ON LOW-DIMENSIONAL GROUP COHOMOLOGY

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ABSTRACT. An elementary discussion of the group extension problem is followed by the algebraic definition of group (co)homology, assuming homological algebra. Thereupon, the rich interplay between algebraic and topological themes in group (co)homology is explored and sample applications, most importantly to group extensions, are discussed. The article concludes with a discussion of functorial properties of group (co)homology, extending so far established methods for computation.

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PREFACE

The subject of group (co)homology, as we explain in section 2.1, was motivated by algebraic topology and is the study of groups via its (co)homology groups. That is, similar to the way (co)homology groups are assigned to a given topological space, we use homological algebra to associate groups H_k (H^k), $k \in \mathbb{N}_0$, to a given group and explore how they encode information about it. In doing so we will experience a rich interplay between algebraic and topological ideas. Low-dimensional (co)homology groups, in particular H^1, H^2, H^3 and H_2 turn out to be related to the classical group extension problem. The latter is discussed in an elementary fashion in section 1 and constitutes a guiding theme throughout the text, hence the title. Later sections assume the reader to be familiar with some algebraic topology [23], [19], category theory [18] and homological algebra [10], [28].

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1. Group Extensions

Generalities. The classification of all finite groups undoubtedly constitutes an utterly desirable goal in group theory. And although it seems hardly viable there is a reasonable elementary, yet general, approach which is being presented below. Most of the material of this section was known in the 1920's, see Schreier [24]. By 1947 it was established completely, see Eilenberg and Mac Lane, [6], [7].

Recall that any finite group may be assembled from finite simple groups in the following sense.

Theorem 1.1. Every finite group G has a normal series with simple factors, i.e. a sequence of subgroups

$$1 = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_{n-1} \trianglelefteq G_n = G$$

such that G_k/G_{k-1} is simple for $k \in \{1, \dots n\}$.

Example 1.2. If $n \geq 5$ then $1 \trianglelefteq A_n \trianglelefteq S_n$ is a normal series of S_n with simple factors.

In the above theorem, subsequent groups G_k and G_{k+1} fit into the exact sequence

$$1 \to G_k \to G_{k+1} \to G_{k+1}/G_k \to 1$$

where G_{k+1}/G_k is simple.

Definition 1.3 (Group extension). An extension of a group N by a group G is an exact sequence $1 \to N \to E \to G \to 1$. An isomorphism of group extensions

$$1 \to N \to E \to G \to 1 \quad \text{and} \quad 1 \to N \to E' \to G \to 1$$

of N by G is an isomorphism $E \cong E'$ such that the following diagram commutes.

$$\begin{tikzcd}
1 \arrow[r] & N \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d] & G \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & N \arrow[r] & E' \arrow[r] & G \arrow[r] & 1
\end{tikzcd}$$

Remark 1.4. We refer to E as an extension of N by G if the sequence is understood.

Remark 1.5. If $1 \to N \xrightarrow{\iota} E \xrightarrow{\pi} G \to 1$ is an extension of N by G then the group $N \cong \operatorname{im}\iota = \ker\pi$ may be regarded as a normal subgroup of E; then $E/N \cong G$.

The completed classification of finite simple groups would yield a description of all finite groups if there was a solution to the problem of finding all extensions of one group by another up to isomorphism. Namely, in the above theorem, G_1 is a simple group, G_2 is an extension of G_1 by the simple group G_2/G_1 and so on.

The remainder of this section is devoted to this extension problem. Before tackling the general case, however, we consider a class of examples, see Brown [4, IV.2].

Split Extensions. The obvious example of a direct product leads us to consider semidirect products. We recall the definition and main properties.

Definition 1.6 (Semidirect product). Given groups N and G as well as an action of G on N by automorphisms $\rho : G \to \operatorname{Aut}(N)$, denoted $\rho(g)(n) =: {}^g n$, the semidirect product $N \rtimes G$ is given by the following group law on the set $N \times G$:

$$(n, g)(n', g') := (n\, {}^g n', gg') \quad \forall n, n' \in N,\ \forall g, g' \in G.$$

Example 1.7. The n-th dihedral group D_n is isomorphic to the semidirect product $\mathbb{Z}_n \rtimes \mathbb{Z}_2$ in which the non-trivial element of $\mathbb{Z}_2$ acts as inversion.

Proposition 1.8. Given N, G and $\rho : G \to \mathrm{Aut}(N)$, the semidirect product $N \rtimes G$ yields the split extension

$$1 \longrightarrow N \xrightarrow{\iota} N \rtimes G \underset{\sigma}{\overset{\pi}{\rightleftarrows}} G \longrightarrow 1$$

where $\iota : n \mapsto (1, n)$, $\pi : (n, g) \mapsto g$, $\sigma : g \mapsto (1, g)$ and $\pi\sigma = \mathrm{id}$. Furthermore, ${}^g n = gng^{-1}$ if we regard N and G as subgroups of $N \rtimes G$ via ι and σ respectively.

Remark 1.9. Any split extension is isomorphic to a semidirect product extension.

In studying the semidirect product $N \rtimes G$ one might want to give a description of $\mathrm{Aut}(N \rtimes G)$. If $N = A$ is abelian, written additively, there are two subgroups of $\mathrm{Aut}(A \rtimes G)$ which are related to our studies.

Definition 1.10. Given A abelian, G and $\rho : G \to \mathrm{Aut}(A)$, define

$$\mathrm{Aut}_{\mathrm{id}}(A \rtimes G) \leq \mathrm{Aut}(A \rtimes G) \quad \text{and} \quad \mathrm{Aut}_A(A \rtimes G) \trianglelefteq \mathrm{Aut}_{\mathrm{id}}(A \rtimes G)$$

to be the subgroups of automorphisms which restrict to the identity on A and $(A \rtimes G)/A \cong G$, and which arise from conjugation by elements of A, respectively.

An automorphism $\varphi \in \mathrm{Aut}_{\mathrm{id}}(A \rtimes G)$ is necessarily given by

$$\varphi : (a, g) \mapsto (a + f(g), g) \quad \text{for some} \quad f : G \to A \quad \text{with} \quad f(1) = 0.$$

Therefore, the following proposition is immediate from the group law.

Definition 1.11. Given A abelian, G and $\rho : G \to \mathrm{Aut}(A)$, make

$$(Z^1) \qquad Z^1_\rho(G, A) := \{f : G \to A \mid f(gh) = {}^g f(h) + f(g)\ \forall g, h \in G\} \quad \text{and}$$

$$(B^1) \qquad B^1_\rho(G, A) := \{f : G \to A \mid \exists a \in A : f(g) = {}^g a - a\ \forall g \in G\}$$

into groups by pointwise addition.

Remark 1.12. The groups Z^k and B^k which arise in the current section will be recovered as part of the general theory in section 3.1.

Proposition 1.13. Given A abelian, G and $\rho : G \to \mathrm{Aut}(A)$, we have

$$\mathrm{Aut}_{\mathrm{id}}(A \rtimes G) \cong Z^1_\rho(G, A) \quad \text{and} \quad \mathrm{Aut}_A(A \rtimes G) \cong B^1_\rho(G, A).$$

Remark 1.14. Another interpretation of the groups $Z^1_\rho(G, A)$ and $B^1_\rho(G, A)$ is the classification of splittings up to A-conjugacy, see Brown [4, IV.2].

General Extensions. With semidirect products as an example at hand we approach the group extension problem in general. A detailed exposition of the following is given by Mac Lane [17, IV.8]. Whereas the descriptions derived here are of theoretical interest mostly, later sections will provide powerful methods for computations.

For arbitrary groups N and G we wish to describe all groups E that fit into an exact sequence S: $1 \to N \xrightarrow{\iota} E \xrightarrow{\pi} G \to 1$. As a first step we associate data to S which depends on N and G only. Motivated by the fact that a semidirect product is given by a homomorphism from G to $\mathrm{Aut}(N)$ which amounts to conjugation in $N \rtimes G$ we note that conjugation of N in E gives rise to a homomorphism $\eta : E \to \mathrm{Aut}(N)$ which maps $N \trianglelefteq E$ to $\mathrm{Inn}(N)$ and thus yields a homomorphism

$$\theta : E/N \cong G \to \mathrm{Out}(N) = \mathrm{Aut}(N)/\,\mathrm{Inn}(N).$$

Conversely, the following questions arise.

Questions. Given a triple $(N, G, \theta : G \to \mathrm{Out}(N))$, is there an extension of N by G which realizes θ? If so, how many are there up to isomorphism?

To answer these questions, assume that S realizing θ is given. In order to examine the group law on E we choose a normalized section $s : G \to E$, i.e. $\pi s = \mathrm{id}$ and $s(1) = 1$. Then each element of E has a unique expression $s(g)n$ where $g \in G$ and $n \in N$. Multiplication behaves as follows:

$$s(g)n = {}^g n s(g) \quad \forall n \in N,\ \forall g \in G \tag{M1}$$

where ${}^g n = \tau(g)(n)$ for some $\tau(g) \in \theta(g)$. We may choose $\tau(1) = \mathrm{id}$. Also,

$$s(g)s(g') = t(g, g')s(gg') \quad \forall g, g' \in G. \tag{M2}$$

for some normalized function $t : G^2 \to N$, i.e. $t(1, g) = 1 = t(g, 1)\ \forall g \in G$.

Turning back to the problem of constructing extensions one might try to mimic (M1) and (M2) for given τ and t. To ensure associativity, τ and t need to be subject to certain conditions.

The associativity of $s(g)s(g')s(g'')$ implies

$${}^g t(g', g'')t(g, g'g'') = t(g, g')t(gg', g'') \quad \forall g, g', g'' \in G. \tag{A1}$$

Eventually, conjugation of N by the two sides of (M2) must yield the same automorphism of N. Hence

$$\tau(g)\tau(g') = \eta(t(g, g'))\tau(gg') \quad \forall g, g' \in G. \tag{A2}$$

Proposition 1.15. Given groups N and G as well as normalized functions $\tau : G \to \mathrm{Aut}(N)$ and $t : G^2 \to N$ satisfying (A1) and (A2), the set $N \times G$ is made into a group by defining

$$(n, g)(n', g') := (n\,{}^g n' t(g, g'), gg') \quad \forall n, n' \in N\ \forall g, g' \in G.$$

With $\iota : N \to N \times G,\ n \mapsto (n, 1)$ and $\pi : G \to N \times G,\ g \mapsto (1, g)$, the set $N \times G$ with the above group law yields the exact sequence $1 \to N \xrightarrow{\iota} N \times G \xrightarrow{\pi} G \to 1$. The associated map $\theta : G \to \mathrm{Out}(N)$ is given by $g \mapsto \pi\tau(g)$.

