**Statistical Parameter Estimation 2024**
**Exercises 2**
*Deadline 24 January 2024, 23:59 Lappeenranta time*

Let us consider a simple exponential decay

$$y(t) = \theta_1 + (A_0 - \theta_1)\exp(-\theta_2 t) + \varepsilon(t) \tag{1}$$

Let the data be

$$\begin{aligned} y(t) &= (0.487, 0.572, 0.369, 0.179, 0.119, 0.0809, 0.104, 0.091, 0.047, 0.051)^\intercal, \\ t &= (1, 2, 3, 4, 5, 6, 7, 8, 9, 10)^\intercal, \\ \varepsilon(t) &\sim \mathcal{N}(0, \sigma^2 I). \end{aligned}$$

1. Assume first that $A_0 = 1$.
   - Write the likelihood density of $\theta$ given $y(t)$.
   - Derive the negative log-likelihood.
2. Maximum likelihood estimation: use the methods in the optimisation toolbox e.g. fminsearch in Matlab, to obtain a numerical optimizer for the maximum likelihood estimator of $\theta$.
3. Compute conditional mean estimate with
   - Metropolis-Hastings,
   - Adaptive Metropolis,
   - Delayed rejection, and,
   - Delayed rejection with adaptation DRAM.
4. Priors
   - Choose a Gaussian prior for $\theta$ and define a posterior distribution. Calculate the negative log posterior. Compute the MAP and CM estimators as above.
   - Do the same as above, but choose a prior distribution that is uniformly distributed.
5. Visualisation and MCMC diagnostics
   - Plot chains, ACFs, densities – compute ESS and OES.
   - Study the chain convergence. How long chains do you need?

- Write down parameter estimates and the Monte Carlo errors of the estimates of posterior mean and posterior covariance.
- Consider different strategies to handle uncertainty in observation, $\sigma$.
- Generate suitable predictive envelopes around the best fit model.
- How would handle the fact that observations are constrained to positive?
- Add $A_0$ as an extra parameter.