ANTONIUS' COMPENDIUM

Derivations, Formulas, Units and Definitions
Volume I
Version 0.001

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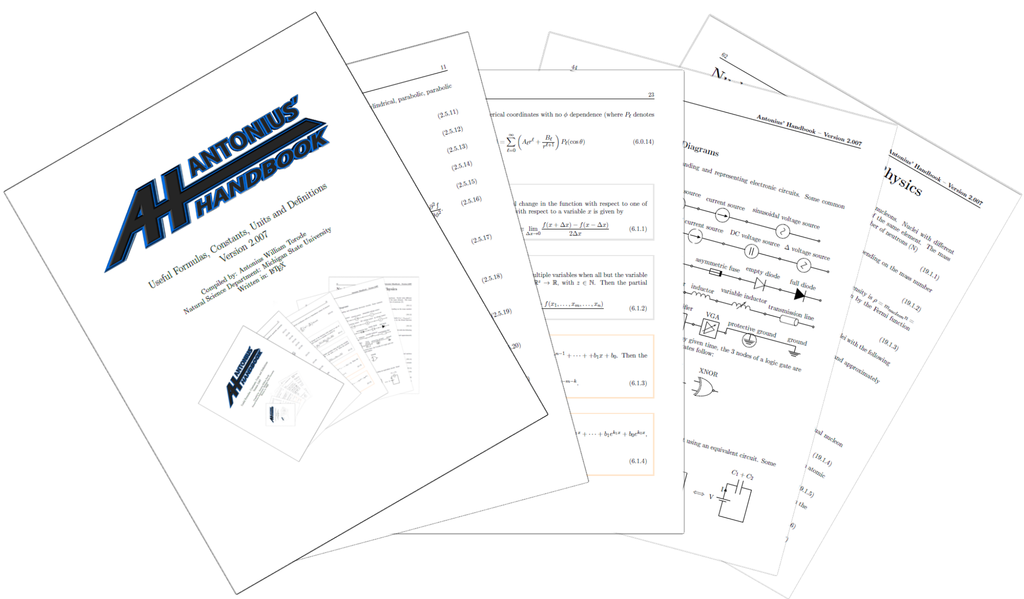

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Preface

This document is a compilation of scratch work, derivations, useful formulas, definitions, constants, and general information used for my own studies as a reference while furthering self education. It's purpose is to provide a complete 'compendium' per say of various mathematical and significant ideas used often. The idea and motivation behind it is to be a quick reference providing easily accessible access to necessary information for either double checking or recalling proper formula for use in various situations due to my own shortcomings in matters of memorization. All the material in this document was either directly copied from one of the references listed at the end or derived from scratch. On occasion *typos may exist* due to human error but will be corrected when discovered.

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Topics Covered In This Book

- topic 1
- topic 2

The information in this book is in no way limited to the topics listed above. They serve as a simple guideline to what you will find within this document. For more information about this book or details about how to obtain your own copy please visit:

https://torodean.github.io/

Disclaimer

This book contains formulas, definitions, and theorems that by nature are very precise. Due to this, some of the material in this book was taken directly from other sources such as but not limited to Wolfram Mathworld. This is only such in cases where a change in wording could cause ambiguities or loss of information quality. Following this, all sources used are listed in the references section and cited when used.

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(Yes, this is a contradiction.)

Contents

Constants and units

1.0.1: Physical Constants

Constant	Symbol	Value	Units
Speed of light in a vacuum	$c \equiv 1/\sqrt{\mu_0\epsilon_0}$	2.99792458×10^8	m/s
Elementary charge	e	$1.602176565(35) \times 10^{-19}$	C
Gravitational constant	G	$6.67384(80) \times 10^{-11}$	$\mathrm{m^3kg^{-1}s^{-2}}$
Avagadro's number	N_a	$6.02214129(27) \times 10^{23}$	$\mathrm{mol}\cdot s^{-1}$
Planck constant	h	$6.62606872(52) \times 10^{-34}$	J·s
		4.135668×10^{-15}	eV·s
	hc	1239.84	eV·nm
Reduced planck constant	$\hbar \equiv h/2\pi$	1.05×10^-34	J·s
Permittivity of the vacuum	ϵ_0	8.854×10^{-12}	$\mathrm{C^2N^{-1}m^{-2}}$
Permeability of the vacuum	μ_0	$4\pi \times 10^{-7}$	$\mathrm{N/A^2}$
Permeability of the vacuum	μ_0	$4\pi \times 10^{-7}$	$\mathrm{N/A^2}$
Boltzmann constant	k_B	$1.38064852 \times 10^{-23}$	J/K
		8.61733×10^{-5}	eV/K
Stefan-Boltzmann constant	$\sigma_\mathrm{B} \equiv \frac{\pi^2 k_B^4}{60\hbar^3 c^3}$	$5.670367(13) \times 10^{-8}$	$\mathrm{W\cdot m^{-2}K^{-4}}$
Thomson cross-section	σ_e	6.652×10^{-29}	m^2
The Bohr Magneton	$\mu_B \equiv \frac{e\hbar}{2m}$	5.788×10^{-5}	eV/T
		9.274×10^{-24}	$\mathrm{Am^2}$
Mass of an electron	m_e	$9.10938291(40) \times 10^{-31}$	kg
		510.9989	keV/c^2
Mass of a proton	m_p	$1.6726218 \times 10^{-27}$	kg
		938.27203	MeV/c^2
Mass of a neutron	m_n	$1.6749274 \times 10^{-27}$	kg
		939.56536	MeV/c^2
Unified amu	u	$1.660538782 \times 10^{-27}$	kg
		931.494028	$\mathrm{MeV/c^2}$

1.0.2: Stellar Data

Spectral Type	T_{eff} (K)	$M/M_\odot$	$L/L_\odot$	$R/R_\odot$	V_{mag}
O5	44,500	60	7.9×10^5	12	-5.7
B5	15,400	5.9	830	3.9	-1.2
A5	8,200	2.0	14	1.7	1.9
F5	6,440	1.4	3.2	1.3	3.4
G5	5,770	0.92	0.79	0.92	4.9
K5	4,350	0.67	0.15	0.72	6.7
M5	3,170	0.21	0.011	0.27	12.3

1.0.3: Astronomical Constants

Constant	Symbol	Value	Units
Mass of Earth	$M_\oplus$	5.974×10^{24}	kg
Mass of Sun	$M_\odot$	1.989×10^{30}	kg
Mass of Moon	$M_☾$	7.36×10^{22}	kg
Equatorial radius of Earth	$R_\oplus$	6.378×10^{6}	m
Equatorial radius of Sun	$R_\odot$	6.6955×10^{8}	m
Equatorial radius of Moon	$R_☾$	1.737×10^{6}	m
Mean density of Earth		5515	kg·m^{-3}
Mean density of Sun		1408	kg·m^{-3}
Mean density of Moon		3346	kg·m^{-3}
Earth-Moon distance		3.84×10^{8}	m
Earth-Sun distance		1.496×10^{11}	m
Luminosity of Sun	$L_\odot$	3.839×10^{26}	W
Effective temp. of Sun		5778	K
Hubble constant	H_0	70 ± 5	km·s^{-1}Mpc^{-1}
Parsec	pc	206264.81	AU
		3.0856776×10^{16}	m
		3.2615638	ly
Astronomical Unit	AU	1.496×10^{11}	m
Light year	ly	9.461×10^{15}	m
1 year on Earth	yr	365.25	days
		3.15576×10^{7}	s

1.0.4: Solar System

Planet	Symbol	Mass (kg)	Radius (m)	Sun-Distance (km)
Mercury	☿	3.285×10^{23}	2.44×10^{6}	5.791×10^{10}
Venus	♀	4.867×10^{24}	6.052×10^{6}	1.082×10^{11}
Mars	♂	6.39×10^{23}	3.390×10^{6}	2.279×10^{11}
Jupiter	♃	1.898×10^{27}	3.83×10^{11}	7.785×10^{11}
Saturn	♄	5.683×10^{26}	5.8232×10^{7}	1.429×10^{12}
Uranus	⛢	8.681×10^{25}	2.5362×10^{7}	2.871×10^{12}
Neptune	♅	1.024×10^{26}	2.4622×10^{7}	4.498×10^{12}
Pluto	♇	1.309×10^{22}	1.187×10^{6}	5.906×10^{12}

1.0.5: Unit conversions

The International System of Units (SI) defines seven units of measure as a basic set from which all other SI units can be derived. These are [length](m), [time](s), [mass](kg), [electric current] $\equiv$ [Ampere](A), [temperature](K), [luminous intensity](cd), [amount of substance](mol).

Unit Symbol	Unit	Equivalence
C	[Coulomb]	[Ampere][time]
N	[Newton]	[mass][length][time]$^{-2}$
P	[Pascal]	[mass][length]$^{-1}$[time]$^{-2}$
J	[Joule]	[mass][length]^{2}[time]$^{-2}$
W	[Watt]	[mass][length]^{2}[time]$^{-3}$
		[Ohm][Ampere]^{2}
		[Volt]^{2}[Ohm]$^{-1}$
V	[Volt]	[mass][length]^{2}[time]$^{-3}$[Ampere]$^{-1}$
Wb	[Weber]	[mass][length]^{2}[time]$^{-2}$[Ampere]$^{-1}$
T	[Tesla]	[mass][time]$^{-2}$[Ampere]$^{-1}$
H	[henry]	[mass][length]^{2}[time]$^{-2}$[Ampere]$^{-2}$
Ω	[Ohm]	[mass][length]^{2}[time]$^{-3}$[Ampere]$^{-2}$
F	[Farad]	[mass]$^{-1}$[length]$^{-2}$[time]^{4}[Ampere]^{2}
Hz	[Hertz]	[time]$^{-1}$

1.0.6: Number Sets ($i \equiv \sqrt{-1}$)

Symbol	Set	Symbol	Set
$\mathbb{R}$	Real numbers	$\emptyset$	$\{\}$
$\mathbb{N} \equiv \mathbb{N}_1$	$\{1,2,3,4,\dots\}$	$\mathbb{Z}$	$\{\dots,-2,1,0,1,2,\dots\}$
$\mathbb{Z}^+ \equiv \mathbb{N}_0$	$\{0,1,2,3,\dots\}$	$\mathbb{Z}^-$	$\{0,-1,-2,-3,-4,\dots\}$
$\mathbb{C}$	$\{x+iy \mid x,y \in \mathbb{R}\}$	$\mathbb{Q}$	$\{\frac{x}{y} \mid x,y \in \mathbb{Z}\}$
$\mathbb{I}$	$\{ix \mid x \in \mathbb{R}\}$	$\mathbb{U}$	Universal Set [a]
$\mathbb{A}$	Algebraic Numbers[b]	$\mathbb{T}$	Transcendental Numbers [c]

[a] Definition: The set containing all objects or elements and of which all other sets are subsets.
[b] Any number that is a solution to a polynomial equation with rational coefficients.
[c] Any number that is not an Algebraic Number.

