Interoperability Between S4 and Int* : Further Work & Proofs

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Abstract

This paper investigates logical interoperability between classical modal logic (S4) and intuitionistic logic. To achieve this, we introduce and formalize a novel intermediary logic, Int*, that extends intuitionistic natural deduction with a constructive box modality and a dual-context judgmental structure. This system allows us to simulate derivations in IS4 within an intuitionistic framework, enabling a direct and constructive interpretation of modal axioms such as K and T, while intentionally restricting axiom 4 to preserve constructivity. We present a translation of derivations from classical S4 to Int*, discuss its soundness, and demonstrate its capability to express modal reasoning constructively.

1 Bridging the Gap

To prove the main assertion in this paper (interoperability between IS4 and Intuitionistic logic) we must show that we can **constructively** simulate every derivable $IS4$ judgment in Int. This hinges entirely on how we translate the box modality and validity judgments in $IS4$. Since we are following a judgmental natural deduction framework of $IS4$ as outlined by Pfenning and Davies [1], we want to prove the following.

$$\text{For every derivation } \Delta;\Gamma \vdash IS4 \text{ A true } \exists[\Delta];[\Gamma] \Rightarrow \text{ Int* [A]}$$

Where $[\cdot]$ is the translation function we define. And $Int*$ is our newly constructed modal logic which bridges the expressibility gap between $IS4$ and Int (original Int).

1.1 Defining Int*

If we want to define a logic that is isomorphic to $IS4$, we can't simply rely on the translation $[\Gamma;\Delta \vdash \Box A] = [\Delta];[\Gamma] \Rightarrow \neg\neg[A]$. From this we would arrive at $\Delta;\Gamma \Rightarrow \neg\neg A \supset A$, which is not provable in Int. Thus, we introduce a necessity modality $\Box$ which is interpreted constructively in $Int*$.

$$\frac{\Delta;\cdot \Rightarrow A}{\Delta;\Gamma \Rightarrow \Box A}\,\Box I \qquad\qquad \frac{\Delta;\Gamma \Rightarrow \Box A \qquad \Delta, A \text{ valid};\Gamma \Rightarrow C}{\Delta;\Gamma \Rightarrow C}\,\Box E$$

This way we can preserve constructivity. Similar to how Pfenning and Davies defined it [1]. Furthermore, we will follow the hypothesis and truth rules[1]. Truth is always derivable.

$$\frac{}{\Delta;\Gamma \Rightarrow \top}\,\top R$$

$$\frac{}{\Delta, A \text{ valid}, \Delta';\Gamma \Rightarrow A \text{ true}}\,hyp* \qquad\qquad \frac{}{\Delta;\Gamma, A \text{ true}, \Gamma' \Rightarrow A \text{ true}}\,hyp$$

The remaining sequent rules for $Int*$ can be defined as follows by adding in our valid context Δ.

$$\frac{\Delta; \Gamma, A \Rightarrow C}{\Delta; \Gamma, A \wedge B \Rightarrow C} \wedge L_1 \qquad \frac{\Delta; \Gamma, B \Rightarrow C}{\Delta; \Gamma, A \wedge B \Rightarrow C} \wedge L_2$$

$$\frac{\Delta; \Gamma \Rightarrow A}{\Delta; \Gamma \Rightarrow A \vee B} \vee R_1 \qquad \frac{\Delta; \Gamma \Rightarrow B}{\Delta; \Gamma \Rightarrow A \vee B} \vee R_2$$

$$\frac{\Delta; \Gamma_1 \Rightarrow A \qquad \Delta; \Gamma_2 \Rightarrow B}{\Delta; \Gamma_1, \Gamma_2 \Rightarrow A \wedge B} \wedge R \qquad \frac{\Delta; \Gamma, A \Rightarrow C \qquad \Delta; \Gamma, B \Rightarrow C}{\Delta; \Gamma, A \vee B \Rightarrow C} \vee L$$

$$\frac{\Delta; \Gamma_1 \Rightarrow A \qquad \Delta; \Gamma_2, B \Rightarrow C}{\Delta; \Gamma_1, \Gamma_2, A \supset B \Rightarrow C} \supset L \qquad \frac{\Delta; \Gamma, A \Rightarrow B}{\Delta; \Gamma \Rightarrow A \supset B} \supset R$$

$$\frac{\Delta; \Gamma \Rightarrow A \supset B \qquad \Delta; \Gamma \Rightarrow A}{\Delta; \Gamma \Rightarrow B \text{ true}} \supset E$$

For structural rules, we only allow weakening and contraction for boxed truths. While exchange is always implicitly valid.

$$\frac{\Delta; \Gamma \Rightarrow A}{\Delta; \Gamma, \Box B \Rightarrow A} W \qquad \frac{\Delta; \Gamma, \Box B, \Box B \Rightarrow A}{\Delta; \Gamma, \Box B \Rightarrow A} C$$

Finally, for cut we need two rules for each respective context. $\Gamma - Cut$ is just standard intuitionistic cut since we are operating in local context. And $\Delta - Cut$ mirrors the $\Box E$ principle; if A is globally provable, then A valid may be used in any derivation.

$$\frac{\Delta; \Gamma_1 \Rightarrow A \qquad \Delta; \Gamma_2, A \Rightarrow C}{\Delta; \Gamma_1, \Gamma_2 \Rightarrow C} \Gamma - Cut \qquad \frac{\Delta; \cdot \Rightarrow A \qquad \Delta, A \text{ valid}; \Gamma \Rightarrow C}{\Delta; \Gamma \Rightarrow C} \Delta - Cut$$

1.2 Defining Classical S4

Since we have now simulated a modal intuitionistic logic $Int*$, we want to reach a translation between Classical S4 and $Int*$. Let's define our inference rules for Classical S4 and outline the key differences between an intuitionistic system and a classical system.

