called an *extended comodule*, or sometimes a *cofree comodule*. Given a Ψ-comodule N, the A-module $\varepsilon_* N$ is called the *underlying module* of N.

It is not true that the category Comod_Ψ is Grothendieck abelian in general. But, it will be the case in the examples we are interested in.

Proposition 0.2.98 ([Hov04, 1.3.1, 1.4.1])**.** *If (A, Ψ) is an Adams Hopf algebroid, then the category Comod_Ψ is a Grothendieck abelian category generated by the dualizable comodules. Furthermore, there is a symmetric monoidal product $- \otimes_\Psi -$, which on the underlying modules is the normal tensor product of A-modules. It has a right adjoint $\underline{\mathrm{Hom}}_\Psi(-,-)$, making Comod_Ψ a closed symmetric monoidal category.*

Example 0.2.99. If R is an Adams type ring spectrum, then Proposition 0.2.98 and Proposition 0.2.94 implies that the category Comod_{R_*R} is Grothendieck abelian, and generated by the dualizable objects. Given any spectrum X, then its R_*-homology R_*X has a coaction

$$R_*X \longrightarrow R_*X \otimes_{R_*} R_*R,$$

which is both coassociative and counital, meaning that the R-homology functor $R_*(-)$ takes values in the more structured category Comod_{R_*R}.

In Section 0.2.3.2 we saw several versions of E-theory, and by Proposition 0.2.71 we know that all their corresponding E-local categories are equivalent. The same occurs for the categories of comodules associated to their respective Adams Hopf algebroid.

Proposition 0.2.100 ([HS05a, 4.2])**.** *If p is a prime and n a positive natural number, then the categories of comodules over the Hopf algebroids associated to $E_{n,p}$, $E_p(n)$ and $A = E_{n,p}^{h\mathbb{F}_p^\times}$ are all equivalent:*

$$\mathrm{Comod}_{E_{n,p*}\mathbb{E}_{n,p}} \simeq \mathrm{Comod}_{E_p(n)_*E_p(n)} \simeq \mathrm{Comod}_{A_*A}.$$

Furthermore, given any Landweber exact v_n-periodic BP*-algebra E, then its associated category of comodules is equivalent to the ones above.*

Notation 0.2.101. When the chromatic height n and the prime p is understood, we will use the common notation Comod_{E_*E} for any of the above categories.

Example 0.2.102. As we will use this functor a lot, we hammer in the fact that the above Morava E-theories are of Adams type. This means that the E-homology functor lands naturally in Comod_{E_*E}. We will mostly focus on the homology theory

$$E_* \colon \mathrm{Sp} \longrightarrow \mathrm{Comod}_{E_*E},$$

where we have restricting the domain to $\mathrm{Sp}_{n,p}$. This is because it is a *conservative* functor, meaning that it detects equivalences.

Remark 0.2.103. In algebraic geometry, Hopf algebroids are usually formulated dually as groupoid objects in affine schemes. The left and right unit maps $A \rightrightarrows \Psi$ induces a presentation of stacks $\mathrm{Spec}(\Psi) \rightrightarrows \mathrm{Spec}(A)$, and the category Comod_Ψ is equivalent to the category of quasi-coherent sheaves on the presented stack, see [Nau07, Thm 8].

Remark 0.2.104. In Example 0.2.89 we saw that the complex cobordism spectrum MU was an Adams type ring spectrum. In light of Remark 0.2.103 we can then state the celebrated result of Quillen, see [Qui69], relating stable homotopy theory to formal groups:

$$\mathrm{Spec}(\mathrm{MU}_*\mathrm{MU}) \rightrightarrows \mathrm{Spec}(\mathrm{MU}*) \to \mathcal{M}_{\mathrm{fg}},$$

where $\mathcal{M}_{fg}$ is the moduli stack of formal groups. Note that we are brushing some technical details under the rug here, but this is correct at least in spirit. Under these ideas, the category Comod_{E_*E} corresponds to quasi-coherent sheaves on the open substack $\mathcal{M}_{\mathrm{fg}}^{\leqslant n} \subset \mathcal{M}_{\mathrm{fg}}$ consisting of formal groups of height less than or equal to n.

As in Section 0.2.2, we have certain objects that are especially important—the compact objects and the dualizable objects. In Grothendieck abelian categories it is, in addition, important to understand the injective objects. This will also become important later in Chapter 1, as we will use injective objects to approximate other objects and to build certain spectral sequences.

Proposition 0.2.105. *Let (A, Ψ) be an Adams Hopf algebroid. A Ψ-comodule M is dualizable if and only if its underlying A-module $\varepsilon_* M$ is dualizable, i.e., it is finitely generated and projective. Similarly, a Ψ-comodule is compact if and only if its underlying A-module is compact, which coincides with being finitely presented.*

Proof. The first claim is [Hov04, 1.3.4] and the second is [Hov04, 1.4.2]. □

Remark 0.2.106. As colimits in Comod_Ψ are exact and are computed in Mod_A, all the dualizable comodules are compact. Hence, the full subcategory of dualizable comodules is a set of compact generators for Comod_Ψ.

Let us turn our attention to the injective objects.

Proposition 0.2.107 ([HS05b, 2.1])**.** *Let (A, Ψ) be an Adams Hopf algebroid. If I is an injective object in Comod_Ψ, then there is an injective A-module Q, such that I is a retract of the extended comodule $\Psi \otimes_A Q$.*

Remark 0.2.108. Note that as Comod_Ψ is Grothendieck abelian, it has enough injective objects. This allows us to construct injective resolutions and thus Ext-groups, which is a highly effective computational technique in stable homotopy theory. For example, the pair $(\mathbb{F}_2, \mathcal{A}_2^*)$ where the latter is the dual Steenrod algebra, is a Hopf algebroid. The groups $\mathrm{Ext}^s_{\mathcal{A}_*}(\mathbb{F}_2, \mathbb{F}_2)$ are used in the Adams spectral sequence to approximate homotopy groups of spheres, see [Ada58].

Given an Adams Hopf algebroid (A, Ψ), we also have an associated derived category. By [Hov04, 2.1.2, 2.1.3] the category of chain complexes of Ψ-comodules, Ch_Ψ, has a cofibrantly generated stable symmetric monoidal model structure—see also [BR11] for a slightly different construction. The homotopy category associated to this model structure is the usual unbounded derived category $\mathrm{D}(\mathrm{Comod}_\Psi)$ associated to the Grothendieck abelian category Comod_Ψ.

Notation 0.2.109. We will use $\mathrm{D}(\Psi)$ as our notation for the underlying symmetric monoidal stable ∞-category associated with the above model structure. The monoidal unit is A, treated as a chain complex in degree 0.

Remark 0.2.110. We warn the reader that some authors use the notation $\mathrm{D}(\Psi)$ to refer to a periodic version of derived category of (A, Ψ), especially in the case of $\Psi = E_*E$. This is the case, for example, in [Pst21]. We will use this periodic category in Chapter 1, and will keep the two derived categories notationally distinct.

The discretization adjunction of Construction 0.2.97 also induces an adjunction on the level of derived categories, allowing us to compare the derived category of (A, Ψ) to the derived category of A.

Proposition 0.2.111. *Let (A, ψ) be an Adams Hopf algebroid. Then the discretization adjunction $(\varepsilon_* \dashv \varepsilon^*)\colon \mathrm{Comod}_\Psi \longrightarrow \mathrm{Mod}_A$ induces an adjunction $(\varepsilon_* \dashv \varepsilon^*)\colon \mathrm{D}(\Psi) \longrightarrow \mathrm{D}(A)$.*

Proof. This follows from the fact that Ψ is flat over A, which implies that both ε_* and ε^* on the abelian categories are exact. □

0.2.4.1 Torsion and completion for comodules

As we have stated, the notion of localizing subcategories will be important in this thesis; not only in the situation of stable ∞-categories, but also in the Grothendieck abelian situation. In the following section we review the construction of a particular kind of localizing subcategory of the Grothendieck abelian category Comod_Ψ associated to the Hopf algebroid (A, Ψ). These are the categories of comodules that are torsion with respect to some nicely behaved ideal $I \subseteq A$. We briefly saw this in Example 0.2.15, but make a more comprehensive treatment here.

We will consider two approaches to torsion in $\mathrm{D}(\Psi)$: one "internal" and one "external". The internal approach uses the classical theory of torsion objects in abelian categories, while the external uses local duality, as in Theorem 0.2.44. These two will luckily be equivalent in the situations we are interested in.

We first review the abelian situation: the internal approach. We follow [BHV18] and [BHV20] in notation and results.

Definition 0.2.112. Let A be a commutative ring and $I \subseteq R$ a finitely generated ideal. The I-power torsion of an A-module M is defined as

$$T_I^A M = \{x \in M \mid I^k x = 0 \text{ for some } k \in \mathbb{N}\}.$$

We say a module M is *I-power torsion* if the natural comparison map $T_I^A M \longrightarrow M$ is an isomorphism. We denote the full subcategory consisting of I-power torsion A-modules by $\mathrm{Mod}_A^{I-\mathrm{tors}}$.

Example 0.2.113. Of particular importance for Chapter 1 is the category $\mathrm{Mod}_{E_*}^{I_n-\mathrm{tors}}$, where E is the Johnson–Wilson spectrum $E_p(n)$, and I_n is the Landweber ideal

$$I_n = (p, v_1, \ldots, v_{n-1}) \subseteq \pi_* E_p(n).$$

Definition 0.2.114. Let A be a commutative ring and $I \subseteq R$ a finitely generated ideal. The I-adic completion of an A-module M is defined as

$$C_I^A M = \lim_k A/I^k \otimes_A M.$$

We say a module M is *I-adically complete* if the natural map $M \longrightarrow C_I^A M$ is an isomorphism.

Remark 0.2.115. The resulting category of I-adically complete modules is not very well-behaved. The I-adic completion functor is usually neither left nor right exact, and the category is often not abelian. To fix these issues, Greenlees and May introduced the notion of L-complete modules in [GM92], using instead the zeroth left derived functor $L = \mathbb{L}_0 C_I^A$. One then defines *I-complete* modules, also called L-complete modules, to be those R-modules such that the natural map $M \longrightarrow LM$ is an equivalence. The full subcategory of I-complete A-modules is denoted by $\mathrm{Mod}_A^{I-\mathrm{comp}}$

Remark 0.2.116. In nice cases the abelian category $\mathrm{Mod}_A^{I-\mathrm{tors}}$ is even Grothendieck. The category $\mathrm{Mod}_A^{I-\mathrm{comp}}$ is abelian, but not Grothendieck in general. It is, however, a locally presentable abelian category with enough projective objects, which is kind of "dual" to being Grothendieck abelian.

Remark 0.2.117. The inclusion of the full subcategory $\mathrm{Mod}_A^{I-\mathrm{tors}} \hookrightarrow \mathrm{Mod}_A$ has a right adjoint, which coincides with the I-power torsion $T_I^A(-)$. This gives the I-power torsion another description as the colimit

$$T_I^A M \cong \mathrm{colim}_k \, \underline{\mathrm{Hom}}_A(A/I^k, M).$$

The means that the category $\mathrm{Mod}_R^{I-\mathrm{tors}}$ is a localizing subcategory of Mod_R, in the sense that it is a full subcategory closed under quotients, subobjects, extensions and arbitrary coproducts.

We want to extend the construction of I-torsion and L-complete modules to general Adams Hopf algebroids (A, Ψ). For this, we need to choose sufficiently nice ideals that interact nicely with the additional comodule structure.

Definition 0.2.118. Let (A, Ψ) be an Adams Hopf algebroid, and I an ideal in A. We say I is an *invariant ideal* if, for any comodule M, the comodule IM is a subcomodule of M. If I is finitely generated by $(x_1, \ldots, x_r)$ and x_i is non-zero-divisor in $R/(x_1, \ldots, x_{i-1})$ for each $i = 1, \ldots, r$, then we say I is *regular*.

Definition 0.2.119. Let (A, Ψ) be an Adams Hopf algebroid and $I \subseteq A$ a regular invariant ideal. The I-power torsion of a comodule M is defined as

$$T_I^{\Psi} M = \{x \in M \mid I^k x = 0 \text{ for some } k \in \mathbb{N}\}.$$

We say a comodule M is *I-power torsion* if the natural map $T_I^{\Psi} M \longrightarrow M$ is an equivalence. The full subcategory of I-power torsion comodules is denoted $\mathrm{Comod}_{\Psi}^{I-\mathrm{tors}}$.

Remark 0.2.120. By [BHV18, 5.10] the full subcategory of I-power torsion comodules is a Grothendieck abelian category. It also inherits a symmetric monoidal structure from Comod_{Ψ}.

Example 0.2.121. Of particular importance for Chapter 1 and Section 3.A is the category $\mathrm{Comod}_{E_*E}^{I_n-\mathrm{tors}}$, where E is some version of Morava E-theory, and I_n is again the Landweber ideal. This example will follow us through the whole thesis, as it is the "abelian version" of the monochromatic category $\mathcal{M}_{n,p}$—we will use a significant amount of time and effort to study their relationship.

Notation 0.2.122. Since $\mathrm{Comod}_{\Psi}^{I-\mathrm{tors}}$ is Grothendieck abelian we have an associated derived stable ∞-category $\mathrm{D}(\mathrm{Comod}_{\Psi}^{I-\mathrm{tors}})$ which we denote simply by $\mathrm{D}(\Psi^{I-\mathrm{tors}})$.

Remark 0.2.123. Unfortunately, the corresponding versions of I-adically complete and L-complete comodules do not in general form abelian categories, as we can have problems with the comodule structure on certain cokernels. However, there are some ways to better the situation, see for example [Bak09].

Remark 0.2.124. As for the case of modules, the inclusion $\mathrm{Comod}_{\Psi}^{I-\mathrm{tors}} \hookrightarrow \mathrm{Comod}_{\Psi}$ has a right adjoint that corresponds to the I-power torsion construction T_I^{Ψ}. This, by [BHV18, 5.5] also has the alternative description

$$T_I^{\Psi} M \cong \mathrm{colim}_k \, \underline{\mathrm{Hom}}_{\Psi}(A/I^k, M).$$

This again means that the category $\mathrm{Comod}_{\Psi}^{I-\mathrm{tors}}$ is a localizing subcategory of Comod_{Ψ}.

Comparing side by side, the constructions of I-power torsion in Mod_A and Comod_Ψ are completely analogous. Hence, one can wonder whether they agree on the underlying modules. This turns out to be the case.

Lemma 0.2.125 ([BHV18, 5.7])**.** *For any Ψ-comodule M there is an isomorphism of A-modules $\varepsilon_* T_I^\Psi M \cong T_I^A \varepsilon_* M$. Furthermore, if an A-module N is I-power torsion, then the extended comodule $\Psi \otimes_A N$ is I-power torsion. In particular, a Ψ-comodule M is I-power torsion if and only if the underlying A-module is I-power torsion.*

As mentioned above, we will later make use of injective objects in $\mathrm{Comod}_\Psi^{I-\mathrm{tors}}$. Hence, we relate some facts about these.

Lemma 0.2.126. *Let (A, Ψ) be an Adams Hopf algebroid and I a regular invariant ideal.*

1. *If J is an injective object in Comod_Ψ, then $T_I^\Psi J$ is an injective object in $\mathrm{Comod}_\Psi^{I-\mathrm{tors}}$.*
2. *There are enough injective objects in $\mathrm{Comod}_\Psi^{I-\mathrm{tors}}$.*
3. *Any injective J' in $\mathrm{Comod}_\Psi^{I-\mathrm{tors}}$ is a retract of an object of the form $T_I^\Psi J$ for an injective Ψ-comodule J.*

Proof. The first point is [BS12, 2.1.4], while the second is a consequence of $\mathrm{Comod}_\Psi^{I-\mathrm{tors}}$ being Grothendieck abelian, as mentioned in Remark 0.2.120. The third point is stated in the proof of [BHV20, 3.16]. □

Remark 0.2.127. Choosing a discrete Hopf algebroid (A, A), Lemma 0.2.126 implies that injective objects in $\mathrm{Mod}_A^{I-\mathrm{tors}}$ are retracts of $T_I^A(Q)$ for some injective A-module Q and that T_I^A preserves injective objects. As noted in Proposition 0.2.107, an injective object in Comod_Ψ is a retract of an extended comodule of the form $\Psi \otimes_A Q$ for an injective A-module Q. This means that any injective object J in $\mathrm{Comod}_\Psi^{I-\mathrm{tors}}$ is a retract of $T_I^\Psi(\Psi \otimes_A Q)$ where Q is an injective A-module.

Remark 0.2.128. As colimits in $\mathrm{Comod}_\Psi^{I-\mathrm{tors}}$ are computed in Comod_Ψ, we have, similar to Proposition 0.2.105, that an I-power torsion Ψ-comodule M is dualizable if and only if its underlying A-module is finitely generated and projective. Similarily, it is compact if and only if the underlying A-module is finitely presented.

We now move to the external approach, using local duality as in Section 0.2.2.

Construction 0.2.129. If (A, Ψ) is an Adams Hopf algebroid and $I \subseteq A$ a regular invariant ideal, then A/I, treated as a complex concentrated in degree zero, is by [BHV18, 5.13] a compact object in $\mathrm{D}(\Psi)$. Thus, $(\mathrm{D}(\Psi), \{A/I\})$ is a local duality context, and we can consider the corresponding local duality diagram

$$\begin{tikzcd}
 & \mathrm{D}(\Psi)^{I-\mathrm{loc}} \arrow[d, shift left] \arrow[ddr, dashed, bend left] & \\
 & \mathrm{D}(\Psi) \arrow[u, shift left, "L_I^\Psi"] \arrow[dl, shift left, "\Gamma_I^\Psi"] \arrow[dr, shift left, "\Delta_I^\Psi"] & \\
\mathrm{D}(\Psi)^{I-\mathrm{tors}} \arrow[ur, shift left] \arrow[uur, dashed, bend left] \arrow[rr, "\simeq"] & & \mathrm{D}(\Psi)^{I-\mathrm{comp}} \arrow[ul, shift left]
\end{tikzcd}$$

where we have used the superscript I instead of A/I for simplicity. This gives, in particular, a definition of I-torsion objects in $\mathrm{D}(\Psi)$ as $\mathrm{D}(\Psi)^{I-\mathrm{tors}}$.

Our goal was to give two constructions and prove that they were equal in the cases we were interested in.

Lemma 0.2.130 ([BHV20, 3.7(2)]). *Let (A, Ψ) be an Adams Hopf algebroid and $I \subseteq A$ a regular invariant ideal. There is an equivalence of categories*

$$\mathrm{D}(\Psi)^{I-\mathrm{tors}} \simeq \mathrm{D}(\Psi^{I-\mathrm{tors}}).$$

Furthermore, an object $M \in \mathrm{D}(\Psi)$ is I-torsion if and only if the homology groups $H_* M$ are I-power torsion Ψ-comodules.

One can wonder whether the same is true for the I-complete derived category, but this is unfortunately not true, as $\mathrm{Comod}_{\Psi}^{I-\mathrm{comp}}$ is not abelian. A partial result can, however, be recovered for discrete Hopf algebroids (A, A). We follow [BHV20] in the following construction.

Construction 0.2.131. Recall that $\mathrm{Mod}_A^{I-\mathrm{comp}}$ denotes the category of I-complete A-modules for $I \subseteq A$ a regular ideal. By [BHV20, 2.11] the category has enough projective objects, hence by [Lur17, 1.3.2] we can associate to it the right bounded category $\mathrm{D}^{-}(\mathrm{Mod}_A^{I-\mathrm{comp}})$. This has a by [Lur17, 1.3.2.19, 1.3.3.16] a left complete t-structure with heart equivalent to $\mathrm{Mod}_A^{I-\mathrm{comp}}$. We can then form its right completion, which we denote $\overline{\mathrm{D}}(\mathrm{Mod}_A^{I-\mathrm{comp}})$, and call the completed derived category of $\mathrm{Mod}_A^{I-\mathrm{comp}}$.

This is what allows for a partial version of Lemma 0.2.130 in the case of I-completion.

Proposition 0.2.132 ([BHV20, 3.7(1)])**.** *Let A be a commutative ring and $I \subseteq A$ a regular ideal. Then, there is an equivalence*

$$\mathrm{D}(\mathrm{Mod}_A)^{I-\mathrm{comp}} \simeq \overline{\mathrm{D}}(\mathrm{Mod}_A^{I-\mathrm{comp}}),$$

where the former category is the full subcategory of A/I-complete objects in $\mathrm{D}(\mathrm{Mod}_A)$ while the latter is the completed derived category of $\mathrm{Mod}_A^{I-\mathrm{comp}}$.

0.3 Summaries

Even though each of the papers contain their own introduction, we include an individual summary using the material from the introduction. The focus here is on themes, intuition and the connection to chromatic homotopy theory, not on overly technical details—there are plenty of these in the rest of the thesis.

0.3.1 Paper I

In the introduction we set up a way to compare the symmetric monoidal stable ∞-category $\mathrm{Sp}_{n,p}$ to the Grothendieck abelian category Comod_{E_*E}, via the E-homology functor

$$\begin{aligned} E_* \colon \mathrm{Sp}_{n,p} &\longrightarrow \mathrm{Comod}_{E_*E} \\ X &\longmapsto \pi_*(E \otimes X) \end{aligned}$$

see Example 0.2.102. There are many uses for such homology theories in general, but this one is particularly nice—it is possible to lift injective resolutions in Comod_{E_*E} to resolutions in $\mathrm{Sp}_{n,p}$. This property allows one to set up an E-based Adams spectral sequence for computing homotopy classes of maps in $\mathrm{Sp}_{n,p}$. For the sphere we obtain a spectral sequence with signature

$$E_2^{s,t} := \mathrm{Ext}^{s,t}_{E_*E}(E_*, E_*) \implies \pi_{t-s} L_{n,p}\mathbb{S},$$

where the Ext-groups are computed in Comod_{E_*E}. At large enough primes $p \gg n$ this spectral sequence collapses to an isomorphism $\pi_* L_{n,p}\mathbb{S} \cong \mathrm{Ext}^{*,*}_{E_*E}(E_*, E_*)$, hinting at the fact that $\mathrm{Sp}_{n,p}$ acts in a more and more algebraic fashion at large primes.

The precise formulation of this conjectured relationship is due to Franke in [Fra96], where he conjectures that at large primes there should be an equivalence between the homotopy categories $h\mathrm{Sp}_{n,p} \simeq h\mathrm{D}^{\mathrm{per}}(\mathrm{Comod}_{E_*E})$—the latter being a certain periodic version of the derived category of the Hopf algebroid E_*E. By work of Pstrągowski and Patchkoria–Pstrągowski, see [Pst21] and [PP21], this conjectured relationship was proven to hold. This equivalence can not be lifted to an equivalence on the level of ∞-categories, hence it is often said to be an *exotic* equivalence.

The resulting sloan is then: *chromatic homotopy theory is exotically algebraic at large primes.*

In the introduction we saw that we have well-behaved local duality theories for both the category $\mathrm{Sp}_{n,p}$ and the category $\mathrm{D}(E_*E)$. These were respectively given by $(\mathrm{Sp}_{n,p}, L_{n,p}F(n))$ for a type n spectrum $F(n)$, and $(\mathrm{D}(E_*E), E_*/I_n)$ for the Landweber ideal $I_n = (p, v_1, \ldots, v_{n-1}) \subseteq E_* = \pi_* E_{n,p}$.

For certain values of n and p, the type n spectrum $F(n)$ can be chosen to be a *Smith–Toda* complex, defined by satisfying $E_*(L_{n,p}F(n)) \cong E_*/I_n$. For example, at $n = 1$ and $p > 2$ we can chose $L_{n,p}F(n) = L_{n,p}\mathbb{S}/p$. Such Smith–Toda complexes do not always exist, but it still hints at a relationship between the two local duality theories: they should be connected via E-homology.

The goal of paper I is to make this connection come to life.

The compact object $L_nF(n) \in \mathrm{Sp}_{n,p}^{\omega}$ generates the localizing $\otimes$-ideal $\mathcal{M}_{n,p}$, and the compact object $E_*/I_n \in \mathrm{D}(E_*E)^{\omega}$ generates the localizing $\otimes$-ideal $\mathrm{D}(E_*E)^{I_n-\mathrm{tors}}$, which by Lemma 0.2.130 is equivalent to $\mathrm{D}(E_*E^{I_n-\mathrm{tors}})$. We prove that E-homology restricts to a well-behaved homology theory

$$E_*\colon \mathcal{M}_{n,p} \longrightarrow \mathrm{Comod}_{E_*E}^{I_n-\mathrm{tors}}$$

which by the general machinery set up in [PP21] proves that also the localizing ideals $\mathcal{M}_{n,p}$ and $\mathrm{D}(E_*E^{I_n-\mathrm{tors}})$ have to be exotically equivalent—up to again switching the latter for a periodic version.

Theorem 0.3.1 (Theorem 1.B)**.** *For $p \gg n$ there is an equivalence of homotopy categories*

$$h\mathcal{M}_{n,p} \simeq h\mathrm{D}^{\mathrm{per}}(E_*E^{I_n-\mathrm{tors}}).$$

By utilizing the local duality equivalences $\mathcal{M}_{n,p} \simeq \mathrm{Sp}_{K_p(n)}$ and $\mathrm{D}(E_*E)^{I_n-\mathrm{tors}} \simeq \mathrm{D}(E_*E)^{I_n-\mathrm{comp}}$ we can then obtain a version of this exotic algebraicity result for the category of $K_p(n)$-local spectra:

Theorem 0.3.2 (Theorem 1.A)**.** *For $p \gg n$ there is an equivalence of homotopy categories*

$$h\mathrm{Sp}_{K_p(n)} \simeq h\mathrm{D}^{\mathrm{per}}(E_*E)^{I_n-\mathrm{comp}}.$$

This gives a new slogan: *monochromatic homotopy theory is exotically algebraic at large primes.*

0.3.2 Paper II

In the introduction we saw that $E_{n,p}$-localization is a smashing localization, see Construction 0.2.58 or [Rav92, 7.5.6]. This gives, via Remark 0.2.38, an equivalence

$$\mathrm{Sp}_{n-1,p} \simeq \mathrm{Mod}_{L_{n-1,p}\mathbb{S}}(\mathrm{Sp}_{n,p}).$$

We know by Proposition 0.2.81 that the kernel of $L_{n-1,p}$ is the category of monochromatic spectra $\mathcal{M}_{n,p}$, hence one can wonder if there is an analogous description of this category in terms of the category of some other type of "module". This turns out to be true: it is equivalent to the category of comodules over the monochromatization of the sphere, $M_{n,p}\mathbb{S}$, treated as a cocommutative coalgebra in $\mathrm{Sp}_{n,p}$.

This means that we have "algebraic", or module-like, descriptions for two out of the three categories appearing in the local duality diagram associated to the local duality context $(\mathrm{Sp}_{n,p}, L_{n,p}F(n))$, see Construction 0.2.80.

Paper II concerns the following question: is there a module-like description of the last part of the local duality diagram, $\mathrm{Sp}_{K_p(n)}$?

One answer is given by taking inspiration from algebra, where one has the concept of a contramodule. These were introduced by Eilenberg and Moore in [EM65], but was not much used or studied until the early 2000's, when Positselski found several important uses for them. Positselski's comodule-contramodule correspondence gives an adjunction between comodules and contramodules over coalgebras in certain categories—like the category of vector spaces over a field. In many nice cases this adjunction is actually a symmetric monoidal equivalence, for example when the coalgebra C is coseparable and cocommutative.

The notion of contramodules has not yet been studied in the context of ∞-categories, so we first need a suitable definition. We define a contramodule over a cocommutative coalgebra C to be a module over the internal hom functor $\underline{\mathrm{Hom}}_{\mathcal{C}}(C, -)$, which is a monad on $\mathcal{C}$. To be certain of the existence of symmetric monoidal structures on the categories of comodules and contramodules, we restrict ourselves to coidempotent coalebras. We then obtain the following ∞-categorical version of Positselski's co-contra correspondence, which we call *Positselski duality*.

Theorem 0.3.3 (Theorem 2.D)**.** *If $\mathcal{C}$ is a presentably symmetric monoidal ∞-category, and $C \in \mathcal{C}$ a cocommutative coidempotent coalgebra, then there is an equivalence*

$$\mathrm{Comod}_C(\mathcal{C}) \simeq \mathrm{Contra}_C(\mathcal{C})$$

of symmetric monoidal ∞-categories.

Now, we can then try to relate this back to the original motivation, which was to have a module-like description of $K_p(n)$-local spectra. In fact, we prove this much more generally, for any local duality context $(\mathcal{C}, \mathcal{K})$.

Theorem 0.3.4 (Theorem 2.E)**.** *If $(\mathcal{C}, \mathcal{K})$ is a local duality context, then there are equivalences*

$$\mathcal{C}^{\mathcal{K}-tors} \simeq \mathrm{Comod}_{\Gamma \mathbb{1}_{\mathcal{C}}}(\mathcal{C}) \text{ and } \mathcal{C}^{\mathcal{K}-comp} \simeq \mathrm{Contra}_{\Gamma \mathbb{1}_{\mathcal{C}}}(\mathcal{C}),$$

where Γ is the smashing colocalization associated to $(\mathcal{C}, \mathcal{K})$. In particular, $\Gamma \mathbb{1}_{\mathcal{C}}$ can be treated as a coidempotent coalgebra in $\mathcal{C}$, hence Positselski duality implies that there is an equivalence

$$\mathcal{C}^{\mathcal{K}-tors} \simeq \mathcal{C}^{\mathcal{K}-comp}$$

of symmetric monoidal stable ∞-categories.

This finally gives the description we were after, and we can conclude that there is an equivalence

$$\mathrm{Sp}_{K_p(n)} \simeq \mathrm{Contra}_{\mathcal{M}_{n,p}\mathbb{S}}(\mathrm{Sp}_{n,p})$$

of symmetric monoidal stable ∞-categories.

This description is all well and good, but there is a conceptual peculiarity at play. It would be more intuitive that $\mathrm{Sp}_{K_p(n)}$ should be dependent on a module-like structure over its unit $L_{K_p(n)}\mathbb{S}$, and not the unit $M_{n,p}\mathbb{S}$ in the dual category $\mathcal{M}_{n,p}$—this is after all the case for the other two categories in the local duality diagram.

We have, as an added bonus for this thesis, added Section 2.A, which included some work on defining contramodules over topological algebras in the ∞-categorical setting. This is not featured in the original paper, but tries to answer some of the questions that arose. We prove that there is an equivalence between comodules over C, and the opposite category of modules over the $\mathcal{C}$-linear dual of C, which is a pro-dualizable commutative algebra in $\mathcal{C}$—which is a way to incorporate a topology on it in the ∞-categorical setting. We also argue why this category deserves to be called the category of contramodules over these pro-dualizable algebras. The main takeaway from this added content is that we do in fact obtain an equivalence between $\mathrm{Sp}_{K(n)}$ and contramodules over $L_{K_p(n)}\mathbb{S}$.

0.3.3 Paper III

The way Patchkoria–Pstrągowski proved the exotic equivalence $h\mathrm{Sp}_{n,p} \simeq h\mathrm{D}^{\mathrm{per}}(E_*E)$, as discussed above, was to construct a "categorification" of the homology theory

$$E_*\colon \mathrm{Sp}_{n,p} \longrightarrow \mathrm{Comod}_{E_*E},$$

consisting—at least intuitively—of formal E-based Adams spectral sequences. This categorification can be interpreted as a local version of Pstrągowski's category of synthetic spectra, LSyn_E. This is a rigidly compactly generated symmetric monoidal stable

∞-category, which incorporates both homotopical information $\mathrm{Sp}_{n,p}$ and algebraic information from E_*E; it has, in particular, a t-structure with heart Comod_{E_*E}.

In the first paper we prove that the E-homology functor above could be restricted to a well behaved homology theory

$$E_*\colon \mathcal{M}_{n,p} \longrightarrow \mathrm{Comod}_{E_*E}^{I_n-\mathrm{tors}}.$$

We know that $\mathcal{M}_{n,p}$ is a localizing subcategory of $\mathrm{Sp}_{n,p}$, and that $\mathrm{Comod}_{E_*E}^{I_n-\mathrm{tors}}$ is a localizing subcategory of Comod_{E_*E}. Hence, we want to show that there is a similar "categorification" of the restricted homology theory, and that the resulting category is a localizing subcategory of Syn_E.

Motivated by the setup above, the goal of paper III is understand the interactions between localizing subcategories in a presentable stable ∞-category $\mathcal{C}$ with a well-behaved t-structure $(\mathcal{C}_{\geqslant 0}, \mathcal{C}_{\leqslant 0})$, and localizing subcategories of the Grothendieck abelian heart, defined as

$$\mathcal{C}^{\heartsuit} = \mathcal{C}_{\geqslant 0} \cap \mathcal{C}_{\leqslant 0}.$$

In particular, we want to classify which localizing subcategories in $\mathcal{C}$ that are uniquely determined by a localizing subcategory in $\mathcal{C}^{\heartsuit}$.

There are two levels to such a classification. A t-structure compatible localizing subcategory $\mathcal{L}$ of $\mathcal{C}$ determines a weak localizing subcategory $\mathcal{L}^{\heartsuit}$ of $\mathcal{C}^{\heartsuit}$, and our first result classifies those $\mathcal{L}$ who are uniquely determined by $\mathcal{L}^{\heartsuit}$.

Theorem 0.3.5 (Theorem 3.3.11)**.** *There is a one-to-one correspondence*

$$\left\{\begin{matrix}\pi\text{-stable localizing}\\ \text{subcategories of } \mathcal{C}\end{matrix}\right\} \simeq \left\{\begin{matrix}\text{weak localizing}\\ \text{subcategories of } \mathcal{C}^{\heartsuit}\end{matrix}\right\},$$

where a localizing ideal $\mathcal{L} \subseteq \mathcal{C}$ is said to be π-stable if $X \in \mathcal{L}$ if and only if $\pi_k^{\heartsuit} X \in \mathcal{L}^{\heartsuit}$ for all $k \in \mathbb{Z}$.

The second level comes from starting with a localizing subcategory $\mathcal{L}^{\heartsuit}$ of $\mathcal{C}^{\heartsuit}$, and try to understand how to lift such a category to a localizing subcategory of $\mathcal{C}$. The difference between a weak localizing subcategory and a localizing subcategory is given by certain exact sequences in $\mathcal{C}^\heartsuit$. One should then perhaps expect that the difference between a classification of localizing subcategories of $\mathcal{C}$ that have a weak localizing heart, compared to a localizing heart, is also detected by certain exact sequences. This is precisely what happens.

Theorem 0.3.6 (Theorem 3.3.35)**.** *There is a one-to-one correspondence*

$$\left\{\begin{matrix} \pi\text{-exact localizing} \\ \text{subcategories of } \mathcal{C} \end{matrix}\right\} \simeq \left\{\begin{matrix} \text{localizing} \\ \text{subcategories of } \mathcal{C}^\heartsuit \end{matrix}\right\},$$

where a localizing subcategory $\mathcal{L} \subseteq \mathcal{C}$ is said to be π-exact if it is π-stable and is the kernel of a t-exact functor on $\mathcal{C}$. This correspondence factors through the correspondence

$$\left\{\begin{matrix} \text{separating localizing} \\ \text{subcategories of } \mathcal{C}_{\geqslant 0} \end{matrix}\right\} \simeq \left\{\begin{matrix} \text{localizing} \\ \text{subcategories of } \mathcal{C}^\heartsuit \end{matrix}\right\}$$

due to Lurie.

The paper itself is written without the explicit goal of understanding the category of synthetic spectra. We have thus added Section 3.A—which is not featured in the original paper—where we focus on synthetic spectra specifically. Therein we prove some added results about compact generation of the π-exact lift, as well as relate these ideas back to their source in the first paper. In particular, we prove that the π-exact lift of $\mathrm{Comod}_{E_*E}^{I_n-\mathrm{tors}}$ in LSyn_E is a compactly generated localizing $\otimes$-ideal, which allows us to compute its deformation theoretical properties: it has generic fiber $\mathcal{M}_{n,p}$ and special fiber Stable_{E_*E}. This is exactly the properties one would expect for it to be the categorification of the homology theory $E_*\colon \mathcal{M}_{n,p} \longrightarrow \mathrm{Comod}_{E_*E}^{I_n-\mathrm{tors}}$.

Paper I

Algebraicity in mono-
chromatic homotopy
theory

To appear in Algebraic &
geometric topology

Paper I

Algebraicity in monochromatic homotopy theory

To appear in Algebraic & geometric topology

Description

The main result of the first paper concerns the behaviour of a class of objects when a parameter is increased. At low values the objects act very topological — which in spirit acts like fluidity, movement, deformation. Increasing the parameter makes the objects behave more and more algebraic, which is more rigid, less fluid, more staccato, more mechanical. Above a certain threshold, the behaviour of the objects is completely algebraic, which is depicted using only straight lines, while the topological behaviour at lower values is depicted with more flowing curved lines.

The colors have no mathematical meaning, and are there only to add visual interst, and to connect to the colors of the papers.

Let p be a large enough prime,
and n, quite small this time.
Chromatic waves of that length,
are in increasing strength,
completely algebraic, how sublime!

– Torgeir Aambø

Abstract:

Using Patchkoria–Pstrągowski's version of Franke's algebraicity theorem, we prove that the category of $K_p(n)$-local spectra is exotically equivalent to the category of derived I_n-complete periodic comodules over the Adams Hopf algebroid (E_*, E_*E) for large primes. This gives a finite prime result analogous to the asymptotic algebraicity for $\mathrm{Sp}_{K_p(n)}$ of Barthel–Schlank–Stapleton.

1.1 Introduction

The central idea in chromatic homotopy theory is to study the symmetric monoidal stable ∞-category of spectra, Sp, via its smaller building blocks. These are the categories $\mathrm{Sp}_{n,p}$ and $\mathrm{Sp}_{K_p(n)}$ of $\mathrm{E}_{n,p}$-local and $K_p(n)$-local spectra, where $E = \mathrm{E}_{n,p}$ is Morava E-theory, and $K_p(n)$ is Morava K-theory—see for example [HS99]. These categories depend on a prime p and an integer n, called the chromatic height. For a fixed height n, increasing the prime p makes both categories behave more algebraically. This manifests itself, for example, in the E-Adams spectral sequence of signature

$$E_2^{s,t}(L_{n,p}\mathbb{S}) = \mathrm{Ext}^{s,t}_{E_*E}(E_*, E_*) \Longrightarrow \pi_{t-s}L_{n,p}\mathbb{S}$$

computing the homotopy groups of the E-local sphere. By the smash product theorem of Ravenel, see [Rav92, 7.5.6], this spectral sequence has a horizontal vanishing line at a finite page. If $p > n+1$, this vanishing line appears already on the second page, where the information is completely described by the homological algebra of Comod_{E_*E}—the Grothendieck abelian category of comodules over the Hopf algebroid (E_*, E_*E).

Increasing the prime p correspondingly increases the distance between objects appearing in the E-Adams spectral sequence. When $2p - 2$ exceeds $n^2 + n$, there is no longer room for any differentials, and the spectral sequence in fact collapses to an isomorphism

$$\pi_*L_{n,p}\mathbb{S} \cong \mathrm{Ext}^{*,*}_{E_*E}(E_*, E_*),$$

for degree reasons. In other words, the homotopy groups are completely algebraic in this range.

A natural question to ask is whether this collapse is a feature solely of the E-Adams spectral sequence or if it is a feature of the category $\mathrm{Sp}_{n,p}$. More precisely, is the entire category of E-local spectra algebraic, in the sense that it is equivalent to a derived category of an abelian category, whenever $2p-2 > n^2+n$? What about the category of $K_p(n)$-local spectra?

At height $n = 0$, both categories $\mathrm{Sp}_{n,p}$ and $\mathrm{Sp}_{K_p(n)}$ is the category of rational spectra $\mathrm{Sp}_{\mathbb{Q}}$, which can be seen to be equivalent to the derived ∞-category of rational vector spaces, but at positive heights $n > 0$, there can never be an equivalence of ∞-categories $\mathrm{Sp}_{n,p} \simeq \mathrm{D}(\mathcal{A})$ or $\mathrm{Sp}_{K_p(n)} \simeq \mathrm{D}(\mathcal{A})$.

However, in [Bou85] Bousfield showed that for $p > 2$ and $n = 1$, that there is an equivalence of homotopy categories

$$h\mathrm{Sp}_{1,p} \simeq h\mathrm{Fr}_{1,p},$$

where $\mathrm{Fr}_{n,p}$ is a certain derived ∞-category of twisted comodules over (E_*, E_*E). As this cannot be lifted to an equivalence of ∞-categories, it is sometimes referred to as an *exotic* equivalence.

Franke expanded upon this in [Fra96] by conjecturing—and attempting to prove—that for $2p-2 > n^2+n$ there should be an equivalence of homotopy categories

$$h\mathrm{Sp}_{n,p} \simeq h\mathrm{Fr}_{n,p}.$$

Unfortunately, a subtle error was discovered in the proof by Patchkoria in [Pat12], but the result was recovered in [Pst21] with a slightly worse bound: $2p-2 > 2n^2+2n$. Pstrągowski also proved that this equivalence gets "stronger" the larger the prime, where we not only get an equivalence of categories but an equivalence of k-categories

$$h_k\mathrm{Sp}_{n,p} \simeq h_k\mathrm{Fr}_{n,p},$$

for $k = 2p-2-2n^2-2n$. Here $h_k\mathcal{C}$ denotes taking the homotopy k-category, given by $(k-1)$-truncating the mapping spaces in $\mathcal{C}$. At $k = 1$, this gives the classical situation of taking the homotopy category $h\mathcal{C}$. Using and developing a more general machinery, Pstrągowski and Patchkoria proved in [PP21] that the above equivalence holds in Franke's conjectured bound:

$$2p-2 > n^2+n.$$

The current belief is that these bounds are optimal. We know this to be true at the prime 2, as Roitzheim proved in [Roi07] that the category $\mathrm{Sp}_{1,2}$ is *rigid*, in the sense that any equivalence of homotopy categories $h\mathrm{Sp}_{1,2} \simeq h\mathcal{C}$ lifts to an equivalence $\mathrm{Sp}_{1,2} \simeq \mathcal{C}$. The $K_p(n)$-local analogue of Roitzheim's result also holds, as Ishak proved in [Ish19] that $\mathrm{Sp}_{K_2(1)}$ is rigid as well. Hence, exotic equivalences for $\mathrm{Sp}_{n,p}$ or $\mathrm{Sp}_{K_p(n)}$ can only exist at primes that are large compared to the height.

The above results imply that increasing the prime p decreases how destructive the k-truncation of the mapping spaces needs to be. In the limit $p \to \infty$, we might expect that there is no need to truncate at all, giving an equivalence of ∞-categories. But, there needs to be an appropriate notion of what "going to the infinite prime" should be. In [BSS20], the authors use a notion of ultraproducts over a non-principal ultrafilter of primes, $\mathcal{F}$, to formalize this limiting process. They use this to prove the existence of a symmetric monoidal equivalence of ∞-categories

$$\prod_{\mathcal{F}} \mathrm{Sp}_{n,p} \simeq \prod_{\mathcal{F}} \mathrm{Fr}_{n,p}.$$

Expanding on their work, Barthel, Schlank, and Stapleton proved in [BSS21] a $K_p(n)$-local version of the above result. More precisely, they show that there is a symmetric monoidal equivalence of ∞-categories

$$\prod_{\mathcal{F}} \mathrm{Sp}_{K_p(n)} \simeq \prod_{\mathcal{F}} \mathrm{Fr}_{n,p}^{I_n-\mathrm{comp}},$$

where the right-hand side consists of derived complete twisted comodules for the naturally occurring Landweber ideal $I_n \subseteq E_*$.

Statement of results

We can summarize the most general of the above algebraicity results in the following table,

A natural question arises: Is there a finite prime exotic algebraicity for $\mathrm{Sp}_{K_p(n)}$? The goal of this paper is to give an affirmative answer. More precisely, we prove the following.

	$p < \infty$	$p \to \infty$
$\mathrm{Sp}_{n,p}$	[PP21]	[BSS20]
$\mathrm{Sp}_{K_p(n)}$		[BSS21]

Theorem 1.A (Theorem 1.4.15)**.** *Let p be a prime and $n \in \mathbb{N}$. If $k = 2p - 2 - n^2 - n > 0$, then there is an equivalence of k-categories*

$$h_k \mathrm{Sp}_{K_p(n)} \simeq h_k \mathrm{Fr}_{n,p}^{I_n-\mathrm{comp}}.$$

In other words, $K_p(n)$-local spectra are exotically algebraic at large primes.

The available tools for proving such a statement require an abelian category with enough injective objects admitting lifts to a stable ∞-category. In lack of such a well-behaved abelian approximation for $\mathrm{Sp}_{K_p(n)}$, we take inspiration from [BSS21] and instead use the dual category $\mathcal{M}_{n,p}$ of monochromatic spectra, which we show has the needed properties. Theorem 1.A will then follow from the following result.

Theorem 1.B (Theorem 1.4.13)**.** *Let p be a prime and n a positive natural number. If $k = 2p - 2 - n^2 - n > 0$, then there is an equivalence*

$$h_k \mathcal{M}_{n,p} \simeq h_k \mathrm{Fr}_{n,p}^{I_n-\mathrm{tors}}$$

as k-categories.

In order to prove Theorem 1.B, we first prove the analogous statement for monochromatic E-modules.

Theorem 1.C (Theorem 1.4.5)**.** *Let p be a prime and n a positive natural number. If $k = 2p-2-n > 0$, then there is an equivalence*

$$h_k \mathrm{Mod}_E^{I_n-\mathrm{tors}} \simeq h_k \mathrm{D}^{\mathrm{per}}(\mathrm{Mod}_{E_*}^{I_n-\mathrm{tors}})$$

as k-categories.

Overview of the paper

Section 1.1 introduces local duality, and the proposed exotic algebraic model using periodic chain complexes of torsion comodules.

Section 1.3 focuses on Franke's algebraicity theorem. Most of the new results of the paper are presented in Section 1.4.1 and Section 1.4.2, where we prove Theorem 1.A, Theorem 1.B and Theorem 1.C. In Section 1.A we prove that Barr–Beck adjunctions interact well with local duality, which is used to prove that periodization, torsion and taking the derived category all commute.

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1.2 The algebraic model

The goal of this section is to set up the necessary background material that will be used throughout the paper. We use these to construct convenient algebraic approximations of categories arising from chromatic homotopy theory.

Some conventions

We freely use the language of ∞-categories, as developed by Joyal [Joy02] and Lurie [Lur09; Lur17]. Even though we are dealing with both classical 1-categories and ∞-categories in this paper, we will sometimes refer to them both as *categories*, hoping that the prefix is clear from the context.

We denote by Pr^L_{st} the ∞-category of presentable stable ∞-categories and colimit preserving functors. Together with the Lurie tensor product, it is a symmetric monoidal ∞-category. The category of commutative algebras $\mathrm{CAlg}(\mathrm{Pr}^L_{st})$ is then the category of presentable stable ∞-categories with a symmetric monoidal structure commuting with colimits separately in each variable.