1.0.7: Mathematical Notation

$\forall$	For all	$\exists$	There exists	$\because$	Because
$\in$	Is an element of	$\notin$	Is not an element of	$\therefore$	Therefore
$\implies$	Implies	$\iff$	Bi conditional	$\approx$	Approximately
$\longrightarrow$	Mapped to	$\not\subset$	Is not a subset of	$\ll$	Much smaller than
$\subset$	Is a subset of	$\subseteq$	Is a subset or equal to	$\gg$	Much greater than
$\propto$	Is proportional to	$\equiv$	Is equivalent to	$\cup/\cap$	Union/Intersection
$\perp$	Is perpendicular to	$\parallel$	Is parallel to	$:$ or $\mid$	Such that

Geometry and Trigonometry

Geometry is a branch of mathematics that deals with the study of shapes, sizes, positions, and properties of space. It explores the relationships and properties of points, lines, angles, surfaces, and solids. In geometry, fundamental concepts include:

1. **Points**: Basic building blocks with no size or dimensions. They are represented by dots and are used to define other geometric elements.

2. **Lines**: Straight paths that extend infinitely in both directions. They are made up of an infinite number of points.

3. **Angles**: The measure of the rotation between two intersecting lines, rays, or line segments.

4. **Polygons**: Closed figures formed by connecting line segments to create shapes like triangles, quadrilaterals, pentagons, etc.

5. **Circles**: A set of points equidistant from a central point, forming a closed curve.

6. **Three-dimensional shapes**: Solids such as cubes, spheres, cylinders, pyramids, etc., with length, width, and height.

7. Higher-dimensional shapes or objects: In addition to the traditional two-dimensional and three-dimensional shapes, geometry also explores mathematical constructs extended to higher dimensions. These higher dimensions go beyond our familiar three-dimensional space and introduce concepts like 4D, 5D, and higher-dimensional shapes. For example, in 4D space, there could be hypercubes, 4D spheres, and other intriguing structures. While challenging to visualize, these higher-dimensional geometries play a crucial role in theoretical mathematics and various scientific fields, offering unique insights into the nature of space and dimensions beyond our immediate perception.

Geometry plays a significant role in various fields, including architecture, engineering, art, design, physics, and many other disciplines. It helps us understand the physical world and solve problems related to spatial relationships and measurements. Euclidean geometry, founded by the ancient Greek mathematician Euclid, is one of the most well-known and widely studied branches of geometry. However, there are also other types of geometries, such as non-Euclidean geometries, which explore different axioms and concepts, leading to intriguing and diverse mathematical systems.

General Mathematics

Mathematics is a systematic field of study that deals with numbers, quantities, shapes, and patterns. It is often regarded as the language of science, as it provides a framework for analyzing and understanding various phenomena in the natural and social world. Mathematics involves a wide range of topics, including arithmetic, algebra, geometry, calculus, statistics, and more. It is based on rigorous logical reasoning and uses symbols and formulas to represent relationships and solve problems. Mathematics plays a crucial role in various fields, such as physics, engineering, economics, computer science, and many other disciplines, making it an essential tool for advancing knowledge and technology.

3.1 Summation

Definition 3.1.1: Summation Notation

sum is the result of addition. The symbol $\sum$ is used to represent the addition of a series of values. Let $..., x_{a-1}, x_a, x_{a+1}, ..., x_{b-1}, x_b, x_{b+1}, ...$ be a fixed series $\mathbb{S}$, where each term $x_i \in \mathbb{S}$, $b, a \in \mathbb{N}$ are index values. Then the sum of values in the series from a to b is given by

$$\sum_{i=a}^{b} x_i = x_a + x_{a+1} + ... + x_{b-1} + x_b \tag{3.1.1}$$

In the context of summing over all of the values of a series for some index i, it is often written in different ways which are typically equivalent.

$$\sum_{i \in \mathbb{S}} x_i \equiv \sum_{i} x_i \tag{3.1.2}$$

If the number of terms in the summation is infinite then it is called an infinite series.

Consider a finite sum of values in a sequence x from index a to b. If we square this sum we would get

$$\begin{aligned}
\left[\sum_{i=a}^{b} x_i\right]^2 &= (x_a + x_{a+1} + ... + x_{b-1} + x_b)^2 && (3.1.3)\\
&= (x_a + x_{a+1} + x_{a+2} + ... + x_{b-1} + x_b)(x_a + x_{a+1} + x_{a+2} + ... + x_{b-1} + x_b) && (3.1.4)\\
&= \quad x_a^2 + 2x_a x_{a+1} + 2x_a x_{a+2} + \cdots + 2x_a x_b\\
&\quad + 2x_{a+1}x_a + x_{a+1}^2 + 2x_{a+1}x_{a+2} + \cdots + 2x_{a+1}x_b\\
&\quad + 2x_{a+2}x_a + 2x_{a+2}a_{a+1} + x_{a+2}^2 + \cdots + 2x_{a+2}x_b\\
&\quad \vdots\\
&\quad + 2x_b x_a + 2x_b a_{a+1} + 2x_b x_{a+2} + \cdots + x_b^2 && (3.1.5)
\end{aligned}$$

From this, a pattern can be observed. There exists an x_i^2 term for each $i \in [a, b]$. There then exists a term $2x_a x_b$ for all $a \neq b$. Therefore, we can write this as

$$\left[\sum_{i=a}^{b} x_i\right]^2 = \sum_{i=a}^{b} x_i^2 + 2\sum_{i=a}^{b}\sum_{j=a}^{b} x_i x_j (1-\delta_{ij}), \tag{3.1.6}$$

where δ_{ij} is the Kronecker delta function.

Definition 3.1.2: Kronecker Delta

he **Kronecker delta** δ_{ij} is a discrete version of the delta function

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases} \implies 1 - \delta_{ij} = \begin{cases} 0, & \text{if } i = j \\ 1, & \text{if } i \neq j \end{cases} \tag{3.1.7}$$

3.2 Binomial Coefficients

To begin with, a useful idea is to sum each term of the binomial coefficient which will be used later. First, by definition of the binomial coefficient, we can write

$$\begin{aligned}
\binom{n}{k} &= \frac{n!}{(n-k)!k!} && (3.2.1)\\
&= \frac{(n-1)!n}{(n-k)!k!} && (3.2.2)\\
&= \frac{(n-1)!n - k(n-1)! + k(n-1)!}{(n-k)!k!} && (3.2.3)\\
&= \frac{(n-1)!(n-k)}{(n-k)!k!} + \frac{k(n-1)!}{(n-k)!k!} && (3.2.4)\\
&= \frac{(n-1)!}{(n-1-k)!k!} + \frac{(n-1)!}{(n-k)!(k-1)!} && (3.2.5)\\
&= \binom{n-1}{k} + \binom{n-1}{k-1} && (3.2.6)\\
&\equiv \binom{m}{k} + \binom{m}{k-1}, \text{ with } m = n-1. && (3.2.7)
\end{aligned}$$

Observe for a moment that (6) can be expanding even further. By the same process of going from (1) to (1.6), we can say

$$\binom{n-1}{k-1} = \binom{n-2}{k-1} + \binom{n-2}{k-2}. \tag{3.2.8}$$

Thus, (1.6) becomes

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-2}{k-1} + \binom{n-2}{k-2}. \tag{3.2.9}$$

If we continue this pattern, we can see that we can write the binomial coefficients as a sum of binomial coefficients with incrementally decreasing numerators (i.e n!, (n-1)!, (n-2)!,...). This gives

$$\begin{aligned}
\binom{n}{k} &= \binom{n-1}{k} + \binom{n-2}{k-1} + \binom{n-3}{k-2} + \cdots + \binom{1}{k+2-n} + \binom{0}{k+1-n} && (3.2.10)\\
&= \sum_{i=1}^{n} \binom{n-i}{k+1-i} && (3.2.11)\\
&= \sum_{i=0}^{n-1} \binom{n-1-i}{k-i}. && (3.2.12)
\end{aligned}$$

Note that we stop the above sequence when the numerator of our factorial sequence has reached zero. If we were to continue the sequence, we would end up having negative factorials in our numerator which would make evaluating the binomial coefficient at that term and the following terms difficult. However, if we happen to have a smaller k than n, it may be such that we end up having negative k numbers. This is ok for now, as it will lead to complex infinities in the denominator of our binomial coefficient expressions hence making those terms zero. This will be demonstrated later. Now, suppose we claim that for all $n \in \mathbb{N}$, that

$$\sum_{i=0}^{n} \binom{n}{i} = \sum_{i=0}^{n} \frac{n!}{(n-i)!i!} = 2^n. \tag{3.2.13}$$

We can then do a proof by induction to prove this is in fact true for all $n, k \in \mathbb{N}$. First, we can see checking the base cases hold as when $n = 0$, we have $1 = 1$, when $n = 1$, we have $2 = 2$, and when $n = 3$ we have $4 = 4$. Next, let's assume for any $n = k$ that (1) holds true. Now, if we let $n = k + 1$ we have from the left hand side of our expression,

$$\begin{aligned}
\sum_{i=0}^{k+1} \binom{k+1}{i} &= \binom{k+1}{0} + \binom{k+1}{1} + \cdots + \binom{k+1}{k} + \binom{k+1}{k+1} && (3.2.14)\\
&= \binom{k}{0} + \binom{k}{-1} + \binom{k}{1} + \binom{k}{0} + \cdots + \binom{k}{k} + \binom{k}{k-1} + \binom{k}{k+1} + \binom{k}{k} && (3.2.15)\\
&= 2\binom{k}{0} + 2\binom{k}{1} + \cdots + 2\binom{k}{k-1} + 2\binom{k}{k} && (3.2.16)\\
&= 2\sum_{i=0}^{k} \binom{k}{i} && (3.2.17)\\
&= 2(2^k) && (3.2.18)\\
&= 2^{k+1}. && (3.2.19)
\end{aligned}$$

Therefore, we can see that for $n = k + 1$ that equation (8) holds true, and thus we conclude by induction that (8) holds for all $n \in \mathbb{N}$. Note that in the above proof, we made use of $\binom{k}{k+1} = \binom{k}{-1} = 0$. If we were to evaluate each of these using the definition of the binomial coefficient above we may notice a slight issue.

