Appendix C

C.1 Derivation of the Discrete Matrix Riccatti equations for the backward sweep of the dynamic programming routine.

The functional f can be written for any arbitrary time step as a qudratic in terms of X defined by,

$$f_N(X) = (X_N, R_N X_N) + (X_N, S_N) + q_N \tag{C.1}$$

Applying the principle of optimality over the *N-1*th time step yields,

$$f_{N-1}(X) = \min_{g_{N-1}}[(QX_{N-1} - d_{N-1}), W(QX_{N-1} - d_{N-1})) + g_{N-1}, Bg_{N-1} + f_N(X)] \tag{C.2}$$

Substituting equation (C.1) into (C.2) and again making use of the fact that,

$$X_N = MX_{N-1} + Pg_{N-1} \tag{C.3}$$

Gives,

$$\begin{aligned}
&(X_{N-1}, R_{N-1}X_{N-1}) + (X_{N-1}, S_{N-1}) + q_{N-1} = \\
&\min_{g_{N-1}}[(QX_{N-1} - d_{N-1}), W(QX_{N-1} - d_{N-1})) + g_{N-1}, Bg_{N-1} + \\
&((MX_{N-1} + Pg_{N-1}), R_N(MX_{N-1} + Pg_{N-1})) + \\
&((MX_{N-1} + Pg_{N-1}), S_N) + q_N]
\end{aligned} \tag{C.4}$$

It is now necessary to fully expand equation (C.4) however; the length of the expansion can be significantly reduced by making use of the identity,

$$(g_{N-1}, Bg_{N-1}) + (g_{N-1}, P^T s_N)/2 + (g_{N-1}, P^T R_N(MX_{N-1} + Pg_{N-1})) = 0 \tag{C.5}$$

Denoting the three terms in equation (C.5) as a, b and c and expanding equation (C.4) accordingly, gives

$$f_{N-1}(X) = X_{N-1}, Q^T WQX_{N-1} - 2X_{N-1}, Q^T Wd_{N-1} + d_{N-1}, Wd_{N-1},$$
$$+g_{N-1}, Bg_{N-1} + ((MX_{N-1} + Pg_{N-1}), R_N(MX_{N-1} + Pg_{N-1})) + \tag{C.6}$$
$$((MX_{N-1} + Pg_{N-1}), S_N) + q_N$$

Completing the expansion and grouping the terms of the identity together yields,

$$f_{N-1}(X) = X_{N-1}, Q^T WQX_{N-1} - 2X_{N-1}, Q^T Wd_{N-1} + d_{N-1}, Wd_{N-1},$$
$$+g_{N-1}, Bg_{N-1} + (g_{N-1}, P^T S_N)/2 + g_{N-1}, P^T R_N(MX_{N-1} + Pg_{N-1}) + \tag{C.7}$$
$$+MX_{N-1}, R_N(MX_{N-1} + Pg_{N-1}) + MX_{N-1}, S_N + (g_{N-1}, P^T S_N)/2 + q_N$$

Completing the expansion yields,

$$f_{N-1}(X) = X_{N-1}, Q^T WQX_{N-1} - 2X_{N-1}, Q^T Wd_{N-1} + d_{N-1}, Wd_{N-1} +$$
$$X_{N-1}, M^T R_N MX_{N-1} + X_{N-1}, M^T R_N Pg_{N-1} + X_{N-1}, M^T S_N + \tfrac{1}{2}(g_{N-1}, P^T S_N) + q_N \tag{C.8}$$

The optimal *g* has already been defined as,

$$g^*_{N-1} = [2B + 2P^T R_N P]^{-1}\{-(P^T S_N + 2P^T R_N MX_{N-1})\} \tag{C.9}$$

The optimal *g* at time *N-1* is contained in two of the terms in equation (B.8) these are,

$$X_{N-1}, M^T R_N Pg_{N-1} \tag{C.10}$$

and

$$\tfrac{1}{2}(g_{N-1}, P^T S_N) \tag{C.11}$$

Equation (C.9) is subsitituted into (C.10) and (C.11) and these are then resubstitued back into equation (C.8). Let

$$D_N = [2B + 2P^T R_N P]^{-1} \tag{C.12}$$

Expanding (C.10) first gives,

$$X_{N-1}, M^T R_N P g_{N-1} = -X_{N-1}, M^T R_N P D_N (P^T S_N + 2P^T R_N M X_{N-1}) \quad \text{(C.13)}$$

$$X_{N-1}, M^T R_N P g_{N-1} = -X_{N-1}, M^T R_N P D_N P^T S_N - 2X_{N-1}, M^T R_N P D_N P^T R_N M X_{N-1} \quad \text{(C.14)}$$

Expanding (C.11) gives,

$$½(g_{N-1}, P^T S_N) = ½ D_N (-P^T S_N + 2P^T R_N M X_{N-1}), P^T S_N \quad \text{(C.15)}$$

$$½(g_{N-1}, P^T S_N) = -½ D_N P^T S_N, P^T S_N + X_{N-1}, (M^T R_N^T P^T D_N^T P^T S_N) \quad \text{(C.16)}$$

Now collecting all terms of *f* at the *N-1* time step gives,

$$\begin{aligned} & f_{N-1}(X) = X_{N-1}, Q^T W Q X_{N-1} - 2X_{N-1}, Q^T W d_{N-1} + d_{N-1}, W d_{N-1} + \\ & X_{N-1}, M^T R_N M X_{N-1} + \\ & -X_{N-1}, M^T R_N P D_N P^T S_N - 2X_{N-1}, M^T R_N P D_N P^T R_N M X_{N-1} + \\ & X_{N-1}, M^T S_N + \\ & ½(-D_N P^T S_N, P^T S_N + X_{N-1}, (M^T R_N^T P^T D_N^T P^T S_N)) \\ & + q_N \end{aligned} \quad \text{(C.17)}$$

Now grouping all like powers of *X* together gives,

$$(X_{N-1}, R_{N-1} X_{N-1}) = X_{N-1}, (Q^T W Q + M^T R_N M - 2M^T R_N P D_N P^T R_N M X_{N-1}) X_{N-1} \quad \text{(C.18)}$$

Let,

$$H_N = 2P^T R_N \quad \text{(C.19)}$$

Substituting equation (C.19) into (C.18) and rearranging gives,

$$(X_{N-1}, R_{N-1}X_{N-1}) = X_{N-1}, (Q^T W Q + M^T (R_N - H_N^T D_N H_N / 2) M) X_{N-1} \quad \text{(C.20)}$$

Therefore

$$R_{N-1} = Q^T W Q + M^T (R_N - H_N^T D_N H_N / 2) M \quad \text{(C.21)}$$

Similarly S at the N-1 time step can be found by grouping the like terms of X,

$$(X_{N-1}, S_{N-1}) = -2X_{N-1}, Q^T W d_{N-1} + X_{N-1}, M^T S_N - X_{N-1}, M^T H_N^T D_N P S_N \quad \text{(C.22)}$$

$$(X_{N-1}, S_{N-1}) = X_{N-1}, (-2Q^T W d_{N-1} + M^T S_N - M^T H_N^T D_N P S_N) \quad \text{(C.23)}$$

Therefore,

$$S_{N-1} = -2Q^T W d_{N-1} + M^T (I - H_N^T D_N P) S_N \quad \text{(C.24)}$$