The Anaconda lemma

Torgeir Aambø Jacob Grevstad

Abstract: We prove a generalization of the classical snake lemma by using the natural associated spectral sequence of a bicomplex. We also explicitly construct the longer connecting morphism.

1 Introduction

In an introductory course on homological algebra one learns about the snake lemma. It's famous for producing results in homological algebra, most notably for producing connecting homomorphisms. The lemma first showed up in D. A. Buchsbaums article "Exact Categories and Duality" from 1955 as lemma 5.8. More and more refined proofs have later arrived, and the standard method for proving it in your typical homological algebra class is to use diagram chasing. Using actual elements can feel cheap sometimes, and invoking the powerful Freyd-Michell's embedding theorem can feel like cheating to get to the point where we can use elements. The snake lemma is intuitive, but for students the connecting homomorphism feels a bit like a magic homomorphism that just happens to show up.

There is an object that is almost entirely built out of connecting homomorphisms, and this is the spectral sequence. The study of spectral sequences arises very naturally when doing algebraic topology and it is one of the standard techniques for computing the homology of extensions, how products of derived functors work and most important for this article, computing the homology of the total complex of a bicomplex. Luckily for us, spectral sequences do not require the use of elements, but in the end of the article, when explicitly constructing the long connecting homomorphism, we will be invoking the Freyd-Michell's embedding theorem.

2 Preliminaries

For the rest of this article, we assume that $\mathcal{A}$ is an abelian category.
Definition 1 (Bicomplex)**.** *A bicomplex $C_{*,*}$ in $\mathcal{A}$ is a bigraded object, or a diagram in $\mathcal{A}$ with morphisms $d^h_n : C_{n,m} \longrightarrow C_{n-1,m}$ and $d^v_m : C_{n,m} \longrightarrow C_{n,m-1}$ such that $d^h \circ d^h = 0 = d^v \circ d^v$ and $d^h \circ d^v + d^v \circ d^h = 0$.*

Note that this is essentially a complex in the category of complexes in $\mathcal{A}$, except that the squares anti-commute instead of commute.

If we have a bicomplex $C_{*,*}$ where all objects $C_{n,m} = 0$ for all $n < 0$ or $m < 0$, then we call C a first quadrant bicomplex.
Definition 2 (Totalization)**.** *The totalization of a bicomplex is the complex*

$$\mathrm{Tot}(C_{*,*}) = (\bigoplus_{a+b=n} C_{a,b} | n \in \mathbb{Z})$$

with the differentials being $d_n = \sum_{a+b=n} d^h_a + d^v_b$.

The totalization of a bicomplex has the natural filtration

$$F_p(\mathrm{Tot}(C_{*,*})_n) = (\bigoplus_{a+b=n} C_{a,b} | a > p).$$

Since we sum along the diagonals when making the totalization, we can interchange the indices in the bicomplex and still be left with the same totalization, i.e.

$$\mathrm{Tot}((C_{a,b}|a, b \in \mathbb{Z})) \cong \mathrm{Tot}((C_{b,a})|a, b \in \mathbb{Z}).$$

Hence we actually get two natural filtrations on the totalization, namely a horisontal one and a vertical one. This is one of the tricks we will use to prove the main theorem.
Definition 3 (Spectral sequence)**.** *A spectral sequence of homological type is a tri-graded object, or a list of bi-graded objects $E^r_{p,q}$ together with morphisms $d_r : E^r_{p,q} \longrightarrow E^r_{p+r,q+r-1}$ for all $r > 0, p, q \in \mathbb{Z}$, and isomorphisms $E^{r+1}_{p,q} \cong H(E^r_{p,q})$.*

This object can be thought of as a book, where for each r, we have a page with a bigraded object $E^r_{*,*}$ together with a set of maps making the bigraded object into a complex. When we flip a page we get a new bigraded object which consists of the homology of the complexes with the maps new maps making this new bigraded object into a complex. The “next page” is sometimes called the derived object of the previous page.

Associated to every bicomplex $C_{*,*}$, we have a spectral sequence whose second page E^2 is the crossed double homology, i.e. $E^2_{p,q} = H^h_p(H^v_q(C))$, where h and v means horizontal and vertical respectively. This associated spectral sequence is a special case of the associated spectral sequence one gets from a filtered complex. In this special case, the complex is the totalization of $C_{*,*}$ and the filtration is one of the two natural filtrations described earlier, i.e. the row and column filtrations. We leave out the description of how one gets a spectral sequence from a filtered complex as how we get it is not important, just that we indeed can.

