T.d.l.c. groups from transcendental field extensions

Reminder: Let X be a set. The permutation topology on $\mathrm{Sym}(X)$ has basis $U_{x,y} := \{g \in \mathrm{Sym}(X) \mid gx_i = y_i \ \forall i \in \{1,\dots,n\}\}$ where $n \in \mathbb{N}$ and $x = (x_1,\dots,x_n)$, $y = (y_1,\dots,y_n) \in X^n$. Then $\mathrm{Sym}(X)$ is a totally disconnected, Hausdorff topological group.

the sets $U_{x,y}$ are also closed:
If $g \notin U_{x,y}$ then $gx_i \neq y_i$ for some $i \in \{1,\dots,n\}$ and $g \in U_{x_i, g(x_i)}$ which intersects $U_{x,y}$ trivially.

$g \neq h \in \mathrm{Sym}(X)$. Say $gx \neq hx$. Then $U_{x,g(x)}$ and $U_{x,h(x)}$ are disjoint open neighbourhoods

---

Let $K \subseteq E$ be a field extension. Then $\mathrm{Aut}_K(E) \leq \mathrm{Sym}(E)$ is closed, tot. disc. and Hausdorff.
When is it (t.s.) c.g. n.d. $\underbrace{\text{t.d.}}_{\checkmark}$ l.c. s.c. ?

Recall: $H \leq \mathrm{Sym}(X)$ is compact if and only if $H \leq \mathrm{Sym}(X)$ is closed and all its orbits are finite.

Prop. If $K \leq E$ is algebraic then $\mathrm{Aut}_K(E)$ is compact.

In general, any extension $K \subseteq E$ can be written in the form
$$K \subseteq K(M) \subseteq E$$
where M is a transcendence basis (so $K \leq K(M)$ is purely transcendental) and $K(M) \subseteq E$ is algebraic.

Prop. Let $K \leq E$ be a field extension. If $[E:K]_{tr} < \infty$ then $\mathrm{Aut}(E:K)$ is locally compact. Conversely, if $\mathrm{Aut}_K(E)$ is locally compact and, in addition, E is algebraically closed, then $\text{tr-deg}(E:K) < \infty$.

Proof If $[E:K]_{tr} < \infty$, let M be a finite transcendence basis...
Conversely, if $\mathrm{Aut}_K(E)$ is locally compact, then $\mathrm{Aut}_K(E)_S$ is compact for some finite $S \subseteq E$, and $\text{tr-deg}(E:K(S))$ is infinite if $\text{tr-deg}(E:K)$ is. Let $(X_i)_{i\in I}$ be a tr.-basis of $E:K(S)$. Look at automorphisms of $K(S)(X_i)_{i\in I}$. They extend to E. So $\mathrm{Aut}_K(E)_S$ has infinite orbits, contradiction.

So we can construct t.d.l.c. groups from extensions $K \subseteq E$ of finite transcendence degree. What about (non-)discreteness?

We have that $\mathrm{Aut}_K(E)$ is discrete if and only if $\mathrm{Aut}_K(E)_S$ is trivial for some finite set $S \subseteq E$.

<u>Ex.</u> Let K be field and $n \in \mathbb{N}$. The automorphism group of $K \subseteq K(X_1, \dots, X_n)$ is the n-th <u>Cremona group</u>. It is t.d.l.c., however, it is also discrete because $\mathrm{Aut}_K(K(X_1,\dots,X_n))_{(X_1,\dots,X_n)} = \{id\}$.
If $n=1$, an automorphism is determined by its image on X. One shows that this image has to be of the form $\frac{aX+b}{cX+d}$ for some $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2,K)$, so $\mathrm{Aut}_K(K(X)) \cong PGL(2,K)$ w/ discrete topology.

Similarly, we see that $\mathrm{Aut}_K(E)$ is t.d.l.c. discrete when $\text{tr-deg}(E:K) < \infty$ and $\deg(E:K(M)) < \infty$. To produce sth. non-discrete, we therefore need an infinite alg. degree. However, this is not enough.

<u>Ex.</u> (Robert '70) Fix $n \in \mathbb{N}_{\geq 2}$ and let K be a field in which no non-trivial element has roots of all orders n^k $(k \in \mathbb{N})$ (e.g. $\mathbb{Q}$). Consider $K \subseteq K(X) \subseteq \overline{K(X)}$. Construct $E^{(n)}$ as the union

$$K \subseteq K(X) \subseteq K(X^{n^{-1}}) \subseteq K(X^{n^{-2}}) \subseteq K(X^{n^{-3}}) \subseteq \dots$$

Then $K \subseteq K(X) \subseteq E^{(n)}$ where $K(X) \subseteq E^{(n)}$ is algebraic of infinite degree.
But $\mathrm{Aut}_K(E^{(n)})$ is discrete because $\mathrm{Aut}_K(E^{(n)})_X = \{id\}$:
Let $\varphi \in \mathrm{Aut}_K(E^{(n)})_X$. Then both $X^{n^{-k}}$ and $\varphi(X^{n^{-k}})$ are roots of $t^{n^k} - X \in K(X)[t]$. Hence $\varphi(X^{n^{-k}}) = \zeta_k X^{n^{-k}}$ for some $\zeta_k \in K$ w/ $\zeta_k^{n^k} = 1$,

$$\zeta_k X^{n^{-k}} = \varphi(X^{n^{-k}}) = \varphi(X^{n^{-k-l}})^{n^l} = (\zeta_{k+l} X^{n^{-k-l}})^{n^l} = \zeta_{k+l}^{n^l} X^{n^{-k}}$$

The assumption on K implies $\zeta_k = 1$ $\forall k \in \mathbb{N}$, so $\varphi = id$.
One can show: $\mathrm{Aut}_K(E^{(n)}) \cong \mathbb{Z} \times C_2$.

<u>Prop.</u> Let $K \subseteq E$ be a field extension. Suppose there is an intermediate extension $K \subseteq F \subseteq E$ s.t. $F \subseteq E$ is non-fin.-gen. Galois. Then $\mathrm{Aut}(E:K)$ is non-discrete.

Proof Show that $\mathrm{Aut}_K(E)_S$ is non-trivial for all $S \subseteq E$ finite. The extension $F(S) \subseteq E$ is non-triv. Galois, ~~so~~ Let $\alpha \in E \setminus F(S)$ and $\bar{\alpha} \in E$ a root of $\mathrm{Irr}(\alpha : F(S))$ distinct from α. Then there is an automorphism of E which sends α to $\bar{\alpha}$ and fixes $F(S)$.

Cor Let $K \subseteq E$ be a field extension s.t. $\text{tr-deg}(E : K) < \infty$ and $K(M) \subseteq E$ is non-fin. gen. Galois. Then $\mathrm{Aut}_K(E)$ is n.d. t.d.l.c.

Example ?

: comp. gen. ?

work in progress

Pro-discrete, elementary ?

Michal 13.07, George 20.07

regionally expansive > compact generation $\rightsquigarrow$ sep $\mathrm{Aut}_K(\overline{K(X)})$

class R (arxiv: dense locally cpct. subgroups P.-E., Ph., C.)

compactly generated subgroups w/ rich action
control over local structure from Galois theory

groups in R that don't involve anything in S ?

Dickson | $\boxed{\text{tree-like graphs}}$ lobe connectivity 1 Simon Rossi

Dunwoody tree from graph, to free products $\rightsquigarrow$ action $\rightsquigarrow$ independence $\rightarrow$ examples