Appendix A

A.1 Solution to the Matrix Differential Equation

The vector matrix differential equation is defined in state space as,

$$\left(\frac{d\{X\}}{dt} - [A]\{X\} \right) = f(t) \tag{A.1}$$

If $e^{-[A]t}$ is used as an integration factor, equation (A.1) can be replaced by

$$e^{-[A]t} \left(\frac{dx}{dt} - [A]\{X\} \right) = e^{-[A]t} \{f(t)\} \tag{A.2}$$

By the chain rule;

$$\frac{d(e^{-[A]t} X)}{dt} = e^{-[A]t} \frac{dx}{dt} - e^{-[A]t} [A]\{X\} \tag{A.3}$$

Hence,

$$\frac{d(e^{-[A]t} X)}{dt} = e^{-[A]t} \left(\frac{dx}{dt} - [A]\{X\} \right) \tag{A.4}$$

and equating equations (A.4) and (A.1) yields a differential equation of the form:

$$\frac{d(e^{-[A]t} X)}{dt} = e^{-[A]t} f(t) \tag{A.5}$$

For a second order differential equation in order to uniquely determine an integral curve of a second order equation, it is necessary to specify not only a point through which it passes but also the slope of the curve at the point (Boyce, bellman, Bang). Although $[A]$ is constant over the total time interval $f(t)$ is not, so the right hand side of equation (2.19) contains an integral of an unknown function. Upon integrating equation (A.5) between 0 and t and making using of the Duhamel integral, Dyke (1990), Boyce and Diprima (1977) gives a solution of the form,

$$e^{-[A]t}X(t) = X(0) + \int_{o}^{t} e^{-[A]\tau} f(\tau) d\tau \tag{A.6}$$

Dividing equation (A.6) by $e^{-[A]t}$ gives a differential equation defined by,

$$X(t) = e^{[A]t}X(0) + \int_{0}^{t} e^{[A](t-\tau)} f(\tau) d\tau \tag{A.7}$$