

Formality of spaces with Lusternik-Schnirelmann category 1

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Abstract

It is a well known fact that formal dg-algebras admit no non-trivial Massey products, and it is possibly equally well known that the converse fails. We prove that by restricting to dg-algebras whose induced product on cohomology is trivial, we do in fact get this converse. This allows us to prove that spaces of Lusternik-Schnirelmann category 1 must be formal spaces.

1 Introduction

Formal dg-algebras, being algebras that are quasi-isomorphic to their cohomology algebra, was introduced in [DGMS75] to solve problems in rational homotopy theory. In the aforementioned paper, the authors remark that having no non-vanishing Massey n -products is a weaker property than being formal, meaning that Massey n -products serve as obstructions to formality. They claim that formality is equivalent to a “uniform vanishing” – a stronger version of just being vanishing. In later times the study of dg-algebras have been explored further using the theory of A_∞ -algebras. Any dg-algebra A can be viewed as a “trivial” A_∞ -algebra, and the general ones can be thought of as homotopy theoretic versions of dg-algebras. In [Kad80] Kadeishvili proved that the cohomology algebra $H(A)$ of a dg-algebra A naturally admits an A_∞ -structure in such a way that there is a quasi-isomorphism of A_∞ -algebras $H(A) \rightarrow A$. The higher products on this A_∞ -structure are often claimed to be the Massey products, but this is not always true as the n ’th product on $H(A)$ is not always a representative of the Massey n -product [BMFM20]. It is however the case that the vanishing of these higher products on $H(A)$ is stronger than the vanishing of the Massey n -products [Val14] – in fact, if all the higher products vanish, then the dg-algebra is formal. Hence having vanishing A_∞ -structure on $H(A)$ is equivalent to A being formal. Using the equivalent definition of formality from [Kel01] this is true almost by definition.

In [BMFM20] the authors prove that even though the higher products on $H(A)$ might not be the Massey products, they are so up to a sum of lower degree products. We use this to prove that vanishing Massey products is equivalent to formality in the case that the cohomology algebra has trivial products. Hence the vanishing Massey products on spaces with vanishing cup products are always uniformly vanishing in the sense of [DGMS75].

There is a property of a space that allows us to know an upper bound for its cup-length, namely the Lusternik-Schnirelmann category of the space. This is an integer describing how a space can be glued together by cones. If we limit our study to spaces with Lusternik-Schnirelmann 1, we are certain that our cup-length is 0, meaning that we have vanishing products on reduced cohomology. We introduce reduced formality and use the above result to show that spaces with Lusternik-Schnirelmann category 1 are reduced formal, and afterward show that the word “reduced” is redundant.

2 A_∞ -algebras

For the rest of the paper we let K be a field of characteristic 0.

Definition 2.1. An A_∞ -algebra (A, m) , over K is a \mathbb{Z} -graded vector space $A = \bigoplus_{i \in \mathbb{Z}} A_i$ together with a family of K -linear maps $m_n : A^{\otimes n} \rightarrow A$ of degree $2 - n$, such that the identities

$$\sum_{r+s+t=n} (-1)^{r+st} m_{r+1+t}(Id^{\otimes r} \otimes m_s \otimes Id^{\otimes t}) = 0$$

hold for all $n, s \geq 1$.

These relations are called the coherence relations, or the Stasheff identities in A . For $n = 1$ the coherence relation simply becomes

$$0 = (-1)^{0+0} m_1(m_1) = m_1^2.$$

This means that A is a cochain complex with differential $d = m_1$ as m_1 is a degree 1 map. For $n = 2$ we get

$$0 = (-1)^1 m_2(Id \otimes m_1) + (-1)^0 m_1(m_2) + (-1)^1 m_2(m_1 \otimes Id)$$

which reduces to $m_1 m_2 = m_2(m_1 \otimes Id + Id \otimes m_1)$. This means that m_1 is a derivation with respect to m_2 as m_2 has degree 2, usually stated as satisfying the Leibniz rule. The standard Leibniz rule comes out of this formula when applying it to elements and using the Koszul grading rule.

The third relation tells us that m_2 is not necessarily associative, and that the associator is given by m_3 . This is usually referred to as m_2 being associative up to homotopy, which gives A_∞ -algebras their other name, strong homotopy associative algebras or sha-algebras for short. If $m_3 = 0$ (or $m_1 = 0$) this relation reduces to the associator being zero, which means that m_2 is an associative product.

