Appendix F

F.1 Soluition to a moving force on a beam

The equation of motion for an Euler-Bernoulli beam subject to a force moving with constant velocity can be described by,

$$\rho\frac{\partial^2 v(x,t)}{\partial t^2}+C\frac{\partial v(x,t)}{\partial t}+EI\frac{\partial^4 v(x,t)}{\partial x^4}=\delta(x-ct)f(t) \qquad \text{(F.1)}$$

Where $v(x,t)$ is the deflection of the beam at a point *x* and a time *t*, *C* is the damping property, *E* is the Young's modulus, *I* is the second moment of inertia, *f(t)* is the force and $\delta(t)$ is the dirac delta function. Subject to the boundary conditions,

$$\begin{aligned} &v(0,t)=0 && v(l,t)=0 \\ &\frac{\partial^2 v(0,t)}{\partial t^2}=0 && \frac{\partial^2 v(L,t)}{\partial t^2}=0 \\ &v(x,0)=0 && \frac{\partial v(x,0)}{\partial t}=0 \end{aligned} \qquad \text{(F.1)}$$

It is assumed that the deflection of the beam can be written as,

$$v(x,t)=\sum_{n=1}^{\infty}\phi_n(x)q_n(t) \qquad \text{(F.2)}$$

where $\phi_n(x)$ is the shape function of the n^{th} mode shape and $q_n(t)$ is the modal response. Substituting equation (F.3) into (F.1) and pre-multiplying both sides of the equation by $\phi_j(x)$ gives an equation of the form,

$$\rho\phi_j(x)\phi_n(x)\frac{\partial^2 q_n(t)}{\partial t^2}+C\phi_j(x)\phi_n(x)\frac{\partial q_n(t)}{\partial t}+EIq_n(t)\phi_j(x)\frac{\partial^4 \phi_n(x)}{\partial x^4}=\delta(x-ct)\phi_j(x)f(t) \qquad \text{(F.3)}$$

Integrating equation (F.4) with respect to *x* between *0* and *L* (Yang et al 2004), results in a decoupled set of equations in modal co-ordinates, provided the mode shapes satisfy certain orthogonality conditions (Green & Cebon 1992, Clough & Penzien 1975

$$\frac{d^2 q_n(t)}{dt^2} + 2\xi_n \omega_n \frac{dq_n(t)}{dt} + \omega_n^2 q_n(t) = \frac{1}{M_n} p_n(t) \tag{F.4}$$

where ξ_n is the damping factor of the n^{th} mode shape and,

$$M_n = \rho L / 2 \tag{F.5}$$

$$\phi_n(x) = \sin(n\pi x / L) \tag{F.6}$$

$$\omega_n = (n^2 \pi^2 / L^2)\sqrt{EI / \rho} \tag{F.7}$$

$$p_n(t) = f(t)\sin(n\pi ct / L) \tag{F.8}$$

Equation (F.4) can then be solved for, by taking a Laplace transform of both sides of the equation, where

$$q_n(s) = L(q_n(t)) \tag{F.9}$$

denotes the Laplace transform, by making use of the derivative property of Laplace transforms whereby,

$$\begin{aligned} L(q_n(t)) &= q_n(s) = \int_0^\infty q_n(t) e^{-st} dt \\ L\{q_n^{'}(t)\} &= -q_n(0) + sq_n(s) \\ L\{q_n^{''}(t)\} &= s^2 q_n(s) - sq_n(0) - q_n(0) \end{aligned} \tag{F.10}$$

and the solution to equation (F.4) in the Laplace domain is defined by,

$$q_n(s) = \frac{1}{M_n} \frac{1}{(s^2 + 2\varepsilon_n \omega_n s + \omega_n^2)} p_n(s) \tag{F.11}$$

The solution to $q_n(t)$ can be found by taking an inverse Laplace of (F.11) and then taking the convolution (Dyke 2000, Clough & Penzien 1975). The convolution of two functions is defined by,

$$q_n(t) * p_n(t) = \int_0^t q_n(\tau) p_n(t - \tau) dt \tag{F.12}$$

Now if

$$L\{q_n(t)\} = \bar{q}_n(s)$$
$$L^{-1}\{\bar{q}_n(s)\,\bar{p}_n(s)\} = q_n(t) * q_n(t) \qquad \text{(F.13)}$$

such that,

$$q_n(t) = \frac{2}{\rho L} L^{-1}\left\{\left(\frac{1}{s^2 + 2\xi_n \omega_n s + \omega_n^2}\right)\left(p_n(s)\right)\right\} \qquad \text{(F.14)}$$

$$q_n(t) = \frac{2}{\rho L}\left\{L^{-1}\left\{\left(\frac{1}{s^2 + 2\xi_n \omega_n s + \omega_n^2}\right)\right\} * L^{-1}\left\{p_n(s)\right\}\right\} \qquad \text{(F.15)}$$

$$q_n(t) = \frac{1}{M_n}\int_0^t h_n(t-\tau)\,p_n(\tau)\,d\tau \qquad \text{(F.16)}$$

where $h_n(t)$ is the unit impulse response function defined by,

$$h_n(t) = \frac{1}{M_n} L^{-1}\left(\frac{1}{s^2 + 2\varepsilon\omega_n s + \omega_n^2}\right) \qquad \text{(F.17)}$$

rewriting equation (F.17) as,

$$h_n(t) = \frac{1}{M_n} L^{-1}\left(\frac{1}{\omega_n\sqrt{1-\varepsilon_n^2}}\left(\frac{\omega_n\sqrt{1-\varepsilon_n^2}}{(s+\varepsilon_n\omega_n)^2 + \left(\omega_n\sqrt{1-\varepsilon_n^2}\right)^2}\right)\right) \qquad \text{(F.18)}$$

and making use of the fact that

$$L^{-1}\left(\frac{a}{s^2 + a^2}\right) = \sin(at)$$
$$a = \omega_n\sqrt{1-\xi_n^2} \qquad \text{(F.19)}$$
$$s = (s + \xi_n\omega_n)^2$$

and the first shift theorem defined by (Dyke 2000),

$$L^{-1}\{e^{-bt}F(t)\} = f(s+b)$$
$$f(s+b) = f(s+\xi_n\omega_n) \tag{F.20}$$

the unit impluse response function can be defined by,

$$h_n(t) = \frac{1}{M_n\omega_n\sqrt{1-\varepsilon_n^2}}\, e^{-\varepsilon_n\omega_n t}\sin\left(\omega_n\left(\sqrt{1-\varepsilon_n^2}\right)t\right) \tag{F.21}$$

Substituting equation (F.21) into (F.16) yields a general solution defined by (Law et al 1997, Chan et al 2001a),

$$q_n(t) = \frac{1}{M_n}\int_0^t \frac{1}{\omega_n\sqrt{1-\varepsilon_n^2}} e^{-\varepsilon_n\omega_n(t-\tau)}\sin(\omega_n\left(\sqrt{1-\varepsilon_n^2}\right)(t-\tau))f(\tau)\sin(n\pi c\tau/L)d\tau \tag{F.22}$$

Now substituting equations (F.22) and (F.6) into (F.2) gives;

$$v(x,t) = \sum_{n=1}^{\infty}\sin(n\pi x/L)\frac{1}{M_n}\int_0^t \frac{1}{\omega_n\sqrt{1-\varepsilon_n^2}} e^{-\varepsilon_n\omega_n(t-\tau)}\sin\left(\left(\omega_n\sqrt{1-\varepsilon_n^2}\right)(t-\tau)\right)f(\tau)\sin(n\pi c\tau/L)d\tau \tag{F.23}$$