Let $\mathcal{C}, \mathcal{D} \in \mathrm{CAlg}(\mathrm{Pr}^L_{st})$. A localization is a functor $f\colon \mathcal{C} \longrightarrow \mathcal{D}$ with a fully faithful right adjoint i. We denote the composite by $L = i \circ f$. The adjoint i identifies $\mathcal{D}$ with a full subcategory of $\mathcal{C}$, which we denote by $\mathcal{C}_L$. We then view L as a functor $L\colon \mathcal{C} \longrightarrow \mathcal{C}_L$, that is left adjoint to the inclusion, and by abuse of notation also call these localizations.

1.2.1 Local duality

The theory of abstract local duality, proved in [HPS97] and generalized to the ∞-categorical setting in [BHV18] will be important for the entire paper. In particular, it is the technology that will allow us to translate Theorem 1.B into Theorem 1.A.

Definition 1.2.1. A pair $(\mathcal{C}, \mathcal{K})$, where $\mathcal{C} \in \mathrm{CAlg}(\mathrm{Pr}^L_{st})$ is compactly generated by dualizable objects, and $\mathcal{K}$ is a subset of compact objects, is called a *local duality context*.

Construction 1.2.2. Let $(\mathcal{C}, \mathcal{K})$ be a local duality context. We define $\mathcal{C}^{\mathcal{K}-\mathrm{tors}}$ to be the localizing tensor ideal generated by $\mathcal{K}$, denoted $\mathrm{Loc}^{\otimes}_{\mathcal{C}}(\mathcal{K})$. Further we define $\mathcal{C}^{\mathcal{K}-\mathrm{loc}}$ to be the right orthogonal complement $(\mathcal{C}^{\mathcal{K}-\mathrm{tors}})^{\perp}$, i.e., the full subcategory consisting of objects $C \in \mathcal{C}$ such that $\mathrm{Hom}_{\mathcal{C}}(T, C) \simeq 0$ for all $T \in \mathcal{C}^{\mathcal{K}-\mathrm{tors}}$. Similarly, define $\mathcal{C}^{\mathcal{K}-\mathrm{comp}}$ to be the right orthogonal complement of $\mathcal{C}^{\mathcal{K}-\mathrm{loc}}$, i.e. $\mathcal{C}^{\mathcal{K}-\mathrm{comp}} = (\mathcal{C}^{\mathcal{K}-\mathrm{loc}})^{\perp}$. These full subcategories are respectively called the $\mathcal{K}$-torsion, $\mathcal{K}$-local and $\mathcal{K}$-complete objects in $\mathcal{C}$. We have inclusions into $\mathcal{C}$, denoted $i_{\mathcal{K}-\mathrm{tors}}$, $i_{\mathcal{K}-\mathrm{loc}}$ and $i_{\mathcal{K}-\mathrm{comp}}$ respectively.

By the adjoint functor theorem, [Lur09, 5.5.2.9], the inclusions $i_{\mathcal{K}-\mathrm{loc}}$ and $i_{\mathcal{K}-\mathrm{comp}}$ have left adjoints $L_{\mathcal{K}}$ and $\Lambda_{\mathcal{K}}$ respectively, while $i_{\mathcal{K}-\mathrm{tors}}$ and $i_{\mathcal{K}-\mathrm{loc}}$ have right adjoints $\Gamma_{\mathcal{K}}$ and $V_{\mathcal{K}}$ respectively. These are then, by definition, localizations and colocalizations. Since the torsion, local and complete objects are ideals, these localizations and colocalizations are compatible with the symmetric monoidal structure of $\mathcal{C}$, in the sense of [Lur17, 2.2.1.7]. In particular, by [Lur17, 2.2.1.9] we get unique induced symmetric monoidal structures such that $L_{\mathcal{K}}$, $\Lambda_{\mathcal{K}}$, $\Gamma_{\mathcal{K}}$ and $V_{\mathcal{K}}$ are symmetric monoidal functors.

For any $X \in \mathcal{C}$, these functors assemble into two cofiber sequences:

$$\Gamma_{\mathcal{K}} X \longrightarrow X \longrightarrow L_{\mathcal{K}} X \quad \text{and} \quad V_{\mathcal{K}} X \longrightarrow X \longrightarrow \Lambda_{\mathcal{K}} X.$$

Note also that these functors only depend on the localizing subcategory $\mathcal{C}^{\mathcal{K}-\mathrm{tors}}$, not on the particular choice of generators $\mathcal{K}$. Thus, when the set $\mathcal{K}$ is clear from the context, we often omit it as a subscript when writing the functors.

The following theorem is a slightly restricted version of the abstract local duality theorem of [HPS97, 3.3.5] and [BHV18, 2.21].

Theorem 1.2.3. *Let $(\mathcal{C}, \mathcal{K})$ be a local duality context. Then*

1. *the functors Γ and L are smashing, meaning that there are natural equivalences*

$$\Gamma X \simeq X \otimes \Gamma \mathbb{1} \textit{ and } LX \simeq X \otimes L\mathbb{1},$$

2. *the functors Λ and V are cosmashing, meaning there are natural equivalences*

$$\Lambda X \simeq \underline{\mathrm{Hom}}(\Gamma \mathbb{1}, X) \textit{ and } VX \simeq \underline{\mathrm{Hom}}(L\mathbb{1}, X),$$

and

3. *the functors*

$$\Gamma \colon \mathcal{C}^{\mathcal{K}-\mathrm{comp}} \longrightarrow \mathcal{C}^{\mathcal{K}-\mathrm{comp}} \textit{ and } \Lambda \colon \mathcal{C}^{\mathcal{K}-\mathrm{tors}} \longrightarrow \mathcal{C}^{\mathcal{K}-\mathrm{comp}}$$

are mutually inverse symmetric monoidal equivalences of categories.

This can be summarized by the following diagram of adjoints

$$
\begin{tikzcd}
 & \mathcal{C}^{\mathcal{K}-\mathrm{loc}} \arrow[d, shift left=1] \arrow[ddr, dashed, bend left] & \\
 & \mathcal{C} \arrow[u, shift left=3, "L"] \arrow[u, shift right=1, "V"'] \arrow[dl, shift left=1, "\Gamma"] \arrow[dr, shift left=1, "\Lambda"] & \\
\mathcal{C}^{\mathcal{K}-\mathrm{tors}} \arrow[ur, shift left=1] \arrow[uur, dashed, bend left] \arrow[rr, "\simeq"] & & \mathcal{C}^{\mathcal{K}-\mathrm{comp}} \arrow[ul, shift left=1]
\end{tikzcd}
$$

Remark 1.2.4. Theorem 1.2.3 implies, in particular, that the symmetric monoidal structure induced by the localization L and the colocalization Γ is just the symmetric monoidal structure on $\mathcal{C}$ restricted to the full subcategories. This is not the case for $\mathcal{C}^{\mathcal{K}-\mathrm{comp}}$, where the symmetric monoidal structure is given by $\Lambda(- \otimes_{\mathcal{C}} -)$. The functor V also induces a symmetric monoidal structure on $\mathcal{C}^{\mathcal{K}-\mathrm{loc}}$, but this coincides with the one induced by L, due to their associated endofunctors on $\mathcal{C}$ defining an adjoint symmetric monoidal monad-comonad pair. Note that we will not need or focus on the functor V, hence it will be omitted from the local duality diagrams for the rest of the paper.

Addendum. An alternative proof of local duality, using a version of Positselski's co/contra correspondence in symmetric monoidal stable ∞-categories, can be found in Chapter 2—more specifically Theorem 2.3.17.

We have two main examples of interest for this paper.

Example 1.2.5. Let (A, Ψ) be an Adams type Hopf algebroid, for example the Hopf algebroid (R_*, R_*R) for an Adams type ring spectrum R—see [Rav86, A.1] and [Hov04] for details. Denote by $\mathrm{D}(\Psi)$ the derived ∞-category associated to the symmetric monoidal Grothendieck abelian category Comod_Ψ. This is defined using the model structure from [BR11]. If $I \subseteq A$ is a finitely generated invariant regular ideal, then $(\mathrm{D}(\Psi), A/I)$ is a local duality context, with associated local duality diagram

$$
\begin{tikzcd}
& \mathrm{D}(\Psi)^{I-\mathrm{loc}} \arrow[d, shift left] \arrow[ddr, dashed, bend left] & \\
& \mathrm{D}(\Psi) \arrow[u, shift left, "L_I^{\Psi}"] \arrow[dl, shift left, "\Gamma_I^{\Psi}"] \arrow[dr, shift left, "\Delta_I^{\Psi}"] & \\
\mathrm{D}(\Psi)^{I-\mathrm{tors}} \arrow[ur, shift left] \arrow[rr, "\simeq"] \arrow[uur, dashed, bend left] & & \mathrm{D}(\Psi)^{I-\mathrm{comp}} \arrow[ul, shift left]
\end{tikzcd}
$$

In Section 1.2.2 we compare $\mathrm{D}(\Psi)^{I-\mathrm{tors}}$ to a more concrete category: the derived category of I-power torsion comodules.

The following example comes from chromatic homotopy theory. For a good introduction, see [BB19].

Example 1.2.6. Let E denote Morava E-theory at prime p and height n. If $F(n)$ is a finite type n spectrum, then the pair $(\mathrm{Sp}_{n,p}, L_{n,p}F(n))$ is a local duality context. The corresponding diagram can be recognized as

$$
\begin{tikzcd}
& \mathrm{Sp}_{n-1,p} \arrow[d, shift left] \arrow[ddr, dashed, bend left] & \\
& \mathrm{Sp}_{n,p} \arrow[u, shift left, "L_{n-1,p}"] \arrow[dl, shift left, "M_{n,p}"] \arrow[dr, shift left, "L_{K_p(n)}"] & \\
\mathcal{M}_{n,p} \arrow[ur, shift left] \arrow[rr, "\simeq"] \arrow[uur, dashed, bend left] & & \mathrm{Sp}_{K_p(n)} \arrow[ul, shift left]
\end{tikzcd}
$$

where $\mathcal{M}_{n,p}$ is the height n monochromatic category and $\mathrm{Sp}_{K_p(n)}$ is the category of spectra localized at height n Morava K-theory $K_p(n)$. The functor $L_{n-1,p}$ is the Bousfield localization at $E_{n-1,p}$, while $L_{K_p(n)}$ is the Bousfield localization at $K_p(n)$, see [Bou79b]. The local duality then exhibits the classical equivalence

$$\mathcal{M}_{n,p} \simeq \mathrm{Sp}_{K_p(n)},$$

see [HS99, 6.19].

Remark 1.2.7. There is also a version of this local duality diagram for modules over E, see [GM95, 4.2, 5.1], or alternatively [BHV18, 3.7] for a version more similar to the above. This gives equivalences

$$\mathcal{M}_{n,p}\mathrm{Mod}_E \simeq \mathrm{Mod}_E^{I_n-\mathrm{tors}} \simeq \mathrm{Mod}_E^{I_n-\mathrm{comp}} \simeq L_{K_p(n)}\mathrm{Mod}_E,$$

where I_n is the Landweber ideal $(p, v_1, \ldots, v_{n-1}) \subseteq E_*$.

Addendum. We have expanded upon the theory of local duality, as well as comodules over a Hopf algebroid and monochromatic spectra in Section 0.2.2, Section 0.2.4 and Section 0.2.3.3 respectively.

1.2.2 The periodic derived torsion category

In this section we identify the category $\mathrm{D}(\Psi)^{I-\mathrm{tors}}$—as obtained in Example 1.2.5—as the derived category of I-power torsion comodules. We also modify the category to exhibit some needed periodicity.

Definition 1.2.8. Let (A, Ψ) be an Adams Hopf algebroid and $I \subseteq A$ a regular invariant ideal. The*I-power torsion* of a comodule M is defined as

$$T_I^\Psi M = \{x \in M \mid I^k x = 0 \text{ for some } k \in \mathbb{N}\}.$$

We say a comodule M is *I-power torsion* if the natural comparison map $T_I^\Psi M \longrightarrow M$ is an equivalence.

Remark 1.2.9. One can similarly define I-power torsion A-modules. If (A, Ψ) is an Adams Hopf algebroid, then a Ψ-comodule M is I-power torsion if and only if its underlying module is I-power torsion, see [BHV18, 5.7].

Remark 1.2.10. By [BHV18, 5.10] the full subcategory of I-power torsion comodules, which we denote $\mathrm{Comod}_\Psi^{I-\mathrm{tors}}$, is a Grothendieck abelian category. It also inherits a symmetric monoidal structure from Comod_Ψ.

The following technical lemma will be needed later.

Lemma 1.2.11. *Let (A, Ψ) be an Adams Hopf algebroid, where A is noetherian and $I \subseteq A$ a regular invariant ideal. Then*

$\mathrm{Comod}_\Psi^{I-\mathrm{tors}}$ *is generated under filtered colimits by the compact I-power torsion comodules.*

Proof. By [BHV20, 3.4] $\mathrm{Comod}_\Psi^{I-\mathrm{tors}}$ is generated by the set

$$\mathrm{Tors}_\Psi^{fp} := \{G \otimes A/I^k \mid G \in \mathrm{Comod}_\Psi^{fp}, k \geqslant 1\},$$

where Comod_Ψ^{fp} is the full subcategory of dualizable Ψ-comodules. Since I is finitely generated and regular, A/I^k is finitely presented as an A-module, hence it is compact in Comod_Ψ by [Hov04, 1.4.2], and also in $\mathrm{Comod}_\Psi^{I-\mathrm{tors}}$, as colimits are computed in Comod_Ψ. Because A was assumed to be noetherian, being finitely generated and finitely presented coincide. The tensor product of finitely generated modules is finitely generated, hence any element in Tors_Ψ^{fp} is compact. □

Remark 1.2.12. The assumption that the ring A is noetherian can most likely be removed, but it makes no difference to the results in this paper.

Notation 1.2.13. Since $\mathrm{Comod}_\Psi^{I-\mathrm{tors}}$ is Grothendieck abelian we have an associated derived stable ∞-category $\mathrm{D}(\mathrm{Comod}_\Psi^{I-\mathrm{tors}})$ which we denote simply by $\mathrm{D}(\Psi^{I-\mathrm{tors}})$.

We e can now compare the torsion category obtained from local duality and the derived category of I-power torsion comodules.

Lemma 1.2.14 ([BHV20, 3.7(2)])**.** *Let (A, Ψ) be an Adams Hopf algebroid and $I \subseteq A$ a regular invariant ideal. There is an equivalence of categories*

$$\mathrm{D}(\Psi)^{I-\mathrm{tors}} \simeq \mathrm{D}(\Psi^{I-\mathrm{tors}}).$$

Furthermore, an object $M \in \mathrm{D}(\Psi)$ is I-torsion if and only if the homology groups H_*M are I-power torsion Ψ-comodules.

Addendum. We can construct an alternative proof of the above result using the theory developed in Chapter 3. In particular, the two categories $\mathrm{D}(\Psi)^{I-\mathrm{tors}}$ and $\mathrm{D}(\Psi^{I-\mathrm{tors}})$ are both π-stable localizing ideals of $\mathrm{D}(\Psi)$ with the same heart $\mathrm{Comod}_\Psi^{I-\mathrm{tors}}$, which by Theorem 3.3.11 means they have to be equivalent.

In order to state both the general algebraicity machinery of [PP21] and our results, we need the respective derived categories to exhibit the periodic nature of the spectra we are interested in. This is done via the periodic derived category. There are several ways to constructing this, but we follow [Fra96] in spirit, using periodic chain complexes.

Definition 1.2.15. Let $\mathcal{A}$ be an abelian category with a local grading, i.e., an auto-equivalence $T\colon \mathcal{A} \longrightarrow \mathcal{A}$, and denote $[1]$ the shift functor on the category of chain complexes $\mathrm{Ch}(\mathcal{A})$ in $\mathcal{A}$. A chain complex $C \in \mathrm{Ch}(\mathcal{A})$ is called *periodic* if there is an isomorphism $\varphi\colon C[1] \longrightarrow TC$. The full subcategory of periodic chain complexes is denoted by $\mathrm{Ch}^{\mathrm{per}}(\mathcal{A})$.

Definition 1.2.16. The forgetful functor $\mathrm{Ch}^{\mathrm{per}}(\mathcal{A}) \longrightarrow \mathrm{Ch}(\mathcal{A})$ has a left adjoint P, called the *periodization*.

Definition 1.2.17. If $\mathcal{A}$ is a locally graded abelian category, then the *periodic derived category* of $\mathcal{A}$, denoted $\mathrm{D}^{\mathrm{per}}(\mathcal{A})$, is defined to be the ∞-category obtained by localizing $\mathrm{Ch}^{\mathrm{per}}(\mathcal{A})$ at the quasi-isomorphism. It is a stable ∞-category by [PP21, 7.8].

Remark 1.2.18. If $\mathcal{A}$ is a symmetric monoidal category with unit $\mathbb{1}$, then $P\mathbb{1}$ is a commutative ring object called the *periodic unit*. By [BR11, 2.3] the category of periodic chain complexes $\mathrm{Ch}^{\mathrm{per}}(\mathcal{A})$ is equivalent to $\mathrm{Mod}_{P\mathbb{1}}(\mathrm{Ch}(\mathcal{A}))$. This descends also to the derived categories, giving an equivalence

$$\mathrm{D}^{\mathrm{per}}(\mathcal{A}) \simeq \mathrm{Mod}_{P\mathbb{1}}(\mathrm{D}(\mathcal{A})),$$

see for example [Pst21, 3.7].

We will also need local duality for the periodic derived category associated to a Hopf algebroid.

Construction 1.2.19. Let (A, Ψ) be an Adams type (graded) Hopf algebroid. Then the shift functor $[1]\colon \mathrm{Comod}_{\Psi} \longrightarrow \mathrm{Comod}_{\Psi}$ defined by $(TM)_k = M_{k-1}$ is a local grading on Comod_{Ψ}. Denote the corresponding periodic derived category by $\mathrm{D}^{\mathrm{per}}(\Psi)$. The pair $(\mathrm{D}^{\mathrm{per}}(\Psi), P(A/I))$ is a local duality context with associated local duality diagram

$$
\begin{tikzcd}
 & \mathrm{D}^{\mathrm{per}}(\Psi)^{I-\mathrm{loc}} \arrow[d, shift left] \arrow[ddr, dashed, bend left] & \\
 & \mathrm{D}^{\mathrm{per}}(\Psi) \arrow[u, shift left, "L_I^{\Psi}"] \arrow[dl, shift left] \arrow[dr, shift left] & \\
\mathrm{D}^{\mathrm{per}}(\Psi)^{I-\mathrm{tors}} \arrow[ur, shift left] \arrow[uur, dashed, bend left] \arrow[rr, "\simeq"] & & \mathrm{D}^{\mathrm{per}}(\Psi)^{I-\mathrm{comp}} \arrow[ul, shift left, "\Lambda_I^{\Psi}"]
\end{tikzcd}
$$

The functors in the diagram are induced by the functors from Example 1.2.5. In fact, there is a diagram

$$
\begin{tikzcd}
\mathrm{D}(\Psi)^{I-\mathrm{tors}} \arrow[r, shift left] \arrow[d, shift right, "P"'] & \mathrm{D}(\Psi) \arrow[l, shift left, "\Gamma_I^{\Psi}"] \arrow[r, shift left, "L_I^{\Psi}"] \arrow[d, shift right, "P"'] & \mathrm{D}(\Psi)^{I-\mathrm{loc}} \arrow[l, shift left] \arrow[d, shift right, "P"'] \\
\mathrm{D}^{\mathrm{per}}(\Psi)^{I-\mathrm{tors}} \arrow[r, shift left] \arrow[u, shift right] & \mathrm{D}^{\mathrm{per}}(\Psi) \arrow[l, shift left, "\Gamma_I^{\Psi}"] \arrow[r, shift left, "L_I^{\Psi}"] \arrow[u, shift right] & \mathrm{D}^{\mathrm{per}}(\Psi)^{I-\mathrm{loc}} \arrow[l, shift left] \arrow[u, shift right]
\end{tikzcd}
$$

that is commutative in all possible directions. Here the unmarked horizontal arrows are the respective fully faithful inclusions.

Remark 1.2.20. In the specific case of $(A, \Psi) = (E_0, E_0E)$ and $I \subseteq E_0$ the Landweber ideal I_n, then the above construction is [BSS21, 3.12].

There is now some ambiguity to take care of for our category of interest $\mathrm{D}^{\mathrm{per}}(\Psi)^{I-\mathrm{tors}}$. In the picture above, we do mean that we take I-torsion objects in $\mathrm{D}^{\mathrm{per}}(\Psi)$, i.e., $[\mathrm{D}^{\mathrm{per}}(\Psi)]^{I-\mathrm{tors}}$, but we could also take the periodization of the category $\mathrm{D}(\Psi^{I-\mathrm{tors}})$ as our model. Luckily, there is no choice, as they are equivalent. This can be thought of as the periodic version of Lemma 1.2.14.

Theorem 1.2.21. *Let (A, Ψ) be an Adams Hopf algebroid and $I \subseteq A$ a finitely generated invariant regular ideal. Then there is an equivalence of stable ∞-categories*

$$[\mathrm{D}^{\mathrm{per}}(\Psi)]^{I-\mathrm{tors}} \simeq \mathrm{D}^{\mathrm{per}}(\Psi^{I-\mathrm{tors}}).$$

Remark 1.2.22. The proof of this uses the fact that Barr–Beck adjunctions commute with local duality. Proving this here disrupts the flow of the paper, so we defer it to Section 1.A.

Proof. As Comod_Ψ is symmetric monoidal we have by Remark 1.2.18 an equivalence

$$\mathrm{D}^{\mathrm{per}}(\Psi) \simeq \mathrm{Mod}_{P\mathbb{1}}(\mathrm{D}(\Psi)),$$

coming from the periodicity Barr–Beck adjunction. By Theorem 1.A.6 this induces a Barr–Beck adjunction on the torsion subcategories, which gives an equivalence

$$[\mathrm{D}^{\mathrm{per}}(\Psi)]^{I-\mathrm{tors}} \simeq \mathrm{Mod}_{\Gamma_I^\Psi(P\mathbb{1})}(\mathrm{D}(\Psi)^{I-\mathrm{tors}}).$$

Since Γ_I^Ψ is a smashing colocalization, and P is given by tensoring with $P(\mathbb{1})$, they do in fact commute. By Lemma 1.2.14 we have $\mathrm{D}(\Psi)^{I-\mathrm{tors}} \simeq \mathrm{D}(\Psi^{I-\mathrm{tors}})$, hence the above equivalence can be rewritten as

$$[\mathrm{D}^{\mathrm{per}}(\Psi)]^{I-\mathrm{tors}} \simeq \mathrm{Mod}_{P(\Gamma_I^\Psi\mathbb{1})}(\mathrm{D}(\Psi^{I-\mathrm{tors}})).$$

Now, also $\mathrm{Comod}_\Psi^{I-\mathrm{tors}}$ is symmetric monoidal, so Remark 1.2.18 gives an equivalence

$$\mathrm{D}^{\mathrm{per}}(\Psi^{I-\mathrm{tors}}) \simeq \mathrm{Mod}_{P(\Gamma_I^\Psi\mathbb{1})}(\mathrm{D}(\Psi^{I-\mathrm{tors}})),$$

which finishes the proof. □

Addendum. This result, and others like it, was one of the inspirations for writing the paper [Aam24b]—see Chapter 3. There we prove some uniqueness results for localizing subcategories that have the property that objects can be detected on the heart. For the above example, both categories have the property that an object $X \in \mathrm{D}^{\mathrm{per}}(\Psi)$ lies in $[\mathrm{D}^{\mathrm{per}}(\Psi)]^{I-\mathrm{tors}}$ or $\mathrm{D}^{\mathrm{per}}(\Psi^{I-\mathrm{tors}})$ if and only if its homology groups H_kX lies in the heart $\mathrm{Comod}_\Psi^{I-\mathrm{tors}}$, which is a localizing subcategory of Comod_Ψ. By Theorem 3.3.35 this means that the categories have to be equivalent, which gives another proof of the above result.

1.3 Exotic algebraic models

We now have two sets of local duality diagrams, one coming from chromatic homotopy theory, see Example 1.2.6, and one from the homological algebra of Adams Hopf algebroids, see Example 1.2.5. We can also pass between these duality theories, by using homology theories. In particular, if we let $E = E_{n,p}$ be height n Morava E-theory at a prime p, then we have the E-homology functor

$$E_* \colon \mathrm{Sp}_{n,p} \longrightarrow \mathrm{Comod}_{E_*E}$$

converting between homotopy theory and algebra. We can, in some sense, say that E_* approximates homotopical information by algebraic information.

The goal of this section is to set up an abstract framework for studying how good such approximations are. The version we recall below was developed in [PP21], taking inspiration from [Fra96] and [Pst23].

1.3.1 Adapted homology theories

Adapted homology theories are particularly well behaved homology theories that have associated Adams type spectral sequences giving computational benefits over other homology theories.

Definition 1.3.1. Let $\mathcal{C}$ be a presentable symmetric monoidal stable ∞-category and $\mathcal{A}$ an abelian category with a local grading [1]. A functor $H \colon \mathcal{C} \longrightarrow \mathcal{A}$ is called a *conservative homology theory* if:

1. H is additive
2. for a cofiber sequence $X \longrightarrow Y \longrightarrow Z$ in $\mathcal{C}$, then the induced sequence $HX \longrightarrow HY \longrightarrow HZ$ is exact in $\mathcal{A}$
3. there is a natural isomorphism $H(\Sigma X) \cong (HX)[1]$ for any $X \in \mathcal{C}$
4. H reflects isomorphisms.

Remark 1.3.2. The first two axioms make H a homological functor, the third makes H into a locally graded functor, i.e., a functor that preserves the local grading, and the last makes it a conservative functor.

Example 1.3.3. Let R be a ring spectrum. Then the functor $\pi_*\colon \mathrm{Mod}_R \longrightarrow \mathrm{Mod}_{R_*}$ defined as $\pi_* M = [\mathbb{S}, M]_*$ is a conservative homology theory.

Example 1.3.4. Let R be a ring spectrum. The R-homology functor $R_*(-)\colon \mathrm{Sp} \longrightarrow \mathrm{Mod}_R$, defined as the composition

$$\mathrm{Sp} \xrightarrow{R\otimes(-)} \mathrm{Mod}_R \xrightarrow{\pi_*} \mathrm{Mod}_{R_*},$$

is a homology theory. If R is of Adams type, then $R_*(-)$ naturally lands in the subcategory Comod_{R_*R}. If we restrict the domain of R_* to the category of R-local spectra, then it is a conservative homology theory. For the rest of the paper we will use R_* to denote the restricted conservative homology theory

$$R_*\colon \mathrm{Sp}_R \longrightarrow \mathrm{Comod}_{R_*R}.$$

Definition 1.3.5. Let $H\colon \mathcal{C} \longrightarrow \mathcal{A}$ be a homology theory and J an injective object in $\mathcal{A}$. An object $\bar{J} \in \mathcal{C}$ is said to be an *injective lift* of J if it represents the functor

$$\mathrm{Hom}_{\mathcal{A}}(H(-), J)\colon \mathcal{C}^{op} \longrightarrow \mathcal{A}b$$

in the homotopy category $h\mathcal{C}$, i.e. $\mathrm{Hom}_{\mathcal{A}}(H(-), J) \cong [-, \bar{J}]$. We call $\bar{J}$ a *faithful lift* if the map $H(\bar{J}) \longrightarrow J$ coming from the identity on $\bar{J}$ is an equivalence.

Definition 1.3.6. A homology theory $H\colon \mathcal{C} \longrightarrow \mathcal{A}$ is said to be *adapted* if $\mathcal{A}$ has enough injective objects, and for any injective $J \in \mathcal{A}$ there is a faithful lift $\bar{J} \in \mathcal{C}$.

Example 1.3.7. We again return to our two guiding examples $\pi_*\colon \mathrm{Mod}_R \longrightarrow \mathrm{Mod}_{R_*}$ and $R_*\colon \mathrm{Sp}_R \longrightarrow \mathrm{Comod}_{R_*R}$, where R is an Adams type ring spectrum. Both functors are conservative adapted homology theories, with faithful lifts provided by Brown representability, see [PP21, 8.2] and [PP21, 8.13] respectively.

Remark 1.3.8. The definition of an adapted homology theory H states that for any injective $J \in \mathcal{A}$, there is some object $\bar{J} \in \mathcal{C}$ together with an equivalence

$$[X, \bar{J}] \simeq \mathrm{Hom}_{\mathcal{A}}(H(X), J)$$

induced by H. Because $\mathcal{A}$ has enough injective objects, we can use these equivalences to approximate homotopy classes of maps by repeatedly mapping into injectives. This gives precisely an associated Adams spectral sequence for the homology theory H. In fact, Patchkoria and Pstrągowski proved that there is a bijection between adapted homology theories and Adams spectral sequences, see [PP21, 3.24, 3.25]. The construction of the Adams spectral sequence associated to an adapted homology theory $H\colon \mathcal{C} \longrightarrow \mathcal{A}$ is given in [PP21, 2.24], or alternatively as a totalization spectral sequence in [PP21, 2.27].

In our particular interest $R = E_{n,p}$, the associated adapted homology theories π_* and E_* are even nicer than a general adapted homology theory. This is because the cohomological information in the associated category of comodules is particularly simple.

Definition 1.3.9. Let $\mathcal{A}$ be a locally graded abelian category with enough injective objects. Then the *cohomological dimension* of $\mathcal{A}$ is the smallest integer $d \in \mathbb{N} \cup \{\infty\}$ such that $\operatorname{Ext}^{s,t}_{\mathcal{A}}(-,-) \cong 0$ for all $s > d$.

Example 1.3.10. Let n be an integer, p a prime such that $p > n+1$ and $E = E_{n,p}$ Morava E-theory at height n. Then by [Pst21, 2.5] the category Comod_{E_*E} has cohomological dimension n^2+n.

Remark 1.3.11. Recall that we are really interested in the category $\mathrm{Sp}_{K_p(n)}$ of $K_p(n)$-local spectra. The spectrum $K_p(n)$ is a field object in Sp, and its homotopy groups $\pi_* K_p(n)$ are graded fields. Hence the homology theory $K_p(n)_*\colon \mathrm{Sp}_{K_p(n)} \longrightarrow \mathrm{Mod}_{K_p(n)_*}$ is too simple to exhibit the algebraicity properties that we want. As $K_p(n)$ is Adams type $K_p(n)_*(-)$ factors through Comod_{K_*K}, but this category is very complicated. In particular, it does not have finite cohomological dimension, a feature we will need later. This can be seen via the following argument of [BP23]. Having finite cohomological dimension would imply that the $K_p(n)$-Adams spectral sequence has a horizontal vanishing line at a finite page. The groups in this spectral sequence are all torsion, hence this would imply that, for example, the homotopy groups of the $K_p(n)$-local sphere is a finite filtration of torsion groups. In particular there could be no rational homotopy groups. But, by [Bar+24] the rational homotopy groups of the $K_p(n)$-local sphere are highly non-trivial, meaning that the original assumption that Comod_{K_*K} has finite cohomological dimension must be wrong.

There is, however, a version of E_*-homology on $\mathrm{Sp}_{K_p(n)}$, defined by sending a $K(n)$-local spectrum X to

$$E_*^\vee(X) := \pi_* L_{K_p(n)}(E \otimes X).$$

The functor does land in a category of comodules, specifically over the L-complete Hopf algebroid $(E_*, E_*^\vee E)$, see [Bak09, 5.3]. However, the category $\mathrm{Comod}_{E_*^\vee E}$ is not abelian. This is the reason for instead using the monochromatic category $\mathcal{M}_{n,p}$ and the category of I_n-power torsion comodules, as these inherit nicer homological properties we can exploit.

For certain Adams type ring spectra R, we get decompositions of the category Comod_{R_*R} into periodic families of subcategories. Such decompositions allows for the construction of partial inverses to the associated homology theories.

Construction 1.3.12. Let R be an Adams-type ring spectrum such that π_*R is concentrated in degrees divisible by some positive number $q+1$, i.e., $\pi_m R = 0$ for all $m \neq 0 \bmod q+1$. Any comodule M in the category Comod_{R_*R} splits uniquely into a direct sum of subcomodules $\bigoplus_{\varphi \in \mathbb{Z}/q+1} M_\varphi$ such that M_φ is concentrated in degrees divisible by φ. Such a splitting induces a decomposition of the full subcategory of injective objects

$$\mathrm{Comod}^{inj}_{R_*R} \simeq \mathrm{Comod}^{inj}_{R_*R,0} \times \mathrm{Comod}^{inj}_{R_*R,1} \times \cdots \times \mathrm{Comod}^{inj}_{R_*R,q}$$

where the category $\mathrm{Comod}^{inj}_{R_*R,\varphi}$ denotes the full subcategory spanned by injective comodules concentrated in degrees divisible by φ.

Let $h_k\mathcal{C}$ denote the homotopy k-category of $\mathcal{C}$, obtained by $k+1$-truncating all the mapping spaces in $\mathcal{C}$. The lift associated with each injective via the Adapted homology theory R_* allows us to construct a partial inverse to R_*, called the Bousfield functor β^{inj} in [PP21]. It is a functor

$$\beta^{inj} \colon \mathrm{Comod}^{inj}_{R_*R} \longrightarrow h_{q+1}\mathrm{Sp}^{inj}_R,$$

where the latter category is the homotopy $(q+1)$-category of the full subcategory of Sp_R containing all spectra X such that R_*X is injective.

In order to mimic this behavior for a general adapted homology theory, Franke introduced the notion of a splitting of an abelian category.

Definition 1.3.13 ([Fra96])**.** Let $\mathcal{A}$ be an abelian category with a local grading [1]. A *splitting* of $\mathcal{A}$ of order $q+1$ is a collection of Serre subcategories $\mathcal{A}_\varphi \subseteq \mathcal{A}$ indexed by $\varphi \in \mathbb{Z}/(q+1)$ satisfying

1. $[k]\mathcal{A}_n \subseteq \mathcal{A}_{n+k \bmod (q+1)}$ for any $k \in \mathbb{Z}$, and

2. the functor $\prod_\varphi \mathcal{A}_\varphi \longrightarrow \mathcal{A}$, defined by $(a_\varphi) \mapsto \oplus_\varphi a_\varphi$, is an equivalence of categories.

Example 1.3.14. As we saw in Construction 1.3.12, the category of comodules over an Adams Hopf algebroid (R_*, R_*R), where R_* is concentrated in degrees divisible by $q+1$, has a splitting of order $q+1$. This, then, also holds for the discrete Hopf algebroid (R_*, R_*), giving the module category Mod_{R_*} a splitting of order $q+1$ as well.

Example 1.3.15. In the case $R = E_p(1)$ this has been written out in detail in [BR11, Section 4]. The Serre subcategories are all copies of the category of p-local abelian groups together with Adams operations ψ^k for $k \neq 0$ in $\mathbb{Z}_{(p)}$. The shift leaves the underlying module unchanged, but changes the Adams operation.

Definition 1.3.16. We will say that objects $A \in \mathcal{A}_\varphi$ are of *pure weight* φ.

Remark 1.3.17. Just as for Comod_{R_*R}, a splitting of order $q+1$ of a locally graded abelian category $\mathcal{A}$ is enough to define, for any adapted homology theory $H\colon \mathcal{C} \longrightarrow \mathcal{A}$, a partial inverse Bousfield functor β^{inj}, see [PP21, Section 7.2].

1.3.2 Exotic homology theories

In order to make some statements about exotic equivalences a bit simpler, we introduce the concept of exotic adapted homology theories. Note that this is not the way similar results are phrased in [PP21], but the notation serves as a shorthand for the criteria that they use.

Definition 1.3.18. Let $H\colon \mathcal{C} \longrightarrow \mathcal{A}$ be a homology theory. We say H is *k-exotic* if H is adapted, conservative, $\mathcal{A}$ has finite cohomological dimension d and a splitting of order $q+1$ such that $k = d + 1 - q > 0$.

One of the remarkable things about the existence of a k-exotic homology theory $H\colon \mathcal{C} \longrightarrow \mathcal{A}$, is that it forces the stable ∞-category $\mathcal{C}$ to be approximately algebraic. Intuitively: As the order of the splitting is greater than the cohomological dimension, the H-Adams spectral sequence is very sparse and well-behaved. There is a partial inverse of H via the Bousfield functor $\beta\colon \mathcal{A}^{inj} \longrightarrow h_{q+1}\mathcal{C}^{inj}$, which forces a certain subcategory of a categorified deformation of H to be equivalent to both $h_k\mathcal{C}$ and $h_k\mathrm{D}^{\mathrm{per}}(\mathcal{A})$. This is the contents of Franke's algebraicity theorem.

Theorem 1.3.19 ([PP21, 7.56])**.** *Let $H\colon \mathcal{C} \longrightarrow \mathcal{A}$ be a k-exotic homology theory. Then there is an equivalence*

$$h_k\mathcal{C} \simeq h_k\mathrm{D}^{\mathrm{per}}(\mathcal{A})$$

of homotopy k-categories

There are several interesting examples of homology theories satisfying Theorem 1.3.19, see Section 8 in [PP21]. We highlight again our two guiding examples but focus specifically on certain Morava E-theories.

Example 1.3.20 ([PP21, 8.7])**.** Let p be a prime, n be a non-negative integer, and E a height n Morava E-theory concentrated in degrees divisible by $2p-2$, for example Johnson-Wilson theory $E_p(n)$. If $k = 2p - 2 - n > 0$, then the functor

$$\pi_*\colon \mathrm{Mod}_E \longrightarrow \mathrm{Mod}_{E_*}$$

is a k-exotic homology theory, giving an equivalence

$$h_k\mathrm{Mod}_E \simeq h_k\mathrm{D}^{\mathrm{per}}(\mathrm{Mod}_{E_*}).$$

Notation 1.3.21. For the following example and the rest of the paper, we follow the notation of [BSS20], [BSS21] and [Bar23] and denote the category $\mathrm{D}^{\mathrm{per}}(\mathrm{Comod}_{E_*E})$ by $\mathrm{Fr}_{n,p}$.

Example 1.3.22 ([PP21, 8.13])**.** Let p be a prime, n be a non-negative integer, and E any height n Morava E-theory. If $k = 2p - 2 - n^2 - n > 0$, then the functor $E_*\colon \mathrm{Sp}_{n,p} \longrightarrow \mathrm{Comod}_{E_*E}$ is a k-exotic homology theory, giving an equivalence

$$h_k\mathrm{Sp}_{n,p} \simeq h_k\mathrm{Fr}_{n,p}.$$

Remark 1.3.23. As noted in [BSS20, 5.29], this equivalence is strictly exotic for all $n \geqslant 1$ and primes p. In other words, it can never be made into an equivalence of stable ∞-categories. In particular, the mapping spectra in $\mathrm{Fr}_{n,p}$ are $H\mathbb{Z}$-linear, while the mapping spectra in $\mathrm{Sp}_{n,p}$ are only $H\mathbb{Z}$-linear for $n = 0$.

Definition 1.3.24. Let $H\colon \mathcal{C} \longrightarrow \mathcal{A}$ be a k-exotic homology theory. The category $\mathrm{D}^{\mathrm{per}}(\mathcal{A})$ is called an *exotic algebraic model* of $\mathcal{C}$ if the equivalence $h_k\mathcal{C} \simeq h_k\mathrm{D}^{\mathrm{per}}(\mathcal{A})$ can not be enhanced to an equivalence of ∞-categories $\mathcal{C} \simeq \mathrm{D}^{\mathrm{per}}(\mathcal{A})$.

Remark 1.3.25. The notion of being exotically algebraic is part of a complex hierarchy of algebraicity levels, see [IRW23] for a great exposé.

Remark 1.3.26. The existence of an exotic algebraic model for a stable ∞-category $\mathcal{C}$ implies that the category is not rigid. This means, in particular, that there cannot exist a k-exotic homology theory with source Sp or $\mathrm{Sp}_{(p)}$ as these are all rigid for all primes, see [Sch07], [SS02] and [Sch01]. The same holds for $\mathrm{Sp}_{1,2}$, as this is rigid by [Roi07], and similarly for $\mathrm{Sp}_{K_2(1)}$ by [Ish19]. This shows that being k-exotic is quite a strong requirement.

1.4 Algebraicity for monochromatic categories

We are now ready to prove our main results. We start by proving Theorem 1.C, which we will later use to prove Theorem 1.B. The main result, Theorem 1.A will then follow by using certain local duality arguments.

1.4.1 Monochromatic modules

For the rest of this section, we assume that E is the height n Johnson–Wilson theory $E_p(n)$. This is an $\mathbb{E}_1$-ring spectrum concentrated in degrees divisible by $2p-2$, with coefficient ring

$$\pi_* E_p(n) \cong \mathbb{Z}_{(p)}[v_1, v_2, \ldots, v_{n-1}, v_n^{\pm 1}],$$

where $|v_i| = 2p^i - 2$. The goal of this section is to prove Theorem 1.C, which we do in three steps. First we show that the functor

$$\pi_*\colon \mathrm{Mod}_E^{I_n-\mathrm{tors}} \longrightarrow \mathrm{Mod}_{E_*}^{I_n-\mathrm{tors}}$$

is a conservative adapted homology theory. We then show that $\mathrm{Mod}_{E_*}^{I_n-\mathrm{tors}}$ has finite cohomological dimension, and lastly that it admits a splitting.

The following lemma is the I_n-power torsion version [BF15, 3.14], and the proof is similar.

Lemma 1.4.1. *If M is an E-module, then $M \in \mathrm{Mod}_E^{I_n-\mathrm{tors}}$ if and only if $\pi_* M \in \mathrm{Mod}_{E_*}^{I_n-\mathrm{tors}}$.*

Proof. Let $X \in \mathrm{Mod}_E^{I_n-\mathrm{tors}}$. By [BHV18, 3.19] there is a strongly convergent spectral sequence of E_*-modules with signature

$$E_2^{s,t} = (H_{I_n}^{-s}\pi_* X)_t \implies \pi_{s+t} M_n X,$$

where $H_{I_n}^{-s}$ denotes local cohomology. By [BS12, 2.1.3(ii)] the E_2-page consist of only I_n-power torsion modules. As $\mathrm{Mod}_{E_*}^{I_n-\mathrm{tors}}$ is abelian, it is closed under quotients and subobjects, as as the higher pages are created from the E_2-page using quotients and subobjects, they must also consist of only I_n-power torsion modules. In particular, the E_∞-page is all I_n-power torsion. By Grothendieck's vanishing theorem, see for example [BS12, 6.1.2], $H^s_{I_n}(-) \cong 0$ for $s > n$, hence the abutment of the spectral sequence $\pi_* M_n X$ is a finite filtration of I_n-power torsion E_*-modules, and is therefore itself an I_n-power torsion module. Since X was assumed to be monochromatic, i.e. $X \in \mathrm{Mod}_E^{I_n-\mathrm{tors}}$, we have $\pi_* M_n X \cong \pi_* X$, and thus $\pi_* X \in \mathrm{Mod}_{E_*}^{I_n-\mathrm{tors}}$.

Assume now $X \in \mathrm{Mod}_E$ such that its homotopy groups are I_n-power torsion. Monochromatization gives a map $\varphi \colon M_n X \longrightarrow X$, and as $\pi_* M_n X$ is I_n-power torsion this map factors on homotopy groups as

$$\pi_* M_n X \longrightarrow H^0_{I_n} \pi_* X \longrightarrow \pi_* X,$$

where the first map is the edge morphism in the above-mentioned spectral sequence. As $\pi_* X$ was assumed to be I_n-power torsion we have $\pi_* X \cong H^0_{I_n} \pi_* X$, and $H^s_{I_n} \pi_* X \cong 0$ for $s > 0$. Hence the spectral sequence collapses to give the isomorphism $\pi_* M_n X \cong H^0_{I_n} \pi_* X$, which shows that $\pi_* \varphi$ is an isomorphism. As π_* is conservative, φ was already an isomorphism, hence we have $X \in \mathrm{Mod}_E^{I_n-\mathrm{tors}}$. □

Lemma 1.4.2. *For any prime p and non-negative integer n, the functor*

$$\pi_* \colon \mathrm{Mod}_E^{I_n-\mathrm{tors}} \longrightarrow \mathrm{Mod}_{E_*}^{I_n-\mathrm{tors}}$$

is a conservative adapted homology theory.

Proof. We first note that the functor $\pi_* \colon \mathrm{Mod}_E \longrightarrow \mathrm{Mod}_{E_*}$ is a conservative adapted homology theory. By Lemma 1.4.1 its restriction to $\mathrm{Mod}_E^{I_n-\mathrm{tors}}$ lands in $\mathrm{Mod}_{E_*}^{I_n-\mathrm{tors}}$, hence automatically $\pi_* \colon \mathrm{Mod}_E^{I_n-\mathrm{tors}} \longrightarrow \mathrm{Mod}_{E_*}^{I_n-\mathrm{tors}}$ is a conservative homology theory.

Let J be an injective I_n-power torsion E_*-module. We can embed $J \longrightarrow Q$ into an injective E_*-module Q, as Mod_{E_*} has enough injective objects. After applying the torsion functor $T^{E_*}_{I_n}$ this map has a section, as $J \cong T^{E_*}_{I_n} J$ is injective. In particular, any injective J is a retract of $T^{E_*}_{I_n}Q$ for some injective E_*-module Q, hence we can assume J to be of that form. By [BS12, 2.1.4] any such $J = T^{E_*}_{I_n}Q$ is injective as an object of Mod_{E_*}.

Now, as π_* is adapted on Mod_E we can chose a faithful injective lift $\bar{J}$ of J to Mod_E, and since $\bar{J}$ was assumed to have I_n-torsion homotopy groups we know by Lemma 1.4.1 that $\bar{J}$ is an object of $\mathrm{Mod}_E^{I_n-\mathrm{tors}}$. In particular, we have faithful lifts for any injective in $\mathrm{Mod}_{E_*}^{I_n-\mathrm{tors}}$, which means that $\pi_*\colon \mathrm{Mod}_E^{I_n-\mathrm{tors}} \longrightarrow \mathrm{Mod}_{E_*}^{I_n-\mathrm{tors}}$ is adapted. □

Let C^{I_n} denote the I_n-adic completion functor on Mod_{E_*}. It is neither left nor right exact, see [HS99, Appendix A.]. As E_* is an integral domain, the higher right derived functors vanish by [GM92, 5.1]. For $i \geqslant 0$ we denote the i'th left derived functor of C^{I_n} by $L_i^{I_n}$. For any $M \in \mathrm{Mod}_E$ there is a natural map $L_0^{I_n}M \longrightarrow C^{I_n}M$. It is always an epimorphism, but usually not an isomorphism.

Lemma 1.4.3. *For any prime p and non-negative integer n, the category $\mathrm{Mod}_{E_*}^{I_n-\mathrm{tors}}$ has cohomological dimension n.*

Proof. Note first that the category Mod_{E_*} has cohomological dimension n, and that Ext-groups in $\mathrm{Mod}_{E_*}^{I_n-\mathrm{tors}}$ are computed in Mod_{E_*}. By [BS12, 2.1.4], this implies that the cohomological dimension of $\mathrm{Mod}_{E_*}^{I_n-\mathrm{tors}}$ cannot be greater than n, so it remains to prove that it is exactly n. We prove this by computing an $\mathrm{Ext}^n_{E_*}$ group that is non-zero.