Definition 2.2. A dg-algebra is an A_∞ -algebra (A, m) where $m_i = 0$ for $i \geq 3$. For simplicity we usually just denote it by A .

By the equations above describing the Stasheff identities, this is an equivalent definition to the classical definition, i.e. a \mathbb{Z} -graded vector space with an associative product and a differential satisfying the Leibniz rule.

Definition 2.3. Let (A, m^A) and (B, m^B) be A_∞ -algebras. A morphism of A_∞ -algebras $f : A \rightarrow B$, also called A_∞ -morphism, is a family of linear maps $f_n : A^{\otimes n} \rightarrow B$ of degree $1 - n$, such that

$$\sum_{n=r+s+t} (-1)^{r+st} f_{r+1+t}(id^{\otimes r} \otimes m_s^A \otimes id^{\otimes t}) = \sum_{k=1}^n \sum_{n=i_1+\dots+i_k} (-1)^{u_k} m_k^B(f_{i_1} \otimes \dots \otimes f_{i_k})$$

where $u_k = \sum_{t=1}^{k-1} t(i_{k-t} - 1)$.

We call f a A_∞ -isomorphism, or a isomorphism of A_∞ -algebras, if f_1 is an isomorphism of chain complexes.

Since any A_∞ -algebra (A, m) has a map m_1 such that $m_1^2 = 0$, we can also create its cohomology algebra, denoted $H(A)$. The cohomology algebra of a dg-algebra is a graded associative algebra with the induced product from A , which we can treat as a dg-algebra by letting it have trivial differential.

Let $f : A \rightarrow B$ be an A_∞ -morphism. We call f a A_∞ -quasi-isomorphism, or a quasi-isomorphism of A_∞ -algebras, if f_1 is a quasi-isomorphism of chain complexes, i.e. it induces an isomorphism on their cohomology algebras.

Notice that if $f_j = 0$ for $j \geq 2$ and $m_i^A = 0 = m_i^B$ for $i \geq 3$, i.e. A and B are dg-algebras, then this definition reduces to the standard quasi-isomorphisms of dg-algebras.

Definition 2.4. Let A be a dg-algebra. We say A is formal if it is A_∞ -quasi-isomorphic to a dg-algebra with trivial differential.

Such a dg-algebra will be isomorphic to $H(A)$, so we can define a dg-algebra to be formal if it is A_∞ -quasi-isomorphic to its cohomology algebra. We also remark that this is not the most classical definition, which uses a zig-zag of dg-quasi-isomorphisms instead of an A_∞ -quasi-isomorphism. This is equivalent to the one we are using here. See [Val14] and [Kel01] for further details.

Formal dg-algebras are very nice algebras that as mentioned have their historical upbringing in rational homotopy theory. Examples include the Sullivan algebras of Kähler manifolds [DGMS75].

3 Massey products

The other important part of this story is the Massey products. These are partially defined higher order cohomology operations on dg-algebras. Since they are only partially defined we need an easy way to package this information in. This is done through defining systems. Let A be a dg-algebra, and denote $\bar{x} = (-1)^{|x|}x$.

Definition 3.1. A defining system for a set of cohomology classes x_1, \dots, x_n in $H(A)$ is a collection $\{a_{i,j}\}$ of cochains in A such that

1. $[a_{i-1,i}] = x_i$

2. $d(a_{i,j}) = \sum_{i < k < j} \overline{a_{i,k}} a_{k,j}$

for all pairs $(i,j) \neq (0,n)$ where $i \leq j$.

Definition 3.2. The Massey n -product of n cohomology classes x_1, \dots, x_n , denoted $\langle x_1, \dots, x_n \rangle$ is defined to be the set of all $[a_{0,n}]$, where

$$a_{0,n} = \sum_{0 < k < n} \overline{a_{0,k}} a_{k,n}$$

such that $\{a_{i,j}\}$ is a defining system.

For $n = 2$ this is just the induced product on cohomology, up to a sign. For $n = 3$ this is the classical triple Massey product. When we use the phrase “all Massey n -products”, we mean all Massey n -products for $n \geq 3$.

The fact that multiple cohomology classes can be in this set means that the product is only partially defined. If this set contains just a single class, then we say the Massey product is uniquely defined. What matters for us is when these products are trivial. Since they are partially defined what we mean is the following.

