Chapter 3

Literature Review

3.1 Introduction

There are three types of problem related to the interaction of a vehicle bridge system. The first is the direct problem, where the parameters of both the vehicle and bridge are known and the response of both the vehicle and bridge are sought. The second is the inverse problem where the properties and response of the bridge are known and the dynamic vehicle loads are required. The third is the system identification problem, and can be defined as the problem of determining the system, given the input and the response.

There has been an extensive amount of research on the direct problem of the vehicle bridge interaction system, see Brady (2004), Li (2005), Li (2006) for an extensive review. This chapter is a review of the current solutions to the inverse problem of identifying the moving forces on a bridge given the properties of the bridge and the response, often referred to as the moving force identification problem. This class of inverse problem is concerned with predicting the complete time history of the vehicle interaction forces given some of the measured bridge responses.

The field of moving force identification can be almost completely attributed either directly or indirectly to Chan and Law with few exceptions (Dempsey 1998, Gonzalez 2001, Jiang 2003 a & b, Pinkaew 2006). The current methods of moving force identification can be loosely broken into three categories. The first is an exact solution method where the bridge is idealised as either an Euler-Bernoulli, Timoshenko beam or an orthotropic plate. The second class are those that employ a finite element model of the bridge and an optimisation method to determine the moving forces. The third method are those commonly referred to as interpretive methods, where the bridge is modelled using either an exact solution or a finite element formulation of the bridge, and an integration or differentiation scheme is used to interpret the relationship between the measured response and those of the model.

3.2 Exact Solution Methods

3.2.1 A Time Domain Method (TDM)

The time domain method of moving force identification (Law et al 1997) is based upon an exact solution. The bridge is modelled as an Euler-Bernoulli beam with constant cross-section *A,* constant mass per unit length *p*, is viscously damped and is subject to forces moving with constant velocity *c*, see figure 3.1. The equation of motion for an Euler-Bernoulli beam subject to a force moving with constant velocity can be described by,

$$\rho\frac{\partial^2 v(x,t)}{\partial t^2}+C\frac{\partial v(x,t)}{\partial t}+EI\frac{\partial^4 v(x,t)}{\partial x^4}=\delta(x-ct)f(t) \tag{3.1}$$

Where $v(x,t)$ is the deflection of the beam at a point *x* and a time *t*, *C* is the damping property, *E* is the Young's modulus, *I* is the second moment of inertia, *f(t)* is the force and $\delta(t)$ is the dirac delta function. It is assumed that the deflection of the beam can be written as,

$$v(x,t)=\sum_{n=1}^{\infty}\phi_n(x)q_n(t) \tag{3.2}$$

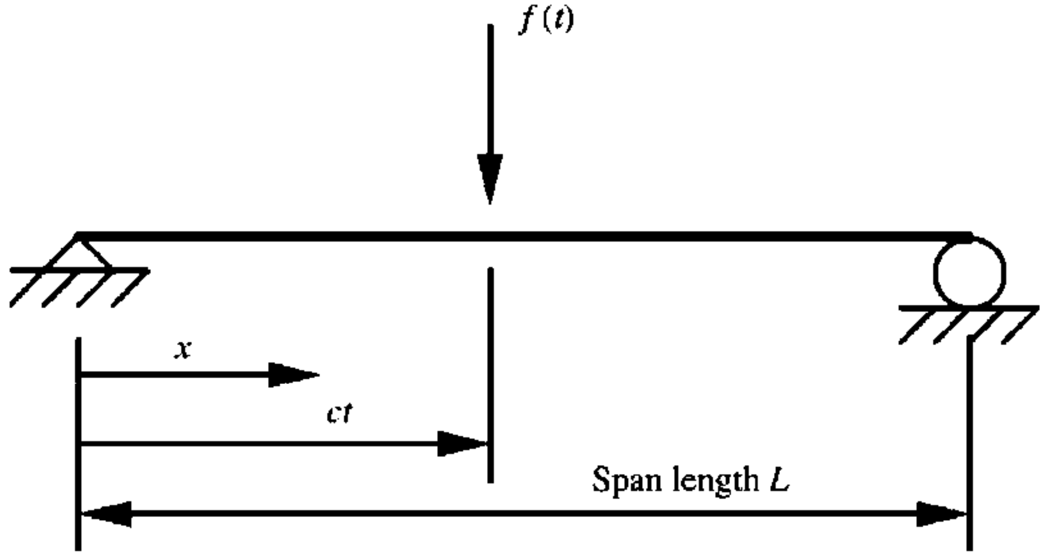


Figure 3.1 - Simply supported beam subject to a moving force *f(t)*
(Chan et al 2001)

and that equation (3.1) can be written in modal coordinates defined by (Law et al 1997),

$$\frac{d^2q_n(t)}{dt^2} + 2\xi_n\omega_n\frac{dq_n(t)}{dt} + \omega_n^2 q_n(t) = \frac{1}{M_n}p_n(t) \tag{3.3}$$

where ξ_n is the damping factor of the n^{th} mode shape and,

$$M_n = \rho L/2 \tag{3.4}$$

$$\phi_n(x) = \sin(n\pi x/L) \tag{3.5}$$

$$\omega_n = (n^2\pi^2/L^2)\sqrt{EI/\rho} \tag{3.6}$$

$$p_n(t) = f(t)\sin(n\pi ct/L) \tag{3.7}$$

The solution as defined by Law et al 1997, Chan et al 2000c, 2001a, Yu & Chan 2003b, to equation (3.3), can be achieved by the convolution integral, see appendix F and is defined by,

$$q_n(t) = \frac{1}{M_n}\int_0^t \frac{1}{\omega_n\sqrt{1-\varepsilon_n^2}} e^{-\varepsilon_n\omega_n(t-\tau)} \sin(\omega_n\left(\sqrt{1-\varepsilon_n^2}\right)(t-\tau)) f(\tau)\sin(n\pi c\tau/L)d\tau \tag{3.8}$$

Now substituting equations (3.8) and (3.6) into (3.2) gives;

$$v(x,t) = \sum_{n=1}^{\infty}\sin(n\pi x/L)\frac{1}{M_n}\int_0^t \frac{1}{\omega_n\sqrt{1-\varepsilon_n^2}} e^{-\varepsilon_n\omega_n(t-\tau)} \sin\left(\left(\omega_n\sqrt{1-\varepsilon_n^2}\right)(t-\tau)\right) f(\tau)\sin(n\pi c\tau/L)d\tau \tag{3.9}$$

The general solution for the displacement in a beam at a point *x* and time *t*, subject to a moving force *f(t)*, defined by equation (3.9), essentially forms the basis for the time domain method of moving force identification (Law et al 1997, Chan et al 2001a).

The inverse problem can now be solved for, in the time domain using bending moments and/or strains as the measurements. If the bending moment of the beam at a point *x* and time *t* is defined by,

$$m(x,t) = -EI\frac{\partial^2 v(x,t)}{\partial x^2} \tag{3.10}$$

substituting equation (3.9) into (3.10) and differentiating twice with respect to x gives a solution for the case of a single moving load, (Law et al 1997, Chan et al 2000c, 2001a, Yu & Chan 2003b)

$$m(x,t) = \sum_{n=1}^{\infty} \frac{2n^2\pi^2}{pL^3} \sin(n\pi x/L) \int_0^{\tau} e^{-\varepsilon\omega_n(t-\tau)} \sin(w(t-\tau)) f(\tau) \sin(n\pi cs/L) d\tau \quad (3.11)$$

If it is assumed that *f (τ)* is a step function over a small time interval, it can be assumed that *f(τ)* is approximately equal to *f(j)* provided the sampling time is small enough. With this assumption the integral over the interval *0* to τ can be written as the sum of the series such that the bending moment at a particular point in time (i) can be rewritten in discrete terms as,

$$m(x,i) = \frac{2EI\pi^2}{\rho l^3} \sum_{n=1}^{\infty} \frac{n^2}{\omega_n} \sin(n\pi x/l) \sum_{j=1}^{i} e^{-\varepsilon_n\omega_n\Delta t(i-j)} \sin(\omega_n \Delta t(i-j)) \sin(n\pi c\Delta tj/l) f(j) \quad (3.12)$$

