The magic of non-principal ultrafilters

(BMath Meetup 16/10/24, 60 minutes)

Let's consider some good old sequences, for $n \in \mathbb{N}$:

$(a_n) = (2, 1, 1, 0, 3, -1, \text{ repeat } \ldots)$ $2, \boxed{1}, 0, \boxed{3}, -1$

$b_n = n \mod 3$ $\boxed{0}, 1, \boxed{2}$

$c_n = (-1)^n$ $\boxed{-1}, \boxed{1}$

$(d_n) = (-1, 2, 0, 1, -2, 0, \text{ repeat} \ldots)$ $-1, \boxed{2}, 0, 1, -2$

None of there are convergent but they all have accumulation points.

Remember:

(x_n) converges to $x \in \mathbb{R} \iff \forall \varepsilon > 0 \ \exists N \in \mathbb{N} : \forall n \geq N : |x - x_n| < \varepsilon$

(x_n) accumulates at $x \in \mathbb{R} \iff \forall \varepsilon > 0 : \{ n \in \mathbb{N} \mid |x - x_n| < \varepsilon \}$ is infinite

(Get the audience to pick an accumulation point of each of the above sequences.)

Does anyone notice relationships between the above sequences?

$$b_n = a_n + c_n \quad \text{and} \quad d_n = b_n \cdot c_n$$

Remember:

If (x_n) and (y_n) converge then so do $z_n := x_n + y_n$ and $w_n := x_n \cdot y_n$

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty} x_n + \lim_{n\to\infty} y_n \quad \text{and} \quad \lim_{n\to\infty} w_n = \left(\lim_{n\to\infty} x_n\right) \cdot \left(\lim_{n\to\infty} y_n\right)$$

What about accumulation points?

If (x_n) accumulates at x and (y_n) accumulates at y then $z_n := x_n + y_n$ does not necessarily accumulate at $x+y$ and $w_n := x_n \cdot y_n$ does not necessarily accumulate at $x \cdot y$. Need not even have accumulation points.

think: $x_n = (-1)^n, \quad y_n = (-1)^{n+1}, \quad z_n = 0, \quad w_n = -1$

or: $x_n = (1, 1, 1, 2, 1, 3, 1, 4, \ldots) \qquad y_n = (1, 1, 2, 1, 3, 1, 4, 1, \ldots)$

$z_n = (2, 2, 3, 3, 4, 4, 5, 5, \ldots) \qquad w_n = (1, 1, 2, 2, 3, 3, 4, 4, \ldots)$

However: Bolzano-Weierstrass: every bounded sequence has an accumulation point.

Let $\Gamma(\mathbb{N},\mathbb{R})$ be the set of all bounded, real-valued sequences.

For example, $(a_n),(b_n),(c_n),(d_n) \in \Gamma(\mathbb{N},\mathbb{R})$.

Is there a map $\varphi : \Gamma(\mathbb{N},\mathbb{R}) \to \mathbb{R}$ such that

- $\varphi((x_n))$ is an accumulation point of (x_n)
- $\varphi((x_n)+(y_n)) = \varphi((x_n)) + \varphi((y_n))$
- $\varphi((x_n)\cdot(y_n)) = \varphi((x_n)) \cdot \varphi((y_n))$

Did the choices of φ from the start work?

$$0 \neq 3 + -1$$

$$2 \neq 0 \cdot -1$$

The magic trick didn't work ... but could it have? For example

$$2 = 1 + 1$$

$$2 = 2 \cdot 1$$

Can we extend this choice to $\Gamma(\mathbb{N},\mathbb{R})$? Or some other choice?

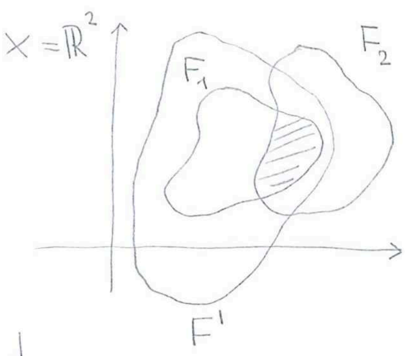
<u>Non-principal ultrafilters</u>

Let X be a set. A filter on X is a collection $\mathcal{F} \subseteq \mathcal{P}(X)$ of subsets of X such that

