Project Instructions

Details concerning the quark DSE can be found in:

Mesons in a Poincare Covariant Bethe-Salpeter Approach
R. Alkofer, P. Watson and H. Weigel, arXiv: hep-ph/0202053

The Dyson-Schwinger equation (DSE) for the quark propagator reads

$$S^{-1}(p) = i\not{p} + m_0 + g^2 \frac{\lambda^a}{2}\frac{\lambda^b}{2} \int \frac{d^4k}{(2\pi)^4} \gamma_\mu S(k) \Gamma_\nu(p,k) G^{ab}_{\mu\nu}(k-p),$$

where the ingredients are given by:

- $m_0 \ldots$ current quark mass
- λ^a, $\lambda^b \ldots$ color matrices
- $\Gamma_\nu(p,k) \ldots$ full quark-gluon vertex (complicated): $\Gamma_\nu(p,k) \to \gamma_\nu$
- γ_μ, $\gamma_\nu \ldots$ Dirac matrices: $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$ (Euclidean metric)
- $G^{ab}_{\mu\nu} \ldots$ gluon propagator.

Because of the full quark-gluon vertex, this equation is too complicated to solve. Thus, in the following a simplified version will be treated

$$S^{-1}(p) = i\not{p} + m_0 + g^2 \frac{\lambda^a}{2}\frac{\lambda^b}{2} \int \frac{d^4k}{(2\pi)^4} \gamma_\mu S(k) \gamma_\nu G^{ab}_{\mu\nu}(k-p).$$

Here, the gluon propagator reads

$$g^2 G^{ab}_{\mu\nu} = 4\pi^2 D\ \delta^{ab} t_{\mu\nu}(q) \frac{q^2}{\omega^2} \exp\left(-\frac{q^2}{\omega^2}\right),$$

where g denotes the coupling between the quarks and the gluons. The transversal projector is given by the expression

$$t_{\mu\nu}(q) = \delta_{\mu\nu} - \frac{q^\mu q^\nu}{q^2} = \frac{1}{q^2}\left(\delta_{\mu\nu} q^2 - q^\mu q^\nu\right)$$

and D and ω are dimensionful parameters. Values for these parameters as well as for the bare mass m_0 are given in the list below.

By plugging an ansatz for the quark propagator

$$S(p) = \frac{1}{i\not{p}A(p^2) + B(p^2)} = \frac{-i\not{p}A(p^2) + B(p^2)}{p^2A^2(p^2) + B^2(p^2)}$$

into the quark DSE one gets

$$\begin{aligned} i\not{p}A(p^2) + B(p^2) &= i\not{p} + m_0 \\ &\quad + \frac{\lambda^a}{2}\frac{\lambda^b}{2}\int \frac{d^4k}{(2\pi)^4}\gamma_\mu \underbrace{\left[\frac{-i\not{k}A(k^2) + B(k^2)}{k^2A^2(k^2) + B^2(k^2)}\right]}_{S(k)} \gamma_\nu \underbrace{g^2 G^{ab}_{\mu\nu}(k-p)}_{\substack{\text{insert gluon} \\ \text{propagator here}}} . \end{aligned}$$

After contracting the Lorentz indices and taking the corresponding Dirac-traces (see presentation and/or given literature) the coupled system of integral equations for the dressing functions A and B reads

$$\begin{aligned} A(x) &= 1 + \frac{D}{\omega^2}\int_0^\infty dy \frac{y\, A(y)}{y\, A^2(y) + B^2(y)} \\ &\times \frac{2}{\pi}\int_{-1}^{1} dz\sqrt{1-z^2}\left[-\frac{2}{3}y + \left(1 + \frac{y}{x}\right)\sqrt{xy}z - \frac{4}{3}yz^2\right]\exp\left\{-\frac{x + y - 2\sqrt{xy}z}{\omega^2}\right\} \\ B(x) &= m_0 + \frac{D}{\omega^2}\int_0^\infty dy \frac{y\, B(y)}{y\, A^2(y) + B^2(y)} \\ &\quad \times \frac{2}{\pi}\int_{-1}^{1} dz\sqrt{1-z^2}\,[x + y - 2\sqrt{xy}z]\exp\left\{-\frac{x + y - 2\sqrt{xy}z}{\omega^2}\right\} . \end{aligned}$$

This system of non-linear coupled integral equations has to be solved iteratively via a discretization of the dressing functions A and B. Here, a convenient quadrature rule has to be chosen in order to perform the radial as well as angular integrals. In the first step the system needs an initialization, where the corresponding starting values can be found in the list below. Another possibility is to introduce modified Bessel functions in order to perform the angular part of the integrals analytically (see literature). However, the treatment of the resulting system is numerically slightly more involved, such that the authors (i.e. Markus and Andreas) recommend a numerical brute force calculation including also the angular integrals.