Proof. Associativity follows from (A1) and (A2). The unit element is given by $(1, 1)$ and the inverse of (n, g) is $((\tau(g))^{-1}(n^{-1}t(g, g^{-1})^{-1}, g^{-1}))$. The remaining assertions are immediate. $\square$

Definition 1.16 (Crossed product). The group defined in the above proposition is denoted by $P(N, G, \tau, t)$ and called the crossed product of N and G with respect to τ and t. The associated extension is called a crossed product extension.

Remark 1.17. Note that $P(N, G, \tau, t) = N \rtimes G$ if $t \equiv 1$. Then τ is a homomorphism.

Conversely, our above analysis readily yields the following.

Proposition 1.18. Given groups N and G as well as a normalized function $\tau : G \to \mathrm{Aut}(N)$, any extension E of N by G realizing $\pi\tau$ is isomorphic to $P(N, G, \tau, t)$ for some normalized $t : G^2 \to N$.

Proof. We may choose a section s such that (M1) holds. Then (M2) holds for a certain normalized t and one checks that $s(g)n \mapsto ({}^g n, g)$ is an isomorphism. $\square$

Combining the above results we obtain the following criterion.

Proposition 1.19. The triple (N, G, θ) admits an extension E if and only if there are normalized functions $\tau : G \to \mathrm{Aut}(N)$ and $t : G^2 \to N$ satisfying (A1) and (A2) as well as $\pi\tau = \theta$ in which case E is isomorphic to $P(N, G, \tau, t)$.

Question. Given (N, G, θ), are there such functions τ and t?

A function τ with the above properties is a normalized set-theoretic lift of θ. And since $\pi(\tau(g)\tau(g')\tau(gg')^{-1}) = 1 \; \forall g, g' \in G$ we may define t by (A2) which then necessarily is normalized. The associativity of $\tau(g)\tau(g')\tau(g'')$ shows that (A1) holds if η is applied to both sides. Since $\ker(\eta) = Z(N)$, the center of N, we have

$$(A3) \qquad {}^{g}t(g', g'')t(g, g'g'') = z(g, g', g'')t(g, g')t(gg', g'') \quad \forall g, g', g'' \in G$$

for some normalized $z : G^3 \to Z(N)$, i.e. $1 = z(1, g', g'') = z(g, 1, g'') = z(g, g', 1)$; and we are facing another question.

Question. How does z depend on the choices τ and t?

The answer is elementary yet tedious to check: A change of τ may be followed by a modification of t such that z remains unchanged. For a given τ, a modification of t replaces z by zb for some $b \in B^3_\varphi(G, Z(N))$ where φ is induced by θ. We therefore have the following result.

Definition 1.20. Given Z abelian, G and $\varphi : G \to \mathrm{Aut}(Z)$, let $B^3_\varphi(G, Z)$ be the group of all $k : G^3 \to Z$ for which there is an $l : G^2 \to Z$ such that

$$(B^3) \qquad k(g, g', g'') = {}^{g}l(g', g'')l(gg', g'')^{-1}l(g, g'g'')l(g, g')^{-1} \quad \forall g, g', g'' \in G.$$

Theorem 1.21. Each triple (N, G, θ), inducing $\varphi : G \to \mathrm{Aut}(Z(N))$, is uniquely assigned a coset of $B^3_\varphi(G, Z(N))$ in $C^3(G, Z(N)) := \{f : G^3 \to Z(N)\}$. There is an extension of N by G realizing θ if and only if that coset is $B^3_\varphi(G, Z(N))$.

Remark 1.22. There is a subgroup $Z^3_\varphi(G, Z(N))$ of $C^3(G, Z(N))$ which contains z of (A3) and has $B^3_\varphi(G, Z(N))$ as a normal subgroup, i.e. each triple (N, G, θ) is uniquely assigned an element of $H^3_\varphi(G, Z(N)) := Z^3_\varphi(G, Z(N))/B^3_\varphi(G, Z(N))$.

Eventually, our above propositions allows us to give a description of the isomorphism classes of extensions, denoted $E(N, G, \theta)$, in the case of existence.

Definition 1.23. Given Z abelian, G and $\varphi : G \to \mathrm{Aut}(Z)$, let $Z^2_\varphi(G, Z)$ be the group of all $l : G^2 \to Z$ such that

$$(Z^2) \qquad {}^{g}l(g', g'')l(gg', g'')^{-1}l(g, g'g'')l(g, g')^{-1} = 0 \quad \forall g, g', g'' \in G;$$

and $B^2_\varphi(G, Z)$ the group of all $l : G^2 \to Z$ for which there is an $m : G \to Z$ with

$$(B^2) \qquad l(g, g') = {}^{g}m(g')m(gg')^{-1}m(g) \quad \forall g, g' \in G.$$

Furthermore, let $H^2_\varphi(G, Z) := Z^2_\varphi(G, Z)/B^2_\varphi(G, Z)$.

Theorem 1.24. If the triple (N, G, θ), inducing φ, admits an extension, there is a one-to-one correspondence between $E(N, G, \theta)$ and elements of $H^2_\varphi(G, Z(N))$.

Proof. Fix a normalized $\tau : G \to \mathrm{Aut}(N)$ lifting θ. Then any extension of N by G realizing θ is isomorphic to $P(N, G, \varphi, t)$ for some t. It can be checked that $P(N, G, \tau, t + l)$ for $l \in Z^2_\varphi(G, Z(N))$ yields all such crossed products and that two are isomorphic if and only if $l \in B^2_\varphi(G, Z(N))$. $\square$

As mentioned above, the description of extensions as crossed products is of theoretical interest mostly. However, in the subsequent sections we develop methods to compute $H^k_\varphi(G, Z(N))$ and therefore derive information about extensions of N by G. For instance, if $N = A$ is abelian and $H^2_\varphi(G, A) = 0$ then any extension of A by G realizing φ is necessarily isomorphic to $A \rtimes_\varphi G$, see e.g. example 4.11.

2. The Algebraic Approach to Group (Co)homology

In section 2.1, a brief historical account of our subject is given. Subsequently, in 2.2, the contemporary algebraic definition is presented and followed by some computations in 2.3, making reference to section 1. As an application of the theory, we classify p-groups with a cyclic subgroup of index p in section 2.4.

2.1. Topological Origin.

An excellent historical treatment of group (co)homology is given by Mac Lane [16]. Remarks are also incorporated in Brown [4, Intr., I, II].

The theory of group (co)homology was originated by algebraic topology, based on the following result proved by Hurewizc [13] in 1936.

Theorem 2.1 (Hurewicz). Let X be a path-connected topological space such that $\pi_k(X) = 0$ for all $k \geq 2$. Then the homotopy class of X is determined by $\pi_1(X)$.

Consequently, (co)homology groups of X may be associated to $\pi_1(X)$. Assume for instance that X is a CW-complex. Then "(co)homology groups of $\pi_1(X)$" may be computed from the cellular chain complex of X; and if X may be assembled from cells in different ways the corresponding different complexes yield isomorphic (co)homology groups. The use of different complexes seems to have been the main ingredient — see Mac Lane [16, Sec. 3], Hopf [11] — to the powerful algebraization of the subsequent years which is being presented below, cf. Weibel [28, Sec. 6].

2.2. Algebraization.

The algebraic definition of group (co)homology may be regarded as an instance of the paradigm that a group G shall be studied via its actions on spaces. Let us consider the abelian category $G-\mathbf{Mod}$ of left G-modules and G-equivariant maps. The following additive functors from $G-\mathbf{Mod}$ to the category of abelian groups $\mathbf{Ab}$ are central to our studies.

Definition 2.2 ((Co)invariants). Let $(-)^G : G-\mathbf{Mod} \to \mathbf{Ab}$ associate to a G-module M the subgroup of invariants, i.e. $M^G = \{m \in M \mid gm = m \ \forall g \in G\}$. Similarly, let $(-)_G : G-\mathbf{Mod} \to \mathbf{Ab}$ associate to M the quotient group of coinvariants, i.e. $M_G = M/\{m - gm \mid m \in M,\ g \in G\}$.

Let $T : \mathbf{Ab} \to G-\mathbf{Mod}$ denote the functor which assigns the trivial G-module to an abelian group. There are adjunctions

$$G-\mathbf{Mod} \underset{\mathrm{T}}{\overset{(-)_G}{\rightleftarrows}} \mathbf{Ab} \underset{(-)^G}{\overset{\mathrm{T}}{\rightleftarrows}} G-\mathbf{Mod}.$$

In particular, $(-)_G$ is right-exact and $(-)^G$ is left-exact. Since $G-\mathbf{Mod}$ has enough projectives and injectives, the extent to which exactness fails may be captured by the left-derived functors $L_k(-)_G$ and right-derived functors $R^k(-)^G$ respectively; and it is the study of these derived functors which constitutes group (co)homology.

The precise definition, however, is stated in a way which often facilitates algebraic manipulations, based on the following proposition. Note, that $G-\mathbf{Mod}$ is isomorphic to $\mathbb{Z}[G]-\mathbf{Mod}$, the category of left $\mathbb{Z}[G]$-modules, where $\mathbb{Z}[G]$ denotes the group ring of G over $\mathbb{Z}$. It is convenient to regard the two as the same category.

Proposition 2.3. Let G be a group. There are isomorphisms of functors

$$(-)^G \cong \hom_{\mathbb{Z}[G]}(\mathbb{Z}, -) \quad \text{and} \quad (-)_G \cong \mathbb{Z} \otimes_{\mathbb{Z}[G]} (-)$$

from $\mathbb{Z}[G]-\mathbf{Mod}$ to $\mathbf{Ab}$ where we regard $\mathbb{Z}$ as a trivial left and right $\mathbb{Z}[G]$-module.

We are now able to define our main objects of study.

Definition 2.4 (Group (co)homology). Let G be a group and let M be a G-module. We define the k-th homology and cohomology group of G with coefficients in M by

$$H_k(G, M) = L_k(\mathbb{Z} \otimes_{\mathbb{Z}[G]} (-))(M) \quad \text{and} \quad H^k(G, M) = R^k(\hom_{\mathbb{Z}[G]}(\mathbb{Z}, -))(M).$$

2.3. **Resolutions via Algebra.** We turn to actually computing (co)homology groups from the definition. In doing so we determine by purely algebraic means the (co)homology of cyclic groups, which is of particular interest in view of section 1, see Evens [8, III.1].

Generalities. As an immediate consequence of the definition of group (co)homology, we have $H_0(G,M) = M_G$ and $H^0(G,M) = M^G$ for every group G and G-module M. To compute the higher (co)homology groups, homological algebra provides various methods. For instance, choose a $\mathbb{Z}[G]$-projective resolution

$$\cdots \xrightarrow{d_3} P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} \mathbb{Z} \to 0 \qquad (\text{in short}, P \to \mathbb{Z})$$

of the trivial right (left) $\mathbb{Z}[G]$-module $\mathbb{Z}$. The (co)homology of G with coefficients in M is then given by the homology of

$$\cdots \xrightarrow{d_{3*}} P_2 \otimes_G M \xrightarrow{d_{2*}} P_1 \otimes_G M \xrightarrow{d_{1*}} P_0 \otimes_G M \to 0$$

and the cohomology of

$$0 \to \hom_G(P_0, M) \xrightarrow{d_1^*} \hom_G(P_1, M) \xrightarrow{d_2^*} \hom_G(P_2, M) \xrightarrow{d_3^*} \cdots .$$

Remark 2.5. We have G-**Mod** $\cong$ **Mod**-G by defining $pg = g^{-1}p$ for all p in a left G-module P. This facilitates switching between left and right module resolutions.