By [HS99, A.2(d)] we have $L_0^{I_n}M \cong \mathrm{Ext}^n_{E_*}(H^n_{I_n}(E_*), M)$ for any E_*module M. In other words, the derived completion of an E_*-module is the n'th derived functor of maps from the I_n-local cohomology of E_* into M. Choosing $M = E_*/I_n$ we get

$$L_0^{I_n}(E_*/I_n) \cong \mathrm{Ext}^n_{E_*}(H^n_{I_n}(E_*), E_*/I_n).$$

As any bounded I_n-torsion E_*-module is I_n-adically complete we have, as remarked in [BH16, 1.4], an isomorphism

$$L_0^{I_n}(E_*/I_n) \cong E_*/I_n.$$

The local cohomology of E_* is also I_n-torsion, in particular $H^n_{I_n} E_* = E_*/I_n^\infty$. Hence we have

$$\mathrm{Ext}^n_{E_*}(E_*/I_n^\infty, E_*/I_n) \cong E_*/I_n \ncong 0,$$

showing that there are two I_n-power torsion E_*-modules with non-trivial n'th Ext, which concludes the proof. $\square$

Lemma 1.4.4. *For any prime p and non-negative integer n, the category $\mathrm{Mod}_{E_*}^{I_n-\mathrm{tors}}$ has a splitting of order $2p-2$.*

Proof. By [PP21, 8.1] the category Mod_{E_*} has a splitting of order $2p-2$. We define the pure weight φ component of $\mathrm{Mod}_{E_*}^{I_n-\mathrm{tors}}$, denoted $\mathrm{Mod}_{E_*,\varphi}^{I_n-\mathrm{tors}}$, to be the essential image of

$$T_{I_n}^{E_*} \colon \mathrm{Mod}_{E_*} \longrightarrow \mathrm{Mod}_{E_*}^{I_n-\mathrm{tors}}$$

restricted to the pure weight φ component $\mathrm{Mod}_{E_*,\varphi}$. We claim that this defines a splitting of order $2p-2$ on $\mathrm{Mod}_{E_*}^{I_n-\mathrm{tors}}$.

As $\mathrm{Mod}_{E_*,\varphi}$ is a Serre subcategory, and being I_n-power torsion is a property closed under sub-objects, quotients, and extensions, also $\mathrm{Mod}_{E_*,\varphi}^{I_n-\mathrm{tors}}$ is a Serre subcategory. As E_* is concentrated in degrees divisible by $2p-2$ every I_n-power torsion module decomposes into its pure weight components. This also gives a decomposition of $\mathrm{Mod}_{E_*}^{I_n-\mathrm{tors}}$. The shift functor on I_n-power torsion modules simply shifts the underlying module, hence shift-invariance follows from the shift-invariance on Mod_{E_*}. $\square$

We can now summarize the above discussion with the first of our main results.

Theorem 1.4.5 (Theorem 1.C)**.** *Let p be a prime and n a non-negative integer. If $k = 2p-2-n > 0$, then the functor*

$$\pi_* \colon \mathrm{Mod}_E^{I_n-\mathrm{tors}} \longrightarrow \mathrm{Mod}_{E_*}^{I_n-\mathrm{tors}}$$

is a k-exotic homology theory, giving an equivalence

$$h_k\mathrm{Mod}_E^{I_n-\mathrm{tors}} \simeq h_k\mathrm{D}^{\mathrm{per}}(\mathrm{Mod}_{E_*}^{I_n-\mathrm{tors}}).$$

In particular, monochromatic E-modules are exotically algebraic at large primes.

Proof. By Lemma 1.4.3 the cohomological dimension of the category $\mathrm{Mod}_{E_*}^{I_n-\mathrm{tors}}$ is n, and by Lemma 1.4.4 we have a splitting on $\mathrm{Mod}_{E_*}^{I_n-\mathrm{tors}}$ of order $2p-2$. Hence, by Lemma 1.4.2 the functor

$$\pi_*\colon \mathrm{Mod}_E^{I_n-\mathrm{tors}} \longrightarrow \mathrm{Mod}_{E_*}^{I_n-\mathrm{tors}}$$

is a k-exotic homology theory for $k = 2p-2-n > 0$, which gives an equivalence

$$h_k\mathrm{Mod}_E^{I_n-\mathrm{tors}} \simeq h_k\mathrm{D}^{\mathrm{per}}(\mathrm{Mod}_{E_*}^{I_n-\mathrm{tors}})$$

by Theorem 1.3.19. □

We can also phrase this dually in terms of $K_p(n)$-local E-modules.

Corollary 1.4.6. *Let p be a prime, n a positive integer and $K_p(n)$ be height n Morava K-theory at the prime p. If $k = 2p-2-n > 0$, then we have a k-exotic algebraic equivalence*

$$h_k L_{K_p(n)}\mathrm{Mod}_E \simeq h_k\mathrm{D}^{\mathrm{per}}(\mathrm{Mod}_{E_*})^{I_n-\mathrm{comp}}.$$

In particular, $K_p(n)$-local E-modules are exotically algebraic at large primes.

Proof. The equivalence is constructed from the equivalences obtained from Remark 1.2.7, Theorem 1.4.5, Theorem 1.2.21 and Construction 1.2.19. In particular, we have

$$\begin{aligned} h_k\mathrm{Mod}_E^{I_n-\mathrm{comp}} &\overset{1.2.7}{\simeq} h_k\,\mathrm{Mod}_E^{I_n-\mathrm{tors}} \\ &\overset{1.4.5}{\simeq} h_k\mathrm{D}^{\mathrm{per}}(\mathrm{Mod}_{E_*}^{I_n-\mathrm{tors}}) \\ &\overset{1.2.21}{\simeq} h_k\mathrm{D}^{\mathrm{per}}(\mathrm{Mod}_{E_*})^{I_n-\mathrm{tors}} \\ &\overset{1.2.19}{\simeq} h_k\mathrm{D}^{\mathrm{per}}(\mathrm{Mod}_{E_*})^{I_n-\mathrm{comp}}, \end{aligned}$$

where we have used that an equivalence of ∞-categories induces an equivalence on homotopy k-categories. □

Now, let HE_* be the Eilenberg–MacLane spectrum of E_*. By Schwede–Shipley's derived Morita theory, see [Lur17, 7.1.1.16], there is a symmetric monoidal equivalence of categories

$$\mathrm{D}(E_*) \simeq \mathrm{Mod}_{HE_*},$$

and we can form a local duality diagram for Mod_{HE_*} corresponding to Example 1.2.5 for the discrete Hopf algebroid (E_*, E_*). By arguments similar to Lemma 1.4.1 and Lemma 1.4.2 one can show that the homotopy groups functor $\pi_* \colon \mathrm{Mod}_{HE_*} \longrightarrow \mathrm{Mod}_{E_*}$ restricts to a conservative adapted homology theory

$$\pi_* \colon \mathrm{Mod}_{HE_*}^{I_n-\mathrm{tors}} \longrightarrow \mathrm{Mod}_{E_*}^{I_n-\mathrm{tors}}.$$

In the same range as Theorem 1.4.5 this is also k-exotic. We can then combine the algebraicity for $\mathrm{Mod}_E^{I_n-\mathrm{tors}}$ and Mod_{HE_*} to get the following statement.

Corollary 1.4.7. *Let p be a prime and n a non-negative integer. If $k = 2p - 2 - n > 0$, then there is an exotic equivalence*

$$h_k\mathrm{Mod}_E^{I_n-\mathrm{tors}} \simeq h_k\mathrm{Mod}_{HE_*}^{I_n-\mathrm{tors}}$$

of homotopy k-categories.

1.4.2 Monochromatic spectra

Having proven that monochromatic E-modules are algebraic at large primes, we now turn to the larger category of all monochromatic spectra $\mathcal{M}_{n,p}$ with the same goal. The strategy is exactly the same as in Section 1.4.1: we first prove that the conservative adapted homology theory $E_* \colon \mathrm{Sp}_{n,p} \longrightarrow \mathrm{Comod}_{E_*E}$ restricts to a conservative adapted homology theory on $\mathcal{M}_{n,p}$, before proving that $\mathrm{Comod}_{E_*E}^{I_n-\mathrm{tors}}$ has a splitting and finite cohomological dimension. This will prove Theorem 1.B, which we then convert into a proof of Theorem 1.A—just as in Corollary 1.4.6.

In this section the choice of v_n-periodic Landweber exact ring spectrum E does not matter, as the categories $\mathrm{Sp}_{n,p}$ and Comod_{E_*E} are equivalent for all such spectra—see [Hov95, 1.12] and [HS05a, 4.2] respectively. However, to make the interaction with Section 1.4.1 as simple as possible we will continue to use the height n Johnson–Wilson spectrum $E_p(n)$.

Lemma 1.4.8. *If X is an E-local spectrum, then $X \in \mathcal{M}_{n,p}$ if and only if $E_*X \in \mathrm{Comod}_{E_*E}^{I_n-\mathrm{tors}}$.*

Proof. Assume first that $X \in \mathcal{M}_{n,p}$. We have $E\otimes X \in \mathrm{Mod}_E^{I_n-\mathrm{tors}}$ as

$$E \otimes X \simeq E \otimes M_n X \simeq M_n(E \otimes X),$$

where the last equivalence follows from M_n being smashing. In particular, the restricted functor $E_*\colon \mathcal{M}_{n,p} \longrightarrow \mathrm{Comod}_{E_*E}$ factors through $\mathrm{Mod}_E^{I_n-\mathrm{tors}}$. By Lemma 1.4.1 and Remark 1.2.9 this means that E_*X is an I_n-power torsion E_*E-comodule.

For the converse, assume that we have $X \in \mathrm{Sp}_{n,p}$ such that $E_*X \in \mathrm{Comod}_{E_*E}^{I_n-\mathrm{tors}}$. Using the monochromatization functor we obtain a comparison map $M_nX \longrightarrow X$, which induces a map on E-modules $E\otimes M_nX \longrightarrow E\otimes X$. This map is an isomorphism on homotopy groups, as E_*X was assumed to be I_n-power torsion. As E_* is conservative on $\mathrm{Sp}_{n,p}$, the original comparison map $M_nX \longrightarrow X$ was an isomorphism, meaning that $X \in \mathcal{M}_{n,p}$. □

Lemma 1.4.9. *For any prime p and non-negative integer n, the functor*

$$E_*\colon \mathcal{M}_{n,p} \longrightarrow \mathrm{Comod}_{E_*E}^{I_n-\mathrm{tors}}$$

is a conservative adapted homology theory.

Proof. First note that the image of $E_*\colon \mathrm{Sp}_{n,p} \longrightarrow \mathrm{Comod}_{E_*E}$, restricted to $\mathcal{M}_{n,p}$, is contained in $\mathrm{Comod}_{E_*E}^{I_n-\mathrm{tors}}$ by Lemma 1.4.8. The functor

$$E_*\colon \mathcal{M}_{n,p} \longrightarrow \mathrm{Comod}_{E_*E}^{I_n-\mathrm{tors}}$$

is then automatically a conservative homology theory. The category $\mathrm{Comod}_{E_*E}^{I_n-\mathrm{tors}}$ has enough injective objects as it is Grothendieck by Remark 1.2.10. Hence, it only remains to prove that we have faithful lifts for all injective objects.

Let J be an injective in $\mathrm{Comod}_{E_*E}^{I_n-\mathrm{tors}}$. As in the proof of Lemma 1.4.2 we can assume that J has the form $J = T_{I_n}^{E_*E} P$ for some injective E_*E-comodule P, as being torsion is a property of the underlying module. By [HS05b, 2.1(c)] any injective E_*E-comodule is a retract of $E_*E \otimes_{E_*} Q$ for some injective E_*-module Q. Hence, we can further assume that J has the form $J = T_{I_n}^{E_*E}(E_*E \otimes_{E_*} Q)$.

From [BHV18, 5.7] it follows that there is a commutative diagram of adjoint functors

$$\begin{tikzcd}
\mathrm{Comod}_{E_*E} \arrow[r, shift left, "\varepsilon_*"] \arrow[d, shift left, "T_{I_n}^{E_*E}"] & \mathrm{Mod}_{E_*} \arrow[l, shift left, "\varepsilon^*"] \arrow[d, shift left, "T_{I_n}^{E_*}"] \\
\mathrm{Comod}_{E_*E}^{I_n-\mathrm{tors}} \arrow[u, shift left] \arrow[r, shift left, "\varepsilon_*"] & \mathrm{Mod}_{E_*}^{I_n-\mathrm{tors}} \arrow[l, shift left, "\varepsilon^*"] \arrow[u, shift left]
\end{tikzcd}$$

where $\varepsilon_* \dashv \varepsilon^*$ is the forgetful-cofree adjunction. In particular, the functor ε^* is given by $E_*E \otimes_{E_*} (-)$. To justify the notation in the bottom row, let us prove that the cofree functor on I_n-power torsion modules is also given by $E_*E \otimes_{E_*} (-)$. In order to do this we prove that for an I_n-power torsion E_*-module M, that

$$T_{I_n}^{E_*E}(E_*E \otimes_{E_*} M) \cong E_*E \otimes_{E_*} M.$$

By [BHV18, 5.5] there is an isomorphism

$$T_{I_n}^{E_*E}(E_*E \otimes_{E_*} M) \cong \mathrm{colim}_k \, \underline{\mathrm{Hom}}_{E_*E}(E_*/I_n^k, E_*E \otimes_{E_*} M),$$

which by [BHV18, 4.4] gives

$$\mathrm{colim}_k \, \underline{\mathrm{Hom}}_{E_*E}(E_*/I_n^k, E_*E \otimes_{E_*} M) \cong \mathrm{colim}_k (E_*E \otimes_{E_*} \mathrm{Hom}_{E_*}(E_*/I_n^k, M)).$$

As the tensor product $- \otimes_{E_*} -$ commutes with filtered colimits separately in each variable, and M was assumed to be I_n-power torsion, the right hand side is $E_*E \otimes_{E_*} M$.

Now, choosing the injective Q in the top right corner and going through the square gives an isomorphism

$$T_{I_n}^{E_*E}(E_*E \otimes_{E_*} Q) \cong E_*E \otimes_{E_*} T_{I_n}^{E_*} Q.$$

By [BS12, 2.1.4] we know that $T^{E_*}_{I_n} Q$ is an injective E_*-module, and by [HS05b, 2.1(a)] the cofree comodule $E_*E \otimes_{E_*} T^{E_*}_{I_n} Q$ is an injective E_*E-comodule. Hence, $J = T^{E_*E}_{I_n}(E_*E \otimes_{E_*} Q)$ is injective also as an object in Comod_{E_*E}.

Finally, as E_* has faithful injective lifts from Comod_{E_*E} to $\mathrm{Sp}_{n,p}$, there is a lift $\bar{J}$ such that $[X, \bar{J}] \simeq \mathrm{Hom}_{E_*E}(E_*X, J)$ and $E_*\bar{J} \simeq J$. By Lemma 1.4.8 we know that $\bar{J} \in \mathcal{M}_{n,p}$, as J was assumed to be I_n-power torsion, hence we have found our faithful injective lift. □

Lemma 1.4.10. *Let p be a prime and n a non-negative integer. If $p-1 \nmid n$, then the category $\mathrm{Comod}^{I_n-\mathrm{tors}}_{E_*E}$ has cohomological dimension n^2+n.*

Proof. The proof follows [Pst21, 2.5] closely, which is itself a modern reformulation of [Fra96, 3.4.3.9]. As in Lemma 1.4.3 we note that also Ext-groups in $\mathrm{Comod}^{I_n-\mathrm{tors}}_{E_*E}$ are computed in Comod_{E_*E}. We start by defining *good targets* to be I_n-power torsion comodules N such that $\mathrm{Ext}^{s,t}_{E_*E}(E_*/I_n, N) = 0$ for all $s > n^2+n$ and *good sources* to be I_n-power torsion comodules M such that $\mathrm{Ext}^{s,t}_{E_*E}(M, N) = 0$ for all $s > n^2+n$ and I_n-torsion comodules N.

By the Landweber filtration theorem, see for example [HS05a, 5.7], we know that any finitely presented comodule M has a finite filtration

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_{s-1} \subseteq M_s = M,$$

where $M_r/M_{r-1} \cong E_*/I_{j_r}[t_r]$ and $j_r \leqslant n$. When M is I_n-power torsion we get $j_r = n$ for all r, as noted in [HS05a, 4.3]. For primes such that $p-1 \nmid n$ Morava's vanishing theorem, see for example [Rav86, 6.2.10], gives us that $\mathrm{Ext}^{s,t}_{E_*E}(E_*, E_*/I_n) = 0$ for all $s > n^2$. As the generators for the ideal I_i form a regular sequence, we get short exact sequences of the form

$$0 \longrightarrow E_*/I_{i-1} \xrightarrow{v_i} E_*/I_{i-1} \longrightarrow E_*/I_i \longrightarrow 0$$

for $0 \leqslant i \leqslant n$. By the induced long exact sequence in Ext-groups, we get that

$$\mathrm{Ext}^{s,t}_{E_*E}(E_*/I_n, E_*/I_n) = 0$$

for $s > n^2+n$, which by the Landweber filtration implies that any finitely presented I_n-power torsion comodule is a good target.

The comodule E_*/I_n has a finite resolution of E_*E-comodules that are projective as modules over E_*. The Ext-functor out of these projective objects can be computed using the cobar complex, see [Rav86, A1.2.12], implying that the functor $\mathrm{Ext}^{s,t}_{E_*E}(E_*/I_n, -)$ commutes with filtered colimits. By Lemma 1.2.11 any I_n-power torsion comodule is a filtered colimit of finitely presented ones, hence any I_n-power torsion comodule is a good target.

Note that the above argument also proves that E_*/I_n is a good source, which by the Landweber filtration argument implies that any finitely presented I_n-torsion comodule is a good source. Again, by Lemma 1.2.11, the category $\mathrm{Comod}^{I_n-\mathrm{tors}}_{E_*E}$ is generated under filtered colimits by finitely presented comodules. Hence, we can apply [Pst21, 2.4] to any injective resolution

$$0 \longrightarrow M \longrightarrow J_0 \longrightarrow J_1 \longrightarrow \cdots$$

to get that the map $J_{n^2+n} \longrightarrow \mathrm{Im}(J_{n^2+n} \longrightarrow J_{n^2+n+1})$ is a split surjection, and that the object $\mathrm{Im}(J_{n^2+n} \longrightarrow J_{n^2+n+1})$ is injective. Hence, any injective resolution can be modified to have length n^2+n, which concludes the proof. □

Remark 1.4.11. In a previous version of this paper, we claimed that the cohomological dimension was n^2. We want to thank Piotr Pstrągowski for pointing out the gap in the proof. This means that $\mathrm{Comod}^{I_n-\mathrm{tors}}_{E_*E}$ has the same cohomological dimension as the non-torsion category Comod_{E_*E}, as seen in Example 1.3.10. However, we do obtain something slightly stronger, as our result holds for all $p-1 \nmid n$, while the analogue in Comod_{E_*E} only holds when $p-1 > n$. In fact, Comod_{E_*E} does not have finite cohomological dimension when $p-1 \leqslant n$, as noted in [Pst21, 2.6]. This difference happens because we only need the Ext^s-groups out of E_*/I_n to vanish for large s, which is given to us by

Morava's vanishing theorem whenever $p-1 \nmid n$. For non-torsion comodules one has to have stronger vanishing results. These can be obtained by using the chromatic spectral sequence, which only gives the vanishing results for $p-1 > n$ instead of for $p-1 \nmid n$.

Lemma 1.4.12. *For any prime p and non-negative integer n, the category $\mathrm{Comod}_{E_*E}^{I_n-\mathrm{tors}}$ has a splitting of order $2p-2$.*

Proof. As E is concentrated in degrees divisible by $2p-2$, [PP21, 8.13] shows that Comod_{E_*E} has a splitting of order $2p-2$. The proof of the induced splitting on the I_n-torsion category is then identical to Lemma 1.4.4. □

We can now summarize the above results with our second main result, which is the monochromatic analogue of Example 1.3.22.

Theorem 1.4.13 (Theorem 1.B)**.** *Let p be a prime and n a non-negative integer. If we have $k = 2p-2-n^2-n > 0$, then the restricted functor $E_*\colon \mathcal{M}_{n,p} \longrightarrow \mathrm{Comod}_{E_*E}^{I_n-\mathrm{tors}}$ is k-exotic. In particular, there is an equivalence*

$$h_k\mathcal{M}_{n,p} \simeq h_k\mathrm{D}^{\mathrm{per}}(E_*E^{I_n-\mathrm{tors}}),$$

meaning that monochromatic homotopy theory is exotically algebraic at large primes.

Proof. By Lemma 1.4.10 we know that the cohomological dimension of $\mathrm{Comod}_{E_*E}^{I_n-\mathrm{tors}}$ is n^2+n, and by Lemma 1.4.12 we have a splitting of order $2p-2$. The restricted functor E_* is then by Lemma 1.4.9 k-exotic whenever $k = 2p-2-n^2-n > 0$, which by Theorem 1.3.19 finishes the proof. □

Remark 1.4.14. By Theorem 1.2.21 there is an equivalence $\mathrm{D}^{\mathrm{per}}(E_*E^{I_n-\mathrm{tors}}) \simeq \mathrm{Fr}_{n,p}^{I_n-\mathrm{tors}}$ and by Example 1.2.6 there is an equivalence $\mathcal{M}_{n,p} \simeq \mathrm{Sp}_{n,p}^{I_n-\mathrm{tors}}$. This means that we can write the equivalence in Theorem 1.4.13 as

$$h_k\mathrm{Sp}_{n,p}^{I_n-\mathrm{tors}} \simeq h_k\mathrm{Fr}_{n,p}^{I_n-\mathrm{tors}},$$

or alternatively,

$$h_k \operatorname{Loc}^{\otimes}_{\mathrm{Sp}_{n,p}}(L_{n,p}F(n)) \simeq h_k \operatorname{Loc}^{\otimes}_{\mathrm{Fr}_{n,p}}(P(E_*/I_n))$$

for $k = 2p - 2 - n^2 - n > 0$. This is more in line with thinking about Theorem 1.4.13 as "coming from" the chromatic algebraicity of Example 1.3.22 on localizing ideals. This formulation is perhaps also easier to connect to the limiting case $p \longrightarrow \infty$ as described using ultra-products in [BSS21], which can be stated informally as

$$\lim_{p \longrightarrow \infty} \mathrm{Sp}_{n,p}^{I_n-\mathrm{tors}} \simeq \lim_{p \longrightarrow \infty} \mathrm{Fr}_{n,p}^{I_n-\mathrm{tors}}.$$

Via Theorem 1.2.3 we can now obtain the associated exotic algebraicity statement for the category of $K_p(n)$-local spectra.

Theorem 1.4.15 (Theorem 1.A)**.** *Let p be a prime, n a non-negative integer and $K_p(n)$ be height n Morava K-theory at the prime p. If $k = 2p-2-n^2 > 0$, then we have a k-exotic algebraic equivalence*

$$h_k \mathrm{Sp}_{K_p(n)} \simeq h_k \mathrm{Fr}_{n,p}^{I_n-\mathrm{comp}}.$$

In other words, $K_p(n)$-local homotopy theory is exotically algebraic at large primes.

Proof. As we did in Corollary 1.4.6, we construct the equivalence from a sequence of equivalences coming from Theorem 1.2.3 and Theorem 1.4.13. More precisely we use equivalences coming from Example 1.2.6, Theorem 1.4.13, Theorem 1.2.21 and Construction 1.2.19, which give

$$\begin{aligned} h_k \mathrm{Sp}_{K_p(n)} &\overset{1.2.6}{\simeq} h_k \mathcal{M}_{n,p} \\ &\overset{1.4.13}{\simeq} h_k \mathrm{D}^{\mathrm{per}}(\mathrm{Comod}_{E_*E}^{I_n-\mathrm{tors}}) \\ &\overset{1.2.21}{\simeq} h_k \mathrm{Fr}_{n,p}^{I_n-\mathrm{tors}} \\ &\overset{1.2.19}{\simeq} h_k \mathrm{Fr}_{n,p}^{I_n-\mathrm{comp}}, \end{aligned}$$

where we again have used that an equivalence of ∞-categories induces an equivalence on homotopy k-categories. $\square$

Remark 1.4.16. As in Remark 1.4.14 we can phrase the equivalence from Theorem 1.4.15 as $h_k\mathrm{Sp}_{n,p}^{I_n-\mathrm{comp}} \simeq h_k\mathrm{Fr}_{n,p}^{I_n-\mathrm{comp}}$.

Addendum. One of the interesting features of the category $\mathrm{Sp}_{K_p(n)}$ is that it is ∞-semiadditive, meaning that limits and colimits indexed over π-finite spaces agree. The 1-semiadditive case is equivalent to the vanishing of the Tate construction in $\mathrm{Sp}_{K_p(n)}$, and the ∞-semiadditivity was shown by Hopkins–Lurie in [HL17]. We thought originally that the above result gave a strong indication that also $\mathrm{Fr}_{n,p}^{I_n-\mathrm{comp}}$ was ∞-semiadditive, or at least 1-semiadditive. But, as we learned from Tomer Schlank, any $\mathbb{Z}$-linear 1-semiadditive category is $\mathbb{Q}$-linear, which destroys this dream outside of $n = 0$.

Some remarks on future work

The reason why Theorem 1.3.19 works so well, is that there is a deformation of stable ∞-categories lurking behind the scenes. One does not need this in order to apply the theorem, but it is there regardless. In the case of some v_n-periodic Landweber exact ring spectrum E, the deformation associated with the conservative adapted homology theory $E_*\colon \mathrm{Sp}_{n,p} \longrightarrow \mathrm{Comod}_{E_*E}$ is equivalent to the category of E-local synthetic spectra, LSyn_E—a variant of the category of E-based synthetic spectra introduced in [Pst23]. As both $\mathrm{Sp}_{n,p}$ and Comod_{E_*E} are invariant under the choice of such E, we conjecture that this is true also for LSyn_E.

Our restricted homology theory $E_*\colon \mathcal{M}_{n,p} \longrightarrow \mathrm{Comod}_{E_*E}^{I_n-\mathrm{tors}}$ should then be associated to a deformation $\mathcal{M}\mathrm{Syn}_E$ coming from a local duality theory for LSyn_E, in the sense that there is a diagram of stable ∞-categories

$$\mathcal{M}_{n,p} \xleftarrow{\tau^{-1}} \mathcal{M}\mathrm{Syn}_E \xrightarrow{C\tau\otimes -} \mathrm{Stable}_{E_*E}^{I_n-\mathrm{tors}}.$$

Since E_* is adapted on $\mathcal{M}_{n,p}$, we abstractly know that there is a deformation $\mathrm{D}^\omega(\mathcal{M}_{n,p})$ arising out of the work of Patchkoria-Pstrągowski in [PP21], called the perfect derived category. This should give an equivalent "internal" approach to monochromatic synthetic spectra, much akin to the equivalences $\mathcal{M}_{n,p} \simeq \mathrm{Sp}_{n,p}^{I_n-\mathrm{tors}}$ and $\mathrm{D}(E_*E)^{I_n-\mathrm{tors}} \simeq \mathrm{D}(E_*E^{I_n-\mathrm{tors}})$.

Addendum. Since writing this paper we have investigated this idea further. We have added a construction of a category we call *monochromatic synthetic spectra* in Section 3.A. There we prove that it comes from a local duality on LSyn_E, that it has the correct deformation properties as above, and that it has a uniqueness property with respect to the abelian category $\mathrm{Comod}_{E_*E}^{I_n-\mathrm{tors}}$, coming from the theory constructed in Chapter 3.

In [Bar23], Barkan provides a monoidal version of Theorem 1.3.19 by using filtered spectra. We conjecture that his deformation $\mathscr{E}_{n,p}$ is equivalent to LSyn_E for big enough primes, which by the above remarks hints towards a monoidal version of Theorem 1.4.13 as well. We originally intended to incorporate such a result into this paper but decided against it in order to keep it free from deformation theory. We do, however, state the conjectured monoidal result, which we hope to pursue in future work.

Conjecture 1.4.17. Let p be a prime and n a natural number. If k is a positive natural number such that $2p-2 > n^2 + (k+3)n + k - 1$, then we have a symmetric monoidal equivalence

$$h_k\mathcal{M}_{n,p} \simeq h_k\mathrm{Fr}_{n,p}^{I_n-\mathrm{tors}}$$

of k-categories.

As Theorem 1.2.3 is monoidal, this would give a similar statement for the $K_p(n)$-local category, i.e., a symmetric monoidal equivalence

$$h_k\mathrm{Sp}_{K_p(n)} \simeq h_k\mathrm{Fr}_{n,p}^{I_n-\mathrm{comp}}.$$

Since E-local synthetic spectra are categorifications of the E-Adams spectral sequence, one should expect the above-mentioned local duality for LSyn_E to give a category $\mathrm{LSyn}_E^{I_n-\mathrm{comp}}$, which categorifies the $K_p(n)$-local E-Adams spectral sequence. We plan to study such categorifications of the $K_p(n)$-local E-Adams spectral sequence in future work joint with Marius Nielsen.

Addendum. We were reminded by Shaul Barkan that Conjecture 1.4.17 holds by a localizing ideal argument. But, this approach does not give any new insight into the deformation underlying said equivalence, which we feel is still an interesting problem, hence the conjecture stands—at least morally.

1.A Barr–Beck for localizing ideals

In this appendix we prove that the monoidal Barr–Beck theorem—a monoidal version of Lurie's ∞-categorical version of the classical Barr–Beck monadicity theorem, see [Lur17, Section 4.7]—interacts nicely with local duality.

Theorem 1.A.1 ([MNN17, 5.29]). *Let $\mathcal{C}$ and $\mathcal{D}$ be presentably symmetric monoidal stable ∞-categories, and $(F \dashv G)\colon \mathcal{C} \longrightarrow \mathcal{D}$ a monoidal adjunction. If in addition*

1. *G is conservative,*
2. *G preserves colimits, and*
3. *the projection formula holds,*

then (F, G) is a monoidally monadic adjunction and the monad GF is equivalent to the monad $G(\mathbb{1}_{\mathcal{D}}) \otimes (-)$. In particular this gives a symmetric monoidal equivalence $\mathcal{D} \simeq \mathrm{Mod}_{G(\mathbb{1}_{\mathcal{D}})}(\mathcal{C})$.

Proof. By [Lur17, 4.7.3.5] the adjunction is monadic by the first two criteria, giving an equivalence $\mathcal{D} \simeq \mathrm{Mod}_{GF}(\mathcal{C})$. The map of monads $G(\mathbb{1}_{\mathcal{D}}) \otimes (-) \longrightarrow GF$ given by [EK20, 3.6], is seen to be an equivalence by applying the projection formula to the unit $\mathbb{1}_{\mathcal{D}}$. $\square$

Definition 1.A.2. When the three criteria above hold for a given monoidal adjunction $(F \dashv G)$, we will say that the adjunction satisfies the monoidal Barr–Beck criteria or that it is a *monoidal Barr–Beck adjunction*. We will sometimes omit the prefix monoidal when it is clear from context.

Let $(\mathcal{C}, \mathcal{K})$ be a local duality context. We wish to prove that the associated local duality diagram is compatible with Theorem 1.A.1. By modifying [BS20, 3.7] slightly, we know that any Barr–Beck adjunction induces a Barr–Beck adjunction on $\mathcal{K}$-local and $\mathcal{K}$-complete objects. Hence, it remains only to prove a similar statement for the $\mathcal{K}$-torsion objects.

Definition 1.A.3. Let $(\mathcal{C}, \mathcal{K})$ and $(\mathcal{D}, \mathcal{L})$ be local duality contexts. A *map of local duality contexts* is a symmetric monoidal colimit-preserving functor $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ such that $F(\mathcal{K}) \subseteq \mathcal{L}$. If, in addition $\mathrm{Loc}^{\otimes}_{\mathcal{D}}(F(\mathcal{K})) = \mathrm{Loc}^{\otimes}_{\mathcal{D}}(\mathcal{L})$, then we say F is a *strict* map of local duality contexts. A monoidal adjunction $(F \dashv G)\colon \mathcal{C} \longrightarrow \mathcal{D}$ such that F is a strict map of local duality contexts is called a *local duality adjunction*, sometimes denoted

$$(F \dashv G)\colon (\mathcal{C}, \mathcal{K}) \longrightarrow (\mathcal{D}, \mathcal{L}).$$

Given a local duality context and an appropriate functor, one can always extend the functor to a strict map of local duality context in the following way.

Construction 1.A.4. Let $(\mathcal{C}, \mathcal{K})$ be a local duality context, $\mathcal{D}$ a presentably symmetric monoidal stable ∞-category and $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ a symmetric monoidal colimit-preserving functor. The image of $\mathcal{K}$ under F generates a localizing ideal $\mathrm{Loc}^{\otimes}_{\mathcal{D}}(F(\mathcal{K}))$ in $\mathcal{D}$, which makes F a strict map of local duality contexts. We call this the local duality context on $\mathcal{D}$ *induced* by $\mathcal{C}$ via F.

The following lemma is essentially the "non-geometric" version of [BS17, 5.11]. The proof is also similar, but as we have phrased it in a different and slightly more general language, we present a full proof.

Lemma 1.A.5. *Let $(F \dashv G)\colon (\mathcal{C}, \mathcal{K}) \longrightarrow (\mathcal{D}, \mathcal{L})$ be a local duality adjunction. Then, the adjunction induces a monoidal adjunction on localizing ideals*

$$\mathrm{Loc}^{\otimes}_{\mathcal{C}}(\mathcal{K}) \underset{G'}{\overset{F'}{\rightleftarrows}} \mathrm{Loc}^{\otimes}_{\mathcal{D}}(\mathcal{L}).$$

Proof. From Remark 1.2.4 we know that the symmetric monoidal structures on $\mathrm{Loc}^{\otimes}_{\mathcal{C}}(\mathcal{K})$ and $\mathrm{Loc}^{\otimes}_{\mathcal{D}}(\mathcal{L})$ is simply the symmetric monoidal structures on $\mathcal{C}$ and $\mathcal{D}$, restricted to the full subcategories.

Since F is a map of local duality contexts, we have an inclusion $F(\mathcal{K}) \subseteq \mathcal{L}$, which gives inclusions

$$F(\mathrm{Loc}^{\otimes}_{\mathcal{C}}(\mathcal{K})) \subseteq \mathrm{Loc}^{\otimes}_{\mathcal{D}}(F(\mathcal{K})) \subseteq \mathrm{Loc}^{\otimes}_{\mathcal{D}}(\mathcal{L}),$$

meaning that the functor F restricts to the torsion objects. In particular we have for any object $X \in \mathcal{C}^{\mathcal{K}-\mathrm{tors}}$ an equivalence $\Gamma_{\mathcal{L}} F(X) \simeq F(X)$. We let $F' = F_{|\,\mathrm{Loc}^{\otimes}_{\mathcal{C}}(\mathcal{K})}$ and define G' to be the composition

$$\mathrm{Loc}^{\otimes}_{\mathcal{D}}(\mathcal{L}) \xrightarrow{i_{\mathcal{L}-\mathrm{tors}}} \mathcal{D} \xrightarrow{G} \mathcal{C} \xrightarrow{\Gamma_{\mathcal{K}}} \mathrm{Loc}^{\otimes}_{\mathcal{C}}(\mathcal{K}),$$

which is an adjoint to F'. We need to show that F is a symmetric monoidal functor, but, as the inclusions $i_{\mathcal{K}-\mathrm{loc}}$ and $i_{\mathcal{L}-\mathrm{loc}}$ are non-unitally monoidal all that remains to be proven is that F' sends the monoidal unit $\Gamma_{\mathcal{K}}\mathbb{1}_{\mathcal{C}}$ to the monoidal unit $\Gamma_{\mathcal{L}}\mathbb{1}_{\mathcal{D}}$.

The localizing ideals $\mathrm{Loc}^{\otimes}_{\mathcal{C}}(\mathcal{K})$ and $\mathrm{Loc}^{\otimes}_{\mathcal{D}}(\mathcal{L})$ are equal to the localizing ideals generated by the respective units, i.e.

$$\mathrm{Loc}^{\otimes}_{\mathcal{C}}(\mathcal{K}) = \mathrm{Loc}^{\otimes}_{\mathcal{C}}(\Gamma_{\mathcal{K}}\mathbb{1}_{\mathcal{C}}) \quad \text{and} \quad \mathrm{Loc}^{\otimes}_{\mathcal{D}}(\mathcal{L}) = \mathrm{Loc}^{\otimes}_{\mathcal{D}}(\Gamma_{\mathcal{L}}\mathbb{1}_{\mathcal{D}}).$$

Since $(F \dashv G)$ is a local duality adjunction we also know that $\mathrm{Loc}^{\otimes}_{\mathcal{D}}(F(\mathcal{K})) = \mathrm{Loc}^{\otimes}_{\mathcal{D}}(\mathcal{L})$, which also means

$$\mathrm{Loc}_{\mathcal{D}}(F(\Gamma_{\mathcal{K}}\mathbb{1}_{\mathcal{C}})) = \mathrm{Loc}^{\otimes}_{\mathcal{D}}(\mathcal{L}).$$

Let $\mathcal{G}$ be the full subcategory of $\mathrm{Loc}^{\otimes}_{\mathcal{D}}(\mathcal{L})$ where $F(\Gamma_{\mathcal{K}}\mathbb{1}_{\mathcal{C}})$ acts as a unit, in other words objects $M \in \mathrm{Loc}^{\otimes}_{\mathcal{D}}(\mathcal{L})$ such that

$$F(\Gamma_{\mathcal{K}}\mathbb{1}_{\mathcal{C}}) \otimes_{\mathcal{D}} M \simeq M.$$

In particular, $F(\Gamma_{\mathcal{K}}\mathbb{1}_{\mathcal{C}})$ is in $\mathcal{G}$. The category $\mathcal{G}$ is closed under retracts, suspension, and colimits, as well as tensoring with objects in $\mathcal{D}$, as we have

$$F(\Gamma_{\mathcal{K}}\mathbb{1}_{\mathcal{C}}) \otimes_{\mathcal{D}} (M \otimes_{\mathcal{D}} D) \simeq (F(\Gamma_{\mathcal{K}}\mathbb{1}_{\mathcal{C}}) \otimes_{\mathcal{D}} M) \otimes_{\mathcal{D}} D \simeq M \otimes_{\mathcal{D}} D$$

for any $M \in \mathcal{G}$ and $D \in \mathcal{D}$. Hence, it is a localizing tensor ideal of $\mathcal{D}$, with a symmetric monoidal structure where the unit is $F(\Gamma_{\mathcal{K}}\mathbb{1}_{\mathcal{C}})$. In particular, $\mathcal{G} = \mathrm{Loc}^{\otimes}_{\mathcal{D}}(F(\Gamma_{\mathcal{K}}\mathbb{1}_{\mathcal{C}}))$, which we already know is equivalent to $\mathrm{Loc}^{\otimes}_{\mathcal{D}}(\mathcal{L})$.

Since the ideals are equivalent, and the unit is unique, we must have $F(\Gamma_{\mathcal{K}}\mathbb{1}_{\mathcal{C}}) \simeq \Gamma_{\mathcal{L}}\mathbb{1}_{D}$, which finishes the proof. $\square$

The key feature for us is that such an induced adjunction inherits the property of being a Barr–Beck adjunction, i.e., that the right adjoint is conservative, preserves colimits, and has a projection formula. An analogous, but not equivalent, statement was proven in [BS20, 4.5]. Another related, but not equivalent statement, is Greenlees–Shipley's cellularization principle, see [GS13].

Theorem 1.A.6. *Let $(F \dashv G)\colon (\mathcal{C},\mathcal{K}) \longrightarrow (\mathcal{D},\mathcal{L})$ be a local duality adjunction. If $(F \dashv G)$ satisfies the Barr–Beck criteria, then the induced monoidal adjunction on localizing $\otimes$-ideals*

$$\mathrm{Loc}^{\otimes}_{\mathcal{C}}(\mathcal{K}) \underset{G'}{\overset{F'}{\rightleftarrows}} \mathrm{Loc}^{\otimes}_{\mathcal{D}}(\mathcal{L})$$

constructed in Lemma 1.A.5, also satisfies the Barr–Beck criteria.

Proof. We need to prove that G' is conservative and colimit-preserving and that the projection formula holds. The first two will both follow from the following computation, showing that also G' is just the restriction of G to $\mathrm{Loc}^{\otimes}_{\mathcal{D}}(\mathcal{L})$.

Let $X \in \mathrm{Loc}^{\otimes}_{\mathcal{D}}(\mathcal{L})$. By definition we have $G'(X) = \Gamma_{\mathcal{K}}G(X)$, where we have omitted the inclusions from the notation for simplicity. Since $\Gamma_{\mathcal{K}}$ is smashing and $(F \dashv G)$ by assumption has a projection formula we have

$$\Gamma_{\mathcal{K}}G(X) \simeq G(X) \otimes_{\mathcal{C}} \Gamma_{\mathcal{K}}\mathbb{1}_{\mathcal{C}} \simeq G(X \otimes_{\mathcal{D}} F(\Gamma_{\mathcal{K}}\mathbb{1}_{\mathcal{C}})).$$

By Lemma 1.A.5 the functor F' is symmetric monoidal, hence there is an equivalence $F(\Gamma_{\mathcal{K}}\mathbb{1}_{\mathcal{C}}) \simeq \Gamma_{\mathcal{L}}\mathbb{1}_{\mathcal{D}}$, which acts on X as the monoidal unit. Thus, we can summarize with

$$G'(X) \simeq G(X \otimes_{\mathcal{D}} F(\Gamma_{\mathcal{K}}\mathbb{1}_{\mathcal{C}})) \simeq G(X \otimes_{\mathcal{D}} \Gamma_{\mathcal{L}}\mathbb{1}_{\mathcal{D}}) \simeq G(X),$$

which shows that also G' is the restriction of G.

Now, as G is both conservative and preserves colimits, and colimits in the localizing ideals are computed in $\mathcal{C}$ and $\mathcal{D}$ respectively, then also G' is conservative and colimit-preserving. The projection formula for $(F' \dashv G')$ also automatically follows from the projection formula for $(F \dashv G)$. $\square$

Paper II

Positselski duality in stable ∞-categories

arXiv:2411.04060

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in stable ∞-categories

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Description

The main result of the second paper concerns a mathematical concept called a duality theory. A duality is a way to view a collection of objects "through a mirror" and study their reflections instead of their direct features. Studying a concept in its mirror image, and then dualizing, should be equivalent to studying the concept directly. This mathematical mirror can be a wide variety of things, but for the drawing we have chosen a classical planar mirroring to signify this effect. The flowing lines of each of the boxes are precisely mirror images of each other, along the symmetry line right in the middle.

The colors again have no mathematical meaning, and are there only to add visual interst, and to connect to the colors of the papers.

Choose a context and take your seat.
I have prepared the following treat.
The hom-tensor adjunction,
makes the following function:
Being contra is equivalent to complete.

– Torgeir Aambø

Abstract:

We introduce the notion of a contramodule over a cocommutative coalgebra in a presentably symmetric monoidal ∞-category $\mathcal{C}$. Based on this we prove that local duality, in the sense of Hovey–Palmieri–Strickland and Dwyer–Greenlees, is equivalent to Positselski's comodule-contramodule correspondence for coidempotent cocommutative coalgebras in compactly generated symmetric monoidal stable ∞-cateogories. This gives new descriptions of the categories of $K_p(n)$-local and $T_p(n)$-local spectra.

2.1 Introduction

Let k be a field and a C a cocommutative coalgebra in the abelian category Vect_k. A *comodule* over C is a vector space V together with a coassociative counital map $V \longrightarrow V \otimes_k C$. These objects were introduced in the seminal paper [EM65] and are categorically dual to modules over algebras. In the same paper Eilenberg and Moore introduced a further dual to comodules, which they called *contramodules*. These are vector spaces V with a map $\mathrm{Hom}_k(C, V) \longrightarrow V$ satisfying similar axioms called contra-associativity and contra-unitality.

While modules and comodules got their fair share of fame throughout the decades following their introduction, contramodules were seemingly lost to history—virtually forgotten—until dug out from their grave of obscurity by Positselski in the early 2000's. Positselski has since developed a considerable body of literature on contramodules, see for example [Pos10; Pos11; Pos16; Pos17b; Pos20] or the survey paper [Pos22].

In [Pos10] Positselski introduced the co/contra correspondence, which is an adjunction between the category of comodules and the category of contramodules over a cocommutative coalgebra C. This correspondence sat existing duality theories on a common footing, for example Serre–Grothendieck duality and Feigin–Fuchs central charge duality. Positselski also introduced the coderived and contraderived categories of C-comodules and C-contramodules respectively, and used this to prove a derived co/contra correspondence of the form

$$\mathrm{D}^{co}(\mathrm{Comod}_C) \simeq \mathrm{D}^{contra}(\mathrm{Contra}_C),$$

generalizing for example Matlis–Greenlees–May duality and Dwyer–Greenlees duality—see [Pos16].

The goal of the present paper is to generalize the co/contra correspondence—which we will refer to as Positselski duality—to cocommutative coalgebras in ∞-categories. We will also use the correspondence in stable ∞-categories, which are natural enhancements of triangulated categories. These serve as the natural place to study similar correspondences and equivalences as in the derived co/contra correspondence. The canonical references for ∞-categories are [Lur09] and [Lur17], and we will throughout the paper freely use their language instead of the more standard language of triangulated categories in the homological algebra literature.

Motivation

Let us try to both make a motivation for the traditional Positselski duality theory and for the connection to coalgebras in stable ∞-categories.

We let X be a separated noetherian scheme, $Z \subset X$ a closed subscheme and $U = X \backslash Z$ its open complement. The derived category of all $\mathcal{O}_X$-modules, $\mathrm{D}(\mathcal{O}_X)$, has a full subcategory $\mathrm{D}(X)$ consisting of complexes with quasi-coherent homology. We define $\mathrm{D}(U)$ similarly. These are all stable ∞-categories, with homotopy categories $h\mathrm{D}(X)$ being the more traditional triangulated derived category.

Letting $i \colon U \longrightarrow X$ be the inclusion we get an induced functor

$$i^* \colon \mathrm{D}(X) \longrightarrow \mathrm{D}(U)$$

by pulling back along i. This has a fully faithful right adjoint $i_* \colon \mathrm{D}(U) \longrightarrow \mathrm{D}(X)$, which itself has a further right adjoint $i^! \colon \mathrm{D}(X) \longrightarrow \mathrm{D}(U)$. The kernels of i^* and $i^!$ determine two equivalent subcategories of $\mathrm{D}(X)$, the former of which is the full subcategory $\mathrm{D}_Z(X) \subseteq \mathrm{D}(X)$ consisting of complexes with homology supported on Z. The fully faithful functor $j_! \colon \mathrm{D}_Z(X) \longrightarrow \mathrm{D}(X)$ has a colimit preserving right adjoint $j^!$. The kernel of $i^!$ is identified with the full subcategory of $\mathrm{D}(X)$ with homology supported on the formal completion of X along Z, which we denote $\mathrm{D}(X)^{\wedge}_Z$. The fully faithful inclusion $j_* \colon \mathrm{D}(X)^{\wedge}_Z \longrightarrow \mathrm{D}(X)$ has a left adjoint j^*.