Suppose we try to evaluate $\binom{n}{-1}$. Using the definition from (1), we would have

$$\binom{n}{-1} = \frac{n!}{(n-(-1))!(-1)!} \tag{3.2.20}$$

$$= \frac{n!}{(n+1)!(-1)!} \tag{3.2.21}$$

$$= \frac{n!}{(n)!(n+1)(-1)!} \tag{3.2.22}$$

$$= \frac{1}{(n+1)(-1)!}. \tag{3.2.23}$$

From above, we have a negative factorial in hte denominator of our expression. Since this is not easily determined as a positive integer factorial would be, we will need to expand this using the Gamma function. The Gamma function Γ is defined by

$$\Gamma(n) = (n-1)! \equiv \int_0^\infty t^{n-1}e^{-t}\mathrm{dt}. \tag{3.2.24}$$

Using this gamma function with $n = 0$ gives

$$\Gamma(0) = \int_0^\infty t^{-1}e^{-t}\mathrm{dt}. \tag{3.2.25}$$

Since $\lim_{t\to 0^+} t^{-1}e^{-t} = \infty$, we can say the integral under the curve from 0 to ∞ will be divergent, and thus ∞. Therefore $\Gamma(0) \equiv \infty$. This allows us to write (18) as

$$\frac{1}{(n+1)(-1)!} = \frac{1}{(n+1)\Gamma(0)} = \lim_{x\to\infty} \frac{1}{x} = 0. \tag{3.2.26}$$

Thus, we can say that $\binom{n}{-1} = 0$. Similarly, by the same process, if we have $\binom{n}{n+1}$ we get

$$\binom{n}{n+1} = \frac{n!}{(n-(n+1))!(n+1)!} \tag{3.2.27}$$

$$= \frac{n!}{(-1)!(n)!(n+1)} \tag{3.2.28}$$

$$= \frac{1}{(-1)!(n+1)} \tag{3.2.29}$$

$$= 0 \tag{3.2.30}$$

Statistics and Probability

Probability is a fundamental concept in mathematics and statistics that measures the likelihood of an event occurring. It is represented as a value between 0 and 1, where 0 indicates the event is impossible, 1 denotes certainty, and values between 0 and 1 represent various degrees of likelihood. In simple terms, probability quantifies how probable or likely it is for an event to happen based on the total number of possible outcomes. It helps us make informed decisions, predict outcomes, and understand uncertainty in various real-world scenarios, such as games of chance, weather forecasts, and medical diagnoses. To understand probability on a mathematical level, some definitions of terminology is needed.

A **state** (or outcome) is particular condition that something is in at a specific time. A **system** is an activity, experiment, process, or model with states or outcomes that are typically subject to uncertainty. A **sample space** of an system is the set of all possible states of a system. An **event** (also may be referred to as a trial or measurement) is any subset or collection of states contained in the sample space of a system. An event is a **simple event** if it consists of exactly one state and a **compound event** if it consistst of more than one state.

Definition 4.0.1: Probability

An **probability** p can be defined as the asymptotic frequency of a system in the state s ($s \in \Omega$, where Ω is the sample space of the system) by the total number of occurrences of that state N_s in the limit of an infinite number of events N.

$$p(s) = \lim_{N\to\infty} \frac{N_s}{N} \tag{4.0.1}$$

$$p(s) \in [0,1] \forall s \in \Omega \tag{4.0.2}$$

For a system with n states, the total probabilities of all states must normalize to one.

$$\sum_{i=0}^{n} p(i) = \sum_{s\in\Omega} p(i) = \lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{n} N_i = 1 \tag{4.0.3}$$

A **Bayesian probability** is defined as a person's knowledge of the outcome of a trial, based on the evidence at their disposal - often accompanied by an associated error. A **model probability** is an assumption or guess for the probability given the possibility of an infinite number of trials.

Some more useful concepts for modeling a system are the mean (average), standard deviation, and standard deviation of the mean.

Definition 4.0.2: Mean

f $x_1, ..., x_N$ denotes N separate measurements of one quantity x, then we define the mean (or average) $\langle x \rangle$ as

$$\langle x \rangle = \frac{1}{N} \sum_{i=1}^{N} x_i \tag{4.0.4}$$

Definition 4.0.3: Standard Deviation

f $x_1, ..., x_N$ denotes N separate measurements of one quantity x, then we define the standard deviation σ_x as

$$\sigma_x = \sqrt{\frac{1}{N-1}\sum_i (x_i - \langle x \rangle)^2} \tag{4.0.5}$$

The standard deviation of the mean can then be found by combining definitions 4.0.3 and 4.0.2. First, we start by squaring (4.0.4) to get

$$\langle x \rangle^2 = \left[\frac{1}{N}\sum_{i=1}^{N} x_i\right]^2 \tag{4.0.6}$$

$$\sigma_x = \sqrt{\frac{1}{N-1}\sum_i (x_i - \langle x \rangle)^2} \tag{4.0.7}$$

$$= \sqrt{\frac{1}{N-1}\sum_i (x_i^2 - 2x_i \langle x \rangle + \langle x \rangle^2)} \tag{4.0.8}$$

Calculus

Calculus is a branch of mathematics that deals with the study of change and motion. It encompasses two main components: differentiation and integration.

- **Differentiation**: This involves finding the rate at which a quantity changes with respect to another variable. The derivative is a fundamental concept in calculus, representing the instantaneous rate of change of a function at a particular point. It helps analyze the behavior of functions, such as finding slopes of curves, determining maximum and minimum points, and understanding the concept of velocity and acceleration.

- **Integration**: Integration is the reverse process of differentiation. It involves calculating the accumulation or total amount of a changing quantity over an interval. The integral of a function represents the area under the curve of the function over a given range. It is used to solve problems involving areas, volumes, and quantities related to accumulation, such as calculating total distance traveled from a velocity function or finding the area of a region bounded by a curve.

Calculus has numerous real-world applications, ranging from physics and engineering to economics and biology. It provides essential tools for understanding how things change, predicting behavior, and solving complex problems that involve continuous change and motion. Developed independently by Isaac Newton and Gottfried Wilhelm Leibniz in the 17th century, calculus remains a fundamental and powerful branch of mathematics used in various scientific and practical domains.

5.1 Definitions and General Differential Equations

The derivative of a function represents an infinitesimal change in the function with respect to one of its variables [14].

Definition 5.1.1

Let $f : \mathbb{R} \to \mathbb{R}$. Then the derivative of f with respect to a variable x is given by

$$\frac{d}{dx} f(x) \equiv f'(x) \equiv \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \equiv \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x - \Delta x)}{2\Delta x}$$

Similarly, partial derivatives are defined as derivatives of a function of multiple variables when all but the variable of interest are held fixed during the differentiation [14].

Definition 5.1.2

Let $f : \mathbb{R}^z \to \mathbb{R}$, with $z \in \mathbb{N}$. Then the partial derivative of f with respect to a variable x_m is given by

$$\frac{\partial}{\partial x_m} f(x_1, \ldots, x_n) \equiv \lim_{\Delta x \to 0} \frac{f(x_1, \ldots, x_m + \Delta x, \ldots, x_n) - f(x_1, \ldots, x_m, \ldots, x_n)}{\Delta x}$$

A few rules from definition 5.1.1 can be defined.

Theorem 5.1.1

Let $g : \mathbb{R} \to \mathbb{R}$, and $h : \mathbb{R} \to \mathbb{R}$ be continuous functions with $f(x) = g(x) + h(x)$. Then the derivative of f with respect to the variable x is

$$\frac{d}{dx}f(x) = \frac{d}{dx}\big(g(x) + h(x)\big) = \frac{d}{dx}g(x) + \frac{d}{dx}g(x).$$

Proof for theorem 5.1.1. Let $g : \mathbb{R} \to \mathbb{R}$, and $h : \mathbb{R} \to \mathbb{R}$ be continuous functions with $f(x) = g(x) + h(x)$. Then, by the definition of a derivative, we have

$$\begin{aligned}
\frac{d}{dx}f(x) &= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{(g+h)(x+\Delta x) - (g+h)(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{g(x+\Delta x) + h(x+\Delta x) - g(x) - h(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \left(\frac{g(x+\Delta x) - g(x)}{\Delta x} + \frac{h(x+\Delta x) - h(x)}{\Delta x} \right) \\
&= \lim_{\Delta x \to 0} \frac{g(x+\Delta x) - g(x)}{\Delta x} + \lim_{\Delta x \to 0} \frac{h(x+\Delta x) - h(x)}{\Delta x} \\
&= \frac{d}{dx}g(x) + \frac{d}{dx}h(x).
\end{aligned}$$

Thus, by simple manipulation we've shown

$$\frac{d}{dx}f(x) = \frac{d}{dx}g(x) + \frac{d}{dx}g(x).$$

□

If we want to take the derivative of a function Given a general polynomial function, we can give the following differentiation rule.

Theorem 5.1.2

Let $f : \mathbb{R} \to \mathbb{R}$ be a function with $f(x) = c_n x^n + c_{n-1}x^{n-1} + \cdots + + C_2 x^2 + c_1 x + c_0$, where c_n is constant for all n. Then

$$\frac{d}{dx}f(x) = f'(x) = nc_n x^{n-1} + (n-1)c_{n-1}x^{n-2} + \cdots + 2c_2 x + c_1$$

Proof for theorem 5.1.2. Let $f : \mathbb{R} \to \mathbb{R}$ be a function with $f(x) = c_n x^n + c_{n-1}x^{n-1} + \cdots + + C_2 x^2 + c_1 x + c_0$, where c_n is constant for all n. Let $g_n(x) = c_n x^n$. By theorem 1.1, we know that $f'(x) = g_n'(x) + g_{n-1}'(x) + \cdots + g_1(x) + g_0(x)$. Therefore, if we can prove $p(x) = Cx^n \implies p'(x) = nCx^n$, then this directly implies each term of $f(x)$ will have a derivative of equivalent form. Starting with p, we must first note that $p(x) = Cx^n$ and $p(x+h) = C(x+h)^n$. Inserting these into the limit definition will give us

$$p'(x) = \lim_{h \to 0} \frac{p(x+h) - p(x)}{h} = \lim_{h \to 0} \frac{C(x+h)^n - Cx^n}{h}.$$

As we can see, we can expand $(x+h)^n$ using the binomial theorem. This gives us

$$C \lim_{h\to 0} \frac{\sum_{k=0}^{n} \binom{n}{k} x^{n-k} h^k - x^n}{h}.$$

If we denote $a_k = \binom{n}{k} = \frac{n!}{k!(n-k)!}$, then expand the summation, we get

$$C \lim_{h\to 0} \frac{\sum_{k=0}^{n} a_k x^{n-k} h^k - x^n}{h} = C \lim_{h\to 0} \frac{a_0 x^n + a_1 x^{n-1} h + a_2 x^{n-2} h^2 + a_3 x^{n-3} h^3 + \cdots + a_k x^{n-k} h^k - x^n}{h}. \tag{5.1.1}$$

If we determine a_0 and a_1 we get

$$a_0 = \frac{n!}{0!n!} = 1 \tag{5.1.2}$$

$$a_1 = \frac{n!}{1!(n-1)!} = \frac{n!(n)}{(n-1)!(n)} = \frac{n!(n)}{n!} = n. \tag{5.1.3}$$

Inserting equations (1.2) and (1.3) into (1.1) yields

$$C \lim_{h\to 0} \frac{x^n + n x^{n-1} h + a_2 x^{n-2} h^2 + a_3 x^{n-3} h^3 + \cdots + a_k x^{n-k} h^k - x^n}{h}.$$

Then, we see that the first and last terms add to zero

$$C \lim_{h\to 0} \frac{n x^{n-1} h + a_2 x^{n-2} h^2 + a_3 x^{n-3} h^3 + \cdots + a_k x^{n-k} h^k}{h}.$$

From here, we simplify by dividing by h and take the limit as $h \to 0$

$$C \lim_{h\to 0} n x^{n-1} + a_2 x^{n-2} h + a_3 x^{n-3} h^2 + \cdots + a_k x^{n-k} h^{k-1} = C n x^{n-1}.$$