Definition 3.3. We say that the Massey n -product vanishes if it contains zero as an element, i.e. $0 \in \langle x_1, \dots, x_n \rangle$.

4 Results on formality

One of the reasons we are interested in Massey products is that they serve as an obstruction to formality. Intuitively, if Massey products exist on the algebra of cochains $C(X)$ on a topological space X , this means that $C(X)$ contains more information about our space than the cohomology ring of X . If a non-trivial Massey product exists on $C(X)$, this means that there will always exist another space Y , not homeomorphic to X , but with the same cohomology ring as X . The most famous example of this is the Borromean rings. These are a set of three rings, every pair of them not linked with the two others, but all three still linked. This has the same cohomology ring as three unlinked circles, but the Borromean rings admits a non-vanishing Massey product, while the three unlinked circles does not. Hence the algebra of forms on the Borromean rings can't be formal. This means that Massey products detects non-formality. This is summarized into the following theorem.

Theorem 4.1. ([DGMS75]). *Let (A, m) be a formal dg-algebra. Then all Massey n -products vanish.*

Unfortunately, knowing a dg-algebra has no non-vanishing Massey products is not enough to determine that it is formal. But using the full A_∞ -algebras we can get closer to some version of this being true.

Theorem 4.2. ([Kad80]). *Let (A, m) be a dg-algebra. Then there exists an (up to A_∞ -isomorphism) unique A_∞ -algebra structure on its cohomology algebra $H(A)$ with $m_1 = 0$, m_2 the induced product from A , and a quasi-isomorphism of A_∞ -algebras $H(A) \rightarrow A$.*

Note that this does not mean that all dg-algebras are formal, as now $H(A)$ is not necessarily just a dg-algebra anymore. We can think of this higher structure on $H(A)$ as measuring how far away A is from being formal. Since $m_1 = 0$ we get that the product m_2 is associative, but not for the reason we mentioned earlier. This means that these higher products no longer are interpreted as homotopies, but instead as something more like Massey products. Hence we call them the “higher products” on $H(A)$.

We said that the A_∞ -structure measures how far away A is from being formal. We noted earlier that an A_∞ -algebra with $m_k = 0$ for $k \geq 3$ is a dg-algebra. This means that if the A_∞ -structure on $H(A)$ has $m_k = 0$ for $k \geq 3$, then A is formal, as we then have an A_∞ -quasi-isomorphism between two dg-algebras. This is rather important so we state it as a theorem.

Theorem 4.3. ([Val14]). *Let (A, m) be a dg-algebra. Then A is formal if and only if all the higher products on $H(A)$ vanish.*

One direction of the proof is by definition, as described above. The other part is because the A_∞ -structure on $H(A)$ is unique up to isomorphism of A_∞ -algebras.

In [DGMS75] the authors say that formality is equivalent to a uniform vanishing of the Massey products. We are intuitively able to choose the zero element as our Massey products in such a nice uniform way that they are a part of an A_∞ -structure, which must mean that we have formality, as above.

We want to use this to get an idea of “how close” the normal Massey products are from being sufficient obstructions. What we mean by this is that we want to find a case where vanishing Massey products mean we have a formal dg-algebra. To get to such a result we first need a theorem that connects the higher products to normal Massey products.

Theorem 4.4. ([BMFM20]). *Let A be a dg-algebra and $x \in \langle x_1, \dots, x_n \rangle$ with $n \geq 3$. Then for any A_∞ -structure on $H(A)$ we have*

$$\epsilon m_n(x_1, \dots, x_n) = x + \Gamma$$

where $\Gamma \in \sum_{j=1}^{n-1} \text{Im}(m_j)$ and $\epsilon = (-1)^{\sum_{j=1}^{n-1} (n-j)|x_j|}$.

Corollary 4.5. ([BMFM20]). *Let A be a dg-algebra and m an A_∞ -structure on its cohomology $H(A)$ such that $m_k = 0$ for all $1 \leq k \leq n-1$. Then the Massey n -product $\langle x_1, \dots, x_n \rangle$ is uniquely defined for any set of cohomology classes x_1, \dots, x_n . Furthermore the unique element in the Massey product is recovered by m_n up to a sign, i.e. $\langle x_1, \dots, x_n \rangle = \epsilon m_n(x_1, \dots, x_n)$, where again $\epsilon = (-1)^{\sum_{j=1}^{n-1} (n-j)|x_j|}$.*

By this we make our first result — the fact that no non-vanishing Massey products is equivalent to formality in the case of trivial induced product in cohomology — as a corollary.

