The scale of (P)-closed groups acting on trees

(Buildings 2025, Louvain-la-Neuve, 30 minutes)

$\rightsquigarrow$ try to classify/parametrise (classes of) groups acting on trees (examples of t.d.l.c. groups)

$\tilde{A}_1$-buildings

joint work with colleague Michal Ferov and our common Ph.D. student Marcus Chijoff | thanks to organisers Bernhard Mühlherr

<u>Def.</u> (Banks-Elder-Willis '15) Let T be a tree (not necessarily locally finite or regular) and $H \leq \mathrm{Aut}(T)$. Let $k \in \mathbb{N}_0$.
The <u>(P_k)-closure</u> of H is

$$H^{(P_k)} := \{ g \in \mathrm{Aut}(T) \mid \forall v \in VT : \exists h \in H : g|_{B(v,k)} = h|_{B(v,k)} \}$$

When $H^{(P_k)} = H$ we say that H is <u>(P_k)-closed</u>. When $k=1$: (P)-closed.

<u>Consequences</u>:

$$\left.\begin{array}{l} \text{(i) } \left(H^{(P_k)}\right)^{(P_k)} = H^{(P_k)}, \text{ so } H^{(P_k)} \text{ is } (P_k)\text{-closed} \\ \text{(ii) } H^{(P_0)} \geq H^{(P_1)} \geq H^{(P_2)} \geq \dots \geq \overline{H} \geq H \\ \text{(iii) } \bigcap_{k=0}^{\infty} H^{(P_k)} = \overline{H} \end{array}\right\} \rightsquigarrow \text{approximation}$$

<u>Examples</u>:

(P_0)-closed: $\mathrm{Aut}(T_d)$, $\mathrm{Aut}(T_d)^+$, $\mathrm{Aut}(T_d)_v$, $\mathrm{Aut}(T_d)_{\{a,\bar{a}\}}$, ...

(P_1)-closed: $\mathrm{Aut}(T_d)_\omega$, Burger-Mozes universal groups $U(F)$, ...

(P_2)-closed: $U_2(\Gamma(S_d)) = \{ g \in \mathrm{Aut}(T_d) \mid g \text{ has constant local action} \}$, ...

(generalisation of Burger-Mozes groups from my PhD thesis)

Not (P_k)-closed for any $k \in \mathbb{N}_0$: $\mathrm{PGL}(2, \mathbb{Q}_p) \leq \mathrm{Aut}(T_{p+1})$

Thm. (Reid–Smith '20)

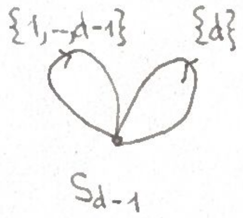
$$\left\{ \begin{array}{c} \text{Pairs } (G,T) \text{ where } G \leq \mathrm{Aut}(T) \\ \text{is (P)-closed} \end{array} \right\} \Big/ \begin{array}{l} \text{action} \\ \text{isomorphism} \end{array} \quad \overset{1:1}{\longleftrightarrow} \quad \left\{ \begin{array}{c} \text{local action} \\ \text{diagrams} \end{array} \right\} \Big/ \cong$$

Def. A local action diagram is a triple $\Delta = (\Gamma, (X_a)_{a \in A\Gamma}, (G(v))_{v \in V\Gamma})$ where

- $\Gamma = (V\Gamma, A\Gamma, o, t, r)$ is a graph
- the X_a are pairwise disjoint sets
- $G(v) \leq \mathrm{Sym}\left(\bigsqcup_{a \in o^{-1}(v)} X_a\right)$ is a permutation group with orbits X_a $(a \in o^{-1}(v))$

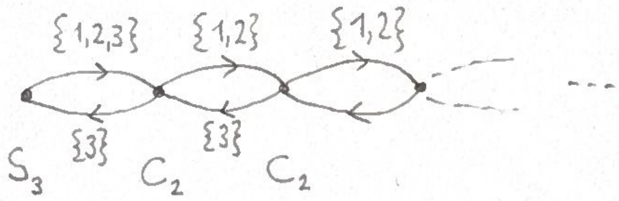
From pairs (G,T) to local action diagrams

$\mathrm{Aut}(T_d)_\omega \rightsquigarrow$



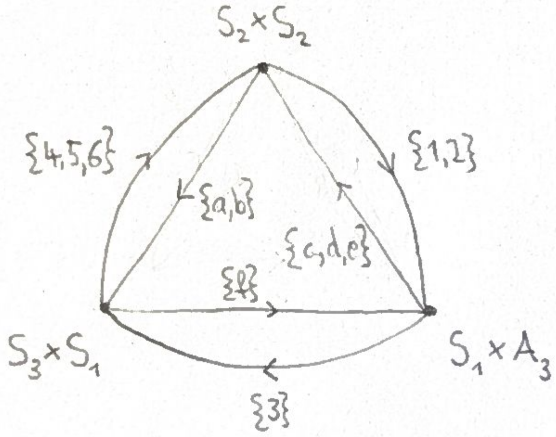
(cf. Burger–Mozes $U(S_{d-1})$)

