On the k-closures of locally transitive groups acting on trees

(UC Louvain, 18.11.19, 50-60 minutes, w/o page 5)

a PhD student is working on representations of k-closed groups

Let T be a locally finite tree and consider $\mathrm{Aut}(T)$ equipped with the permutation topology for its action on the vertex set $V(T)$. This is a totally disconnected, locally compact (t.d.l.c.) group which plays an important role in the theory of all locally compact groups.

[diagram if necessary]

I want to relate two constructions of subgroups of $\mathrm{Aut}(T)$.

① Banks – Elder – Willis k-closures ('15)

Given $H \leq \mathrm{Aut}(T)$ and $k \in \mathbb{N}$, define the k-closure of H by

$$H^{(k)} = \{ g \in \mathrm{Aut}(T) \mid \forall x \in V(T)\ \exists h \in H : g|_{B(x,k)} = h|_{B(x,k)} \}.$$

The following properties of k-closures are easy to prove:

(i) $H^{(k)} \leq \mathrm{Aut}(T)$ is closed.

[Look at the complement. There is a vertex ...]

(ii) $H^{(1)} \geq H^{(2)} \geq \dots \geq H^{(k)} \geq \dots \geq H$.

(iii) $\bigcap_{k \in \mathbb{N}} H^{(k)} = \overline{H}$

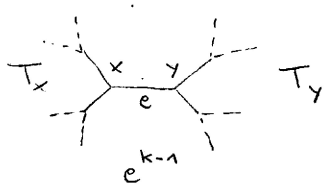
[The intersection is closed and contains H, so contains $\overline{H}$. Conversely show that any open nbhd. of $g \in \bigcap_{k \in \mathbb{N}} H^{(k)}$ contains some $h \in H$.]

The k-closures of H satisfy a generalisation of Tits' Property P.

Def. Let $H \leq \mathrm{Aut}(T)$ be closed. Then H satisfies P_k if for every edge $e = (x,y) \in E(T)$ with associated half-trees T_x, T_y we have

$$H_{e^{k-1}} = H_{e^{k-1}, T_y} \cdot H_{e^{k-1}, T_x}$$

"Independence Property"



"enlargement of e"

Note that P_1 is P.

Prop. Let $H \leq \mathrm{Aut}(T)$. Then $H^{(k)}$ satisfies Property P_k.

[Direct proof, use that $id \in H$.]

Similar to the context of Tits' Property P, define the group

$$H^{+_k} := \langle \{ H_{e^{k-1}} \mid e \in E(T) \} \rangle \leq H. \qquad (H^{+_1} = H^+)$$

Thm. Let $H \leq \mathrm{Aut}(T)$ be geometrically dense (does not preserve a proper subtree, nor fixes an end) and satisfy P_k. Then H^{+_k} is either trivial or simple.

Banks-Elder-Willis went on to find groups $H \leq \mathrm{Aut}(T)$ with infinitely many distinct k-closures, $H^{(k)}$, such as $PGL(2, \mathbb{Q}_p) \leq \mathrm{Aut}(T_{p+1})$, and showed that the associated simple groups $(H^{(k)})^{+_k} \leq \mathrm{Aut}(T)$ are pairwise non-conjugate. They ended with a 2-fold question:

(i) Given H, find an "algebraic description" of $H^{(k)}$.

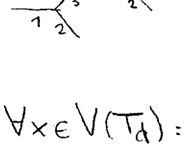
(ii) Use this description to tell whether $(H^{(k)})^{+_k}$ and $(H^{(l)})^{+_l}$ are isomorphic as topological groups.

[Story about Zurich '14 / '15]

② Universal groups (Burger-Mozes '00, T. '18)

Let T_d be the d-regular tree, and let Ω be a set of cardinality d. Let $l: E(T_d) \to \Omega$ be a legal labelling of T_d. For $k \in \mathbb{N}$, fix a finite tree $B_{d,k}$ which is isomorphic to a ball of radius k around a vertex in the labelled tree T_d. For every $x \in V(T_d)$, let $l_x^k : B(x,k) \to B_{d,k}$ be the unique label-preserving isomorphism. Consider

$$\sigma_k : \mathrm{Aut}(T_d) \times V(T_d) \longrightarrow \mathrm{Aut}(B_{d,k})$$
$$(\alpha, x) \longmapsto l_{\alpha x}^k \circ \alpha \circ (l_x^k)^{-1}$$



$B_{3,2}$

Def. Let $F \leq \mathrm{Aut}(B_{d,k})$. Define $U_k^{(l)}(F) := \{ g \in \mathrm{Aut}(T_d) \mid \forall x \in V(T_d) : \sigma_k(g,x) \in F \}$.