Hence, we have determined that $p'(x) = Cnx^{n-1}$ which directly implies

$$\frac{d}{dx} f(x) = f'(x) = n c_n x^{n-1} + (n-1) c_{n-1} x^{n-2} + \cdots + 2 c_2 x + c_1$$

□

Consider the basic product rule of a differential equation. The product rule is given by

Theorem 5.1.3

Let $f : \mathbb{R} \to \mathbb{R}$ such that $f(x)$ and $g(x)$ are continuous functions of the variable x. We can then say

$$\frac{d}{dx} f(x) g(x) = g(x) \frac{d}{dx} f(x) + f(x) \frac{d}{dx} g(x) = f'(x) g(x) + f(x) g'(x)$$

Proof for theorem 5.1.3. Let $f : \mathbb{R} \to \mathbb{R}$ such that $f(x)$ and $g(x)$ are continuous functions of the variable x. By the definition of the derivative, we have

$$\begin{aligned}
\frac{d}{dx} f(x)g(x) &= \lim_{\Delta x \to 0} \frac{f(x+\Delta x)g(x+\Delta x) - f(x)g(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{f(x+\Delta x)g(x+\Delta x) - f(x+\Delta x)g(x) + f(x+\Delta x)g(x) - f(x)g(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{f(x+\Delta x)\big(g(x+\Delta x) - g(x)\big) + g(x)\big(f(x+\Delta x) - f(x)\big)}{\Delta x} \\
&= \lim_{\Delta x \to 0} f(x+\Delta x)\frac{g(x+\Delta x) - g(x)}{\Delta x} + \lim_{\Delta x \to 0} g(x)\frac{f(x+\Delta x) - f(x)}{\Delta x} \\
&= f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x) \\
&= f'(x)g(x) + f(x)g'(x).
\end{aligned}$$

□

It is thus clear that given two continuous functions of x, say $f(x)$ and $g(x)$, the derivative follows as

$$\begin{aligned}
\frac{d}{dx} f(x)g(x) &= g(x)\frac{d}{dx}f(x) + f(x)\frac{d}{dx}g(x) && (5.1.4) \\
&= f(x)g'(x) + g(x)f'(x). && (5.1.5)
\end{aligned}$$

Following this, the second derivative is given by

$$\begin{aligned}
\frac{d^2}{dx^2} f(x)g(x) &= \frac{d}{dx}\left(g(x)\frac{d}{dx}f(x)\right) + \frac{d}{dx}\left(f(x)\frac{d}{dx}g(x)\right) && (5.1.6) \\
&= \frac{d}{dx}f(x)\frac{d}{dx}g(x) + g(x)\frac{d^2}{dx^2}f(x) + f(x)\frac{d^2}{dx^2}g(x) + \frac{d}{dx}f(x)\frac{d}{dx}g(x) && (5.1.7) \\
&= 2\frac{d}{dx}f(x)\frac{d}{dx}g(x) + g(x)\frac{d^2}{dx^2}f(x) + f(x)\frac{d^2}{dx^2}g(x) && (5.1.8) \\
&= f(x)g''(x) + 2f'(x)g'(x) + f''(x)g(x). && (5.1.9)
\end{aligned}$$

Similarly, the third derivative is given by

$$\begin{aligned}
\frac{d^3}{dx^3} f(x)g(x) &= \frac{d}{dx}\left(2\frac{d}{dx}f(x)\frac{d}{dx}g(x) + g(x)\frac{d^2}{dx^2}f(x) + f(x)\frac{d^2}{dx^2}g(x)\right) && (5.1.10) \\
&= \frac{d}{dx}\left(2\frac{d}{dx}f(x)\frac{d}{dx}g(x)\right) + \frac{d}{dx}\left(g(x)\frac{d^2}{dx^2}f(x)\right) + \frac{d}{dx}\left(f(x)\frac{d^2}{dx^2}g(x)\right). && (5.1.11)
\end{aligned}$$

This derivative can be done in parts to help keep track of what is being done. Let us denote each term in the above sum as $h_1(x)$, $h_2(x)$ and $h_3(x)$ respectively. Then we have

$$\begin{aligned}
h_1(x) &= \frac{d}{dx}\left(2\frac{d}{dx}f(x)\frac{d}{dx}g(x)\right) && (5.1.12) \\
h_2(x) &= \frac{d}{dx}\left(g(x)\frac{d^2}{dx^2}f(x)\right) && (5.1.13) \\
h_3(x) &= \frac{d}{dx}\left(f(x)\frac{d^2}{dx^2}g(x)\right). && (5.1.14)
\end{aligned}$$

Simplifying these expressions then gives

$$h_1(x) = \left(2\frac{d^2}{dx^2}f(x)\frac{d}{dx}g(x) + 2\frac{d}{dx}f(x)\frac{d^2}{dx^2}g(x)\right) \tag{5.1.15}$$

$$h_2(x) = \left(\frac{d}{dx}g(x)\frac{d^2}{dx^2}f(x) + g(x)\frac{d^3}{dx^3}f(x)\right) \tag{5.1.16}$$

$$h_3(x) = \left(\frac{d}{dx}f(x)\frac{d^2}{dx^2}g(x) + f(x)\frac{d^3}{dx^3}g(x)\right). \tag{5.1.17}$$

Then, adding these three expressions together gives us

$$\frac{d^3}{dx^3}f(x)g(x) = f(x)\frac{d^3}{dx^3}g(x) + 3\frac{d}{dx}f(x)\frac{d^2}{dx^2}g(x) + 3\frac{d^2}{dx^2}f(x)\frac{d}{dx}g(x) + g(x)\frac{d^3}{dx^3}f(x) \tag{5.1.18}$$

$$= f(x)g'''(x) + 3f'(x)g''(x) + 3f''(x)g'(x) + f'''(x)g(x). \tag{5.1.19}$$

Using the product rule, a nice pattern seems to be emerging as we continually apply it to the same function. From equation (2), (6) and (16), we can see that it appears to be a similar pattern as when applying the binomial expansion theorem. As a reminder, the binomial expansion theorem is

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k \tag{5.1.20}$$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}. \tag{5.1.21}$$

If we observe the first three terms of this (with $n = 1, 2, 3$), we have

$$(x+y)^1 = x + y \tag{5.1.22}$$

$$(x+y)^2 = x^2 + 2xy + y^2 \tag{5.1.23}$$

$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3. \tag{5.1.24}$$

Comparing this to the three equations in questions allows us to notice interesting similarities. If we expand this to n we have

$$(x+y)^n = x^n y^0 + \binom{n}{1} x^{n-1} y^1 + \cdots + \binom{n}{n-1} x y^{k-1} + x^0 y^k. \tag{5.1.25}$$

Now, let $x = \frac{d}{dx}g(x)$ and $y = \frac{d}{dx}f(x)$. If we define the following conditions

$$\left(\frac{d}{dx}h(x)\right)^n := \frac{d^n}{dx^n}h(x) \tag{5.1.26}$$

$$\frac{d}{dx}g(x) + \frac{d}{dx}f(x) := \frac{d}{dx}g(x)f(x), \tag{5.1.27}$$

then inserting x and y into (21) gives

$$\left[\frac{d}{dx}g(x) + \frac{d}{dx}f(x)\right]^n = \left(\frac{d}{dx}g(x)\right)^n \left(\frac{d}{dx}f(x)\right)^0 + \binom{n}{1}\left(\frac{d}{dx}g(x)\right)^{n-1}\left(\frac{d}{dx}f(x)\right)^1 + \cdots \tag{5.1.28}$$

$$\cdots + \binom{n}{n-1}\left(\frac{d}{dx}g(x)\right)\left(\frac{d}{dx}f(x)\right)^{n-1} + \left(\frac{d}{dx}g(x)\right)^0\left(\frac{d}{dx}f(x)\right)^n. \tag{5.1.29}$$

Simplifying the above expression gives us

$$\frac{d^n}{dx^n}g(x)f(x) = f(x)\frac{d^n}{dx^n}g(x) + \binom{n}{1}\frac{d^{n-1}}{dx^{n-1}}g(x)\frac{d}{dx}f(x) + \cdots \tag{5.1.30}$$

$$\cdots + \binom{n}{n-1}\frac{d}{dx}g(x)\frac{d^{n-1}}{dx^{n-1}}f(x) + g(x)\frac{d^n}{dx^n}f(x). \tag{5.1.31}$$

Therefore, we can see that with our re-definitions in (23) and (24), it appears that the n^{th} derivative of something using the product rule can be expressed using the binomial expansion theorem. At least to $n = 3$ we can see from above that this is accurate. If we want to determine if this works for all values of n, we can try a proof by induction. Before going any further, since this may be a large amount of work for such a simple idea, let us quickly have a sanity check. We claim that the terms of the product rule can be expanded using the binomial theorem. It is important to note that when taking the product rule of an expression, you get two new expressions. Then, taking the product rule again will double those two into four, and then those four into eight, and so on. So we can see that each product rule doubles the number of terms we have. The binomial coefficient is used when we have like terms to combine them. If we were to add up each binomial coefficient in the sum, we should then see that each sum of coefficients is the same number of terms we would get as if we doubled our expression each time. If this were not so then our situation would have a flaw. In other words, it must hold true that

$$\sum_{i=0}^{n}\binom{n}{k} = 2^n. \tag{5.1.32}$$

This indeed is true. See section 1 for the proof on this. Now, we know from (1), (5) and (15) that cases $n = 1, 2, 3$ all hold. Next, if we assume that for any $n = k$ that

$$\begin{aligned}\frac{d^k}{dx^k}g(x)f(x) &= f(x)\frac{d^k}{dx^k}g(x) + \binom{k}{1}\frac{d^{k-1}}{dx^{k-1}}g(x)\frac{d}{dx}f(x) + \cdots \\ &\cdots + \binom{k}{k-1}\frac{d}{dx}g(x)\frac{d^{k-1}}{dx^{k-1}}f(x) + g(x)\frac{d^k}{dx^k}f(x) \quad (5.1.33)\\ &= \sum_{i=0}^{k}\binom{k}{i}\left[\frac{d^i}{dx^i}f(x)\right]\left[\frac{d^{k-i}}{dx^{k-i}}g(x)\right] \quad (5.1.34)\end{aligned}$$

holds true, we can then determine if it holds true for $n = k + 1$. This gives

$$\begin{aligned}\frac{d^{k+1}}{dx^{k+1}}g(x)f(x) &= f(x)\frac{d^{k+1}}{dx^{k+1}}g(x) + \binom{k+1}{1}\frac{d^k}{dx^k}g(x)\frac{d}{dx}f(x) + \cdots \\ &\cdots + \binom{k+1}{k}\frac{d}{dx}g(x)\frac{d^k}{dx^k}f(x) + g(x)\frac{d^{k+1}}{dx^{k+1}}f(x) \quad (5.1.35)\\ &= \sum_{i=0}^{k+1}\binom{k+1}{i}\left[\frac{d^i}{dx^i}f(x)\right]\left[\frac{d^{k+1-i}}{dx^{k+1-i}}g(x)\right]. \quad (5.1.36)\end{aligned}$$