The required resolutions may be constructed as follows: Choose a projective, e.g. a free $\mathbb{Z}[G]$-module P_0 and a surjection $\varepsilon : P_0 \to \mathbb{Z}$; then choose a projective $\mathbb{Z}[G]$-module P_1 and a surjection $d_1 : P_1 \to \ker(\varepsilon) \subseteq P_0$. Inductively, given P_n choose a projective $\mathbb{Z}[G]$-module P_{n+1} and a surjection $d_{n+1} : P_{n+1} \to \ker(d_n) \subseteq P_n$.

Cyclic Groups. The (co)homology of the infinite cyclic group is readily computed.

Proposition 2.6. Let $G = \langle g \rangle$ and let M be a G-module. Then, for $k \neq 0$,

$$H_k(G,M) = 0 = H^k(G,M).$$

Proof. Use the resolution

$$\cdots \to 0 \to \mathbb{Z}[G] \xrightarrow{g-1} \mathbb{Z}[G] \xrightarrow{\varepsilon} \mathbb{Z}$$

where ε maps g to $1 \in \mathbb{Z}$ and $g-1$ denotes multiplication by $g-1$ (from either side). Then, note that $\mathbb{Z}[G] \otimes_{\mathbb{Z}[G]} M \cong M \cong \hom_{\mathbb{Z}[G]}(\mathbb{Z}[G], M)$ whence the (co)homology of G with coefficients in M may be computed from

$$\cdots \to 0 \to M \xrightarrow{g-1} M \quad \text{and} \quad M \xrightarrow{g-1} M \to 0 \to \cdots .$$

□

In a similar fashion, the (co)homology of finite cyclic groups is computed.

Proposition 2.7. Let $G = \langle g \mid g^n = 1 \rangle$ and $S = \sum_{g \in G} g \in \mathbb{Z}[G]$. Then

$$H_k(G,M) = \left\{ \begin{matrix} M^G/SM & k \text{ odd} \\ {}_SM/(g-1)M & k \text{ even} \end{matrix} \right\} = H^{k+1}(G,M)$$

in non-zero dimensions where ${}_SM = \{m \in M \mid Sm = 0\}$.

Proof. Use the infinite resolution $\cdots \xrightarrow{g-1} \mathbb{Z}[G] \xrightarrow{S} \mathbb{Z}[G] \xrightarrow{g-1} \mathbb{Z}[G] \xrightarrow{\varepsilon} \mathbb{Z} \to 0.$ □

Example 2.8. In view of how the group extension problem might help classifying finite groups, let us consider extensions of $N := \mathbb{Z}_p$ by $G := \mathbb{Z}_q$ where p and q are distinct primes. Assume for instance, that $\theta : G \to \mathrm{Out}(N) \cong \mathrm{Aut}(N)$ is trivial. Then $N^G = N$ and $SN = qN = N$. Hence, by the above, $H^2(G,N) = 0$ and any untwisted extension of N by G is isomorphic to $N \times G$. We continue this discussion in section 4.1, example 4.11.

2.4. **Classification of p-groups with a Cyclic Subgroup of Index p.** We shall further illustrate the usefulness of the preceding algebraization by classifying all p-groups with a cyclic subgroup of index p with the help of proposition 2.7, see Brown [4, IV.4]. The following list of examples of such groups may come to mind.

Theorem 2.9. Let G be a p-group with a cyclic subgroup of index p. Then G is isomorphic to one of the following groups:

(i) $\mathbb{Z}_q$ $(q = p^n, n \geq 1)$.
(ii) $\mathbb{Z}_q \times \mathbb{Z}_p$ $(q = p^n, n \geq 1)$.
(iii) $\mathbb{Z}_q \rtimes \mathbb{Z}_p$ $(q = p^n, n \geq 2)$.
  (iii.1) The generator of $\mathbb{Z}_p$ acts as multiplication by $1 + p^{n-1}$.
  (iii.2) $p = 2$ and the generator of $\mathbb{Z}_2$ acts as multiplication by -1.
  (iii.3) $p = 2$ and the generator of $\mathbb{Z}_2$ acts as multiplication by $-1 + 2^{n-1}$.
(iv) $Q_m := \langle x, y \mid x^m = y^2,\ yxy^{-1} = x^{-1}\rangle$ $(m = 2^n, n \geq 1)$.

Proposition 2.10. Let $\mathbb{H} := \mathbb{R} \oplus \mathbb{R}\, i \oplus \mathbb{R}\, j \oplus \mathbb{R}\, k$ denote the quaternion algebra. Then Q_m $(m \in \mathbb{N})$ is isomorphic to $\langle \exp(\pi i/m), j\rangle \subset \mathbb{H}^*$ for $m \in \mathbb{N}$ and has $4m$ elements.

Definition 2.11. The groups $Q_m := \langle x, y \mid x^m = y^2,\ yxy^{-1} = x^{-1}\rangle$ $(m \in \mathbb{N})$ are called generalized quaternion.

The case of semidirect products is based on the following lemma from elementary number theory.

Lemma 2.12. Let a be an integer such that $a^p \equiv 1 \bmod p^n$ for some $n \geq 2$ and a prime p. If $p \neq 2$, then $a \equiv 1 \bmod p^{n-1}$. If $p = 2$ then $a \equiv \pm 1 \bmod 2^{n-1}$.

Proof. (Theorem 2.9). Recall that a subgroup of index the smallest prime divisor of the group is necessarily normal. The assumptions on G then imply that there is an extension $E : 0 \to \mathbb{Z}_q \to G \to \mathbb{Z}_p \to 0$ where $q = p^n$ for some $n \geq 0$.

If $\mathbb{Z}_p$ acts trivially on $\mathbb{Z}_q$ then G is abelian by (M1) of section 1. Finite abelian group theory then implies that G is of type (i) or (ii).

If $\mathbb{Z}_p$ acts non-trivially on $\mathbb{Z}_q$ then necessarily $n \geq 2$ and the action is given by some embedding $\iota : \mathbb{Z}_p \hookrightarrow \mathbb{Z}_q^* = \mathrm{Aut}(\mathbb{Z}_q)$.

If $p \neq 2$, lemma 2.12 implies $\operatorname{im}\iota = \{1 + b \mid b \in \mathbb{Z}\, p^{n-1}/\mathbb{Z}\, p^n\}$. Therefore, some generator of $\mathbb{Z}_p$ acts as multiplication by $1 + p^{n-1}$. In order to apply proposition 2.7 we compute $\mathbb{Z}_q^{\mathbb{Z}_p} = \mathbb{Z}\, p/\mathbb{Z}\, p^n$ and $S\,\mathbb{Z}_q = (\sum_b 1 + b)\,\mathbb{Z}_q = p\,\mathbb{Z}_q = \mathbb{Z}\, p/\mathbb{Z}\, p^n$. Therefore $H_2(\mathbb{Z}_p, \mathbb{Z}_q) = 0$, the extension E splits and G is of type (iii.1).

If $p = 2$, lemma 2.12 implies $\operatorname{im}\iota \leq \{\pm 1 + b \mid b \in \mathbb{Z}\, 2^{n-1}/\mathbb{Z}\, 2^n\}$. Hence the image $a \in \mathbb{Z}_q^*$ of the generator of $\mathbb{Z}_2$ is an element of $\{-1, -1 + 2^{n-1}, 1 + 2^{n-1}\}$.

If $a = -1$ then $\mathbb{Z}_q^{\mathbb{Z}_2} = \mathbb{Z}\, 2^{n-1}/\mathbb{Z}\, 2^n$ and $S\,\mathbb{Z}_q = (1 + (-1))\,\mathbb{Z}_q = 0$. Therefore $H_2(\mathbb{Z}_2, \mathbb{Z}_q) \cong \mathbb{Z}_2$ and there are two inequivalent extensions of $\mathbb{Z}_q$ by $\mathbb{Z}_2$ realizing the given action. These are the cases (iii.2) and (iv).

The other two possibilites for a yield trivial second cohomology groups and hence the cases (iii.3) and (iii.1). □

The following group-theoretic result may be viewed as a corollary to the classification of p-groups with a cyclic subgroup of index p.

Theorem 2.13. If G is a p-group which has a unique subgroup of order p then G is either cyclic or generalized quaternion.

Sketch of Proof. By induction, every proper subgroup of G is cyclic or generalized quaternion. By the first Sylow theorem we may choose a subgroup H of G of index p. If H is cyclic then G is either cyclic or generalized quaternion by theorem 2.9; for these are the only groups in (i)-(iv) with a unique subgroup of order p. Otherwise H is generalized quaternion and $p = 2$. In this case one may construct a cyclic subgroup of G of index 2, see Brown [4, IV.4]. Hence the assertion.

3. Topological Methods in Group (Co)homology

In section 3.1 we discuss various topological methods to obtain $\mathbb{Z}[G]$-projective resolutions of the trivial $\mathbb{Z}[G]$-module $\mathbb{Z}$, section 3.2 then introduces the Baer sum. In section 3.3 we find an algebraic description of $H_2(G, \mathbb{Z})$ in terms of G which is applied in section 3.4.

3.1. **Resolutions via Topology.**

Topological considerations as in section 2.1 yield various methods of constructing resolutions of the trivial $\mathbb{Z}[G]$-module $\mathbb{Z}$, see Brown [4, I.3,4,5]. By exploring these, we rediscover the groups that occurred in section 1. For a background in algebraic topology see e.g. Rotman [23] and May [19].

G-CW Complexes. The augmented cellular chain complex

$$\cdots \to C_n(X) \xrightarrow{\partial} C_{n-1}(X) \to \cdots \to C_1(X) \xrightarrow{\partial} C_0(X) \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

of a CW-complex X may, under certain assumptions, be made into a free whence projective resolution of $\mathbb{Z}$ over $\mathbb{Z}[G]$.

Definition 3.1 (G-CW-complex). A G-CW-complex is a CW-complex on which G acts by cellular homeomorphisms, hence permutes the cells in each dimension.

In particular, the cellular chain complex of a G-CW-complex is a complex of G-modules which in fact are free $\mathbb{Z}[G]$-modules if G acts freely on the cells of X.

Proposition 3.2. Let X be a free G-set and S a set of representatives for the orbits. Then $\mathbb{Z}[X]$, the free abelian group on X, is a free $\mathbb{Z}[G]$-module with basis S.