As mentioned these two categories are equivalent, and the equivalence is given by the composite $j^* j_!$ with inverse $j^! j_*$. In fact we get a stable recollement

$$
\begin{tikzcd}
\mathrm{D}_Z(X) \arrow[dd, "\simeq"] \arrow[dr, shift left, "j_!"] & & \\
& \mathrm{D}(X) \arrow[ul, shift left, "j^!"] \arrow[dl, shift right, "j^*"'] \arrow[r, bend left, "i^*"] \arrow[r, bend right, "i^!"'] & \mathrm{D}(U) \arrow[l, "i_*"'] \\
\mathrm{D}(X)^{\wedge}_{Z} \arrow[ur, shift right, "j_*"'] & &
\end{tikzcd}
$$

This equivalence does not on the surface have anything to do with comodules or contramodules, so let us fix this. For simplicity we assume that $X = \mathrm{Spec}(\mathbb{Z})$, such that $\mathrm{D}(X) \simeq \mathrm{D}(\mathbb{Z})$. Any prime p determines a closed subscheme P of X. With this setup we can identify $\mathrm{D}_P(X) \simeq \mathrm{D}(\mathrm{Comod}_{\mathbb{Z}/p^\infty})$ and $\mathrm{D}(X)^{\wedge}_{Z} \simeq \mathrm{D}(\mathrm{Contra}_{\mathbb{Z}/p^\infty})$, where $\mathbb{Z}/p^\infty$ is the p-Prüfer coalgebra of $\mathbb{Z}$. It is the Pontryagin dual of the p-adic completion of $\mathbb{Z}$, often denoted $\mathbb{Z}_p$.

Remark. There is a more familiar description of $\mathrm{Comod}_{\mathbb{Z}/p^\infty}$ as the p-power torsion objects in Ab and $\mathrm{Contra}_{\mathbb{Z}/p^\infty}$ as the L-complete objects in Ab. The above then reduces to the derived version of Grothendieck local duality by Dwyer–Greenlees, showing that this is a certain version of Positselski duality. In [Pos17a, 2.2(1), 2.2(3)] Positselski proves that the derived complete modules also correspond to a suitably defined version of contramodules over an adic ring. For the above example this is precisely the p-adic integers $\mathbb{Z}_p$. The comodules over $\mathbb{Z}/p^\infty$ then correspond to discrete $\mathbb{Z}_p$-modules, see [Pos22, Sec. 1.9, Sec. 1.10].

The above motivates the classical co/contra correspondence, so let us now see how we wish to abstract this.

As i^* is a symmetric monoidal localization the category $\mathrm{D}_Z(X)$ is a localizing ideal. By [Rou08, 6.8] there is a compact object $F \in \mathrm{D}(X)$ with homology supported on Z such that F generates $\mathrm{D}_Z(X)$ under colimits. Now, as $\mathrm{D}_Z(X)$ is a compactly generated localizing ideal of a compactly generated symmetric monoidal stable ∞-category, the right adjoint $j^*\colon \mathrm{D}(X) \longrightarrow \mathrm{D}_Z(X)$ is smashing, hence given as $j_*j^*(\mathbb{1}) \otimes_{\mathrm{D}(X)} (-)$, where $\mathbb{1}$ denotes the unit in $\mathrm{D}(X)$. In $\mathrm{D}(X)$ the object $j_*j^*(\mathbb{1})$ is the fiber of the unit map $\mathbb{1} \longrightarrow i_*i^*(\mathbb{1})$. In fact, $i_*i^*(\mathbb{1})$ is an idempotent commutative algebra in $\mathrm{D}(X)$, hence the fiber of the unit map, i.e. $j_*j^*(\mathbb{1})$, is a coidempotent cocommutative coalgebra.

Using a dual version of Barr–Beck monadicity, see Section 2.2.3, one can prove that

$$\mathrm{D}_Z(X) \simeq \mathrm{Comod}_{j_*j^*(\mathbb{1})}(\mathrm{D}(X)).$$

Similarly, there is an equivalence

$$\mathrm{D}(X)^{\wedge}_{Z} \simeq \mathrm{Contra}_{j_*j^*(\mathbb{1})}(\mathrm{D}(X)),$$

which, put together gives us an instance of Positselski duality for stable ∞-categories:

$$\mathrm{Comod}_{j_*j^*(\mathbb{1})}(\mathrm{D}(X)) \simeq \mathrm{Contra}_{j_*j^*(\mathbb{1})}(\mathrm{D}(X)).$$

This is a special case of our second main theorem, Theorem 2.E, which is an application of the Positselski duality for commutative coalgebras set up in Theorem 2.D.

Overview of results

As mentioned, the main goal of this paper is to introduce the notion of comodules and contramodules in ∞-categories. Our main result is the following.

Theorem 2.D (Theorem 2.3.11)**.** *Let $\mathcal{C}$ be a presentably symmetric monoidal ∞-category. For any coidempotent cocommutative coalgebra C, there are mutually inverse equivalences*

$$\mathrm{Comod}_C(\mathcal{C}) \rightleftarrows \mathrm{Contra}_C(\mathcal{C})$$

given by the free contramodule and cofree comodule functor respectively.

Our main application of this is to give an alternative perspective on local duality, in the sense of [HPS97] and [BHV18].

Theorem 2.E (Theorem 2.3.17)**.** *Let $(\mathcal{C}, \mathcal{K})$ be a pair consisting of a rigidly compactly generated symmetric monoidal stable ∞-category $(\mathcal{C}, \otimes, \mathbb{1})$ and a set of compact objects $\mathcal{K} \subseteq \mathcal{C}$. Let Γ be* *the right adjoint to the fully faithful inclusion of the localizing tensor ideal generated by* $\mathcal{K}$*, i.e.* $i\colon \mathcal{C}^{\mathcal{K}-\mathrm{tors}} := \mathrm{Loc}^{\otimes}_{\mathcal{C}}(\mathcal{K}) \hookrightarrow \mathcal{C}$*. Then Positselski duality for the* $\mathbb{E}_\infty$*-coalgebra* $i\Gamma\mathbb{1}$*, recovers the local duality equivalence* $\mathcal{C}^{\mathcal{K}-\mathrm{loc}} \simeq \mathcal{C}^{\mathcal{K}-\mathrm{comp}}$*.*

As an example of why the two theorems above might be interesting, we have the following descriptions of the categories $\mathrm{Sp}_{K(n)}$ and $\mathrm{Sp}_{T(n)}$ in chromatic homotopy theory.

Corollary. There are equivalences $\mathrm{Sp}_{K_p(n)} \simeq \mathrm{Contra}_{M_{n,p}\mathbb{S}}(\mathrm{Sp}_{n,p})$ and $\mathrm{Sp}_{T_p(n)} \simeq \mathrm{Contra}_{M^f_{n,p}\mathbb{S}}(\mathrm{Sp}^f_{n,p})$ of symmetric monoidal stable ∞-categories.

Acknowledgements and personal remarks

The contents of this paper go back to one of the first ideas I had at the beginning of my PhD. I had my two favorite mathematical hammers—local duality and the monoidal Barr–Beck theorem—and was trying to see if these were really one and the same tool. Local duality consists of three parts: local objects, torsion objects and complete objects. The core idea came from the fact that the local objects are modules over an idempotent algebra, and I thus wanted a similar description of the other two parts. Drew Heard's guidance led me to a dual monoidal Barr–Beck result, checking off the torsion part. I got the first hints of the last piece after an email correspondence with Marius Nielsen, where we discussed a local duality type statement for mapping spectra. The solution clicked into place during a research visit to Aarhus University. During my stay Sergey Arkhipov gave two talks on contramodules, for completely unrelated reasons, and I immediately knew this was the last piece of the puzzle. Greg Stevenson taught me some additional details, solidifying my ideas, which led me to conjecture one of the main results of the present paper during my talk in their seminar. The crowd nodded in approval, thus, being satisfied I knew the answer, I naturally spent almost two years not writing it up.

I want to thank all of the people mentioned above for their in-

sights and pathfinding skills, without which this project would still have been a rather simple-minded idea in the optimistic brain of a young PhD student.

2.2 General preliminaries

The goal of this section is to introduce comodules and contramodules over an $\mathbb{E}_\infty$-coalgebra in some ∞-category $\mathcal{C}$. In order to do this we first review some basic facts about coalgebras, monads and comonads.

We will for the rest of this section work in some fixed presentably symmetric monoidal ∞-category $\mathcal{C}$. In other words, $\mathcal{C}$ is an commutative algebra object in Pr^L, the category of presentable ∞-categories and left adjoint functors. In particular, the monoidal product, which we denote by $- \otimes - \colon \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$ preserves colimits in both variables. We denote the unit of the monoidal structure by $\mathbb{1}$.

2.2.1 Coalgebras, monads and comonads

We denote the category of commutative algebras in $\mathcal{C}$ by $\mathrm{CAlg}(\mathcal{C})$. These are the coherently commutative ring objects in $\mathcal{C}$. By [Lur17, 2.4.2.7] there is a symmetric monoidal structure on $\mathcal{C}^{\mathrm{op}}$, and we define the category of $\mathbb{E}_\infty$-coalgebras in $\mathcal{C}$ to be the category $\mathrm{cCAlg}(\mathcal{C}) := \mathrm{CAlg}(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}}$. We will from now on omit the prefix $\mathbb{E}_\infty$ and refer to objects in $\mathrm{cCAlg}(\mathcal{C})$ as *commutative coalgebras*, or simply just *coalgebras*. Classical coalgebras will be referred to as *discrete* in order to avoid confusion.

Proposition 2.2.1. *The following properties hold for the category* $\mathrm{cCAlg}(\mathcal{C})$*.*

1. *The forgetful functor* $U\colon \mathrm{cCAlg}(\mathcal{C}) \longrightarrow \mathcal{C}$ *is conservative and creates colimits.*

2. *The categorical product of two coalgebras* C, D *is given by the tensor product of their underlying objects* $C \otimes D$*.*

3. *The category* $\mathrm{cCAlg}(\mathcal{C})$ *is presentably symmetric monoidal when equipped with the cartesian monoidal structure. In particular, this means that the forgetful functor* U *is symmetric monoidal.*

4. *The forgetful functor* U *has a lax-monoidal right adjoint* $cf\colon \mathcal{C} \longrightarrow \mathrm{cCAlg}(\mathcal{C})$*. The image of an object* $X \in \mathcal{C}$ *is called the cofree coalgebra on* X*.*

Proof. The presentability and creation of colimits by the forgetful functor is proven in [Lur18a, 3.1.2] and [Lur18a, 3.1.4]. The cartesian symmetric monoidal structure on $\mathrm{cCAlg}(\mathcal{C})$ follows from [Lur17, 3.2.4.7]. The last item follows from the first three together with the adjoint functor theorem, [Lur09, 5.5.2.9]. □

Given any ∞-category $\mathcal{D}$, the category of endofunctors $\mathrm{Fun}(\mathcal{D}, \mathcal{D})$ can be given the structure of a monoidal category via composition of functors.

Definition 2.2.2. A monad M on $\mathcal{D}$ is an $\mathbb{E}_1$-algebra in $\mathrm{Fun}(\mathcal{D}, \mathcal{D})$, and a comonad C is an $\mathbb{E}_1$-coalgebra in $\mathrm{Fun}(\mathcal{D}, \mathcal{D})$.

Example 2.2.3. Any adjunction of ∞-categories $F : \mathcal{D} \rightleftarrows \mathcal{E} : G$ gives rise to a monad $GF\colon \mathcal{D} \longrightarrow \mathcal{D}$ and a comonad $FG\colon \mathcal{E} \longrightarrow \mathcal{E}$. We call these the *adjunction monad* and *adjunction comonad* of the adjunction $F \dashv G$.

The category $\mathcal{D}$ is left tensored over $\mathrm{Fun}(\mathcal{D}, \mathcal{D})$ via evaluation of functors. Hence, for any monad M on $\mathcal{D}$ we get a category of left modules over M in $\mathcal{D}$.

Definition 2.2.4. Let $\mathcal{D}$ be an ∞-category and M a monad on $\mathcal{D}$. We define the *Eilenberg–Moore category* of M to be the category of left modules $\mathrm{LMod}_M(\mathcal{D})$. Objects in $\mathrm{LMod}_M(\mathcal{D})$ are referred to as *modules over M*.

Remark 2.2.5. Dually, any comonad C on $\mathcal{D}$ gives rise to a category of left comodules over C in $\mathcal{D}$. We also call this the Eilenberg–Moore category of C, and denote it by $\mathrm{LComod}_C(\mathcal{D})$. Its objects are referred to as *comodules* over C.

Given a monad M on $\mathcal{D}$ we have a forgetful functor

$$U_M \colon \mathrm{LMod}_M(\mathcal{D}) \longrightarrow \mathcal{D}.$$

By [Lur17, 4.2.4.8] this functor admits a left adjoint $F_M \colon \mathcal{D} \longrightarrow \mathrm{LMod}_M(\mathcal{D})$ given by $X \longmapsto MX$. We call this the free module functor. The adjunction $F_M \dashv U_M$ is called the *free-forgetful* adjunction of M.

Definition 2.2.6. An adjunction is said to be *monadic* if it is equivalent to the free-forgetful adjunction $F_M \dashv U_M$ of a monad M. A functor $G \colon \mathcal{D} \longrightarrow \mathcal{E}$ is called *monadic* if it is equivalent to the right adjoint U_M for some monadic adjunction.

The existence of the free-forgetful adjunction for a monad M implies that any monad is the adjunction monad of some adjunction. However, there can be more than one adjunction $F \dashv G$ such that M is the adjunction monad for this adjunction.

Definition 2.2.7. Let $\mathcal{D}$ be an ∞-category and M a monad on $\mathcal{D}$. A left M-module $B \in \mathrm{LMod}_M(\mathcal{D})$ is *free* if it is equivalent to an object in the image of F_M. The full subcategory of free modules is called the *Kleisli category* of M, and is denoted $\mathrm{LMod}^{\mathrm{fr}}_M(\mathcal{D})$.

The free-forgetful adjunction restricts to an adjunction on the Kleisli Category: $F_M : \mathcal{D} \rightleftarrows \mathrm{LMod}^{\mathrm{fr}}_M(\mathcal{D}) : U^{\mathrm{fr}}_M$. By [Chr23, 1.8] this is the minimal adjunction with adjunction monad equivalent to M. Using Lurie's ∞-categorical version of the Barr–Beck theorem we can also identify the free-forgetful adjunction as the maximal adjunction with adjunction monad M.

Theorem 2.2.8 ([Lur17, 4.7.3.5])**.** *A functor $G \colon \mathcal{D} \longrightarrow \mathcal{E}$ of ∞-categories is monadic if and only if*

1. *G admits a left adjoint,*
2. *G is conservative, and*
3. *the category $\mathcal{D}$ admits colimits of G-split simplicial objects, and these are preserved under G.*

Remark 2.2.9. By definition, if a functor $G\colon \mathcal{D} \longrightarrow \mathcal{E}$ is monadic, then there is an equivalence of categories $\mathcal{E} \simeq \mathrm{LMod}_{GF}(\mathcal{D})$, where F is the left adjoint of G.

Definition 2.2.10. Dually, given any comonad C on an ∞-category $\mathcal{D}$, there is a forgetful functor $U_C\colon \mathrm{LComod}_C(\mathcal{D}) \longrightarrow \mathcal{D}$, which admits a right adjoint

$$F_C\colon \mathcal{D} \longrightarrow \mathrm{LComod}_C(\mathcal{D}).$$

We call this the *cofree comodule functor*, and hence the adjunction $U_C \dashv F_C$ is called the *cofree-forgetful* adjunction of C. Any adjunction equivalent to a cofree-forgetful adjunction for some comonad C is said to be *comonadic*. A functor $G\colon \mathcal{D} \longrightarrow \mathcal{E}$ is said to be *comonadic* if it is equivalent to the left adjoint of a comonadic adjunction.

Remark 2.2.11. The essential image of F_C in $\mathrm{LComod}_C(\mathcal{D})$ determines the Kleisli category $\mathrm{LComod}^{\mathrm{fr}}_C(\mathcal{D})$ of cofree coalgebras. The cofree-forgetful adjunction restricts to an adjunction on cofree comodules, $U^{\mathrm{fr}}_C : \mathrm{LComod}^{\mathrm{fr}}_C(\mathcal{D}) \rightleftarrows \mathcal{D} : F_C$, which is the minimal adjunction whose adjunction comonad is C.

2.2.2 Comodules and contramodules

Recall that we have fixed a presentably symmetric monoidal ∞-category $\mathcal{C}$. Let us now construct the monads and comonads of interest for this paper.

Example 2.2.12. Let $A \in \mathrm{CAlg}(\mathcal{C})$ be a commutative algebra object in $\mathcal{C}$. The endofunctor $A \otimes (-)\colon \mathcal{C} \longrightarrow \mathcal{C}$ is a monad on $\mathcal{C}$. By [Chr23, 1.17] the Eilenberg–Moore category of this monad is equivalent to the category of modules over A in $\mathcal{C}$. As A is commutative we denote this by $\mathrm{Mod}_A(\mathcal{C})$. As $\mathcal{C}$ is presentable and the monad $A \otimes (-)$ preserves colimits, there is a right adjoint $\underline{\mathrm{Hom}}(A, -)\colon \mathcal{C} \longrightarrow \mathcal{C}$. This is a comonad on $\mathcal{C}$. Since these form an adjoint monad-comonad pair, their Eilenberg–Moore categories are equivalent,

$$\mathrm{Mod}_A(\mathcal{C}) \simeq \mathrm{LMod}_{A\otimes(-)}(\mathcal{C}) \simeq \mathrm{LComod}_{\underline{\mathrm{Hom}}(A,-)}(\mathcal{C}),$$

see [MM94, V.8.2] in the 1-categorical situation. The ∞-categorical version is exactly the same, and follows from the monadicity and comonadicity of the adjunctions.

The above example changes in an interesting way when replacing the algebra A with a coalgebra C.

Example 2.2.13. Let $C \in \mathrm{cCAlg}(\mathcal{C})$ be a cocommutative coalgebra in $\mathcal{C}$. By an ∞-categorical version of [HJR23, 2.5] the endofunctor $C \otimes (-) \colon \mathcal{C} \longrightarrow \mathcal{C}$ is a comonad on $\mathcal{C}$. By an argument dual to [Chr23, 1.17] the Eilenberg–Moore category of this comonad is equivalent to the category of comodules over the coalgebra C, which we denote by $\mathrm{Comod}_C(\mathcal{C})$. Since $C \otimes (-)$ preserves colimits there is a right adjoint $\underline{\mathrm{Hom}}(C, -)$, and this functor is a comonad on $\mathcal{C}$, again by [HJR23, 2.5]. Note that the category $\mathrm{Comod}_C(\mathcal{C})$ is presentable by [Ram24, 3.8], as $C \otimes (-)$ is accessible.

Notice that the pair $C \otimes (-) \dashv \underline{\mathrm{Hom}}(C, -)$ is not an adjoint monad-comonad pair—it is now an an adjoint comonad-monad pair. This means, in particular, that their Eilenberg–Moore categories might not be equivalent. This possible non-equivalence is the *raison d'être* for contramodules, which we can then define as follows.

Definition 2.2.14. Let $C \in \mathrm{cCAlg}(\mathcal{C})$ be a cocommutative coalgebra. A *contramodule* over C is a module over the internal hom-monad $\underline{\mathrm{Hom}}_{\mathcal{C}}(C, -) \colon \mathcal{C} \longrightarrow \mathcal{C}$. The category of contramodules over C in $\mathcal{C}$ is the corresponding Eilenberg–Moore category, which will be denoted $\mathrm{Contra}_C(\mathcal{C})$.

Notation 2.2.15. Since we are working in a fixed category $\mathcal{C}$ we will often simply write Contra_C for the category of contramodules, and Comod_C for the category of comodules.

Remark 2.2.16. We also mention that the hom-tensor adjunction is an *internal adjunction*, in the sense that there is an equivalence of internal hom-objects

$$\underline{\mathrm{Hom}}(X \otimes Y, Z) \simeq \underline{\mathrm{Hom}}(X, \underline{\mathrm{Hom}}(Y, Z)).$$

This follows from the hom-tensor adjunction together with a Yoneda argument.

Notation 2.2.17. We denote the mapping space in Comod_C by Hom_C and the mapping space in Contra_C by Hom^C. Similarly, the forgetful functors will be denoted $U_C\colon \mathrm{Comod}_C \longrightarrow \mathcal{C}$ and $U^C\colon \mathrm{Contra}_C \longrightarrow \mathcal{C}$ respectively, while their adjoints—the cofree and free functors—will be denoted $C \otimes (-)\colon \mathcal{C} \longrightarrow \mathrm{Comod}_C$ and $\underline{\mathrm{Hom}}(C, -)\colon \mathcal{C} \longrightarrow \mathrm{Contra}_C$, hoping that it is clear from context whether we use them as above or as endofunctors on $\mathcal{C}$.

The following proposition is standard for monads and comonads, see for example [RV15, 5.7].

Proposition 2.2.18. *Let C be a cocommutative coalgebra in $\mathcal{C}$. The forgetful functor $U_C\colon \mathrm{Comod}_C \longrightarrow \mathcal{C}$ creates colimits, while the forgetful functor $U^C\colon \mathrm{Contra}_C \longrightarrow \mathcal{C}$ creates limits.*

2.2.3 The dual monoidal Barr–Beck theorem

Lurie's version of the Barr–Beck monadicity theorem, see [Lur17, Section 4.7], allows us to recognize monadic functors from simple criteria. Combined with a recognition theorem for when a monoidal monadic functor is equivalent to $R\otimes -$ for some commutative ring R, Mathew–Neumann–Noel extended the Barr–Beck theorem to a monoidal version. In this short section we prove a categorical dual version of their result.

Let $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ pair of adjoint functors between symmetric monoidal categories, such that the left adjoint F is symmetric monoidal. This means that the right adjoint G is lax-monoidal, and does in particular preserve algebra objects. There is for any two objects $X \in \mathcal{C}$ and $Y \in \mathcal{D}$, a natural map

$$F(G(Y) \otimes_{\mathcal{C}} X) \xrightarrow{\simeq} FG(Y) \otimes_{\mathcal{D}} F(X) \longrightarrow Y \otimes_{\mathcal{D}} F(X)$$

where the last map is given by the adjunction counit. By the adjunction property, there is an adjoint map

$$G(Y) \otimes_{\mathcal{C}} X \longrightarrow G(Y \otimes_{\mathcal{D}} F(X)).$$

Definition 2.2.19. An adjoint pair $F \dashv G$ as above is said to satisfy the *monadic projection formula* if the map

$$G(Y) \otimes_{\mathcal{C}} X \longrightarrow G(Y \otimes_{\mathcal{D}} F(X))$$

is an equivalence for all $X \in \mathcal{C}$ and $Y \in \mathcal{D}$.

We now state the monoidal Barr–Beck theorem of Mathew–Neumann–Noel.

Theorem 2.2.20 ([MNN17, 5.29])**.** *Let $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ be an adjunction of presentably symmetric monoidal ∞-categories, such that the left adjoint F is symmetric monoidal. If, in addition*

1. *G is conservative,*
2. *G preserves arbitrary colimits, and*
3. *$F \dashv G$ satisfies the monadic projection formula,*

then the adjunction is monadic, and there is an equivalence of monads

$$GF \simeq G(\mathbb{1}_{\mathcal{D}}) \otimes_{\mathcal{C}} (-).$$

In particular, there is an equivalence $\mathcal{D} \simeq \mathrm{Mod}_{G(\mathbb{1}_{\mathcal{D}})}(\mathcal{C})$ of symmetric monoidal ∞-categories.

Remark 2.2.21. Note that this result is stated only for stable ∞-categories in [MNN17], but also holds unstably by a combination of Lurie's ∞-categorical Barr–Beck theorem, Theorem 2.2.8, together with the fact that the monadic projection formula applied to the unit gives an equivalence of monads by [EK20, 3.6].

There is also a dual version of the classical Barr–Beck theorem, see for example [BM23, 4.5]. We wish to extend this to a monoidal version.

Let $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ pair of adjoint functors between symmetric monoidal categories, such that the right adjoint G is symmetric monoidal. This means that the left adjoint F is op-lax-monoidal, and does in particular preserve coalgebra objects. There is for any two objects $X \in \mathcal{C}$ and $Y \in \mathcal{D}$, a natural map

$$G(F(X) \otimes_{\mathcal{D}} Y) \xrightarrow{\simeq} GF(X) \otimes_{\mathcal{C}} G(Y) \longrightarrow X \otimes_{\mathcal{C}} G(Y)$$

where the last map is given by the adjunction unit. By the adjunction property, there is an adjoint map

$$F(X) \otimes_{\mathcal{D}} Y \longrightarrow F(X \otimes_{\mathcal{C}} G(Y)).$$

Definition 2.2.22. An adjoint pair $F \dashv G$ as above is said to satisfy the *comonadic projection formula* if the map

$$F(X) \otimes_{\mathcal{D}} Y \longrightarrow F(X \otimes_{\mathcal{C}} G(Y))$$

is an equivalence for all $X \in \mathcal{C}$ and $Y \in \mathcal{D}$.

Theorem 2.2.23. *Let $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ be an adjunction of presentably symmetric monoidal ∞-categories, such that the right adjoint G is symmetric monoidal. If, in addition*

1. *F is conservative,*
2. *F preserves arbitrary limits, and*
3. *$F \dashv G$ satisfies the comonadic projection formula,*

then the adjunction is comonadic, and there is an equivalence of comonads

$$FG \simeq F(\mathbb{1}_C) \otimes_{\mathcal{D}} (-)$$

In particular, this gives an equivalence $\mathcal{C} \simeq \mathrm{Comod}_{F(\mathbb{1}_{\mathcal{C}})}(\mathcal{D})$.

Remark 2.2.24. Before the proof we just remark why the above statement makes sense. The unit $\mathbb{1}_{\mathcal{C}}$ in a presentably symmetric monoidal ∞-category $\mathcal{C}$ is both a commutative algebra and a cocommutative coalgebra. In the above adjunction we have that the right adjoint G is symmetric monoidal, hence its left adjoint F is op-lax monoidal. In particular, it sends coalgebras to coalgebras, meaning that $F(\mathbb{1}_{\mathcal{C}})$ is an $\mathbb{E}_\infty$-coalgebra in $\mathcal{D}$. By Example 2.2.13 tensoring with $F(\mathbb{1}_{\mathcal{C}})$ is a comonad, not a monad, as for Theorem 2.2.20.

Proof. By [BM23, 4.5] the adjunction is comonadic. A dual version of [EK20, 3.6] shows that there is a map of comonads $\varphi\colon FG \longrightarrow F(\mathbb{1}_{\mathcal{C}}) \otimes_{\mathcal{D}} (-)$, and consequently an adjunction

$$\mathrm{Comod}_{FG}(\mathcal{D}) \underset{\varphi^*}{\overset{\varphi_*}{\rightleftarrows}} \mathrm{Comod}_{F(\mathbb{1}_{\mathcal{C}})\otimes_{\mathcal{D}}(-)}(\mathcal{D})$$

By applying the projection formula to the unit $\mathbb{1}_{\mathcal{C}}$ we get that φ is a natural equivalence, which means that the adjunction (φ_*, φ^*) is an adjoint equivalence. By Example 2.2.13 the Eilenberg–Moore category of the comonad $F(\mathbb{1}_{\mathcal{C}}) \otimes_{\mathcal{D}} (-)$ is equivalent to the category of comodules over the cocommutative coalgebra $F(\mathbb{1}_{\mathcal{C}})$, finishing the proof. □

Remark 2.2.25. We want to specify when the above equivalence is an equivalence of symmetric monoidal categories. We could hope for the existence of a symmetric monoidal structure on Comod_C for a cocommutative coalgebra $C \in \mathcal{C}$. For the category $\mathrm{Mod}_R(\mathcal{C})$ of modules over a commutative algebra $R \in \mathcal{C}$ this is done by Lurie's relative tensor product, see [Lur17, Section 4.5.2]. But, for such a relative monoidal product to exist on Comod_C one needs the tensor product in $\mathcal{C}$ to commute with cosifted limits, which is rarely the case. But, as we will see in Section 2.3.1 we sometimes get a monoidal structure, and when this is the case, the equivalence in Theorem 2.2.23 is symmetric monoidal.

2.3 Positselski duality

Classical Positselski duality, usually called the co-contra correspondence, is an adjunction between comodules and contramodules over a discrete R-coalgebra C, where R is an algebra over a field k. In particular, the categories involved are abelian, which makes some constructions easier. For example, the monoidal structure on Mod_R induces monoidal structures on Comod_C via the relative tensor construction—given by a certain equalizers. For ∞-categories the relative tensor construction is more complicated, as we need the monoidal structure to behave well with all higher coherencies, as mentioned in Remark 2.2.25. We can, however, restrict our attention to a certain type of coalgebra, fixing these issues. This also puts us in the setting we are interested in regarding local duality—see Section 2.3.2.

2.3.1 Coidempotent coalgebras

We now restrict our attention to the special class of coalgebras that we will focus on for the remainder of the paper.

Definition 2.3.1. A cocommutative coalgebra $C \in \mathrm{cCAlg}(\mathcal{C})$ is said to be *coseparable* if the comultiplication map

$$\Delta \colon C \longrightarrow C \otimes C^{\mathrm{op}}$$

admits a (C, C)-bicomodule section $s \colon C \otimes C \longrightarrow C$. It is *coidempotent* if Δ is an equivalence.

Remark 2.3.2. Any coidempotent coalgebra is in particular separable, see [Ram23, 1.6(1)] for a formally dual statement.

The first reason for our focus on coidempotent coalgebras is that their categories of comodules inherit a symmetric monoidal structure from $\mathcal{C}$, which is rarely the case for general coalgebras, see Remark 2.2.25.

Lemma 2.3.3. *Let C be a coidempotent cocommutative coalgebra in $\mathcal{C}$. The category of C-comodules Comod_C inherits the structure of a presentably symmetric monoidal ∞-category making the cofree comodule functor a symmetric monoidal smashing colocalization.*

Proof. The category Comod_C is presentable by [Hol20, 2.1.11]. Let M and N be two comodules. Their relative tensor product in Comod_C is defined by the the two sided co-bar construction,

$$M \otimes_C N := \lim_n (M \otimes C^{\otimes n} \otimes N),$$

but, as C is coidempotent this is just the object $N \otimes C \otimes M$, which is the cofree comodule on the underlying object of $M \otimes N$. This means that the relative tensor product is defined for all comodules. The unit for the monoidal structure $- \otimes_C -$ is C, and the monoidal structure is symmetric monoidal as the monoidal structure in $\mathcal{C}$ is.

The endofunctor $C \otimes (-) \colon \mathcal{C} \longrightarrow \mathcal{C}$ is idempotent when C is coidempotent. Hence, as the forgetful functor $U_C \colon \mathrm{Comod}_C \longrightarrow \mathcal{C}$ is fully faithful whenever C is coidempotent, the cofree comodule functor $C \otimes (-) \colon \mathcal{C} \longrightarrow \mathrm{Comod}_C$ is a smashing colocalization of $\mathcal{C}$. Hence it is a symmetric monoidal functor by a dual version of [Lur17, 2.2.1.9], as it is obviously compatible with the symmetric monoidal structure in $\mathcal{C}$, due to the coidempotency of C. $\square$

Lemma 2.3.4. *The symmetric monoidal structure on the category* Comod_C *is closed.*

Proof. As the cofree-forgetful adjunction creates colimits in Comod_C the functor

$$- \otimes_C - \simeq C \otimes (- \otimes -) \colon \mathrm{Comod}_C \times \mathrm{Comod}_C \longrightarrow \mathrm{Comod}_C$$

preserves colimits separately in each variable. In particular, the functor $M \otimes_C (-)$ preserves colimits, hence has a right adjoint $\underline{\mathrm{Hom}}_C(M, -)$ for any comodule M by the adjoint functor theorem, [Lur09, 5.5.2.9]. This determines a functor

$$\underline{\mathrm{Hom}}_C(-,-) \colon \mathrm{Comod}_C^{\mathrm{op}} \times \mathrm{Comod}_C \longrightarrow \mathrm{Comod}_C$$

making Comod_C a closed symmetric monoidal category. $\square$

Remark 2.3.5. This adjunction, being a hom-tensor adjunction is also internally adjoint in the sense of Remark 2.2.16. Hence we have an equivalence

$$\underline{\mathrm{Hom}}_C(M \otimes_C N, A) \simeq \underline{\mathrm{Hom}}_C(M, \underline{\mathrm{Hom}}_C(N, A))$$

for all comodules M, N and A.

Another important reason for using coidempotent coalgebras in this paper is the following result. Recall that a comodule over a coalgebra C is called cofree, if it is of the form $M \otimes C$ for some $M \in \mathcal{C}$. These are precisely the comodules in the image of the right adjoint to the forgetful functor $U_C\colon \mathrm{Comod}_C \longrightarrow \mathcal{C}$, when C is coidempotent. This is a slightly weaker coalgebraic version of [Ram23, 1.13, 1.14]. See also [Brz10, 3.6] for a related 1-categorical version.

Lemma 2.3.6. *Every comodule over a coidempotent coalgebra C is a retract of a cofree comodule. In particular, there is an equivalence*

$$\mathrm{Comod}_C(\mathcal{C}) \simeq \mathrm{Comod}_C^{\mathrm{fr}}(\mathcal{C})$$

between the Eilenberg–Moore category and the Kleisli category of the comonad $C \otimes (-)$ on $\mathcal{C}$.

Proof. As coidempotent coalgebras are separable, see Remark 2.3.2, the result will follow from the fact that the forgetful functor $U_C\colon \mathrm{Comod}_C \longrightarrow \mathcal{C}$ is separable, in the sense that the adjunction unit map

$$\mathrm{Id}_{\mathrm{Comod}_C} \longrightarrow C \otimes U(-)$$

has a $\mathcal{C}$-linear section, whenever C is separable. The section is given by

$$C \otimes M \xrightarrow{\simeq} (C \otimes C) \otimes_C M \longrightarrow C \otimes_C M \xleftarrow{\simeq} M$$

for any comodule M. □

Remark 2.3.7. The fact that C is coidempotent implies that any C-comodule M has a unique comodule structure. In particular, if M is a comodule, then the cofree comodule $C \otimes M$ is equivalent to M.

We get a similar statement for contramodules over C. Recall that a contramodule is said to be free if it is of the form $\underline{\mathrm{Hom}}(C, M)$ for some $M \in \mathcal{C}$.

Proposition 2.3.8. *Let $C \in \mathrm{cCAlg}(\mathcal{C})$ be a separable coalgebra. Then every contramodule over C is a retract of a free contramodule. In particular, there is an equivalence*

$$\mathrm{Contra}_C(\mathcal{C}) \simeq \mathrm{Contra}_C^{\mathrm{fr}}(\mathcal{C})$$

between the Eilenberg–Moore category and the Kleisli category of the monad $\underline{\mathrm{Hom}}(C,-)$ *on* $\mathcal{C}$.

Proof. We can prove this by showing that the section for the separable coalgebra C gives a section of the forgetful functor $U^C\colon \mathrm{Contra}_C \longrightarrow \mathcal{C}$. The section is, for a contramodule X, given by the adjoint map $M \longrightarrow \underline{\mathrm{Hom}}(C,M)$ to the section of the forgetful functor U_C on Comod_C from Lemma 2.3.6. □

We know from Lemma 2.3.3 that the cofree comodule functor

$$C\otimes(-)\colon \mathcal{C} \longrightarrow \mathrm{Comod}_C$$

can be given the structure of a symmetric monoidal functor when the coalgebra C is coidempotent. Naturally we want a similar statement for the free contramodule functor

$$\underline{\mathrm{Hom}}(C,-)\colon \mathcal{C} \longrightarrow \mathrm{Contra}_C.$$

Remark 2.3.9. Let M be a C-comodule and V any object in $\mathcal{C}$. The structure map $\rho_M\colon M \longrightarrow C\otimes M$ induces a C-contramodule structure on the internal hom-object $\underline{\mathrm{Hom}}(M,V)$, via

$$\underline{\mathrm{Hom}}(C,\underline{\mathrm{Hom}}(M,V)) \simeq \underline{\mathrm{Hom}}(C\otimes M,V) \xrightarrow{-\circ\rho_M} \underline{\mathrm{Hom}}(M,V).$$

Lemma 2.3.10. *Let C be a coidempotent cocommutative coalgebra in $\mathcal{C}$. The category of C-contramodules* Contra_C *inherits the structure of a presentably symmetric monoidal ∞-category making the free contramodule functor a symmetric monoidal localization.*

Proof. The functor $\underline{\mathrm{Hom}}(C,-)\colon \mathcal{C} \longrightarrow \mathcal{C}$ is an idempotent functor, as we have

$$\underline{\mathrm{Hom}}(C,\underline{\mathrm{Hom}}(C,-)) \simeq \underline{\mathrm{Hom}}(C\otimes C,-) \simeq \underline{\mathrm{Hom}}(C,-)$$

by the internal adjunction property together with the coidempotency of C. The forgetful functor $U^C\colon \mathrm{Contra}_C \longrightarrow \mathcal{C}$ is a fully faithful functor, which means that the free contramodule functor $\underline{\mathrm{Hom}}(C, -)\colon \mathcal{C} \longrightarrow \mathrm{Contra}_C$ is a localization.

In order to determine that it induces a symmetric monoidal structure on Contra_C we need to check that the free functor is compatible with the monoidal structure in $\mathcal{C}$. By [Nik16, 2.12(3)] it is enough to check that $\underline{\mathrm{Hom}}(V, X) \in \mathrm{Contra}_C$ for any $X \in \mathrm{Contra}_C$ and $V \in \mathcal{C}$. By Proposition 2.3.8 we can assume that $X \simeq \underline{\mathrm{Hom}}(C, A)$ for some $A \in \mathcal{C}$. By the hom-tensor adjunction we get

$$\underline{\mathrm{Hom}}(A, \underline{\mathrm{Hom}}(C, V)) \simeq \underline{\mathrm{Hom}}(C \otimes V, A).$$

The latter is a C-contramodule by Remark 2.3.9, as $C \otimes V$ is a C-comodule.

We can then apply [Lur09, 2.2.1.9], which tells us that the free contramodule functor $\underline{\mathrm{Hom}}(C, -)\colon \mathcal{C} \longrightarrow \mathrm{Contra}_C$ can be given the structure of a symmetric monoidal functor. As Contra_C is a localization of a presentably symmetric monoidal category by an accessible functor, it is also presentably symmetric monoidal. □

We can now deduce our main result, namely that Positselski duality is a symmetric monoidal equivalence for coidempotent coalgebras.

Theorem 2.3.11. *Let $\mathcal{C}$ be a presentably symmetric monoidal category and $C \in \mathcal{C}$ a coidempotent cocommutative coalgebra. In this situation there are mutually inverse symmetric monoidal functors*

$$\mathrm{Comod}_C(\mathcal{C}) \underset{C \otimes (-)}{\overset{\underline{\mathrm{Hom}}(C,-)}{\rightleftarrows}} \mathrm{Contra}_C(\mathcal{C})$$

given on the underlying objects by the free contramodule functor and the cofree comodule functor respectively.

Proof. By Lemma 2.3.6 and Proposition 2.3.8 every C-comodule (resp. C-contramodule) is a retract of a cofree comodule (resp. free contramodule). Hence, it is enough to prove that the functors are mutually inverse equivalences between cofree and free objects.

Let A be any object in $\mathcal{C}$. Denote by $C \otimes A$ the corresponding cofree comodule and $\underline{\mathrm{Hom}}(C, A)$ the corresponding free contramodule. A simple adjunction argument, using both the cofree-forgetful adjunction and the hom-tensor adjunctions in $\mathcal{C}$ and Comod_C, shows that there is an equivalence

$$\underline{\mathrm{Hom}}_C(M, C \otimes A) \simeq C \otimes \underline{\mathrm{Hom}}(U_C M, A)$$

for any comodule M. In other words, the internal comodule hom is determined by the underlying internal hom in $\mathcal{C}$. For $M = C$ we get

$$C \otimes \underline{\mathrm{Hom}}(C, A) \simeq \underline{\mathrm{Hom}}_C(C, C \otimes A)$$

which is equivalent to $C \otimes A$ as C is the unit in Comod_C.

We wish to show that $\underline{\mathrm{Hom}}(C, C \otimes A) \simeq \underline{\mathrm{Hom}}(C, A)$. We do this by showing that the cofree-forgetful functor is an internal adjunction, in the sense of Remark 2.2.16.

Let B be an arbitrary object in $\mathcal{C}$, and recall our notation $\mathrm{Hom}(-,-)$ for the mapping space in $\mathcal{C}$. By the hom-tensor adjunction in $\mathcal{C}$ we have

$$\mathrm{Hom}(B, \underline{\mathrm{Hom}}(C, C \otimes A)) \simeq \mathrm{Hom}(C \otimes B, C \otimes A).$$

Both of these are in the image of the forgetful functor

$$U_C \colon \mathrm{Comod}_C \longrightarrow \mathcal{C}.$$

As it is fully faithful whenever C is coidempotent we get

$$\mathrm{Hom}(C \otimes B, C \otimes A) \simeq \mathrm{Hom}_C(C \otimes B, C \otimes A),$$

where we recall that the latter denotes maps of comodules. By the cofree-forgetful adjunction we have

$$\mathrm{Hom}_C(C \otimes B, C \otimes A) \simeq \mathrm{Hom}(C \otimes B, A),$$

which by the hom-tensor adjunction in $\mathcal{C}$ finally gives

$$\mathrm{Hom}(C \otimes B, A) \simeq \mathrm{Hom}(B, \underline{\mathrm{Hom}}(C, C \otimes A)).$$

Summarizing the equivalences we have

$$\mathrm{Hom}(B, \underline{\mathrm{Hom}}(C, C \otimes A)) \simeq \mathrm{Hom}(B, \underline{\mathrm{Hom}}(C, A)),$$

which by a Yoneda argument implies that there is an equivalence of internal hom-objects $\underline{\mathrm{Hom}}(C, C \otimes A) \simeq \underline{\mathrm{Hom}}(C, A)$.

We know by Lemma 2.3.3 and Lemma 2.3.10 that the cofree comodule functor and the free contramodule functor are both symmetric monoidal. By the arguments above, we know that the equivalence $\mathrm{Comod}_C \simeq \mathrm{Contra}_C$ is given by the compositions

$$\mathrm{Comod}_C \underset{C \otimes -}{\overset{U_C}{\rightleftarrows}} \mathcal{C} \underset{U^C}{\overset{\underline{\mathrm{Hom}}(C,-)}{\rightleftarrows}} \mathrm{Contra}_C$$

The composition from left to right is an op-lax symmetric monoidal functor, and the composition from right to left is a lax symmetric monoidal functor. Since they are both equivalences they are necessarily also symmetric monoidal. $\square$

Remark 2.3.12. We do believe that the above result to hold more generally. In fact, we believe it should hold for all separable cocommutative coalgebras, as this holds in the 1-categorical situation. However, it will in general not be a monoidal equivalence, due to the lack of monoidal structures.

2.3.2 Local duality

Our main interest for constructing an ∞-categorical version of Positselski duality is related to local duality, in the sense of [HPS97] and [BHV18]. In this section we use Theorem 2.3.11 to to construct an alternative proof of [BHV18, 2.21]. We first recall the construction of local duality—see also Section 0.2.2 for more details.

Let $(\mathcal{C}, \otimes, \mathbb{1})$ be a presentably symmetric monoidal ∞-category. The tensor product $\otimes$ preserves filtered colimits separately in each variable, which by the adjoint functor theorem ([Lur09, 5.5.2.9]) means that the functor $X \otimes (-)$ has a right adjoint

$\underline{\mathrm{Hom}}(X, -)$, making $\mathcal{C}$ a closed symmetric monoidal category. From this internal hom-object we get a functor

$$(-)^\vee = \underline{\mathrm{Hom}}(-, \mathbb{1})\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathcal{C},$$

which we call *the linear dual*.

Definition 2.3.13. An object $X \in \mathcal{C}$ is *compact* if the functor $\mathrm{Hom}(X, -)$ preserves filtered colimits, and it is *dualizable* if the natural map $X^\vee \otimes Y \longrightarrow \mathrm{Hom}(X, Y)$ is an equivalence for all $Y \in \mathcal{C}$.

The category $\mathcal{C}$ is said to be *rigidly compactly generated* if it is compactly generated by dualizable objects, and the unit $\mathbb{1}$ is compact. In this situation, the collection of compact objects is also the collection of dualizable objects.

Definition 2.3.14. A *local duality context* is a pair $(\mathcal{C}, \mathcal{K})$, where $\mathcal{C}$ is a rigidly compactly generated symmetric monoidal stable ∞-category and $\mathcal{K} \subseteq \mathcal{C}$ is a set of compact objects.

Construction 2.3.15. Let $(\mathcal{C}, \mathcal{K})$ be a local duality context. We denote the localizing ideal generated by $\mathcal{K}$ by $\mathcal{C}^{\mathcal{K}-\mathrm{tors}} = \mathrm{Loc}^{\otimes}_{\mathcal{C}}(\mathcal{K})$. The *right orthogonal complement* of $\mathcal{C}^{\mathcal{K}-\mathrm{tors}}$, in other words those objects $Y \in \mathcal{C}$ such that $\mathrm{Hom}(X, Y) \simeq 0$ for all $X \in \mathcal{C}^{\mathcal{K}-\mathrm{tors}}$ is denoted by $\mathcal{C}^{\mathcal{K}-\mathrm{loc}}$. By [BHV18, 2.17] this category is also a compactly generated localizing subcategory of $\mathcal{C}$. Lastly, we define the category $\mathcal{C}^{\mathcal{K}-\mathrm{comp}}$ to be the right orthogonal complement to $\mathcal{C}^{\mathcal{K}-\mathrm{loc}}$.

The fully faithful inclusion $i_{\mathcal{K}-\mathrm{tors}}\colon \mathcal{C}^{\mathcal{K}-\mathrm{tors}} \hookrightarrow \mathcal{C}$ has a right adjoint $\Gamma\colon \mathcal{C} \longrightarrow \mathcal{C}^{\mathcal{K}-\mathrm{tors}}$, This means, in particular, that Γ is a colocalization. The fully faithful inclusions $i_{\mathcal{K}-\mathrm{loc}}\colon \mathcal{C}^{\mathcal{K}-\mathrm{loc}} \hookrightarrow \mathcal{C}$ and $i_{\mathcal{K}-\mathrm{comp}}\colon \mathcal{C}^{\mathcal{K}-\mathrm{comp}} \hookrightarrow \mathcal{C}$ both have left adjoints $L\colon \mathcal{C} \longrightarrow \mathcal{C}^{\mathcal{K}-\mathrm{loc}}$ and $\Lambda\colon \mathcal{C} \longrightarrow \mathcal{C}^{\mathcal{K}-\mathrm{comp}}$ respectively, making them localizations.

Remark 2.3.16. Note that in the paper [BHV18] referenced above, they use the term *left orthogonal complement* instead of right. Both of these are used throughout the literature, but we decided on using *right*, as it felt more natural to the author.