The evaluated values for the dressing functions at the Gauss-Legendre nodes can be re-used within the integrals such that there is no need to do the calculations using splines or whatsoever.

Because of the square root in the angular integrals a Tschebyshev quadrature is a good choice, but also Gauss-Legendre or tanh-sinh quadratures can be used. For the numerical implementation using CUDA it is recommended to perform the calculations sequentially on a conventional CPU in a first step (i.e. as a simple host code using C/C++/Fortran) and then, subsequently, put the relevant parts of the code onto the GPU device. This would also have the benefits to compare the two solutions in the end.

In order to solve the problem, the following steps should be performed:

- take the coupled system of equations
- discretize the dressing functions
  - $A(x) \to A(x_i) \rightsquigarrow A[i]$
  - $B(x) \to B(x_i) \rightsquigarrow B[i]$
- re-use the evaluated values within the integrals
  - $A(y) \to A(y_i) \rightsquigarrow A[n]$
  - $B(y) \to B(y_i) \rightsquigarrow B[n]$
- the calculations should be performed within the range: $x_i, y_i \in [10^{-4}, 5 \cdot 10^4]$
- parameters for the calculations
  - $D = 16\, GeV^{-2}$
  - $\omega = 0.5\, GeV$
  - $m_0 = 0$ & $m_0 = 0.115\, GeV$
- initial values: $A_0(x_i) = 1$, $B_0(x_i) = 0.4$
- possible quadratures
  - radial part: Gauss-Legendre (mapping !!! - see below)
  - angular part: Gauss-Tschebyshev (no mapping needed)

It is very important to map the radial Gauss-Legendre nodes/weights to the corresponding region with a non-linear mapping. Here, the following mapping function can be applied:

- node mapping: $\tilde{x}_i = a + s\dfrac{e^{gx_i} - 1}{1 + e - e^{x_i}}$
- weight mapping: $\tilde{w}_i = w_i \dfrac{s\, g\, e^{gx_i} + (\tilde{x}_i - a)e^{x_i}}{1 + e - e^{x_i}}$
- $g = \ln\left(1 + \dfrac{b-a}{s}\right)$
- $a = 10^{-4}$, $b = 5 \cdot 10^4 \ldots$ IR/UV cutoff (in principle: $a, b \geq 0 \wedge a > b$)
- $s = 1 \ldots$ mapping parameter (nodes/weights density maximum)
- It is important to generate the Gauss-Legendre nodes within the interval $[0, 1]$!!! Otherwise the mapping function will return wrong values.

In the end one should get the following plots for the dressing functions A and B. Note that M is defined via $M(x) = B(x)/A(x)$.

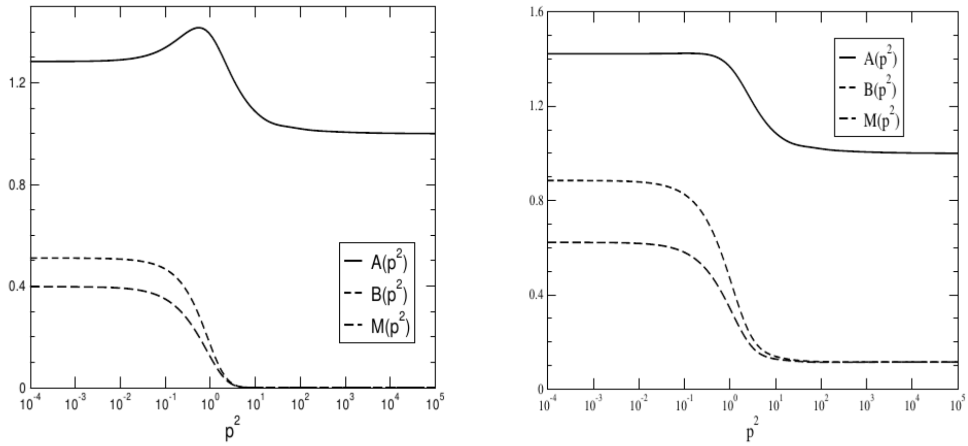

FIG. 1. Plot of the spacelike chiral quark functions (as a function of the momentum squared). The parameters are $\omega = 0.5$GeV, $D = 16.0\text{GeV}^{-2}$. Left panel: $m_0 = 0$, right panel: $m_0 = 0.115$ GeV. All units are GeV.

Group Assignment

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