If we were to manually evaluate the $n = k + 1$ term of this expression we would have

$$\frac{d}{dx}\frac{d^k}{dx^k}g(x)f(x) = \frac{d}{dx}\sum_{i=0}^{k}\binom{k}{i}\left[\frac{d^i}{dx^i}f(x)\right]\left[\frac{d^{k-i}}{dx^{k-i}}g(x)\right]. \tag{5.1.37}$$

Since differentiating is distributive (by theorem 0.1), we can evaluate each term of this sum and compare it to (32) and (33). Doing this, we can use the product rule to first get

$$\frac{d^{k+1}}{dx^{k+1}}g(x)f(x) = \sum_{i=0}^{k}\binom{k}{i}\left[\frac{d^{i+1}}{dx^{i+1}}f(x)\right]\left[\frac{d^{k-i}}{dx^{k-i}}g(x)\right] + \binom{k}{i}\left[\frac{d^{i}}{dx^{i}}f(x)\right]\left[\frac{d^{k+1-i}}{dx^{k+1-i}}g(x)\right]. \tag{5.1.38}$$

From here, writing out the series explicitly allows us to combine terms in a useful matter. This would obviously be tedious to write out so let us denote each term by a_i, for any i^{th} term in the summation. Writing out a few of the terms then gives

$$a_0 = \binom{k}{0}\left[\frac{d^1}{dx^1}f(x)\right]\left[\frac{d^k}{dx^k}g(x)\right] + \binom{k}{0}f(x)\left[\frac{d^{k+1}}{dx^{k+1}}g(x)\right] \tag{5.1.39}$$

$$a_1 = \binom{k}{1}\left[\frac{d^2}{dx^2}f(x)\right]\left[\frac{d^{k-1}}{dx^{k-1}}g(x)\right] + \binom{k}{1}\left[\frac{d}{dx}f(x)\right]\left[\frac{d^k}{dx^k}g(x)\right] \tag{5.1.40}$$

$$a_2 = \binom{k}{2}\left[\frac{d^3}{dx^3}f(x)\right]\left[\frac{d^{k-2}}{dx^{k-2}}g(x)\right] + \binom{k}{2}\left[\frac{d^2}{dx^2}f(x)\right]\left[\frac{d^{k-1}}{dx^{k-1}}g(x)\right] \tag{5.1.41}$$

$$\vdots \tag{5.1.42}$$

$$a_{k-2} = \binom{k}{k-2}\left[\frac{d^{k-1}}{dx^{k-1}}f(x)\right]\left[\frac{d^2}{dx^2}g(x)\right] + \binom{k}{k-2}\left[\frac{d^{k-2}}{dx^{k-2}}f(x)\right]\left[\frac{d^3}{dx^3}g(x)\right] \tag{5.1.43}$$

$$a_{k-1} = \binom{k}{k-1}\left[\frac{d^{k}}{dx^{k}}f(x)\right]\left[\frac{d}{dx}g(x)\right] + \binom{k}{v}\left[\frac{d^{k-1}}{dx^{k-1}}f(x)\right]\left[\frac{d^2}{dx^2}g(x)\right] \tag{5.1.44}$$

$$a_k = \binom{k}{k}\left[\frac{d^{k+1}}{dx^{k+1}}f(x)\right]g(x) + \binom{k}{k}\left[\frac{d^{k}}{dx^{k}}f(x)\right]\left[\frac{d}{dx}g(x)\right]. \tag{5.1.45}$$

At this point, it becomes obvious as to why the $f^{(n)}(x)$ notation of a derivative was invented, and switching over will greatly help in our next step. Changing our notation gives us

$$a_0 = \binom{k}{0}f^{(1)}(x)g^{(k)}(x) + \binom{k}{0}f(x)g^{(k+1)}(x) \tag{5.1.46}$$

$$a_1 = \binom{k}{1}f^{(2)}(x)g^{(k-1)}(x) + \binom{k}{1}f^{(1)}(x)g^{(k)}(x) \tag{5.1.47}$$

$$a_2 = \binom{k}{2}f^{(3)}(x)g^{(k-2)}(x) + \binom{k}{2}f^{(2)}(x)g^{(k-1)}(x) \tag{5.1.48}$$

$$\vdots \tag{5.1.49}$$

$$a_{k-2} = \binom{k}{k-2}f^{(k-1)}(x)g^{(2)}(x) + \binom{k}{k-2}f^{(k-2)}(x)g^{(3)}(x) \tag{5.1.50}$$

$$a_{k-1} = \binom{k}{k-1}f^{(k)}(x)g^{(1)}(x) + \binom{k}{k-1}f^{(k-1)}(x)g^{(2)}(x) \tag{5.1.51}$$

$$a_k = \binom{k}{k}f^{(k+1)}(x)g(x) + \binom{k}{k}f^{(k)}(x)g^{(1)}(x). \tag{5.1.52}$$

Our end goal is to get this expression to be in the same form as in equation (33). If we first re-write (33) using $f^{(n)}(x)$ notation, we can easily see what our end goal is. This is

$$\sum_{i=0}^{k+1}\binom{k+1}{i}f^{(i)}(x)g^{(k+1-i)}(x). \tag{5.1.53}$$

The next step is then summing expressions (4.43) through (4.49), then determining if what we get is equivalent to (4.50). Let $\Upsilon = a_0 + a_1 + \cdots + a_{k-1} + a_k$, then we have

$$\Upsilon = \binom{k}{0} f(x)g^{(k+1)}(x) + \binom{k}{0} f^{(1)}(x)g^{(k)}(x) + \binom{k}{1} f^{(1)}(x)g^{(k)}(x) + \cdots \tag{5.1.54}$$

$$\cdots + \binom{k}{k} f^{(k)}(x)g^{(1)}(x) + \binom{k}{k-1} f^{(k)}(x)g^{(1)}(x) + \binom{k}{k} f^{(k+1)}(x)g(x) \tag{5.1.55}$$

$$= \binom{k}{0} f(x)g^{(k+1)}(x) + \left[\binom{k}{0} + \binom{k}{1}\right] f^{(1)}(x)g^{(k)}(x) + \cdots \tag{5.1.56}$$

$$\cdots + \left[\binom{k}{k-1} + \binom{k}{k}\right] f^{(k)}(x)g^{(1)}(x) + \binom{k}{k} f^{(k+1)}(x)g(x) \tag{5.1.57}$$

$$= \sum_{i=0}^{k+1} \left[\binom{k}{i-1} + \binom{k}{i}\right] f^{(i)}(x)g^{(k+1-i)}(x). \tag{5.1.58}$$

Using the definition of the binomial coefficient and the derivation in (7), if we let $k = m-1 \implies m = k+1$, we can re-write the binomial coefficient in (55) as follows.

$$\binom{k}{i-1} + \binom{k}{i} = \binom{m-1}{i-1} + \binom{m-1}{i} = \binom{m}{i} = \binom{k+1}{i}. \tag{5.1.59}$$

Thus, (5.55) becomes

$$\Upsilon = \sum_{i=0}^{k+1} \binom{k+1}{i} f^{(i)}(x)g^{(k+1-i)}(x). \tag{5.1.60}$$

Now, $\Upsilon = a_0 + a_1 + \cdots + a_{k-1} + a_k =$ (35) which we have shown is equivalent to (55). Therefore, we can conclude that (28) holds for all $n \in \mathbb{N}$, which allows us to tidy our work up as a theorem.

Theorem 5.1.4

Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be continuous and differentiable functions of the variable x. Then the m^{th} derivative of $f(x)g(x)$ with respect to x is

$$\frac{d^m}{dx^m} f(x)g(x) = \sum_{i=0}^{n} \binom{n}{i} \left[\frac{d^i}{dx^i} f(x)\right] \left[\frac{d^{n-i}}{dx^{n-i}} g(x)\right].$$

5.2 Ordinary Linear Homogeneous Differential Equations

Consider the following linear homogeneous differential equation with constants A and B

$$\ddot{x} + 2A\dot{x} + Bx = 0. \tag{5.2.1}$$

We can similarly write this as

$$\frac{d^2}{dt^2} x(t) + \frac{d}{dt} 2Ax(t) + Bx(t) = 0. \tag{5.2.2}$$

Next, we can simplify the expression by dividing both sides by $x(t)$. Note that this can only be done when $x(t) \neq 0$. We get

$$\frac{d^2}{dt^2} + \frac{d}{dt}2A + B = 0. \tag{5.2.3}$$

Next, notice that the expression is of a quadratic form $af^2 + bf + c = 0$, with $f = \frac{d}{dt}$, $a = 1$, $b = 2A$, and $c = B$. Thus, using the quadratic formula we can say

$$f = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \Longleftrightarrow \frac{d}{dt} = \frac{-2A \pm \sqrt{(2A)^2 - 4(B)}}{2}. \tag{5.2.4}$$

From here, we can simplify the expression to achieve

$$\frac{d}{dt} = \big(- A \pm \sqrt{A^2 - B}\big). \tag{5.2.5}$$

Notice that we have a differential on the left side of the expression, yet no function. To fix this, suppose we multiply both sides by $x(t)$, which is our original function of t in our differential equation. Then we get

$$\frac{d}{dt}x(t) = \big(- A \pm \sqrt{A^2 - B}\big)x(t). \tag{5.2.6}$$

In a case where we differentiate a function, and get the same function multiplied by a constant, we would have $x(t) = C_1 e^{kt}$ as a solution, where C_1 and k are constants. Note that taking $\frac{d}{dt}x(t)$ in this case yields kC_1e^{kt}, which if matched to our differential yields $k = -A \pm \sqrt{A^2 - B}$. Therefore, we can say a solution to our differential equation will be of the form

$$x(t) = C_1 e^{(-A \pm \sqrt{A^2 - B})t}, \tag{5.2.7}$$

or if written more generally as two separate solutions $x_1(t)$ and $x_2(t)$, we have

$$x_1(t) = C_1 e^{(-A + \sqrt{A^2 - B})t} \tag{5.2.8}$$

$$x_2(t) = C_2 e^{(-A - \sqrt{A^2 - B})t} \tag{5.2.9}$$

Note that we must change the constant coefficients in front of each solution to be arbitrary related to each other because both possible cases are a unique solution and each case would be independent of the other. Finally, if we want to verify these solution, we can do so by differentiating twice and plugging our functions into our original differential equation. Doing this (while letting $k = -A + \sqrt{A^2 - B}$ and keeping our $x(t)$ in the general form of Ce^{kt}), we achieve

$$x(t) = Ce^{kt} \tag{5.2.10}$$

$$\frac{d}{dt}x(t) = (-A \pm \sqrt{A^2 - B})Ce^{kt} \tag{5.2.11}$$

$$\frac{d^2}{dt^2}x(t) = (-A \pm \sqrt{A^2 - B})^2 Ce^{kt} \tag{5.2.12}$$

$$= (A^2 \mp 2A\sqrt{A^2 - B} + (A^2 - B))Ce^{kt}. \tag{5.2.13}$$

Plugging these expressions into our differential equation gives us

$$(A^2 \mp 2A\sqrt{A^2 - B} + (A^2 - B))Ce^{kt} + 2A(-A \pm \sqrt{A^2 - B})Ce^{kt} + BCe^{kt} = 0. \tag{5.2.14}$$

