Appendix B

B.1 Proof by induction of the quadratic nature of the Riccatti equations

The following notation is used throughout,

$$(x, y) = y^T x = \sum_{i=1}^{N} x_i y_i \tag{B.1}$$

$$(x, y) = (y, x) \tag{B.2}$$

For a scalar *a,*

$$(ax, y) = a(x, y) = (x, ay) \tag{B.3}$$

For a matrix A,

$$(x, Ay) = A^T y, x = \sum_{i,j=1}^{N} y_i a_{ij} x_j \tag{B.4}$$

$$Ax, y = x, A^T y \tag{B.5}$$

$$[AB]^T = B^T A^T \tag{B.6}$$

$$[A + B]^T = A^T + B^T \tag{B.7}$$

Similarly

$$[ABC]^T = [(AB)(C)]^T = C^T B^T A^T \tag{B.8}$$

At the last time step of the dynamic programming routine the functional f_N is defined by,

$$f_N(X) = \min_{g_N}[(QX_N - d_N), W(QX_N - d_N)) + (g_N, Bg_N)] \qquad \text{(B.9)}$$

It is assumed that the optimal g at the n^{th} time step is zero; therefore equation (B.9) reduces upon expansion to,

$$f_N(X) = X_N, Q^T WQX_N - X_N, Q^T Wd_N - X_N, Q^T Wd_N + d_N, Wd_N \quad \text{(B.10)}$$

Equation (A.10) is a quadratic in terms of X, defined by,

$$f_N(X) = X_N, Q^T WQX_N - 2X_N, Q^T Wd_N + d_N, Wd_N \qquad \text{(B.11)}$$

It is now necessary to show that the f_{N-1} is a quadratic in terms of X_{N-1} and after that assume that all of the f_N's are quadratics in terms of X. The proof begins by defining the functional for f at the N-1^{th} time step,

$$f_{N-1}(X) = (QX_{N-1} - d_{N-1}), W(QX_{N-1} - d_{N-1}) + g_{N-1}, Bg_{N-1} + f_N(X_N) \quad \text{(B.12)}$$

where the last term in equation (B.12) denotes the cost of the previous time step, making use of the fact that,

$$X_N = MX_{N-1} + Pg_{n-1} \qquad \text{(B.13)}$$

and then substituting equation (B.13) into (B.11) gives

$$\begin{aligned} f_N(X) &= \left(MX_{N-1} + Pg_{n-1}\right), Q^T WQ(MX_{N-1} + Pg_{n-1}) \\ &-2(MX_{N-1} + Pg_{n-1}), Q^T Wd_N + d_N, Wd_N \end{aligned} \qquad \text{(B.14)}$$

This gives the functional for the N^{th} time step in terms of X and g at the N-1^{th} time step, now substituting equation (B.14) into equation (B.12) gives,

$$f(X_{N-1}) = X_{N-1}, Q^T W Q X_{N-1} - 2X_{N-1}, Q^T W d_{N-1} + d_{N-1}, W d_{N-1}$$
$$+ g_{N-1}, B g_{N-1} + \left(M X_{N-1} + P g_{n-1}\right), Q^T W Q (M X_{N-1} + P g_{n-1}) \qquad \text{(B.15)}$$
$$-2(M X_{N-1} + P g_{n-1}), Q^T W d_N + d_N, W d_N$$

Let the unexpanded terms in equation (B.15) be denoted by,

$$\alpha = \left(M X_{N-1} + P g_{n-1}\right), Q^T W Q (M X_{N-1} + P g_{n-1}) \qquad \text{(B.16)}$$
$$\beta = 2(M X_{N-1} + P g_{n-1}), Q^T W d_N \qquad \text{(B.17)}$$

Expanding α with respect to the vector products gives,

$$\begin{aligned} \alpha = {} & X_{N-1}, M^T Q^T W Q M X_{N-1} + X_{N-1}, M^T Q^T W Q P g_{N-1} + \\ & g_{N-1}, P^T Q^T W Q M X_{N-1} + g_{N-1}, P^T Q^T W Q P g_{N-1} \end{aligned} \qquad \text{(B.18)}$$

Now expanding β gives,

$$\beta = 2X_{N-1}, M^T Q^T W d_N + 2g_{N-1}, P^T Q^T W d_N \qquad \text{(B.19)}$$

Now substituting α and β into equation (B.15) and collecting the terms as vector products in the following order, (X_{N-1}, X_{N-1}) (g_{N-1}, X_{N-1}) (d_N, d_N) (d_{N-1}, d_{N-1}) (g_{N-1}, d_{N-1}) and (X_{N-1}, d_N) gives,

$$\begin{aligned} f_{N-1}(X) = {} & X_{N-1}, [Q^T W Q + M^T Q^T W Q M] X_{N-1} - \\ & 2X_{N-1}(M^T Q^T W d_N + Q^T W d_{N-1}) + d_N, W d_N + \\ & d_{N-1}, W d_{N-1} + g_{N-1}[P^T Q^T W Q P + B] g_{N-1} + \\ & 2g_{N-1}, [P^T Q^T W d_N - P^T Q^T W Q M X_{N-1}] \end{aligned} \qquad \text{(B.20)}$$