Proof. Denote the orbit of $s \in S$ by X_s. Then

$$\mathbb{Z}[X] \cong \mathbb{Z}\left[\coprod\nolimits_{s\in S} X_s\right] \cong \coprod\nolimits_{s\in S} \mathbb{Z}[X_s] \cong \coprod\nolimits_{s\in S} \mathbb{Z}[G/\operatorname{Stab}_G(s)] \cong \coprod\nolimits_{s\in S} \mathbb{Z}[G]$$

with coproducts formed in G-**Set** and G-**Mod** respectively. $\square$

As an immediate consequence we obtain a topological source of resolutions.

Proposition 3.3. Let X be a contractible free G-CW-complex. Then the augmented cellular chain complex of X is a free resolution $\mathbb{Z}$ over $\mathbb{Z}[G]$.

In turn, a source of free G-CW-complexes is given by the theory of covering spaces which we recall presently, cf. May [19, Sec. 3]. For a group G denote by $\mathcal{O}(G)$ the category of canonical orbits of G which has objects the G-sets G/H for arbitrary subgroups H of G and morphisms G-equivariant maps. For a topological space X denote by $\mathbf{Cov}(B)$ the category of coverings of B.

Theorem 3.4. Let B be connected, locally path-connected and semi-locally simply connected. Choose a basepoint $b \in B$ and let $G = \pi_1(B, b)$. There is a functor

$$E(-) : \mathcal{O}(G) \to \mathbf{Cov}(B)$$

which is an equivalence of categories. For $H \le G$ the cover $p : E(G/H) \to B$ has a canonical basepoint e in its fiber F_b over b such that $p_*((E(G/H)), e) = H$. Moreover, $F_b \cong G/H$ as a G-set.

Example 3.5. The projections $\mathbb{R} \to S^1$ as well as $S^n \to \mathbb{R}\mathbb{P}^n$ $(n \ge 2)$ are universal covers. If $b \in \mathbb{R}\mathbb{P}^n$ then F_b has two elements. Therefore, $\pi_1(\mathbb{R}\mathbb{P}^n) \cong \mathbb{Z}_2$ for $n \ge 2$.

For instance, the above may be applied to a connected CW-complex X. Its universal cover $E(\pi_1(X, x))$ admits a natural CW-structure and is being acted on freely and transitively by $\pi_1(X, x)$. Hence we are lead to study CW-complexes X which are connected, satisfy $\pi_1(X) = G$ for a given G and whose universal cover E is contractible. The latter is equivalent to both $H_k(E) = 0$ for $k \ge 2$ and $\pi_k(X) = 0$ for $k \ge 2$; hence we are studying the same spaces as Hurewicz, cf. section 2.1.

Definition 3.6 (Eilenberg-Mac Lane Complex). A connected CW-complex which satisfies $\pi_n(X) = G$ and $\pi_k(X) = 0$ if $k \neq n$ for a given group G is called an Eilenberg-Mac Lane complex of type (G, n), denoted $K(G, n)$.

Remark 3.7. A $K(G, n)$-complex may be constructed for every choice of G and n.

Proposition 3.8. Let X be a $K(G, 1)$-complex with universal cover E. Then the augmented cellular chain complex of E is a free resolution of $\mathbb{Z}$ over $\mathbb{Z}[G]$; also $H_*(G, M) \cong H_*(X, M)$ and $H^*(G, M) \cong H^*(X, M)$ for every abelian group M.

Proof. For the second assertion, regard M as a trivial $\mathbb{Z}[G]$-module and note that $C(E) \otimes_G M \cong C(X) \otimes_{\mathbb{Z}} M$ and $\hom_G(C(E), M) \cong \hom_{\mathbb{Z}}(C(X), M)$. $\square$

Remark 3.9. Proposition 3.8 holds in fact for G-modules M, the (co)homology of X being the (co)homology with local coefficients M.

Free Groups. As an illustration of the above, let $G = F(S)$, the free group on a set S. It is known that the bouquet of circles $X := \bigvee_{s \in S} S^1_s$ is a $K(F(S), 1)$-complex. Denote its universal cover by E. Choose a vertex e^0 in the fiber over the single vertex x^0 of X; it represents the unique G-orbit of vertices of E and thus generates $C_0(E)$ as a free $\mathbb{Z}[G]$-module. As a basis of $C_1(E)$ we may, for each $s \in S$, choose a 1-cell e^1_s lying over $S^1_s - \{x^0\}$ with initial vertex e^0; its terminal vertex is se^0. Hence

$$0 \to \coprod_{s \in S} \mathbb{Z}[G] \xrightarrow{\partial} \mathbb{Z}[G] \xrightarrow{\varepsilon} \mathbb{Z} \to 0 \quad \text{where} \quad \partial : e^1_s \mapsto s - 1$$

from which a description of the (co)homology of free groups may be obtained.

Free Actions on Spheres. A further instance of the rich interplay between algebra and topology in group (co)homology is provided by groups whose (co)homology groups exhibit periodicity. Namely, we have the following proposition.

Proposition 3.10. Let X be a free G-CW-complex that is homeomorphic to an odd-dimensional sphere S^{2k-1}. Then G has $2k$-periodic (co)homology.

Proof. From the augmented cellular chain complex of X we obtain

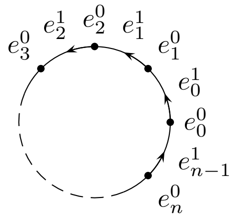
$$0 \to \mathbb{Z} = H_{2k-1}(X, \mathbb{Z}) \xrightarrow{\iota} C_{2k-1}(X) \to \cdots \to C_1(X) \to C_0(X) \xrightarrow{\varepsilon} \mathbb{Z} \to 0.$$

The Lefschetz fixed-point theorem, see below, implies that G acts trivially on $H_{2k-1}(X, \mathbb{Z})$. Therefore, we may concatenate copies of the above sequence to obtain $\cdots \to C_1(X) \to C_0(X) \xrightarrow{\iota\varepsilon} C_{2k-1}(X) \to \cdots \to C_1(X) \to C_0(X) \xrightarrow{\varepsilon} \mathbb{Z} \to 0.$ $\square$

Example 3.11. As an example we rediscover the resolution used in 2.7 to compute the (co)homology of a cyclic group. Namely, let $G = \langle g \mid g^n = 1 \rangle$. Then G acts freely on $S^1 \subset \mathbb{C}$, regarded as a CW-complex X with the following cells:

$$e^0_k := \exp(2\pi i k/n) \quad , \quad e^1_k := \{\exp(i\varphi) \mid \varphi \in (2\pi k/n, 2\pi(k+1)/n)\}$$

for $k \in \{0, \ldots, n-1\}$; the action being rotation, i.e. $ge^i_k := e^i_{k+1 \bmod n}$ for $i \in \{0, 1\}$.



As before, let $S = \sum_{g \in G} g \in \mathbb{Z}[G]$. Then, $\ker(\partial_1 : C_1(X) \to C_0(X)) = \langle Se^1_0 \rangle$. With the help of proposition 3.2 we thus obtain the 2-periodic resolution

$$\cdots \to \mathbb{Z}[G] \xrightarrow{g-1} \mathbb{Z}[G] \xrightarrow{S} \mathbb{Z}[G] \xrightarrow{g-1} \mathbb{Z}[G] \xrightarrow{\varepsilon} \mathbb{Z} \to 0.$$

Theorem 3.12 (Lefschetz). Let X be a compact CW-complex and let $f : X \to X$ be a continuous cellular map. If $\sum_{k=0}^{\infty} \operatorname{tr} f_{k*} \neq 0$, where $f_{k*} : H_k(X, \mathbb{Z}) \to H_k(X, \mathbb{Z})$ is induced by f, then f has a fixed point.

Corollary 3.13. Let X be a free G-CW-complex that is homeomorphic to an odd-dimensional sphere S^{2k-1}. Then G acts trivially on $H_{2k-1}(G, \mathbb{Z})$.

Proof. Recall that $H_k(S^n, \mathbb{Z}) = \mathbb{Z}$ for $k \in \{0, n\}$ and $H_k(S^n, \mathbb{Z}) = 0$ otherwise for $n \geq 1$. Now apply theorem 3.12 to the maps $g : X \to X$ for $g \in G$. □

The Simplicial Complex of a Group. Yet another example of how topological ideas help constructing resolutions via proposition 3.2 is the following intrinsic one which contains the description of (co)homology groups that occurred in section 1.

To every group G a simplicial complex $K(G)$ is associated, cf. Brown [4, I.5].

Definition 3.14. For a group G, the simplicial complex $K(G)$ with vertices G and simplices the finite subsets of G is called the simplicial complex associated to G.

The action of G on itself by left translation induces an action of G on $K(G)$ by simplicial maps. To ensure that G acts freely on the homogenous components of the corresponding simplicial chain complex we consider ordered tuples.

Definition 3.15. Let G be a group with associated simplicial complex $K(G)$. We define the ordered simplicial chain complex of $K(G)$ by letting $C_n(K(G))$ be the free abelian group on the ordered $(n+1)$-tuples $(g_0, g_1, \dots, g_n)$, $g_i \in G \ \forall i \in \{1, \dots, n\}$ and the boundary operators $\partial_n : C_n(K(G)) \to C_{n-1}(K(G))$,

$$\partial_n = \sum\nolimits_{k=0}^{n} (-1)^k d_{k,n} \quad \text{where} \quad d_{k,n}(g_0, \dots, g_n) = (g_0, \dots, \widehat{g_i}, \dots, g_n).$$

By Proposition 3.2, the augmented complex $C(K(G)) \xrightarrow{\varepsilon} \mathbb{Z}$ is a free resolution of the trivial left $\mathbb{Z}[G]$-module $\mathbb{Z}$. To make this explicit we choose the tuples with first component 1 as $\mathbb{Z}[G]$-bases; it is convient to put them in the form:

$$(1, g_1, g_1g_2, \dots, g_1g_2\cdots g_n) =: [g_1|g_2|\cdots|g_n].$$

The following formulae for $d_{k,n}$ are then readily checked:

$$d_k[g_1, \dots, g_n] = \begin{cases} g_1[g_2|\dots|g_n] & k = 0 \\ [g_1|\cdots|g_{k-1}|g_kg_{k+1}|g_{k+2}|\cdots|g_n] & 0 < k < n \\ [g_1|\cdots|g_{n-1}] & k = n \end{cases}.$$

Remark 3.16. The complex $C(K(G))$ may also be arrived at as the cellular chain complex of a free contracible G-CW-complex, see Brown [4, I.5 Ex. 3]. It may as well be treated purely algebraically, see Evens [8, Sec. 2.3].