Theorem 2.3.17. *For any local duality context* $(\mathcal{C}, \mathcal{K})$*,*

1. *the functor* L *is a smashing localization,*
2. *the functor* Γ *is a smashing colocalization,*
3. *the functors* $\Lambda \circ i_{\mathcal{K}-\mathrm{tors}}$ *and* $\Gamma \circ i_{\mathcal{K}-\mathrm{comp}}$ *are mutually inverse equivalences, and*
4. *the functors* (Γ, Λ)*, viewed as endofunctors on* $\mathcal{C}$ *form an adjoint pair.*

In particular, there are equivalences

$$\mathcal{C}^{\mathcal{K}-\mathrm{tors}} \simeq \mathcal{C}^{\mathcal{K}-\mathrm{comp}}$$

of symmetric monoidal stable ∞*-categories.*

Remark 2.3.18. The result will essentially follow from recognizing (Γ, Λ), viewed as endofunctors on $\mathcal{C}$ as the adjoint comonad-monad pair $C \otimes (-) \dashv \underline{\mathrm{Hom}}(C, -)$ for a certain coidempotent cocommutative coalgebra C, and then applying Theorem 2.3.11.

Proof. By [HPS97, 3.3.3] the functor L is smashing, as it is a finite localization away from $\mathcal{K}$. By construction the functor Γ is determined by the kernel of the localization $X \longrightarrow LX$, hence is also smashing. The functor L has a fully faithful right adjoint, hence is a localization—similarly for Γ.

As Γ is smashing it is given by $\Gamma X \simeq \Gamma \mathbb{1} \otimes X$, and as $\mathcal{C}^{\mathcal{K}-\mathrm{tors}}$ is an ideal, it inherits a symmetric monoidal structure from $\mathcal{C}$, making Γ a symmetric monoidal functor. In particular, the object $\Gamma \mathbb{1}$ is the unit in $\mathcal{C}^{\mathcal{K}-\mathrm{tors}}$. The unit in a compactly generated symmetric monoidal stable ∞-category is both an $\mathbb{E}_\infty$-algebra and an $\mathbb{E}_\infty$-coalgebra. The inclusion $i_{\mathcal{K}-\mathrm{tors}} \colon \mathcal{C}^{\mathcal{K}-\mathrm{tors}} \hookrightarrow \mathcal{C}$ is oplax monoidal, as it is the left adjoint of a symmetric monoidal functor, meaning that it preserves coalgebras. In particular, $\Gamma \mathbb{1}$ treated as an object in $\mathcal{C}$ is a cocommutative coalgebra. Since Γ is a smashing colocalization $\Gamma \mathbb{1}$ is a coidempotent coalgebra.

By Theorem 2.3.11 we then get an equivalence of categories

$$\mathrm{Comod}_{\Gamma \mathbb{1}}(\mathcal{C}) \simeq \mathrm{Contra}_{\Gamma \mathbb{1}}(\mathcal{C})$$

given by the mutually inverse equivalences

$$\underline{\mathrm{Hom}}(\Gamma\mathbb{1}, -)\colon \mathrm{Comod}_{\Gamma\mathbb{1}}(\mathcal{C}) \longrightarrow \mathrm{Contra}_{\Gamma\mathbb{1}}(\mathcal{C})$$

and

$$\Gamma\mathbb{1} \otimes -\colon \mathrm{Contra}_{\Gamma\mathbb{1}}(\mathcal{C}) \longrightarrow \mathrm{Comod}_{\Gamma\mathbb{1}}(\mathcal{C}).$$

By Theorem 2.2.23 there is an equivalence $\mathcal{C}^{\mathcal{K}-\mathrm{tors}} \simeq \mathrm{Comod}_{\Gamma\mathbb{1}}(\mathcal{C})$, so it remains to show that $\mathcal{C}^{\mathcal{K}-\mathrm{comp}} \simeq \mathrm{Contra}_{\Gamma\mathbb{1}}(\mathcal{C})$. This follows from [BHV18, 2.2], just as in the proof of [BHV18, 2.21(4)], as it gives a sequence of equivalences

$$\underline{\mathrm{Hom}}(\Gamma X, Y) \simeq \underline{\mathrm{Hom}}(\Lambda X, \Lambda Y) \simeq \underline{\mathrm{Hom}}(X, \Lambda Y)$$

which reduces to $\underline{\mathrm{Hom}}(\Gamma\mathbb{1}, Y) \simeq \Lambda Y$ when applied to $X = \mathbb{1}$. $\square$

Remark 2.3.19. The author feels that the equivalence

$$\mathcal{C}^{\mathcal{K}-\mathrm{comp}} \simeq \mathrm{Contra}_{\Gamma\mathbb{1}}$$

should be a formal consequence of a "contramodular" Barr–Beck theorem, but such a result has so far escaped our grasp.

Remark 2.3.20. If the more general version of Positselski duality from Remark 2.3.12 holds, one could be able to generalize local duality to slightly more exotic situations, where the functors are not localizations.

The motivation for proving local duality in this setup was to have the following visually beautiful description of local duality.

$$\begin{tikzcd}
& \mathrm{Mod}_{L\mathbb{1}} \arrow[d, shift left] \arrow[rdd, dotted, bend left] & \\
& \mathcal{C} \arrow[u, shift left] \arrow[ld, shift left] \arrow[rd, shift left] & \\
\mathrm{Comod}_{\Gamma\mathbb{1}} \arrow[ru, shift left] \arrow[rr, shift left, "\simeq"] \arrow[ruu, dotted, bend left] & & \mathrm{Contra}_{\Gamma\mathbb{1}} \arrow[lu, shift left] \arrow[ll, shift left]
\end{tikzcd}$$

Here the dotted arrows correspond to taking the right-orthogonal complement.

Remark 2.3.21. A visual, and intuitional, problem with the above picture is that the contramodule category is dependent on the coalgebra $\Gamma\mathbb{1}$ and not on its unit $\Lambda\mathbb{1}$. In the abelian situation, there is a notion of contramodule over a topological ring which would perfectly fix this issue, as one can show that $\Lambda\mathbb{1}$ is always an "adic" commutative algebra. We plan to explore this connection in future work. The above comment in Remark 2.3.19 on a Barr–Beck result for contramodules might then be more easily accessible in this case, as one does not have to construct the unit in a dual but equal category. It might also be possible directly by using Positselski's notion of a *dedualizing complex*, see [Pos16].

Remark 2.3.22. In local duality there is another functor, that we did not really consider here, which is the right adjoint to the inclusion $\mathcal{C}^{\mathcal{K}-\mathrm{loc}} \hookrightarrow \mathcal{C}$. This functor is given by $V = \underline{\mathrm{Hom}}(L\mathbb{1}, -)$. As discussed in Example 2.2.12 it is a comonadic functor, and its category of comodules is equivalent to $\mathrm{Mod}_{L\mathbb{1}}$. We can think of the objects in $\mathrm{Comod}_{\underline{\mathrm{Hom}}(L\mathbb{1},-)}$ as "co-contramodules". Adding these to the picture gives

$$\begin{tikzcd}
\mathrm{Mod}_{L\mathbb{1}} \arrow[rr, "\simeq"] \arrow[dr, shift left] & & \mathrm{Cocontra}_{L\mathbb{1}} \arrow[dl, shift left] \arrow[dd, dotted] \\
& \mathcal{C} \arrow[ul, shift left] \arrow[ur, shift left] \arrow[dl, shift left] \arrow[dr, shift left] & \\
\mathrm{Comod}_{\Gamma\mathbb{1}} \arrow[rr, "\simeq"] \arrow[ur, shift left] \arrow[uu, dotted] & & \mathrm{Contra}_{\Gamma\mathbb{1}} \arrow[ul, shift left]
\end{tikzcd}$$

which also makes this story enticingly connected to 4-periodic semi-orthogonal decompositions and spherical adjunctions—see [Dyc+24, Section 2.5].

2.3.3 Examples

Our main interest in Theorem 2.3.17 is related to chromatic homotopy theory and derived completion of rings. We will not present comprehensive introductions to these topics here, hence the interested reader is referred to [BB19] for details on the former and

[BHV20] for the latter.

Chromatic homotopy theory

The category of spectra, Sp, is the initial presentably symmetric monoidal stable ∞-category. Fixing a prime p, one can describe chromatic homotopy theory as the study of p-local spectra together with a *chromatic filtration*, coming from the height filtration of formal groups. In such a filtration there is a filtration component corresponding to each natural number n, which we will refer to as the n-th component. There are, at least, two different chromatic filtrations on Sp, and their conjectural equivalence was recently disproven in [Bur+23]. For simplicity we will distinguish these two by referring to them as the *compact filtration* and the *finite filtration*. This latter is a bit misleading, as it is not a finite filtration—the word finite corresponds to a certain finite spectrum. The n-th filtration component in the compact filtration is controlled by the Morava K-theory spectrum $K_p(n)$, and the n-th filtration component in the finite filtration is controlled by the telescope spectrum $T_p(n)$.

We denote the n-th component of the compact filtration by $\mathrm{Sp}_{n,p}$ and the n-th component of the finite filtration by $\mathrm{Sp}^f_{n,p}$. The different components are related by smashing localization functors $L_{n-1,p} \colon \mathrm{Sp}_{n,p} \longrightarrow \mathrm{Sp}_{n-1,p}$ and $L^f_{n,p} \colon \mathrm{Sp}^f_{n,p} \longrightarrow \mathrm{Sp}^f_{n-1,p}$ respectively.

In the light of local duality, the category $\mathrm{Sp}_{n-1,p}$ is the category of local objects in $\mathrm{Sp}_{n,p}$ for a compact object $L_{n,p}F(n) \in \mathrm{Sp}_{n,p}$. The torsion objects with respect to $L_{n,p}F(n)$ is the category of *monochromatic spectra*, denoted $\mathcal{M}_{n,p}$ and the category of complete objects are the $K_p(n)$-local spectra, $\mathrm{Sp}_{K(n)}$. For more details on monochromatic and $K_p(n)$-local spectra, see [HS99], and for the relationship to local duality, see [BHV18, Section 6.2].

Proposition 2.3.23. *For any non-negative integer n, there are symmetric monoidal equivalences $\mathcal{M}_{n,p} \simeq \mathrm{Comod}_{M_{n,p}\mathbb{S}}(\mathrm{Sp}_{n,p})$ and $\mathrm{Sp}_{K_p(n)} \simeq \mathrm{Contra}_{M_{n,p}\mathbb{S}}(\mathrm{Sp}_{n,p})$.*

Proof. This follows directly from Theorem 2.3.17, as $L_{n,p}F(n)$ is compact in $\mathrm{Sp}_{n,p}$, making the pair $(\mathrm{Sp}_{n,p}, \{L_{n,p}F(n)\})$ a local duality context. □

We also have a similar description of the objects coming from the finite chromatic filtration.

Proposition 2.3.24. *For any non-negative integer n, there are symmetric monoidal equivalences $\mathcal{M}^f_{n,p} \simeq \mathrm{Comod}_{M^f_{n,p}\mathbb{S}}(\mathrm{Sp}^f_{n,p})$ and $\mathrm{Sp}_{T_p(n)} \simeq \mathrm{Contra}_{M^f_{n,p}\mathbb{S}}(\mathrm{Sp}^f_{n,p})$.*

Proof. As the functor $M^f_{n,p}\colon \mathrm{Sp}^f_{n,p} \longrightarrow \mathcal{M}^f_{n,p}$ is a smashing colocalization, Theorem 2.2.23 gives an equivalence $\mathcal{M}^f_{n,p} \simeq \mathrm{Comod}_{M^f_{n,p}\mathbb{S}}(\mathrm{Sp}^f_{n,p})$. As there is an equivalence $\mathcal{M}^f_{n,p} \simeq \mathrm{Sp}_{T_p(n)}$ the claim of the result is then a formal consequence of Theorem 2.3.11. □

Remark 2.3.25. In light of Remark 2.3.21 we would really like to have a description of $\mathrm{Sp}_{K_p(n)}$ and $\mathrm{Sp}_{T_p(n)}$ via certain contramodules over their respective units, which are the $K_p(n)$-local and $T_p(n)$-local spheres respectively. These spheres are both naturally $\mathbb{E}_\infty$-algebras in $\mathrm{Pro}(\mathrm{Sp}^\omega_{n,p})$ and $\mathrm{Pro}(\mathrm{Sp}^{f,\omega}_{n,p})$ respectively, hence a natural starting point is to take advantage of this fact. We will investigate this in joint work with Florian Riedel.

Derived completion

Let R be a commutative noetherian ring and $I \subseteq R$ an ideal generated by a finite regular sequence. The I-adic completion functor $C^I\colon \mathrm{Mod}_R \longrightarrow \mathrm{Mod}_R$, defined by $C^I(M) = \lim_k M/I^k$ is neither a left, nor right exact functor. However, by [GM92, 5.1] the higher right derived functors vanish. We denote the higher left derived functors of C^I by L^I_i. An R-module M is said to be *I-adically complete* if the natural map $M \longrightarrow C^I(M)$ is an isomorphism. It is said to be *L-complete* if the natural map $M \longrightarrow L^I_0(M)$ is an isomorphism.

The map $M \longrightarrow C^I(M)$ factors through $L^I_0(M)$, and the map $L^I_0(M) \longrightarrow C^I(M)$ is always an epimorphism, but usually not an isomorphism. The full subcategory consisting of the L-complete modules form an abelian category $\mathrm{Mod}_R^{I-\mathrm{comp}}$. The full subcategory of I-adically complete modules, $\mathrm{Mod}_R^{\wedge}$ is usually not abelian.

The I-power torsion submodule of an R-module M is defined to be

$$T_I(M) := \{m \in M \mid I^k m = 0 \text{ for some } k \geqslant 0\}.$$

We say an R-module M is I-power torsion if the natural map $T_I(M) \longrightarrow M$ is an isomorphism. The full subcategory of I-power torsion R-modules form a Grothendieck abelian category, denoted $\mathrm{Mod}_R^{I-\mathrm{tors}}$.

The object R/I is compact in $\mathrm{D}(R)$, which is a rigidly compactly generated symmetric monoidal stable ∞-category. Hence, $(\mathrm{D}(R), R/I)$ is a local duality context. The category $\mathrm{D}(R)^{R/I-\mathrm{tors}}$ is by [BHV20, 3.7(2)] equivalent to $\mathrm{D}(\mathrm{Mod}_R^{I-\mathrm{tors}})$, the derived category I-power torsion modules. The category $\mathrm{D}(R)^{R/I-\mathrm{comp}}$ is by [BHV20, 3.7(1)] equivalent to the right completion of the derived category of $\mathrm{Mod}_R^{I-\mathrm{comp}}$.

The functors Γ and Λ coming from this local duality context can by [BHV18, 3.16] be identified with the total right derived functor $\mathbb{R}T_I$ and the total left derived functor $\mathbb{L}C^I$ respectively. By Theorem 2.3.11 we know that these are the cofree comodule functor and the free contramodule functor, hence we can conclude with the following.

Proposition 2.3.26. *There are symmetric monoidal equivalences*

$$\mathrm{D}(R)^{R/I-tors} \simeq \mathrm{D}(\mathrm{Mod}_R^{I-\mathrm{tors}}) \simeq \mathrm{Comod}_{\mathbb{R}T_I(R)}$$

and

$$\mathrm{D}(R)^{R/I-\mathrm{comp}} \simeq \mathrm{D}(\mathrm{Mod}_R^{I-\mathrm{comp}}) \simeq \mathrm{Contra}_{\mathbb{R}T_I(R)}$$

Interestingly, there are also descriptions of the category $\mathrm{Mod}_R^{I-\mathrm{comp}}$ as a category of contramodules, and $\mathrm{Mod}_R^{I-\mathrm{tors}}$ as a category of comodules. This makes the above example into an example of the derived co-contra-correspondence, see for example [Pos16].

As for the $K(n)$-local case described above, the equivalences

$$\mathrm{D}(R)^{R/I-\mathrm{comp}} \simeq \mathrm{D}(\mathrm{Mod}_R^{I-\mathrm{comp}}) \simeq \mathrm{Contra}_{\mathbb{R}T_I(R)}$$

is somewhat unsatisfactory, as we would really like to have equivalences

$$\mathrm{D}(R)^{R/I-\mathrm{comp}} \simeq \mathrm{D}(\mathrm{Mod}_R^{I-\mathrm{comp}}) \simeq \mathrm{Contra}_{\mathbb{L}C^I(R)}$$

instead. We hope that the before-mentioned future joint work with Florian Riedel will shed some light on this, hopefully giving such an equivalence.

2.A Addendum: Modules over pro-algebras

We ended the last section by wishing for a way to construct a well behaved category of contramodules over the $K_p(n)$-local and $T_p(n)$-local spheres, $L_{K_p(n)}\mathbb{S}$ and $L_{T_p(n)}\mathbb{S}$. The current section is not a part of the paper [Aam24c], but we wanted to include some progress on the above question.

The category of contramodules over a topological ring R is defined by Positselski to be the category of modules over a certain bracketing-opening monad on the category of sets, determined by R, see [Pos22]. However, in nice enough situations this can be compared to other descriptions, which more easily generalize to ∞-categories.

Remark 2.A.1. Let R be the I-adic completion of a commutative noetherian ring. In this situation there is an equivalence between the category of contramodules over R and I-complete R-modules in the sense of Section 2.3.3. Similarly, there is an equivalence between discrete R-modules and I-torsion R-modules, and a further equivalence between discrete R-modules and the opposite category of pro-finite R-modules via Pontryagin duality. To summarize we have

$$\mathrm{Contra}_R \simeq \mathrm{Mod}_R^{I-\mathrm{comp}} \not\simeq \mathrm{Mod}_R^{I-\mathrm{tors}} \simeq \mathrm{Pro}\,\mathrm{Mod}_R^{\mathrm{op}}$$

where we have highlighted that the complete and torsion objects are not equivalent as abelian categories. However, by local duality this issue is fixed when passing to the associated derived categories, as described in the previous section. Similarly, it is fixed by using ring spectra rather than discrete rings. This gives a heuristic definition in the context of ∞-categories: contramodules over a "topological algebra" $R \in \mathrm{CAlg}(\mathcal{C})$ is a "pro-finite module over R", and maps are reversed module morphisms.

Let us make this heuristic more precise. For the rest of this section we let $\mathcal{C}$ be a symmetric monoidal stable ∞-category, compactly generated by dualizable objects. In particular, there are symmetric monoidal equivalences

$$\mathcal{C} \simeq \mathrm{Ind}(\mathcal{C}^{\omega}) \simeq \mathrm{Ind}(\mathcal{C}^{\mathrm{dual}}).$$

Definition 2.A.2. A *pro-dualizable algebra* in $\mathcal{C}$ is a commutative algebra in $\mathrm{Pro}(\mathcal{C}^{\mathrm{dual}})$.

Definition 2.A.3. Given a pro-dualizable algebra R, we define the category of contramodules over R to be the category opposite to modules over R, i.e., $\mathrm{Contra}_R := \mathrm{Mod}_R(\mathrm{Pro}(\mathcal{C}^{\mathrm{dual}}))^{\mathrm{op}}$.

Remark 2.A.4. These definitions make sense more generally as well—we do not need the dualizable objects to generate $\mathcal{C}$. But, the examples we are interested in will have this feature, and we will not attempt to introduce the most general possible framework here.

2.A.1 Dualities

For any algebra $A \in \mathrm{CAlg}(\mathcal{C})$ we have a lax monoidal functor $\underline{\mathrm{Hom}}(A, -)\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathcal{C}$, adjoint to $A \otimes (-)\colon \mathcal{C} \longrightarrow \mathcal{C}$. Fixing the other input defines another functor $\underline{\mathrm{Hom}}(-, A)$, which induces a functor

$$\underline{\mathrm{Hom}}(-, A)\colon \mathrm{cCAlg}(\mathcal{C})^{\mathrm{op}} \longrightarrow \mathrm{CAlg}(\mathcal{C})$$

as it sends coalgebras to algebras. Choosing $A = \mathbb{1}_{\mathcal{C}}$ defines a functor

$$(-)^{\vee} := \underline{\mathrm{Hom}}(-, \mathbb{1}_{\mathcal{C}})\colon \mathrm{cCAlg}(\mathcal{C})^{\mathrm{op}} \longrightarrow \mathrm{CAlg}(\mathcal{C})$$

which we call the $\mathcal{C}$-*linear dual*. Colimits of coalgebras, as well as limits of algebras are computed underlying, hence the functor $(-)^\vee$ sends colimits to limits. In particular we get a right adjoint $(-)^\circ$ by the adjoint functor theorem:

$$\mathrm{cCAlg}(\mathcal{C})^{\mathrm{op}} \underset{(-)^\circ}{\overset{(-)^\vee}{\rightleftarrows}} \mathrm{CAlg}(\mathcal{C})$$

The right adjoint functor is rather opaque, but on dualizable coalgebras we have the following.

Lemma 2.A.5 ([Rie24, 2.16])**.** *If C is a dualizable coalgebra, then $C^\circ \simeq C^\vee$.*

Remark 2.A.6. As the forgetful functor $U\colon \mathrm{cCAlg}(\mathcal{C}) \longrightarrow \mathcal{C}$ is symmetric monoidal, and hence preserves dualizable objects, we can see that a coalgebra $C \in \mathrm{cCAlg}(\mathcal{C})$ is dualizable if and only if it is dualizable as an object in $\mathcal{C}$. In particular we get an equivalence $\mathrm{cCAlg}(\mathcal{C})^{\mathrm{dual}} \simeq \mathrm{cCAlg}(\mathcal{C}^{\mathrm{dual}})$. Note that, even in the setting where $\mathcal{C}$ is rigidly compactly generated, a dualizable coalgebra is usually not compact as an object in $\mathrm{cCAlg}(\mathcal{C})$ itself, see for example [Lur18b, 1.2.15].

Lemma 2.A.7. *The duality functor $(-)^\vee = \underline{\mathrm{Hom}}_{\mathcal{C}}(-, \mathbb{1}_{\mathcal{C}})$ gives an equivalence*

$$\mathrm{cCAlg}(\mathcal{C})^{\mathrm{op}} \simeq \mathrm{CAlg}(\mathrm{Pro}(\mathcal{C}^{\mathrm{dual}})),$$

between cocommutative coalgebras and pro-dualizable algebras.

Proof. The symmetric monoidal equivalence $\mathcal{C} \simeq \mathrm{Ind}(\mathcal{C}^{\mathrm{dual}})$ induces an equivalence

$$\mathrm{cCAlg}(\mathcal{C}) \simeq \mathrm{cCAlg}(\mathrm{Ind}(\mathcal{C}^{\mathrm{dual}}))$$

on coalgebras. By [Lur18a, 3.2.4] we get an equivalence

$$\mathrm{cCAlg}(\mathrm{Ind}(\mathcal{C}^{\mathrm{dual}})) \simeq \mathrm{cCAlg}(\mathrm{Ind}((\mathcal{C}^{\mathrm{dual}})^{\mathrm{op}})),$$

which, via the equivalence $\mathrm{Ind}(\mathcal{D}^{\mathrm{op}}) \simeq \mathrm{Pro}(\mathcal{D})^{\mathrm{op}}$ for any small category with finite limits, gives

$$\mathrm{cCAlg}(\mathrm{Ind}((\mathcal{C}^{\mathrm{dual}})^{\mathrm{op}})) \simeq \mathrm{cCAlg}(\mathrm{Pro}(\mathcal{C}^{\mathrm{dual}})^{\mathrm{op}}).$$

By definition we have

$$\mathrm{cCAlg}(\mathrm{Pro}(\mathcal{C}^{\mathrm{dual}})^{\mathrm{op}}) \simeq \mathrm{CAlg}(\mathrm{Pro}(\mathcal{C}^{\mathrm{dual}}))^{\mathrm{op}},$$

which, upon putting the above equivalences together gives exactly

$$\mathrm{cCAlg}(\mathcal{C})^{\mathrm{op}} \simeq \mathrm{CAlg}(\mathrm{Pro}(\mathcal{C}^{\mathrm{dual}})),$$

as wanted. □

Remark 2.A.8. Analyzing the definitions one can see that the $\mathcal{C}$-linear dual $(-)^\vee$ sends a coalgebra C presented by an ind-system "colim"$_j C_j$ to the algebra $C^\vee$ presented by the pro-tower "lim"$_j C_j^\vee$. The inverse to $(-)^\vee$ is given by applying $(-)^\vee$ point-wise to the dualizable objects in the pro-tower, and taking the limit in the opposite category, i.e. the colimit in $\mathcal{C}$.

The goal is to recover more global information about general objects in $\mathcal{C}$, not just algebras and coalgebras. In order to do this we utilize a duality between stabilization and costabilization of ∞-categories.

Proposition 2.A.9. *For any cocommutative coalgebra* $C \in \mathrm{cCAlg}(\mathcal{C})$ *there is an equivalence*

$$\mathrm{Comod}_C^{\mathrm{op}}(\mathcal{C}) \simeq \mathrm{Mod}_{C^\vee}(\mathrm{Pro}(\mathcal{C}^{\mathrm{dual}})).$$

Proof. The equivalence from Lemma 2.A.7 induces an equivalence of slice categories

$$(\mathrm{cCAlg}(\mathcal{C})^{\mathrm{op}})_{/C} \simeq \mathrm{CAlg}(\mathrm{Pro}(\mathcal{C}^{\mathrm{dual}}))_{/C^\vee},$$

see [Lur09, 5.2.5.1]. The slice category over C of the opposite category is equivalent to the opposite category of the coslice category over C, giving

$$(\mathrm{cCAlg}(\mathcal{C})^{\mathrm{op}})_{/C} \simeq \mathrm{cCAlg}(\mathcal{C})^{\mathrm{op}}_{C/}.$$

Taking spectrum objects on both sides induces an equivalence

$$\mathrm{Sp}(\mathrm{cCAlg}(\mathcal{C})^{\mathrm{op}}_{C/}) \simeq \mathrm{Sp}(\mathrm{CAlg}(\mathrm{Pro}(\mathcal{C}^{\mathrm{dual}}))_{/C^\vee})$$

The left hand side is equivalent to the opposite category of cospectrum objects $\mathrm{cCAlg}(\mathcal{C})$, in other words,

$$\mathrm{Sp}(\mathrm{cCAlg}(\mathcal{C})^{\mathrm{op}}_{C/}) \simeq \mathrm{coSp}(\mathrm{cCAlg}(\mathcal{C})_{C/})^{\mathrm{op}},$$

which together gives an equivalence

$$\mathrm{coSp}(\mathrm{cCAlg}(\mathcal{C})_{C/})^{\mathrm{op}} \simeq \mathrm{Sp}(\mathrm{CAlg}(\mathrm{Pro}(\mathcal{C}^{\mathrm{dual}}))_{/C^\vee}).$$

The left hand side is equivalent to $\mathrm{Comod}_C(\mathcal{C})^{\mathrm{op}}$ by [Che24, 1.0.3], while the right hand side is equivalent to $\mathrm{Mod}_{C^\vee}(\mathrm{Pro}(\mathcal{C}^{\mathrm{dual}}))$ by [Lur17, 7.3.4.13], which finishes the proof. □

Remark 2.A.10. This should be viewed as an ∞-categorical analog of the anti-equivalence between comodules over a cocommutative coalgebra $C \in \mathrm{Vect}_k$ and the category of linearly compact modules over $C^\vee$, see [Lef42, II.29].

Using the version of local duality in Theorem 2.E, this means that we have the following visually beautiful depiction of local duality:

$$\begin{tikzcd}
 & \mathrm{Mod}_{L\mathbb{1}} \arrow[d, shift left] \arrow[rdd, dotted, bend left] & \\
 & \mathcal{C} \arrow[u, shift left] \arrow[ldd, shift left] \arrow[rdd, shift left] & \\
\mathrm{Comod}_{\Gamma\mathbb{1}} \arrow[ruu, dotted, bend left] \arrow[ru, shift left] \arrow[rr, shift left, "\simeq"] & & \mathrm{Contra}_{\Lambda\mathbb{1}} \arrow[lu, shift left] \arrow[ll, shift left]
\end{tikzcd}$$

2.A.2 Monochromatic contramodules

Let us now turn our attention to the setting of chromatic homotopy theory. As $M_{n,p}\mathbb{S}$ is a cocommutative coalgebra we know that its dual $M_{n,p}\mathbb{S}^\vee$ is a commutative ring spectrum in $\mathrm{Sp}_{n,p}$. By [BHV18, 2.21(4)] the $K_p(n)$-localization functor, treated as an endofunctor on $\mathrm{Sp}_{n,p}$, is equivalent to $\underline{\mathrm{Hom}}(M_{n,p}\mathbb{S}, -)$. Applied to the $E_{n,p}$-local sphere $L_{n,p}\mathbb{S}$ this is exactly the dual of $M_{n,p}\mathbb{S}$

$$M_{n,p}\mathbb{S}^\vee \simeq \underline{\mathrm{Hom}}(M_{n,p}\mathbb{S}, L_{n,p}\mathbb{S}) \simeq L_{K_p(n)}L_{n,p}\mathbb{S} \simeq L_{K_p(n)}\mathbb{S}.$$

Note that, as an endofunctor on $\mathrm{Sp}_{n,p}$, the $K_p(n)$-localization functor is not symmetric monoidal. But, as the inclusion $\mathrm{Sp}_{K_p(n)} \hookrightarrow \mathrm{Sp}_{n,p}$ is lax symmetric monoidal it sends algebras to algebras, which in particular means that $L_{K_p(n)}\mathbb{S}$ is also an $\mathbb{E}_\infty$-ring spectrum in $\mathrm{Sp}_{n,p}$.

Let X be any spectrum. By [HS99, 7.10] there is a tower of generalized Moore spectra

$$\cdots \longrightarrow M_2 \longrightarrow M_1 \longrightarrow M_0$$

such that $L_{K_p(n)}X \simeq \lim_j(L_{n,p}X \otimes M_j)$. In particular, any $K_p(n)$-local spectrum is a pro-spectrum indexed by the generalized Moore tower. Now, consider the pro-tower "$\lim$"$_j(L_{n,p}X \otimes M_j)$ for $X = \mathbb{S}$. As $E_{n,p}$-localization is smashing, there is an equivalence

$$\text{“}\lim\text{”}_j(L_{n,p}\mathbb{S} \otimes M_j) \simeq \text{“}\lim\text{”}_j L_{n,p}M_j.$$

Lemma 2.A.11. *The pro-tower "$\lim$"$_j L_{n,p}M_j$ is a commutative algebra in* $\mathrm{Pro}(\mathrm{Sp}_{n,p}^{\mathrm{dual}})$*.*

Proof. By [DL14, 6.3] the Moore tower "$\lim$"$_j M_j$ is a commutative algebra in $\mathrm{Pro}(\mathrm{Sp})$. As each generalized Moore spectrum is a finite spectrum it is dualizable, and as $E_{n,p}$-localization is symmetric monoidal it preserves dualizable objects. Hence, the tower "$\lim$"$_j L_{n,p}M_j$ is a commutative algebra in $\mathrm{Pro}(\mathrm{Sp}_{n,p}^{\mathrm{dual}})$. □

Lemma 2.A.12. *The ind-system "colim"$_j L_{n,p}M_j^\vee$ dual to the pro-tower "$\lim$"$_j L_{n,p}M_j$ is an cocommutative coalgebra in* $\mathrm{Ind}(\mathrm{Sp}_{n,p}^{\mathrm{dual}})$*, and taking the colimit gives an equivalence*

$$\mathrm{colim}_j\, L_{n,p}M_j^\vee \simeq M_{n,p}\mathbb{S}$$

of cocommutative coalgebras in $\mathrm{Sp}_{n,p}$*.*

Proof. By incorporating the multiplicative structure on Moore spectra constructed by Burklund in [Bur22], Li–Zhang proves in [LZ23, 2.1.4] that we can choose the pro-tower in such a way that each M_j is an $\mathbb{E}_j$-M_k-algebra for any $2 \geqslant j \geqslant k$. In particular, it is an $\mathbb{E}_j$-ring spectrum in $\mathrm{Sp}_{n,p}^{\mathrm{dual}}$. This implies that the similar statement holds for the localized pro-tower "$\lim$"$_j L_{n,p} M_j$. In particular, $L_{n,p} M_j \in \mathrm{Alg}_{\mathbb{E}_j}(\mathrm{Sp}_{n,p}^{\mathrm{dual}})$.

By the $\mathbb{E}_j$-operadic version of the equivalence in Lemma 2.A.5 as proved in [Pér22, 2.21], the dual of each generalized Moore spectrum M_j is an $\mathbb{E}_j$-coalgebra in $\mathrm{Sp}_{n,p}^{\mathrm{dual}}$. Hence, the pro-tower "$\lim$"$_j L_{n,p} M_j$ gets sent to an ind-system "colim"$_j L_n M_j^\vee$ in $\mathrm{Sp}_{n,p}^{\mathrm{dual}}$ under the linear dual functor. By [HS99, 7.10(c)] this ind-system is a presentation of $M_{n,p}\mathbb{S}$, as there is an equivalence

$$M_{n,p}\mathbb{S} \simeq \mathrm{colim}_j\, M_j^\vee \otimes L_{n,p}\mathbb{S} \simeq \mathrm{colim}_j\, L_{n,p} M_j^\vee .$$

Hence it remains to show that "colim"$_j L_{n,p} M_j^\vee$ is a cocommutative coalgebra in $\mathrm{Ind}(\mathrm{Sp}_{n,p}^{\mathrm{dual}})$ and that the equivalence above is an equivalence of cocommutative coalgebras.

By Lemma 2.A.11 the pro-tower "$\lim$"$_j L_{n,p} M_j$ is a commutative algebra in $\mathrm{Pro}(\mathrm{Sp}_{n,p}^{\mathrm{dual}})$, hence Remark 2.A.8 implies that the linear dual, which is "colim"$_j L_{n,p} M_j^\vee$, is a cocommutative coalgebra in $\mathrm{Ind}(\mathrm{Sp}_{n,p}^{\mathrm{dual}})$. As the equivalence $\mathrm{Ind}(\mathrm{Sp}_{n,p}^{\mathrm{dual}}) \simeq \mathrm{Sp}_{n,p}$ is symmetric monoidal, and given by taking the colimit of the ind-systems, the equivalence $\mathrm{colim}_j\, L_{n,p} M_j^\vee \simeq M_{n,p}\mathbb{S}$ is an equivalence of cocommutative coalgebras. □

Remark 2.A.13. Li–Zhang also proved in [LZ23, 2.1.5] that there is an equivalence $L_{K_p(n)}\mathbb{S} \simeq \lim_j M_j$ as $K_p(n)$-local ring spectra. As the inclusion $\mathrm{Sp}_{K_p(n)} \hookrightarrow \mathrm{Sp}_{n,p}$ is lax symmetric monoidal, these are also equivalent as $E_{n,p}$-local ring spectra, which is where our story takes place. Alternatively one can prove this directly using [LZ23, 2.1.6].

We can now summarize the discussion above as follows. The monochromatic sphere $M_{n,p}\mathbb{S}$ is a cocommutative coalgebra in $\mathrm{Sp}_{n,p}$. It is presented by the ind-system "colim"$_j L_{n,p} M_j^\vee$, which is a cocommutative coalgebra in $\mathrm{Ind}(\mathrm{Sp}_{n,p}^{\mathrm{dual}})$. The linear dual is a pro-tower "$\lim$"$_j L_{n,p} M_j$ which is a commutative algebra in $\mathrm{Pro}(\mathrm{Sp}_{n,p}^{\mathrm{dual}})$. Its materialization is the $K_p(n)$-local sphere as an commutative algebra in $\mathrm{Sp}_{n,p}$.

This allows us to conclude with our wanted description of the category of $K_p(n)$-local spectra.

Theorem 2.A.14. *There is a symmetric monoidal equivalence*

$$\mathrm{Sp}_{K_p(n)} \simeq \mathrm{Mod}_{\text{“}\lim\text{”}_j L_{n,p}M_j}(\mathrm{Pro}(\mathrm{Sp}_{n,p}^{\mathrm{dual}}))^{\mathrm{op}}$$

between $K_p(n)$-local spectra and modules over the pro-dualizable algebra “$\lim$”$_j L_{n,p}M_j$ presenting the $K_p(n)$-local sphere $L_{K_p(n)}\mathbb{S}$.

Proof. Proposition 2.A.9 gives an equivalence between the category of comodules over the monochromatic sphere, $\mathrm{Comod}_{M_{n,p}\mathbb{S}}(\mathrm{Sp}_{n,p})$, and modules over its linear dual. By choosing the ind-presentation of $M_{n,p}\mathbb{S}$ constructed in Lemma 2.A.12, the linear dual is precisely the pro-tower “$\lim$”$_j L_{n,p}M_j$, which is a commutative algebra in $\mathrm{Pro}(\mathrm{Sp}_{n,p}^{\mathrm{dual}})$. Hence we have a symmetric monoidal equivalence

$$\mathrm{Comod}_{M_{n,p}\mathbb{S}}(\mathrm{Sp}_{n,p}) \simeq \mathrm{Mod}_{\text{“}\lim\text{”}_j L_{n,p}M_j}(\mathrm{Pro}(\mathrm{Sp}_{n,p}^{\mathrm{dual}}))^{\mathrm{op}}.$$

By Theorem 2.2.23 there is a symmetric monoidal equivalence $\mathrm{Comod}_{M_{n,p}\mathbb{S}}(\mathrm{Sp}_{n,p}) \simeq \mathcal{M}_{n,p}$, and by [HS99, 6.19] there is a symmetric monoidal equivalence $\mathcal{M}_{n,p} \simeq \mathrm{Sp}_{K_p(n)}$. □

Remark 2.A.15. By definition this implies that we have a description of $K_p(n)$-local spectra as contramodules over the $K_p(n)$-local sphere, $\mathrm{Sp}_{K_p(n)} \simeq \mathrm{Contra}_{L_{K_p(n)}\mathbb{S}}$. This might seem ad-hoc, but we feel that this is justifiable by Remark 2.A.1.

Remark 2.A.16. The above construction works more generally for any local duality context, and for any Positselski-duality in the sense of Theorem 2.3.11. In particular, there is a similar pro-presentation of $L_{T_p(n)}\mathbb{S}$, giving rise to an equivalence $\mathrm{Sp}_{T_p(n)} \simeq \mathrm{Contra}_{L_{T_p(n)}\mathbb{S}}$.

Paper III

Classification of localizing subcategories along t-structures

arXiv:2412.09391

Paper III

Classification of localizing subcategories along t-structures

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Description

The main result of the third paper concerns a connection between three different related mathematical worlds, called the stable, the prestable and the abelian world. These three worlds are all connected by a mathematical concept called a t-structure. One could think of this setup as follows: in the stable world we have an infinite list of things, labeled by every positive and negative whole number. In the prestable world we only have things labeled by positive numbers, which still gives us infinitely many things, but only in one direction. In the abelian world we have only the part labeled by 0—the only part right between the positive and the negative. The t-structure allows us to remove all of the information in any degree. For example, it can first remove all negative numbers, and then all positive numbers, leaving us only with 0. This is precisely how it connects the stable, the prestable and the abelian worlds. I have tried to signify this passing in the drawing, where one the left we have free flowing information in all directions, while in the middle half of it is restricted in one direction. In the final rightmost part, information is restricted in both directions, leaving us only with straight lines—no geometry, no topology.

The colors again have no mathematical meaning, and are there only to add visual interst, and to connect to the colors of the papers.

Let be stable with a t
—as nice as it needs to be.
If the heart has a localization,
then I have a declaration:
There is a unique π-exact lift to $\mathcal{C}$.

– Torgeir Aambø

Abstract:

We study the interplay between localizing subcategories in a stable ∞-category $\mathcal{C}$ with t-structure $(\mathcal{C}_{\geqslant 0}, \mathcal{C}_{\leqslant 0})$, the prestable ∞-category $\mathcal{C}_{\geqslant 0}$ and the abelian category $\mathcal{C}^{\heartsuit}$. We prove that weak localizing subcategories of $\mathcal{C}^{\heartsuit}$ are in bijection with the localizing subcategories of $\mathcal{C}$ where object-containment can be checked on the heart. This generalizes similar known correspondences for noetherian rings and bounded t-structures. We also prove that this restricts to a bijection between localizing subcategories of $\mathcal{C}^{\heartsuit}$, and localizing subcategories of $\mathcal{C}$ that are kernels of t-exact functors—lifting Lurie's correspondence between localizing subcategories in $\mathcal{C}_{\geqslant 0}$ and $\mathcal{C}^{\heartsuit}$ to the stable category $\mathcal{C}$.

3.1 Introduction

The concept of a t-structure on a triangulated category was introduced in [BBD82], and in a way axiomatizes the concept of taking the homology of a chain complex in the derived category of a ring. Most interesting triangulated categories arise as the homotopy category of a stable ∞-category, and the concept of a t-structure lifts to this setting. Having a t-structure allows us to naturally compare features of a stable ∞-category $\mathcal{C}$ to features of an abelian category $\mathcal{C}^{\heartsuit}$, called the heart of the given t-structure.

In order to understand the internal structure of a stable ∞-category, is its important to understand its *localizing subcategories*. A full subcategory is called localizing if it is a stable full subcategory closed under colimits. The goal of this paper is to classify the localizing subcategories of $\mathcal{C}$ that interact well with t-structures. These are the localizing subcategories $\mathcal{L} \subseteq \mathcal{C}$ that inherit a t-structure, and you can check if an object X is in $\mathcal{L}$ by checking whether $\pi_n^{\heartsuit} X \in \mathcal{L}^{\heartsuit}$. We call these the *$\pi$-stable localizing subcategories*—see Definition 3.3.2.

We want to compare these localizing subcategories of $\mathcal{C}$ to subcategories of $\mathcal{C}^{\heartsuit}$. The abelian analog of localizing subcategories of a stable ∞-category, are the *weak Serre subcategories* closed under coproducts. We call these the *weak localizing subcategories*. Our first main result is the following classification of π-stable localizing subcategories in $\mathcal{C}$ via the heart construction. This generalizes a similar correspondence due to Takahashi ([Tak09]) for commutative noetherian rings, see Corollary 3.3.13.

Theorem 3.F (Theorem 3.3.11)**.** *Let $\mathcal{C}$ be a stable ∞-category with a t-structure. If the t-structure is right complete and compatible with filtered colimits, then the map $\mathcal{L} \longmapsto \mathcal{L}^{\heartsuit}$ gives a one-to-one correspondence between π-stable localizing subcategories of $\mathcal{C}$ and weak localizing subcategories in $\mathcal{C}^{\heartsuit}$.*

The above theorem also holds when we exclude the existence of coproducts, giving a one-to-one correspondence between π-stable thick subcategories of $\mathcal{C}$ and weak Serre subcategories of $\mathcal{C}^{\heartsuit}$. This generalizes the similar result of Zhang–Cai ([ZC17]) to the setting of unbounded t-structures, see Proposition 3.3.17 and Corollary 3.3.18.

We also want a way to study the analog of (non-weak) Serre subcategories of $\mathcal{C}^\heartsuit$ closed under coproducts—called the *localizing subcategories* of $\mathcal{C}^\heartsuit$—in the stable ∞-category $\mathcal{C}$. In order to do this we use the bridge between stable ∞-categories with a t-structure and prestable ∞-categories, as developed mainly by Lurie in [Lur16, App. C]. A prestable ∞-category acts as the connected part of the t-structure, denoted $\mathcal{C}_{\geqslant 0}$, and they allow us to study t-structures on $\mathcal{C}$ indirectly, without carrying around extra data.

Lurie introduced the notion of localizing subcategories of the prestable ∞-category $\mathcal{C}_{\geqslant 0}$, which more closely mimics the construction of localizing subcategories of abelian categories. The analog of π-stable localizing subcategory in this situation are called *separating localizing subcategories* by Lurie. Using the heart construction for prestable ∞-categories, Lurie classified the separating localizing subcategories of $\mathcal{C}_{\geqslant 0}$ in [Lur16, C.5.2.7], by proving that there is a one-to-one correspondence

$$\left\{\begin{matrix}\text{separating localizing}\\ \text{subcategories of } \mathcal{C}_{\geqslant 0}\end{matrix}\right\} \simeq \left\{\begin{matrix}\text{localizing}\\ \text{subcategories of } \mathcal{C}^\heartsuit\end{matrix}\right\}.$$

Our second main theorem provides an extension of this correspondence to the stable ∞-category $\mathcal{C}$, allowing us to strengthen Theorem 3.F to non-weak localizing sucategories. This interacts well with existing classifications of localizing subcategories in modules over noetherian rings and quasicoherent sheaves on noetherian schemes.

Theorem 3.G (Theorem 3.3.35)**.** *Let $\mathcal{C}$ be a stable category with a t-structure. If the t-structure is right complete and compatible with filtered colimits, then the map $\mathcal{L} \longmapsto \mathcal{L}^\heartsuit$ gives a one-to-one correspondence between localizing subcategories of $\mathcal{C}^\heartsuit$, and π-stable localizing subcategories of $\mathcal{C}$ that are kernels of a t-exact localization.*

Note that any stable ∞-category is prestable, hence the above result might at first glance seem to follow trivially from Luries's classification. But, any separating localizing subcategory of a stable ∞-category $\mathcal{C}$, viewed as a prestable one, is the whole category $\mathcal{C}$ by [Lur16, C.1.2.14, C.5.2.4]. This means that the stable situation needs its own separate treatment, hence the existence of the current paper.

The results of the paper can be summarized in the following diagram, showcasing the bijections ($\simeq$) and the inclusions ($\subseteq$) between the different types of subcategories.

$$
\begin{tikzcd}
\left\{\begin{matrix}t\text{-stable localizing}\\ \text{subcategories of } \mathcal{C}\end{matrix}\right\} \arrow[drr, "(-)^{\heartsuit}"] & & \\
\left\{\begin{matrix}\pi\text{-stable localizing}\\ \text{subcategories of } \mathcal{C}\end{matrix}\right\} \arrow[u, hook, "\subseteq"'] \arrow[rr, "\simeq"] & & \left\{\begin{matrix}\text{weak localizing}\\ \text{subcategories of } \mathcal{C}\end{matrix}\right. \\
\left\{\begin{matrix}\pi\text{-exact localizing}\\ \text{subcategories of } \mathcal{C}\end{matrix}\right\} \arrow[u, hook, "\subseteq"'] \arrow[r, "\simeq"] \arrow[d, hook, "\subseteq"] & \left\{\begin{matrix}\text{separating localizing}\\ \text{subcategories of } \mathcal{C}_{\geqslant 0}\end{matrix}\right\} \arrow[r, "\simeq"] \arrow[d, hook, "\subseteq"] & \left\{\begin{matrix}\text{localizing}\\ \text{subcategories of } \mathcal{C}\end{matrix}\right. \arrow[u, hook, "\subseteq"'] \\
\left\{\begin{matrix}t\text{-exact localizing}\\ \text{subcategories of } \mathcal{C}\end{matrix}\right\} \arrow[r, "(-)_{\geqslant 0}"] & \left\{\begin{matrix}\text{localizing}\\ \text{subcategories of } \mathcal{C}_{\geqslant 0}\end{matrix}\right\} \arrow[ur, "(-)_{\leqslant 0}"'] &
\end{tikzcd}
$$

Linear overview: We start Section 3.2 with some recollections on t-structures, prestable ∞-categories, and their interactions, before we introduce the notion of localizing subcategories in Section 3.2.2. We then study some further interactions between these, which we use to prove Theorem 3.F in Section 3.3.1 and Theorem 3.G in Section 3.3.2. We finish the paper by looking at some consequences and applications of our results.

