# The cohomology ring of $G_2(\mathbb{C}^4)$

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## 1 Introduction

In this little write-up I want to calculate the cohomology ring of $G_2(\mathbb{C}^4)$, i.e. the Grassmannian manifold consisting of all 2-dimensional subspaces of $\mathbb{C}^4$. To do this we use the Serre spectral sequence we get from the fibration

$$U(2) \longrightarrow V_2(\mathbb{C}^4) \longrightarrow G_2(\mathbb{C}^4)$$

where $V_2(\mathbb{C}^4)$ is the Stiefel manifold, i.e. the collection of 2-frames in $\mathbb{C}^4$, and $U(2)$ is the unitary group. I won't do the calculation of the cohomology rings of $V_2(\mathbb{C}^4)$ and $U(2)$, but the latter we can easily get from the homeomorphism $U(n) \simeq U(1) \times SU(n-1)$, giving in particular $U(2) \simeq S^1 \times S^3$, and the former can be calculated by induction and the following fibration

$$S^{2(n-k)+1} \longrightarrow V_k(\mathbb{C}^n) \longrightarrow V_{k-1}(\mathbb{C}^n).$$

We then have

$$H^*(V_2(\mathbb{C}^4)) = \Lambda(a_5, a_7) \text{ and } H^*(U(2)) = \Lambda(a_1, a_3)$$

where $a_i$ lies in degree $i$.

From the fibration

$$U(2) \longrightarrow V_2(\mathbb{C}^4) \longrightarrow G_2(\mathbb{C}^4)$$

we get a spectral sequence $E_r$ which uses the cohomology of $G_2(\mathbb{C}^4)$ with coefficients from the cohomology of $U(2)$ to calculate the cohomology of $V_2(\mathbb{C}^4)$. Since we now already know the cohomology ring of $V_2(\mathbb{C}^4)$ we can guess what the cohomology of $G_2(\mathbb{C}^4)$ has to be in order for the spectral sequence to calculate the correct cohomology ring. This is the approach of this write-up.

## 2 Calculation

We start on the $E_2$ page. On the zeroth column, we get only the cohomology of $U(2)$, hence we at least see

$$
\begin{array}{c|ccccccccc}
4 & a_1a_3 & & & & & & & & \\
3 & a_3 & & & & & & & & \\
2 & 0 & & & & & & & & \\
1 & a_1 & & & & & & & & \\
0 & 1 & & & & & & & & \\
\hline
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8
\end{array}
$$

Since we know that $H^*(V_2(\mathbb{C}^4))$ has no elements in degree 1, we know that $H^2(G_2(\mathbb{C}^4)) \cong H^1(V_2(\mathbb{C}^4))$, as the generator $a_1$ has to be killed by a differential. Call the generator for the group $H^2(G_2(\mathbb{C}^4))$ that is the image of $a_1$ for $x$. By multiplications by the other generators and itself we then know we have

$$
\begin{tikzcd}
4 & a_1a_3 & & a_1a_3x & & a_1a_3x^2 & & a_1a_3x^3 & & a_1a_3x^4 \\
3 & a_3 & & a_3x & & a_3x^2 & & a_3x^3 & & a_3x^4 \\
2 & & & & & & & & & \\
1 & a_1 \arrow[drr, "\cong"] & & a_1x & & a_1x^2 & & a_1x^3 & & a_1x^4 \\
0 & 1 & & x & & x^2 & & x^3 & & x^4 \\
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8
\end{tikzcd}
$$

We know that the differential is a derivation, hence we get that $d_2(a_1x) = d_2(a_1)x + (-1)^{|x|}a_1d_2(x) = d_2(a_1)x = x^2$ since $d_2(x) = 0$. The same calculation for higher powers of $x$ and for $a_1a_3$ and $a_1a_3x$, gives us isomorphisms all the way to the right

$$
\begin{tikzcd}
4 & a_1a_3 \arrow[drr, "\cong"] & & a_1a_3x \arrow[drr, "\cong"] & & a_1a_3x^2 \arrow[drr, "\cong"] & & a_1a_3x^3 \arrow[drr, "\cong"] & & a_1a_3x^4 \\
3 & a_3 & & a_3x & & a_3x^2 & & a_3x^3 & & a_3x^4 \\
2 & & & & & & & & & \\
1 & a_1 \arrow[drr, "\cong"] & & a_1x \arrow[drr, "\cong"] & & a_1x^2 \arrow[drr, "\cong"] & & a_1x^3 \arrow[drr, "\cong"] & & a_1x^4 \\
0 & 1 & & x & & x^2 & & x^3 & & x^4 \\
& 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8
\end{tikzcd}
$$