As we can see, each term has a common factor of Ce^{kt}, so we can divide both sides by Ce^{kt} giving

$$(A^2 \mp 2A\sqrt{A^2 - B} + (A^2 - B)) + 2A(-A \pm \sqrt{A^2 - B}) + B = 0. \tag{5.2.15}$$

This removes all of the C constants which verifies that they can be arbitrary and the function will still work. From here, we can just apply algebraic manipulation to achieve

$$A^2 \mp 2A\sqrt{A^2 - B} + A^2 - B - 2A^2 \pm 2A\sqrt{A^2 - B} + B = 0. \tag{5.2.16}$$

Which we can clearly see is a true statement by canceling terms. Thus, we can say the following:

Theorem 5.2.1

Let A, B and C_n be constants and $x(t)$ be a function of t. Given any homogeneous ordinary differential equation of the form

$$\frac{d^2}{dt^2}x(t) + \frac{d}{dt}2Ax(t) + Bx(t) = 0,$$

There will exist a solution of the form

$$x(t) = C_1 e^{(-A \pm \sqrt{A^2 - B})t}.$$

If we further the thought of this differential equation having more than one solution, we can easily understand why. First, consider an ordinary linear homogeneous differential equation given in terms of a function of t ($x(t)$) with constants C_n of the form

$$\frac{d^n}{dt^n}x(t) + \frac{d^{n-1}}{dt^{n-1}}C_{n-1}x(t) + \cdots + \frac{d}{dt}C_1 x(t) + C_0 x(t) = 0. \tag{5.2.17}$$

Assume there exists multiple solutions to this equation, say $x_1(t), x_2(t), \ldots, x_n(t)$ and let $x_g(t) = x_1(t) + x_2(t) + \cdots + x_n(t)$. Notice that differentiating $x_g(t)$ with respect to t can be written as (By Theorem 1.1)

$$\frac{d}{dt}x_g(t) = \frac{d}{dt}\big(x_1(t) + x_2(t) + \cdots + x_n(t)\big) \tag{5.2.18}$$

$$= \frac{d}{dt}x_1(t) + \frac{d}{dt}x_2(t) + \cdots + \frac{d}{dt}x_n(t). \tag{5.2.19}$$

Similarly, differentiating a second time gives

$$\frac{d^2}{dt^2}x_g(t) = \frac{d}{dt}\left(\frac{d}{dt}x_1(t) + \frac{d}{dt}x_2(t) + \cdots + \frac{d}{dt}x_n(t)\right) \tag{5.2.20}$$

$$= \frac{d^2}{dt^2}x_1(t) + \frac{d^2}{dt^2}x_2(t) + \cdots + \frac{d^2}{dt^2}x_n(t). \tag{5.2.21}$$

And so forth so that differentiating n times gives us

$$\frac{d^n}{dt^n}x_g(t) = \frac{d^n}{dt^n}x_1(t) + \frac{d^n}{dt^n}x_2(t) + \cdots + \frac{d^n}{dt^n}x_n(t). \tag{5.2.22}$$

Thus, looking at our original differential equation, since we initially claimed that $x_n(t)$ is a solution to our differential equation, we can say

$$\frac{d^n}{dt^n}x_n(t) + \frac{d^{n-1}}{dt^{n-1}}C_{n-1}x_n(t) + \cdots + \frac{d}{dt}C_1x_n(t) + C_0x_n(t) = 0 \tag{5.2.23}$$

$$\frac{d^n}{dt^n}x_{n-1}(t) + \frac{d^{n-1}}{dt^{n-1}}C_{n-1}x_{n-1}(t) + \cdots + \frac{d}{dt}C_1x_{n-1}(t) + C_0x_{n-1}(t) = 0 \tag{5.2.24}$$

$$\vdots$$

$$\frac{d^n}{dt^n}x_0(t) + \frac{d^{n-1}}{dt^{n-1}}C_{n-1}x_0(t) + \cdots + \frac{d}{dt}C_1x_n(t) + C_0x_0(t) = 0 \tag{5.2.25}$$

Furthermore, we can add these solutions together and factor the expression to get

$$\frac{d^n}{dt^n}\big(x_n(t) + \cdots x_0(t)\big) + \cdots + \frac{d}{dx}C_1\big(x_n(t) + \cdots x_0(t)\big) + C_0\big(x_n(t) + \cdots x_0(t)\big) = 0, \tag{5.2.26}$$

or simply

$$\frac{d^n}{dt^n}x_g(t) + \frac{d^{n-1}}{dt^{n-1}}C_{n-1}x_g(t) + \cdots + \frac{d}{dt}C_1x_g(t) + C_0x_g(t) = 0. \tag{5.2.27}$$

Thus, we have showed that a sum of solutions to this form of linear homogeneous differential equation is also a solution to the equation. This means we can say

Theorem 5.2.2

Let $x(t)$ be a function of t and C_n be constant for all n. Given any ordinary homogeneous differential equation of the form

$$\frac{d^n}{dt^n}x(t) + \frac{d^{n-1}}{dt^{n-1}}C_{n-1}x(t) + \cdots + \frac{d}{dt}C_1x(t) + C_0x(t) = 0,$$

if n solutions $x_n(t)$ exist, then another solution can be formed from summing any combination of those solutions and $x_g(t)$ will be the most general solution with

$$x_g(t) = \sum_{k=1}^{n} x_k(t).$$

5.3 Linear Ordinary Differential Equations

Now let us consider some differential equations that are not homogeneous and see if we can come up with a similar way of deriving a solution. Again, let $x(t)$ be some function of t with A and B as constants. In addition, let $g(t)$ be a function of t as well. Similar to earlier, consider the following differential equation

$$\frac{d^2}{dt^2}x(t) + \frac{d}{dt}2Ax(t) + Bx(t) = g(t) \tag{5.3.1}$$

In order for this equation to be satisfied, we must first notice that whatever function $g(t)$ is in, the left hand of the expression must also be that form.

5.3.1 n-Degree Polynomial Particular Functions

For example, let's consider $g(t) = at^2 + bt + c$, where a, b, and c are constants. We can see then that if the equation is satisfied, we will have t^2, t and constant terms on the left side after differentiating $x(t)$ the appropriate number of times and plugging it in. Let us work through this example completely. Consider

$$\frac{d^2}{dt^2}x(t) + \frac{d}{dt}2Ax(t) + Bx(t) = at^2 + bt + c. \tag{5.3.2}$$

We can start by considering theorem 1.2 which tells us that the first derivative of $g(t)$ will no longer have any t^2 terms, and the second derivative will no longer have any t or t^2 terms. Therefore, by a careful observation, we can see that if we let $x(t)$ be a second degree polynomial, then $\frac{d^2}{dx^2}x(t)$ will be a constant polynomial and $\frac{d}{dx}x(t)$ will be a polynomial of degree 1. This allows us to see that the degree of the left side of this expression (given that $x(t)$ is a polynomial of degree 2) will be of degree 2 as well. Hence, by choosing $x(t) = C_1t^2 + C_2t + C_3$, we will end up getting a second degree polynomial equal to a second degree polynomial, so choosing appropriate constants will allow this to be a solution. Let's demonstrate and check our answer by choosing

$$x(t) = C_1t^2 + C_2t + C_3. \tag{5.3.3}$$

From here, we have the appropriate derivatives as

$$\frac{d}{dx}x(t) = 2C_1t + C_2 \tag{5.3.4}$$

$$\frac{d^2}{dx^2}x(t) = 2C_1. \tag{5.3.5}$$

Then, substituting this into our differential equation yields

$$2C_1 + 2A(2C_1t + C_2) + B(C_1t^2 + C_2t + C_3) = at^2 + bt + c. \tag{5.3.6}$$

Consolidating terms then gives

$$(BC_1)t^2 + (BC_2 + 4AC_1)t + (2C_1 + 2AC_2 + BC_3) = at^2 + bt + c. \tag{5.3.7}$$

From here Since these must be equal, the coefficients in front of the t^2 terms must be equal, and so forth. This then gives us the following simple set of equations to solve:

$$BC_1 = a \tag{5.3.8}$$

$$BC_2 + 4AC_1 = b \tag{5.3.9}$$

$$2C_1 + 2AC_2 + BC_3 = c. \tag{5.3.10}$$

Solving for $C_1, C_2, andC_3$ then gives

$$C_1 = \frac{a}{B} \tag{5.3.11}$$

$$C_2 = \frac{b}{B} - \frac{4Aa}{B^2} = \frac{bB - 4Aa}{B^2} \tag{5.3.12}$$

$$C_3 = \frac{8A^2a}{B^3} - \frac{2a}{B^2} - \frac{2Ab}{B^2} + \frac{c}{B} = \frac{8aA^2 - 2aB - 2AbB + B^2c}{B^3}. \tag{5.3.13}$$

Therefore, a solution that would work for our differential is

$$x(t) = \frac{a}{B}t^2 + \frac{bB - 4Aa}{B^2}t + \frac{8aA^2 - 2aB - 2AbB + B^2c}{B^3}. \tag{5.3.14}$$

It is simple enough to verify that this is indeed a solution. To begin, we can take the derivative twice, and plug the function in. Taking the first and second derivatives give us

$$\frac{d}{dx}x(t) = \frac{2a}{B}t + \frac{bB - 4Aa}{B^2} \tag{5.3.15}$$

$$\frac{d^2}{dx^2}x(t) = \frac{2a}{B}. \tag{5.3.16}$$

Then, plugging these into our equation gives

$$\frac{2a}{B} + 2A\left(\frac{2a}{B}t + \frac{bB - 4Aa}{B^2}\right) + B\left(\frac{a}{B}t^2 + \frac{bB - 4Aa}{B^2}t + \frac{8aA^2 - 2aB - 2AbB + B^2c}{B^3}\right), \tag{5.3.17}$$

which should simplify to $at^2 + bt + c$. Foiling terms and canceling like terms gives

$$\frac{Ba}{B}t^2 + \frac{bB^2}{B^2}t + \frac{B^3c}{B^3} \equiv at^2 + bt + c. \tag{5.3.18}$$

Thus, we have verified that our solution is valid. Now by the same principles used in this above example, we can generalize the form of our solution. Let C_n and a_ℓ be constant for all n, ℓ. Consider any differential equation of the form

$$\frac{d^n}{dt^n}x(t) + \frac{d^{n-1}}{dt^{n-1}}C_{n-1}x(t) + \cdots + \frac{d}{dt}C_1x(t) + C_0x(t) = g(t) \tag{5.3.19}$$

$$g(t) = a_\ell t^\ell + a_{\ell-1}t^{\ell-1} + \cdots + a_1t + a_0. \tag{5.3.20}$$