Some properties of $U_k(F)$ to warm up:

(i) closed in $\mathrm{Aut}(T_d)$

(ii) vertex-transitive

[$U_1(\{id\}) \leq U_k(F)$ is vertex-transitive]

(iii) compactly generated

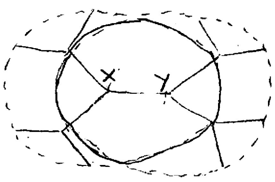
[show that $U_1(\{id\})$ is finitely generated]

(iv) satisfies Property P_k

A fruitful line of thought is that an automorphism $\alpha \in \mathrm{Aut}(T_d)$ is completely determined by its image $\alpha(b)$ on a base vertex b and all its k-local actions $\{\sigma_k(\alpha, x) \mid x \in V(T_d)\}$. Conversely, compatible such collections define automorphisms.

Def. A group $F \leq \mathrm{Aut}(B_{d,k})$ satisfies the compatibility condition (C) if $U_k(F)$ is locally action isomorphic to F, i.e. $(U_k(F)_x \curvearrowright B(x,k)) \cong (F \curvearrowright B_{d,k}) \quad \forall x \in V.$

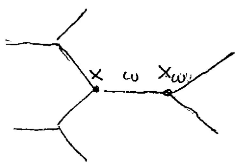
The compatibility / extension problem:



There is a natural isomorphism $\mathrm{Aut}(B_{d,1}) \cong \mathrm{Sym}(\Omega)$. We let

$$\mathrm{Aut}(B_{d,k+1}) \lesssim \mathrm{Aut}(B_{d,k}) \ltimes \prod_{\omega \in \Omega} \mathrm{Aut}(B_{d,k})$$

$$\alpha \longmapsto \left(\sigma_k(\alpha, b), \left(\sigma_k(\alpha, b_\omega)\right)_{\omega \in \Omega}\right)$$

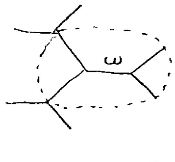


Condition (C) becomes an interchangeability condition: Two elements $\alpha = (a, (a_\omega)_\omega)$ and $\beta = (b, (b_\omega)_\omega)$ are compatible in direction ω if $b = a_\omega$ and $b_\omega = a$. ("swapping a and a_ω"). Condition (C) is:

(C) $\forall \alpha \in F \; \forall \omega \in \Omega \; \exists \alpha_\omega \in F$: α and α_ω are compatible in direction ω.
$(\alpha_\omega \in C_F(\alpha, \omega))$

Prop. Let $F \leq \mathrm{Aut}(T_d)$ satisfy (C). Then $U_k(F)$ is discrete if and only if F satisfies

(D) $\forall \omega \in \Omega$: $F_{T_\omega} = \{id\}$ (where $T_\omega = B(b,k) \cap B(b_\omega, k)$)



Examples Let $k=2$. Then $\mathrm{Aut}(B_{d,2}) = \{(a,(a_\omega)_\omega) \mid a, a_\omega \in \mathrm{Sym}(\Omega),\ \forall \omega\in\Omega: a(\omega)=a_\omega(\omega)\}$.
Let $F \le \mathrm{Sym}(\Omega)$. The following groups $\tilde{F} \le \mathrm{Aut}(B_{d,2})$ satisfy (C) and $\pi(\tilde{F}) = F$.

(i) $\Gamma(F) := \{(a,(a,\dots,a)) \mid a\in F\} \cong F$. Then $U_2(\Gamma(F)) = D(F)$ is discrete and elements have constant 1-local action.
[Illustrate (C) and (D)]

(ii) $\Phi(F) := \{(a,(a_\omega)_\omega) \mid a, a_\omega \in F,\ \forall\omega\in\Omega: a_\omega(\omega)=a(\omega)\} \cong F \ltimes \prod_{\omega\in\Omega} F_\omega$.
Then $U_2(\Phi(F)) = U_1(F)$.

Common generalisation: Suppose F preserves a partition $\mathcal{P}: \Omega = \bigsqcup_{i\in I} \Omega_i$.

(iii) $\Phi(F,\mathcal{P}) := \{(a,(a_\omega)_\omega) \mid (a_\omega)_\omega \text{ constant on blocks of } \mathcal{P}\}$

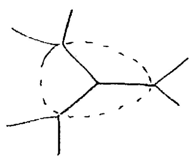
There are more, e.g. discrete groups $\Delta(F,C)$ for $C \le Z(F_{\omega_0})$ and $\Phi(F,N)$ for $N \trianglelefteq F_{\omega_0}$. See paper.