Now, given a G-module M, applying the functor $\hom_G(-, M)$ to $C(K(G))$ yields a complex which encodes the cohomology of G with coefficients in M:

$$0 \to \hom_G(C_0, M) \xrightarrow{\partial_1^*} \hom_G(C_1, M) \xrightarrow{\partial_2^*} \hom_G(C_2, M) \to \cdots.$$

It rarely allows itself for computations but provides an easy way to identify low-dimensional cohomology groups. As a matter of fact, the reader may convince himself that the groups $Z^k(G, M) := \ker \partial_{k+1}^*$ and $B^k(G, M) := \operatorname{im} \partial_k^*$ are essentially the groups that occurred in section 1. For instance, we have

$$(\partial_2^* f)[g_1|g_2] = g_1 f[g_2] - f[g_1g_2] + f[g_1] \quad \text{for} \quad f \in \hom_G(C_1, M).$$

Remark 3.17. One might ask whether (co)homology groups of a given group vanish in sufficiently high dimensions. Whereas this is not at all clear from the above resolution, the topological description of proposition 3.8 might help: if G admits a finite-dimensional $K(G, 1)$-complex then proposition 3.8 shows that the (co)homology of G may be computed from a resolution of finite length; see Brown [4, VIII] for more.

3.2. **The Baer Sum.** In section 1 we have established a bijection between isomorphism classes $E(N, G, \theta)$ of extensions of N by G realizing θ, if existent, and elements of $H^2(G, Z(N))$. This allows us to transfer the group structure from the latter to the former. Whereas the bijection is not canonical in the general case, see theorem 1.24, it is when $N = A$ is abelian. The corresponding group-theoretic construction in $E(A, G, \varphi)$ was first discussed by Baer [1]; the author follows Beyl and Tappe [3, I] who base their work on Mac Lane [17, III.2, IV.4 Ex.].

Proposition 3.18. Let a group G, a G-module A and an extension E of A by G be given. For a further group G' and a G-module A' as well as homomorphisms $\alpha : A \to A'$ and $\gamma : G' \to G$, there are extensions $E\gamma$ and αE of A by G' and A' by G, universally fitting into

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, equal] & E \arrow[r] & G \arrow[r] & 1 \\ 0 \arrow[r] & A \arrow[r] & E\gamma \arrow[r] \arrow[u] & G' \arrow[r] \arrow[u, "\gamma"'] & 1 \end{tikzcd} \quad , \quad \begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, "\alpha"'] & E \arrow[r] \arrow[d] & G \arrow[r] \arrow[d, equal] & 1 \\ 0 \arrow[r] & A' \arrow[r] & \alpha E \arrow[r] & G \arrow[r] & 1. \end{tikzcd}$$

If E corresponds to $[f] \in H^2(G, A)$, where $f : G \times G \to A$, then $E\gamma$ corresponds to $[f \circ (\gamma \times \gamma)] \in H^2(G', A)$ and αE corresponds to $[\alpha \circ f] \in H^2(G, A')$.

Sketch of Proof. Use the following explicit constructions for $E\gamma$ and αE:

$$E\gamma := \{(e, g') \subset E \times G' \mid \pi(e) = \gamma(g')\} \quad , \quad \alpha E := (A' \rtimes E)/\{(\alpha(a)^{-1}, a) \mid a \in A\}.$$

The induced group structure of $H^2(G, A)$ on $E(G, A)$ is now given as follows.

Definition 3.19. Let G be a group and A a G-module. The Baer sum $[E] + [E']$ of extensions $[E], [E'] \in E(G, A, \varphi)$ is defined as $[\nabla((E \times E')\Delta)]$,

$$\begin{tikzcd} 0 \arrow[r] & A \oplus A \arrow[r] \arrow[d, equal] & E \times E' \arrow[r] & G \times G \arrow[r] & 1 \\ 0 \arrow[r] & A \oplus A \arrow[r] \arrow[d, "\nabla"'] & (E \times E')\Delta \arrow[r] \arrow[u] \arrow[d] & G \arrow[r] \arrow[u, "\Delta"'] \arrow[d, equal] & 1 \\ 0 \arrow[r] & A \arrow[r] & \nabla((E \times E')\Delta) \arrow[r] & G \arrow[r] & 1, \end{tikzcd}$$

where $\Delta : g \mapsto (g, g)$ for all $g \in G$ and $\nabla : (a, b) \mapsto a + b$ for all $(a, b) \in A \oplus A$.

With the help of proposition 3.18 one proves the subsequent proposition which gives $E(G, A, \varphi)$ the desired group structure with neutral element the semidirect product and the inverse of $[0 \to A \xrightarrow{\iota} E \to G \to 1]$ being $[0 \to A \xrightarrow{-\iota} E \to G \to 1]$.

Proposition 3.20. Let G be a group and A a G-module. If $[E], [E'] \in E(G, A, \varphi)$ correspond to $[f], [f'] \in H^2(G, A)$ then $[E]+[E']$ corresponds to $[f+f'] \in H^2(G, A)$.

Example 3.21. Let us consider central, cf. definition 3.24, i.e. untwisted extensions of $\mathbb{Z}$ by the fundamental group $G := \pi_1(\Sigma_g)$ of a genus g surface Σ_g, $g \geq 2$,

$$\pi_1(\Sigma_g) = \left\langle a_1, b_1, \ldots, a_g, b_g \,\middle|\, \prod\nolimits_{k=1}^{g} [a_k, b_k] = 1 \right\rangle .$$

As a matter of fact, Σ_g is a $K(G, 1)$-complex, see Brown [4, II.4]. Therefore, $H^2(G, \mathbb{Z}) = H^2(\Sigma_g, \mathbb{Z}) = \mathbb{Z}$. In particular, $E(\mathbb{Z}, G, \mathrm{id})$ is, as a group with the Baer sum, generated by a single extension. Now, it is a non-trivial fact that $E(\mathbb{Z}, G, \mathrm{id})$ is exhausted by the homotopy exact sequences $0 \to \pi_1(S^1) \to \pi_1(M_k) \to \pi_1(\Sigma_g) \to 1$ of certain S^1-bundles M_k, $k \in \mathbb{Z}$, over Σ_g, see Orlik et al. [21], [22]. Here,

$$\pi_1(M_k) = \left\langle a_1, b_1, \ldots, a_g, b_g, z \,\middle|\, [a_i, z] = [b_i, z] = 1 \ \forall i, \prod\nolimits_{i=1}^{k} [a_i, b_i] = z^k \right\rangle =: E_k.$$

However, the groups E_k may be identified as central extensions of $\mathbb{Z}$ by G by more explicit means: note that $\langle z\rangle$ is normal in E_k and that $E_k/\langle z\rangle \cong G$. It remains to be proven that $z \in E_k$ has infinite order. This holds for $E_0 \cong \mathbb{Z}\times G$. For the general case, let $D_i := \langle a_i, b_i, z \mid [a_i, z] = [b_i, z] = 1\rangle$ for $i \in \{1,\ldots,g\}$. Then $E_k \cong D_1 *_{\langle z\rangle} \cdots *_{\langle z\rangle} D_{g-1} *_H D_g$ where $H = \langle z, \prod_{i=1}^{g-1}[a_i, b_i]\rangle$ in $D_1 *_{\langle z\rangle} \cdots *_{\langle z\rangle} D_{g-1}$ and $H = \langle z, ([a_g, b_g]z^{-k})^{-1}\rangle$ in D_g are free abelian subgroups of rank two. The fact that z is of infinite order is therefore inherited from the factors.

3.3. **Hopf's Theorem.** Let X be a path-connected topological space whose higher homotopy groups vanish, i.e. $\pi_k(X) = 0$ for $k \geq 2$. Hurewicz' theorem 2.1 suggests that there are descriptions of its (co)homology groups in terms of $\pi_1(X)$. In view of Proposition 3.8 we would obtain descriptions of the (co)homology of $\pi_1(X)$. For instance, it is a classical result that

$$H_1(\pi_1(X), \mathbb{Z}) = H_1(X, \mathbb{Z}) = \pi_1(X)_{\mathrm{ab}}$$

where $\pi_1(X)_{\mathrm{ab}}$ denotes the abelianization of $\pi_1(X)$; the reader is invited to check that this formula indeed holds in our so far computations, e.g. in proposition 2.7.

In this section we recover a result of Hopf, see [12, Sec. 2], which gives a description of $H_2(\pi_1(X), \mathbb{Z}) = H_2(X, \mathbb{Z})$, by employing the methods developed in section 3.1, cf. Brown [4, II.5].

Theorem 3.22 (Hopf). If $G = F/R$ where F is free, then $H_2(G, \mathbb{Z}) \cong R\cap[F, F]/[F, R]$.

Proof. Let $F = F(S)$ and $X = \bigvee_{s\in S} S^1_s$. Furthermore let $p : E \to X$ be the cover which corresponds to $R \trianglelefteq F$. In particular, $\pi_1(E) \cong R$ and E is being acted on freely by $F/R = G$. Therefore, the augmented cellular chain complex of E provides a partial resolution of $\mathbb{Z}$ over $\mathbb{Z}[G]$, $C_1(E) \to C_0(E) \to \mathbb{Z} \to 0$ and homological algebra implies $H_2(G, \mathbb{Z}) \cong \ker(H_1(E, \mathbb{Z})_G \to H_1(X, \mathbb{Z}))$. To obtain a more explicit description, note that $H_1(E, \mathbb{Z}) \cong \pi_1(E)_{\mathrm{ab}} \cong R_{\mathrm{ab}}$. By checking the corresponding definitions, it can be seen that the composite isomorphism $H_1(E, \mathbb{Z}) \cong R_{\mathrm{ab}}$ is an isomorphism of G-modules if G acts on R_{ab} by the conjugation induced by F and that we may compute $H_2(G, \mathbb{Z})$ from the right column, induced by $R \hookrightarrow F$, of

$$\begin{tikzcd}
H_1(E,\mathbb{Z})_G \arrow[r, "\cong"] \arrow[d] & (R_{\mathrm{ab}})_G \arrow[r, equal] \arrow[d] & R/[F,R] \arrow[d] \\
H_1(X,\mathbb{Z}) \arrow[r, "\cong"] & F_{\mathrm{ab}} \arrow[r, equal] & F/[F,F].
\end{tikzcd}$$

Therefore, $H_2(G, \mathbb{Z}) \cong \ker(R/[F, R] \to F/[F, F]) = R \cap [F, F]/[F, R]$. ☐

Remark 3.23. There is a purely algebraic proof of Hopf's Theorem which utilizes the five-term sequence of a spectral sequence, see 4.14 and Weibel [28, Sec. 6.8].

3.4. **Central Extensions.** It turns out that $H_2(G, \mathbb{Z})$ is related to group extensions as well, namely to central extensions. The discussion of these sheds some more light on Hopf's formula for $H_2(G, \mathbb{Z})$. Refer to Weibel [28, Sec. 6.9] and Brown [4, IV.3 Ex. 7] for the material of this section.

