E

RUNGE-KUTTA METHOD

The solution of a first order differential equation by the method of Runge-Kutta is presented. Then, the method of Runge-Kutta is applied to a second order differential equation. The latter is solved by transforming the second order differential equation into two first order equations.

E.1 FIRST ORDER DIFFERENTIAL EQUATION

Equation E.1 represents a first order differential equation.

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$$\frac{dz}{dt} = g(t,z) \tag{E.1}$$

z(t) can be expressed in terms of the Taylor series in the neighborhood of z_i. Given a time increment $h = \Delta t$, if the first derivative is replaced by an average slope and the higher order derivatives are ignored, when a value z_i is known, the next increment z_{i+1} is given as shown in Equation E.2:

$$z_{i+1} = z_i + \left(\frac{dz}{dt}\right)_{i\,av} h \tag{E.2}$$

If Runge-Kutta method is used, the average slope in the interval *h* is made of four estimates of the increment as shown in Equation E.3:

$$z_{i+1} = z_i + \frac{h}{6}(Y_1 + 2Y_2 + 2Y_3 + Y_4) \tag{E.3}$$

where

$$Y_1 = hg(t_i, z_i) \tag{E.4}$$

$$Y_2 = hg(t_i + \frac{h}{2}, z_i + \frac{Y_1}{2}) \tag{E.5}$$

$$Y_3 = hg(t_i + \frac{h}{2}, z_i + \frac{Y_2}{2}) \tag{E.6}$$

$$Y_4 = hg(t_i + h, z_i + Y_3) \tag{E.7}$$

E.2 SECOND ORDER DIFFERENTIAL EQUATION

For example, the differential equation for a single degree of freedom system has the form given by Equation E.1:

$$m\frac{d^2z(t)}{dt^2} + c\frac{dz(t)}{dt} + kz(t) = f(t) \tag{E.8}$$

where m, c and k are constants, and $z(t)$ and $f(t)$ are functions depending on variable t.

By re-arranging Equation E.1, it is obtained:

$$\frac{d^2z(t)}{dt^2} = \frac{1}{m}\left[f(t) - kz(t) - c\frac{dz(t)}{dt} \right] \tag{E.9}$$

By re-writting $y(t) = \frac{dz(t)}{dt}$,

$$\frac{d^2z(t)}{dt^2} = \frac{1}{m}\left[f(t) - kz(t) - cy(t) \right] \tag{E.10}$$

and if re-naming $F(z,y,t)=\frac{d^2z}{dt^2}$:

$$F(z,y,t)=\frac{1}{m}\left[f(t)-kz(t)-cy(t)\right] \tag{E.11}$$

So, Equation E.8 has been reduced to the following two first order equations:

$$y(t)=\frac{dz(t)}{dt} \tag{E.12}$$

$$\frac{dy(t)}{dt}=F(z,y,t) \tag{E.13}$$

If Runge-Kutta method is used, the average slope in the interval h is made of four terms and four values (t, z, y and F) as shown in the following recurrence formulae:

$$z_{i+1}=z_i+\frac{h}{6}(Y_1+2Y_2+2Y_3+Y_4) \tag{E.14}$$

$$y_{i+1}=y_i+\frac{h}{6}(F_1+2F_2+2F_3+F_4) \tag{E.15}$$

The values in Equations E.14 and E.15 are obtained as shown in Table E.1.

Table E.1 – Runge-Kutta terms

t	z	$y = \frac{dz}{dt}$	$F = \frac{dy}{dt} = \frac{d^2 z}{dt^2}$
$T_1 = t_i$	$Z_1 = z_i$	$Y_1 = y_i$	$F_1 = f(T_1, Z_1, Y_1)$
$T_2 = t_i + \frac{h}{2}$	$Z_2 = z_i + Y_1 \frac{h}{2}$	$Y_2 = y_i + F_1 \frac{h}{2}$	$F_2 = f(T_2, Z_2, Y_2)$
$T_3 = t_i + \frac{h}{2}$	$Z_3 = z_i + Y_2 \frac{h}{2}$	$Y_3 = y_i + F_2 \frac{h}{2}$	$F_3 = f(T_3, Z_3, Y_3)$
$T_4 = t_i + h$	$Z_4 = z_i + Y_3 h$	$Y_4 = y_i + F_3 h$	$F_4 = f(T_4, Z_4, Y_4)$

Displacement (z), velocity $\left(\frac{dz}{dt}\right)$ and acceleration $\left(\frac{d^2 z}{dt^2}\right)$of the system can be calculated for each iteration i. The initial values are $z_{t=t_0}$ and $\left(\frac{dz}{dt}\right)_{t=t_0}$ However, in the case of a truck crossing a bridge, there is not force applied at t = 0 and the initial conditions are zero. Then, the computation can be started by assuming the acceleration varies linearly from $\left(\frac{d^2 z}{dt^2}\right)_{t=t_0} = 0$ to $\left(\frac{d^2 z}{dt^2}\right)_{t=t_1}$ during the first time interval.