Law et al (1997), Chan et al 200a & Chan et al (2001a), show how equation (3.12) can be reduced to matrix form defined by,

$$[B]\{f\} = \{m\} \quad (3.13)$$

where *[B]* is the system matrix, *{f}* is the vector of forces to be determined and *{m}* is the vector of measured bending moments. If the number of sampling points N is equal to the number of forces to be determined, then the matrix [B] of equation (3.13) is a lower triangular matrix and the system can be solved by forward substitution. However if the number of measuring points is greater than one, or the number of sampling points is greater than the number of forces to be determined equation (3.13) is then solved by the least squares minimisation (Law et al 1997, Chan et al 200a Yu & Chan 2003a). It is also assumed that both the force at the entry and exit of the vehicle are equal to zero, and that the system is at rest before the entry of the first force such that the bending moment of the first two sampling points is also equal to zero. For the case of multiple force identification, equation (3.13) can be modified using the principle of linear superposition to derive a similar relationship (see Law et al 1997, Zhu 2001, Chan et al 2001a). The moving forces can also be identified using accelerations as the measurements. In this case, equation (3.9) is differentiated twice with respect to time, and the accelerations can be related to the forces via a matrix operator defined by

$$\{\ddot{v}_n\} = [A]\{f\} \tag{3.14}$$

where $\{\ddot{v}_n\}$ is the vector of measured accelerations, $\{f\}$ is the vector of forces to be determined and $[A]$ is the system matrix relating the forces to the accelerations. Equation (3.14) can be solved using a least squares minimisation. Further to this both accelerations and bending moments can be used at the same time to solve for the unknown forces. This can be achieved by normalising both the measurement vectors and their equivalent matrix operators with respect to the Euclidian norm of the measurement vector. This relationship is then defined by (Law et al 1997, Yu 2001, Hung 1999, Chan et al 2001(a)):

$$\begin{bmatrix} B/\|m\| \\ A/\left\|\ddot{v}\right\| \end{bmatrix}\{f\} = \begin{Bmatrix} m/\|m\| \\ \ddot{v}/\left\|\ddot{v}\right\| \end{Bmatrix} \tag{3.15}$$

For the time domain method (TDM), Law et al (1997) define some specific aspects in the implementation of an algorithm. The ratio of the sampling frequency to the highest analysis frequency of interest should be larger than 5; this is to satisfy the assumptions made in the discrete integration of equation (3.12). It is recommended that both accelerations and bending moments be used in the vector of response, as the static component of the force cannot be identified from the acceleration alone. The measurement location is also important in the inverse solution. If only one measurement location is used, it should be at quarter span, and if two measurement locations are used, they should be at quarter and mid-span of the bridge.

Law et al (1997) state that for the cases of both single and multiple moving force identification, the results from bending moments alone do not achieve acceptable results. Acceptable results can be achieved for the case of a single moving force using accelerations and/or bending moments at quarter and mid span of the beam. However for the case of two moving forces, the algorithm exhibits large fluctuations in the predicted force at both the start and end of the time history, although acceptable results are achieved for the time period when both forces are on the beam.

There have been extensive studies into the accuracy of the TDM, both theoretically and experimentally (Chan et al 2000c, 2001a & b, Law et al 2001, Zhu & Law 2002 a & c, Yu 2001, Yu & Chan 2003b). The conclusions and results from these studies can be summarised as follows. It was concluded by Chan et al (2000c) that for the TDM, the minimum number of modes used in the algorithm should be four and that at least three to four measurement stations should be used in the analysis. It was also found that the accuracy of the TDM was independent of the velocity of the vehicle (Chan 2000c), and that for small errors in axle spacing and vehicle velocity, the TDM still gave acceptable results (Chan 2001b). However the TDM is computationally very expensive and highly noise sensitive (Zhu 2001, Zhu & Law 2002c). Chan et al (2000c) state the TDM requires approximately one hour to complete a case study using a Pentium III, and this is for a two axle vehicle. The computational time would obviously increase as the number of axles increase.

The solution to the TDM method can be found using the normal equations or pseudo-inverse (PI) solution defined by,

$$\{f\} = ([B]^T[B])^{-1}[B]^T\{m\} \tag{3.16}$$

where only bending moments are used as the measured response. Yu & Chan (2003b) and Yu (2001), state that, where the TDM employs the PI, it may fail in some cases to accurately identifiy the moving forces, due to the near rank deficiency/ill-conditioning of the matrix $([B]^T[B])^{-1}$. Yu & Chan (2003b) and Yu (2001) demonstrate the robustness of the use of the Singular value decomposition (SVD) (Golub & Van Loan 1989, Lawson & Hanson 1974) in the solution to equation (3.13). The SVD can decompose any $(m \times n)$ rectangular matrix $[B]$ into;

$$B = USV^T \tag{3.17}$$

It is sufficient to say that $[U]$ is an $(m\times m)$ orthogonal matrix, $[V]$ is an $(n\times n)$ orthogonal matrix and $[S]$ is an $(m\times n)$ diagonal matrix of real positive elements, known as the singular values of $[B]$. The solution to equation (3.13) using the SVD can be defined by,

$$\{f\} = [V][S]^{-1}[U]^{T}\{m\} \tag{3.18}$$

Yu & Chan (2003b) and Yu (2001) state that the use of the SVD results in more stable identification of moving forces in cases where the solutions using the PI have failed. However it should be noted that from the comparative studies (Yu 2001), that the difference in errors between identified forces using either PI or SVD are negligible once the number of measuring stations is greater than 3 and the number of modes used in the solution is greater than four.

In general the TDM exhibits large fluctuations in the predicted forces at the start and end of the identified time histories. This is due to the ill-conditioning of the system in the discretization of the inverse problem. The method of Tikhonov regularisation has been employed in the TDM to reduce the errors of the solution (see Zhu 2001, Zhu and Law 2002a, 2002c, 2003c). The TDM method can be reformulated as a least squares minimisation with Tikhonov regularisation defined by the function,

$$J(\{f\},\lambda) = \min\{\|[B]\{f\}-\{m\}\|_2 + \lambda\|[H]\{f\}\|_2\} \tag{3.19}$$

if bending moments alone are used as the measured inputs. The solution to equation (3.13) is then defined by,

$$\{f_\lambda\} = [B^T B + \lambda H^T H]^{-1} B^T\{m\} \tag{3.20}$$

where the solution is now defined for each value of λ. Similar solutions can be derived depending on the nature of the measured input, ie measured bending moments, accelerations or bending moments and accelerations. The optimal regularisation parameter is found using Hansens L-curve method or Generalised Cross validation (Hansen 1992 & 1998). It was found that both methods provide similar values of the regularisation parameter. Zhu and Law (2002c) report that in general the regularisation procedure provides more accurate results than the TDM alone. In particular it reduces the errors calculated at the starts and ends of the time histories of the forces. It is also reported that similar to the TDM, accelerations give better results than bending moments or strains alone. However it should be noted that in the numerical studies, the noise is added to the simulated response as a percentage of the standard deviation of the particular response being used in the inverse analysis. This could result in significantly different signal to noise ratios, and hence one response providing more accurate identified forces than another.

The TDM of moving force identification has also been applied to two span continuous bridges both theoretically and in the laboratory experiments (Zhu 2001, Zhu & Law 2002c & 2006, Chan & Ashebo 2004, 2006 a & b). The application of the TDM to two span continuous bridges is formulated in the exact same manner as outlined at the start of this section. The only significant difference is that the mode shapes and the frequencies must be solved using either the eigen-function of a multiple span continuous Euler-Bernoulli beam (Hayashikawa & Wantanabe 1981, Zhu & Law 2001, 2006 Chan & Ashebo 2006a) or assumed modes of vibration (Lee 1994).

Zhu and law (2002c & 2006) and Chan and Ashebo (2006 a & b) have shown that acceptable results can be obtained using the TDM for the time period when the forces to be predicted are between supports. However for the time periods when the forces are near to or on the support, the predicted forces are inaccurate due to the fact that the numerical values of the modal shape functions approach zero. This in turn affects the conditioning of the system matrix generated from equation (3.19) and thus, as the number of supports and the number of forces increases, the accuracy of the TDM decreases. Chan & Ashebo (2006 a & b), show that more acceptable results can be achieved by applying the TDM to a single span of a continuous bridge, and identifying the moving forces on this span using only measurements from this span.

Chan et al (2000a) have also accounted for the effect of the prestressing in concrete bridges, on the mode shapes and natural frequencies. The effect of prestressing is accounted for in the Euler-Bernoulli equation of motion and the TDM can be implemented as discussed previously. Chan et al (2000a) experimentaly validated the TDM using an existing prestressed concrete bridge in Hong Kong.