Thm. (Universality) Let $H \le \mathrm{Aut}(T_d)$ be locally transitive and contain an involutive inversion. Then there is a labelling l of T_d such that

$$U_1^{(l)}(F^{(1)}) \ge U_2^{(l)}(F^{(2)}) \ge \dots \ge U_k^{(l)}(F^{(k)}) \ge \dots \ge H \ge U_1^{(l)}(\{id\})$$

where $F^{(k)} \le \mathrm{Aut}(B_{d,k})$ is action isomorphic to the k-local action of H.

[Fix $x\in V$. By assumption, there is an involutive inversion ι_ω for every edge (x,x_ω). Use these to define $l: E(T_d) \to \Omega$ as follows: Choose a bijection $l_x: E(x) \to \Omega$. For $e \in E(x,n+1)\setminus E(n)$ define $l(e) := l(\iota_\omega(e))$ where x_ω is part of the unique reduced path from x to e.



Then the ι_ω $(\omega\in\Omega)$ generate $U_1^{(l)}(\{id\}) \le H$.
Let $F^{(k)} := l_x^k \circ H_x|_{B(x,k)} \circ (l_x^k)^{-1}$. This works.]

(i) **Thm.** Let $H \le \mathrm{Aut}(T_d)$ be locally transitive and contain an involutive inversion. Then $H^{(k)} = U_k^{(l)}(F^{(k)})$ for some l and $F^{(k)} \le \mathrm{Aut}(B_{d,k})$ with (C).

["$\le$": all k-local actions of $g\in H^{(k)}$ come from $H \le U_k(F^{(k)})$.
"$\ge$": $g\in U_k(F^{(k)})$, $x\in V$. Pick $h'' \in U_k(F^{(k)})_x$ w/ $\sigma_k(h'',x) = \sigma_k(g,x)$ and $h' \in U_1(\{id\})$ w/ ...

Part (ii) of BEW's question relies on the following, inspired by Radu.

Prop. Let $H, H' \leq \mathrm{Aut}(T_d)$ be closed and locally transitive w/ distinct point stabilisers. Then H, H' are isomorphic as top. groups if and only if they are conjugate in $\mathrm{Aut}(T_d)$.

[Recover the tree from the top. group structure:

Every compact subgroup of H is either contained in a vertex-stabiliser, or, when $H \not\leq \mathrm{Aut}(T_d)^+$ in a geometric edge stabiliser $H_{\{e,\bar{e}\}}$ $(e \in E)$. Since H is loc. transitive, these are pairwise distinct.

Vertex stabilisers are max. compact $K \leq H$ w/o max. compact K' such that $[K : K \cap K'] = 2$: For $e = (x,y) \in E(T_d)$ we have

$$[H_{\{e,\bar{e}\}} : H_{\{e,\bar{e}\}} \cap H_x] = 2$$

whereas

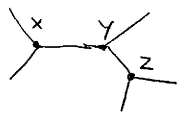
$$[H_x : H_x \cap H_y] \ , \ [H_x : H_x \cap H_{\{e,\bar{e}\}}] \geq 3$$

by the orbit-stabiliser theorem, as $d \geq 3$ and H is loc. tran.

Adjacency can also be expressed in terms of indices: Let $x, y \in V(T_d)$ be distinct. Then $(x,y) \in E$ if and only if

$$\underbrace{[H_x : H_x \cap H_y]}_{=d} \leq \underbrace{[H_x : H_x \cap H_z]}_{>d} \quad \text{for all } z \in V(T_d) \setminus \{x\}$$

b/c point stabilisers in local action are distinct



Let $\Phi : H \to H'$ be an isomorphism of top. groups. Then Φ preserves max. cpct. subgroups and indices, so there is $\varphi \in \mathrm{Aut}(T_d)$ with $\Phi(H_x) = H'_{\varphi(x)}$. Then

$$H'_{\varphi h \varphi^{-1}(x)} = \Phi(H_{h\varphi^{-1}(x)}) = \Phi(h H_{\varphi^{-1}(x)} h^{-1}) = \Phi(h) H'_x \Phi(h^{-1}) = H'_{\Phi(h)x} .$$

Thus $\varphi h \varphi^{-1} = \Phi(h)$ $\forall h \in H$ since point stabilisers in H' are distinct.

Then use the understanding of k-closures of loc. tran. groups w/ involutive inversions as universal groups $H^{(k)} = U_k(F^{(k)})$ to apply this criterion to the $(H^{(k)})^{+_k}$, e.g. for $H = \mathrm{PGL}(2, \mathbb{Q}_p)$, but not exclusively.