Here we see a problem. The generator $a_3$ never gets killed by a differential, but the cohomology of $V_2(\mathbb{C}^4))$ has no elements in degree 3, so it cant survive. The only degree a differential starting from $a_3$ can land in is 4, and the only possible non-zero cohomology in degree 4 comes from $H^4(G_2(\mathbb{C}^4))$, which by the likes of it so far has already been killed by a $d_2$ differential. Hence we got to have a $d_4$ differential hitting another generator in $H^4(G_2(\mathbb{C}^4))$, hence we need higher dimensional cohomology in this degree. We call this new generator which is the image of $a_3$ under the $d_3$ differential for $y$. Looking back at the $E_2$ page, we then have new elements

$$
\begin{tikzcd}
4 & a_1a_3 \arrow[drr, "\cong"] & & a_1a_3x \arrow[drr, "\cong"] & & a_1a_3x^2, a_1a_3y \arrow[drr, "\cong"] & & a_1a_3x^3, a_1a_3xy \arrow[drr, "\cong"] & & a_1a_3x^4, a_1a_3y^2 \\
3 & a_3 & & a_3x & & a_3x^2, a_3y & & a_3x^3, a_3xy & & a_3x^4, a_3y^2 \\
2 & & & & & & & & & \\
1 & a_1 \arrow[drr, "\cong"] & & a_1x \arrow[drr, "\cong"] & & a_1x^2, a_1y \arrow[drr, "\cong"] & & a_1x^3, a_1xy \arrow[drr, "\cong"] & & a_1x^4, a_1y^2 \\
0 & 1 & & x & & x^2, y & & x^3, xy & & x^4, y^2 \\
& 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8
\end{tikzcd}
$$

Now, what does this new information and new elements give us? By the same calculation as previously, we get that $d_2(a_1y) = xy$, but here have more information. We know that $H^*(V_2(\mathbb{C}^4))$ has a generator in degree 5, and this new generator $a_1y$ is the only generator left in total degree 5 in the spectral sequence. Thus it can't die by a $d_2$ differential. And since it's image is $xy$, this image has to be zero, which gives us the first of two relations on the cohomology ring of our Grassmannian, namely $xy = 0$. Hence, as far as we know so far, our $E_2$ page looks like this:

$$
\begin{tikzcd}
4 & a_1a_3 \arrow[drr, "\cong"] & & a_1a_3x \arrow[drr, "\cong"] & & a_1a_3x^2, a_1a_3y \arrow[drr, "\cong"] & & a_1a_3x^3 \arrow[drr, "\cong"] & & a_1a_3x^4, a_1a_3y^2 \\
3 & a_3 & & a_3x & & a_3x^2, a_3y & & a_3x^3 & & a_3x^4, a_3y^2 \\
2 & & & & & & & & & \\
1 & a_1 \arrow[drr, "\cong"] & & a_1x \arrow[drr, "\cong"] & & a_1x^2, a_1y \arrow[drr, "\cong"] & & a_1x^3 \arrow[drr, "\cong"] & & a_1x^4, a_1y^2 \\
0 & 1 & & x & & x^2, y & & x^3 & & x^4, y^2 \\
& 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8
\end{tikzcd}
$$

As far as I'm aware, we can't squeeze any more information out of the $E_2$ page yet. Let us pass to the $E_3$ page and see which generator we are left with after all the isomorphisms kill the generators. We have

$$
\begin{array}{cccccccccc}
4 & & & & & a_1a_3y & & & & a_1a_3y^2 \\
3 & a_3 & & & & a_3y & & & & a_3y^2 \\
2 & & & & & & & & & \\
1 & & & & & a_1y & & & & a_1y^2 \\
0 & 1 & & & & y & & & & y^2 \\
& 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8
\end{array}
$$

and we see that all the $d_3$ differentials either start or land in places which are zero, hence $E_3 = E_4$. On the $E_4$ page however, we know that we at least have one differential, namely the one we used to justify the generator $y$'s existence. By the derivation property of the differentials, we also get that $d_4(a_1a_3y) = a_1y^2$, hence we at least have an $E_4$ page looking like

$$
\begin{tikzcd}
4 & & & & & a_1a_3y \arrow[dddrrrr, "\cong"] & & & & a_1a_3y^2 \\
3 & a_3 \arrow[dddrrrr, "\cong"] & & & & a_3y & & & & a_3y^2 \\
2 & & & & & & & & & \\
1 & & & & & a_1y & & & & a_1y^2 \\
0 & 1 & & & & y & & & & y^2 \\
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8
\end{tikzcd}
$$