3.2.2 A Frequency Time Domain Method (FTDM)

The frequency time domain method (FTDM) (Law et al 1999, Yu & Chan 2003a) of moving force identification is also based on an exact solution method. Similar to the TDM the structure is assumed to be an Euler-Bernoulli beam subject to moving forces. Again the equation of motion is reduced to a decoupled set of equations in modal co-ordinates as defined in equation (3.4). In the frequency time domain method the modal coordinates are mapped to the frequency domain using a Fourier transform, and consequently the inverse Fourier transform can be used to map the solution in the frequency domain back to the time domian. The Fourier transform which maps the modal coordinate $q_n(t)$ to the frequency domain $Q_n(\omega)$, is defined by,

$$Q_n(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} q_n(t)e^{-i\omega t}dt \tag{3.21}$$

If the solution to the n^{th} modal response defined by the convolution integral of equation (3.11) is substituted into equation (3.21), the response in the frequency domain can be defined by (Green & Cebon 1992, Newland 1989),

$$Q_n(\omega) = \frac{1}{M_n}\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_n(t-\tau)p_n(\tau)d\tau dt \tag{3.22}$$

By making a change of,

$$t = \beta + \tau \tag{3.23}$$

the Fourier transform defined in equation (3.22) can be reduced to,

$$Q_n(\omega) = \frac{1}{M_n}\frac{1}{2\pi}\int_{-\infty}^{\infty} h_n(\beta)e^{-i\omega\beta}d\beta\int_{-\infty}^{\infty} p_n(\tau)e^{-i\omega\tau}d\tau \tag{3.24}$$

$$Q_n(\omega) = \frac{1}{M_n} H_n(\omega) P_n(\omega) \tag{3.25}$$

where

$$P_n(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} p_n(\tau) e^{-i\omega t} d\tau \tag{3.26}$$

and

$$H_n(\omega) = \int_{-\infty}^{\infty} h_n(\beta) e^{-i\omega\beta} d\beta = \frac{1}{\omega_n^2 - \omega^2 + 2\xi_n i\omega_n \omega} \tag{3.27}$$

where $H_n(\omega)$ is the frequency response function (see Clough & Penzien 1975, Newland 1989). The Fourier transform of the dynamic deflection can be found by substituting equation (3.25) into (3.2) which yields (Law et al 1999, Yu & Chan 2003a),

$$V(x,\omega) = \sum_{n=1}^{\infty} \frac{1}{M_n} \phi_n(x) H_n(\omega) P_n(\omega) \tag{3.28}$$

By differentiating equation (3.28) twice with respect to ω, the Fourier transform of the acceleration of the beam at a point x can be defined by,

$$\ddot{V}(x,\omega) = -\omega^2 \sum_{n=1}^{\infty} \frac{1}{M_n} \phi_n(x) H_n(\omega) P_n(\omega) \tag{3.29}$$

In a similar manner to the time domain method, the relationship between the measurements and the forces must be reformulated as a discrete relationship defined by,

$$\ddot{V}(x,\omega) = -\sum_{k=0}^{N-1} \sum_{n=1}^{\infty} \frac{\Delta f^3 m^2}{M_n} \Phi_n(x) H_n(m) \Psi_n(m-k) P_n(\omega) \tag{3.30}$$

where m is the particular time step, $N\text{-}1$ is the total number of time steps sampled in the FFT, Ψ_n is the Fourier transform of the n^{th} mode shape and Δf is the frequency resolution. Chan et al (2001), Yu & Chan (2003 a & b) and Law et al (2001b), have demonstrated how equation (3.30) can be written in matrix form, in the frequency domain, containing both real and imaginary parts. The solution is obtained by separating the real and imaginary parts coupled with the inverse Fourier transform, to transform the forces from the frequency domain back to the time domain. In general equation (3.30) reduces to an overdetermined system of equations defined by,

$$\{\ddot{V}\}_{(N+2)\times 1} = [A]_{(N+2)\times(N+2)}\{f\}_{(N+2)\times 1} \tag{3.31}$$

where $\{V\}$ is the vector of measured accelerations, $[A]$ is the system matrix resulting from the discretization of equation (3.31) and $\{f\}$ is the vector of forces to be determined.

The solution to equation (3.31) can be solved using accelerations and or bending moments as the measured response. As is the case with the TDM when both bending moments and accelerations are used at the same time, it is necessary to normalise both the measurement vector and the system matrix, relating these measurements to the forces. The procedure described above is for a single moving force, and the solution must be modified in order to allow for the presence of multiple moving forces on the bridge. This relationship can be achieved through the principle of linear superposition (see Law et al 2001b, Yu & Chan 2003a). The solution to equation (3.32) is generally found using the least squares minimisation, which can be solved using the standard normal equations. However it has been shown (Yu & Chan 2003a, Yu & Chan 2004), that the use of the normal equations in the solution to the least squares minimisation can sometimes fail as a robust solution to the FTDM. In order to overcome the instabilities encountered with the normal equations, the singular value decomposition (SVD), is recommended as a more robust solution see Golub and Von Loan (1989). However it should be noted that the SVD results in a 60% increase in the computation time compared to the normal equations. However it should also be noted, that once the number of measurement stations is greater than or equal to 5, the difference in errors in identified forces using either the PI or SVD is negligible.

Chan et al (2000a & b) make some recommendations in the implementation of the FTDM. It is recommended that the minimum number of modes used in the solution be greater than or equal to 4. The FTDM is sensitive to sensor location and these locations require careful selection. The minimum number of sensors for the solution of 2 forces should be at least 4. However it is unclear whether or not this is an indication of the number of sensors required per force, whether or not 3 forces would require 6 sensors. The accuracy of the FTDM is also dependent on the sampling frequency, as the sampling frequency of the data increases the accuracy of the FTDM reduces. Chan et al (2000) recommend that the sampling frequency for the FTDM be less than 333Hz, as the errors in identified forces become significantly worse after this point. They also recommend the use of accelerations over bending moments as they provide more acceptable results.

Similar studies have considered for levels of noise of the order of 5 or 10% added to simulated data as a product of the standard deviation of the measurement vector (Law et al 2001b, Zhu 2001, Zhu & Law 2002c). It has been found that errors in the identified forces can exceed 1000% where bending moments alone are used and 70% where accelerations alone are used. Zhu and Law (2001b) apply a regularisation method to the FTDM. The regularised form can be defined by the function:

$$f_{\lambda} = \min\{\| [A]f - \ddot{V} \|_2 + \lambda \| [H]f \|_2\} \tag{3.32}$$

where λ is the regularisation parameter, $\{P\}$ is the vector of forces to be predicted and $[A]$ is the system matrix relating the forces to the accelerations $\{\ddot{V}\}$. The solution to equation (3.32) is of a similar form to that of equation (3.20). Law et al (2001b) state that the optimal regularisation parameter can be calculated using either the S-curve (Trujillo & Busby 1997) or the L-curve. The S-curve as defined by Trujillo & Busby (1997) is a plot of the error between the true and identified forces against the regularisation parameter. However in inverse problems clearly the true force is not known. The S-curve can be only used as a numerical assessment of the chosen method to calculate the optimal regularisation parameter. Zhu & Law (2001b) also use the L-curve method to calculate the optimal regularisation parameter. However, instead of plotting the norm of the solution against the residual norm of the error, they plot the seminorm of the difference between estimated forces for two consective values of the regularisation parameter:

$$E1 = \| f_{\lambda_{j+1}}^{identify} - f_{\lambda_j}^{identify} \|_2 \tag{3.33}$$

Law et al (2001b) refer to this as first order regularisation. However this is not first order regularisation as defined in this thesis or that defined by Trujillo & Busby (1997), which predicts the derivative of the forces $\{r\}$ and plots the seminorm of the derivative for each regularisation parameter. It is also unclear what advantage is to be gained from plotting the norm of the difference in predicted forces for consecutive regularisation parameters over that of just plotting the norm of the solution. Nevertheless the regularised form of the FTDM provides significantly better results than that the unregularised version as it reduces the effect of noise in the measured data Zhu and Law (2001b).

3.2.3 Timoshenko Beam Model and Lagrange Formulation

The use of a Timoshenko beam model instead of the Euler-Bernoulli beam model has been suggested by Zhu & Law (1999), Law & Zhu (2000) and Zhu (2001), for the identification of moving forces on multi-span continuous bridges. The Timoshenko beam model is employed, so that the effects of shear deformation and rotary inertia can be considered in the inverse solution. The multi-span continuous bridge is modelled as a continuous Timoshenko beam with (Q-1) intermediate supports subject to N_p moving loads with constant axle spacing and velocity as shown in figure 3.2.