Conventions: We will work in the setting of ∞-categories, as developed by Lurie in [Lur09] and [Lur17]. We will restrict our attention to presentable stable ∞-categories, which we will just call *stable categories*. Given a stable category $\mathcal{C}$ with a nice t-structure, its associated prestable category will be denoted $\mathcal{C}_{\geqslant 0}$ and its heart by $\mathcal{C}^{\heartsuit}$. We assume all t-structures to be accessible.

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3.2 Prestable and stable categories

For the rest of the paper we fix a stable category $\mathcal{C}$. We wish to equip this with a t-structure, which will allow us to always have a comparison from $\mathcal{C}$ to an abelian category. The main reference for t-structures in this setting is [Lur17, Sec 1.2.1]. Note that, as opposed to much of the homological algebra literature, we follow Lurie's homological indexing convention.

Definition 3.2.1. A *t-structure* on $\mathcal{C}$ is a pair of full subcategories $(\mathcal{C}_{\geqslant 0}, \mathcal{C}_{\leqslant 0})$ such that:

1. The mapping space $\mathrm{Map}_{\mathcal{C}}(X, Y[-1]) \simeq 0$ for all $X \in \mathcal{C}_{\geqslant 0}$ and $Y \in \mathcal{C}_{\leqslant 0}$;
2. There are inclusions $\mathcal{C}_{\geqslant 0}[1] \subseteq \mathcal{C}_{\geqslant 0}$ and $\mathcal{C}_{\leqslant 0}[-1] \subseteq \mathcal{C}_{\leqslant 0}$;
3. For any $Y \in \mathcal{C}$ there is a fiber sequence $X \longrightarrow Y \longrightarrow Z$ such that $X \in \mathcal{C}_{\geqslant 0}$ and $Z[1] \in \mathcal{C}_{\leqslant 0}$.

This is equivalent to choosing a t-structure on the homotopy category $h\mathcal{C}$, which is a triangulated category. Hence the contents of this paper should be equally useful to those familiar with t-structures on triangulated categories.

We will assume all t-structures to be accessible, in the sense that the connected part $\mathcal{C}_{\geqslant 0}$ is presentable. By [Lur17, 1.2.16] the inclusions $\mathcal{C}_{\geqslant 0} \longrightarrow \mathcal{C}$ and $\mathcal{C}_{\leqslant 0} \longrightarrow \mathcal{C}$ have a right adjoint $\tau_{\geqslant 0}$ and a left adjoint $\tau_{\leqslant 0}$ respectively. We denote $\mathcal{C}_{\geqslant n} := \mathcal{C}_{\geqslant 0}[n]$ and $\mathcal{C}_{\leqslant n} := \mathcal{C}_{\leqslant 0}[n]$.

Definition 3.2.2. The heart of a t-structure $(\mathcal{C}_{\geqslant 0}, \mathcal{C}_{\leqslant 0})$ on $\mathcal{C}$ is defined as the full subcategory $\mathcal{C}^{\heartsuit} := \mathcal{C}_{\geqslant 0} \cap \mathcal{C}_{\leqslant 0}$.

The heart $\mathcal{C}^{\heartsuit}$ is always equivalent to the nerve of its homotopy category $h\mathcal{C}^{\heartsuit}$, which was proven in [BBD82] to be an abelian category. It is standard to follow [Lur17, 1.2.1.12] and identify the two.

Definition 3.2.3. The composite functor

$$\tau_{\geqslant 0} \circ \tau_{\leqslant 0} \simeq \tau_{\leqslant 0} \circ \tau_{\geqslant 0} \colon \mathcal{C} \longrightarrow \mathcal{C}^{\heartsuit}$$

is denoted by $\pi_0^{\heartsuit}$ and its composition with the shift functor $X \longrightarrow X[-n]$ by $\pi_n^{\heartsuit}$. These are called the *heart-valued homotopy groups* of X.

The last definition we will need, before going on to prestable categories is the following niceness condition.

Definition 3.2.4. A t-structure $(\mathcal{C}_{\geqslant 0}, \mathcal{C}_{\leqslant 0})$ on a stable category $\mathcal{C}$ is said to be *compatible with filtered colimits* if $\mathcal{C}_{\leqslant 0}$ is closed under all filtered colimits in $\mathcal{C}$.

We now recall the notion of prestable ∞-categories, which, similarly to the stable ∞-categories, we will simply call *prestable categories*. The theory of prestable categories was developed by Lurie in [Lur16, App. C], and has since been applied in a varied range of areas. We define these as follows.

Definition 3.2.5. An ∞-category $\mathcal{D}$ is *prestable* if there exists a stable category $\mathcal{C}$ with a t-structure $(\mathcal{C}_{\geqslant 0}, \mathcal{C}_{\leqslant 0})$, such that $\mathcal{D} \simeq \mathcal{C}_{\geqslant 0}$.

Remark 3.2.6. This is not the most general, nor the standard, definition of a prestable category—see [Lur16, C.1.2.1]—but by [Lur16, C.1.2.9] the above definition describes all prestable categories admitting finite limits, hence it is not a very severe restriction. The category $\mathcal{D}$ is also not unique, see [Lur16, C.1.2.10], but we will mostly focus on the choice

$$\mathcal{D} = \mathrm{Sp}(\mathcal{C}_{\geqslant 0}) = \mathrm{colim}(\cdots \xrightarrow{\Omega} \mathcal{C}_{\geqslant 0} \xrightarrow{\Omega} \mathcal{C}_{\geqslant 0}).$$

Since we will discuss both stable and prestable categories, and their interactions, we will try to consequently denote prestable categories by $\mathcal{C}_{\geqslant 0}$ and stable categories by $\mathcal{C}$.

Remark 3.2.7. Any stable category $\mathcal{C}$ is prestable, as seen by choosing the trivial t-structure $(\mathcal{C}, 0)$. This is both a blessing, as it allows us to talk about both in a common language, and a curse, as using common language can be rather confusing when trying to study their interactions.

We will restrict our attention to Grothendieck prestable categories, which are prestable categories that work well with colimits. There are numerous different equivalent definition of these, see [Lur16, C.1.4.1], but the one best related to the above definition of a prestable category is the following.

Definition 3.2.8. A prestable category $\mathcal{C}_{\geqslant 0}$ is *Grothendieck* if the t-structure on its associated stable category $\mathcal{C}$ is compatible with filtered colimits.

The following example is perhaps the main reason for the naming convention.

Example 3.2.9. For any Grothendieck abelian category $\mathcal{A}$, the derived category category $\mathrm{D}(\mathcal{A})$ has a natural t-structure with heart $\mathcal{A}$. The connected component $\mathrm{D}(\mathcal{A})_{\geqslant 0}$, which consists of complexes $X_\bullet$ such that $H_i(X_\bullet) = 0$ for $i < 0$ is a Grothendieck prestable category.

We also have some examples showing up in stable homotopy theory.

Example 3.2.10. Let Sp be the stable ∞-category of spectra. This has a natural t-structure with heart Ab. The connected component $\mathrm{Sp}_{\geqslant 0}$, consisting of connective spectra, is a Grothendieck prestable category.

Example 3.2.11. Important for modern homotopy theory is the category of E-based synthetic spectra Syn_E for some Landweber exact homology theory E, see [Pst23]. This has a naturally occurring t-structure with heart Comod_{E_*E}, and its connected component $\mathrm{Syn}_E^{\geqslant 0}$ is Grothendieck prestable. This example is one of our main motivations for this work, and we plan to study the applications of the contents in this paper to synthetic spectra in future work.

Remark 3.2.12. If the prestable category $\mathcal{C}_{\geqslant 0}$ is compactly generated, then it is automatically Grothendieck, see [Lur16, C.1.4.4]. A stable ∞-category $\mathcal{C}$ is, as mentioned above, also prestable. It is in fact Grothendieck if and only if it is presentable.

Definition 3.2.13. We say a t-structure on a stable category $\mathcal{C}$ is *right complete* if the natural functor $\operatorname{colim}_n \mathcal{C}_{\geqslant -n} \xrightarrow{\simeq} \mathcal{C}$ is an equivalence.

Remark 3.2.14. For any Grothendieck prestable category $\mathcal{C}_{\geqslant 0}$ the functor $\mathrm{Sp}(-)$, sending $\mathcal{C}_{\geqslant 0}$ to its stabilization, $\mathrm{Sp}(\mathcal{C}_{\geqslant 0})$, provides a one-to-one correspondence between Grothendieck prestable categories and stable categories equipped with a right complete t-structure compatible with filtered colimits. This is one of the main reasons to study prestable categories, as being prestable is a property, while having a t-structure is extra structure.

Remark 3.2.15. If $\mathcal{C}$ is a stable category with a t-structure compatible with filtered colimits, then the heart-valued homotopy groups functors $\pi_n^{\heartsuit}$ preserve filtered colimits.

3.2.1 Bridging the gap

In this section we study the passage from stable to prestable and vice versa. In particular we look into when they determine each other.

If $\mathcal{C}$ is a stable category with a right complete t-structure $(\mathcal{C}_{\geqslant 0}, \mathcal{C}_{\leqslant 0})$, we can reconstruct it from its connected component.

Lemma 3.2.16 ([Lur16, C.1.2.10])**.** *Let $\mathcal{C}$ be a stable category. If $\mathcal{C}$ has a right complete t-structure, then there is an equivalence $\mathrm{Sp}(\mathcal{C}_{\geqslant 0}) \simeq \mathcal{C}$.*

This fact also extends to equivalences of categories, as proven by

Antieau.

Lemma 3.2.17 ([Ant21, 6.1])**.** *Let $\mathcal{C}$ and $\mathcal{D}$ be stable categories equipped with right complete t-structures. If $\mathcal{C}_{\geqslant 0} \simeq \mathcal{D}_{\geqslant 0}$, then also $\mathcal{C} \simeq \mathcal{D}$.*

Remark 3.2.18. In particular both the above results hold for any $\mathcal{C}$ such that $\mathcal{C}_{\geqslant 0}$ is Grothendieck.

We can also naturally go in the other direction. If we have an equivalence of stable categories $\mathcal{C} \simeq \mathcal{D}$, that is compatible with the t-structures, then we get an induced equivalence on the connected components. The precise definition of being compatible with the t-structures is as follows.

Definition 3.2.19. Let $\mathcal{C}, \mathcal{D}$ be stable categories with t-structures. An exact functor $F \colon \mathcal{C} \longrightarrow \mathcal{D}$ is *right t-exact* if $F(\mathcal{C}_{\geqslant 0}) \subseteq \mathcal{D}_{\geqslant 0}$. The notion of *left t-exactness* is defined similarly. If F satisfies both, we say that it is a *t-exact functor*.

Remark 3.2.20. This convention might seem wrong to readers with a background in homological algebra, as the role of left and right t-exact functors are usually the opposite. This flip is a consequence of using the homological indexing convention rather than cohomological indexing.

The above can then be made precise as follows.

Lemma 3.2.21. *Let $\mathcal{C}, \mathcal{D}$ be stable categories with t-structures. If $F \colon \mathcal{C} \longrightarrow \mathcal{D}$ is a right t-exact functor, then we have an induced functor of prestable categories $F_{\geqslant 0} \colon \mathcal{C}_{\geqslant 0} \longrightarrow \mathcal{D}_{\geqslant 0}$. If F is an equivalence, then so is $F_{\geqslant 0}$.*

For the rest of the paper we will use the following terminology.

Definition 3.2.22. A *t-stable category* is a stable category $\mathcal{C}$ together with a choice of a right complete t-structure compatible with filtered colimits.

Example 3.2.23. Let us see some examples of t-stable categories.

1. For every commutative noetherian ring R, the derived category $\mathrm{D}(R)$ together with its natural t-structure, is a t-stable category.

2. The category of spectra, together with its natural t-structure, is a t-stable category.

3. The category of synthetic spectra, Syn_E, together with its natural t-structure is a t-stable category.

4. For a noetherian scheme X, its associated derived category of quasi-coherent $\mathcal{O}_X$-modules, $\mathrm{D}_{qc}(X)$, is t-stable.

Remark 3.2.24. Let $\mathcal{C}$ be a t-stable category. By definition we have that the connective part, $\mathcal{C}_{\geqslant 0}$, is a Grothendieck prestable category, and that the heart $\mathcal{C}^{\heartsuit}$ is a Grothendieck abelian category. Hence t-stable categories serve as a natural place to study the interactions between these three types of categories.

Remark 3.2.25. In [Lur16, Section C.3.1] Lurie constructs a category of t-stable categories. If we denote this by $t\mathrm{Cat}$ then the contents of Remark 3.2.14 can be described as an adjoint pair of equivalences

$$\mathrm{Groth}_\infty \underset{(-)_{\geqslant 0}}{\overset{\mathrm{Sp}(-)}{\rightleftarrows}} t\mathrm{Cat}.$$

This should, however, be viewed as a heuristic rather than a very precise statement, as the right hand category is a bit tricky to define.

3.2.2 Localizing subcategories

We now turn our attention to localizing subcategories. As we are working in three interconnected settings—stable, prestable and abelian—and all settings use the same terminology, we feel that this section is very ripe for confusions to occur . In an attempt to clarify which setting we are in, we will usually refer to localizing subcategories of stable categories as *stable localizing subcategories*, localizing subcategories of prestable categories as *prestable localizing subcategories* and localizing subcategories of abelian categories as *abelian localizing subcategories*. We will, however, sometimes omit the categorical prefix when we feel that it is clear from context.

Definition 3.2.26. Let $\mathcal{C}$ be a stable category. A full subcategory $\mathcal{L} \subseteq \mathcal{C}$ is said to be *thick* if it is a full stable subcategory closed under finite colimits. In particular, it is closed under extensions and desuspensions. We say $\mathcal{L}$ is a *stable localizing subcategory* if it is thick and closed under filtered colimits.

Stable localizing subcategories are uniquely determined by localization functors on $\mathcal{C}$, hence their name. This is a standard fact about localizations, but we include a sketch of the proof for convenience.

Lemma 3.2.27. *A full subcategory $\mathcal{L}$ of a stable category $\mathcal{C}$, is a stable localizing subcategory if and only if there is a stable category $\mathcal{D}$, and an exact localization $L\colon \mathcal{C} \longrightarrow \mathcal{D}$, such that $\mathcal{L}$ is the kernel of L.*

Proof. Let $\mathcal{L}$ be an arbitrary localizing subcategory of $\mathcal{C}$. The right-orthogonal complement of $\mathcal{L}$, denoted

$$\mathcal{L}^{\perp} = \{C \in \mathcal{C} \mid \operatorname{Hom}(X, C) \simeq 0, \forall X \in \mathcal{C}\},$$

is closed under all limits in $\mathcal{C}$, meaning that the fully faithful inclusion $\mathcal{L}^{\perp} \hookrightarrow \mathcal{C}$ has a left adjoint L. This is an exact localization of stable ∞-categories, and the kernel is precisely $\mathcal{L}$.

For the converse, assume we are given an exact localization functor $L\colon \mathcal{C} \longrightarrow \mathcal{D}$ such that $\mathcal{L} = \operatorname{Ker} L$. Then $\mathcal{L}$ is a stable category by the exactness of L, which is in addition closed under colimits as L preserves these by being a left adjoint. □

The definition of a localizing subcategory of a prestable category is very similar in nature to its stable brethren, but there is a slight variation.

Definition 3.2.28. Let $\mathcal{C}_{\geqslant 0}$ be a Grothendieck prestable category and C an object in $\mathcal{C}$. Another object $C' \in \mathcal{C}$ is said to be a subobject of C if there is a map $f\colon C' \longrightarrow C$ with $\operatorname{Cofib}(f) \in \mathcal{C}^{\heartsuit}$.

Remark 3.2.29. For Grothendieck prestable categories, this is equivalent to the assertion that C' is a (-1)-truncated object in $\mathcal{C}_{/C}$ via the map f, which is the more standard definition of being a sub-object—see [Lur16, C.2.3.4]

Definition 3.2.30 ([Lur16, C.2.3.3])**.** Let $\mathcal{C}_{\geqslant 0}$ be a Grothendieck prestable category. A full subcategory $\mathcal{L}_{\geqslant 0} \subseteq \mathcal{C}_{\geqslant 0}$ is a *prestable localizing subcategory* if it is accessible and closed under coproducts, cofiber sequences and sub-objects.

Remark 3.2.31. Any prestable localizing subcategory $\mathcal{L}_{\geqslant 0}$ of a Grothendieck prestable category $\mathcal{C}_{\geqslant 0}$ is by [Lur16, C.5.2.1] itself a Grothendieck prestable category. This means, in particular, that $\mathcal{L}_{\geqslant 0}$ is the connected part of a colimit-compatible t-structure on a stable category, hence using the notation $\mathcal{L}_{\geqslant 0}$ is not abusive.

Remark 3.2.32. Recall from Remark 3.2.7 that any stable category $\mathcal{C}$ can be treated as a prestable category. By [Lur16, C.2.3.6] a full subcategory $\mathcal{L}$ of $\mathcal{C}$ is a prestable localizing subcategory if and only if it is a stable localizing subcategory.

As in the stable situation we have a description of prestable localizing subcategories via localization functors.

Proposition 3.2.33 ([Lur16, C.2.3.8])**.** *A full subcategory $\mathcal{L}_{\geqslant 0}$ of a Grothendieck prestable category $\mathcal{C}_{\geqslant 0}$ is localizing if and only if there is a Grothendieck prestable category $\mathcal{D}_{\geqslant 0}$, and left exact localization $L\colon \mathcal{C}_{\geqslant 0} \longrightarrow \mathcal{D}_{\geqslant 0}$, such that $\mathcal{L}_{\geqslant 0}$ is the kernel of L.*

As prestable localizing subcategories are again prestable, we know that there is some stable category with a t-structure presenting it as its connected component. The prestable localizing subcategories hence naturally encodes a sort of induced t-structure. This does not happen automatically for stable categories, hence we need to make some additional requirements in order to successfully move between the prestable and stable situation.

Definition 3.2.34. Let $\mathcal{C}$ be a t-stable category. A full subcategory $\mathcal{L} \subseteq \mathcal{C}$ is said to be a *t-stable localizing subcategory* if it is localizing, and for any $X \in \mathcal{L}$ we have $\tau_{\geqslant 0}X \in \mathcal{L}$ and $\tau_{\leqslant 0}X \in \mathcal{L}$.

Remark 3.2.35. We hope that using both the names t-stable categories and t-stable localizing subcategories does not cause confusion. We decided to use this terminology, as a t-stable localizing subcategory is itself a t-stable category, as we will see in Lemma 3.2.46.

Remark 3.2.36. Let $\mathcal{L}$ be a t-stable localizing subcategory of $\mathcal{C}$. As localizing subcategories are stable under (de)suspension, this means that also all $\tau_{\geqslant n}X$ and $\tau_{\leqslant n}$ lie in $\mathcal{L}$ for all n. In particular, the homotopy groups $\pi_n^{\heartsuit}X$ lie in $\mathcal{L}$ for all n.

Remark 3.2.37. This definition is motivated by [BBD82, 1.3.19], where the authors prove that such a full subcategory inherits a t-structure given by

$$(\mathcal{L}_{\geqslant 0}, \mathcal{L}_{\leqslant 0}) = (\mathcal{C}_{\geqslant 0} \cap \mathcal{L}, \mathcal{C}_{\leqslant 0} \cap \mathcal{L})$$

with heart $\mathcal{C}^{\heartsuit} \cap \mathcal{L}$. In other words, a t-stable localizing subcategory has a “sub t-structure”, such that the inclusion is t-exact. In particular, the truncation functors $\tau_{\geqslant n}$ and $\tau_{\leqslant n}$ are the same as those in $\mathcal{C}$, hence also the homotopy group functors $\pi_n^{\heartsuit}$ are the same in $\mathcal{C}$ and $\mathcal{L}$.

We will from now on assume that a t-stable localizing subcategory is equipped with the above t-structure.

Proposition 3.2.38. *Let $\mathcal{C}$ be a stable category with a right complete t-structure and let $\mathcal{L} \subseteq \mathcal{C}$ be a localizing subcategory. If $\mathcal{L}$ is t-stable, then the induced t-structure on $\mathcal{L}$ is right complete.*

Proof. This follows immediately from the fact that the truncation functors are the same as in $\mathcal{C}$, and that colimits in $\mathcal{L}$ are the same as those in $\mathcal{C}$. □

The last thing to introduce in this section are the abelian analogs of the above definitions.

Definition 3.2.39. A full subcategory $\mathcal{T}$ of a Grothendieck abelian category $\mathcal{A}$ is called a *weak Serre subcategory*, if for any exact sequence

$$A_1 \longrightarrow A_2 \longrightarrow A_3 \longrightarrow A_4 \longrightarrow A_5$$

in $\mathcal{A}$ such that A_1, A_2, A_4, A_5 are all in $\mathcal{T}$, then also $A_3 \in \mathcal{T}$. It is a *abelian weak localizing subcategory* it it is a weak Serre subcategory closed under arbitrary coproducts.

Remark 3.2.40. A full subcategory is a weak Serre subcategory if it is closed under kernels, cokernels and extensions. In particular it is an abelian subcategory, and the fully faithful inclusion functor $\mathcal{T} \hookrightarrow \mathcal{A}$ is exact.

Definition 3.2.41. A full subcategory $\mathcal{T}$ of a Grothendieck abelian category $\mathcal{A}$ is called a *Serre subcategory* if for any short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

in $\mathcal{A}$, we have $B \in \mathcal{T}$ if and only if $A, C \in \mathcal{T}$. It is an *abelian localizing subcategory* if it is a Serre subcategory closed under arbitrary coproducts.

Remark 3.2.42. A full subcategory is a Serre subcategory if it is closed under sub-objects, quotients and extensions. This means that all Serre subcategories are weak Serre subcategories, and that all abelian localizing subcategories are abelian weak localizing subcategories. In particular they are all abelian subcategories with exact inclusions into $\mathcal{A}$.

Remark 3.2.43. Weak Serre subcategories seem to also be called *thick* or *wide* subcategories in the homological algebra literature. But, to make the connection with abelian localizing subcategories clearer we chose to use this terminology.

As one perhaps should expect at this point, Abelian localizing subcategories are also determined by localization functors—as above, so below.

Proposition 3.2.44 ([Lur16, C.5.1.1, C.5.1.6])**.** *A full subcategory $\mathcal{T}$ of a Grothendieck abelian category $\mathcal{A}$ is an abelian localizing subcategory if and only if there is an exact localization functor $L\colon \mathcal{A} \longrightarrow \mathcal{B}$, where $\mathcal{B}$ is a Grothendieck abelian category, such that $\mathcal{T}$ is the kernel of L.*

3.2.3 Stable and prestable comparisons

The first thing we need is to be able to recognize stable localizing subcategories by their connected part, as we did for stable categories in Lemma 3.2.16.

Corollary 3.2.45. *Let $\mathcal{C}$ be a stable category with a right complete t-structure and $\mathcal{L}$ a t-stable localizing subcategory. In this situation there is an equivalence $\mathcal{L} \simeq \mathrm{Sp}(\mathcal{L}_{\geqslant 0})$.*

Proof. This follows directly from Proposition 3.2.38 and Lemma 3.2.16. □

Using this we can increase the strength of Proposition 3.2.38 by also incorporating compatibility with filtered colimits. Recall that we use the name t-stable category for a stable category with a right complete t-structure compatible with filtered colimits.

Lemma 3.2.46. *Let $\mathcal{C}$ be a t-stable category and $\mathcal{L}$ a localizing subcategory. If $\mathcal{L}$ is t-stable, then $\mathcal{L}$ is itself a t-stable category.*

Proof. By Proposition 3.2.38 we know that the induced t-structure on $\mathcal{L}$ is right complete. By [Lur16, C.5.2.1(1)] $\mathcal{L}_{\geqslant 0}$ is Grothendieck prestable, hence the t-structure on its stabilization $\mathrm{Sp}(\mathcal{L}_{\geqslant 0})$ is compatible with filtered colimits by definition, see [Lur16, C.1.4.1]. This stabilization is by Corollary 3.2.45 equivalent to $\mathcal{L}$, completing the proof. □

Recall that any stable localizing subcategory $\mathcal{L} \subseteq \mathcal{C}$ is equivalently determined as the acyclic objects to an exact localization functor $L\colon \mathcal{C} \longrightarrow \mathcal{D}$. We want a similar fact to hold for the t-stable ones. The naïve guess could perhaps be that $\mathcal{L}$ is t-stable if and only if the localization functor L is t-exact. This turns out to be too strong of a condition on the nose, but a very interesting condition nonetheless.

Lemma 3.2.47. *Let $L\colon \mathcal{C} \longrightarrow \mathcal{D}$ be a localization of stable categories with t-structures. If L is t-exact, then $\mathrm{Ker}(L)$ is a t-stable localizing subcategory.*

Proof. Let $X \in \mathrm{Ker}(L)$. Since L is t-exact we have

$$L(\tau_{\geqslant 0}X) \simeq \tau_{\geqslant 0}L(X) \simeq 0,$$

hence also $\tau_{\geqslant 0}X$ is in $\mathrm{Ker}(L)$. We have $\tau_{\leqslant 0}X \in \mathrm{Ker}(L)$ by an identical argument. □

We can then relate this to the prestable situation via the following lemma.

Lemma 3.2.48 ([Lur16, C.2.4.4])**.** *If $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ is an t-exact functor between t-stable categories, then the induced functor of Grothendieck prestable categories $F_{\geqslant 0}\colon \mathcal{C}_{\geqslant 0} \longrightarrow \mathcal{D}_{\geqslant 0}$ is left exact.*

Remark 3.2.49. Since prestable localizing subcategories are determined by left exact localization functors, see Proposition 3.2.33, Lemma 3.2.48 means that if $\mathcal{L}$ is a stable localizing subcategory determined by a t-exact localization functor $\mathcal{C} \longrightarrow \mathcal{D}$, then the connected part $\mathcal{L}_{\geqslant 0}$ is a prestable localizing subcategory of $\mathcal{C}_{\geqslant 0}$.

We also want a converse to this statement.

Lemma 3.2.50. *If $\mathcal{L}_{\geqslant 0}$ is a prestable localizing subcategory of a Grothendieck prestable category $\mathcal{C}_{\geqslant 0}$, then its stabilization* $\mathrm{Sp}(\mathcal{L}_{\geqslant 0})$ *is the kernel of a t-exact localization L on* $\mathrm{Sp}(\mathcal{C}_{\geqslant 0})$.

Proof. By Proposition 3.2.33 we know that $\mathcal{L}_{\geqslant 0}$ is the kernel of a left exact localization $L_{\geqslant 0}\colon \mathcal{C}_{\geqslant 0} \longrightarrow \mathcal{D}_{\geqslant 0}$. This is a colimit preserving functor, hence the induced functor $\mathrm{Sp}(\mathcal{L}_{\geqslant 0})\colon \mathrm{Sp}(\mathcal{C}_{\geqslant 0}) \longrightarrow \mathrm{Sp}(\mathcal{D}_{\geqslant 0})$ is then left t-exact by [Lur16, C.3.2.1] and right t-exact by [Lur16, C.3.1.1]. □

Remark 3.2.51. In particular, by Lemma 3.2.47 the stabilization $\mathrm{Sp}(\mathcal{L}_{\geqslant 0})$ is a t-stable localizing subcategory.

In light of the above results we introduce the following definition.

Definition 3.2.52. A stable localizing subcategory $\mathcal{L} \subseteq \mathcal{C}$ is said to be *t-exact* if it is the kernel of a t-exact localization.

Remark 3.2.53. As we will have several definitions for different kinds of localizing subcategories, we will have a recurring remark about their dependencies. In this first such remark, we note that there is an implication

$$t\text{-exact} \implies t\text{-stable}$$

by Lemma 3.2.47.

We can then conclude this section with the following bijection.

Corollary 3.2.54. *For any t-stable category $\mathcal{C}$, there is a bijection between the collection of t-exact stable localizing subcategories $\mathcal{L} \subseteq \mathcal{C}$, and prestable localizing subcategories of $\mathcal{C}_{\geqslant 0}$, given by the mutually inverse functors $(-)_{\geqslant 0}$ and $\mathrm{Sp}(-)$.*

Proof. From Remark 3.2.49 and Lemma 3.2.50 we have maps

$$\begin{Bmatrix} t\text{-exact localizing} \\ \text{subcategories of } \mathcal{C} \end{Bmatrix} \xrightarrow{(-)_{\geqslant 0}} \begin{Bmatrix} \text{localizing} \\ \text{subcategories of } \mathcal{C}_{\geqslant 0} \end{Bmatrix}$$

and

$$\begin{Bmatrix} \text{localizing} \\ \text{subcategories of } \mathcal{C}_{\geqslant 0} \end{Bmatrix} \xrightarrow{\mathrm{Sp}(-)} \begin{Bmatrix} t\text{-exact localizing} \\ \text{subcategories of } \mathcal{C} \end{Bmatrix}$$

These are mutually inverse functors by Corollary 3.2.45, and the fact that any prestable localizing subcategory of a Grothendieck prestable category is itself a Grothendieck prestable category, see Remark 3.2.31. □

Remark 3.2.55. The above corollary gives us a t-exact approximation result for t-stable localizing subcategories. Suppose we have a t-stable localizing subcategory $\mathcal{L} \subseteq \mathcal{C}$. We can choose the smallest prestable localizing subcategory of $\mathcal{C}_{\geqslant 0}$ containing $\mathcal{L}_{\geqslant 0}$, which we denote $\mathrm{Loc}_{\geqslant 0}(\mathcal{L}_{\geqslant 0})$. Upon stabilization we obtain by Corollary 3.2.54 a stable localizing subcategory $\mathcal{L}^t$ that is the kernel of a t-exact functor. As $\mathrm{Sp}(\mathcal{L}_{\geqslant 0}) \simeq \mathcal{L}$, we know that $\mathcal{L} \subseteq \mathcal{L}^t$, making $\mathcal{L}^t$ a t-exact approximation of $\mathcal{L}$. It is also the smallest such approximation, and, naturally, $\mathcal{L}$ is t-exact if and only if $\mathcal{L} \simeq \mathcal{L}^t$.

3.3 The correspondences

The goal of this section is to prove our two main results. We start with the classification of weak localizing subcategories, before proving the non-weak case. The former does not need any of the connections to prestable categories, hence can also be viewed as a self contained argument. The latter, however, relies on Lurie's correspondence between certain prestable localizing subcategories of $\mathcal{C}_{\geqslant 0}$ and localizing subcategories of $\mathcal{C}^{\heartsuit}$.

3.3.1 Classification of weak localizing subcategories

The goal of this section is to prove Theorem 3.F, and the following lemma is the first step for obtaining the wanted correspondence.

Lemma 3.3.1. *Let $\mathcal{C}$ be a t-stable category. If $\mathcal{L}$ is a t-stable localizing subcategory, then $\mathcal{L}^{\heartsuit}$ is a weak localizing subcategory of $\mathcal{C}^{\heartsuit}$.*

Proof. As $\mathcal{L}$ is t-stable we know that the fully faithful inclusion $\mathcal{L} \longrightarrow \mathcal{C}$ is t-exact. By [AGH19, 2.19] the induced functor $\mathcal{L}^{\heartsuit} \longrightarrow \mathcal{C}^{\heartsuit}$ is exact and fully faithful, and $\mathcal{L}^{\heartsuit}$ is closed under extensions. In particular, $\mathcal{L}^{\heartsuit}$ is an abelian subcategory closed under extensions, so it remains only to show that $\mathcal{L}^{\heartsuit}$ is closed under coproducts.

As $\mathcal{L}^{\heartsuit} \subseteq \mathcal{L}$ we can include a coproduct of objects in $\mathcal{L}^{\heartsuit}$ into $\mathcal{L}$. The inclusion and $\pi_n^{\heartsuit}$ preserves coproducts for all n. Hence, as $\mathcal{L}$ is localizing it is closed under coproducts, implying that also $\mathcal{L}^{\heartsuit}$ is. $\square$

This means that the heart construction $\mathcal{C} \longmapsto \mathcal{C}^{\heartsuit}$ determines a map

$$\left\{\begin{matrix} t\text{-stable localizing} \\ \text{subcategories of } \mathcal{C} \end{matrix}\right\} \xrightarrow{(-)^{\heartsuit}} \left\{\begin{matrix} \text{weak localizing} \\ \text{subcategories of } \mathcal{C}^{\heartsuit} \end{matrix}\right\}$$

for any t-stable category $\mathcal{C}$.

This map is in general not injective, meaning we have to restrict our domain. As described in the introduction, we will use the localizing subcategories where objects can be identified by their heart-valued homotopy groups. The precise definition is as follows.

Definition 3.3.2. Let $\mathcal{C}$ be a stable category with a t-structure. A stable localizing subcategory $\mathcal{L}$ is said to be π-*stable* if $X \in \mathcal{L}$ if and only if $\pi_n^{\heartsuit} X \in \mathcal{L}^{\heartsuit}$ for all n.

Remark 3.3.3. The terminology is motivated by, and generalizes, Takahashi's definition of H-stable subcategories of the unbounded derived category of a commutative noetherian ring, see [Tak09, 2.11]. These are subcategories of derived categories where one can detect containment by checking on homology. Letting $\mathcal{C} = \mathrm{D}(R)$ for a Noetherian commutative ring R considered with the natural t-structure, then we have $\pi_n^{\heartsuit} = H_n$, meaning that being π-stable is equivalent to being H-stable. Note, however, that the homological algebra literature often uses cohomological indexing, while we follow Lurie's convention of using the homological one.

Remark 3.3.4. The above definition is equivalent to Zhang–Cai's generalization of Takahashi's H-stable subcategories, see [ZC17]. Note that the authors of loc. cit. do not consider the subcategories themselves to have t-structures, but rather just includes the image of $\pi_k^{\heartsuit}$ back into the stable category.

Example 3.3.5. Let R be a commutative noetherian ring and $I \subseteq R$ a finitely generated regular ideal. Then the full subcategory of I-power torsion modules, $\mathrm{Mod}_R^{I-tors} \subseteq \mathrm{Mod}_R$ is an abelian weak localizing subcategory. It is in particular a Grothendieck abelian category, hence has a derived category $\mathrm{D}(\mathrm{Mod}_R^{I-tors})$. We can also form the derived I-torsion category $\mathrm{D}(R)^{I-tors}$, which is the localizing subcategory generated by A/I. The categories $\mathrm{D}(R)^{I-tors}$ and $\mathrm{D}(\mathrm{Mod}_R^{I-tors})$ are both π-stable localizing subcategories of $\mathrm{D}(R)$ with heart Mod_R^{I-tors}—see [GM92] or [BHV18] for more details. These categories are equivalent, seemingly implying that having the same heart is enough for the stable categories to be equivalent as well. This also generalizes to other similar situations, see for example [BHV20, 3.15, 3.17] or Theorem 1.2.21. Such equivalences were one of the main inspirations for this paper, where the author wanted an easier way of checking similar statements, which led to the main result Theorem 3.F.

Proposition 3.3.6. *Let $\mathcal{L}$ be a localizing subcategory of $\mathcal{C}$. If $\mathcal{L}$ is π-stable, then $\mathcal{L}$ is t-stable.*

Proof. Let $X \in \mathcal{L}$. We need to show that $\tau_{\geqslant 0}X \in \mathcal{L}$ and $\tau_{\leqslant 0}X \in \mathcal{L}$. The proofs are similar, hence we only cover the former.

We have $\pi_n^\heartsuit \tau_{\geqslant 0}X \simeq \pi_n^\heartsuit X$ for all $n \geqslant 0$ and $\pi_n^\heartsuit \tau_{\geqslant 0}X \simeq 0$ for all $n < 0$. This means that $\pi_n^\heartsuit \tau_{\geqslant 0}X \in \mathcal{L}^\heartsuit$ for all n, which implies $\tau_{\geqslant 0}X \in \mathcal{L}$ by the assumption that $\mathcal{L}$ was π-stable. ☐

Remark 3.3.7. In light of Proposition 3.3.6 we can continue our recurring remark (see Remark 3.2.53) about the dependencies of the different definitions. We now have implications

$$\begin{tikzcd} & \pi\text{-stable} \arrow[d, Rightarrow] \\ t\text{-exact} \arrow[r, Rightarrow] & t\text{-stable} \end{tikzcd}$$

for any localizing subcategory $\mathcal{L}$ of a t-stable category $\mathcal{C}$.

Remark 3.3.8. In particular, if $\mathcal{L}$ is a π-stable localizing subcategory then Lemma 3.2.46 implies that $\mathcal{L}$ is itself a t-stable category. This is rather convenient, as it allows us to treat nested pairs of π-stable localizing subcategories $\mathcal{L}_2 \subseteq \mathcal{L}_1 \subseteq \mathcal{C}$ either as both being subcategories of $\mathcal{C}$, or as $\mathcal{L}_2$ being a π-stable localizing subcategory of $\mathcal{L}_1$.

Proposition 3.3.6 implies that the heart construction $\mathcal{L} \longmapsto \mathcal{L}^\heartsuit$ gives a map

$$\left\{\begin{matrix}\pi\text{-stable localizing} \\ \text{subcategories of } \mathcal{C}\end{matrix}\right\} \xrightarrow{(-)^\heartsuit} \left\{\begin{matrix}\text{weak localizing} \\ \text{subcategories of } \mathcal{C}^\heartsuit\end{matrix}\right\}$$

as the heart of any t-stable localizing subcategory $\mathcal{L} \subseteq \mathcal{C}$ is an abelian weak localizing subcategory $\mathcal{L}^\heartsuit \subseteq \mathcal{C}^\heartsuit$ by Lemma 3.3.1. The claim of Theorem 3.F is that this map is a bijection.

It turns out that the π-stable localizing subcategories are the largest localizing subcategories with a given heart. This is the stable analog of [Lur16, C.5.2.5] for prestable categories.

Lemma 3.3.9. *Let $\mathcal{C}$ be a t-stable category. Given two t-stable localizing subcategories $\mathcal{L}_0$ and $\mathcal{L}_1$, where $\mathcal{L}_1$ is π-stable, then $\mathcal{L}_0 \subseteq \mathcal{L}_1$ if and only if $\mathcal{L}_0^\heartsuit \subseteq \mathcal{L}_1^\heartsuit$.*

Proof. First, notice that as both categories are t-stable the truncation functors and the homotopy groups functors $\pi_k^\heartsuit$ are the same, see Remark 3.2.37.

Assume $\mathcal{L}_0^\heartsuit \subseteq \mathcal{L}_1^\heartsuit$ and $X \in \mathcal{L}_0$. Then $\pi_k^\heartsuit X \in \mathcal{L}_0^\heartsuit \subseteq \mathcal{L}_1^\heartsuit$ for all k. This implies that $X \in \mathcal{L}_1$ by the assumption that it is π-stable.

For the converse, assume $\mathcal{L}_0 \subseteq \mathcal{L}_1$. As the truncation functors are the same in $\mathcal{L}_0$ and $\mathcal{L}_1$ we have that $\mathcal{L}_0$ is a t-stable localizing subcategory of the t-stable category $\mathcal{L}_1$, see Remark 3.3.8. In particular, $\mathcal{L}_0^\heartsuit = \mathcal{L}_1^\heartsuit \cap \mathcal{L}_0$, hence we have $\mathcal{L}_0^\heartsuit \subseteq \mathcal{L}_1^\heartsuit$. $\square$

This immediately implies the injectivity of our proposed one-to-one correspondence.

Corollary 3.3.10. *For any t-stable category $\mathcal{C}$, the map*

$$\left\{ \begin{matrix} \pi\text{-stable localizing} \\ \text{subcategories of } \mathcal{C} \end{matrix} \right\} \xrightarrow{(-)^\heartsuit} \left\{ \begin{matrix} \text{weak localizing} \\ \text{subcategories of } \mathcal{C}^\heartsuit \end{matrix} \right\}$$

is injective.

Proof. Let $\mathcal{L}_0$ and $\mathcal{L}_1$ be π-stable localizing subcategories such that $\mathcal{L}_0^\heartsuit \simeq \mathcal{L}_1^\heartsuit$ as subcategories of $\mathcal{C}^\heartsuit$. In particular, they are contained in each other, hence Lemma 3.3.9 implies that $\mathcal{L}_0 \subseteq \mathcal{L}_1$ and $\mathcal{L}_1 \subseteq \mathcal{L}_0$ as they are both π-stable. $\square$

It remains to show that the map is also surjective.

Theorem 3.3.11 (Theorem 3.F)**.** *Let $\mathcal{C}$ be a t-stable category. In this situation, the map*

$$\left\{\begin{matrix}\pi\text{-stable localizing} \\ \text{subcategories of } \mathcal{C}\end{matrix}\right\} \xrightarrow{(-)^{\heartsuit}} \left\{\begin{matrix}\text{weak localizing} \\ \text{subcategories of } \mathcal{C}^{\heartsuit}\end{matrix}\right\}$$

is a bijection.

Proof. We know by Corollary 3.3.10 that the map is injective, hence it remains to prove surjectivity. To do this we follow the proof of [Lur16, C.5.2.7], adapted to the stable setting.

Let $\mathcal{A}$ be a weak localizing subcategory of $\mathcal{C}^{\heartsuit}$. Define $\mathcal{L} \subseteq \mathcal{C}$ to be the full subcategory spanned by objects X such that $\pi_n^{\heartsuit} X \in \mathcal{A}$. We prove that it is a stable localizing subcategory—it will obviously be π-stable by definition. In particular we prove that it is closed under cofiber sequences, desuspension and colimits.

Let $A \to B \to C$ be a cofiber sequence in $\mathcal{C}$. We need to show that if two of the objects A, B, C is in $\mathcal{L}$, then also the last one is. The long exact sequence of heart-valued homotopy groups has the form

$$\cdots \to \pi_n^{\heartsuit} A \to \pi_n^{\heartsuit} B \to \pi_n^{\heartsuit} C \to \pi_{n+1}^{\heartsuit} A \to \pi_{n+1}^{\heartsuit} B \to \cdots$$

Assuming that A, B are in $\mathcal{L}$ we get by the definition of $\mathcal{L}$ that the four objects $\pi_n^{\heartsuit} A, \pi_n^{\heartsuit} B, \pi_{n+1}^{\heartsuit} A, \pi_{n+1}^{\heartsuit} B$ are in $\mathcal{A}$. Hence, as $\mathcal{A}$ is a weak Serre subcategory we have $\pi_n^{\heartsuit} C \in \mathcal{A}$. This works for all n, hence we must have $C \in \mathcal{L}$ as well. The proof is identical in the case that A, C or B, C are in $\mathcal{L}$.

The full subcategory $\mathcal{L}$ is also closed under desuspension, as we have $\pi_n^{\heartsuit}(\Omega X) \simeq \pi_{n+1}^{\heartsuit}(X)$ by the long exact sequence in heart-valued homotopy groups. Hence $\mathcal{L}$ is a full stable subcategory of $\mathcal{C}$. In particular this means it is closed under finite colimits. Now, as $\pi_n^{\heartsuit}$ preserves coproducts, and $\mathcal{A}$ is closed under coproducts, we also get that $\mathcal{L}$ is closed under coproducts. This implies that $\mathcal{L}$ is closed under colimits, which finishes the proof. □

Remark 3.3.12. It is somewhat unfortunate that the terminology does not align perfectly in these two situations—meaning that we had to add a prefix "weak" for the abelian case. As both are inspired by the existence of localization functors, they are the natural terminology in their respective settings, and we should perhaps not expect everything to always agree perfectly. In Theorem 3.G we will use the abelian localizing subcategories, and then again be left with a choice of a different prefix for the stable version.

Theorem 3.F recovers, and generalizes, a theorem by Takahashi for commutative noetherian rings. Note that Takahashi does not refer to the abelian subcategories as weak localizing, but as thick subcategories closed under coproducts.

Corollary 3.3.13 ([Tak09])**.** *If R is a commutative noetherian ring, then there is a bijection between the set of H-stable localizing subcategories of $\mathrm{D}(R)$ and the set of weak localizing subcategories in Mod_R.*

A theorem of Krause—see [Kra08, 3.1]—shows that these two collections are also in bijection with certain subsets of $\operatorname{Spec} R$, which Krause calls the *coherent subsets*. In light of Theorem 3.F we can generalize Takahashi's result to a noetherian scheme X, and we conjecture that these are also in bijection with the coherent subsets of X—generalizing the result by Krause.

Corollary 3.3.14. *If X noetherian scheme, then there is a bijection between the set of stable localizing subcategories of $\mathrm{D}_{qc}(X)$ closed under homology, and the set of weak localizing subcategories in $\mathrm{QCoh}(X)$.*

Conjecture 3.3.15. For a noetherian scheme X, there is a bijection between the collection of coherent subsets of X and weak localizing subcategories of $\mathrm{QCoh}(X)$.

Remark 3.3.16. A hint towards the truth of this conjecture comes from a theorem by Gabriel ([Gab62, VI.2.4(b)]), where he shows that the above proposed bijection restricts to a bijection between specialization closed subsets of X and localizing subcategories of $\mathrm{QCoh}(X)$.

Now, we want to mention that we also obtain a classification of weak Serre subcategories of $\mathcal{C}$. This is done by recognizing that the proofs of Lemma 3.3.1, Corollary 3.3.10 and Theorem 3.F also holds without the assumption about coproducts. The proofs treats coproducts as a separate part, hence just omitting it from the proofs gives the following result.

Proposition 3.3.17. *Let $\mathcal{C}$ be a t-stable category. In this situation, the map*

$$\left\{ \begin{matrix} \pi\text{-stable thick} \\ \text{subcategories of } \mathcal{C} \end{matrix} \right\} \xrightarrow{(-)^{\heartsuit}} \left\{ \begin{matrix} \text{weak Serre} \\ \text{subcategories of } \mathcal{C}^{\heartsuit} \end{matrix} \right\}$$

is a bijection.

This recovers the following classification of weak Serre subcategories in the case where the t-structure on $\mathcal{C}$ is bounded, due to Zhang–Cai, see [ZC17].

Corollary 3.3.18. *Let $\mathcal{C}$ be a triangulated category with a bounded t-structure. In this situation there is a bijection between π-stable subcategories of $\mathcal{C}$ and weak Serre subcategories of $\mathcal{C}^{\heartsuit}$.*

We can summarize the contents of this section with half of the diagram from the introduction.

$$\begin{tikzcd}
\left\{ \begin{matrix} t\text{-stable localizing} \\ \text{subcategories of } \mathcal{C} \end{matrix} \right\} \arrow[drr, "(-)^{\heartsuit}"] & & \\
\left\{ \begin{matrix} \pi\text{-stable localizing} \\ \text{subcategories of } \mathcal{C} \end{matrix} \right\} \arrow[u, hook, "\subseteq"'] \arrow[rr, "\simeq"] & & \left\{ \begin{matrix} \text{weak localizing} \\ \text{subcategories of } \mathcal{C}^{\heartsuit} \end{matrix} \right\}
\end{tikzcd}$$

Digression on Grothendieck homology theories

There is a slight generalization of the surjectivity result above, which we decided to include here for future reference. The generalization comes from realizing that there are other functors that have similar properties to the heart valued homotopy group functor $\pi_*^{\heartsuit} \colon \mathcal{C} \longrightarrow \mathcal{C}^{\heartsuit}$.

Let $\mathcal{C}$ be a presentable stable ∞-category and $\mathcal{A}$ be a graded Grothendieck abelian category—meaning it comes equipped with an autoequivalence $[1]\colon \mathcal{A} \longrightarrow \mathcal{A}$, which we think of as a grading shift functor.