Now minimising the functional f with respect to g to find the optimal g at the N-1 time step gives,

$$\frac{\delta f_{N-1}(X)}{\delta g_{N-1}} = 2g_{N-1}{,}[P^T Q^T WQP + B] + 2P^T Q^T WQMX_{N-1} - 2P^T Q^T Wd_N = 0$$
(B.21)

The optimal g at the N-1 time step can now be defined by,

$$g^*_{N-1} = [P^T Q^T WQP + B]^{-1}[P^T Q^T Wd_N - P^T Q^T WQMX_{N-1}] \qquad \text{(B.22)}$$

The optimal g at the N-1 time step is now substituted into equation (B.20), which gives,

$$\begin{aligned}
&f_{N-1}(X) = X_{N-1}{,}[Q^T WQ + M^T Q^T WQM]X_{N-1} - \\
&2X_{N-1}(M^T Q^T Wd_N + Q^T Wd_{N-1}) + d_N{,}Wd_N + d_{N-1}{,}Wd_{N-1} + \\
&[P^T Q^T WQP + B]^{-1}[P^T Q^T Wd_N - P^T Q^T WQMX_{N-1}], \\
&[P^T Q^T WQP + B][P^T Q^T WQP + B]^{-1}[P^T Q^T Wd_N - P^T Q^T WQMX_{N-1}] + \\
&2[P^T Q^T WQP + B]^{-1}[P^T Q^T Wd_N - P^T Q^T WQMX_{N-1}], \\
&[P^T Q^T Wd_N - P^T Q^T WQMX_{N-1}]
\end{aligned} \qquad \text{(B.23)}$$

At this stage for ease of derivation it is convenient to make some substitutions for equation (B.23), this substitutions are defined as follows,

$$a = d_N{,}Wd_N + d_{N-1}{,}Wd_{N-1} \qquad \text{(B.24)}$$

$$b = Q^T WQ + M^T Q^T WQM \qquad \text{(B.25)}$$

$$c = M^T Q^T Wd_N + Q^T Wd_{N-1} \qquad \text{(B.26)}$$

$$d = P^T Q^T Wd_N \qquad \text{(B.27)}$$

$$e = P^T Q^T WQM \qquad \text{(B.28)}$$

$$f = [P^T Q^T WQP + B] \qquad \text{(B.29)}$$

Substituting equations (B.24) through (B.29) into equation (B.23) gives,

$$\begin{aligned}
&f_{N-1}(X) = X_{N-1},(b)X_{N-1} - 2X_{N-1},(c) + a + \\
&[f]^{-1}[d - eX_{N-1}],[f][f]^{-1}(d - eX_{N-1}) - \\
&2[f]^{-1}[d - eX_{N-1}],[eX_{N-1} - d]
\end{aligned} \tag{B.30}$$

Equation (B.30) can then be reduced to,

$$\begin{aligned}
&f_{N-1}(X) = X_{N-1},(b)X_{N-1} - 2X_{N-1},(c) + a + ... \\
&-[f]^{-1}[d - eX_{N-1}],[d - eX_{N-1}]
\end{aligned} \tag{B.31}$$

Expanding equation (B.31) yields,

$$\begin{aligned}
&f_{N-1}(X) = X_{N-1},(b)X_{N-1} - 2X_{N-1},(c) + a + \\
&-[f^{-1}d, d - f^{-1}d, eX_{N-1} - f^{-1}eX_{N-1}, d + f^{-1}eX_{N-1}, eX_{N-1}]
\end{aligned} \tag{B.32}$$

Rearranging equation (B.32) and equating like powers of X gives,

$$f_{N-1}(X) = \{X_{N-1}\},[b - e^T f^{-1} e]\{X_{N-1}\} + 2X_{N-1}[-c + d^T f^{-1} e] + a - d^T f^{-1} d \tag{B.33}$$

This shows that the functional f is also a quatratic in terms of X at the *N-1* time step, the final solution is found by substituting the abbreviations defined in equations (B.24) to (B.29) which yields the final solution in quadratic terms defined by,

$$\begin{aligned}
&f_{N-1}\{X\} = \{X_{N-1}\},(Q^T WQ + M^T Q^T WQM - [P^T Q^T WQM]^T \\
&[P^T Q^T WQP + B]^{-1}[P^T Q^T WQM])\{X_{N-1}\} + \\
&2\{X_{N-1}\}(-[M^T Q^T WQP + Q^T Wd_{N-1}] + \\
&[[P^T Q^T Wd_N]^T [P^T Q^T WQP + B]^{-1}[P^T Q^T Wd_N]) + \\
&d_N, Wd_N + d_{N-1}, Wd_{N-1} - \\
&([P^T Q^T Wd_N]^T, [P^T Q^T WQP + B]^{-1}[P^T Q^T Wd_N])
\end{aligned} \tag{B.34}$$