W.T.S. : $\exp\left[-\frac{1}{2}\left(\left(\sum_{i=1}^{n}\phi(x_i-\theta)^2\right)+\tau(\theta-\theta_0)^2\right)\right] \propto \exp\left(-\frac{1}{2}(\tau+n\phi)\left(\theta-\frac{1}{\tau+n\phi}\left(\tau\theta_0+\phi\sum_{i=1}^{n}x_i\right)\right)^2\right)$

- We assume given data(x_i), τ, θ_0, ϕ

- Following the hint 1, we try to expand everything.

$$\exp\left[-\frac{1}{2}\left(\underbrace{\left(\sum_{i=1}^{n}\phi(x_i-\theta)^2\right)+\tau(\theta-\theta_0)^2}_{①}\right)\right]$$

Expand ①:

$$\sum_{i=1}^{n}\phi(x_i^2-2x_i\theta+\theta^2)+\tau(\theta^2-2\theta\theta_0+\theta_0^2)$$

$$=\sum_i\phi x_i^2-\sum_i 2\phi\theta x_i+n\phi\theta^2+\tau\theta^2-2\tau\theta\theta_0+\tau\theta_0^2$$

$$=(\tau+n\phi)\left(\frac{\phi\sum x_i^2}{\tau+n\phi}-\frac{2\phi\theta\sum x_i}{\tau+n\phi}+\theta^2-\frac{2\tau\theta\theta_0}{\tau+n\phi}+\frac{\tau\theta_0^2}{\tau+n\phi}\right)$$

$$=\underbrace{(\tau+n\phi)}_{X}\left(\underbrace{\theta^2-\frac{2\theta}{\tau+n\phi}\left(\tau\theta_0+\phi\sum_i x_i\right)}_{Y}+\underbrace{\frac{\phi\sum x_i^2}{\tau+n\phi}+\frac{\tau\theta_0^2}{\tau+n\phi}}_{Z}\right)$$

-Now, we bring back to exponetial equation, we can simplify function as following.

$$\exp\left[-\frac{1}{2}\left(\left(\sum_{i=1}^{n}\phi(x_i-\theta)^2\right)+\tau(\theta-\theta_0)^2\right)\right]=\exp\left[-\frac{1}{2}X(Y+Z)\right]$$

$$\propto\exp\left[-\frac{1}{2}XY\right]$$ where $\exp(-\frac{1}{2}XZ)$ is a constant

$$\propto\exp\left[-\frac{1}{2}X(Y+Z')\right]$$ where $\exp(-\frac{1}{2}XZ')$ is a constant. and $Z'=\frac{(\tau\theta_0+\phi\sum^{n}x_i)^2}{(\tau+n\phi)^2}$

$$=\exp\left(-\frac{1}{2}(\tau+n\phi)\left(\theta^2-\frac{2\theta}{\tau+n\phi}(\tau\theta_0+\phi\sum x_i)+\frac{(\tau\theta_0+\phi\sum_i x_i)^2}{(\tau+n\phi)^2}\right)\right)$$

$$=\exp\left(-\frac{1}{2}(\tau+n\phi)\left(\theta-\frac{1}{\tau+n\phi}\left(\tau\theta_0+\phi\sum_i x_i\right)\right)^2\right)$$

$\blacksquare$