Appendix E

E.1 Newmark-Beta direct integration scheme

The Newmark beta integration scheme is an implicit direct integration scheme, where the dynamic response of the structure is solved at a particular point in time, using values of the dynamic variables at both the current and previous time steps. The size of the time step is governed by the desired accuracy of the solution. The equilibrium equation of motion for an n_{dof} degree of freedom system is defined by,

$$[M_g]\{\ddot{u}(t)\} + [C_g]\{\dot{u}(t)\} + [K_g]\{u(t)] = F(t) \tag{E.1}$$

the following assumptions are made on the nature of the system;

$$\dot{u}(t+\Delta t) = \dot{u}(t) + [(1-\delta)\ddot{u}(t) + \delta\ddot{u}(t+\Delta t)]\Delta t \tag{E.2}$$

$$u(t+\Delta t) = u(t) + \dot{u}(t)\Delta t + [(1-\alpha)\ddot{u}(t) + \alpha\ddot{u}(t+\Delta t)]\Delta t^2 \tag{E.3}$$

In the above definitions α and δ are constants to be determined, for the Newmark beta integration scheme, α and δ are derived from the trapezoidal rule for integration and are defined as ½ and ¼ for α and δ respectively. The integration was programmed in Matlab to allow for damping to be included in the solution, damping can be included if Raleigh damping is assumed where,

$$[C_g] = a[M_g] + b[K_g] \tag{E.4}$$

For ease of derivation the following change of variables are made,

$$\begin{aligned} u &= U \\ \dot{u} &= V \\ \ddot{u} &= A \end{aligned} \tag{E.5}$$

Such that,

$$[M_g]\{A(t)\}+[C_g]\{V(t)\}+[K_g]\{U(t)\}=F(t) \tag{E.6}$$

$$[M_g]\{A(t+\Delta t)\}+[C_g]\{V(t+\Delta t)\}+[K_g]\{U(t+\Delta t)\}=F(t+\Delta t) \tag{E.7}$$

Rewriting equation (E.7) as,

$$[M_g]\{A(t+\Delta t)\}+[a[M_g]+b[K_g]]\{V(t+\Delta t)\}+[K_g]\{U(t+\Delta t)\}=F(t+\Delta t) \tag{E.8}$$

Substituting equations (E.2) and (E.3) into (E.8) yields the general equation of motion at time $t+\Delta t$ as,

$$\begin{aligned}&[M_g]\{A(t+\Delta t)\}+[a[M_g]+b[K_g]]\{\dot{u}(t)+[(1-\delta)A(t)+\delta A(t+\Delta t)]\Delta t\}+\ldots\\&[K_g]\{U(t)+V(t)\Delta t+[(1-\alpha)A(t)+\alpha A(t+\Delta t)]\Delta t^2\}=F(t+\Delta t)\end{aligned} \tag{E.9}$$

Rearrange equation (E.9) such that all terms of index $t+\Delta t$ are brought to the left hand side of the equals sign and all terms of index t are brought to the right hand side of the equals sign. However the vector of forces at time $t+\Delta t$ remains on the right hand side, the resulting equation is,

$$\begin{aligned}&[M_g]A(t+\Delta t)+\Delta t\delta a[M_g]A(t+\Delta t)+\Delta t\delta b[K_g]A(t+\Delta t)+\Delta t^2\alpha[K_g]A(t+\Delta t)=\\&F(t+\Delta t)-a[M_g]V(t)-b[K_g]V(t)-\Delta t[K_g]V(t)-\\&\Delta ta[M_g](1-\delta)A(t)-\Delta tb[K_g](1-\delta)A(t)+\Delta t^2(\tfrac{1}{2}-\alpha)[K_g]A(t)-[K_g]U(t)\end{aligned} \tag{E.10}$$

Grouping all terms of equal index together in A, V and U equation (E.10) can be rearranged as,

$$\begin{bmatrix} [M_g]+ \\ \Delta t\delta a[M_g]+ \\ \Delta t\delta b[K_g]+ \\ \Delta t^2\alpha[K_g] \end{bmatrix}\{A(t+\Delta t)\} =$$

$$F(t+\Delta t)-\begin{bmatrix} a[M_g]+ \\ b[K_g]+ \\ \Delta t[K_g] \end{bmatrix}\{V(t)\}-\begin{bmatrix} \Delta ta[M_g](1-\delta)+ \\ \Delta tb[K_g](1-\delta)+ \\ \Delta t^2(\tfrac{1}{2}-\alpha)[K_g] \end{bmatrix}\{A(t)\}-[K_g]\{U(t)\} \tag{E.11}$$

The acceleration at time $t+\Delta t$ can now be calculated as,

$$\{A(t+\Delta t)\} =$$

$$\begin{bmatrix} [M_g]+ \\ \Delta t\delta a[M_g]+ \\ \Delta t\delta b[K_g]+ \\ \Delta t^2\alpha[K_g] \end{bmatrix}^{-1}\left\{F(t+\Delta t)-\begin{bmatrix} a[M_g]+ \\ b[K_g]+ \\ \Delta t[K_g] \end{bmatrix}\{V(t)\}-\begin{bmatrix} \Delta ta[M_g](1-\delta)+ \\ \Delta tb[K_g](1-\delta)+ \\ \Delta t^2(\tfrac{1}{2}-\alpha)[K_g] \end{bmatrix}\{A(t)\}-[K_g]\{U(t)\}\right\} \tag{E.12}$$

For ease of derivation let,

$$X=[M_g]+\Delta t\delta a[M_g]+\Delta t\delta b[K_g]+\Delta t^2\alpha[K_g] \tag{E.13}$$

$$Y=a[M_g]+b[K_g]+\Delta t[K_g] \tag{E.14}$$

$$Z=\Delta ta[M_g](1-\delta)+\Delta tb[K_g](1-\delta)+\Delta t^2(\tfrac{1}{2}-\alpha)[K_g] \tag{E.15}$$

Substituting equations (E.13) to (E.14) into (E.12) gives the general solution to the equilibrium equation of motion with damping at time $t+\Delta t$ as,

$$\{A(t+\Delta t)\}=[X]^{-1}\{F(t+\Delta t)-[Y]\{V(t)\}-[Z]\{A(t)\}-[K_g]\{U(t)\} \tag{E.16}$$

If zero damping is assumed a and b are equal to zero and equation (E.16) reduces to,

$$\{A(t+\Delta t) =$$
$$[[M_g] + \Delta t^2 \alpha [K_g]]^{-1} \{F(t+\Delta t) - \Delta t [K_g]\{V(t)\} - \Delta t^2 (1/2 - \alpha)[K_g]\{A(t)\} - [K_g]\{U(t)\}\}$$

(E.17)

To initialise the Newmark Beta direct integration scheme it is assumed that at time zero when the forcing function is initially applied, both the displacement and the velocity are equal to zero, therefore,

$$A(0) = [M_g]^{-1}\{F(t)\} \tag{E.18}$$

The acceleration at time $t+\Delta t$ can now be calculated from equation (E.16) and the velocities and displacements from equations (E.2) and (E.3).