Lemma 1. *Suppose $C_{*,*}$ is a first quadrant bicomplex. Then the associated spectral sequence with respect to both of the natural filtrations converge to the homology of the total complex, i.e.*

$$\begin{aligned} E^2_{p,q} = H_p H_q(C_{*,*}) &\implies H_{p+q}(\mathrm{Tot}(C_{*,*})) \\ D^2_{p,q} = H_q H_p(C_{*,*}) &\implies H_{p+q}(\mathrm{Tot}(C_{*,*})) \end{aligned}$$

Proof. We won't go through the proof, but we refer the reader to [Wei94, theorem 5.5.1]. □

Now we have the tools to tackle the main theorem.

3 Main theorem

Our goal is to get a snake lemma type connecting homomorphism for bigger diagrams than the standard 2×3. The idea of the proof is to take a bicomplex which is exact almost everywhere. In this way, when we construct its associated spectral sequence, we mostly get 0's everywhere. We leave room for the bicomplex to be non-exact at places such that when we "flip" to the n'th and final page, we have a single morphism which is then forced to be an isomorphism. Then we use that isomorphism to construct the long connecting homomorphism.

Theorem 1 (The anaconda lemma)**.** *Let $n \geq 2$ and*

$$\begin{tikzcd}
 & C_{n+1,n} \arrow[r, "d^h_{n+1,n}"] \arrow[d, "d^v_{n+1,n}"] & C_{n,n} \arrow[r, "d^h_{n,n}"] \arrow[d, "d^v_{n,n}"] & \cdots \arrow[r] & C_{1,n} \arrow[r] \arrow[d, "d^v_{1,n}"] & 0 \\
0 \arrow[r] & C_{n+1,n-1} \arrow[r, "d^h_{n+1,n-1}"] \arrow[d] & C_{n,n-1} \arrow[r] \arrow[d] & \cdots \arrow[r] & C_{1,n-1} \arrow[r] \arrow[d] & 0 \\
\vdots & \vdots \arrow[d] & \vdots \arrow[d] & & \vdots \arrow[d] & \\
0 \arrow[r] & C_{n+1,2} \arrow[r, "d^h_{n+1,2}"] \arrow[d, "d^v_{n+1,2}"] & C_{n,2} \arrow[r] \arrow[d, "d^v_{n,2}"] & \cdots \arrow[r] & C_{1,2} \arrow[r] \arrow[d, "d^v_{1,2}"] & 0 \\
0 \arrow[r] & C_{n+1,1} \arrow[r] & C_{n,1} \arrow[r] & \cdots \arrow[r, "d^h_{2,1}"] & C_{1,1} &
\end{tikzcd}$$

be a bicomplex with n exact rows and $n+1$ exact columns. Then we have an exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ker}(d^h_{n+1,n}) \arrow[r] & \mathrm{Ker}(d^v_{n+1,n}) \arrow[r] & \cdots \arrow[r] & \mathrm{Ker}(d^v_{1,n}) \arrow[dllll, rounded corners] & \\
\mathrm{Cok}(d^v_{n+1,2}) \arrow[r] & \cdots \arrow[r] & \mathrm{Cok}(d^v_{1,2}) \arrow[r] & \mathrm{Cok}(d^h_{2,1}) \arrow[r] & 0
\end{tikzcd}$$

Proof. By inserting a kernel, a cokernel and two zeroes, we can make the bicomplex above into the following bicomplex.

$$\begin{tikzcd}
\mathrm{Ker}(d^h_{n+1,n}) \arrow[r] \arrow[d] & C_{n+1,n} \arrow[r] \arrow[d] & C_{n,n} \arrow[r] \arrow[d] & \cdots \arrow[r] & C_{1,n} \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] \arrow[d] & C_{n+1,n-1} \arrow[r] \arrow[d] & C_{n,n-1} \arrow[r] \arrow[d] & \cdots \arrow[r] & C_{1,n-1} \arrow[r] \arrow[d] & 0 \arrow[d] \\
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & & \vdots \arrow[d] & \vdots \arrow[d] \\
0 \arrow[r] & C_{n+1,1} \arrow[r] & C_{n,1} \arrow[r] & \cdots \arrow[r] & C_{1,1} \arrow[r] & \mathrm{Cok}(d^h_{2,1})
\end{tikzcd}$$

