Introduction to buildings

22/09/25

(Summer School Willebadessen, $\approx$ 45 minutes)

Syllabus: Coxeter systems, types A_n, $\tilde{A}_1$, $\tilde{A}_2$, buildings: W-metric approach,
examples: trees w/o leafs, Bruhat-Tits tree of SL_2(local field)

Buildings are highly symmetric objects. They admit both a geometric and a combinatorial definition. Every building is of a certain "type" – a Coxeter system.

Def. A Coxeter system is a pair (W,S) consisting of a group W with a finite generating set $S = \{s_1, \dots, s_n\}$ of the special form

$$W = \langle S \mid \forall i \in \{1,\dots,n\}: s_i^2 = 1, \ \forall i,j \in \{1,\dots,n\}: (s_i s_j)^{m_{ij}} = 1 \rangle$$

for some $m_{ij} \in \mathbb{N}_{\geq 2} \cup \{\infty\}$, where "$\infty$" means no relation.

One does not need to look too far for examples.

Ex.

(i) For $n \geq 2$ consider the symmetric group S_n with generators $s_i = (i, i+1)$ for $i \in \{1, \dots, n-1\}$. Notice that

- $s_i^2 = 1$ for all i,
- if $|i-j| = 1$ then $(s_i s_j)^3 = 1$, so $m_{ij} = 3$,
- if $|i-j| \geq 2$ then s_i and s_j are unrelated, so $m_{ij} = \infty$.

In fact, $S_n \cong \langle S \mid s_i^2 = 1, (s_i s_j)^{m_{ij}} = 1 \rangle$.

number of generators

The Coxeter system (S_n, S) is said to have type A_{n-1}.

(ii) The above group is finite. This need not be the case.
Consider $D_\infty := \langle s, t \mid s^2 = 1, t^2 = 1 \rangle \cong \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$. Type $\tilde{A}_1$.
Here, $m_{st} = \infty$.

(iii) Yet another example: $W := \langle s_1, s_2, s_3 \mid s_i^2 = 1, \forall i \neq j: (s_i s_j)^3 = 1 \rangle \cong \mathbb{Z}^2 \rtimes S_3$

Type $\tilde{A}_2$ $\rightsquigarrow$ tesselation of plane by equilateral triangles

translations

Def. Let (W,S) be a Coxeter system. A building of type (W,S) is a pair (Δ,δ) consisting of a set Δ, whose elements are called chambers, and a function $\delta : \Delta \times \Delta \longrightarrow W$, called W-metric, such that for all $C, D \in \Delta$ we have

(i) $\delta(C,D) = 1 \iff C = D$

(ii) If $\delta(C,D) = w$ and $C' \in \Delta$ satisfies $\delta(C',C) = s \in S$ then $\delta(C',D) \in \{sw, w\}$.
If, additionally, $l(sw) = l(w)+1$ then $\delta(C',D) = sw$.

(iii) If $\delta(C,D) = w$ then for any $s \in S$ there is $C' \in \Delta$ such that $\delta(C',C) = s$ and $\delta(C',D) = sw$.

One can prove that, as a consequence, $\delta(C,D) = \delta(D,C)^{-1}$.

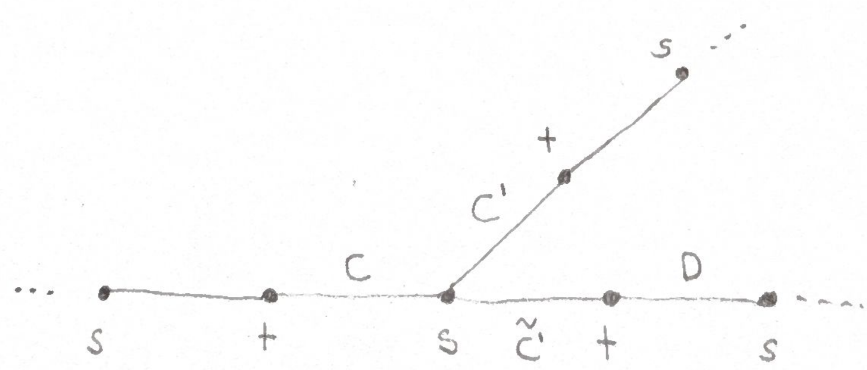
Example Let $T = (VT, ET)$ be an undirected tree without leaves (necessarily infinite). Then $\Delta := ET$ can be turned into a building of type $(D_\infty, \{s,t\})$. Fix a bipartition $VT = V_1 \sqcup V_2$. Whenever $e, e' \in ET$ are adjacent, define

$$\delta(e,e') = \begin{cases} s & \text{if } e \cap e' \in V_1 \\ t & \text{if } e \cap e' \in V_2 \end{cases} \qquad (e, e' \text{ adjacent})$$

Then, for any $e, e' \in ET$, let $e = e_0, e_1, e_2, \dots, e_n$ be the unique simple path from e to e' and define

$$\delta(e,e') = \prod_{i=0}^{n-1} \delta(e_i, e_{i+1}) \qquad \text{(some alternating product)}$$

Visually:



$\delta(C,D) = st = w$
$\delta(C',C) = s$
$\delta(C',D) = st = w$

<u>The Bruhat-Tits tree of SL_2</u>

(over non-Archimedean local fields: (finite extensions of) $\mathbb{Q}_p$ and $\mathbb{F}_q((t))$)