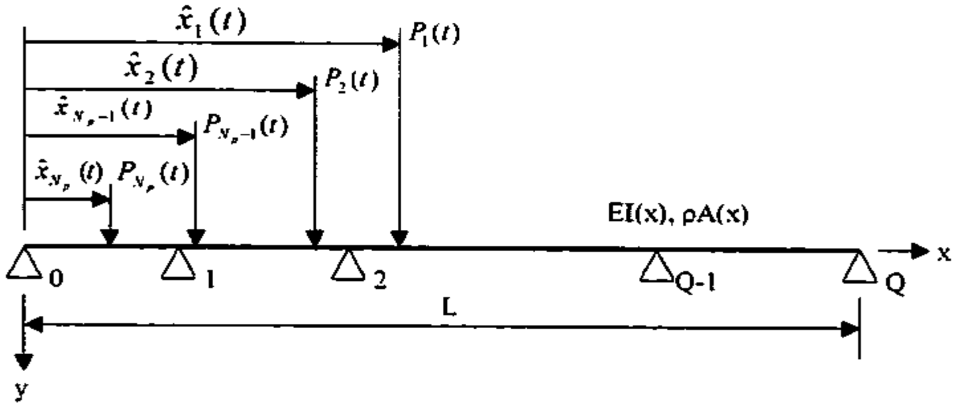


Figure 3.2 - A continuous beam with Q supports subject to N_p moving forces (Zhu and Law 1999)

The bending moment and transverse shear force for a Timoshenko can be defined by, (see Timoshenko and Gere 1961).

$$
\begin{aligned}
M(x) &= EI\frac{\partial\psi(x,t)}{\partial x} \\
V(x) &= \kappa GA[\frac{\partial w(x,t)}{\partial x} - \psi(x,t)]
\end{aligned}
\tag{3.34}
$$

where E is Youngs modulus, I is the second moment of inertia, κ is the shear correction factor, G is the shear modulus of the beam and A is the cross section area. A beam formulation of this type differs from the Euler-Bernoulli formulation in that, the rotation $\psi(x,t)$ at a point x and time t is not equal to the derivative of the translation $w(x,t)$ as would be the case for the Euler-Bernoulli formulation. Zhu & Law 1999, Law & Zhu 2000 and Zhu 2001, state that for the identifcation of moving forces on multi-span continuos bridges of large cross-section, the Euler-Bernoulli beam theory is not sufficent to accuratley model both the large cross-section and the change in cross-section of the bridge. To this end they employ a Timoshenko beam, with non-uniform cross section, where the dynamic properties of the bridge using a Timoshenko beam formualtion cannot be ignored.

This essentially means that both the translation and rotation need to be formulated independently. The functions $\psi(x,t)$ and $w(x,t)$ define the rotation and deflection of the beam at a point x and time t respectively. The total potential energy for the Timoshenko beam can be written as the sum of the kinetic energy (T), the strain energy due to bending and shear, and in the current model intermediate supports are idealised as very stiff linear springs, so the potential energy due to the intermediate supports must also be considered. The total potential energy of the system is equal to the external work done by the moving loads.

If it is assumed that both the translation and rotation response of the beam can be expressed in modal coordinates as,

$$
w(x,t) = \sum_{i=1}^{n} q_i(t)W_i(x) \tag{3.35}
$$

$$\psi(x,t) = \sum_{i=1}^{n} q_i(t)\phi_i(x) \tag{3.36}$$

where q is the modal response and W and ϕ are the assumed modes of vibration for the displacements and the rotations (see Zhu 2001, Zhu & Law 1999). Zhu & Law (1999), Law & Zhu (2000) and Zhu (2001) demonstrate how the Timonshenko beam model be reduced to the equilibrium equation of motion in modal coordinates in matrix form, see appendix G;

$$M\ddot{q}(t) + C\dot{q}(t) + Kq(t) = F(t) \tag{3.37}$$

where

$$\begin{aligned}
&M = \{m_{ij}, i = 1,2,...n; j = 1,2,.....n\} \\
&C = \{c_{ij}, i = 1,2,...n; j = 1,2,...n\} \\
&K = \{k_{ij}^{e} + k_{ij}^{Q}, i = 1,2,...n; j = 1,2,...n\} \\
&q(t) = \{q_1(t), q_2(t),...q_n(t)\} \\
&F(t) = \{f_1(t), f_2(t),...f_n(t)\}
\end{aligned} \tag{3.38}$$

The matrix equation defined in equation (3.37) can be formulated using assumed mode shapes defined by general forms of the vibration modes (Zhu & Law 1999, Zhu 2001) or analytical vibration modes derived from the eigenfunction of a continuous *Q-1* span beam (Hayashikawa & Wantanabe 1981, Zhu & Law 2001). The inverse solution to the problem of moving forces on a Timoshenko beam can be derived with respect to a discrete number of measured displacements on the beam. However it should be noted that both displacements and/or strains can be used as the measured response. If the measured displacements at each point in time is defined by a vector d of size (n_s x 1) where n_s is the number of measurement locations, these measurements can be related to the modal co-ordinates in matrix form as,

$$\{d\}_{n_s \times 1} = [W]_{n_s \times n} \{q\}_{n \times 1} \tag{3.39}$$

where $[W]$ is a matrix containing the assumed modes of vibration. The vector of generalised coordinates can be solved, using the normal equations,

$$\{q\} = [W^T W]^{-1} W^T \{d\} \tag{3.40}$$

The modal velocities and accelerations are then obtained by numerically differentiaiting the modal coordinates calculated from the least squares minimisation defined in equation (3.40). The values obtained for the vectors of modal coordinates, velocities and accelerations are then substituted directly into equation (3.37) to obtain the vector of generalised forces *F(t)*. The vector of generalised forces can be related to the applied forces via in matrix form as

$$\{F\}_{n\times 1} = [B]_{n\times N_f} \{P\}_{N_f \times 1} \tag{3.41}$$

where in this case $[B]$ represents the modal components of the applied load, defined by

$$B = [W_i(x_j(t))]\{i = 1,2,...n; j = 1,2,...N_f\} \tag{3.42}$$

The force at each time step can now be calculated form the least squares minimisation as,

$$\{P\} = [B^T B]^{-1} B^T \{F\} \tag{3.43}$$

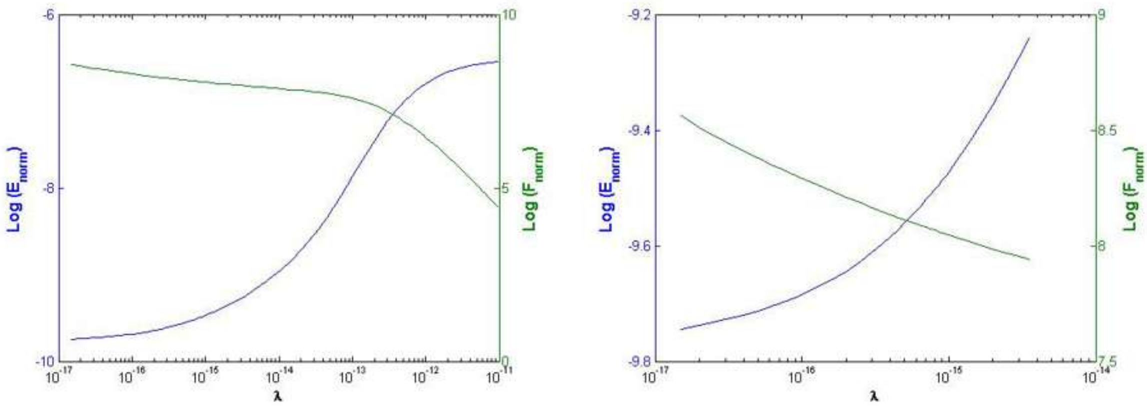
However as stated by Zhu (2001), and Zhu and Law (1999), the least squares minimisation alone is not sufficient to obtain acceptable results. To this end as with the other methods discussed, the method of Tikhonov regularisation is applied to equation (3.63) to yield a smoother solution. However it should be noted that the regularised solution is solved for every point in time, which means that the regularisation parameter is not constant over the total time interval but unique to each point in time. This implies that neither the L-curve nor Generalised cross validation can be used to calculate the optimal regularisation parameter, over the total time interval.

Zhu (2001) states that for the simulation studies of the Timoshenko beam model, the error between the true and estimated forces are minimised for a specific range of the regularisation parameterλ. For practical cases where the true force is not known, it is the error in identified forces between successive incremental values of λ that is minimised (Zhu 2001), which in any case is meaningless unless there is some knowledge of the force.

Zhu (2001) and Law & Zhu (2000) also present a modified L-curve where the semi norm of the solution and the norm of the error are plotted on the same graph against the regularisation parameter. The optimal regularisation parameter occurs at the intersection of the two norms. This modified L-curve appears to have no mathematical basis as in general when these norms are plotted on the same graph, the semi-norm of the solution will in general have negative slope as λ increases and the error norm will have a positive slope. This means that over any non-negative range of λ, the two norms will always intersect at a particular but different regularisation parameter, depending on the range (see figure 3.3). Nevertheless it should be noted that the algorithm does give acceptable results both theoretically and experimentally in the laboratory (Zhu & Law 1999), and the results show a significant improvement of the regularisation over that of the stand alone least squares approach.