Corollary 4.6. *Let A be a dg-algebra and $H(A)$ its cohomology algebra. If the induced product on $H(A)$ is trivial and all Massey n -products on A vanish, then A is formal.*

Proof. By Kadeishvili's theorem we know that $H(A)$ can be equipped with the structure $\{m_i\}$ of an A_∞ -algebra such that $m_1 = 0$ and m_2 is the product induced from A , which is assumed to be trivial. We claim that $m_k = 0$ for all $k \geq 3$ as well, and hence that A is formal by theorem 4.3. We prove this claim by induction.

Since $m_2 = 0$ we know that all Massey triple products are defined. By the below induction argument all the higher Massey products will be defined as well. Since $m_2 = 0$ we already have our base case.

Assume $m_k = 0$ for $1 \leq k \leq n-1$. By corollary 4.5 we know that $\langle x_1, \dots, x_n \rangle$ consists of a unique element for all choices of classes x_1, \dots, x_n . This element is by assumption the zero class, as we assumed all Massey products to be vanishing. This class is recovered up to a sign by m_n , which means $m_n(x_1, \dots, x_n) = 0$ for all choices of x_1, \dots, x_n . Hence $m_n = 0$ and we are done. \square

This proof shows that when we have a trivial induced product on $H(A)$, the vanishing Massey products neatly forms a trivial A_∞ -structure on $H(A)$, which we earlier said was the way of interpreting the uniform vanishing.

It is tempting to think that having trivial product in cohomology also makes every attempt to build and produce a Massey product impossible. This feels true intuitively, but there are examples of this not being the case. One example is the free loop space of an even-dimensional sphere. Its cohomology algebra has trivial product, and it is shown in [Bas15] to have non-zero Massey products. Hence it can't be formal. We also mentioned the Borromean rings earlier, which gives another example.

Question. Is there a more general procedure for choosing elements in all the Massey products in such a way that they form an A_∞ -structure?

5 The Lusternik-Schnirelmann category

As we have just developed a new way to test formality, it would be nice to test it out on some examples. Our gut feeling is that this criteria of having trivial induced product in cohomology is pretty strong. If we are hoping for results applicable to topological spaces this is especially troubling. Say we have a path-connected topological space X — then it has zeroth cohomology $H^0(X; K) \cong K$. Hence the requirement to have trivial cup product reduces to requiring $H^i(X; K) = 0$ for $i > 0$, which is really

limiting. One solution to this is looking at reduced cohomology.

Definition 5.1. Let X be a topological space and $C^*(X; K)$ its cochain complex (treated here as a dg-algebra). We define its augmented cochain dg-algebra, denoted $\tilde{C}^*(X; K)$ by adding a copy of the ground field K injectively farthest to the left, i.e.

$$\cdots \longrightarrow 0 \longrightarrow K \xrightarrow{\epsilon} C^0(X) \longrightarrow C^1(X) \longrightarrow \cdots \longrightarrow C^n(X) \longrightarrow \cdots$$

The cohomology algebra of the augmented cochain complex is called the reduced cohomology algebra of X and is denoted $\tilde{H}^*(X; K)$. If the space X is connected, then $\tilde{H}^0(X; K) = 0$, meaning that we have completely removed the problem described above.

Spaces with trivial cup product are not in abundance, but examples include the spheres and more generally the suspended spaces. There are ways to make sure that a space has a trivial cup product — one of these using the Lusternik-Schnirelmann category.

Definition 5.2. Let X be a topological space. The Lusternik-Schnirelmann category of X , denoted $cat_{LS}(X)$, is the least integer n such that there is a covering of X by $n + 1$ open subsets U_i that are all contractible in X , i.e. their inclusion into X is null-homotopic.

This invariant was originally developed in [LS34] as an invariant on manifolds to be a lower bound for the number of critical points any real valued function on it could have. It has since become a useful – but very difficult to calculate – invariant of topological spaces.

So why do we introduce this invariant? Recall that the cup-length of a topological space X , denoted $cl(X)$ is the largest integer n such that a chain $[x_1] \cup \cdots \cup [x_n]$ of cohomology classes with $\deg |x_i| \geq 1$ is non-zero. We have the following fundamental relation between the cup length and the Lusternik-Schnirelmann category of X [CLOT03].