Definition 3.3.19. A functor $H\colon \mathcal{C} \longrightarrow \mathcal{A}$ is called a *Grothendieck homology theory* if it satisfies the following properties:

1. It is additive.
2. It sends cofiber sequences $X \to Y \to Z$ to exact sequences $HX \to HY \to HZ$.
3. It is a graded functor, i.e. $H(\Sigma X) \cong (HX)[1]$.
4. It preserves coproducts.

Remark 3.3.20. The first two criteria defines H to be what is usually called a homological functor. Adding the third criteria makes H a homology theory, and the last is what makes it Grothendieck.

The main example of these come from the category of spectra, Sp, where the associated homology theory to any spectrum is a Grothendieck homology theory.

Example 3.3.21. Let $\mathcal{C} = \mathrm{Sp}$ and R be a graded commutative ring. The Eilenberg–MacLane spectrum HR is a commutative ring spectrum, and the associated homology theory $HR_* := [\mathbb{S}, HR \otimes (-)]_*\colon \mathrm{Sp} \longrightarrow \mathrm{Mod}_R$ is a Grothendieck homology theory. This homology theory is equivalent to singular homology with R coefficients.

The above example holds more generally as well.

Example 3.3.22. If $\mathcal{C}$ is monoidal and the unit $\mathbb{1}$ is compact, then for any $H \in \mathcal{C}$ the associated functor

$$\begin{aligned} H_*\colon \mathcal{C} &\longrightarrow \mathrm{Ab}^{\mathrm{gr}} \\ X &\longmapsto [\mathbb{1}, H \otimes X]_* \end{aligned}$$

is a Grothendieck homology theory.

Proposition 3.3.23. *Let $H\colon \mathcal{C} \longrightarrow \mathcal{A}$ be a Grothendieck homology theory and $\mathcal{T}$ a weak localizing subcategory of $\mathcal{A}$. In this situation, the full subcategory $\mathcal{L} \subseteq \mathcal{C}$ consisting of objects X such that $HX \in \mathcal{T}$, is a localizing subcategory of $\mathcal{C}$.*

Proof. This holds by using the same surjectivity argument from Theorem 3.3.11, just exchanging $\pi_n^\heartsuit(-)$ with $H(-)[n]$. □

This gives a commutative diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, "H"] & \mathcal{A} \\ \mathcal{L} \arrow[u] \arrow[r, "H"] & \mathcal{T} \arrow[u] \end{tikzcd}$$

where both of the fully faithful vertical functors have right adjoints. Note that the adjoint diagram might not commute.

Remark 3.3.24. In addition to being a localizing subcategory, we have by definition that we can check containment of $\mathcal{L}$ on the associated Grothendieck abelian category $\mathcal{T}$. This means that $\mathcal{L}$ also has a certain π-stability property, which one might call being H-stable, generalizing both Definition 3.3.2 and Takahashi's notion of H-stability.

3.3.2 Classification of localizing subcategories

The goal of this section is to prove Theorem 3.G, and that it interacts well with both Lurie's classification via prestable categories, and Theorem 3.F. As in Section 3.3.1 we start by proving that the wanted map of sets exists.

Lemma 3.3.25. *Let $\mathcal{C}$ be a t-stable category. If $\mathcal{L}$ is a t-exact localizing subcategory, then $\mathcal{L}^\heartsuit$ is an abelian localizing subcategory of $\mathcal{C}^\heartsuit$.*

Proof. The t-exact localization $L\colon \mathcal{C} \longrightarrow \mathcal{D}$ and its right adjoint i induces an adjunction

$$\mathcal{C}^\heartsuit \underset{\longleftarrow}{\xrightarrow{L^\heartsuit}} \mathcal{D}^\heartsuit$$

on the corresponding hearts. As L was t-exact, the functor $L^\heartsuit$ is exact. In particular, the heart $\mathcal{L}^\heartsuit$ is the kernel of $L^\heartsuit$, which by Proposition 3.2.44 means that $\mathcal{L}^\heartsuit$ is an abelian localizing subcategory of $\mathcal{C}^\heartsuit$. □

This means that we have a map

$$\left\{\begin{matrix} t\text{-exact localizing} \\ \text{subcategories of } \mathcal{C} \end{matrix}\right\} \xrightarrow{(-)^\heartsuit} \left\{\begin{matrix} \text{localizing} \\ \text{subcategories of } \mathcal{C}^\heartsuit \end{matrix}\right\}$$

Just as for the non-t-exact case, this map is not injective in general, meaning we have to restrict to a type of subcategory with more structure.

Definition 3.3.26. A localizing subcategory $\mathcal{L}$ of a t-stable category $\mathcal{C}$ is said to be a π-exact localizing subcategory if

1. it is π-stable, and
2. it is the kernel of a t-exact localization.

Remark 3.3.27. We continue our recurring remark about the dependencies of the different kinds of localizing subcategories introduced in the paper, see Remark 3.2.53 and Remark 3.3.7. We now have implications

$$\begin{array}{ccc} \pi\text{-exact} & \Longrightarrow & \pi\text{-stable} \\ \Downarrow & & \Downarrow \\ t\text{-exact} & \Longrightarrow & t\text{-stable} \end{array}$$

for any localizing subcategory $\mathcal{L}$ of a t-stable category $\mathcal{C}$.

Remark 3.3.28. The above remark also shows how the classification results are related. By Theorem 3.F we know that π-stable corresponds to abelian weak localizing subcategories, and by Corollary 3.2.54 we know that t-exact corresponds to prestable localizing subcategories. By Lurie's classification, see Theorem 3.3.34, we should expect the combination of the two to yield a correspondence between π-exact localizing subcategories and abelian localizing subcategories.

As π-exact localizing subcategories are by definition t-exact, we immediately get that the map $(-)^\heartsuit$ restricts to a map

$$\begin{Bmatrix} \pi\text{-exact localizing} \\ \text{subcategories of } \mathcal{C} \end{Bmatrix} \xrightarrow{(-)^\heartsuit} \begin{Bmatrix} \text{localizing} \\ \text{subcategories of } \mathcal{C}^\heartsuit \end{Bmatrix}$$

The claim of Theorem 3.G is that this map is a bijection.

The π-exact localizing subcategories are the stable analogs of Lurie's notion of separating prestable localizing subcategories, defined as follows.

Definition 3.3.29. Let $\mathcal{C}_{\geqslant 0}$ be a Grothendieck prestable category. A prestable localizing subcategory $\mathcal{L}_{\geqslant 0} \subseteq \mathcal{C}_{\geqslant 0}$ is *separating* if for every $X \in \mathcal{C}_{\geqslant 0}$ such that $\pi_n^\heartsuit X \in \mathcal{L}^\heartsuit$ for all n, then $X \in \mathcal{L}_{\geqslant 0}$.

What we mean by saying that these are the stable analogs, is that the bijection

$$\begin{Bmatrix} t\text{-exact localizing} \\ \text{subcategories of } \mathcal{C} \end{Bmatrix} \xrightarrow{(-)_{\geqslant 0}} \begin{Bmatrix} \text{localizing} \\ \text{subcategories of } \mathcal{C}_{\geqslant 0} \end{Bmatrix}$$

from Corollary 3.2.54 restricts to a bijection between π-exact stable localizing subcategories and separating prestable localizing subcategories. We prove this in two steps.

Lemma 3.3.30. *Let $\mathcal{C}$ be a t-stable category. If $\mathcal{L}$ is a π-exact localizing subcategory of $\mathcal{C}$, then $\mathcal{L}_{\geqslant 0}$ is a separating localizing subcategory of $\mathcal{C}_{\geqslant 0}$.*

Proof. By Corollary 3.2.54 we know that $\mathcal{L}_{\geqslant 0}$ is a prestable localizing subcategory of $\mathcal{C}_{\geqslant 0}$, so it remains to check that it is separating. Assume $X \in \mathcal{C}_{\geqslant 0}$ and $\pi_n^\heartsuit X \in \mathcal{L}^\heartsuit$ for all $n \geqslant 0$. Treating X as an object in $\mathcal{C}$ via the inclusion $\mathcal{C}_{\geqslant 0} \hookrightarrow \mathcal{C}$ we have $\pi_i^\heartsuit X = 0$ for all $i < 0$. Hence,by the assumption that $\mathcal{L}$ is π-stable, we must have $X \in \mathcal{L}$. This means that $X \in \mathcal{C}_{\geqslant 0} \cap \mathcal{L} = \mathcal{L}_{\geqslant 0}$, which finishes the proof. □

Lemma 3.3.31. *If $\mathcal{L}_{\geqslant 0}$ is a separating prestable localizing subcategory of $\mathcal{C}_{\geqslant 0}$, then $\mathrm{Sp}(\mathcal{L}_{\geqslant 0})$ is a π-exact localizing subcategory of $\mathcal{C}$.*

Proof. We know by Corollary 3.2.54 that $\mathrm{Sp}(\mathcal{L}_{\geqslant 0})$ is a t-exact localizing subcategory of $\mathcal{C}$, so it remains to show that it is π-stable.

For the sake of a contradiction, assume that there is some $X \in \mathcal{C}$ with $\pi_n^{\heartsuit} X \in \mathcal{L}^{\heartsuit}$ for all n, but $X \notin \mathcal{L}$. Using a suspension argument, it is enough to assume that X is not coconnective. As the corresponding localization functor $L\colon \mathcal{C} \longrightarrow \mathcal{D}$ is t-exact we get $L\tau_{\geqslant 0} X \simeq \tau_{\geqslant 0} LX$, which is by assumption non-zero, as X was not in $\mathcal{L}$. This means, however, that there is an object $Y = \tau_{\geqslant 0} X$ in $\mathcal{C}_{\geqslant 0}$ with $\pi_n^{\heartsuit} Y \in \mathcal{L}^{\heartsuit}$ but Y not in $\mathcal{L}_{\geqslant 0}$, which contradicts $\mathcal{L}_{\geqslant 0}$ begin separating. □

We are now ready to prove Theorem 3.G. As for Theorem 3.F we prove that the map $(-)^{\heartsuit}$ is both injective and surjective, starting with the former.

Lemma 3.3.32. *Let $\mathcal{C}$ be a t-stable category. Given two t-exact localizing subcategories $\mathcal{L}_0$ and $\mathcal{L}_1$, where $\mathcal{L}_1$ is π-exact, then we have $\mathcal{L}_0 \subseteq \mathcal{L}_1$ if and only if $\mathcal{L}_0^{\heartsuit} \subseteq \mathcal{L}_1^{\heartsuit}$.*

Proof. This immediately follows from the non-t-exact case from Lemma 3.3.9, as $\mathcal{L}_1$ is π-stable and $\mathcal{L}_0$ is t-stable. □

As before, this implies that the wanted map is injective.

Corollary 3.3.33. *For any t-stable category $\mathcal{C}$, the map*

$$\left\{\begin{matrix}\pi\text{-exact localizing} \\ \text{subcategories of } \mathcal{C}\end{matrix}\right\} \xrightarrow{(-)^{\heartsuit}} \left\{\begin{matrix}\text{localizing} \\ \text{subcategories of } \mathcal{C}^{\heartsuit}\end{matrix}\right\}$$

is injective.

It remains then to show that the map is also surjective. In order to do this we invoke Lurie's correspondence. The author originally wanted to have a proof not relying on the prestable case.

But, we currently do not know how to directly lift an abelian subcategory to a kernel of a t-exact functor, without passing through the bijection from Corollary 3.2.54. There is a more direct approach in certain contexts—for example if the t-structure is bounded, see [AGH19, 2.20], or the inclusion $\mathcal{L} \subseteq \mathcal{C}$ preserves compacts, see [HPV16, 2.7]—but as far as the author is aware, there is no general way to know when the localization determined by a localizing subcategory $\mathcal{L}$ is t-exact.

Theorem 3.3.34 ([Lur16, C.5.2.7])**.** *For any Grothendieck prestable category $\mathcal{C}_{\geqslant 0}$, there is a bijection*

$$\left\{ \begin{matrix} \textit{separating localizing} \\ \textit{subcategories of } \mathcal{C}_{\geqslant 0} \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} \textit{localizing} \\ \textit{subcategories of } \mathcal{C}^{\heartsuit} \end{matrix} \right\}$$

given by $\mathcal{L}_{\geqslant 0} \longmapsto \mathcal{L}^{\heartsuit}$.

Using this, together with Lemma 3.3.31 we finally get our wanted one-to-one correspondence.

Theorem 3.3.35 (Theorem 3.G)**.** *Let $\mathcal{C}$ be a t-stable category. There is a bijective map*

$$\left\{ \begin{matrix} \pi\textit{-exact localizing} \\ \textit{subcategories of } \mathcal{C} \end{matrix} \right\} \xrightarrow{(-)^{\heartsuit}} \left\{ \begin{matrix} \textit{localizing} \\ \textit{subcategories of } \mathcal{C}^{\heartsuit} \end{matrix} \right\}$$

given by $\mathcal{L} \longmapsto \mathcal{L}^{\heartsuit}$.

Proof. The map is injective by Corollary 3.3.33, so it remains only to show surjectivity. Let $\mathcal{A} \subseteq \mathcal{C}^{\heartsuit}$ be an abelian localizing subcategory. By Theorem 3.3.34 there is a unique separating prestable localizing subcategory $\mathcal{L}_{\geqslant 0} \subseteq \mathcal{C}_{\geqslant 0}$ such that $\mathcal{L}^{\heartsuit} \simeq \mathcal{A}$. By Lemma 3.3.31 the spectrum objects in this category, $\mathrm{Sp}(\mathcal{L}_{\geqslant 0})$ is a π-exact stable localizing subcategory of $\mathcal{C}$ with heart $\mathcal{A}$. Hence, the map is also surjective. $\square$

From this we again obtain some natural corollaries. The first one is a partial converse to [Tak09, 2.13].

Corollary 3.3.36. *Let R be a commutative noetherian ring and equip $\mathrm{D}(R)$ with its natural t-structure. In this situation there is a bijection between the collection of smashing localizing subcategories and the collection of π-exact localizing subcategories in $\mathrm{D}(R)$.*

Proof. A theorem of Gabriel, see [Gab62, VI.2.4(b)], shows that there is a bijection between the collection of localizing subcategories of Mod_R and specialization closed subsets of $\operatorname{Spec} R$. Further, Neeman shows in [Nee92, 3.3] that there is a bijection between specialization closed subsets of $\operatorname{Spec} R$ and smashing localizing subcategories of $\mathrm{D}(R)$. The result then follows from these, together with Theorem 3.3.35. $\square$

We can also obtain an extension of Corollary 3.3.36 to noetherian schemes X. Recall that we denote the abelian category of quasi-coherent sheaves on X by $\mathrm{QCoh}(X)$, and its associated derived category of quasi-coherent $\mathcal{O}_X$-modules by $\mathrm{D}_{qc}(X)$.

Lemma 3.3.37. *For any noetherian scheme X, there are bijections*

$$\left\{\begin{matrix}\textit{smashing subcategories}\\ \textit{of } \mathrm{D}_{qc}(X)\end{matrix}\right\} \simeq \left\{\begin{matrix}\textit{specialization closed}\\ \textit{subsets of } X\end{matrix}\right\} \simeq \left\{\begin{matrix}\textit{localizing subcategories}\\ \textit{of } \mathrm{QCoh}(X)\end{matrix}\right\}$$

Proof. The latter bijection is again due to Gabriel—[Gab62, VI.2.4(b)]. By [AJS04, 4.13] the telescope conjecture holds for noetherian schemes. In particular, this means that there is a bijection between subsets of X and localizing $\otimes$-ideals in $\mathrm{D}_{qc}(X)$, see [Ste13, 8.13], which restricts to a bijection

$$\left\{\begin{matrix}\text{smashing subcategories}\\ \text{of } \mathrm{D}_{qc}(X)\end{matrix}\right\} \simeq \left\{\begin{matrix}\text{specialization closed}\\ \text{subsets of } X\end{matrix}\right\},$$

giving the first bijection. $\square$

Utilizing this, together with Theorem 3.3.35, we obtain the following generalization.

Corollary 3.3.38. *Let X be a noetherian scheme and equip $\mathrm{D}_{qc}(X)$ with its natural t-structure. In this situation, there is a bijection*

$$\left\{\begin{matrix}\textit{smashing subcategories}\\ \textit{of } \mathrm{D}_{qc}(X)\end{matrix}\right\} \simeq \left\{\begin{matrix}\pi\textit{-exact localizing}\\ \textit{subcategories of } \mathrm{D}_{qc}(X)\end{matrix}\right\}.$$

Using Corollary 3.3.14 we then get a partial extension of the two bottom rows in the main result of [Tak09] to the case of noetherian schemes.

$$\begin{tikzcd}
\left\{\begin{matrix}\pi\text{-stable localizing}\\ \text{subcategories of } \mathrm{D}_{qc}(X)\end{matrix}\right\} \arrow[r, "\simeq"] \arrow[rr, bend left, "\simeq"] & \left\{\begin{matrix}\text{coherent subsets}\\ \text{of } X\end{matrix}\right\} \arrow[r, "\simeq"] & \left\{\begin{matrix}\text{weak localiz}\\ \text{subcategor}\\ \text{of QCoh(}X\end{matrix}\right. \\
\left\{\begin{matrix}\text{smashing subcategories}\\ \text{of } \mathrm{D}_{qc}(X)\end{matrix}\right\} \arrow[r, "\simeq"] \arrow[u, hook, "\subseteq"'] & \left\{\begin{matrix}\text{specialization closed}\\ \text{subsets of } X\end{matrix}\right\} \arrow[r, "\simeq"] \arrow[u, hook, "\subseteq"'] & \left\{\begin{matrix}\text{localizing}\\ \text{subcategor}\\ \text{of QCoh(}X\end{matrix}\right. \arrow[u, hook, "\subseteq"']
\end{tikzcd}$$

Here the grey color indicates the conjectured relationship from Conjecture 3.3.15.

We can also use the proof of the telescope conjecture for certain algebraic stacks, due to Hall–Rydh ([HR17]), to extend the above corollary even further. We leave the details of this to the interested reader.

By work of Kanda we can almost extend this to the locally noetherian setting. In particular, for X a locally noetherian scheme, Kanda proves in [Kan15, 1.4] that there is a bijection between localizing subcategories of $\mathrm{QCoh}(X)$ and specialization closed subsets of X. However, as the telescope conjecture is—to the best of our knowledge—currently unresolved for locally noetherian schemes, we do not get a bijection to smashing localizing subcategories. The best we can obtain is then the following corollary.

Corollary 3.3.39. *For X a locally noetherian scheme, there are bijections*

$$\left\{\begin{matrix}\pi\text{-}\textit{exact localizing}\\ \textit{subcategories of } \mathrm{D}_{qc}(X)\end{matrix}\right\} \simeq \left\{\begin{matrix}\textit{specialization closed}\\ \textit{subsets of } X\end{matrix}\right\} \simeq \left\{\begin{matrix}\textit{localizing subcategories}\\ \textit{of } \mathrm{QCoh}(X)\end{matrix}\right.$$

Remark 3.3.40. It would be very interesting to have a more direct proof for the fact that π-exact localizing subcategories of $\mathrm{D}(R)$ and $\mathrm{D}_{qc}(X)$ corresponds to smashing localizations. Having a direct proof would allow for a new proof of the telescope conjecture for commutative noetherian rings and noetherian schemes, and could shed some new light on the currently unsolved telescope conjecture for locally noetherian schemes.

Remark 3.3.41. We also want to highlight other work of Kanda, where he shows that localizing subcategories of a locally noetherian Grothendieck abelian category $\mathcal{A}$ are classified by the *atom spectrum* of $\mathcal{A}$, see [Kan12, 5.5]. It would be interesting to see if these atomic methods could provide new insight also into the stable ∞-category $\mathcal{C}$.

To summarize this section, we construct the bottom part of the diagram from the introduction. By Lemma 3.3.30 the bijection from Theorem 3.G factors through the bijection of Theorem 3.3.34. In particular, we get bijections

$$\left\{\begin{matrix}\pi\text{-exact localizing}\\ \text{subcategories of } \mathcal{C}\end{matrix}\right\} \underset{\mathrm{Sp}(-)}{\overset{(-)_{\geqslant 0}}{\rightleftarrows}} \left\{\begin{matrix}\text{separating localizing}\\ \text{subcategories of } \mathcal{C}_{\geqslant 0}\end{matrix}\right\} \xrightarrow{(-)_{\leqslant 0}} \left\{\begin{matrix}\text{localizing}\\ \text{subcategories of } \mathcal{C}\end{matrix}\right.$$

such that the composite map from the left to the right is the map $(-)^{\heartsuit}$ from Theorem 3.G. This finally gives the wanted diagram.

$$\begin{tikzcd}
\left\{\begin{matrix}\pi\text{-exact localizing}\\ \text{subcategories of } \mathcal{C}\end{matrix}\right\} \arrow[r, "\simeq"] \arrow[d, hook, "\subseteq"] & \left\{\begin{matrix}\text{separating localizing}\\ \text{subcategories of } \mathcal{C}_{\geqslant 0}\end{matrix}\right\} \arrow[r, "\simeq"] \arrow[d, hook, "\subseteq"] & \left\{\begin{matrix}\text{localizing}\\ \text{subcategories of } \mathcal{C}\end{matrix}\right. \\
\left\{\begin{matrix}t\text{-exact localizing}\\ \text{subcategories of } \mathcal{C}\end{matrix}\right\} \arrow[r, "(-)_{\geqslant 0}"] & \left\{\begin{matrix}\text{localizing}\\ \text{subcategories of } \mathcal{C}_{\geqslant 0}\end{matrix}\right\} \arrow[ur, "(-)_{\leqslant 0}"'] &
\end{tikzcd}$$

3.3.3 Comparing stable categories with the same heart

We round off the paper by proving some easy corollaries of Theorem 3.F and Theorem 3.G for stable categories with t-structures with the same heart. The first immediate corollary is the following.

Corollary 3.3.42. *Let $\mathcal{A}$ be any Grothendieck abelian category. For any two t-stable categories $\mathcal{C}$ and $\mathcal{D}$ with $\mathcal{C}^\heartsuit \simeq \mathcal{A} \simeq \mathcal{D}^\heartsuit$ there are one-to-one correspondences*

$$\left\{\begin{matrix} \pi\text{-stable localizing} \\ \text{subcategories of } \mathcal{C} \end{matrix}\right\} \longrightarrow \left\{\begin{matrix} \pi\text{-stable localizing} \\ \text{subcategories of } \mathcal{D} \end{matrix}\right\}$$

and

$$\left\{\begin{matrix} \pi\text{-exact localizing} \\ \text{subcategories of } \mathcal{C} \end{matrix}\right\} \longrightarrow \left\{\begin{matrix} \pi\text{-exact localizing} \\ \text{subcategories of } \mathcal{D} \end{matrix}\right\}.$$

The above correspondence might not be induced by a functor between $\mathcal{C}$ and $\mathcal{D}$, but is just an abstract isomorphism. However, in the case when there is a functor, the π-stable localizing subcategories are also functorially related. We can set this up as follows.

Lemma 3.3.43. *Let $\mathcal{C}$ and $\mathcal{D}$ be t-stable categories with $\mathcal{A}^\heartsuit \subseteq \mathcal{C}^\heartsuit$ and $\mathcal{T}^\heartsuit \subseteq \mathcal{D}^\heartsuit$ abelian weak localizing subcategories of the respective hearts. If there is a t-exact functor $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ such that the functor on hearts $F^\heartsuit\colon \mathcal{C}^\heartsuit \longrightarrow \mathcal{D}^\heartsuit$ restricts to a functor*

$$F^\heartsuit_{|\mathcal{A}^\heartsuit} : \mathcal{A}^\heartsuit \longrightarrow \mathcal{T}^\heartsuit,$$

then the functor F restricts to the unique π-stable localizing subcategories $F_{|\mathcal{A}}\colon \mathcal{A} \longrightarrow \mathcal{T}$.

Proof. As F is t-exact we have $F(\pi^\heartsuit_{\mathcal{C},n}X) \simeq \pi^\heartsuit_{\mathcal{D},n}F(X)$. By assumption we know that $F(\pi^\heartsuit_{\mathcal{C},n}X) \simeq F^\heartsuit(\pi^\heartsuit_{\mathcal{C},n}X) \in \mathcal{T}^\heartsuit$, hence any Y in the image of F has $\pi^\heartsuit_{\mathcal{D},n}Y \in \mathcal{T}^\heartsuit$ for any n. Since $\mathcal{T}$ is π-stable this implies that $Y \in \mathcal{T}$, proving the claim. □

Let $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ be a t-exact functor of t-stable categories such that the induced functor $F^\heartsuit\colon \mathcal{C}^\heartsuit \xrightarrow{\simeq} \mathcal{D}^\heartsuit$ is an equivalence. Assume further that $\mathcal{A}$ is an abelian weak localizing subcategory of $\mathcal{C}^\heartsuit$, and that $F^\heartsuit$ restricts to a functor $F^\heartsuit_{|\mathcal{A}}\colon \mathcal{A} \longrightarrow \mathcal{A}$. By Lemma 3.3.43 we get restricted functors $F_{|\mathcal{A}_\mathcal{C}}\colon \mathcal{A}_\mathcal{C} \longrightarrow \mathcal{A}_\mathcal{D}$, where $\mathcal{A}_\mathcal{C}$ and $\mathcal{A}_\mathcal{D}$ respectively denote the unique π-stable localizing subcategories of $\mathcal{C}$ and $\mathcal{D}$ obtained via Theorem 3.F.

Corollary 3.3.44. *If F is an equivalence, then every such restricted functor $F_{|\mathcal{A}_\mathcal{C}}$ is an equivalence.*

One interesting feature of the ∞-categorical framework is the existence of realization functors in reasonable generalities. If $\mathcal{C}$ is a t-stable category, then a realization functor for $\mathcal{C}$ is a functor $R\colon \mathrm{D}(\mathcal{C}^\heartsuit) \longrightarrow \mathcal{C}$, extending the inclusion of the heart. In particular, R restricts to the identity on $\mathrm{D}(\mathcal{C}^\heartsuit)^\heartsuit \simeq \mathcal{C}^\heartsuit$. These realization functors are rarely equivalences, even rarely full or faithful, but we can still apply Lemma 3.3.43 to functorially relate the π-stable localizing subcategories. Note that as R restricts to the identity in hearts, we do not even need to assume or prove that the functor R is t-exact, as the proof of Lemma 3.3.43 goes through regardless.

The following argument is due to Maxime Ramzi.

Lemma 3.3.45. *Let $\mathcal{C}$ be a t-stable category and $\mathrm{D}(\mathcal{C}^\heartsuit)$ the derived category of its heart. In this situation there is a realization functor $R\colon \mathrm{D}(\mathcal{C}^\heartsuit) \longrightarrow \mathcal{C}$ extending the inclusion $\mathcal{C}^\heartsuit \hookrightarrow \mathcal{C}$.*

Proof. The inclusion of the heart $\mathcal{C}^\heartsuit \hookrightarrow \mathcal{C}$ extends to a functor $\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{C}^\heartsuit) \longrightarrow \mathcal{C}$ via geometric realization, which preserves weak equivalences by [Lur17, 1.2.4.4, 1.2.4.5]. Via the Dold–Kan correspondence this gives a essentially unique colimit preserving functor $\mathrm{D}(\mathcal{C}^\heartsuit)_{\geqslant 0} \longrightarrow \mathcal{C}$, which extends uniquely to a functor $\mathrm{D}(\mathcal{C}^\heartsuit) \longrightarrow \mathcal{C}$ by [Lur17, 1.4.4.5], as $\mathcal{C}$ is stable. This functor preserves both colimits and the heart $\mathcal{C}^\heartsuit$. $\square$

We can then functorially relate the π-stable localizing subcategories of $\mathrm{D}(\mathcal{C}^\heartsuit)$ and $\mathcal{C}$ via the realization functor.

Corollary 3.3.46. *Let $\mathcal{C}$ be a t-stable category and let*

$$R\colon \mathrm{D}(\mathcal{C}^\heartsuit) \longrightarrow \mathcal{C}$$

be the realization functor from Lemma 3.3.45. For any weak localizing subcategory $\mathcal{A} \subseteq \mathcal{C}^\heartsuit$, the functor R restricts to a functor

$$R\colon \mathcal{A}_{\mathrm{D}(\mathcal{C}^\heartsuit)} \longrightarrow \mathcal{A}_{\mathcal{C}},$$

where the former category denotes the unique π-stable lift of $\mathcal{A}$ to $\mathrm{D}(\mathcal{C}^\heartsuit)$, and the latter the unique π-stable lift of $\mathcal{A}$ to $\mathcal{C}$.

Proof. This follows immediately from Lemma 3.3.43, the π-stability of $\mathcal{A}_{\mathcal{C}}$ and the fact that the identity restricts to the identity functor $\mathcal{A} \simeq \mathcal{A}^\heartsuit_{\mathrm{D}(\mathcal{C}^\heartsuit)} \longrightarrow \mathcal{A}^\heartsuit_{\mathcal{C}} \simeq \mathcal{A}$. □

Remark 3.3.47. By Proposition 3.3.6 the π-stable localizing subcategory $\mathcal{A}_{\mathcal{C}}$ is also t-stable, with heart $\mathcal{A}$. Hence, there is also a realization functor $R'\colon \mathrm{D}(\mathcal{A}) \longrightarrow \mathcal{A}_{\mathcal{C}}$, and a natural question to ask is wether this coincides with the above restricted functor $R\colon \mathcal{A}_{\mathrm{D}(\mathcal{C}^\heartsuit)} \longrightarrow \mathcal{A}_{\mathcal{C}}$. There is an inclusion $\mathrm{D}(\mathcal{A}) \subseteq \mathcal{A}_{\mathrm{D}(\mathcal{C}^\heartsuit)}$, as the latter is a π-stable localizing subcategory of $\mathrm{D}(\mathcal{C}^\heartsuit)$, but we do not know if this is always an equivalence. In particular, we don't know whether $\mathcal{D}(\mathcal{A})$, treated as a subcategory of $\mathrm{D}(\mathcal{C}^\heartsuit)$, is always a π-stable localizing subcategory.

Addendum. To add some credibility to the above remark we recall the localizing subcategory $\mathrm{Comod}_{E_*E}^{I_n-\mathrm{tors}}$ of I_n-torsion comodules in Comod_{E_*E}, that we studied in Chapter 1. We know that $\mathrm{D}(\mathcal{A})$ comes equipped with a natural t-structure for any Grothendieck abelian category $\mathcal{A}$, making $\mathrm{D}(\mathcal{A})$ into a t-stable category. In particular, this holds for $\mathrm{D}(\mathrm{Comod}_{E_*E})$. By Theorem 3.3.35 we know that there is a unique π-exact localizing subcategory $\mathcal{L} \subseteq \mathrm{D}(\mathrm{Comod}_{E_*E})$ such that $\mathcal{L}^\heartsuit \simeq \mathrm{Comod}_{E_*E}^{I_n-\mathrm{tors}}$. By [BHV20, 3.7(2)] this localizing subcategory is precisely

$$\mathrm{D}(\mathrm{Comod}_{E_*E})^{I_n-tors} \simeq \mathrm{D}(\mathrm{Comod}_{E_*E}^{I_n-\mathrm{tors}}),$$

as it is π-stable, hence equivalent to $\mathcal{L}$ by a maximality argument—see Lemma 3.3.9.

By Theorem 1.2.21 this example also holds for the periodic derived category $\mathrm{D}^{\mathrm{per}}(\mathrm{Comod}_{E_*E}^{I_n-\mathrm{tors}})$, which we also studied in Chapter 1.

3.A Addendum: Subcategories of synthetic spectra

One of the main motivations for Theorem 3.3.35 was to understand localizing subcategories of Pstrągowski's category of E-based synthetic spectra—the main reference is [Pst23]. This section is not part of the paper [Aam24b], but is added to flesh out this example further, and to relate this paper to Chapter 1.

We will focus only on the case of synthetic spectra based on height n Morava E-theory $E = E_{n,p}$ in this section, even though the results we cover will also work in more general setups. We will *not* use the standard category of synthetic spectra, but instead a "local" variant, that for familiar readers lie somewhere between E-based synthetic spectra and its hypercompletion.

Definition 3.A.1. A finite spectrum $P \in \mathrm{Sp}^{\omega}$ is said to *E-finite projective* if its E-homology is finitely generated and projective as an E_*-module. The full subcategory of E-finite projective spectra is denoted $\mathrm{Sp}^{\mathrm{fp}}$.

Remark 3.A.2. By Proposition 0.2.105, a finite spectrum P is E-finite projective if and only if E_*P is a dualizable comodule.

We can equip the category $\mathrm{Sp}^{\mathrm{fp}}$ with a Grothendieck topology, defined by covers being single maps $P \longrightarrow P'$ such that the induced map on E-homology is an epimorphism. This makes $\mathrm{Sp}^{\mathrm{fp}}$ an excellent ∞-site—see [Pst23, Section 2.3] for details.

Definition 3.A.3. An *synthetic spectrum* is an additive sheaf $X\colon \mathrm{Sp}^{\mathrm{fp,op}} \longrightarrow \mathrm{Sp}$. The category of synthetic spectra will be denoted $\mathrm{Syn}_E := \mathrm{P}_\Sigma(\mathrm{Sp}^{\mathrm{fp}}_{n,p}; \mathrm{Sp})$.

Localizing all of the finite projective spectra at E gives us another excellent ∞-site, which we denote by $\mathrm{Sp}^{\mathrm{fp}}_{n,p}$. To be clear, objects in $\mathrm{Sp}_{n,p}^{\mathrm{fp}}$ are of the form $L_{n,p}P$, for $P \in \mathrm{Sp}^{\mathrm{fp}}$.

Definition 3.A.4. An E-local synthetic spectrum is an additive sheaf $X\colon \mathrm{Sp}_{n,p}^{\mathrm{fp;op}} \longrightarrow \mathrm{Sp}$. The category of E-local synthetic spectra will be denoted $\mathrm{LSyn}_E := \mathrm{P}_\Sigma(\mathrm{Sp}_{n,p}^{\mathrm{fp}}; \mathrm{Sp})$.

Remark 3.A.5. As mentioned, the category of E-local synthetic spectra is slightly different from the categories already appearing in the literature. It should be thought of as living somewhere between normal synthetic spectra and hypercomplete synthetic spectra. This category is used to avoid the bad properties of the hypercompletion functor, which for example, is not smashing, meaning compact generation for hypercomplete synthetic spectra is a bit tricky.

Remark 3.A.6. A slightly more high-brow perspective on E-local synthetic spectra is the following. One reason normal synthetic spectra based on MU work so well is that the sphere $\mathbb{S}$ is MU-nilpotent complete, meaning that the Adams–Novikov spectral sequence converges to $\pi_*\mathbb{S}$. For E this is not the case, the sphere is not E-nilpotent complete. But, as E-localization is smashing, the E-nilpotent completion of the sphere is precisely the E-local sphere $L_{n,p}\mathbb{S}$. Hence one can think about LSyn_E as the natural category of formal "conditionally convergent" E-Adams spectral sequences, similarly to how $\mathrm{Syn}_{\mathrm{MU}}$ is the category of formal "conditionally convergent" Adams–Novikov spectral sequences.

Most of the theory for synthetic spectra work straight out of the box also for LSyn_E, and their proofs are completely analogous. We mention only the most important ones.

There is a map of excellent ∞-sites $f\colon \mathrm{Sp}^{\mathrm{fp}} \longrightarrow \mathrm{Sp}_{n,p}^{\mathrm{fp}}$ given by E-localizaton, which satisfies the covering lifting property of [Pst23, A.12]. Hence, the associated adjunction

$$f^* : \mathrm{Syn}_E \rightleftarrows \mathrm{LSyn}_E : f_*$$

has $f_*\colon \mathrm{LSyn}_E \longrightarrow \mathrm{Syn}_E$ a colimit preserving t-exact functor by [Pst23, 2.22, 2.23], given by precomposing with f. As E-

localization does not alter the E_*-homology of a spectrum, the t-structure on LSyn_E is also Comod_{E_*E}, and the functor f_* extends the identity on the level of hearts.

Similarily to [Pst23, 4.4, 4.38], there is a lax monoidal fully faithful functor $\nu\colon \mathrm{Sp}_{n,p} \longrightarrow \mathrm{LSyn}_E$ called the synthetic analog. The category LSyn_E is compactly generated by the objects $\nu L_{n,p}P$ for $P \in \mathrm{Sp}^{\mathrm{fp}}$, which are also dualizable, just as in [Pst23, 4.14]. The unit is also compact, which implies that LSyn_E is rigidly compactly generated.

The functor f^* is the unique colimit preserving functor such that $f^*\nu P \simeq \nu f P$, and as noted above, f_* is given by precomposition. As both f^* and f_* preserve colimits, we can easily check that f_* is fully faithful, via checking that the counit map $f^*f_*\nu P \longrightarrow \nu P$ is an equivalence. By the description of f^* above, this is obvious, as E-localization is an idempotent functor. It follows then, that LSyn_E is a smashing localization of Syn_E—the fact that f^* is compatible with the monoidal structure follows from the fact that it preserves colimits, and ν is symmetric monoidal on $\mathrm{Sp}^{\mathrm{fp}}$.

Remark 3.A.7. As the hypercompletion functor $(-)^\wedge$, which agrees with νE-localization by [Pst23, 5.4] on normal synthetic spectra is not smashing—a fact following from η not being nilpotent in the homotopy groups of spheres—this means that our category of E-local synthetic spectra is not equivalent to hypercomplete E-based synthetic spectra.

The functor ν induces a deformation parameter τ on any synthetic spectrum X, see [Pst23, Section 4.3], making LSyn_E act as a one-parameter deformation between $\mathrm{Sp}_{n,p}$ and Stable_{E_*E}, as in the following result.

Theorem 3.A.8. *Inverting the deformation parameter τ gives an equivalence*

$$\mathrm{LSyn}_E[\tau^{-1}] \simeq \mathrm{Sp}_{n,p}$$

of symmetric monoidal stable ∞-categories. Furthermore, killing τ, via tensoring with its cofiber, gives an equivalence

$$\mathrm{Mod}_{C\tau}(\mathrm{LSyn}_E) \simeq \mathrm{Stable}_{E_*E}$$

of symmetric monoidal stable ∞-categories.

Proof. The proof of the first equivalence is completely analogous to the setting of normal syntehtic spectra, see [Pst23, 4.37, 4.40], except that the associated spectral Yoneda embedding used to give the equivalence starts in E-local spectra, i.e., $Y\colon \mathrm{Sp}_{n,p} \longrightarrow \mathrm{LSyn}_E[\tau^{-1}]$.

For the second we note that E-localization does not alter the E_*-homology. Via [Pst23, 2.22, 4.43] this gives an adjunction

$$\mathrm{LSyn}_E \rightleftarrows \mathrm{Stable}_{E_*E},$$

which by following the rest of [Pst23, Section 4.5] is an equivalence. □

In a certain sense, this makes LSyn_E a categorification of the *conservative* adapted homology theory $E_{n*}\colon \mathrm{Sp}_{n,p} \longrightarrow \mathrm{Comod}_{E_*E}$, instead of normal synthetic spectra Syn_E being a categorification of $E_*\colon \mathrm{Sp} \longrightarrow \mathrm{Comod}_{E_*E}$. The main result for this addendum is to construct a categorification of the restricted adapted homology theory $E_*\colon \mathcal{M}_{n,p} \longrightarrow \mathrm{Comod}_{E_*E}^{I_n-\mathrm{tors}}$ that we studied in Chapter 1, with associated natural deformation properties as above.

Conjecture 3.A.9. The connected part of the t-structure on LSyn_E is equivalent to Patchkoria–Pstrągowski's separated perfect derived category $\mathrm{D}^\omega(\mathrm{Sp}_{n,p})$, see [PP21, 6.49]. We know this to be true at large primes $p-1>n$, due to the argument in [PP21, 6.57], together with the fact that the category Comod_{E_*E} has finite cohomological dimension in this range, implying that both the categories are already hypercomplete.

3.A.1 Monochromatic synthetic spectra

One of the reason we chose to work with $E = E_{n,p}$ specifically, rather than more general Adams type ring spectra, is that we have a very good understanding of localizing subcategories of the heart of the natural t-structure on LSyn_E. In fact, using the partial classification of localizing subcategories in $\mathrm{Comod}_{\mathrm{BP}_*\mathrm{BP}}$, due to Hovey–Strickland in [HS05a], Barthel and Heard was able to classify localizing subcategories in Comod_{E_*E} completely.

Proposition 3.A.10 ([BH18, 2.17])**.** *Let $\mathcal{T}$ be a localizing subcategory in a Grothendieck abelian category $\mathcal{A}$, and $\Psi\colon \mathcal{A} \longrightarrow \mathcal{A}/\mathcal{T}$ the associated Gabriel quotient. If S is a localizing subcategory in $\mathcal{A}/\mathcal{T}$, then there is a localizing subcategory $\overline{S}$ in $\mathcal{A}$ containing $\mathcal{T}$ such that $\Psi(\overline{S}) = S$.*

The above result is precisely what gives the classification of localizing subcategories of Comod_{E_*E}. Recall that for any $0 \leqslant k \leqslant n$ we have an ideal $I_k \subseteq \pi_* E_n$, called the Landweber ideals of E. More precisely these are given by $I_k = (p, v_1, v_2, \ldots, v_{k-1})$. These ideals are finitely generated regular invariant ideals, hence Section 0.2.4.1 gives us for any such k a localizing subcategory $\mathrm{Comod}_{E_*E}^{I_k-\mathrm{tors}} \subseteq \mathrm{Comod}_{E_*E}$, called the category of I_k-power torsion comodules.

Theorem 3.A.11 ([BH18, 2.21])**.** *If $\mathcal{T}$ is a localizing subcategory in Comod_{E_*E}, then there is an integer $0 \leqslant k \leqslant n$ such that $\mathcal{T} \simeq \mathrm{Comod}_{E_*E}^{I_k-\mathrm{tors}}$.*

Remark 3.A.12. By the above result we do, in fact, get a chain of localizing subcategories

$$\mathcal{T}_0 \subseteq \mathcal{T}_1 \subseteq \cdots \subseteq \mathcal{T}_n$$

in Comod_{E_*E}, corresponding each to one of the generators in the Landweber ideal $I_n = (p, v_1, v_2, \ldots, v_{n-1}) \subseteq \pi_* E$. Hence this result also classifies the localizing subcategories in the torsion categories $\mathrm{Comod}_{E_*E}^{I_k-\mathrm{tors}}$ themselves.

For simplicity we will focus only on the case when $k = n$, giving us the category $\mathrm{Comod}_{E_*E}^{I_n-\mathrm{tors}}$. Via the homology theory E_*, which the heart-valued homotopy groups in LSyn_E is supposed to generalize, we know that I_n-torsion comodules correspond to monochromatic spectra, as introduced in Section 0.2.3.3. Hence, we make the following definition.

Definition 3.A.13. The unique π-exact lift of the localizing subcategory $\mathrm{Comod}_{E_*E}^{I_n-\mathrm{tors}}$ is denoted $\mathcal{M}\mathrm{Syn}_E$. We call it the category of height n *monochromatic synthetic spectra.*

The justification for this name also in synthetic spectra is due to the following result.

Lemma 3.A.14. *For a spectrum $X \in \mathrm{Sp}_{n,p}$ we have $\nu X \in \mathcal{M}\mathrm{Syn}_E$ if and only if $X \in \mathcal{M}_{n,p}$.*

Proof. By definition we have

$$\nu X \in \mathcal{M}\mathrm{Syn}_E \iff \pi_k^{\heartsuit} \nu X \in \mathrm{Comod}_{E_*E}^{I_n-\mathrm{tors}} \text{ for all } k \in \mathbb{Z}.$$

By [Pst23, 4.21, 4.22] there is an isomorphism of E_*E-comodules $\pi_k^{\heartsuit} \nu X \simeq E_*X[-k]$, meaning that the E_*-homology of X is I_n-torsion. By Lemma 1.4.8 this is the case if and only if $X \in \mathcal{M}_{n,p}$, finishing the proof. $\square$

3.A.2 Compact generation

By our result Lemma 1.2.11 from Chapter 1 we know that the category $\mathrm{Comod}_{E_*E}^{I_n-\mathrm{tors}}$ is compactly generated, with an explicit set of compact generators given by

$$\mathrm{Tors}_{E_*E}^{\mathrm{fp}} := \{G \otimes E_*/I_n^k \mid G \in \mathrm{Comod}_{E_*E}^{\mathrm{fp}}, k \geqslant 1\}.$$

For the moment we want to avoid any near-lying problematic telescopes in E-local synthetic spectra, and prove that the unique lift of $\mathrm{Comod}_{E_*E}^{I_n-\mathrm{tors}}$—which was our definition of $\mathcal{M}\mathrm{Syn}_E$—is also a compactly generated localizing subcategory. This will require a more refined analysis compared to just lifting the localizing subcategories via Theorem 3.3.35.

Let us start with the prestable case. The functor

$$\pi_0 \colon \mathrm{LSyn}_{E,\geqslant 0} \longrightarrow \mathrm{Comod}_{E_*E}$$

preserves compact objects, so instead of pulling back to the whole category $\mathrm{LSyn}_{E,\geqslant 0}$ we can instead pull back to compact objects. This will not give us a prestable localizing subcategory, but a very similar category closed under finite colimits instead of all colimits.

Definition 3.A.15. A full subcategory $\mathcal{T}_{\geqslant 0} \subseteq \mathrm{LSyn}^{\omega}_{E,\geqslant 0}$ is *thick* if it is closed under finite coproducts, cofiber sequences and subobjects.

Remark 3.A.16. This is exactly the definition of a localizing subcategory of a prestable ∞-category, just with finite coproducts rather than all coproducts. This distinction is then similar to the distinction between localizing and Serre subcategories of abelian categories, and localizing vs. thick subcategories of stable ∞-categories.

We can further sharpen the analogy between Serre subcategories and thick subcategories.

Lemma 3.A.17. *If $\mathcal{T}^{\heartsuit}$ is a Serre subcategory of $\mathrm{Comod}^{\omega}_{E_*E}$, then the full subcategory $\mathcal{T}_{\geqslant 0} \subseteq \mathrm{LSyn}^{\omega}_{E,\geqslant 0}$ such that $t \in \mathcal{T}_{\geqslant 0}$ if and only if $\pi^{\heartsuit}_k t \in \mathcal{T}^{\heartsuit}$ for all $k \geqslant 0$, is a thick subcategory of $\mathrm{LSyn}^{\omega}_{E,\geqslant 0}$.*

Proof. The proof is identical to [Lur16, C.5.2.7], just with finite coproducts rather than all coproducts. □

Definition 3.A.18. The category $\mathcal{T}_{\geqslant 0}$ associated to a Serre subcategoru $\mathcal{T}$ is called the *prestable lift* of $\mathcal{T}$.

Definition 3.A.19. Similarly to Lurie's classification of abelian localizing subcategories, see Theorem 3.3.34, one gets a one-to-one correspondence between separating thick subcategories and Serre subcategories. Hence, the prestable lift is unique, as it is separating by definition.

After lifting to the prestable category, the next step—as before—is to stabilize. In order to do this in the case where we only have compact objects, we utilize another "small" stabilization instead of using $\mathrm{Sp}(-)$.