By the same reasoning as before, the left side of the equation will equate to a polynomial of degree less than or equal to ℓ. This allows us to let $x(t)$ equal an arbitrary polynomial of degree ℓ similarly to above, and then we would be able to solve for the coefficients in order to make the equation a solution. Thus, let b_ℓ be constant for all ℓ and

$$x(t) = b_\ell t^\ell + b_{\ell-1}t^{\ell-1} + \cdots + b_4t^4 + b_3t^3 + b_2t^2 + b_1t + b_0. \tag{5.3.21}$$

We would then solve for our coefficients b_ℓ in the same manner as above so they are in terms of a_ℓ and C_n, for some n, ℓ. Let's look at what happens when we do this. First, let's differentiate $x(t)$ an appropriate number of times. This gives

$$\frac{d}{dt}x(t) = \ell b_\ell t^{\ell-1} + (\ell-1)b_{\ell-1}t^{\ell-2} + \cdots + 4b_4t^3 + 3b_3t^2 + 2b_2t + b_1 \tag{5.3.22}$$

$$\frac{d^2}{dt^2}x(t) = \ell(\ell-1)b_\ell t^{\ell-2} + (\ell-1)(\ell-2)b_{\ell-1}t^{\ell-3} + \cdots + 12b_4t^2 + 6b_3t + 2b_2 \tag{5.3.23}$$

$$\frac{d^3}{dt^3}x(t) = \ell(\ell-1)(\ell-2)b_\ell t^{\ell-3} + (\ell-1)(\ell-2)(\ell-3)b_{\ell-1}t^{\ell-4} + \cdots + 24b_4t + 6b_3 \tag{5.3.24}$$

$$\vdots$$

$$\frac{d^\ell}{dt^\ell}x(t) = \ell!b_\ell. \tag{5.3.25}$$

As we can see, a pattern begins to emerge that allows us to write the m^{th} derivative as

$$\frac{d^m}{dt^m}x(t) = \frac{\ell!}{(\ell-m)!}b_\ell t^{\ell-m} + \frac{(\ell-1)!}{(\ell-m-1)!}b_{\ell-1}t^{\ell-m-1} + \cdots + (m+1)!b_{m+1}t + m!b_m. \tag{5.3.26}$$

Since ℓ is fixed as the degree of $g(t)$, we can write this m^{th} derivative expression as a sum of terms from above. This then gives us

$$\frac{d^m}{dx^m}x(t) = \sum_{k=0}^{\ell-m} \frac{(\ell-k)!}{(\ell-m-k)!} b_{\ell-k} t^{\ell-m-k}. \tag{5.3.27}$$

Thus, to formalize this into something useful, we can state it as a theorem.

Theorem 5.3.1

Let $f : \mathbb{R} \to \mathbb{R}$ such that $f(t) \in \mathbb{P}_n$ of the form $f(t) = b_n t^n + b_{n-1}t^{n-1} + \cdots + +b_1 t + b_0$. Then the m^{th} derivative of $f(t)$ is given by

$$\frac{d^m}{dx^m}f(t) = f^{(m)}(t) = \sum_{k=0}^{n-m} \frac{(n-k)!}{(n-m-k)!} b_{n-k} t^{n-m-k}.$$

What we have done above is find a solution to the differential equation that correlates to an expression on the right hand side of our equation. This solution is called a particular solution and is generally denoted with a subscript of p, such as $x_p(t)$. We have also shown that it is fairly simple to determine a particular solution when our differential equation equates to a polynomial. In some cases, the particular solution is not as simple to compute. Since $g(t)$ is a generic function of t, we know it can be in the form of any possible function. As it is seemingly impossible to determine the infinite possible particular solutions for all of these differential equations, we will try to generalize a few and make some rules for combining solutions.

5.3.2 Simple Exponential Particular Functions

An even easier case for particular solutions when we have exponential functions. Consider the case where $g(t) = ae^{kt}$, with a and k being constants. Then we have may have some differential of a form similar to

$$\frac{d^2}{dt^2}x(t) + \frac{d}{dt}2Ax(t) + Bx(t) = ae^{kt}. \tag{5.3.28}$$

When given a homogeneous differential equation, it is sometimes possible to derive a solution in a completely analytical way. In some cases for particular solutions especially, it is easiest to make a keen observation, start with a possible solution, and verify from there. The most important observation here is that when an exponential has a derivative of the same form as the original exponential only with a different constant out front. Notice for example that $\frac{d}{dt}g(t) = ake^{kt}$ and $\frac{d^2}{dt^2}g(t) = ak^2e^{kt}$. Thus, if $x(t)$ were to also be an exponential of the form be^{kt} (with b being a constant), we would have all terms on the left side of our expression in the form of a constant multiplied by e^{kt}. Therefore, let us choose $x(t) = be^{kt}$ and check to see if it could be a solution. First, we have

$$\frac{d}{dt}x(t) = bke^{kt} \tag{5.3.29}$$

$$\frac{d^2}{dt^2}x(t) = bk^2e^{kt} \tag{5.3.30}$$

Then substituting this into our differential yields

$$bk^2e^{kt} + 2ACke^{kt} + Bbke^{kt} = ae^{kt}. \tag{5.3.31}$$

From here, we can divide out the e^{kt}, factor and solve for our constant as follows

$$b(k^2 + 2Ak + B) = a \implies b = \frac{a}{k^2 + 2Ak + B}. \tag{5.3.32}$$

Hence, we have that the particular solution to our differential equation is

$$x_p(t) = \frac{ae^{kt}}{k^2 + 2Ak + B}. \tag{5.3.33}$$

Checking this solution is trivial and can be done simply by taking the first and second derivatives and plugging into our differential equation. The differential equation should then evaluate as true. As we can see, exponential particular solutions are very simple, and if we were to have something such as $g(t) = a_1e^{kt} + a_2e^{kt}$, then we would solve the particular in exactly the same manner. This is because we would be able to factor and say $g(t) = (a_1 + a_2)e^{kt}$, with $a_1 + a_2$ being our constant in $g(t)$. Let us consider a more general case however. If for example we have $g(t) = a_\ell e^{k_\ell t} + a_{\ell-1}e^{k_{\ell-1}t} + \cdots + a_1e^{k_1t} + a_0e^{k_0t}$, where a_ℓ and k_ℓ are constant, we would use the same methods, but a combination of multiple solutions. As theorem 1.1 states, we can see that $g(t)$ is a combination of functions $h_i(t) = a_ie^{k_it}$, and thus, the derivatives can be taken separately then added. A good way to represent this is

$$g(t) = \sum_{i=0}^{\ell} h_i(t) = \sum_{i=0}^{\ell} a_ie^{k_it} \tag{5.3.34}$$

Consider as well the following differential equation again:

$$\frac{d^n}{dt^n}x(t) + \frac{d^{n-1}}{dt^{n-1}}C_{n-1}x(t) + \cdots + \frac{d}{dt}C_1x(t) + C_0x(t) = g(t). \tag{5.3.35}$$

Similarly, this can be written as

$$\frac{d^n}{dt^n}x(t) + \frac{d^{n-1}}{dt^{n-1}}C_{n-1}x(t) + \cdots + \frac{d}{dt}C_1x(t) + C_0x(t) = \sum_{i=0}^{\ell} a_ie^{k_it}. \tag{5.3.36}$$

Let $g_\ell(t)$ denote the corresponding $b_\ell e^{k_\ell t}$ term. We know by the above example that if we were to choose $x_n(t) = b_ne^{k_nt}$, it would be a solution for $g_\ell(t) = a_\ell e^{k_\ell t}$. Similarly, if we were to choose $x(t) = x_0(t) + x_1(t) = b_0e^{k_1t} + b_0e^{k_1t}$, we can show that it is a solution to $g(t) = g_0(t) + g_1(t) = a_0e^{k_0t} + a_1e^{k_1t}$ using theorem 1.1. First we say

$$\frac{d^n}{dt^n}\big(x_0(t) + x_1(t)\big) + \cdots + \frac{d}{dt}C_1\big(x_0(t) + x_1(t)\big) + C_0\big(x_0(t) + x_1(t)\big) = g_0(t) + g_1(t). \tag{5.3.37}$$

This then becomes

$$\frac{d^n}{dt^n}x_0(t) + \frac{d^n}{dt^n}x_1(t) + \cdots + \frac{d}{dt}C_1x_0(t) + \frac{d}{dt}C_1x_1(t) + C_0x_0(t) + C_0x_1(t) = g_0(t) + g_1(t). \tag{5.3.38}$$

Then by grouping these we have

$$\left(\frac{d^n}{dt^n}x_0(t) + \cdots + \frac{d}{dt}C_1x_0(t) + C_0x_0(t)\right) + \left(\frac{d^n}{dt^n}x_1(t) + \cdots + \frac{d}{dt}C_1x_1(t) + C_0x_1(t)\right) = g_0(t) + g_1(t).$$

As we can see, this is a solution since $x_0(t)$ is a solution to

$$\frac{d^n}{dt^n}x_0(t) + \cdots + \frac{d}{dt}C_1x_0(t) + C_0x_0(t) = g_0(t) \tag{5.3.39}$$

and $x_1(t)$ is a solution to

$$\frac{d^n}{dt^n}x_1(t) + \cdots + \frac{d}{dt}C_1x_1(t) + C_0x_1(t) = g_1(t). \tag{5.3.40}$$

Similarly, this method will work for any number for terms in $g(t)$. Therefore, if we had

$$g(t) = \sum_{i=0}^{\ell} a_i e^{k_i t}, \tag{5.3.41}$$

We will have a solution of the form

$$x(t) = \sum_{i=0}^{\ell} b_i e^{k_i t}. \tag{5.3.42}$$

We can then verify this by plugging our above equations into the differential as follows:

$$\frac{d^n}{dt^n}\sum_{i=0}^{\ell} b_i e^{k_i t} + \frac{d^{n-1}}{dt^{n-1}}C_{n-1}\sum_{i=0}^{\ell} b_i e^{k_i t} + \cdots + \frac{d}{dt}C_1\sum_{i=0}^{\ell} b_i e^{k_i t}x(t) + C_0\sum_{i=0}^{\ell} b_i e^{k_i t} = \sum_{i=0}^{\ell} a_i e^{k_i t}. \tag{5.3.43}$$

Similar to theorem 4.1, we need a simple way of determining the m^{th} derivative for $x(t)$. We can first express $x(t)$ as a sum of all of it's components

$$x(t) = b_\ell e^{k_\ell t} + b_{\ell-1}e^{k_{\ell-1}t} + \cdots + b_1 e^{k_1 t} + b_0 e^{k_0 t}. \tag{5.3.44}$$

From here, it is simple to see that the derivatives follow as

$$\frac{d}{dx}x(t) = b_\ell k_\ell e^{k_\ell t} + b_{\ell-1}k_{\ell-1}e^{k_{\ell-1}t} + \cdots + b_1 k_1 e^{k_1 t} + b_0 k_0 e^{k_0 t} \tag{5.3.45}$$

$$\frac{d^2}{dx^2}x(t) = b_\ell k_\ell^2 e^{k_\ell t} + b_{\ell-1}k_{\ell-1}^2 e^{k_{\ell-1}t} + \cdots + b_1 k_1^2 e^{k_1 t} + b_0 k_0^2 e^{k_0 t} \tag{5.3.46}$$

$$\vdots \tag{5.3.47}$$

$$\frac{d^n}{dx^n}x(t) = b_\ell k_\ell^n e^{k_\ell t} + b_{\ell-1}k_{\ell-1}^n e^{k_{\ell-1}t} + \cdots + b_1 k_1^n e^{k_1 t} + b_0 k_0^n e^{k_0 t}. \tag{5.3.48}$$

