Logical Interoperability Between S4 and Intuitionistic Logic using Adjoint Modalities

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Date: April 23, 2025

The relationship between modal and intuitionistic logics touches on fundamental questions about the nature of truth, proof, and necessity. Modal logics, such as S4, augment classical systems with additional operators for necessity ($\Box$) and possibility ($\Diamond$), allowing the formal expression of concepts like provability and temporality. Intuitionistic logic, grounded in a constructive notion of truth, provides a computational interpretation of proofs, yet struggles to express modal concepts directly. Our project is driven by a central question: *Can we embed the reasoning principles of classical modal logic S4 into a constructive framework, such that derivability and meaning are preserved?*

To address this, we attempt to augment Intuitionistic logic into a new logic $Int*$ which simulates $IS4$; however with less expressibility. We proceeded by the following steps.

1. From classical modal logic **S4** to its intuitionistic variant **IS4**, via Gödel's translation.
2. From **IS4** into our system **Int***, which internalizes modal reasoning into intuitionistic logic using judgmental modalities.

Our goal was to establish a robust form of **logical interoperability**. Defining a translation that enables modal proofs and theorems, originally formulated within the confines of classical S4, to be reconstructed, interpreted, and verified within $Int*$. By doing so, we aimed to provide semantic integrity and derivability of the original results.

This paper presents our motivation, formal approach, and the hurdles faced—especially with respect to modeling validity, context-dependence, and the T axiom constructively—offering insights for future mechanization attempts.

Related Work

This project draws on several foundational threads:

- **Gödel's 1933 Translation**: A well-known translation of S4 done by prefixing subformulas with $\Box$. Which forces constructivity through the $\Box$ modalities.
- **Pfenning and Davies (2001)** developed a *judgmental reconstruction of modal logic*, introducing contextual modal type theory and judgmental modalities. This served as our foundation for studying and defining intuitionistic modal logics.
- **Goré and Thomson (2019)** corrected earlier work on translating S4 to intuitionistic logic by providing a provably correct polynomial translation. Our work is inspired by their structure, though we adopt a syntactic and judgmental style rather than a model-theoretic one.

Proposed Approach and Contribution

Our work attempts to provide a mechanized framework for translating classical modal reasoning (specifically S4) into a constructive setting grounded in intuitionistic logic. We formalize the syntax and inference rules of the systems S4 and $Int*$ within the Beluga proof environment.

The translation from S4 to IS4 follows a Gödel-style embedding where classical disjunction and negation are replaced with their intuitionistic counterparts. For instance, classical negation ($\neg A$) is mapped to either $\neg\neg A$ or to $A \to \bot$, preserving constructivity. IS4 acts as a bridge logic based on the judgmental modal framework of Pfenning and Davies. From IS4, we define a mutually recursive translation into Int* using Beluga's functional language. In Int*, necessity is not modeled as $\neg\neg A$, which fails to support the T axiom constructively, but as a judgmental modality with explicit introduction and elimination rules.

We intended to realize the correctness condition:

$$\textit{If} \quad \Delta; \Gamma \vdash_{IS4} A \text{ true}, \quad \text{ then } [\Delta]; [\Gamma] \vdash_{Int^*} [A]$$

where $[\cdot]$ is our translation function.

To implement $\Box$ constructively, we adopted the following rules:

$$\frac{\Delta; \cdot \Rightarrow A}{\Delta; \Gamma \Rightarrow \Box A} \Box I \qquad\qquad \frac{\Delta; \Gamma \Rightarrow \Box A \qquad \Delta, A \text{ valid}; \Gamma \Rightarrow C}{\Delta; \Gamma \Rightarrow C} \Box E$$

Our Beluga implementation defined the syntactic structure, typing rules, and translation functions for the three systems. Int* extends intuitionistic logic by separating valid assumptions (Δ) from truth assumptions (Γ), supporting contextual necessity. However, several obstacles emerged. We found it quite difficult to express the dependency of $\Box E$ on the validity of A in the extended context. The mutually recursive definition of the translation function led to consistent typing and substitution issues.

Discussion and Future Work

Although our project did not culminate in a complete mechanization of the S4 $\to$ IS4 $\to$ Int* translation in Beluga, our experience highlighted several key insights that may guide future research in this space. One of the central conceptual hurdles was faithfully modeling the axiom T ($\Box A \to A$) within a constructive setting. Interpreting $\Box A$ as $\neg\neg A$ leads to a derivability gap, as the intuitionistic framework does not support the principle of double-negation elimination. This issue underlined the fundamental divergence between classical and constructive reasoning about necessity. Our solution, adopting a judgmental modality with explicit $\Box$ introduction and elimination rules, provides a viable alternative[1], but it demands a more sophisticated treatment of context structure and dependencies in the proof assistant. From a mechanization standpoint, expressing nested validity contexts and managing dependencies between Δ and Γ introduced complexity in the typing and substitution mechanisms. Another area ripe for exploration is proof term enrichment. Currently, Int* is designed as a judgmental logic, with limited attention paid to the computational content of modal proofs. Adding proof terms—possibly via a dependent type-theoretic encoding—would allow us to reason not only about the provability of modal formulas, but also about the structure and normalization behavior of the corresponding proofs. This would open the door to formalizing meta-theorems like subject reduction, normalization, and the substitution lemma within Int*. Finally, we note that our construction of Int* may have applications beyond the specific goal of S4 to Int translation. The separation of valid and truth contexts, along with explicit modal introduction and elimination, resembles systems used in staged computation, epistemic logic, and even proof-carrying code. Exploring these connections could both enrich the theory and motivate practical extensions of Int* as a modal subsystem within larger constructive meta-theories.

Conclusion

Our attempt illustrates both the promise and the difficulty of embedding classical modal logic into constructive frameworks. Despite falling short of full mechanization, we developed a coherent and semantically informed translation architecture, grounded in modern modal type theory. The judgmental treatment of necessity and the formal distinction of contexts bring us closer to a general framework for logical interoperability. We hope our contributions lay groundwork for future mechanization efforts in modal-constructive logic integration. In doing so, we also aim to inspire deeper exploration into how proof assistants can better accommodate rich modal reasoning. The theoretical clarity we achieved, despite practical limitations, reinforces the importance of foundational infrastructure in enabling cross-logical translation.

[1] See Pfenning and Davies' reconstruction of modal logic for the foundational treatment of judgmental modalities in IS4 [1].

References

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