(F1) $\emptyset \notin \mathcal{F}$, $X \in \mathcal{F}$

(F2) $F_1, F_2 \in \mathcal{F} \Rightarrow F_1 \cap F_2 \in \mathcal{F}$

(F3) $F \in \mathcal{F}$, $F' \subseteq X$, $F' \supseteq F \Rightarrow F' \in \mathcal{F}$



A filter is an <u>ultrafilter</u> if it is maximal with respect to inclusion. For example, for $x \in \mathbb{R}^2$ and $\varepsilon > 0$:

$\mathcal{F}_1 := \{A \subseteq \mathbb{R}^2 \mid A \supseteq B(x,\varepsilon)\}$ is a filter on $\mathbb{R}^2$. It is non-principal.

$\mathcal{F}_2 := \{A \subseteq \mathbb{R}^2 \mid x \in A\}$ is an ultrafilter on $\mathbb{R}^2$ (can't make bigger – take complement)

{A filter is non-principal if it contains no finite sets.} comments about topology: Hausdorff/compact

Suppose $\mathcal{F}$ is a non-principal ultrafilter on $\mathbb{N}$.

Given a sequence $(x_n) \in \Gamma(\mathbb{N}, \mathbb{R})$ define the set

$$\mathcal{F}\text{-}\lim(x_n) := \{ x \in \mathbb{R} \mid \forall \varepsilon > 0 : \{ n \in \mathbb{N} \mid |x - x_n| < \varepsilon \} \in \mathcal{F} \}$$

Lemma: If $(x_n) \in \Gamma(\mathbb{N}, \mathbb{R})$ then $\mathcal{F}\text{-}\lim(x_n)$ contains exactly one point $x \in \mathbb{R}$ and this x is an accumulation point of (x_n).

Proof: Suppose $x \in \mathcal{F}\text{-}\lim(x_n)$. Then for every $\varepsilon > 0$ the set $\{ n \in \mathbb{N} \mid |x - x_n| < \varepsilon \}$ is in $\mathcal{F}$ and therefore infinite because $\mathcal{F}$ is non-principal. So x is an accumulation point of (x_n).

Now suppose $x, y \in \mathcal{F}\text{-}\lim(x_n)$ and $x \neq y$. Choose $\varepsilon > 0$ so that $|x - y| \geq \varepsilon$. Then $F_1 := \{ n \in \mathbb{N} \mid |x - x_n| < \frac{\varepsilon}{2} \}$ and $F_2 := \{ n \in \mathbb{N} \mid |y - x_n| < \frac{\varepsilon}{2} \}$ are in $\mathcal{F}$. Hence so is $F_1 \cap F_2$ by (F2) but $F_1 \cap F_2 = \emptyset$ in contradiction to (F1).

Going back, $\boxed{\text{define}}$ $\varphi : \Gamma(\mathbb{N}, \mathbb{R}) \to \mathbb{R}$ by $\varphi((x_n)) = \mathcal{F}\text{-}\lim(x_n)$.

By Lemma, $\varphi((x_n))$ is an accumulation point of (x_n).

Do sums and products work out?

Let $(x_n), (y_n) \in \Gamma(\mathbb{N}, \mathbb{R})$ and $\mathcal{F}\text{-}\lim(x_n) = x$, $\mathcal{F}\text{-}\lim(y_n) = y$. Put $z_n := x_n + y_n$.

We show that $\mathcal{F}\text{-}\lim(z_n) = x + y$. Let $\varepsilon > 0$. We need that

$F_\varepsilon := \{ n \in \mathbb{N} \mid |x + y - z_n| < \varepsilon \} \in \mathcal{F}$.

The set F_ε contains $\underbrace{\{ n \in \mathbb{N} \mid |x - x_n| < \frac{\varepsilon}{2} \}}_{\in \mathcal{F}} \cap \underbrace{\{ n \in \mathbb{N} \mid |y - y_n| < \frac{\varepsilon}{2} \}}_{\in \mathcal{F}}$ and

hence is contained in $\mathcal{F}$ by (F2) and (F3).

Products work out similarly.

Magical! (to construct $\mathcal{F}$, apply Zorn's Lemma to the filter of cofinite subsets of $\mathbb{N}$)