Appendix D

D.1 Derivation of the Stiffness and Consistent mass matrices of a Euler Bernoulli beam.

The differential equation for an Euler-Bernoulli beam can be defined as follows,

$$\rho \frac{\delta^2 v}{\delta t^2} + \frac{\delta^2}{\delta x^2}\left(EI \frac{\delta^2 v}{\delta x^2} \right) = q(x,t) \tag{D.1}$$

Where *v(x,t)* is the transverse displacement of the beam and *q(x,t)* is the externally applied pressure. The shear force *V* and bending moment *M* can be defined as follows,

$$V = -EI\left(\frac{\delta^3 v}{\delta x^3} \right) \tag{D.2}$$

$$M = EI\left(\frac{\delta^2 v}{\delta x^2} \right) \tag{D.3}$$

The finite element model of the beam uses a two noded bar element with two degrees of freedom per node, a transverse displacement and a rotation. As there are four degrees of freedom in the model it is assumed tht a cubic polynomial can adequatly represent the displacement of the beam at a distance *x* from the left hand side. The displacement *v* is defined by,

$$v(x) = \alpha_0 + \alpha_1 x + \alpha_3 x^2 + \alpha_4 x^3 \tag{D.4}$$

The rotation is defined by,

$$\theta(x) = \frac{\delta v(x)}{\delta x} = \alpha_1 + 2\alpha_2 x + 3\alpha_3 x^2 \tag{D.5}$$

Defining a vector of displacements containing v and θ as,

$$\begin{bmatrix} v \\ \theta \end{bmatrix} = \begin{bmatrix} 1 & x & x^2 & x^3 \\ 0 & 1 & 2x & 3x \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} \tag{D.6}$$

Applying the boundary conditions at nodes one and two, at node one x is equal to zero and at node two x is equal to L. The vector of elemental dispacements can be defined by,

$$\begin{bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & L & L^2 & L^3 \\ 0 & 1 & 2L & 3L^2 \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} \tag{D.7}$$

If the vector of elemental displacements is defined by d and the vector of coeffiecents by α, equation (D.7) can be written as,

$$\{d\} = [A]\{\alpha\} \tag{D.8}$$

The vector of coefficents can then be solved by inverting the matrix A where,

$$[A]^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{3}{L^2} & -\frac{2}{L} & \frac{3}{L^2} & -\frac{1}{L} \\ \frac{2}{L^3} & \frac{1}{L^2} & -\frac{2}{L^3} & \frac{1}{L^2} \end{bmatrix} \tag{D.9}$$

$$\begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{3}{L^2} & -\frac{2}{L} & \frac{3}{L^2} & -\frac{1}{L} \\ \frac{2}{L^3} & \frac{1}{L^2} & -\frac{2}{L^3} & \frac{1}{L^2} \end{bmatrix} \begin{bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{bmatrix} \tag{D.10}$$

Now substituting the coefficents from equation (D.10) into equation (D.4) gives,

$$v(x) = v_1 + \theta_1 x + \left(-\frac{3}{L^2} v_1 - \frac{2}{L}\theta_1 + \frac{3}{L^2} v_2 - \frac{1}{L}\theta_2 \right) x^2 + \left(\frac{2}{L^3} v_1 + \frac{1}{L^2}\theta_1 - \frac{2}{L^3} v_2 + \frac{1}{L^2}\theta_2 \right) x^3 \tag{D.11}$$

Seperating the varibales into v_1, θ_1, v_2 and θ_2 gives,

$$v(x) = v_1 \left(1 - \frac{3x^2}{L^2} + \frac{2x^3}{L^3} \right) + \theta_1 \left(x - \frac{2x^2}{L} + \frac{x^3}{L^2} \right) + v_2 \left(\frac{3x^2}{L^2} - \frac{2x^3}{L^3} \right) + \theta_2 \left(\frac{-x^2}{L} + \frac{x^3}{L^2} \right) \tag{D.12}$$