The Lagrange formulation of the moving force identification algorithm for the Timoshenko beam model has also been applied to the Euler-Bernoulli beam model. Jiang (2003), Jiang et al (2002), Law & Zhu (2000) and Zhu (2001) use the Euler-Lagrange function to formulate the Euler-Bernoulli Beam model as an equation in generalised modal co-ordinates.

Jiang (2003) employs accelerations as the measured response, and assumed shape functions are used for the modes of vibration. The measured accelerations are related to the accelerations of the generalised coordinates by differentiating the assumed modes of vibration. The moving forces at each time step are then solved using the singular value decomposition of the matrix, relating the vector of generalised forces to the applied forces. Jiang (2003) reports that acceptable results can be obtained in the identification of a single moving force. However the accuracy of the algorithm reduces as the number of forces increases.



(a) – Range 1 **(b) –** Range 2

Figure 3.3 – The modified L-curve over two different ranges of λ

Law & Zhu (2000) and Zhu (2001) have also developed a similar method for the Euler-Bernoulli beam model. The equilibrium equation of motion is derived using the Euler-Lagrange formulation of an Euler-Bernoulli beam as described in this section for the Timoshenko beam. The measured displacements at a discrete number of locations are used as the input to the MFI algorithm. The modal displacements, velocities and accelerations are calculated using a set of orthogonal functions (Law & Zhu 2000, Zhu & Law 2001a), instead of numerical differentiation, as this tends to amplify the measurement noise. The moving forces are then identified in the same manner as that outlined for the Timoshenko beam. Law & Zhu (2000) and Zhu & Law (2001a) demonstrate how the use of the orthogonal functions to calculate the modal displacements, velocities and accelerations means that the regularisation procedure does not have any smoothing effect on the identified results.

Law & Zhu (2000) also state that the Timoshenko model generally gives better results than the Euler-Bernoulli beam model. However it should be stressed that in the comparative study of the Euler-Bernoulli and Timoshenko beam models (Law & Zhu 2000), the simulation of moving forces is carried out on a Timoshenko beam model. Hence better identified results are achieved with the Timoshenko model. In reality the use of either beam model would depend on the actual bridge in question, and what theoretical model better represents the physical behaviour of the bridge.

3.2.4 Moving Force Identification on an Orthotropic Plate

Much of the attention on MFI theory has focused on the use of beam models to represent the bridge dynamics. In many cases the idealisation of the bridge as either an Euler-Bernoulli or Timoshenko beam, is not an accurate representation of bridge characteristics as torsional and lateral modes of vibration can have a significant effect on the overall behaviour of the structure. Zhu and Law have developed two methods to identify moving loads on an orthotorpic plate (see Zhu and Law 2001b, 2002b, 2003a & b).

In both of these methods the bridge is idealised as an orthotropic plate subject to N_f moving forces (see figure 3.4), defined by the differential equation,

$$D_x \frac{\partial^4 w}{\partial x^4} + 2D_{xy} \frac{\partial^4 w}{\partial^2 x \partial^2 y} + D_y \frac{\partial^4 w}{\partial y^4} + C\frac{\partial w}{\partial t} + \rho h \frac{\partial^2 w}{\partial t^2} = \sum_{l=1}^{N_f} p_l(t)\delta(x - x_l(t))\delta(y - y_l(t)) \tag{3.44}$$

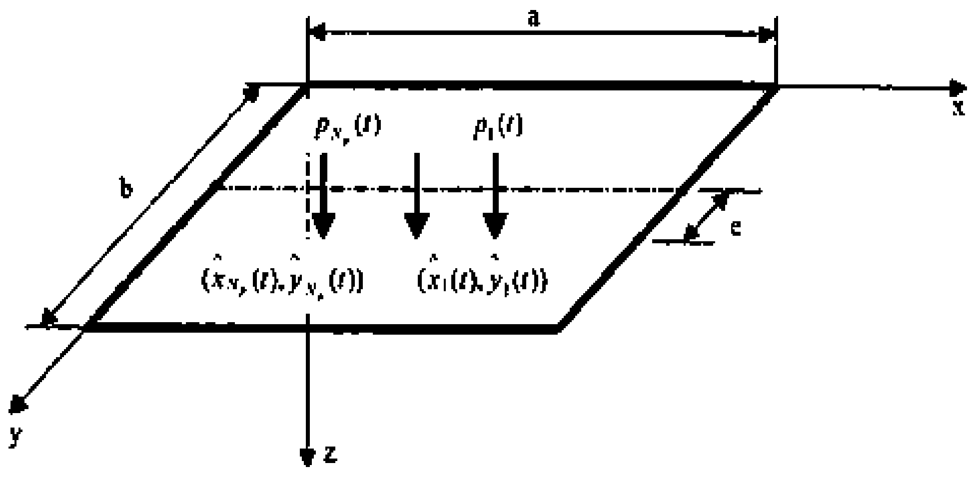


Figure 3.4 – An Orthortropic plate subject to moving forces
Zhu (2001)

where D_x, D_y and D_{xy} are the flexural rigidities in the x, y and xy planes, w is the displacement in the z direction, C is the damping co-efficient h is the plate thickness and ρ is the density. The plate is subject to N_f moving forces, where the truck is idealised as either a group of moving axles or moving individual wheels, $(x_l(t), y_l(t))$ give the position of the l^{th} force with respect to time and δ is the Dirac delta function. The solution to equation (3.44) is based upon the assumed mode shapes of the bridge deck for particular support conditions of the bridge. It is assumed that the plate is simply supported along the edges $x = 0$ and $x = a$, and the displacement of the orthotropic plate can be approximated by,

$$w(x,y,t) = \sum_{m}^{\infty}\sum_{n}^{\infty} W_{mn}(x,y)q_{mn}(t) \tag{3.45}$$

where,

$$W_{mn} = Y_{mn}(y)\sin\left(\frac{m\pi x}{a}\right) \tag{3.46}$$

and where W_{mn} is the mode shape corresponding to the m^{th} mode in the x direction and the n^{th} mode in the y direction. The solution to obtaining the natural frequencies and their equivalent mode shapes is based upon a classification system developed by Jayaraman et al (1990), or the method developed by Zhu and Law (2003d). The assumed displacement function is substituted into equation (3.44), and in a similar manner to that outlined in section 3.2.1 the differential equation for an orthotropic plate can be reduced to an equation in modal coordinates defined by,

$$\ddot{q}_{mn}(t) + 2\xi_{mn}\omega_{mn}\dot{q}_{mn}(t) + \omega_{mn}q_{mn}(t) = \frac{1}{M_{mn}}\sum_{l=1}^{N_f} p_l(t)Y_{mn}(y_l(t))\sin\left(\frac{m\pi\, x(t)}{a}\right) \tag{3.47}$$

Equation (3.67) can be solved in the time domain using the Duhamel integral (see Zhu 2001, Zhu & Law 2001, 2002, 2003), and the displacements can be found using the assumed displacement field defined by,

$$w(x,y,t) = z\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{m\pi}{a}\right)^2 Y_{mn}(y)\sin\left(\frac{m\pi x}{a}\right)\times\frac{1}{M_{mn}}\int_0^t H_{mn}(t-\tau)f_{mn}(\tau)d\tau \tag{3.48}$$

where

$$M_{mn} = \frac{\rho h a}{2}\int_0^b Y_{mn}^2(y)dy \tag{3.49}$$

$$H_{mn}(t) = \frac{1}{\omega_{mn}^{'}}e^{-\xi_{mn}\omega_{mn}^{'}}\sin(\omega_{mn}^{'}t) \tag{3.50}$$

$$\omega_{mn}^{'} = \omega_{mn}\sqrt{1-\xi_{mn}^2} \tag{3.51}$$

$$f_{mn}(t) = \sum_{l=1}^{N_f} p_l(t)Y_{mn}(y_l(t))\sin\left(\frac{m\pi x_l(t)}{a}\right) \tag{3.52}$$

The strains in the *x* direction at a point (*x, y*) and time *t* are derived from equation (3.48) as,

$$\varepsilon_x(x,y,t) = z\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{m\pi}{a}\right)^2 Y_{mn}(y)\sin\left(\frac{m\pi x}{a}\right)\frac{1}{M_{mn}}\int_0^t H_{mn}(t-\tau)f_{mn}(\tau)d\tau \tag{3.53}$$