Assume that G is a perfect group, i.e. $G = [G, G]$ or, equivalently, $H_1(G) = 0$. Then theorem 4.16 implies an isomorphism $H^2(G, A) \cong \hom(H_2(G, \mathbb{Z}), A)$ for every abelian group A, suggesting that $H_2(G, \mathbb{Z})$ is related to extensions of A by G by fitting into a diagram

$$\begin{tikzcd}
 & H_2(G,\mathbb{Z}) \arrow[d] & & & \\
0 \arrow[r] & A \arrow[r] & E \arrow[r] & G \arrow[r] & 1.
\end{tikzcd}$$

We elaborate this observation in the subsequent discussion.

Definition 3.24 (Central extension). A central extension of an abelian group A by a group G is an exact sequence $0 \to A \xrightarrow{\iota} E \xrightarrow{\pi} G \to 1$ such that $\operatorname{im}\iota$ is contained in the center of E. A morphism of extensions $0 \to A \to E \to G \to 1$ and $0 \to A \to E \to G \to 1$ by G is a homomorphism $E \to E'$ such that

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & E & \longrightarrow & G & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & A' & \longrightarrow & E' & \longrightarrow & G & \longrightarrow & 1. \end{array}$$

commutes. A central extension by G is called universal if it admits a unique morphism to any other central extension by G.

As an immediate consequence, universal central extensions are unique up to isomorphism if they exist; and as a first necessary criterion for the existence of a universal central extension by G we obtain the following.

Lemma 3.25. Let G be a group. If $0 \to A \to E \to G \to 1$ is a universal extension by G then both G and E are perfect.

Proof. If E is perfect then G is perfect as a quotient of a perfect group. Otherwise, $E^{(1)} = E/[E,E]$ is non-zero abelian and $0 \to E^{(1)} \to E^{(1)} \times G \to G \to 1$ is a central extension. The maps $(0,\pi), (\pi,\pi) : E \to E^{(1)} \times G$ are then two distinct morphisms, in contradiction to E being a universal central extension by G. □

The above remarks suggest that $H_2(G,\mathbb{Z})$ might be part of a universal central extension. In fact we have the following theorem which one might guess, given Hopf's formula for $H_2(G,\mathbb{Z})$.

Theorem 3.26. Let G be a group. There is a universal central extension by G if and only if G is perfect; in which case the universal central extension is given by

$$0 \to H_2(G,\mathbb{Z}) \to [F,F]/[F,R] \xrightarrow{\pi} G \to 1$$

where $G = F/R$ is a presentation of G.

Proof. If there is a universal central extension by G then G is perfect by lemma 3.25. Conversely, assume that G is perfect. Check that

$$0 \to H_2(G,\mathbb{Z}) \to [F,F]/[F,R] \to G \to 1$$

is indeed a central extension and that $[F,F]/[F,R]$ is perfect. Next, let

$$0 \to A \to E \xrightarrow{\rho} G \to 1$$

be an arbitrary central extension by G. Then the map $\pi : F \to G$ lifts to a map $h : F \to E$ as F is free. And since $\rho \circ h(R) = 1$ we deduce $h(R) \leq A \leq Z(E)$ whence $h([F,R]) = 1$. Therefore, we obtain a map $f : [F,F]/[F,R] \hookrightarrow F/[F,R] \to E$ which is a morphism. If f' is another such morphism then f and f' differ by a homomorphism $[F,F]/[F,R] \to A$ which is necessarily trivial since the former group is perfect and the latter is abelian. □

Example 3.27. The smallest perfect group is A_5. Its universal central extension uses the isomorphism $A_5 \cong \mathrm{PSL}_2(\mathbb{F}_5)$, cf. Suzuki [27],

$$0 \to \{\pm I\} \to \mathrm{SL}_2(\mathbb{F}_5) \to \mathrm{PSL}_2(\mathbb{F}_5) \to 1.$$

Actually computing $H_2(G,\mathbb{Z})$ from Hopf's Theorem is not easy. As a matter of fact one often computes $H_2(G)$ by constructing a universal central extension, cf. Brown [4, IV.3 Ex. 7].

4. Functorial Properties of Group (Co)homology

Given a group G, we have introduced the (co)homology functors

$$H_k(G,-), H^k(G,-) : G\text{-}\mathbf{Mod} \to \mathbf{Ab}.$$

In this section, we include functoriality with respect to the group in a sense which is made precise below. Whereas in section 4.1 we explore how these functors behave on subgroups and quotients, section 4.2 deals with products and coproducts. See Brown [4, III.8, III.10] and Stammbach [10, VI.2].

Definition 4.1. Let $\mathbf{Grp}{\downarrow\downarrow}\mathbf{Mod}$ have objects the pairs (G, M) where G is a group and M is a G-module and morphisms $(\varphi, f) : (G, M) \to (G', M')$ where

$$\varphi \in \hom_{\mathbf{Grp}}(G, G'),\ f \in \hom_{\mathbf{Ab}}(M, M'),\ f(gm) = \varphi(g)f(m)\ \forall g \in G\ \forall m \in M.$$

Similarly, let $\mathbf{Grp}{\downarrow\uparrow}\mathbf{Mod}$ have objects the pairs (G, M) where G is a group and M is a G-module but morphisms $(\varphi, f) : (G, M) \to (G', M')$ where

$$\varphi \in \hom_{\mathbf{Grp}}(G, G'),\ f \in \hom_{\mathbf{Ab}}(M', M),\ f(\varphi(g)m') = gf(m')\ \forall g \in G\ \forall m' \in M'.$$

As a mnemonic device, note that we have the following diagrams:

$$\begin{tikzcd} G \arrow[r, bend left, hook] \arrow[d, "\varphi"'] & M \arrow[d, "f"] \\ G' \arrow[r, bend left, hook] & M' \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, bend left, hook] \arrow[d, "\varphi"'] & M \\ G' \arrow[r, bend left, hook] & M'. \arrow[u, "f"'] \end{tikzcd}$$

The group (co)homology functors now become (covariant) functors

$$H_k : \mathbf{Grp}{\downarrow\downarrow}\mathbf{Mod} \to \mathbf{Ab} \quad \text{and} \quad H^k : \mathbf{Grp}{\downarrow\uparrow}\mathbf{Mod}^{\mathrm{op}} \to \mathbf{Ab}.$$

A pair (G, M) is assigned the (co)homology of G with coefficients in M and a morphism $(\varphi, f) : (G, M) \to (G', M')$ in, say, $\mathbf{Grp}{\downarrow\uparrow}\mathbf{Mod}$ is assigned a morphism of abelian groups $(\varphi, f)^* : H^k(G', M') \to H^k(G, M)$ as follows: Choose projective resolutions $P \to \mathbb{Z}$ and $P' \to \mathbb{Z}$ of $\mathbb{Z}$ over $\mathbb{Z}[G]$ and $\mathbb{Z}[G']$. In regarding P' as a complex of G-modules via φ, homological algebra provides an augmentation-preserving chain map of $\mathbb{Z}[G]$-modules τ,

$$\begin{tikzcd} \cdots \arrow[r] & P_2 \arrow[r] \arrow[d, "\tau_2"] & P_1 \arrow[r] \arrow[d, "\tau_1"] & P_0 \arrow[r] \arrow[d, "\tau_0"] & 0 \\ \cdots \arrow[r] & P_2' \arrow[r] & P_1' \arrow[r] & P_0' \arrow[r] & 0. \end{tikzcd}$$

This in turn yields a chain map $\tau^* = \hom_G(\tau, f)$,

$$\begin{tikzcd} 0 \arrow[r] & \hom_G(P_0, M) \arrow[r] & \hom_G(P_1, M) \arrow[r] & \hom_G(P_2, M) \arrow[r] & \cdots \\ 0 \arrow[r] & \hom_{G'}(P_0', M') \arrow[r] \arrow[u, "\tau_0^*"] & \hom_{G'}(P_1', M') \arrow[r] \arrow[u, "\tau_1^*"] & \hom_{G'}(P_2', M') \arrow[r] \arrow[u, "\tau_2^*"] & \cdots, \end{tikzcd}$$

which induces $(\varphi, f)^*$ in cohomology. Similarly, a map (φ, f) in $\mathbf{Grp}{\downarrow\downarrow}\mathbf{Mod}$ induces a map between homology groups. Namely, τ yields the chain map $\tau_* = \tau \otimes_G f$

$$\begin{tikzcd} \cdots \arrow[r] & P_2 \otimes_G M \arrow[r] \arrow[d, "\tau_{2*}"] & P_1 \otimes_G M \arrow[r] \arrow[d, "\tau_{1*}"] & P_0 \otimes_G M \arrow[r] \arrow[d, "\tau_{0*}"] & 0 \\ \cdots \arrow[r] & P_2' \otimes_{G'} M \arrow[r] & P_1' \otimes_{G'} M \arrow[r] & P_0' \otimes_{G'} M \arrow[r] & 0. \end{tikzcd}$$

which induces $(\varphi, f)_*$ in homology.

4.1. **Subgroups and Quotients.** Subsequently we investigate how the cohomology of a subgroup and a quotient relate to the cohomology of the group.

(Co)induction of Modules. Let H be a subgroup of a given group G. A result due to Eckmann [5] states that the (co)homology of H with coefficents in an H-module N is isomorphic to the (co)homology of G with coefficients in a certain G-module. To be precise, consider, as in representation theory, the following two canonical ways of turning an H-module into a G-module.

Definition 4.2 ((Co)induction). For groups $H \leq G$ define

$$\mathrm{ind}_H^G(-) : H\text{−}\mathbf{Mod} \to G\text{−}\mathbf{Mod},\ N \mapsto \mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} N$$

where $\mathbb{Z}[G]$ is regarded as a $(\mathbb{Z}[G], \mathbb{Z}[H])$-bimodule; and

$$\mathrm{coind}_H^G(-) : H\text{−}\mathbf{Mod} \to G\text{−}\mathbf{Mod},\ N \mapsto \hom_{\mathbb{Z}[H]}(\mathbb{Z}[G], N)$$

where $\mathbb{Z}[G]$ is regarded as a $(\mathbb{Z}[H], \mathbb{Z}[G])$-bimodule.

Induced and coinduced modules are not necessarily isomorphic. They are, however, if H is of finite index in G.

Proposition 4.3. Let groups $H \leq G$ satisfying $[G : H] < \infty$ be given. Then there is an isomorphism $\mathrm{ind}_H^G(N) \cong \mathrm{coind}_H^G(N)$ for all H-modules N.

Proof. Choose a set $S \subset G$ of left-coset representatives of H in G. Then the map

$$\mathrm{coind}_H^G N \to \mathrm{ind}_H^G N,\ f \mapsto \sum\nolimits_{g \in S} g \otimes_{\mathbb{Z}[H]} f(g^{-1})$$

is an isomorphism of G-modules. □

Eckmann's result may now be stated as follows.