Defining the displacement v by,

$$v(x) = N_1 v_1 + N_2 \theta_1 + N_3 v_2 + N_4 \theta_2 \tag{D.13}$$

Where N are the Hermite shape functions describing the behaviour of the beam,

$$N_1(x) = \left(1 - \frac{3x^2}{L^2} + \frac{2x^3}{L^3} \right) \tag{D.14}$$

$$N_2(x) = \left(x - \frac{2x^2}{L} + \frac{x^3}{L^2} \right) \tag{D.15}$$

$$N_3(x) = \left(\frac{3x^2}{L^2} - \frac{2x^3}{L^3} \right) \tag{D.16}$$

$$N_4(x) = \left(\frac{-x^2}{L} + \frac{x^3}{L^2} \right) \tag{D.17}$$

The strain at a particulat point in the element can be defined by,

$$\varepsilon_x = \frac{\delta^2 v}{\delta x^2} \tag{D.18}$$

The strain can be defined in terms of the shape functions and the vector of elemental displacements as,

$$\varepsilon_x = \frac{\delta^2}{\delta x^2}\left([N]\{d\}\right) = \frac{\delta^2 [N]}{\delta x^2}\{d\} \tag{D.19}$$

Rewritting equation (D.19) as,

$$\varepsilon_x = [B]\{d\} \tag{D.20}$$

where [B] is the strain displacement matrix, relating nodal displacements to element strains. The strain in the element can now be written as,

$$\varepsilon_x = [B]\{d\} = \left\{ \frac{\delta^2 N_1(x)}{\delta x^2} \quad \frac{\delta^2 N_2(x)}{\delta x^2} \quad \frac{\delta^2 N_3(x)}{\delta x^2} \quad \frac{\delta^2 N_4(x)}{\delta x^2} \right\} \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix} \tag{D.21}$$

from the principle of minimum potential energy the elemental stiffnes matrix can be defined as,

$$K^e = \int_{\sigma}[B]^T EI[B\}dvol = EI\int_0^L [B]^T[B]dx \tag{D.22}$$

$$K^e = EI\int_0^L \begin{Bmatrix} \frac{6}{L^2}-\frac{12x}{L^3} \\ \frac{4}{L}-\frac{6x}{L^2} \\ \frac{-6}{L^2}+\frac{12x}{L^3} \\ \frac{2}{L}-\frac{6x}{L^2} \end{Bmatrix} \begin{Bmatrix} \frac{6}{L^2}-\frac{12x}{L^3} & \frac{4}{L}-\frac{6x}{L^2} & \frac{-6}{L^2}+\frac{12x}{L^3} & \frac{2}{L}-\frac{6x}{L^2} \end{Bmatrix} dx$$

(D.23)

Check K(1,1),

$$K_{11}^e = EI\int_0^L \left(\frac{6}{L^2}-\frac{12x}{L^3}\right)^2 dx = \frac{12EI}{L^3} \tag{D.24}$$

The stiffness matrix for the beam element can then be defined by,

$$K^e = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6l \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \tag{D.25}$$

The consistent mass matrix can be defined by,

$$M^e = \int_0^L \rho A[N]^T[N]dx \tag{D.26}$$

$$M^{e} = \rho A \int_{0}^{L} \begin{Bmatrix} 1 - \frac{3x^2}{L^2} + \frac{2x^3}{L^3} \\ x - \frac{2x^2}{L} + \frac{x^3}{L^2} \\ \frac{3x^2}{L^2} - \frac{2x^3}{L^3} \\ \frac{-x^2}{L} + \frac{x^3}{L^2} \end{Bmatrix} \begin{Bmatrix} 1 - \frac{3x^2}{L^2} + \frac{2x^3}{L^3} \\ x - \frac{2x^2}{L} + \frac{x^3}{L^2} \\ \frac{3x^2}{L^2} - \frac{2x^3}{L^3} \\ \frac{-x^2}{L} + \frac{x^3}{L^2} \end{Bmatrix}^{T} dx \qquad \text{(D.27)}$$

Check M(1,1),

$$M_{44}^{e} = \rho A \int_{0}^{L} \left[\frac{-x^2}{L} + \frac{x^3}{L^2} \right]^2 dx = \frac{x^5}{5L^2} + \frac{x^7}{7L^4} - \frac{2x^5}{L^3} \Big|_{0}^{L} = \frac{\rho A L^3}{105} = \frac{4L^3 \rho A}{420} \qquad \text{(D.28)}$$

The consistent mass matrix for the beam element can now be defined by,

$$M_e = \rho A L / 420 \begin{pmatrix} 156 & 22L & 54 & -13L \\ 22L & 4L^2 & 13L & -3L^2 \\ 54 & 13L & 156 & -22L \\ -13L & -3L^2 & -22L & 4L^2 \end{pmatrix} \qquad \text{(D.29)}$$