Equation (3.53) can be written in discrete from, if a discrete number of modes are assumed in the analysis and if *f(τ)* can be assumed to be a step function over a small time interval. Under these assumptions, Zhu and Law (2001b, 2002b, 2003b) demonstrate how equation (3.73) can be reduced to a linear system of equations, which in matrix form is defined by,

$$\{\varepsilon_x\} = [B]\{P\} \tag{3.54}$$

where ε_x is the $\{N_s \times N\}$ vector of measured strains, and $\{P\}$ is the $\{N_p \times N\}$ vector of moving forces to be predicted. Both $\{\varepsilon_x\}$ and $\{P\}$ are essentially partitioned vectors with respect to time where,

$$\{\varepsilon_x\} = \{\varepsilon_x(x_{1,}y_1,1), \varepsilon_x(x_{2,}y_2,1), \ldots \varepsilon_x(x_{N_s,}y_{N_s},1), \ldots\ldots\ldots \varepsilon_x(x_{N_s,}y_{N_s},N)\}^T \quad (3.55)$$

$$\{P\} = \{p_1(0), p_2(0), \ldots p_{N_f}(0), \ldots\ldots, p_{N_f}(N-1)\}^T \quad (3.56)$$

and $[B]$ is the [$(N_s \times N) \times (N_p \times N)$] matrix relating the strains to the forces. Equation (3.54) could be solved using a least squares minimisation, which would give;

$$\{P\} = [B^T B]^{-1} B^T \{\varepsilon_x\} \quad (3.57)$$

However Zhu and Law (2001b) state that, as $\{P\}$ is not a continuous function of the measured data; a regularisation term is added to the least squares to provide a smoother solution to the ill-posed problem. The moving loads can also be determined from the measured accelerations (see Zhu 2001 and Zhu & Law 2001b). In both cases the moving loads are determined using the method of Tikhonov regularisation. The optimal regularisation parameter is determined using either the error in identified forces, the method of generalised cross validation or the L-curve.

Zhu and Law (2003b) state that for acceptable results to be obtained, the number of measurement locations used should be greater than or equal to the number of vibration modes used in the response. Also, for acceptable results to be obtained, the number of modes used should be such that the last mode exceeds the frequency range of excitation of the forces. The method gives good results when accelerations or strains are used as the measured response, but in general using accelerations gives more accurate results than measured strain (Zhu 2001).

The method also gives acceptable results for forces moving along the centreline of the bridge, or for forces moving at a small eccentricity, although forces moving at a small eccentricity are slightly less accurate than those moving along the centreline of the bridge deck. There are large errors in the identified forces at the start and end of the time histories due to the discontinuity of the forces at the entry and exit of the vehicle. However the identified forces between these points are quite acceptable.

Zhu and Law (2003a & b), present a variation on the above method, which they define as moving load identification based on the finite element method. The measured displacements at each point in time are defined by a vector d of size ($n_s \times 1$) where n_s is the number of measurement locations. These measurements can be related to the modal co-ordinates in matrix form as,

$$\{d\}_{n_s \times 1} = [W]_{n_s \times n}\{q\}_{n \times 1} \tag{3.58}$$

where $[W]$ is the $[N_s \times mn]$ matrix relating the vector of measured displacements to the modal coordinates defined by;

$$W = \begin{bmatrix} W_{11}(x_1, y_1) & W_{12}(x_1, y_1) & . & . & W_{mn}(x_1, y_1) \\ W_{11}(x_2, y_2) & W_{12}(x_2, y_2) & . & . & W_{mn}(x_2, y_2) \\ . & & & & . \\ . & & & & . \\ W_{11}(x_{n_s}, y_{n_s}) & W_{12}(x_{n_s}, y_{n_s}) & . & . & W_{mn}(x_{n_s}, y_{n_s}) \end{bmatrix} \tag{3.59}$$

The modal displacements can be found from the least squares minimisation of equation (3.78), which gives:

$$\{q\} = [W^T W]^{-1}[W^T]\{d\} \tag{3.60}$$

Zhu (2001) states that if the displacements are measured, the velocities and accelerations can be obtained directly using a dynamic programming filter (Trujillo and Busby 1990). If the accelerations and velocities are known, the modal velocities and accelerations can be obtained directly from equation (3.60). These are then substituted directly into equation (3.47). The results is a matrix equation of the form,

$$\{B\} = [S]\{P\} \tag{3.61}$$

where

$$\{B\} = \begin{Bmatrix} \ddot{q}_{11}(t) + 2\xi_{11}\omega_{11}\dot{q}_{11}(t) + \omega_{11}q_{11}(t) \\ \ddot{q}_{12}(t) + 2\xi_{12}\omega_{12}\dot{q}_{12}(t) + \omega_{12}q_{12}(t) \\ . \\ \ddot{q}_{1n}(t) + 2\xi_{1n}\omega_{1n}\dot{q}_{1n}(t) + \omega_{1n}q_{1n}(t) \\ . \\ . \\ \ddot{q}_{mn}(t) + 2\xi_{mn}\omega_{mn}\ddot{q}_{mn}(t) + \omega_{mn}q_{mn}(t) \end{Bmatrix} \tag{3.62}$$

and [S] is essentially the right hand side of equation (3.47) defined by,

$$[S] = \begin{bmatrix} \dfrac{2\sin\left(\dfrac{\pi x_1(t)}{a}\right)Y_{11}\left(y_1(t)\right)}{pha\int_0^b Y_{11}^2(y)dy} & . \quad . & \dfrac{2\sin\left(\dfrac{\pi x_{N_f}(t)}{a}\right)Y_{11}\left(y_{N_f}(t)\right)}{pha\int_0^b Y_{11}^2(y)dy} \\ . & & \\ . & & \\ \dfrac{2\sin\left(\dfrac{m\pi x_1(t)}{a}\right)Y_{mn}\left(y_1(t)\right)}{pha\int_0^b Y_{mn}^2(y)dy} & . \quad . & \dfrac{2\sin\left(\dfrac{\pi x_{N_f}(t)}{a}\right)Y_{mn}\left(y_{N_f}(t)\right)}{pha\int_0^b Y_{mn}^2(y)dy} \end{bmatrix} \tag{3.63}$$

Equation (3.61) can be solved using the straight forward least squares. However the solution is again subject to ill-conditioning, so a regularisation term is added to improve it (Zhu 2001, Zhu &Law 2003a). Similar results are obtained using either algorithm; the finite element method in general requires more measured information. However according to Zhu and Law (2003b), both methods fail to identify loads with a large eccentricity.

3.3 Interpretive Methods of Moving Force Identification

3.3.1 Interpretive Method I (IMI)

O'Connor and Chan (1988 a & b) developed a moving force identification algorithm which is commonly referred to as interpretive method I, Chan et al (2000) and (2001). The predictive analysis simulates the structural response caused by a set of loads, which can move, over time, across the bridge, followed by an interpretve analysis to recover the dynamic loads form the predictive analysis. The bridge is modelled as *n* lumped masses interconnected by massless elastic beams, see figure 3.5. The method is based upon the principle that the measured nodal responses are equal to the equivalent static response caused by the loads, less the inertial and damping forces. This is defined by:

$$MNR = NRA - NRI - NRD \tag{3.64}$$

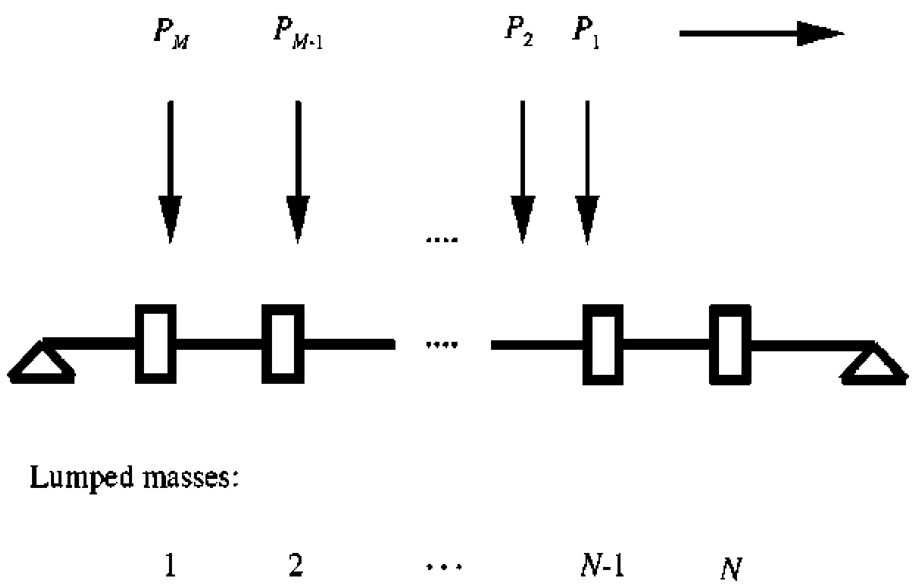


Figure 3.5 - Bridge model for interpretive method I
(Chan et al 2000)

where the abbreviations in equation (3.64) are the measured nodal response (MNR), nodal response due to acting loads (NRA), nodal response due to inertial loads (NRI) and nodal response due to damping loads (NRD). The nodal response due to acting loads can be related to the *m* applied loads via the relationship,