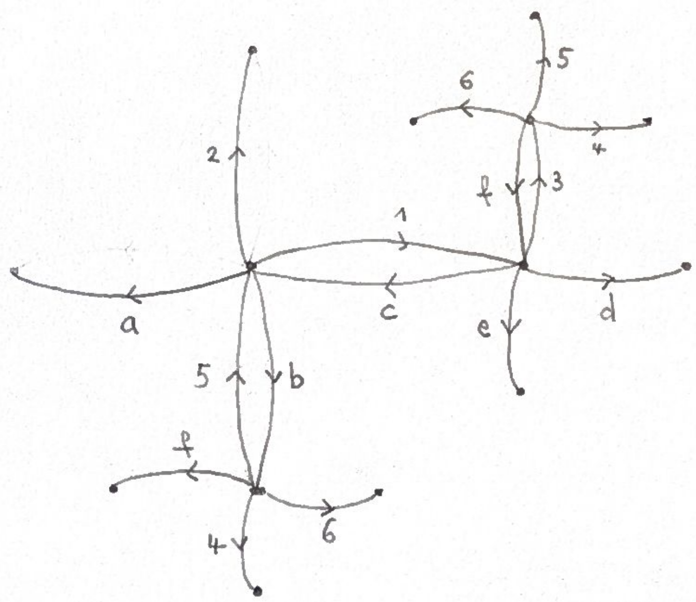
$\mathrm{Aut}(T_d)_v \rightsquigarrow$



From local action diagrams to pairs (G,T)

Δ $\qquad\qquad$ $(T, \pi, \mathcal{L})$





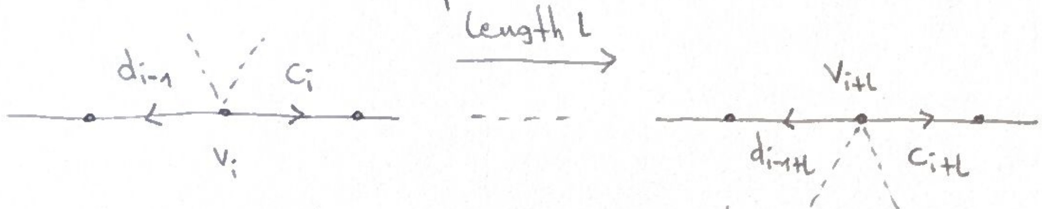
$$U(\Delta) := \{ g \in \mathrm{Aut}_\pi(T) \mid \forall v \in VT: \ \sigma_{\mathcal{L}}(g, v) \in G(\pi(v)) \}$$

Correspondence

- G has compact vertex stabilisers $\iff$ X_a is finite for all $a \in A\Gamma$
- G fixes a vertex $\iff$ Δ has a single vertex cotree
- G compactly generated $\iff$ ...
- G locally compact $\iff$ ...
- G discrete $\iff$ ...

Scale function

Assume that G has compact vertex stabilisers $\rightsquigarrow$ translations



Def. Δ local action diagram. A translatable circuit of length l in Δ is a tuple $(a_i, S_i)_{i=0}^{l-1}$ where $(a_i)_{i=0}^{l-1}$ is a closed walk in Γ and $S_i \subseteq X_{\overline{a_{i-1}}} \times X_{a_i}$ is a non-diagonal orbit of $G(o(a_i))$.

Thm. Δ local action diagram, $G := U(\Delta)$. Then there is a 1-1 correspondence

$$G \backslash \mathrm{Axes}(G) \underset{\text{"lift"}}{\overset{\text{"quotient"}}{\rightleftarrows}} \{\text{translatable circuits of } \Delta\}/\sim$$

Prop. The scale of a translation of length l associated to a translatable circuit of length l is, in the above notation

$$s(g) = \prod_{i=0}^{l-1} |G(\pi(v_i))_{c_i} \cdot d_{i-1}| \quad , \quad s(g^{-1}) = \prod_{i=0}^{l-1} |G(\pi(v_i))_{d_{i-1}} \cdot c_i|$$

$\rightsquigarrow$ uniscalar $\iff$ restrictions of local actions to translatable circuits are semiregular ($\iff$ $G \cong$ profinite $\rtimes$ discrete)

Thm. Δ local action diagram, $G := U(\Delta)$. Then

G unimodular $\iff$ for every oriented fundamental cycle $(a_i)_{i=0}^{l-1}$:
$$\prod_{i=0}^{l-1} |X_{a_i}| = \prod_{i=0}^{l-1} |X_{\overline{a_i}}|$$

Ex. 

$\rightsquigarrow \mathrm{Aut}(T_3)_\omega$ not unimodular

$\longrightarrow U(C_2 \leq S_3)$ unimodular

Example from before:

$2 \cdot 1 \cdot 3 = 3 \cdot 1 \cdot 2$ ✓

unimodular