From here we see that all rows are exact, so applying homology in the horizontal direction first in our spectral sequence gives us that the E^∞-page should be all zeroes. Starting with vertical homology we get the E^1-page to be

$$\begin{tikzcd}
\mathrm{Ker}(d^h_{n+1,n}) \arrow[r, "i"] & \mathrm{Ker}(d^v_{n+1,n}) \arrow[r, "g_{n+1}"] & \mathrm{Ker}(d^v_{n,n}) \arrow[r, "g_n"] & \cdots \arrow[r, "g_2"] & \mathrm{Ker}(d^v_{1,n}) \arrow[r] & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 0 \\
0 \arrow[r] & \mathrm{Cok}(d^v_{n+1,2}) \arrow[r, "h_{n+1}"] & \mathrm{Cok}(d^v_{n,2}) \arrow[r, "h_n"] & \cdots \arrow[r, "h_2"] & \mathrm{Cok}(d^v_{1,2}) \arrow[r, "j"] & \mathrm{Cok}(d^h_{2,1})
\end{tikzcd}$$

And then the E_2 page to be:

$$
\begin{array}{ccccccc}
\mathrm{Ker}(i) & \dfrac{\mathrm{Ker}(g_{n+1})}{\mathrm{Im}(i)} & \dfrac{\mathrm{Ker}(g_n)}{\mathrm{Im}(g_{n+1})} & \cdots & \dfrac{\mathrm{Ker}(g_2)}{\mathrm{Im}(g_3)} & \mathrm{Cok}(g_2) & \\
 & 0 & 0 & \cdots & 0 & 0 & \\
 & \vdots & \vdots & & \vdots & & \\
 & 0 & 0 & \cdots & 0 & 0 & \\
 & \mathrm{Ker}(h_{n+1}) & \dfrac{\mathrm{Ker}(h_{n+1})}{\mathrm{Im}(h_n)} & \cdots & \dfrac{\mathrm{Ker}(h_2)}{\mathrm{Im}(h_3)} & \dfrac{\mathrm{Ker}(j)}{\mathrm{Im}(h_2)} & \mathrm{Cok}(j)
\end{array}
$$

Since the differentials on page r is a morphism from $E^r_{p,q}$ to $E^r_{p+r,q+r-1}$, there can be no non-zero differentials on the following pages, except on the E_n-page. This is because we see that $E^r_{p,q} = 0$ for all $q \in \{1, \dots, n\}, p \in \{2, \dots, n-1\}$. This means that all of the differentials either pass out from, or into a 0. Hence all the differentials are zero, and the second page is equal to the third, and the fourth, and so on, until we get to the n-th page. Here, finally we have a differential long enough to reach from the top row to the bottom one. This differential, $\Psi : \mathrm{Ker}(h_{n+1}) \longrightarrow \mathrm{Cok}(g_2)$ is the longest differential we are going to get.

$$
\begin{tikzcd}
\mathrm{Ker}(i) & \dfrac{\mathrm{Ker}(g_{n+1})}{\mathrm{Im}(i)} & \dfrac{\mathrm{Ker}(g_n)}{\mathrm{Im}(g_{n+1})} & \cdots & \dfrac{\mathrm{Ker}(g_2)}{\mathrm{Im}(g_3)} & \mathrm{Cok}(g_2) & \\
 & 0 & 0 & \cdots & 0 & 0 & \\
 & \vdots & \vdots & & \vdots & & \\
 & 0 & 0 & \cdots & 0 & 0 & \\
 & \mathrm{Ker}(h_{n+1}) \arrow[uuuurrrr, "\Psi"] & \dfrac{\mathrm{Ker}(h_{n+1})}{\mathrm{Im}(h_n)} & \cdots & \dfrac{\mathrm{Ker}(h_2)}{\mathrm{Im}(h_3)} & \dfrac{\mathrm{Ker}(j)}{\mathrm{Im}(h_2)} & \mathrm{Cok}(j)
\end{tikzcd}
$$