$\rightsquigarrow$ highly symmetric (regular) tree on which SL_2 (and also GL_2) acts, i.e. a $\tilde{A}_1$-building; can be generalised to SL_n and a higher-dimensional building

See Serre's "Trees" for full generality (difficult to read). We consider the case of $\mathbb{Q}_p$. Take the vector space $V := \mathbb{Q}_p^2$. A <u>lattice</u> in V is a finitely generated $\mathbb{Z}_p$-submodule of V which generates V as a vector space over $\mathbb{Q}_p$ (think integer lattices in $\mathbb{R}^2$). These are of the form $\{av + bw \mid a, b \in \mathbb{Z}_p;\ v, w \in \mathbb{Q}_p^2 \text{ linearly independent}\}$. We obtain maps

$$GL(2,\mathbb{Q}_p) \xrightarrow[\text{vectors}]{\text{column}} \{\text{lattices in } V\}$$

equivariant

$$GL(2,\mathbb{Q}_p)/GL(2,\mathbb{Z}_p) \rightarrowtail\!\!\!\!\rightarrow \{\text{lattices in } V\}$$

Identify lattices that are scalar multiples of each other ($\mathbb{Q}_p^*$)

$$\underbrace{GL(2,\mathbb{Q}_p)/\mathbb{Q}_p^* GL(2,\mathbb{Z}_p)}_{\cong\ PGL(2,\mathbb{Q}_p)/PGL(2,\mathbb{Z}_p)} \rightarrowtail\!\!\!\!\rightarrow \{\text{classes of lattices in } V\} =: VT$$

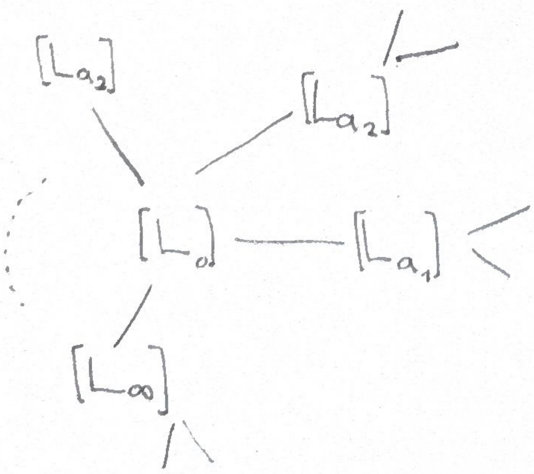
(vertices of tree)

What about edges?

<u>Prop.</u> Let L_1, L_2 be lattices in V. There is a basis (v,w) of L_1 and a pair $m \le n$ of integers such that $(p^m v, p^n w)$ is a basis of L^2. The difference $n-m =: d(L_1, L_2)$ depends only on the classes of L_1, L_2.

$\rightsquigarrow$ classes Λ_1, Λ_2 are adjacent if $d(\Lambda_1, \Lambda_2) = 1$.

Concretely, let $L_0 := \left\langle \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix} \right\rangle$, $L_a := \left\langle \begin{pmatrix}p\\0\end{pmatrix}, \begin{pmatrix}a\\1\end{pmatrix} \right\rangle$ $(a \in \mathbb{F}_p)$, $L_\infty := \left\langle \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\p\end{pmatrix} \right\rangle$



$PGL(2,\mathbb{Q}_p) \le \mathrm{Aut}(T)$

$PGL(2,\mathbb{Q}_p)_{[L_0]} = PGL(2,\mathbb{Z}_p)$

Prop. There is a $PGL(2,\mathbb{Z}_p)$-invariant bijection

$$P^1(\mathbb{Z}_p/p^n\mathbb{Z}_p) \longrightarrow S(\Lambda_0, n), \quad [(x,y)] \longmapsto \left[\mathbb{Z}_p\begin{bmatrix} x \\ y \end{bmatrix} + p^n L_0\right]$$

In particular, the local action of $PGL(2,\mathbb{Q}_p) \leq Aut(T)$ is given by size $p+1$

$$(PGL(2,\mathbb{Z}_p) \curvearrowright P^1(\mathbb{Z}_p/p\mathbb{Z}_p)) \cong (PGL(2,p) \curvearrowright P^1(\mathbb{F}_p) = \mathbb{F}_p^2\setminus\{0\}/\mathbb{F}_p^*)$$

Prop. $\mathrm{Fix}_{PGL(2,\mathbb{Q}_p)}(S(\Lambda_0,n)) = \{A \in PGL(2,\mathbb{Q}_p) \mid A \equiv \mathrm{Id} \mod p^n\mathbb{Z}_p\}$. $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ $\begin{pmatrix} 1 & 4 \\ 4 & 5 \end{pmatrix}$

Prop. $\mathrm{Fix}_{PGL(2,\mathbb{Q}_p)}((\Lambda_0, \Lambda_\infty)) = \left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in PGL(2,\mathbb{Z}_p) \;\middle|\; c \in p\mathbb{Z}_p\right\}$. $\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$

Visualisation tool: ariymarkowitz.github.io/Bruhat-Tits-Tree-Visualiser

$$P^1(R) := \{[(x,y)] \in R^2/R^* \mid xR + yR = R\}$$ projective line over a ring