Definition 3.A.20. Let $\mathcal{E}$ be a pointed category with finite limits. The *Spanier–Whitehead category* of $\mathcal{E}$ is defined to be the colimit of the diagram

$$\mathcal{E} \xrightarrow{\Sigma} \mathcal{E} \xrightarrow{\Sigma} \mathcal{E} \xrightarrow{\Sigma} \cdots$$

where Σ is the functor given by the cofiber of the map $0 \to x$ for $x \in \mathcal{E}$.

The first thing we need is to compare prestable and stable compact objects.

Theorem 3.A.21. *The is an equivalence* $\mathrm{SW}(\mathrm{LSyn}^{\omega}_{E,\geqslant 0}) \simeq \mathrm{LSyn}^{\omega}_{E}$ *of symmetric monoidal stable ∞-categories.*

Proof. The category $\mathrm{LSyn}^{\omega}_{E,\geqslant 0}$ is a prestable category closed under finite limits, hence it is the connected part of a t-structure on some stable ∞-category, which is precicely the Spanier–Whitehead category $\mathrm{SW}(\mathrm{LSyn}^{\omega}_{E,\geqslant 0})$, see [Lur16, C.1.1, C.1.2].

By [Lur16, C.1.1.6] there is a commutative diagram of ∞-categories

$$\begin{tikzcd}
\mathrm{Cat}^{\mathrm{rex}}_{\infty} \arrow[r, "\mathrm{SW}(-)"] \arrow[d, "\mathrm{Ind}(-)"'] & \mathrm{Cat}^{\mathrm{rex}}_{\infty} \arrow[d, "\mathrm{Ind}(-)"] \\
\mathrm{Pr}^{L} \arrow[r, "\mathrm{Sp}(-)"'] & \mathrm{Pr}^{L}
\end{tikzcd}$$

meaning that there is an equivalence

$$\mathrm{Ind}(\mathrm{SW}(\mathrm{LSyn}^{\omega}_{E,\geqslant 0})) \simeq \mathrm{Sp}(\mathrm{Ind}(\mathrm{LSyn}^{\omega}_{E,\geqslant 0})).$$

As all functors are symmetric monoidal, the equivalence is also symmetric monoidal. The category $\mathrm{Ind}(\mathrm{LSyn}^{\omega}_{E,\geqslant 0})$ is $\mathrm{LSyn}_{E,\geqslant 0}$—as it is compactly generated—which we know stabilizes to LSyn_{E}. This category we know has a collection of compact generators, $\mathrm{LSyn}^{\omega}_{E}$, which is a small stable ∞-category, giving an equivalence $\mathrm{Ind}(\mathrm{LSyn}^{\omega}_{E}) \simeq \mathrm{LSyn}_{E}$ by definition. As the functor Ind is an equivalence between small stable ∞-categories and compactly generated ∞-categories, we get our wanted equivalence $\mathrm{SW}(\mathrm{LSyn}^{\omega}_{E,\geqslant 0}) \simeq \mathrm{LSyn}^{\omega}_{E}$. $\square$

This now allows us to finally define the lift of a Serre subcategory to the stable ∞-world.

Definition 3.A.22. Given a Serre subcategory $\mathcal{T}^\heartsuit$, we define its *stable lift* $\mathcal{T}$ to be the Spanier–Whitehead category of its prestable lift $\mathcal{T} := \mathrm{SW}(\mathcal{T}_{\geqslant 0})$.

Remark 3.A.23. Intuitively one should think about this as the "small" version of the construction from Chapter 3, where one lifts an abelian localizing subcategory through the t-structure by first lifting to the prestable category and then stabilizing. The Spanier–Whitehead construction is the natural version of stabilization for small categories, as is made clear by the commutative diagram in the above proof.

We have now defined our lift, and it remains to prove that it has the expected properties: it should in particular be a thick subcategory—in the stable sense.

Lemma 3.A.24. *Let $\mathcal{T}^\heartsuit \subseteq \mathrm{Comod}^\omega_{E_*E}$ be a Serre subcategory. The stable lift $\mathcal{T}$ is a thick subcategory of LSyn^ω_E.*

Proof. We have a fully faithful inclusion $\mathcal{T}_{\geqslant 0} \hookrightarrow \mathrm{LSyn}^\omega_{E,\geqslant 0}$, which gives a fully faithful inclusion

$$\mathcal{T} = \mathrm{SW}(\mathcal{T}_{\geqslant 0}) \hookrightarrow \mathrm{SW}(\mathrm{LSyn}^\omega_{E,\geqslant 0}) \simeq \mathrm{LSyn}^\omega_E$$

by Theorem 3.A.21. As $\mathcal{T}$ is a stable ∞-category by definition, we need only to check that it is closed under finite colimits in LSyn^ω_E.

Given a finite colimit in $\mathcal{T}$, it factors through $\mathcal{T}_{\geqslant 0}$ at some finite stage in the diagram

$$\mathcal{T}_{\geqslant 0} \xrightarrow{\Sigma} \mathcal{T}_{\geqslant 0} \xrightarrow{\Sigma} \mathcal{T}_{\geqslant 0} \xrightarrow{\Sigma} \cdots$$

As $\mathcal{T}_{\geqslant 0}$ is closed under finite colimits in $\mathrm{LSyn}^\omega_{E,\geqslant 0}$, together with the fact that

$$\begin{tikzcd}
\mathcal{T}_{\geqslant 0} \arrow[r, "\Sigma"] \arrow[d] & \mathcal{T}_{\geqslant 0} \arrow[d] \\
\mathrm{LSyn}^\omega_{E,\geqslant 0} \arrow[r, "\Sigma"'] & \mathrm{LSyn}^\omega_{E,\geqslant 0}
\end{tikzcd}$$

commutes, and lastly that all of the maps $\mathcal{T}_{\geqslant 0} \longrightarrow \mathrm{SW}(\mathcal{T}_{\geqslant 0}) \simeq \mathcal{T}$ and

$$\mathrm{LSyn}^{\omega}_{E,\geqslant 0} \longrightarrow \mathrm{SW}(\mathrm{LSyn}^{\omega}_{E,\geqslant 0}) \simeq \mathrm{LSyn}^{\omega}_{E}$$

preserve finite colimits—see [Lur16, C.1.1.5]—this implies that also the fully faithful inclusion $\mathcal{T} \subseteq \mathrm{LSyn}^{\omega}_{E}$ preserves finite colimits, finishing the proof. $\square$

We know that there is a unique π-exact lift of $\mathrm{Loc}(B)$ via Theorem 3.3.35, which we denote by $\mathcal{L}$. We now prove that this lift $\mathcal{L}$ is in fact uniquely determined by B.

Lemma 3.A.25. *For any Serre subcategory* $\mathcal{T}^{\heartsuit} \subseteq \mathrm{Comod}_{E_*E}$*, we have* $\mathcal{L} \cap \mathrm{LSyn}^{\omega}_{E} = \mathcal{T}$*.*

Proof. As $\mathcal{T}^{\heartsuit} \subseteq \mathrm{Loc}(\mathcal{T}^{\heartsuit})$ we also have $\mathcal{T} \subseteq \mathcal{L}$ by Lemma 3.3.9, as the latter is π-stable. This gives the first of the inclusions:

$$\mathcal{T} = \mathcal{T} \cap \mathrm{LSyn}^{\omega}_{E} \subseteq \mathcal{L} \cap \mathrm{LSyn}^{\omega}_{E}.$$

Let l be an object in $(\mathcal{L} \cap \mathrm{LSyn}^{\omega}_{E})_{\geqslant 0}$. This means that $l \in \mathcal{L}_{\geqslant 0}$ and $l \in \mathrm{LSyn}^{\omega}_{E,\geqslant 0}$. Hence, $\pi_k l \in \mathrm{Loc}(\mathcal{T}^{\heartsuit}) \cap \mathrm{Comod}^{\omega}_{E_*E} \simeq \mathcal{T}^{\heartsuit}$ for all $k \geqslant 0$, which by definition implies $l \in \mathrm{LSyn}^{\omega}_{E,\geqslant 0}$, giving

$$(\mathcal{L} \cap \mathrm{LSyn}^{\omega}_{E})_{\geqslant 0} \subseteq \mathcal{T}_{\geqslant 0}.$$

This gives the other inclusion upon taking Spanier–Whitehead categories. $\square$

For the uniqueness of $\mathcal{T}$ we will need the following lemma, stating that the lift of a compactly generated abelian localizing subcategory is a compactly generated stable localizing subcategory.

Lemma 3.A.26. *There is an equivalence of localizing subcategories* $\mathrm{Loc}(\mathcal{T}) = \mathcal{L}$*.*

Proof. By Lemma 3.A.25 these have the same compact objects, hence $\mathrm{Loc}(\mathcal{T}) \subseteq \mathcal{L}$. If we can prove that $\mathrm{Loc}(\mathcal{T})$ is a π-stable localizing subcategory with heart $\mathrm{Loc}(\mathcal{T}^{\heartsuit})$, then we are done by the uniqueness of the lift $\mathcal{L}$.

Now, $\mathcal{T}$ is π-stable, hence we have $\pi_k t \in \mathrm{Loc}(\mathcal{T}^\heartsuit)$ if and only if $t \in \mathrm{Loc}(\mathcal{T})$ for all compact t. We know that $\mathrm{Loc}(\mathcal{T})$ is generated by $\mathcal{T}$ under filtered colimits, which means, as π_0 preserves filtered colimits and $\mathrm{Loc}(\mathcal{T}^\heartsuit)$ is closed under these, that also $\pi_k t \in \mathrm{Loc}(\mathcal{T})$ if and only if $t \in \mathrm{Loc}(\mathcal{T})$ for all (not necessarily compact) objects t. It also follows from this that

$$\mathrm{Loc}(\mathcal{T})^\heartsuit = \mathrm{Loc}(\mathcal{T}^\heartsuit),$$

hence we get $\mathrm{Loc}(\mathcal{T}) = \mathcal{L}$ by uniqueness of the lift. □

We can now finally prove that the lift $\mathcal{T}$ is unique.

Theorem 3.A.27. *Given a Serre subcategory $\mathcal{T} \subseteq \mathrm{Comod}^\omega_{E_*E}$, the lift $\mathcal{T}$ is unique.*

Proof. Let $\mathcal{T}'$ be another stable lift of $\mathcal{T}^\heartsuit$, in other words it is a π-stable thick subcategory with heart $\mathcal{T}^\heartsuit$. By the same arguments as in Lemma 3.A.26, we get two π-stable localizing subcategories $\mathrm{Loc}(\mathcal{T})$ and $\mathrm{Loc}(\mathcal{T}')$, which necessarily must have the same heart $\mathrm{Loc}(\mathcal{T}^\heartsuit)$. By uniqueness of the lift $\mathcal{L}$ we must then have $\mathrm{Loc}(\mathcal{T}) = \mathcal{L} = \mathrm{Loc}(\mathcal{T}')$. By Lemma 3.A.25 we conclude that

$$\mathcal{T} = \mathrm{Loc}(\mathcal{T}) \cap \mathrm{LSyn}^\omega_E = \mathrm{Loc}(\mathcal{T}') \cap \mathrm{LSyn}^\omega_E = \mathcal{T}',$$

finishing the proof. □

As a consequence we get that the category of monochromatic synthetic spectra $\mathcal{M}\mathrm{Syn}_E$ is compactly generated, as it is equivalent to the category $\mathrm{Loc}(\mathcal{T})$ associated to the stable lift $\mathcal{T}$ of the Serre subcategory of compact objects in $\mathrm{Comod}^{I_n-\mathrm{tors}}_{E_*E}$.

Corollary 3.A.28. *The category of monochromatic synthetic spectra $\mathcal{M}\mathrm{Syn}_E$ is compactly generated.*

We now wish to find a good description of these compact generators. The category of monochromatic spectra $\mathcal{M}_{n,p}$ is compactly generated by the E_n-localization of any finite type n spectrum $F(n)$, see Definition 0.2.79 and Proposition 0.2.81. One natural guess for the compact generators of monochromatic synthetic spectra could then be to lift these to the synthetic setting.

Construction 3.A.29. By [Pst23, 4.23] we can lift the fiber sequence $L_{n,p}\mathbb{S} \xrightarrow{p} L_{n,p}\mathbb{S} \longrightarrow L_{n,p}\mathbb{S}/p$ to a fiber sequence

$$\nu L_{n,p}\mathbb{S} \xrightarrow{\widetilde{p}} \nu L_{n,p}\mathbb{S} \longrightarrow \nu(L_{n,p}\mathbb{S}/p)$$

in synthetic spectra LSyn_E, as it induces a short exact sequence

$$0 \longrightarrow E_* \xrightarrow{\cdot p} E_* \longrightarrow E_*/p \longrightarrow 0$$

on E_*-homology. In particular, $\nu(L_{n,p}\mathbb{S}/p) \simeq (\nu L_{n,p}\mathbb{S})/\widetilde{p}$. Similar ideas were used by Burklund to prove the existence of $\mathbb{E}_1$ structures on Moore spectra in [Bur22].

Now, as $\nu(L_{n,p}\mathbb{S}/p)$ is a finite number of cones away from the synthetic sphere, it is a compact object in LSyn_E. We can iterate this construction to lift generalized Moore spectra into the synthetic setting. These are then compact synthetic objects that behave similarily to finite type n spectra.

Lemma 3.A.30. *There is a finite type n spectrum $F(n)$, whose E-local synthetic analog $\nu L_{n,p}F(n)$ is compact.*

Proof. Let I_n be the Landweber ideal $(p, v_1, v_2, \ldots, v_{n-1})$. By [HS99, 4.14] there is a finite type n generalized Moore spectrum $L_{n,p}\mathbb{S}/J$ for $J = (p^{i_0}, v_1^{i_1}, \ldots, v_{n-1}^{i_{n-1}})$ constructed by iterated fiber sequences. These fiber sequences all induce short exact sequences on E_*-homology, hence we can lift them to fiber sequences in synthetic LSyn_E by [Pst23, 4.23]. In particular we have $\nu(L_{n,p}\mathbb{S}/J) \simeq (\nu L_{n,p}\mathbb{S})/\widetilde{J}$ for $\widetilde{J} = (\widetilde{p}^{i_0}, \widetilde{v}_1^{i_1}, \ldots, \widetilde{v}_{n-1}^{i_{n-1}})$. Since we used a finite number of shifts and fiber sequences, $\nu(L_{n,p}\mathbb{S}/J)$ is a compact object in LSyn_E. □

Definition 3.A.31. An E-local synthetic spectrum X is said to be of *synthetic type n*, if it is compact, and $X \simeq \nu L_{n,p}F(n)$ for a finite type n spectrum $F(n)$.

Our goal is to show that such a synthetic type n spectrum $\nu L_{n,p}F(n)$ does indeed generate $\mathcal{M}\mathrm{Syn}_E$ as a localizing subcategory.

Lemma 3.A.32. *There localizing subcategory $\mathcal{M}_{n,p}\mathrm{LSyn}_E$ is compactly generated by a synthetic type n spectrym. In other words, there is an equivalence $\mathrm{Loc}(\nu L_{n,p}F(n)) \simeq \mathcal{M}\mathrm{Syn}_E$ of stable ∞-categories.*

Proof. As the spectrum $L_{n,p}F(n)$ is monochromatic, we have by Lemma 3.A.14 that $\nu L_{n,p}F(n)$ lies in $\mathcal{M}\mathrm{Syn}_E$. In particular we have

$$\mathrm{Loc}(\nu L_{n,p}F(n)) \subseteq \mathcal{M}\mathrm{Syn}_E.$$

By [Nee92, 2.2] the compact objects in $\mathcal{M}\mathrm{Syn}_E$ are precisely those compact objects in LSyn_E that lie in $\mathcal{M}\mathrm{Syn}_E$. As we have shown that $\mathcal{M}\mathrm{Syn}_E$ is compactly generated, we must then have

$$\mathcal{M}\mathrm{Syn}_E \simeq \mathrm{Loc}(\{\nu M_{n,p}P\}),$$

for $P \in \mathrm{Sp}^{\mathrm{fp}}$ all the E-finite projective spectra. This is because $\nu L_{n,p}P$ compactly generate LSyn_E, and these lie in $\mathcal{M}\mathrm{Syn}_E$ precisely when $L_{n,p}P \in \mathcal{M}_{n,p}$, again by Lemma 3.A.14.

Now, as ν preserves filtered colimits we have

$$\nu\,\mathrm{Loc}(L_{n,p}F(n)) \subseteq \mathrm{Loc}(\nu L_{n,p}F(n)).$$

As $L_{n,p}F(n)$ generates $\mathcal{M}_{n,p}$ under filtered colimits, this implies that

$$\mathrm{Loc}(\{\nu M_{n,p}P\}) \subseteq \mathrm{Loc}\,\{\nu M\} \subseteq \mathrm{Loc}(\nu L\nu F(n)),$$

where $\mathrm{Loc}\,\{\nu M\}$ denotes the localizing subcategory generated by the synthetic analogs of all monochromatic spectra $M \in \mathcal{M}_{n,p}$. As we have shown that $\mathrm{Loc}(\{\nu M_{n,p}P\}) \simeq \mathcal{M}\mathrm{Syn}_E$, this finishes the proof. $\square$

3.A.3 Deformation properties

We now round off this addendum by showing that $\mathcal{M}\mathrm{Syn}_E$ has the desired deformation properties. First we need to show that it is in fact a localizing $\otimes$-ideal, and not just a localizing subcategory.

Lemma 3.A.33. *The synthetic type n spectrum $nuL_{n,p}F(n)$ generates the category $\mathcal{M}\mathrm{Syn}_E$ as a localizing $\otimes$-ideal. In other words, there is an equivalence $\mathcal{M}\mathrm{Syn}_E \simeq \mathrm{Loc}^{\otimes}(\nu L_{n,p}F(n))$ of symmetric monoidal stable ∞-categories.*

Proof. By Lemma 3.A.32 it is enough to show that $\mathcal{M}\mathrm{Syn}_E$ is closed under tensoring with objects of LSyn_E. As the tensor product commutes with colimits separately in each variable, and the category $\mathcal{M}\mathrm{Syn}_E$ is closed under colimits, it is enough to check this on generators of LSyn_E and $\mathcal{M}\mathrm{Syn}_E$, namely $\nu L_{n,p}P$ and $\nu L_{n,p}F(n)$ respectively. By [Pst23, 4.24] we have an equivalence

$$\nu L_{n,p}P \otimes \nu L_{n,p}F(n) \simeq \nu(L_{n,p}P \otimes L_{n,p}F(n)).$$

As $\mathcal{M}_{n,p}$ is a localizing ideal of $\mathrm{Sp}_{n,p}$, the spectrum $L_{n,p}P \otimes L_{n,p}F(n)$ is monochromatic. This means that its synthetic analog is a synthetic monochromatic spectrum by Lemma 3.A.14. $\square$

The computations for the generic and special fibers of the deformation parameter τ now follow quite easily from the results in Section 1.A. Let us start with the special fibre.

Recall the definition of the derived I_n-torsion stable category of comodules in [BHV20, 2.4] as $\mathrm{Stable}_{E_*E}^{I_n-\mathrm{tors}} := \mathrm{Loc}^{\otimes}(E_*/I_n)$ as a localizing subcategory of Stable_{E_*E}. We furthermore recall the definition of the stable category of I_n-power torsion comodules, see [BHV20, 3.5], as

$$\mathrm{Stable}(\mathrm{Comod}_{E_*E}^{I_n-\mathrm{tors}}) := \mathrm{Ind}(\mathrm{Thick}^{\otimes}(\mathrm{Tors}_{E_*E}^{\mathrm{fp}}),$$

where $\mathrm{Tors}_{E_*E}^{\mathrm{fp}}$ denotes a collection of compact generators of $\mathrm{Comod}_{E_*E}^{I_n-\mathrm{tors}}$, see Lemma 1.2.11.

Theorem 3.A.34. *There is an equivalence*

$$\mathrm{Mod}_{C\tau}(\mathcal{M}\mathrm{Syn}_E) \simeq \mathrm{Stable}(\mathrm{Comod}_{E_*E}^{I_n-\mathrm{tors}})$$

of symmetric monoidal stable ∞-categories.

Proof. By Theorem 3.A.8 there is a monoidal Barr–Beck adjunction

$$\mathrm{LSyn}_E \rightleftarrows \mathrm{Stable}_{E_*E}.$$

As $\nu X \otimes C\tau \simeq E_*X$, we get a local duality adjunction

$$(\mathrm{LSyn}_E, \nu L_{n,p}F(n)) \rightleftarrows (\mathrm{Stable}_{E_*E}, E_*F(n)).$$

By Theorem 1.A.6 there is an induced monoidal Barr–Beck adjunction

$$\mathrm{Loc}^{\otimes}(\nu L_{n,p}F(n)) \rightleftarrows \mathrm{Loc}^{\otimes}(E_*F(n)).$$

The left hand side is equivalent to $\mathcal{M}\mathrm{Syn}_E$ by Lemma 3.A.32, so we need only to identify the right. The localizing ideal is only dependent on the radical of the ideal J used to construct the type n synthetic spectrum $F(n)$, and the radical of J is equivalent to the radical of I_n, meaning that the right hand side can be identified with E_*/I_n, which gives

$$\mathrm{Loc}^{\otimes}(E_*/I_n) \simeq \mathrm{Stable}_{E_*E}^{I_n-\mathrm{tors}} \simeq \mathrm{Stable}(\mathrm{Comod}_{E_*E}^{I_n-\mathrm{tors}}),$$

where the last equivalence is due to [BHV20, 3.17]. Hence, as the above adjunction is Barr–Beck, we get

$$\mathrm{Mod}_{C\tau}(\mathcal{M}\mathrm{Syn}_E) \simeq \mathrm{Stable}(\mathrm{Comod}_{E_*E}^{I_n-\mathrm{tors}}),$$

finishing the proof. □

The computation of the generic fibre is very similar to Theorem 3.A.34, but we include it for completion.

Theorem 3.A.35. *Inverting the deformation parameter τ gives an equivalence $\mathcal{M}\mathrm{Syn}_E[\tau^{-1}] \simeq \mathcal{M}_{n,p}$ of symmetric monoidal ∞-categories.*

Proof. It follows from Theorem 3.A.8 that there is a local duality adjunction

$$(\mathrm{LSyn}_E, \nu L_{n,p}F(n)) \rightleftarrows (\mathrm{Sp}_{n,p}, \tau^{-1}\nu L_{n,p}F(n))$$

which by Theorem 1.A.6 induces a Barr–Beck adjunction on the respective localizing $\otimes$-ideals. By [Pst23, 4.40] there is an equivalence $\tau^{-1}\nu X \simeq X$, hence we get a monoidal Barr–Beck adjunction

$$\mathrm{Loc}^{\otimes}(\nu L_{n,p}F(n)) \rightleftarrows \mathrm{Loc}^{\otimes}(L_{n,p}F(n)).$$

The former is again $\mathcal{M}\mathrm{Syn}_E$, and the latter is $\mathcal{M}_{n,p}$. As the adjunction is Barr–Beck we get

$$\mathcal{M}\mathrm{Syn}_E[\tau^{-1}] \simeq \mathcal{M}_{n,p},$$

just as wanted. $\square$

This proves in essence that $\mathcal{M}\mathrm{Syn}_E$ is the correct deformation underlying the adapted homology theory

$$E_*\colon \mathcal{M}_{n,p} \longrightarrow \mathrm{Comod}_{E_*E}^{I_n-\mathrm{tors}},$$

showing that the deformation theory plays well with the classification result for π-exact localizing subcategories in Theorem 3.3.35.

Bibliography

[Aam24a] Torgeir Aambø. “Algebraicity in monochromatic homotopy theory”. In: *To appear in Algebraic & Geometric topology* (2024). arXiv:2403.07725 (Page v).

[Aam24b] Torgeir Aambø. *Classification of localizing subcategories along t-structures.* 2024. arXiv: 2412.09391 [math.AT] (Pages v, 71, 181).

[Aam24c] Torgeir Aambø. *Positselski duality in stable ∞-categories.* 2024. arXiv: 2411.04060 [math.AT] (Pages v, 132).

[Ada58] John F. Adams. “On the structure and applications of the Steenrod algebra”. In: *Commentarii Mathematici Helvetici* 32.1 (1958), pp. 180–214 (Page 38).

[Ada74] John F. Adams. *Stable Homotopy and Generalised Homology.* Chicago Lectures in Mathematics. ISBN: 978-0-226-00524-9. Chicago, IL: University of Chicago Press, 1974 (Page 32).

[AJS04] Leovigildo Alonso Tarrío, Ana Jeremías López, and María José Souto Salorio. “Bousfield localization on formal schemes”. In: *Journal of Algebra* 278.2 (2004), pp. 585–610 (Page 175).

[Ang11] Vigleik Angeltveit. “Uniqueness of Morava K-theory”. In: *Compositio Mathematica* 147.2 (2011), pp. 633–648 (Page 22).

[Ant21] Benjamin Antieau. *On the uniqueness of infinity-categorical enhancements of triangulated categories.* 2021. arXiv: 1812.01526 [math.AG] (Page 153).

[AGH19] Benjamin Antieau, David Gepner, and Jeremiah Heller. "K-theoretic obstructions to bounded t-structures". In: *Inventiones Mathematicae* 216 (2019), pp. 241–300 (Pages 162, 174).

[Baa73a] Nils A. Baas. "On bordism theory of manifolds with singularities". In: *Mathematica Scandinavica* 33.2 (1973), pp. 279–302 (Page 24).

[Baa73b] Nils A. Baas. "On formal groups and singularities in complex cobordism theory". In: *Mathematica Scandinavica* 33.2 (1973), pp. 303–313 (Page 24).

[Bak09] Andrew Baker. "L-complete Hopf algebroids and their comodules". In: *Contemporary Mathematics* 504 (2009), pp. 1–22 (Pages 41, 75).

[BS17] Paul Balmer and Beren Sanders. "The spectrum of the equivariant stable homotopy category of a finite group". In: *Inventiones Mathematicae* 208.1 (2017), pp. 283–326 (Page 94).

[Bar23] Shaul Barkan. *Chromatic Homotopy is Monoidally Algebraic at Large Primes*. 2023. arXiv: `2304.14457 [math.AT]` (Pages 26, 78, 92).

[BR11] David Barnes and Constanze Roitzheim. "Monoidality of Franke's exotic model". In: *Advances in mathematics* 228.6 (2011), pp. 3223–3248 (Pages 38, 65, 69, 76).

[BB19] Tobias Barthel and Agnès Beaudry. "Chromatic structures in stable homotopy theory". In: *Handbook of Homotopy Theory*. Chapman and Hall/CRC, 2019 (Pages 17, 66, 128).

[BF15] Tobias Barthel and Martin Frankland. "Completed power operations for Morava E-theory". In: *Algebraic & Geometric Topology* 15.4 (2015), pp. 2065–2131 (Page 79).

[BH16] Tobias Barthel and Drew Heard. “The E_2-term of the $K(n)$-local E_n-Adams spectral sequence”. In: *Topology and its Applications* 206 (2016), pp. 190–214 (Page 81).

[BH18] Tobias Barthel and Drew Heard. “Algebraic chromatic homotopy theory for BP_*BP-comodules”. In: *Proceedings of the London Mathematical Society* 117.6 (2018). arXiv:1708.09261 [math], pp. 1135–1180 (Page 185).

[BHV18] Tobias Barthel, Drew Heard, and Gabriel Valenzuela. “Local duality in algebra and topology”. In: *Advances in Mathematics* 335 (2018), pp. 563–663 (Pages 11, 15, 20, 30, 35, 39, 41–43, 63, 64, 66, 67, 79, 86, 107, 124, 125, 127, 129, 131, 136, 163).

[BHV20] Tobias Barthel, Drew Heard, and Gabriel Valenzuela. “Derived completion for comodules”. In: *Manuscripta Mathematica* 161.3-4 (2020), pp. 409–438 (Pages 39, 42–44, 68, 129, 131, 164, 180, 194, 195).

[BP23] Tobias Barthel and Piotr Pstrągowski. “Morava K-theory and filtrations by powers”. In: *Journal of the Institute of Mathematics of Jussieu* (2023), pp. 1–77 (Pages 25, 74).

[BSS20] Tobias Barthel, Tomer M. Schlank, and Nathaniel Stapleton. “Chromatic homotopy theory is asymptotically algebraic”. In: *Inventiones Mathematicae* 220.3 (2020), pp. 737–845 (Pages 60, 61, 78).

[BSS21] Tobias Barthel, Tomer M. Schlank, and Nathaniel Stapleton. “Monochromatic homotopy theory is asymptotically algebraic”. In: *Advances in Mathematics* 393 (2021) (Pages 60, 61, 70, 78, 90).

[Bar+24] Tobias Barthel, Tomer M. Schlank, Nathaniel Stapleton, and Jared Weinstein. *On the rationalization of the $K(n)$-local sphere.* 2024. arXiv: `2402.00960 [AT]` (Page 75).

[BS20] Mark Behrens and Jay Shah. “C_2-equivariant stable homotopy from real motivic stable homotopy”. In: *Annals of K-Theory* 5.3 (2020), pp. 411–464 (Pages 94, 96).

[BBD82] Alexander Beilinson, Joseph Bernstein, and Pierre Deligne. “Faisceaux pervers”. In: *Astérisque* 100 (1982) (Pages 146, 150, 157).

[Bou79a] Aldridge K. Bousfield. “The Boolean algebra of spectra”. In: *Commentarii Mathematici Helvetici* 54.1 (1979), pp. 368–377 (Page 28).

[Bou79b] Aldridge K. Bousfield. “The localization of spectra with respect to homology”. In: *Topology* 18.4 (1979), pp. 257–281 (Pages 18, 20, 66).

[Bou85] Aldridge K. Bousfield. “On the homotopy theory of K-local spectra at an odd prime”. In: *American Journal of Mathematics* 107.4 (1985), pp. 895–932 (Page 59).

[Bou96] Aldridge K. Bousfield. “Unstable localization and periodicity”. In: *Algebraic Topology: New Trends in Localization and Periodicity*. Basel: Birkhäuser Basel, 1996, pp. 33–50 (Page 28).

[BM23] Lukas Brantner and Akhil Mathew. *Deformation Theory and Partition Lie Algebras*. 2023. arXiv: `1904.07352 [math.AG]` (Pages 115, 116).

[BS12] Markus Brodmann and Rodney Sharp. *Local Cohomology: An Algebraic Introduction with Geometric Applications*. 2nd ed. Cambridge Studies in Advanced Mathematics. ISBN: 978-0-521-51363-0. Cambridge: Cambridge University Press, 2012 (Pages 42, 79–81, 87).

[Bro62] Edgar H. Brown. “Cohomology theories”. In: *Annals of Mathematics* 75.3 (1962), pp. 467–484 (Page 24).

[Brz10] Tomasz Brzeziński. “The structure of corings: Induction functors, Maschke-type theorem, and Frobenius and Galois-type properties”. In: *Algebras and representation theory* 5 (2010), pp. 389–410 (Page 120).

[Bur22] Robert Burklund. *Multiplicative structures on Moore spectra*. 2022. arXiv: `2203.14787 [math.AT]` (Pages 137, 192).

[Bur+23] Robert Burklund, Jeremy Hahn, Ishan Levy, and Tomer M. Schlank. *K-theoretic counterexamples to Ravenel's telescope conjecture*. 2023. arXiv: `2310.17459 [math.AT]` (Pages 20, 129).

[Che24] Fei Yu Chen. *Costability of Comodules*. 2024. arXiv: `2404.08082 [math.CT]` (Page 136).

[Chr23] Merlin Christ. “Spherical monadic adjunctions of stable infinity categories”. In: *International Mathematics Research Notices* 2023 (2023) (Pages 111–113).

[DL14] Daniel G. Davis and Tyler Lawson. “Commutative ring objects in pro-categories and generalized Moore spectra”. In: *Geometry and Topology* 18 (2014), pp. 103–140 (Page 137).

[DH04] Ethan S. Devinatz and Michael J. Hopkins. “Homotopy fixed point spectra for closed subgroups of the Morava stabilizer groups”. In: *Topology* 43.1 (2004) (Page 26).

[Div79] Joy Division. *Unknown Pleasures*. Factory Records. 1979 (Page v).

[Dyc+24] Tobias Dyckerhoff, Mikhail Kapranov, Vadim Schechtman, and Yan Soibelman. “Spherical adjunctions of stable ∞-categories and the relative S-construction”. In: *Mathematische Zeitschrift* 307.4 (2024), p. 73 (Page 128).

[EM65] Samuel Eilenberg and John Moore. “Foundations of relative homological algebra”. In: *Memoirs of the American Mathematical Society* 55 (1965) (Pages 47, 104).

[EK20] Elden Elmanto and Håkon Kolderup. “On modules over motivic ring spectra”. In: *Annals of K-theory* 5 (2020), pp. 327–355 (Pages 24, 93, 115, 116).

[Fra96] Jens Franke. “Uniqueness for certain categories with an Adams SS”. In: *K-theory preprint archives* 139 (1996) (Pages 45, 59, 69, 72, 76, 87).

[Gab62] Pierre Gabriel. “Des catégories abéliennes”. In: *Bulletin de la Société Mathématique de France* 90 (1962), pp. 323–448 (Pages 167, 175).

[GH04] Paul G. Goerss and Michael J. Hopkins. “Moduli spaces of commutative ring spectra”. In: *Structured ring spectra*. London Mathematical Society Lecture Note Series. Cambridge: Cambridge University Press, 2004, pp. 151–200 (Page 25).

[GM92] John P. C. Greenlees and Jon P. May. “Derived functors of I-adic completion and local homology”. In: *Journal of Algebra* 149.2 (1992), pp. 438–453 (Pages 40, 81, 130, 163).

[GM95] John P. C. Greenlees and Jon P. May. “Completions in algebra and topology”. In: *Handbook of algebraic topology*. North-Holland, Amsterdam, 1995, pp. 255–276 (Page 66).

[GS13] John P. C. Greenlees and Brooke Shipley. “The Cellularization Principle for Quillen adjunctions”. In: *Homology, Homotopy and Applications* 15.2 (2013), pp. 173–184 (Page 96).

[HR17] Jack Hall and David Rydh. “The telescope conjecture for algebraic stacks”. In: *Journal of Topology* 10.3 (2017), pp. 776–794 (Page 176).

[HPV16] Benjamin Hennion, Mauro Porta, and Gabriele Vezzosi. *Formal gluing along non-linear flags*. 2016. arXiv: `1607.04503 [math.AG]` (Page 174).

[Hol20] Maximilien Thomas Holmberg-Peroux. “Highly Structured Coalgebras and Comodules”. In: (2020). PhD thesis, University of Illinois (Page 118).

[HL17] Michael J. Hopkins and Jacob Lurie. *On Brouer groups of Lubin–Tate spectra I*. Available at the authors website. 2017 (Pages 25, 91).

[HS98] Michael J. Hopkins and Jeffrey H. Smith. “Nilpotence and Stable Homotopy Theory II”. In: *Annals of Mathematics* 148.1 (1998), pp. 1–49 (Page 28).

[Hov95] Mark Hovey. “Bousfield localization functors and Hopkins’ chromatic splitting conjecture”. In: *The ech centennial (Boston, MA, 1993)*. Vol. 181. Contemporary Mathematics. Amer. Math. Soc., Providence, RI, 1995, pp. 225–250 (Pages 26, 84).

[Hov04] Mark Hovey. “Homotopy theory of comodules over a Hopf algebroid”. In: *Homotopy theory: relations with algebraic geometry, group cohomology, and algebraic K-theory*. Vol. 346. Contemporary mathematics. Providence, RI: American Mathematical Society, 2004, pp. 261–304 (Pages 32, 34, 35, 37, 38, 65, 68).

[HPS97] Mark Hovey, John H. Palmieri, and Neil P. Strickland. *Axiomatic Stable Homotopy Theory*. ISBN: 978-0-8218-0624-1. American Mathematical Society, 1997 (Pages 7, 11, 15, 63, 64, 107, 124, 126).

[HS99] Mark Hovey and Neil P. Strickland. *Morava K-theories and localisation*. Vol. 139. Memoirs of the American Mathematical Society. ISBN: 978-0-8218-1079-8. American Mathematical Society, 1999 (Pages 21, 22, 29, 30, 58, 66, 81, 129, 137–139, 192).

[HS05a] Mark Hovey and Neil P. Strickland. “Comodules and Landweber exact homology theories”. In: *Advances in Mathematics* 192.2 (2005), pp. 427–456 (Pages 36, 84, 87, 185).

[HS05b] Mark Hovey and Neil P. Strickland. “Local cohomology of BP_*BP-comodules”. In: *Proceedings of the London Mathematical Society* 90.2 (2005), pp. 521–544 (Pages 37, 86, 87).

[HJR23] Katerina Hristova, John Johes, and Dmitriy Rumynin. “General comodule-contramodule correspondence”. In: *São Paulo Journal of Mathematical Sciences* (2023) (Page 113).

[Ish19] Jocelyne Ishak. “Rigidity of the $K(1)$-local stable homotopy category”. In: *Homology, Homotopy and Applications* 21.2 (2019), pp. 261–278 (Pages 60, 78).

[IRW23] Jocelyne Ishak, Constanze Roitzheim, and Jordan Williamson. “Levels of algebraicity in stable homotopy theories”. In: *Journal of the London Mathematical Society* 108.2 (2023), pp. 545–577 (Page 78).

[JW75] David C. Johnson and Walter S. Wilson. “BP operations and Morava’s extraordinary K-theories”. In: *Mathematische Zeitschrift* 144.1 (1975), pp. 55–75 (Page 24).

[Joy02] André Joyal. “Quasi-categories and Kan complexes”. In: *Journal of Pure and Applied Algebra*. Special volume celebrating the 70th birthday of professor Max Kelly 175.1 (2002), pp. 207–222 (Pages 5, 62).

[Kan12] Ryo Kanda. “Classifying Serre subcategories via atom spectrum”. In: *Advances in Mathematics* 231.3 (2012), pp. 1572–1588 (Page 177).

[Kan15] Ryo Kanda. “Classification of categorical subspaces of locally noetherian schemes”. In: *Documenta Mathematica* 20 (2015), pp. 1403–1465 (Page 176).

[Kra08] Henning Krause. “Thick subcategories of modules over commutative noetherian rings (with an appendix by Srikanth Iyengar)”. In: *Mathematische Annalen* 340.4 (2008), pp. 733–747 (Page 167).

[Lan76] Peter S. Landweber. "Homological Properties of Comodules Over $MU^*(MU)$ and $BP^*(BP)$". In: *American Journal of Mathematics* 98.3 (1976), pp. 591–610 (Page 24).

[Lef42] Solomon Lefschetz. *Algebraic topology*. American Mathematical Society, 1942 (Page 136).

[LZ23] Guchuan Li and Ningchuan Zhang. *The inverse limit topology and profinite descent on Picard groups in $K(n)$-local homotopy theory*. 2023. arXiv: `2309.05039 [math.AT]` (Pages 137, 138).

[LT66] Jonathan Lubin and John Tate. "Formal moduli for one-parameter formal Lie groups". In: *Bulletin de la Société Mathématique de France* 94 (1966), pp. 49–59 (Page 23).

[Lur09] Jacob Lurie. *Higher topos theory*. ISBN: 978-0-691-14049-0. Princeton University Press, 2009 (Pages 5, 12, 15, 62, 63, 105, 110, 119, 122, 124, 135, 148).

[Lur16] Jacob Lurie. *Spectral algebraic geometry*. Available at the authors website. 2016 (Pages 147, 148, 150–152, 154, 156, 158–160, 165, 166, 174, 187, 188, 190).

[Lur17] Jacob Lurie. *Higher algebra*. Available at the authors website. 2017 (Pages 5, 12, 13, 44, 62–64, 84, 93, 105, 109–111, 114, 117, 119, 136, 148–150, 179).

[Lur18a] Jacob Lurie. *Elliptic cohomology I: Spectral abelian varieties*. 2018 (Pages 110, 134).

[Lur18b] Jacob Lurie. *Elliptic cohomology II: Orientations*. 2018 (Page 134).

[MM94] Saunders MacLane and Ieke Moerdijk. *Sheaves in geometry and logic: A first introduction to topos theory*. Universitext. Springer Verlag, 1994 (Page 113).

[MNN17] Akhil Mathew, Niko Naumann, and Justin Noel. "Nilpotence and descent in equivariant stable homotopy theory". In: *Advances in Mathematics* 305 (2017), pp. 994–1084 (Pages 93, 115).

[Mil58] John Milnor. "The Steenrod Algebra and Its Dual". In: *The Annals of Mathematics* 67.1 (Jan. 1958), pp. 150–171 (Page 33).

[Nau07] Niko Naumann. "The stack of formal groups in stable homotopy theory". In: *Advances in Mathematics* 215.2 (2007), pp. 569–600 (Page 36).

[Nee92] Amnon Neeman. "The connection between the \$K\$-theory localization theorem of Thomason, Trobaugh and Yao and the smashing subcategories of Bousfield and Ravenel". In: *Annales scientifiques de l'École Normale Supérieure* 25.5 (1992), pp. 547–566 (Pages 175, 193).

[Nik16] Thomas Nikolaus. *Stable ∞-Operads and the multiplicative Yoneda lemma*. 2016. arXiv: `1608.02901 [math.AT]` (Page 122).

[Pat12] Irakli Patchkoria. "On the algebraic classification of module spectra". In: *Algebraic & Geometric Topology* 12.4 (2012), pp. 2329–2388 (Page 59).

[PP21] Irakli Patchkoria and Piotr Pstrągowski. *Adams spectral sequences and Franke's algebraicity conjecture*. 2021. arXiv: `2110.03669 [math.AT]` (Pages 45, 46, 59, 61, 69, 72–78, 82, 89, 91, 184).

[Pér22] Maximilien Péroux. "The coalgebraic enrichment of algebras in higher categories". In: *Journal of Pure and Applied Algebra* 226 (2022) (Page 138).

[Pos10] Leonid Positselski. *Homological algebra of semimodules and semicontramodules: Semi-infinite homological algebra of associative algebraic structures*. Vol. 70. Monografie Matematyczne. Birkhäuser, 2010 (Page 104).

[Pos11] Leonid Positselski. "Two kinds of derived categories, Koszul duality, and comodule-contramodule correspondence". In: *Memoirs of the American Mathematical Society* 212 (2011) (Page 104).

[Pos16] Leonid Positselski. "Dedualizing complexes and MGM duality". In: *Journal of Pure and Applied Algebra* 220 (2016) (Pages 104, 128, 131).

[Pos17a] Leonid Positselski. *Abelian right perpendicular subcategories in module categories.* 2017. arXiv: `1705.04960 [math.CT]` (Page 106).

[Pos17b] Leonid Positselski. "Contraadjusted modules, contramodules, and reduced cotorsion modules". In: *Moscow Mathematical Journal* 17 (2017) (Page 104).

[Pos20] Leonid Positselski. "Smooth duality and co-contra correspondence". In: *Journal of Lie Theory* 30 (2020) (Page 104).

[Pos22] Leonid Positselski. *Contramodules.* 2022. arXiv: `1503.00991 [math.CT]` (Pages 104, 106, 132).

[Pst21] Piotr Pstrągowski. "Chromatic homotopy theory is algebraic when $p > n^2 + n + 1$". In: *Advances in Mathematics* 391 (2021), p. 107958 (Pages 38, 45, 59, 69, 74, 87, 88).

[Pst23] Piotr Pstrągowski. "Synthetic spectra and the cellular motivic category". In: *Inventiones Mathematicae* 232.2 (2023), pp. 553–681 (Pages 72, 91, 151, 181–184, 186, 192, 194, 196).

[PV22] Piotr Pstrągowski and Paul VanKoughnett. "Abstract Goerss-Hopkins theory". In: *Advances in Mathematics* 395 (2022) (Page 25).

[Qui69] Daniel Quillen. "On the formal group laws of unoriented and complex cobordism theory". In: *Bulletin of the American Mathematical Society* 75.6 (1969), pp. 1293–1298 (Pages 18, 36).

[Ram23] Maxime Ramzi. *Separability in homotopical algebra.* 2023. arXiv: `2305.17236 [math.AT]` (Pages 118, 120).

[Ram24] Maxime Ramzi. *Dualizable presentable ∞-categories.* 2024. arXiv: `2410.21537 [math.CT]` (Page 113).

[Rav86] Douglas C. Ravenel. *Complex cobordismc and stable homotopy groups of spheres*. 2nd ed. ISBN: 978-1-4704-7293-1. AMS Chelsea Publishing, 1986 (Pages 18, 23, 34, 35, 65, 87, 88).

[Rav92] Douglas C. Ravenel. *Nilpotence and periodicity in stable homotopy theory*. ISBN: 978-0-691-08792-4. Princeton University Press, 1992 (Pages 22, 28, 47, 58).

[Rie24] Florian Riedel. *On the deformation theory of $\mathbb{E}_\infty$-coalgebras*. 2024. arXiv: `2303.12958 [math.AT]` (Page 134).

[RV15] Emily Riehl and Dominic Verity. "Completeness results for quasi-categories of algebras, homotopy limits, and related general constructions". In: *Homology, Homotopy and Applications* 17.1 (June 2015). Publisher: International Press of Boston, pp. 1–33 (Page 114).

[Rog08] John Rognes. *Galois extensions of structured ring spectra*. Vol. 192. Memoirs of the American Mathematical Society. ISBN: 978-0-8218-4076-4. American Mathematical Society, 2008 (Page 26).

[Rog23] John Rognes. *Chromatic homotopy theory*. MAT4580/MAT9580 Lecture notes. 2023 (Page 23).

[Roi07] Constanze Roitzheim. "Rigidity and exotic models for the K-local stable homotopy category". In: *Geometry & Topology* 11.4 (2007), pp. 1855–1886 (Pages 59, 78).

[Rou08] Raphaël Rouquier. "Dimensions of triangulated categories". In: *Journal of K-theory* 1 (2008) (Page 106).

[Sch01] Stefan Schwede. "The stable homotopy category has a unique model at the prime 2". In: *Advances in Mathematics* 164.1 (2001), pp. 24–40 (Page 78).

[Sch07] Stefan Schwede. "The stable homotopy category is rigid". In: *Annals of Mathematics. Second Series* 166.3 (2007), pp. 837–863 (Page 78).

[SS02] Stefan Schwede and Brooke Shipley. “A uniqueness theorem for stable homotopy theory”. In: *Mathematische Zeitschrift* 239.4 (2002), pp. 803–828 (Page 78).

[Ste13] Greg Stevenson. “Support theory via actions of tensor triangulated categories”. In: *Journal für die reine und angewandte Mathematik* 2013.681 (2013), pp. 219–254 (Page 175).

[Sul05] Dennis Sullivan. *Geometric topology: localization, periodicity and Galois symmetry: The 1970 MIT Notes*. ISBN: 978-1-4020-3511-1. Springer, 2005 (Page 20).

[Tak09] Ryo Takahashi. “On localizing subcategories of derived categories”. In: *Kyoto Journal of Mathematics* 49 (2009), pp. 771–783 (Pages 146, 163, 167, 174, 176).

[ZC17] Chao Zhang and Hongyan Cai. “A note on thick subcategories and wide subcategories”. In: *Homology, Homotopy and Applications* 19.2 (2017), pp. 131–139 (Pages 147, 163, 168).

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