We are also lucky enough to know that $H^*(V_2(\mathbb{C}^4))$ has a generator in degree 7, which we see is $a_3y$ since it is the only one left in the correct total degree. But what happens to the differential $d_4(a_3y) = y^2$ you ask? Do we have $y^2 = 0$ then, same as last time? Actually, no. Since we know that $a_1y$ and $a_3y$ both are non-zero generators in $H^*(V_2(\mathbb{C}^4))$, their product $a_1a_3y^2$ is also a generator in degree 12, hence it has to be non-zero, and then $y^2$ has to be non-zero in $H^8(G_2(\mathbb{C}^4))$. But, it still has to die in the spectral sequence though, and the solution is the second relation on the cohomology ring, namely $x^4 = y^2$. This ensures that $y^2$ is already killed on the $E_2$ page, and hence that $a_3y$ isn't killed on the $E_4$ page.

To retrospectively correct some things now that we have all the information, we first note that we were lucky we didn't bother writing more that 8 columns, because everything in row 0 and columns above 8 either is a product of $x$'s and $y$'s, or contain powers of $x$ greater or equal to 5 or powers of $y$ greater or equal to 3, making $x^5 = xx^4 = xy^2 = (xy)y = 0$ and $y^3 = yx^4 = 0$. Hence our generators $a_1x^4$ and $a_1a_3x^4$ never was killed by any $d_2$ differentials, since their images are $x^5 = 0$ and $a_3x^5 = 0$ respectively. Hence we still have $a_1x^4 = a_1y^2$ and $a_1a_3x^4 = a_1a_3y^2$ on the $E_4$ page. Also, we get that $a_3x^4 = a_3y^2$, which means that $a_3y^2$ was killed by a $d_2$ differential on the $E_2$ page, and hence it does not show up on the $E_4$ page. The final form of the $E_2$ page is then

$$
\begin{tikzcd}
4 & a_1a_3 \arrow[drr] & & a_1a_3x \arrow[drr] & & a_1a_3x^2, a_1a_3y \arrow[drr] & & a_1a_3x^3 \arrow[drr] & & a_1a_3x^4 = a_1a_3y^2 \\
3 & a_3 & & a_3x & & a_3x^2, a_3y & & a_3x^3 & & a_3x^4 = a_3y^2 \\
2 & & & & & & & & & \\
1 & a_1 \arrow[drr] & & a_1x \arrow[drr] & & a_1x^2, a_1y \arrow[drr] & & a_1x^3 \arrow[drr] & & a_1x^4 = a_1y^2 \\
0 & 1 & & x & & x^2, y & & x^3 & & x^4 = y^2 \\
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8
\end{tikzcd}
$$

and the final form of the $E_4$ page is

$$
\begin{tikzcd}
4 & & & & & a_1a_3y \arrow[dddrrrr, "\cong"] & & & & a_1a_3y^2 \\
3 & a_3 \arrow[dddrrrr, "\cong"] & & & & a_3y & & & & \\
2 & & & & & & & & & \\
1 & & & & & a_1y & & & & a_1y^2 \\
0 & 1 & & & & y & & & & \\
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8
\end{tikzcd}
$$

Since all higher degree differentials, i.e. $d_5$ and above miss any generators due to being too long, we have $E_5 = E_\infty$. We can also see this because we have arrived at the correct cohomology ring for $V_2(\mathbb{C}^4)$, namely the final page

$$
\begin{array}{cccccccccc}
4 & & & & & & & & & a_1a_3x^4 = a_1a_3y^2 \\
3 & & & & & a_3y & & & & \\
2 & & & & & & & & & \\
1 & & & & & a_1y & & & & \\
0 & 1 & & & & & & & & \\
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8
\end{array}
$$

Hence we are done, and we have found the correct cohomology ring for our Grassmannian $G_2(\mathbb{C}^4)$. By all our extensive calculations and work, we finally can say that $H^*(G_2(\mathbb{C}^4)) = \mathbb{Z}(x,y)/(xy, x^4 - y^2)$ or if we add the degrees into the names of the generators, $H^*(G_2(\mathbb{C}^4)) = \mathbb{Z}(a_2, a_4)/(a_2a_4, a_2^4 - a_4^2)$. Hence the calculation is finished, and we are done.