Since our spectral sequence is a first quadrant spectral sequence from a bicomplex, we know that it converges to the homology of the totalization of the bicomplex with respect to both of the natural filtrations. We showed above that the spectral sequence converges to zero when we start with the horizontal homology, so then it has to converge to zero when starting with the vertical homology as well. This means that we have $E^{n+1}_{*,*} = E^{\infty}_{*,*} = 0$. For this to be the case we must have that everything on the $E^{n}_{*,*}$-page, except $\mathrm{Ker}(h_{n+1})$ and $\mathrm{Cok}(g_2)$ to be zero, and that the differential Ψ must be an isomorphism. Since all the homology groups in the top and bottom row are zero, this means that we have exact sequences

$$0 \longrightarrow \mathrm{Ker}(d^h_{n+1,n}) \xrightarrow{i} \mathrm{Ker}(d^v_{n+1,n}) \xrightarrow{g_{n+1}} \cdots \xrightarrow{g_3} \mathrm{Ker}(d^v_{2,n}) \xrightarrow{g_2} \mathrm{Ker}(d^v_{1,n})$$

and

$$\mathrm{Cok}(d^v_{n+1,2}) \xrightarrow{h_{n+1}} \mathrm{Cok}(d^v_{n,2}) \xrightarrow{h_n} \cdots \xrightarrow{h_2} \mathrm{Cok}(d^v_{1,2}) \xrightarrow{j} \mathrm{Cok}(d^h_{2,1}) \longrightarrow 0$$

The connecting homomorphism comes from inverting our differential Ψ. We then have an exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ker}(d^h_{n+1,n}) \arrow[r] & \cdots \arrow[r] & \mathrm{Ker}(d^v_{2,n}) \arrow[r] & \mathrm{Ker}(d^v_{1,n}) \arrow[dllll, dashed, "\partial"] \\
\mathrm{Cok}(d^v_{n+1,2}) \arrow[r] & \mathrm{Cok}(d^v_{n,2}) \arrow[r] & \cdots \arrow[r] & \mathrm{Cok}(d^h_{2,1}) \arrow[r] & 0
\end{tikzcd}$$

If we let π be the epimorphism $\pi : \mathrm{Ker}(d^v_{1,n}) \to \mathrm{Cok}(g_2)$ and ι be the monomorphism $\iota : \mathrm{Ker}(h_{n+1}) \to \mathrm{Cok}(d^v_{n+1,2})$, then we have $\partial = \iota \circ \psi^{-1} \circ \pi$. We have then joined together our two exact sequences into one longer exact sequence, which is the sequence we wanted. $\square$

Corollary 1.1 (Snake lemma)**.** *The normal snake lemma follows from setting $n = 2$. We do remark that this proof of the snake lemma is circular, as the snake lemma is needed to prove the convergence of the spectral sequence.*

Corollary 1.2. *Every surjection-injection composition can be realised as a connecting homomorphism in a diagram.*

Proof. This is inspired by the fact that we can realise each morphism f between two objects A and B in an abelian category $\mathscr{A}$ as the connecting homomorphism in the following diagram.

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & A \arrow[r, "id_A"] \arrow[d, "f"] & A \arrow[d] \\
B \arrow[r, "id_B"] & B \arrow[r] & 0
\end{tikzcd}$$

We can then take a composition of morphisms $A \xrightarrow{f} B \xrightarrow{g} C$ and construct a diagram realising $g \circ f$ as the connecting homomorphism by using these as building blocks. For the morphisms f and g separately, we get diagrams

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & A \arrow[r, "id_A"] \arrow[d, "f"] & A \arrow[d] \\ B \arrow[r, "id_B"] & B \arrow[r] & 0 \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} 0 \arrow[r] \arrow[d] & B \arrow[r, "id_B"] \arrow[d, "g"] & B \arrow[d] \\ C \arrow[r, "id_C"] & C \arrow[r] & 0 \end{tikzcd}$$

which we can stack on top of each other to form the following staircase diagram.

$$\begin{tikzcd} & 0 \arrow[r] \arrow[d] & A \arrow[r, "id_A"] \arrow[d, "f"] & A \arrow[d] \\ 0 \arrow[r] \arrow[d] & B \arrow[r, "id_B"] \arrow[d, "g"] & B \arrow[r] \arrow[d] & 0 \\ C \arrow[r, "id_C"] & C \arrow[r] & 0 & \end{tikzcd}$$

Since f is surjective and g is injective, we get that the diagram is an exact bicomplex, and by theorem 1 we get a homomorphism $\partial : A \longrightarrow C$, which by going down the staircase in the diagram is the same as the composition $g \circ f$.

