Appendix H

H.1 Classical Plate Bending theory

The classical Kirchhoff plate bending theory is based on the following assumptions (Mansfield 1989, Reddy 1993).

(i) A straight line normal to the midplane of the plate before deformation remains normal after deformation.

(ii) The stresses normal to the midplane of the plate are negligible in comparison with those in the plane of the plate

This is equivalent to defining the conditions that transverse deformation is neglected and the strains in the *z*, *yz* and *xz* directions are equal to zero. The transverse shear strain can then be neglected and the inplane strains can be written as,

$$\{\varepsilon_x, \varepsilon_y, \varepsilon_{xy}\} = -z\{\kappa_x, \kappa_y, \kappa_{xy}\} \tag{H.1}$$

where $\varepsilon_x, \varepsilon_y$ and ε_{xy} are the strains in the x, y and xy directions/planes and κ are the curvatures in these directions/planes respectively, z is the distance from the centroid. The in-plane displacements u in the x direction and v in the y direction can be defined by their slopes about the x and y axes, such that the displacement field is defined by,

$$\begin{aligned} u &= -z\frac{\partial w}{\delta x} \\ v &= -z\frac{\partial w}{\delta y} \\ w &= w \end{aligned} \tag{H.2}$$

where u and v are the displacements in the x and y directions, while w is the vertical displacement in the z-direction. The strains in the x-y directions/planes now can be defined by,

$$\{\varepsilon_x, \varepsilon_y, \varepsilon_{xy}\} = \left\{\frac{\partial u}{\partial x} \quad \frac{\partial v}{\partial y} \quad \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right\} = \left\{-z\frac{\partial^2 w}{\partial x^2} \quad -z\frac{\partial^2 w}{\partial y^2} \quad 2\frac{\partial^2 w}{\partial x \delta y}\right\} \tag{H.3}$$

Assuming plane stress conditions, the constitutive equation for a thin plate can be defined by,

$$\{\sigma\} = -z[D]\{\kappa\} \tag{H.4}$$

where

$$\{\sigma\} = \{\sigma_x, \sigma_y, \sigma_{xy}\}^T \tag{H.5}$$

and $[D]$ is the constitutive matrix for an orthotropic plate defined by (Reddy 1993, Zienkiewicz 1991)

$$D = \begin{bmatrix} \frac{E_x}{(1-\nu_x\nu_y)} & \frac{\nu_y E_x}{(1-\nu_x\nu_y)} & 0 \\ \frac{\nu_x E_y}{(1-\nu_x\nu_y)} & \frac{E_y}{(1-\nu_x\nu_y)} & 0 \\ 0 & 0 & G_{xy} \end{bmatrix} = \begin{bmatrix} D_{11} & D_{12} & 0 \\ D_{21} & D_{22} & 0 \\ 0 & 0 & D_{66} \end{bmatrix} \tag{H.6}$$

The moments about the plate are defined by,

$$\{M\} = \{M_x, M_y, M_{xy}\}^T = \int_{-t/2}^{t/2} \{\sigma\} z dz \tag{H.7}$$

where t is the thickness of the plate. Substituting equation (H.4) and (H.5) into equation (H.7), the moments can be defined by,

$$\{M\} = [D] \begin{Bmatrix} \frac{\partial^2 w}{\partial x^2} \\ \frac{\partial^2 w}{\partial y^2} \\ 2\frac{\partial^2 w}{\partial x \delta y} \end{Bmatrix} \int_{-t/2}^{t/2} -z^2 dz \tag{H.8}$$

Integration of equation (H.8) gives the moments as,

$$\begin{aligned} M_x &= -\frac{t^3}{12}\left\{D_{11}\frac{\partial^2 w}{\partial x^2} + D_{12}\frac{\partial^2 w}{\partial y^2}\right\} \\ M_y &= -\frac{t^3}{12}\left\{D_{21}\frac{\partial^2 w}{\partial x^2} + D_{22}\frac{\partial^2 w}{\partial y^2}\right\} \\ M_{xy} &= -\frac{t^3}{12} D_{66} 2\frac{\partial^2 w}{\partial x \partial y} \end{aligned} \tag{H.9}$$

The equilibrium equations can then be found from the free body diagram (Mansfield 1989, Reddy 1993). The moments about the x and y axes respectively are in equilibrium. Taking forces about the w axes, if the higher order terms are neglected the equilibrium equations become (Reddy 1993, Zienkiewicz 1977, and Mansfield 1983),

$$\frac{\partial M_x}{\partial x}+\frac{\partial M_{xy}}{\partial y}-Q_x=0 \tag{H.10}$$

$$\frac{\partial M_{xy}}{\partial x}+\frac{\partial M_y}{\partial y}-Q_y=0 \tag{H.11}$$

$$\frac{\partial Q_x}{\partial x}+\frac{\partial Q_y}{\partial y}+p=0 \tag{H.12}$$

Now substituting equations (H.10) and (H.11) into (H.12) gives,

$$\frac{\partial}{\partial x}\left(\frac{\partial M_x}{\partial x}+\frac{\partial M_{xy}}{\partial x \partial y}\right)+\frac{\partial}{\partial y}\left(\frac{\partial M_{xy}}{\partial x}+\frac{\partial M_y}{\partial y}\right)+p=0 \tag{H.13}$$

Expanding equation (H.13) gives,

$$\frac{\partial^2 M_x}{\partial x^2}+2\frac{\partial^2 M_{xy}}{\partial x \partial y}+\frac{\partial^2 M_y}{\partial y^2}+p=0 \tag{H.14}$$

Substituting M_x, M_y and M_{xy} as defined in equation (H.9) gives,

$$-\frac{\partial^2}{\partial x^2}\left[D_{11}\frac{\partial^2 w}{\partial x^2}+D_{12}\frac{\partial^2 w}{\partial y^2}\right]-2\frac{\partial^2}{\partial x \partial y}\left[2D_{66}\frac{\partial^2 w}{\partial x \partial y}\right]-\frac{\partial^2}{\partial y^2}\left[D_{21}\frac{\partial^2 w}{\partial x^2}+D_{22}\frac{\partial^2 w}{\partial y^2}\right]+p=0 \tag{H.15}$$

Expanding equation (H.15) gives

$$D_{11}\frac{\partial^4 w}{\partial x^4}+D_{12}\frac{\partial^4 w}{\partial x^2 \partial y^2}+4D_{66}\frac{\partial^4 w}{\partial x^2 \partial y^2}+D_{21}\frac{\partial^4 w}{\partial x^2 \partial y^2}+D_{22}\frac{\partial^4 w}{\partial y^4}=p \tag{H.16}$$

Equation (H.16) is the governing differential equation for a plate in bending also known as the bi-harmonic operator (Mansfield 1983).