Lemma 5.3. Let X be a topological space. Then the cup length of X is a lower bound for its Lusternik-Schnirelmann category, i.e. $cl(X) \leq cat_{LS}(X)$.

Thus, choosing spaces with $cat_{LS}(X) = 1$ means we have trivial product on reduced cohomology. Examples of such spaces are again the suspended spaces, as they are the union of two cones. The last thing we need to know is whether spaces with Lusternik-Schnirelmann category 1 admit any non-vanishing Massey n -products. Luckily, by Rudyak we know that they don't.

Theorem 5.4. ([Rud99]). Let X be a topological space with $cat_{LS}(X) \leq 1$ and $x_1, \dots, x_n \in \tilde{H}^*(X)$. If the Massey n -product $\langle x_1, \dots, x_n \rangle$ is defined then $\langle x_1, \dots, x_n \rangle = 0$.

Call spaces with formal reduced cochain algebras reduced formal spaces. This means now that spaces with Lusternik-Schnirelmann category 1 are reduced formal by the above theorem together with the earlier result on formality.

Corollary 5.5. Let X be a topological space with $cat_{LS}(X) = 1$. Then $\tilde{C}^*(X)$ is a formal dg-algebra.

We still have to deal with the degree 0 cochains in order to call X formal and not just reduced formal. We need that either its piece-wise linear De Rham algebra or its full – non-reduced – cochain algebra is formal.

Theorem 5.6. *Any reduced formal topological space X is formal.*

Proof. Since X is reduced formal we know that there is a span of dg-quasi-isomorphisms $\tilde{H}^*(X) \leftarrow B \rightarrow \tilde{C}^*(X)$ for some dg-algebra B :

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & k & \xrightarrow{\epsilon} & C^0(X) & \xrightarrow{d^0} & C^1(X) & \xrightarrow{d^1} & C^2(X) & \longrightarrow & \cdots \\
& & \uparrow q^{-1} & & \uparrow q^0 & & \uparrow q^1 & & \uparrow q^2 & & \\
\cdots & \longrightarrow & B^{-1} & \xrightarrow{d_B^{-1}} & B^0 & \xrightarrow{d_B^0} & B^1 & \xrightarrow{d_B^1} & B^2 & \longrightarrow & \cdots \\
& & \downarrow & & \downarrow p^0 & & \downarrow p^1 & & \downarrow p^2 & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & H^1(X) & \xrightarrow{0} & H^2(X) & \longrightarrow & \cdots
\end{array}$$

By changing the diagram slightly at the left-most side, we get the following new diagram:

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & 0 & \longrightarrow & C^0(X) & \xrightarrow{d^0} & C^1(X) & \xrightarrow{d^1} & C^2(X) & \longrightarrow & \cdots \\
& & \uparrow & & \uparrow [q^0, 0] & & \uparrow q^1 & & \uparrow q^2 & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & B^0 \oplus B^{-1} & \xrightarrow{d_B^0} & B^1 & \xrightarrow{d_B^1} & B^2 & \longrightarrow & \cdots \\
& & \downarrow & & \downarrow [0, p^0] & & \downarrow p^1 & & \downarrow p^2 & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & k & \longrightarrow & H^1(X) & \xrightarrow{0} & H^2(X) & \longrightarrow & \cdots
\end{array}$$

which gives a span of dg-quasi-isomorphisms $H^*(X) \leftarrow B' \rightarrow C^*(X)$. Hence X is formal. \square

We can then conclude with our result as a corollary.

Corollary 5.7. *Let X be a space with $\text{cat}_{LS}(X) = 1$. Then X is formal.*

This result is certainly already known by specialists, but this seems to be a new method of proving it. Alternatively one can use that any space X with $\text{cat}_{LS}(X) = 1$ is a co-H-space [Hes07], and then that any co-H-space is a wedge spheres [Hen83]. Formality is preserved under the wedge product [Hes07], and since spheres are formal, we know that any co-H-space, and thus any space X with $\text{cat}_{LS}(X) = 1$ is a formal space. The special case of suspended spaces being formal was also proven in [FHT01].

Question. Are there any formal spaces with trivial cup product and Lusternik-Schnirelmann category greater than 1?

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