For comparison with our sequent-based constructive system (Int^*), we express the core rules of S4 as derivable sequents. Notably, Classical S4 treats all contexts as reusable (structural rules are implicit), and truth is bivalent (not constructively justified).

We include standard modal inference rules, which mirror the modal axioms:

$$\frac{\Gamma \vdash A \supset B}{\Gamma \vdash \Box A \supset \Box B} \text{K} \qquad \frac{\Gamma \vdash \Box A}{\Gamma \vdash A} \text{T}$$

$$\frac{\Gamma \vdash \Box A}{\Gamma \vdash \Box\Box A} 4 \qquad \frac{\Gamma \vdash A}{\Gamma \vdash \Box A} \Box I$$

We also define the core classical structural and logical rules, where assumptions may be reused, contracted, or weakened arbitrarily (unlike in Int^*):

$$\frac{\Gamma, A \vdash B}{\Gamma, A, A \vdash B} \text{C} \qquad \frac{\Gamma, A, B \vdash C}{\Gamma, A \vdash C} \text{W}$$

In Classical S4, box elimination is not an explicit rule as it is in IS4. Instead, the T axiom ($\Box A \to A$) and Modus Ponens simulate it

2 Contrasting IS4 and Int*

Our $Int*$ validates only the K and T modal axioms, unlike IS4. This is due to the constructive nature of our $\Box$ modality. We can not just apply $\Box I$ again to reach axiom 4, since $\Box A$ must be true without any local assumptions. As for K and T, the derivations are as follows.

Deriving T

$$
\dfrac{\dfrac{\dfrac{}{\Delta; \Box A \Rightarrow \Box A}\,hyp \qquad \dfrac{}{\Delta, A \text{ valid}; \Box A \Rightarrow A}\,hyp*}{\Delta; \Box A \Rightarrow A}\,\Box E}{\Delta; \cdot \Rightarrow \Box A \supset A}\supset R
$$

Deriving K

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Delta; \Box(A \supset B), \Box A \Rightarrow \Box(A \supset B)}\,hyp \quad \dfrac{}{\Delta, A \supset B \text{ valid} \Rightarrow A \supset B}\,hyp*}{\Delta; \Box(A \supset B), \Box A \Rightarrow A \supset B}\,\Box E \quad \dfrac{\dfrac{}{\Delta; \Box(A \supset B), \Box A \Rightarrow \Box A}\,hyp \quad \dfrac{}{\Delta, A \text{ valid} \Rightarrow A}\,hyp*}{\Delta; \Box(A \supset B), \Box A \Rightarrow A}\,\Box E}{\Delta; \Box(A \supset B), \Box A \Rightarrow B}\supset E}{\Delta; \Box(A \supset B), \Box A \Rightarrow \Box B}\,\Box I}{\Delta; \Box(A \supset B) \Rightarrow \Box A \supset \Box B}\supset R}{\Delta; \cdot \Rightarrow \Box(A \supset B) \supset (\Box A \supset \Box B)}\supset R
$$

3 Translating Classical S4

Using the Gödel translation $G(A)$ defined as follows.

If p an atom:

$$G(p) ::= \Box p$$

For connectives:

$$
\begin{aligned}
G(A \wedge B) &:= \Box(G(A) \wedge G(B)) \\
G(A \vee B) &:= \Box(G(A) \vee G(B)) \\
G(A \rightarrow B) &:= \Box(G(A) \rightarrow G(B)) \\
G(\Box A) &:= \Box G(A)
\end{aligned}
$$

This translation is sound, satisfying the property

$$S4 \vdash A \Rightarrow IS4 \vdash G(A)$$

The following are examples of translating $S4 \rightarrow IS4$, particularly the modal inference rules.

D ends in $\Box I_c$:

$$
\dfrac{\begin{matrix}\mathcal{D}' \\ \cdot \vdash A \text{ true}\end{matrix}}{\cdot \vdash \Box A \text{ true}}\,\Box I_c
$$

Defining our context translation to be $[\cdot] \mapsto \Delta; \cdot$ and $[\Gamma] \mapsto \Delta; \Gamma$

$$
\dfrac{\begin{matrix}\mathcal{IH}(\mathcal{D}') \\ \Delta; \cdot \vdash \Box A\end{matrix}}{\Delta; \Gamma \vdash \Box\Box A}\,\Box I_{int}
$$

D ends in $\Box E_c$:

$$
\dfrac{\begin{matrix}\mathcal{D}' \\ \cdot \vdash \Box A\end{matrix}}{\cdot \vdash A}\,\Box E_c
$$

Then our goal is $G(A) ::= \Box A$:

$$\dfrac{\overset{\mathcal{IH}(\mathcal{D}')}{\Delta;\Gamma \vdash \Box\Box A} \qquad \dfrac{}{\Delta, \Box A \text{ valid}; \Gamma \vdash \Box A}\,hyp*}{\Delta;\Gamma \vdash \Box A}\,\Box E_{int}$$

References

[1] Frank Pfenning, Rowan Davies, "A judgmental reconstruction of modal logic," https://www.cs.cmu.edu/~fp/papers/mscs00.pdf, 2000, accessed: April 15, 2025.

[2] P.-M. Osera and S. Zdancewic, "Type-and-example-directed program synthesis."

[3] G. Matthews, "Intuitionistic modal logic," 2016, lecture notes for PHIL 511, University of Massachusetts Amherst. [Online]. Available: https://courses.umass.edu/phil511-gmh/pdf/C05-2016.pdf

[4] Stanford Encyclopedia of Philosophy, "Intuitionistic logic," https://plato.stanford.edu/archives/spr2023/entries/logic-intuitionistic/, 2023, accessed: April 2, 2025.