$$\{NRA\} = [Y_A]\{W\} \tag{3.65}$$

where $[Y_A]$ is the $[m \times n]$ matrix relating the *n* nodal responses to the *m* applied loads $\{W\}$. The columns of $[Y_A]$ represent the influence ordinates for a particular node due to a unit wheel load not necessarily at the node. The nodal response due to inertial loads can be expressed as,

$$NRI = [Y^f][M_g]\{\ddot{Y}\} \tag{3.66}$$

where $[Y^f]$ is the flexibility matrix connecting nodal displacements and nodal loads, $[M_g]$ is the $(n \times n)$ lumped mass matrix and $\{\ddot{Y}\}$ is the vector of nodal accelerations. Similarly, if Raleigh damping is assumed, the nodal response due to damping loads can be expressed as,

$$NRD = [Y^f][C]\{\dot{Y}\} \tag{3.67}$$

where $[C]$ is the damping matrix and $\{\dot{Y}\}$ is the vector of nodal accelerations. Substituting equations (3.65) through (3.67) into (3.64) results in the equilibrium equation of motion expressed as the measured nodal response at any instant in time by,

$$\{Y\} = [Y_A]\{W\} - [Y^f][M_g]\{\ddot{Y}\} - [Y^f][C]\{\dot{Y}\} \tag{3.68}$$

The problem can also be expressed as a function of the measured bending moment defined by,

$$\{BM\} = [BM_A]\{W\} - [BM^f][M_f]\{\ddot{Y}\} - [BM^f][C]\{\dot{Y}\} \tag{3.69}$$

where $[BM_A]$ and $[BM^f]$ have meanings similar to $[Y_A]$ and $[Y^f]$ respectively. The applied forces can then be solved using measured strains, displacements, velocities or accelerations. If the measured displacements are known for all interior nodes at all time steps, the nodal velocities and accelerations can be obtained by numerical differentiation

(O'Connor and Chan 1988 a & b). Equation (3.68) or (3.69) becomes a set of overdetermined linear equations which can be solved for $\{W\}$ at each time instant.

If accelerations are measured, nodal velocities and displacements at each time instant can be approximated by numerical integration and then substituted into equation (3.68). O'Connor and Chan state that the use of accelerometer data to predict dynamic wheel loads may result in larger errors, as opposed to strain data as there may be high frequency noise in the measured data. This high frequency noise could induce significant errors in the numerical integration of acceleration data to obtain nodal velocities and displacements.

O'Connor and Chan state that predicted loads are much more sensitive to an error in the measurement of displacements, than an error in the measurement of strains. Hence, strains are the preferred input data of O'Connor & Chan in the dynamic algorithm. However, the use of strains reqiuires the establishment of a relationship between nodal displacements and nodal bending moments. This relationship between displacement and strain requires the unknown loads, W. These loads are approximated by the values in the preceding time step. Then, velocities and accelerations are derived from the nodal displacements. If accelerations, displacements or bending moment are known for all interior nodes at all times, then the number of knowns n will exceed the number of unknown applied forces, which will require a solution through a least-squares minimisation technique.

3.3.2 Interpretive Method II (IMII)

Chan et al (1999, 2000c, 2001a &b) develop a method of moving force identification commonly referred to as Interpretive Method II. The method is similar to IMI: however in IMII a simply supported Euler-Bernoulli beam model is used instead of the lumped mass finite element model used in IMI. The equilibrium equation of motion is reduced to an equation in modal coordinates as defined in equation (3.3). If there are assumed to be k moving loads, at spacing x_k from the first load on the beam, equation (3.3) can be written in matrix form as,

$$\begin{Bmatrix} \dfrac{d^2q_1(t)}{dt^2} \\ \dfrac{d^2q_2(t)}{dt^2} \\ . \\ . \\ . \\ \dfrac{d^2q_n(t)}{dt^2} \end{Bmatrix} + \begin{Bmatrix} 2\xi_1\omega_1\dfrac{dq_1(t)}{dt} \\ 2\xi_2\omega_2\dfrac{dq_2(t)}{dt} \\ . \\ . \\ . \\ 2\xi_n\omega_n\dfrac{dq_n(t)}{dt} \end{Bmatrix} + \begin{Bmatrix} \omega_1^2 q_1(t) \\ \omega_2^2 q_2(t) \\ . \\ . \\ . \\ \omega_n^2 q_n(t) \end{Bmatrix} = \frac{2}{\rho l}[\Phi]\begin{Bmatrix} P_1 \\ P_2 \\ . \\ . \\ . \\ P_k \end{Bmatrix} \tag{3.70}$$

where,

$$[\Phi] = \begin{bmatrix} \sin\left(\dfrac{\pi(ct-\hat{x}_1)}{l}\right) & \sin\left(\dfrac{\pi(ct-\hat{x}_2)}{l}\right) & . \; . \; . & \sin\left(\dfrac{\pi(ct-\hat{x}_k)}{l}\right) \\ \sin\left(\dfrac{2\pi(ct-\hat{x}_1)}{l}\right) & \sin\left(\dfrac{2\pi(ct-\hat{x}_2)}{l}\right) & . \; . \; . & \sin\left(\dfrac{2\pi(ct-\hat{x}_k)}{l}\right) \\ . & & & . \\ . & & & . \\ . & & & . \\ \sin\left(\dfrac{n\pi(ct-\hat{x}_1)}{l}\right) & \sin\left(\dfrac{n\pi(ct-\hat{x}_2)}{l}\right) & . \; . \; . & \sin\left(\dfrac{n\pi(ct-\hat{x}_k)}{l}\right) \end{bmatrix} \tag{3.71}$$

Chan et al (1999) and (2001a) present a closed form solution for the case of moving constant loads. However, an interpretive method is used to predict time varying moving forces. The interpretive method requires that the measured bridge responses at various locations be transformed into modal values (see Chan et al 1999). The measured displacements at a particular location on the beam can be expressed as,

$$v(x,t) = \left\{ \sin\left(\frac{\pi x}{L}\right), \sin\left(\frac{2\pi x}{L}\right), ..., \sin\left(\frac{n\pi x}{L}\right) \right\} \{q_1, q_2, ..., q_n\}^T \tag{3.72}$$

The central difference method is then used to evaluate the corresponding modal velocities and accelerations at the various locations on the beam. Substituting the numerical values for the modal displacements, velocities and accelerations into equation (3.70) results in a set of linear equations similar to that defined in equation (3.61), which can be solved to find the load vector $\{P\}$ at each time instant.

Chan et al (1999), state that either displacements or bending moments can be used as the measured input. However, in general, bending moments from strain generally give better results. The algorithm gives accurate results when no noise is added to the simulated measurements. However when the simulated data contains noise, as would be the case in the field, IMII requires that a filtering scheme be applied to the measurents. Chan et al recommend that a low pass filter be applied to the measured data to reduce the identification errors in IMII. Chan et al also state that the span of the bridge and the axle spacing can significantly affect the accuracy of the algorithm. For bridge spans less than or equal to 10m, they found the error in identified axles always to be less than 5%. For a bridge span equal to 27.4m and axle spacing less than 4m, the error in identified axles was always greater than 10%. Chan et al state that acceptable results can be obtained using IMII for short span bridges with large axle spacing.

Dempsey (1997, 1998) independently developed a dynamic B-WIM algorithm similar to IMII. Although theoretically the algorithms are almost identical, the algorithm developed by Dempsey uses a dynamic bridge model to determine the vehicle axle weights using only the first modal component of the applied force. The mid-span strain is measured due to the passage of a vehicle; the bending moment is then integrated to calculate the displacements. Finally the velocities and accelerations at the measurement position are obtained using the central difference method. The axle weights can be determined by a least squares error function, which minimises the squared difference between the theoretical and experimental approximations of the first modal component of the applied load. The first modal component can be calculated in the same way as outlined for IMII.

3.4 Moving Force Identification using the Finite Element Method

Much of the research on moving force identification has focused on the use of exact solution methods where the bridge is idealised as an Euler-Bernoulli or Timoshenko beam or plate. The use of finite element modelling in the field of moving force identification is much less commonplace, there are currently two methods, excluding IMI, in the literature in which the finite element method is utilised to model the bridge and hence predict the moving forces.