Proposition 4.4 (Eckmann). Let groups $H \leq G$ and an H-module N be given. Then

$$H_k(H, N) \cong H_k(G, \mathrm{ind}_H^G N) \quad \text{and} \quad H^k(H, N) \cong H^k(G, \mathrm{coind}_H^G N)$$

for every k.

Proof. The assertion for homology follows from the facts that a $\mathbb{Z}[G]$-projective resolution P of $\mathbb{Z}$ is also a $\mathbb{Z}[H]$-projective resolution of $\mathbb{Z}$ and that $P \otimes_{\mathbb{Z}[H]} N \cong P \otimes_{\mathbb{Z}[G]} \mathrm{ind}_H^G N$. Analogous facts hold for cohomology. □

(Co)restriction, (Co)transfer, (Co)inflation. We now investigate several maps in (co)homology which are related to inclusions $H \hookrightarrow G$ and projections $G \twoheadrightarrow G/N$. In relating these to each other, Eckmann's result will imply a particularly useful statement about (co)homology groups. For a start, there are obvious maps induced by inclusion of a subgroup in $\mathbf{Grp}\downarrow\downarrow\mathbf{Mod}$ and $\mathbf{Grp}\downarrow\uparrow\mathbf{Mod}$ respectively.

Definition 4.5 ((Co)restriction). Let $H \leq G$ and a G-module M be given. In regarding M as an H-module by applying the corresponding (exact) forgetful functor define for all k

$$\mathrm{res}_H^G := (\iota, \mathrm{id})^* : H^k(G, M) \to H^k(H, M)$$

and analogously

$$\mathrm{cores}_H^G := (\iota, \mathrm{id})_* : H_k(G, M) \to H_k(H, M).$$

Now, for groups $H \leq G$ and a G-module M, regarded as an H-module where necessary, there are canonical maps

$$\mathrm{ind}_H^G M = \mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} M \twoheadrightarrow M \quad \text{and} \quad M \hookrightarrow \hom_{\mathbb{Z}[H]}(\mathbb{Z}[G], M) = \mathrm{coind}_H^G M.$$

Applying $H_k(G, -)$ to the former yields $\mathrm{cores}_H^G : H_k(H, M) \to H_k(G, M)$; applying $H^k(G, -)$ to the latter yields $\mathrm{res}_H^G : H^k(G, M) \to H^k(H, M)$; and if $[G : H] < \infty$, i.e. $\mathrm{coind}_H^G M \cong \mathrm{ind}_H^G M$, we obtain maps in the other direction.

Definition 4.6 ((Co)transfer). Let $H \leq G$ and a G-module M, regarded as an H-module where necessary, be given. Define for all k

$$\mathrm{tr}_H^G := H^k(H, M) \cong H^k(G, \mathrm{coind}_H^G M) \xrightarrow{(\mathrm{coind}_H^G M \cong \mathrm{ind}_H^G M \twoheadrightarrow M)_*} H^k(G, M)$$

and, analogously,

$$\mathrm{cotr}_H^G := H_k(G, M) \xrightarrow{(M \hookrightarrow \mathrm{coind}_H^G \cong \mathrm{ind}_H^G M)_*} H_k(G, \mathrm{ind}_H^G M) \cong H_k(H, M).$$

Remark 4.7. The required naturality of Eckmann's isomorphism for definition 4.6 is proved in Stammbach [10, IV.12.3].

Remark 4.8. There are several other descriptions of the (co)transfer map, including the following topological one: In view of proposition 3.8, let X be $K(G,1)$-space. As a matter of fact, the covering space Y of X corresponding to $H \leq G$ is a $K(H,1)$-space. Therefore the (co)homologies of H and G may be computed from the cellular chain complexes of X and Y. The transfer map then is induced by $f \mapsto (\sigma \mapsto \sum_{\tau \in F_\sigma} f(\tau))$ where σ is an oriented cell of X and $\tau \in F_\sigma$ denotes a cell in the fiber of σ. Similarly, the cotransfer map is induced by $\sigma \otimes m \mapsto \sum_{\tau \in F_\sigma} \tau \otimes m$.

The concatenations $\mathrm{tr}_H^G \circ \mathrm{res}_H^G$ and $\mathrm{cotr}_H^G \circ \mathrm{cores}_H^G$ have a useful property following from definition 4.6 and the proof of proposition 4.3. As we are mostly interested in cohomology, we state the result for this case only.

Proposition 4.9. Let $H \leq G$ such that $[G : H] < \infty$. Then for every G-module M and $k \geq 1$, $\mathrm{tr}_H^G \circ \mathrm{res}_H^G : H^k(G, M) \to H^k(G, M)$ is multiplication by $[G : H]$.

As a corollary we obtain the following.

Proposition 4.10. If G is a finite group and M is a G-module, then $|G|H^k(G, M) = 0$ for $k \geq 1$. If multiplication by $|G|$ is an isomorphism of M, then $H^k(G, M) = 0$.

Proof. Choose $H = 1$ in proposition 4.9 and note that $H^k(H, M) = 0$ for $k \geq 1$. For the second assertion, note that multiplication by $|G|$ is an isomorphism of $H^k(G, M)$ if and only if it is an isomorphism of M. □

Example 4.11. We return to the discussion begun in example 2.8: let $N = \mathbb{Z}_p$ and $G = \mathbb{Z}_q$ where p and q are distinct primes. Since multiplication by $|\,\mathbb{Z}_q\,| = q$ is an isomorphism of $\mathbb{Z}_p$, proposition 4.10 implies that $H^2(G, N) = 0$. Therefore, every extension of N by G is isomorphic to a semidirect product $N \rtimes G$.

Corollary 4.12. Let G be a finite group. If M is finitely generated as an abelian group then $H^k(G, M)$ is finite for $k \geq 1$.

Proof. This is a consequence of proposition 4.10 if we keep in mind that any finite group admits a resolution of finitely generated modules, see definition 3.15. □

For the sake of completeness we mention that the (co)homology of a group G is determined by the (co)homologies of a normal subgroup N and the quotient group G/N in the following sense, see e.g. Weibel [28, 6.8].

Definition 4.13 (Inflation). Let $N \trianglelefteq G$ and a G-module M be given. Then M^N is a G/N-module. Define

$$\mathrm{inf}_{G/N}^G := (\pi, \iota)^* : H^k(G/N, M^N) \to H^k(G, M).$$

Theorem 4.14 (Lyndon-Hochschild-Serre). Let $N \trianglelefteq G$ and a G-module M be given and let $Q = G/N$. There is a first-quadrant spectral sequence

$$E_{pq}^2 = H^p(Q, H^q(N, M)) \Rightarrow H^{p+q}(G, M).$$

The related five-term sequence is an inflation restriction sequence:

$$0 \to H^1(Q, M^N) \to H^1(G, M) \to H^1(N, M)^Q \to H^2(Q, M^N) \to H^2(G, M).$$

4.2. **Products and Coproducts.** Beyond subgroups and quotients we turn to the question whether the (co)homology of a (co)product of groups may be computed from the (co)homologies of its factors. For the material of this section refer to Hilton and Stammbach [10, VI.14, 15].

Products. As a frist step in the case of products, the Universal Coefficient Theorem, which we recall presently, reduces questions about the (co)homology of a group G with trivial coefficients M to questions about its integral homology, i.e. homology with trivial coefficients $\mathbb{Z}$, denoted $H_k(G)$.

Theorem 4.15 (Universal Coefficient Theorem)*.* Let R be a principal ideal domain, F a chain complex of free R-modules, and M an R-module. Then for $n \geq 0$ there are sequences

$$0 \to H_n(F) \otimes_R M \to H_n(F \otimes_R M) \to \mathrm{Tor}_1^R(H_{n-1}(F), M) \to 0,$$
$$0 \to \mathrm{Ext}_R^1(H_{n-1}(F), M) \to H^n(\hom_R(F, M)) \to \hom_R(H_n(F), M) \to 0$$

which are exact and split (non-naturally).

The application of theorem 4.15 to group cohomology reads as follows.

Theorem 4.16*.* Let G be a group and M a trivial G-module. Then for $n \geq 0$ there are sequences

$$0 \to H_n(G) \otimes_{\mathbb{Z}} M \to H_n(G, M) \to \mathrm{Tor}_1^{\mathbb{Z}}(H_{n-1}(G), M) \to 0,$$
$$0 \to \mathrm{Ext}_{\mathbb{Z}}^1(H_{n-1}(G), M) \to H^n(G, M) \to \hom_{\mathbb{Z}}(H_n(G), M) \to 0$$

which are exact and split (non-naturally).

Proof. Let P be a $\mathbb{Z}[G]$-projective resolution of $\mathbb{Z}$ and let $F = P_G = P \otimes_{\mathbb{Z}[G]} \mathbb{Z}$. Then F is a complex of free abelian groups and $P \otimes_{\mathbb{Z}[G]} M \cong F \otimes_{\mathbb{Z}} M$as well as $\hom_{\mathbb{Z}[G]}(P, M) \cong \hom_{\mathbb{Z}}(F, M)$. Therefore, theorem 4.15 implies the assertion. □

Remark 4.17*.* The Ext and Tor groups which occur in theorem 4.16 may be viewed as obstructions to the other two groups being isomorphic; for if the former vanish the latter are isomorphic by exactness.

Next, a Künneth Theorem from homological algebra facilitates computing the (co)homology of a product of groups.

Theorem 4.18 (Künneth)*.* Let R be a principal ideal domain, F a chain complex of free R-modules and C an arbitrary chain complex of R-modules. Then for $n \geq 0$ there is a sequence

$$0 \to \bigoplus_{k+l=n} H_k(F) \otimes_R H_l(C) \to H_n(F \otimes_R C) \to \bigoplus_{k+l=n-1} \mathrm{Tor}_1^R(H_k(F), H_l(C)) \to 0$$

which is exact and splits (non-naturally).

Theorem 4.19*.* Let $G = G_1 \times G_2$. Then for $n \geq 0$ there is a sequence

$$0 \to \bigoplus_{k+l=n} H_k(G_1) \otimes_{\mathbb{Z}} H_l(G_2) \to H_n(G) \to \bigoplus_{k+l=n-1} \mathrm{Tor}_1^{\mathbb{Z}}(H_k(G_1), H_l(G_2)) \to 0$$

which is exact and splits (non-naturally).

Proof. Let $P^{(i)}$ be a $\mathbb{Z}[G_i]$-projective resolution of $\mathbb{Z}$ for $i \in \{1, 2\}$. Then the complex $P^{(1)} \otimes P^{(2)}$ is naturally made into a complex of G-modules which is seen to be a $\mathbb{Z}[G]$-projective resolution of $\mathbb{Z}$. Furthermore,

$$H_n(G) = H_n((P^{(1)} \otimes P^{(2)}) \otimes_{\mathbb{Z}[G]} \mathbb{Z}) = H_n((P^{(1)} \otimes P^{(2)})_G) = H_n(P_{G_1}^{(1)} \otimes P_{G_2}^{(2)}).$$

And since the complexes $P_G^{(i)}$, $i \in \{1, 2\}$ are complexes of free abelian groups, theorem 4.18 yields the assertion. □

In principle, theorems 4.16 and 4.19 allow us to compute the (co)homology of any finite product of groups with trivial coefficients in terms of the (co)homologies of the factors.