$\square$

4 Constructing the connecting homomorphism

Let $n \geq 2$ and

$$\begin{tikzcd} & C_{n+1,n} \arrow[r, "d^h_{n+1,n}"] \arrow[d, "d^v_{n+1,n}"] & C_{n,n} \arrow[r, "d^h_{n,n}"] \arrow[d, "d^v_{n,n}"] & \cdots \arrow[r, "d^h_{2,n}"] & C_{1,n} \arrow[r] \arrow[d, "d^v_{1,n}"] & 0 \\ 0 \arrow[r] & C_{n+1,n-1} \arrow[r, "d^h_{n+1,n-1}"] \arrow[d] & C_{n,n-1} \arrow[r] \arrow[d] & \cdots \arrow[r] & C_{1,n-1} \arrow[r] \arrow[d] & 0 \\ \vdots & \vdots \arrow[d] & \vdots \arrow[d] & & \vdots \arrow[d] & \\ 0 \arrow[r] & C_{n+1,2} \arrow[r, "d^h_{n+1,2}"] \arrow[d, "d^v_{n+1,2}"] & C_{n,2} \arrow[r] \arrow[d, "d^v_{n,2}"] & \cdots \arrow[r] & C_{1,2} \arrow[r] \arrow[d, "d^v_{1,2}"] & 0 \\ 0 \arrow[r] & C_{n+1,1} \arrow[r] & C_{n,1} \arrow[r] & \cdots \arrow[r, "d^h_{2,1}"] & C_{1,1} & \end{tikzcd}$$

be an exact bicomplex. We want to construct a map from $\operatorname{Ker} d^v_{1,n}$ to $\operatorname{Cok} d^v_{n+1,2}$. Let x_0 be in $\operatorname{Ker} d^v_{1,n}$. Since $d^h_{2,n}$ is surjective there is an $x_1 \in C_{2,n}$ such that

$x_1 + \operatorname{Ker} d^h_{2,n} = x_1 + \operatorname{Im} d^h_{3,n}$ is the preimage of x_0 by $d^h_{2,n}$. Let $x_2 := d^v_{2,n}(x_1)$, then

$$\begin{aligned} d^v_{2,n}(x_1 + \operatorname{Im} d^h_{3,n}) &= x_2 + d^v_{2,n}(\operatorname{Im} d^h_{3,n}) \\ &= x_2 + \operatorname{Im}(d^v_{2,n} \circ d^h_{3,n}) \\ &= x_2 + \operatorname{Im}(d^h_{3,n-1} \circ d^v_{3,n}) \end{aligned}$$

Since x_0 is in $\operatorname{Ker} d^v_{1,n}$ and the squares commute it follows that x_2 is in $\operatorname{Ker} d^h_{2,n-1} = \operatorname{Im} d^h_{3,n-1}$. Thus we can find x_3 in $C_{3,n-1}$ such that $x_3 + \operatorname{Im} d^v_{3,n} + \operatorname{Ker} d^h_{3,n-1}$ is the preimage of $x_2 + \operatorname{Im}(d^h_{3,n-1} \circ d^v_{3,n})$ by $d^h_{3,n-1}$. Let $x_4 := d^v_{3,n-1}(x_3)$, then

$$\begin{aligned} d^v_{3,n-1}(x_3 + \operatorname{Im} d^v_{3,n} + \operatorname{Ker} d^h_{3,n-1}) &= d^v_{3,n-1}(x_3 + \operatorname{Ker} d^v_{3,n-1} + \operatorname{Im} d^h_{4,n-1}) \\ &= x_4 + \operatorname{Im}(d^v_{3,n-1} \circ d^h_{4,n-1}) \\ &= x_4 + \operatorname{Im}(d^h_{4,n-2} \circ d^v_{4,n-1}) \end{aligned}$$

We see that this is simply a shift in index from our previous equation and thus we can iterate this process until we reach the bottom of the diagram. This will lead us to $x_{2n-2} + \operatorname{Im}(d^h_{n+1,1} \circ d^v_{n+1,2}) \subseteq \operatorname{Ker} d^h_{N,1}$. Since $d^h_{N+1,1}$ is injective we may consider x_{2n-2} as an element of $C_{n+1,1}$. Then the preimage of $x_{2n-2} + \operatorname{Im}(d^h_{n+1,1} \circ d^v_{n+1,2})$ by $d^h_{n,1}$ is $x_{2n-2} + \operatorname{Im} d^v_{n+1,2}$ which gives a well-defined element of $\operatorname{Cok} d^v_{n+1,2}$.

References

[Wei94] Charles A. Weibel. *An introduction to homological algebra*. Cambridge university press, 1994.