Appendix G

G.1 Lagrange Formulation of Timoshenko beam

The bending moment and transverse shear force for a Timoshenko can be defined by, (see Timoshenko and Gere 1961).

$$\begin{aligned} M(x) &= EI\frac{\partial \psi(x,t)}{\partial x} \\ V(x) &= \kappa GA[\frac{\partial w(x,t)}{\partial x} - \psi(x,t)] \end{aligned} \tag{G.1}$$

where E is Youngs modulus, I is the second moment of inertia, κ is the shear correction factor, G is the shear modulus of the beam and A is the cross section area. A beam formulation of this type differs from the Euler-Bernoulli formulation in that, the rotation at a point x and time t is not equal to the derivative of the translation as would be the case for the Euler-Bernoulli formulation. Zhu & Law 1999, Law & Zhu 2000 and Zhu 2001, state that for the identifcation of moving forces on multi-span continuos bridges of large cross-section, the Euler-Bernoulli beam theory is not sufficent to accuratley model both the large cross-section and the change in cross-section of the bridge. To this end they employ a Timoshenko beam, with non-uniform cross section, where the dynamic properties of the bridge using a Timoshenko beam formualtion cannot be ignored.

This essentially means that both the translation and rotation need to be formulated independently. The functions $\psi(x,t)$ and $w(x,t)$ define the rotation and deflection of the beam at a point x and time t respectively. The total potential energy for the Timoshenko beam can be written as the sum of the kinetic energy (T), the strain energy due to bending and shear, and in the current model intermediate supports are idealised as very stiff linear springs, so the potential energy due to the intermediate supports must also be considered. The total potential energy of the system is equal to the external work done by the moving loads. The kinetic energy of the system can be defined as,

$$T = \frac{1}{2}\int_0^L pA\left[\left(\frac{\partial w(x,t)}{\partial t}\right)^2 + \gamma^2\left(\frac{\partial \psi(x,t)}{\partial t}\right)^2\right] \tag{G.2}$$

This is the kinetic energy of the system due to both inertia and rotary inertia. In the above definition γ is the radius of gyration of the cross section. The potential energy due to bending and shear can be defined by,

$$U_e = \frac{1}{2}\int_0^L [EI\left(\frac{\partial \psi(x,t)}{\partial x}\right)^2 + \kappa GA\left(\frac{\partial w(x,t)}{\partial x} - \psi(x,t)\right)^2] \tag{G.3}$$

Equation (G.3) only considers the potential energy in the beam at a point x and time t. However, for the total potential, the energy stored in each spring representing the intermediate supports must also be include to evaluate the total potential energy of the system. The potential energy in the intermediate supports is defined by,

$$U_Q = \frac{1}{2}\sum_{i=1}^{Q-1} k_i w(x,t)^2 \tag{G.4}$$

where k is the equivalent stiffness of each support. The work done by the external forces can be defined by,

$$W = \int_0^L \sum_{i=1}^{N_f} \delta(x - x_i(t)) P_i(t) w(x,t) dx \tag{G.5}$$

If it is assumed that both the translation and rotation response of the beam can be expressed in modal coordinates as,

$$w(x,t) = \sum_{i=1}^{n} q_i(t) W_i(x) \tag{G.6}$$

$$\psi(x,t) = \sum_{i=1}^{n} q_i(t) \phi_i(x) \tag{G.7}$$

where q is the modal response and W and ϕ are the assumed modes of vibration for the displacements and the rotations (see Zhu 2001, Zhu and Law 1999). Substitution of the assumed displacement functions into equations (G.2) through to (G.5) results in the following definitions for the kinetic and potential energies and the external work.

$$T = \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{2}\dot{q}_i(t)m_{ij}\dot{q}_j(t) \tag{G.8}$$

$$U_e = \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{2}q_i(t)k_{ij}^{e}q_j(t) \tag{G.9}$$

$$U_Q = \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{2}q_i(t)k_{ij}^{Q}q_j(t) \tag{G.10}$$

$$W = \sum_{i=1}^{n}q_i(t)f_i(t) \tag{G.11}$$

where

$$m_{ij} = \int_0^L \rho A[W_i(x)W_j(x) + \gamma^2\phi_i(x)\phi_j(x)]dx \tag{G.12}$$

and

$$k_{ij}^{e} = \int_0^L [EI\frac{\partial\phi_i(x)}{\partial x}\frac{\partial\phi_j(x)}{\partial x} + \kappa GA[\frac{\partial W_i(x)}{\partial x} - \phi_i(x)][\frac{\partial W_j(x)}{\partial x} - \phi_j(x)]]dx \tag{G.13}$$

$$k_{ij}^{Q} = \sum_{i=1}^{Q-1}k_iW_i(x)W_j(x) \tag{G.14}$$

$$f_i = \sum_{i=1}^{N_f}P_i(t)W_i(x_i(t)) \tag{G.15}$$

The Lagrange equation of motion can be defined in terms of generalised coordinates (Clough and Penzien 1971) as,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial U}{\partial q_i} = W_i \tag{G.16}$$

The virtual work performed by the external forces W can be replaced with an equivalent statement, which includes the damping forces (Clough and Penzien 1971),

$$W_i = f_i(t) - \sum_{j=1}^{n}c_{ij}\dot{q}_j(t) \tag{G.17}$$

Substituting equations (G.8) through (G.10) and equation (G.17) into the Lagrange equation results in the standard equilibrium equation of motion in matrix form:

$$M\ddot{q}(t) + C\dot{q}(t) + Kq(t) = F(t) \tag{G.18}$$

where

$$M = \{m_{ij}, i = 1, 2, ...n; j = 1, 2, ...n\}$$
$$C = \{c_{ij}, i = 1, 2, ...n; j = 1, 2, ...n\}$$
$$K = \{k_{ij}^{e} + k_{ij}^{Q}, i = 1, 2, ...n; j = 1, 2, ...n\}$$
$$q(t) = \{q_1(t), q_2(t), ...q_n(t)\}$$
$$F(t) = \{f_1(t), f_2(t), ...f_n(t)\}$$