Example 4.20. As an example, let us use theorems 4.16 and 4.19 to compute the integral (co)homology of $\mathbb{Z}_k \oplus \mathbb{Z}_l$ for intergers k and l. To this end, let $d = \gcd(k, l)$. If $d = 1$ then $\mathbb{Z}_k \oplus \mathbb{Z}_l \cong \mathbb{Z}_{kl}$ and proposition 2.7 yields the result; namely,

$$H_n(\mathbb{Z}_k) = \left\{ \begin{matrix} \mathbb{Z}_k & n \text{ odd} \\ 0 & n \text{ even} \end{matrix} \right\} = H^{n+1}(\mathbb{Z}_k)$$

in non-zero dimensions. Otherwise, we have for $n \geq 0$:

$$H_n(\mathbb{Z}_k \oplus \mathbb{Z}_l) \cong \left(\bigoplus_{i+j=n} H_i(\mathbb{Z}_k) \otimes_{\mathbb{Z}} H_j(\mathbb{Z}_l) \right) \oplus \left(\bigoplus_{i+j=n-1} \mathrm{Tor}_1^{\mathbb{Z}}(H_i(\mathbb{Z}_k), H_j(\mathbb{Z}_l)) \right).$$

The involved tensor products and Tor groups are readily computed: for every abelian group A,

$$\mathbb{Z} \otimes A \cong A \otimes \mathbb{Z} \cong A, \ \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}, A) = \mathrm{Tor}_1^{\mathbb{Z}}(A, \mathbb{Z}) = 0 = \mathrm{Tor}_1^{\mathbb{Z}}(0, A) = \mathrm{Tor}_1^{\mathbb{Z}}(A, 0).$$

Furthermore, $\mathbb{Z}_k \otimes \mathbb{Z}_l \cong \mathbb{Z}_d$ and $\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}_k, \mathbb{Z}_l) = {}_k\mathbb{Z}_l \cong \mathbb{Z}_d$. Therefore,

$$H_n(\mathbb{Z}_k \oplus \mathbb{Z}_l) \cong \left\{ \begin{matrix} \mathbb{Z}_k \oplus \mathbb{Z}_l \oplus \mathbb{Z}_d^{(n-1)/2} & k \text{ odd} \\ \mathbb{Z}_d^{(n/2)} & n \text{ even} \end{matrix} \right\} = H^{n+1}(\mathbb{Z}_k \oplus \mathbb{Z}_l)$$

in non-zero dimensions. For the cohomology case we used that $\hom_{\mathbb{Z}}(G, \mathbb{Z}) = 0$ for any finite group G; and $\mathrm{Ext}^1_{\mathbb{Z}}(\bigoplus_{\alpha \in A} G_\alpha, \mathbb{Z}) \cong \prod_{\alpha \in A} \mathrm{Ext}^1_{\mathbb{Z}}(G_\alpha, \mathbb{Z})$ as well as $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}_n, \mathbb{Z}) \cong \mathbb{Z}_n$, cf. Weibel [28, 3.3].

Coproducts. We shall only sketch the case of coproducts. Recall that coproducts of groups, or, more generally, amalgamated coproducts, i.e. pushouts, arise naturally in the Seifert-van Kampen theorem which computes the fundamental group of a topological space in terms of the fundamental groups of the elements of a cover, see May [19, Sec. 2.7]. Computing the (co)homology of a pushout features a common theme in (co)homology theories, the Mayer-Vietoris sequence.

Theorem 4.21. Given a pushout diagram in **Grp** and a G-module M,

$$\begin{tikzcd} H \arrow[r, tail, "\kappa_1"] \arrow[d, tail, "\kappa_2"'] \arrow[dr, phantom, "\text{po}"] & G_1 \arrow[d, tail, "\iota_1"] \\ G_2 \arrow[r, tail, "\iota_2"'] & G_1 *_H G_2, \end{tikzcd}$$

there is a long exact Mayer-Vietoris sequence

$$\begin{array}{l} 0 \longrightarrow H^0(G,M) \xrightarrow{\iota^*} H^0(G_1,M) \oplus H^0(G_2,M) \xrightarrow{\kappa^*} H^0(H,M) \longrightarrow \\ \longrightarrow H^1(G,M) \xrightarrow{\iota^*} H^1(G_1,M) \oplus H^1(G_2,M) \xrightarrow{\kappa^*} H^1(H,M) \longrightarrow \\ \cdots \\ \longrightarrow H^n(G,M) \xrightarrow{\iota^*} H^n(G_1,M) \oplus H^n(G_2,M) \xrightarrow{\kappa^*} H^n(H,M) \longrightarrow \end{array}$$

where $\iota^* = ((\iota_1, \mathrm{id})^*, (\iota_2, \mathrm{id})^*)$ and $\kappa^* = \langle (\kappa_1, \mathrm{id})^*, -(\kappa_2, \mathrm{id})^* \rangle$, see definition 4.1. An analogous result holds for homology.

An algebraic proof of the above theorem can be found in Stammbach [26, II.6]. However, there is, once more, a topological interpretation, see e.g. Brown [4, II.7] and Gelfand and Manin [9]: to the above pushout diagram in **Grp** corresponds a pushout diagram of $K(-,1)$-complexes in **Top**

$$\begin{tikzcd} X_1 \cap X_2 = Y \arrow[r, tail] \arrow[d, tail] \arrow[dr, phantom, "\mathrm{po}"] & X_1 \arrow[d, tail] \\ X_2 \arrow[r, tail] & X_1 \sqcup_Y X_2. \end{tikzcd}$$

The associated (co)homological Mayer-Vietoris sequence from algebraic topology yields the integral (co)homology case of theorem 4.21 via proposition 3.8.

Specializing theorem 4.21 to coproducts, i.e. the case $H = 1$, shows that, in a way, $H^k(-,M)$ preserves coproducts for $k \geq 2$.

Corollary 4.22. Let $G = G_1 * G_2$ and M a G-module. Then the map

$$((\iota_1, \mathrm{id})^*, (\iota_2, \mathrm{id})^*) : H^k(G,M) \to H^k(G_1,M) \oplus H^k(G_2,M)$$

is an isomorphism for $k \geq 2$.

Proof. If $H = 1$ then $H^k(H,M) = 0$ for $k \geq 2$ and the assertion is immediate from the exactness of the Mayer-Vietoris sequence. □

Remark 4.23. The assertion of corollary 4.22 may fail for $k \in \{0,1\}$, see Hilton and Stammbach [10, Sec. 6.14].

Example 4.24. Let $\mathrm{PSL}_2(\mathbb{Z}) \cong \langle s,t \mid s^2 = t^3 = 1\rangle \cong \mathbb{Z}_2 * \mathbb{Z}_3$ with generators

$$s = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad t = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}.$$

Applying the homology sequence, the results 2.7 and 4.16 as well as the formula $H_1(\mathrm{PSL}_2(\mathbb{Z})) \cong \mathrm{PSL}_2(\mathbb{Z})_{\mathrm{ab}} \cong \langle s,t \mid s^2 = t^3 = [s,t] = 1\rangle \cong \mathbb{Z}_2 \oplus \mathbb{Z}_3$ yields

$$H_k(\mathrm{PSL}_2(\mathbb{Z})) = \begin{Bmatrix} \mathbb{Z}_2 \oplus \mathbb{Z}_3 & k \text{ odd} \\ 0 & k \text{ even} \end{Bmatrix} = H^{k+1}(\mathrm{PSL}_2(\mathbb{Z}))$$

in non-zero dimensions. With generators as above we have $\mathrm{SL}_2(\mathbb{Z}) = \mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_6$ and the Mayer-Vietoris sequences encode, cf. [15, Sec 4.1],

$$H_k(\mathrm{SL}_2(\mathbb{Z})) = \begin{Bmatrix} \mathbb{Z}_{12} & k \text{ odd} \\ 0 & k \text{ even} \end{Bmatrix} = H^{k+1}(\mathrm{SL}_2(\mathbb{Z}))$$

in non-zero dimensions. Indeed, the homological sequence reads

$$\begin{array}{l} 0 \longleftarrow H_0(\mathrm{SL}_2(\mathbb{Z})) \xleftarrow{\iota_*} \mathbb{Z} \oplus \mathbb{Z} \xleftarrow{\kappa_*} \mathbb{Z} \longleftarrow \\ \longleftarrow H_1(\mathrm{SL}_2(\mathbb{Z})) \xleftarrow{\iota_*} \mathbb{Z}_4 \oplus \mathbb{Z}_6 \xleftarrow{\kappa_*} \mathbb{Z}_2 \longleftarrow \\ \longleftarrow H_2(\mathrm{SL}_2(\mathbb{Z})) \xleftarrow{\iota_*} 0 \xleftarrow{\kappa_*} 0 \longleftarrow \\ \longleftarrow H_3(\mathrm{SL}_2(\mathbb{Z})) \xleftarrow{\iota_*} \mathbb{Z}_4 \oplus \mathbb{Z}_6 \xleftarrow{\kappa_*} \mathbb{Z}_2 \longleftarrow \end{array}.$$

With $\kappa_* = ((\kappa_1, \mathrm{id})_*, -(\kappa_2, \mathrm{id})_*)$ being injective from the second row on we obtain $H_k(\mathrm{SL}_2(\mathbb{Z})) = 0$ if $k \geq 2$ is even and $H_k(\mathrm{SL}_2(\mathbb{Z})) = \mathbb{Z}_{12}$ if $k \geq 3$ is odd. Also,

$$H_1(\mathrm{SL}_2(\mathbb{Z})) \cong \mathrm{SL}_2(\mathbb{Z})_{\mathrm{ab}} \cong \langle s,t \mid s^4 = t^6 = [s,t] = 1,\ a^2 = b^3\rangle \cong \mathbb{Z}_{12}\,.$$

The assertion for cohomology then follows from proposition 4.16 as in example 4.20.

Ramifications of Group (Co)homology

(Co)homology theories are ubiquitous in mathematics. Group (co)homology in particular is a vast subject which ramifies into many different areas of mathematics. For instance, given that the definition of a G-module generalizes classical representation theory of groups it is not surprising that our subject exhibits connections to the latter, see e.g. Benson [2]. If the groups under consideration come with an additional structure such as Lie groups or Lie algebras we reach subjects which arose from analytical considerations, see [14]. Also, groups may have a particular interpretation such as being Galois groups, cf. [25]. The (co)homological study of these leads to (algebraic) number theory, see Neukrich et. al. [20].

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