3.4.1 Load Identification using the Finite Element Method

Law et al (2004) developed a moving force identification algorithm based on the finite element method and the dynamic condensation technique. The bridge is discretized into a finite number of beam elements and the complete dynamic model is defined by,

$$[M_g]\{\ddot{y}\}+[C_g]\{\dot{y}\}+[K_g]\{y\}=[L]\{P_{\text{int}}\} \tag{3.73}$$

where $[L]$ is the location matrix and $\{P\}$ is the vector of applied interaction forces to be determined. The solution begins by defining the strain at a distance x and time t in the finite element model given by,

$$\varepsilon(x,t)=-z\frac{\partial^2[N]}{\partial x^2}\{y(t)\} \tag{3.74}$$

as defined in appendix C, where N is the vector of shape functions of the hermite beam element, z is the distance form the neutral axis, y is the vector of displacements. Equation (3.74) can be redefined as,

$$\varepsilon(x,t)=[B]\{y(t)\} \tag{3.75}$$

where $[B]$ is the strain matrix which selects the dof's from the vector $\{y(t)\}$, that calculates the strain at a point x and time t. The strain is then approximated by a set of generalised orthogonal functions (Zhu 2001, Zhu and Law 2001, and Law et al 2004) as,

$$\varepsilon(x,t)=\sum_{i=1}^{N_f}T_i(t)C_i(x) \tag{3.76}$$

where $\{T_i(t), i = 1,2......N_f\}$ is the generalised orthogonal function, N_f is the number of terms used in the polynomial expansion, and $\{C_i(x), i = 1,2....N_f\}$ is the vector of coefficients to be determined. If there are N_s measurements stations, equation (3.76) can be expressed in matrix form as,

$$\varepsilon = [C]\{T\} \tag{3.77}$$

$$\varepsilon = \{\varepsilon(x_1,t), \varepsilon(x_2,t), ..., \varepsilon(x_{N_s},t)\}^T \tag{3.78}$$

$$T = \{T_0(t), T_1(t), ...T_{N_f}(t)\}^T \tag{3.79}$$

$$C = \begin{bmatrix} C_{11}(x_1) & . & . & C_{1N_f}(x_1) \\ . & & & . \\ . & & & . \\ C_{N_s1}(x_{N_s}) & . & . & C_{N_sN_f}(x_{N_s}) \end{bmatrix} \tag{3.80}$$

If the measurements taken on the structure are strains, the vector of coefficients can be obtained using a least squares minimisation:

$$C = \varepsilon T^T [T^T T]^{-1} \tag{3.81}$$

Law et al (2004) state that if equation (3.75) is substituted into equation (3.77) provided the coefficient matrix has been previously found, the degree of freedom vector *y(t)* can be directly solved using a least squares minimisation again, defined by,

$$\{y\} = (G^T G)^{-1} G^T C T \tag{3.82}$$

The [G] matrix above is essentially a matrix representation of the coefficients in the [B] matrix of equation (3.75) for all of the measurement points. The nodal velocities can then be obtained by directly differentiating equation (3.82) with respect to [T] which yields,

$$\{\dot{y}\} = (G^T G)^{-1} G^T C \dot{T} \tag{3.83}$$

and

$$\{\ddot{y}\} = (G^T G)^{-1} G^T C \ddot{T} \tag{3.84}$$

The nodal response can then be directly substituted into equation (3.73), and the vehicle interaction forces can then solved directly using a least squares minimisation. However Law et al (2004) state that for the above formulation, measurements would be required for all the degrees of freedom in the model, and thus the approach is impractical for real structures. Law et al utilise a condensation technique to address this problem by reducing the dynamic degrees of freedom in the FE model.

The condensation technique used is a variation on the Guyan reduction (Craig 1994, Bathe 1996), and is known as the improved reduced system (IRS) technique (O'Callahan 1989). The system is partitioned into what are known as master and slave dof's. Each of the matrices in the equilibrium equation of motion are rearranged in accordance with the new partitioned system; the slave degrees of freedom are condensed out, leaving a smaller system equal in size to the number of master dof's.

The moving force identification scheme outlined in this section can then be implemented using the reduced system. Law et al (2004) state that the number of master dof's used in the inverse analysis should be made less than or equal to the number of measurements. If the least squares alone is used to predict the interaction forces, large errors may arise if the system is ill-conditioned. Law et al (2004) use a regularisation technique to reduce the errors that would be inherent in the least squares minimisation.

Law et al (2004) show that the algorithm is relatively insensitive to sampling frequency, measurement noise, and the velocity of the vehicle. They also state that as the algorithm is based on the finite element method, it could be potentially applied to any finite element model.

However the algorithm is limited in its application. The model employed by Law et al (2004) uses 8 beam elements, and in general 3,5 or 7 measurements points are used in the analysis, so the global stiffness and mass matrices are reduced from [16x16] to a system equal in size to the number of measurement positions. For models of this size, the reduction to a new condensed system is not too large and the main low frequencies of the model will be retained.

For very large systems where the number of dof's is of the order of, 1000 or more, the condensation of the system to a size equal to the number of measurements will inevitable results in errors, as the higher frequencies of the condensed model will be highly inaccurate (Friswell et al 1995, Bouhaddi & Fillod 1996, Lin & Xia 2003). Even for smaller systems, calculation of accurate frequencies using reduction techniques requires iterated IRS schemes and this is only to calculate the frequencies. The reduced system model prior to the iterated IRS schemes will still have significant errors in the higher frequencies of the model. This would obviously have a greater impact. Depending on the number of measurements used in the inverse analysis, a very larger model with few measurement points would not be feasible. A smaller model with many measuring points as applied by Law et al (2004) would obviously produce good results, but is again limited in its application as complex finite element modelling of the structure is not possible.

3.4.2 Optimal State Estimation Approach

Law and Fang (2001) developed a moving force identification algorithm based on dynamic programming and Tikhonov regularisation, similar to the method of force identification outlined in chapter 2. The algorithm developed by Law and Fang is based on the zeroth order regularisation and essentially forms a basis for the research in this thesis. The algorithm of Law and Fang is detailed in chapter 4.

Pinkaew (2006) presents a variation on the algorithm developed by Law & Fang, which uses an updated static component algorithm to iteratively predict the moving vehicle interaction forces. The algorithm uses zeroth order Tikhonov regularisation in conjunction with dynamic programming to give an initial estimate of the moving interaction forces. The static components of the axles are then calculated by time averaging. The system is simulated for these constant loads, and the dynamic strain component is determined from this simulation. The dynamic loads that induce the dynamic strain component are then identified using the algorithm of Law & Fang.

Finally the initial total estimate of the applied load is updated using the predicted dynamic component and the static axle loads are again calculated by time averaging. The process is repeated until the difference between two successive predictions has reached a predefined tolerance.

The algorithm is tested using the simulated strain from a vehicle bridge interaction system, which employs a finite element model of the bridge, a truck with 2 degrees of freedom and a random road profile. However no measurement noise is added. Whilst this may be fair as an initial test of an inverse algorithm, the results are meaningless in terms of the accuracy, as essentially the algorithm has not been tested. Nevertheless, the errors in identified axle forces are as high as 40% for a vehicle travelling at 30m/s on a rough road. In fact the algorithm is velocity dependent, as the velocity reduces and the behaviour of the bridge tends towards the static response, the errors in identified forces reduce. Clearly if the algorithm exhibits errors of this magnitude in theory where no noise has been added, it would be highly unlikely that the reality would be any better and hence there would appear little point in attempting to implement this experimentally.

3.5 Conclusions

A detailed theoretical review of the current methods of moving force identification (MFI) has been presented. The methods can be loosely broken into three categories as follows

The first are those methods that employ and exact solution method coupled with a system identification technique. These are the time domain method (TDM), applied to an Euler-Bernoulli beam model or a plate model, the frequency time domain method (FTDM) applied to an Euler-Bernoulli beam. A Lagrange formulation of the equilibrium equation of motion, where the bridge is formulated as either a Timoshenko or Euler-Bernoulli beam model and the moving loads are identified using a least squares formulation of the modal components of the applied loads.

The second category of moving force identification algorithms are those commonly referred to as interpretive methods of moving force identification namely IMI and IMII. The interpretive methods model the bridge as either, an assembly of lumped masses interconnected by massless elastic beams or as an Euler-Bernoulli beam reduced to *n* decoupled equations of motion in modal coordinates. In both of these cases the measurements can be strains, displacements, velocities or accelerations of either the nodes, or the modal coordinates of the model. An interpretive analysis is then used to relate the measurements to the particular model, using a differentiation or integration scheme. From the inferred displacements, velocities and accelerations of the model, the equivalent dynamic loads at each point in time can be identified using a predictive analysis.

The final identification scheme is that which employs a finite element model of the bridge to identify the vehicle interaction forces. To this author's knowledge there are currently only two algorithms in the literature, which explicitly use a finite element formulation of the bridge, and use this model in the identification process. These are the moving force identification algorithm using an optimal state estimation approach and the condensation technique.

In all of the algorithms outlined, there has been an emphasis placed on the theoretical formulation of the MFI algorithms and much less attention given to the actual computer implementation of these methods, which in some cases (FTDM) is a complex non-trivial process.