From here we can clearly see that

$$\frac{d^n}{dx^n}x(t) = \sum_{i=0}^{\ell} b_{\ell-i}k_{\ell-i}^n e^{k_{\ell-i}t}. \tag{5.3.49}$$

This can then be generalized for use as

Theorem 5.3.2

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f(t) = b_n e^{k_n t} + b_{n-a}e^{k_{n-1}t} + \cdots + b_1 e^{k_1 t} + b_0 e^{k_0 t}$, with b_n and k_n as constants. The m^{th} derivative is then given by

$$\frac{d^m}{dx^m}f(t) = \frac{d^m}{dx^m}\sum_{i=0}^{n} b_i e^{k_i t} = \sum_{i=0}^{n} b_{n-i}k_{n-i}^m e^{k_{n-i}t}.$$

From here we can verify our differential equation by inserting the derivatives into our expression. Thus, equation (3.43) becomes

$$\sum_{i=0}^{\ell} b_{\ell-i}k_{\ell-i}^{m}e^{k_{\ell-i}t} + C_{n-1}\sum_{i=0}^{\ell} b_{\ell-i}k_{\ell-i}^{m-1}e^{k_{\ell-i}t} + \cdots + C_0\sum_{i=0}^{\ell} b_{\ell-i}e^{k_{\ell-i}t} = \sum_{i=0}^{\ell} a_i e^{k_i t}. \tag{5.3.50}$$

Starting with the left hand side of the expression (let us denote it by $F(t)$), this can be shown to be equivalent to the right as follows:

$$F(t) = \sum_{i=0}^{\ell}\left[b_{\ell-i}k_{\ell-i}^{m}e^{k_{\ell-i}t} + C_{n-1}b_{\ell-i}k_{\ell-i}^{m-1}e^{k_{\ell-i}t} + \cdots + C_0 b_{\ell-i}e^{k_{\ell-i}t}\right] \tag{5.3.51}$$

$$= \sum_{i=0}^{\ell}\left[b_{\ell-i}e^{k_{\ell-i}t}\big(k_{\ell-i}^{m} + C_{n-1}k_{\ell-i}^{m-1} + \cdots + C_0\big)\right]. \tag{5.3.52}$$

Since $b_{\ell-i}(k_{\ell-i}^{m} + C_{n-1}k_{\ell-i}^{m-1} + \cdots + C_0)$ is constant for all values of i; we can denote it by $C_\ell(i)$, since it is dependent on the value of i. We can then express $F(t)$ as

$$F(t) \equiv \sum_{i=0}^{\ell} C_\ell(i)e^{k_{\ell-i}t}. \tag{5.3.53}$$

If we write out the terms of this sum, we get

$$F(t) \equiv C_\ell(0)e^{k_\ell t} + C_\ell(1)e^{k_{\ell-1}} + \cdots + C_\ell(\ell-1)e^{k_1 t} + C_\ell(\ell)e^{k_0 t} \tag{5.3.54}$$

$$\equiv C_\ell(\ell)e^{k_0 t} + C_\ell(\ell-1)e^{k_1 t} + \cdots + C_\ell(1)e^{k_{\ell-1}} + C_\ell(0)e^{k_\ell t} \tag{5.3.55}$$

$$\equiv \sum_{i=0}^{\ell} C_\ell(\ell-i)e^{k_i t}. \tag{5.3.56}$$

Now, since $C_\ell(\ell-i)$ is still an arbitrary constant for all i, we can let $C_\ell(\ell-i) = a_i$ and then we see

$$F(t) \equiv \sum_{i=0}^{\ell} a_i e^{k_i t} \tag{5.3.57}$$

which verifies that $x(t)$ (equation 3.42) is a solution to our differential equation. From here, we can also solve for our constants in $x(t)$ in terms of the given constants C_n and a_n. First, we have (from equation 3.50)

$$\sum_{i=0}^{\ell} b_{\ell-i}\big(k_{\ell-i}^{m} + C_{n-1}k_{\ell-i}^{m-1} + \cdots + C_0\big)e^{k_{\ell-i}t} = \sum_{i=0}^{\ell} a_i e^{k_i t}. \tag{5.3.58}$$

Writing the terms of this gives (let "$\rightsquigarrow$" denote "$+\cdots+$" to save some space)

$$b_\ell\big(k_\ell^{m} \rightsquigarrow C_0\big)e^{k_\ell t} + b_{\ell-1}\big(k_{\ell-1}^{m} \rightsquigarrow C_0\big)e^{k_{\ell-1}t} \rightsquigarrow b_1\big(k_1^{m} \rightsquigarrow C_0\big)e^{k_1 t} + b_0\big(k_0^{m} \rightsquigarrow C_0\big)e^{k_0 t} = a_0 e^{k_0 t} \rightsquigarrow a_\ell e^{k_\ell t}. \tag{5.3.59}$$

From this, we can then match up the corresponding $e^{k_n t}$ terms on both sides to solve for our constants. This gives us

$$b_\ell\big(k_\ell^m \rightsquigarrow C_0\big) = a_\ell \tag{5.3.60}$$
$$b_{\ell-1}\big(k_{\ell-1}^m \rightsquigarrow C_0\big) = a_{\ell-1} \tag{5.3.61}$$
$$\vdots \tag{5.3.62}$$
$$b_0\big(k_0^m \rightsquigarrow C_0\big) = a_0. \tag{5.3.63}$$

Thus, we notice that for all values of i,

$$b_i\big(k_i^m \rightsquigarrow C_0\big) = a_i \iff b_i = \frac{a_i}{k_i^m \rightsquigarrow C_0}. \tag{5.3.64}$$

Therefore, we can express our solution to our differential as

$$x(t) = \sum_{i=0}^{\ell} \frac{a_i e^{k_i t}}{k_i^m \rightsquigarrow C_0} = \sum_{i=0}^{\ell} \frac{a_i e^{k_i t}}{k_{\ell-i}^m + C_{n-1} k_{\ell-i}^{m-1} + \cdots + C_0} \tag{5.3.65}$$

Theorem 5.3.3

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous and differential function and let C_n, a_n, b_n and k_n be constant for all n. Given a differential equation of the form

$$\frac{d^n}{dt^n} f(t) + \frac{d^{n-1}}{dt^{n-1}} C_{n-1} f(t) + \cdots + \frac{d}{dt} C_1 f(t) + C_0 f(t) = \sum_{i=0}^{\ell} a_i e^{k_i t},$$

A solution of the following form exists:

$$f(t) = \sum_{i=0}^{\ell} b_i e^{k_i t} = \sum_{i=0}^{\ell} \frac{a_i e^{k_i t}}{k_{\ell-i}^n + C_{n-1} k_{\ell-i}^{n-1} + \cdots + C_0}$$

Linear Algebra

Linear algebra is a branch of mathematics that focuses on the study of vector spaces, linear transformations, and systems of linear equations. It provides a powerful framework for representing and solving problems involving linear relationships between variables. Key concepts in linear algebra include:

- **Vectors**: Vectors are mathematical objects that represent magnitude and direction. They can be represented as ordered lists of numbers and are used to describe quantities with both magnitude and direction, such as velocity and force.
- **Vector Spaces**: A vector space is a set of vectors that satisfy certain properties under addition and scalar multiplication. It forms the foundation of linear algebra and allows for the study of linear combinations and transformations.
- **Matrices**: Matrices are rectangular arrays of numbers or elements, organized into rows and columns. They are used to represent linear transformations and to solve systems of linear equations.
- **Linear Transformations**: Linear transformations are functions that preserve the structure of vector spaces. They map vectors from one vector space to another while maintaining properties like linearity and preservation of the origin.
- **Eigenvalues** and **Eigenvectors**: In linear algebra, eigenvalues and eigenvectors are associated with linear transformations. Eigenvectors are special vectors that remain in the same direction after a linear transformation, and eigenvalues represent how much the eigenvectors are scaled during the transformation.

Linear algebra finds applications in various fields, including physics, computer graphics, engineering, data science, and economics. It plays a fundamental role in solving systems of linear equations, understanding linear transformations, and providing tools to analyze complex systems with multiple variables and interactions. Moreover, it forms the basis for more advanced mathematical concepts and techniques in areas like optimization, machine learning, and numerical analysis.

Classical Mechanics

Classical mechanics is a branch of physics that describes the motion of objects and systems under the influence of forces. It forms the foundation of mechanics before the advent of quantum mechanics and relativistic physics. Classical mechanics is based on Newton's laws of motion and the concept of conservation of energy and momentum. Key principles and concepts of classical mechanics include:

- **Newton's Laws of Motion**: Sir Isaac Newton formulated three fundamental laws that govern the motion of objects. The first law (Law of Inertia) states that an object at rest remains at rest, and an object in motion continues to move at a constant velocity unless acted upon by an external force. The second law describes how the acceleration of an object is directly proportional to the net force applied and inversely proportional to its mass. The third law states that for every action, there is an equal and opposite reaction.
- **Conservation of Energy**: The principle of energy conservation states that the total energy of an isolated system remains constant over time. Energy can transform from one form to another (e.g., kinetic energy to potential energy), but the total amount of energy remains unchanged.
- **Conservation of Momentum**: The principle of momentum conservation states that the total momentum of an isolated system remains constant, provided no external forces act on it. Momentum is the product of an object's mass and velocity.
- **Gravitation**: Classical mechanics includes the study of gravitational forces between objects, described by Newton's law of universal gravitation. This law explains how objects attract each other with a force proportional to their masses and inversely proportional to the square of the distance between them.
- **Harmonic Motion**: The study of harmonic motion involves oscillations and vibrations of systems, such as a pendulum or a mass-spring system. These motions follow simple harmonic motion equations and exhibit periodic behavior.

Classical mechanics provides accurate and practical predictions for a wide range of everyday scenarios and macroscopic systems. While it is highly effective in describing the behavior of objects at non-relativistic speeds, it becomes less accurate when dealing with extremely high speeds or microscopic particles, where quantum mechanics and relativistic physics are more appropriate. Nonetheless, classical mechanics remains a crucial and fundamental branch of physics, forming the basis for understanding the motion of everyday objects and engineering applications.

Statistical Mechanics & Thermodynamics

Statistical mechanics is a branch of physics that aims to explain the macroscopic properties of a system (such as temperature, pressure, and entropy) by understanding the behavior and interactions of its microscopic constituents at the atomic or molecular level. It bridges the gap between the microscopic world of particles and the macroscopic world we observe in everyday life. Key concepts and principles of statistical mechanics include:

- **Microstates** and **Macrostates**: In statistical mechanics, a system's microstate refers to the specific arrangement and energy distribution of its individual particles. A macrostate, on the other hand, represents the observable properties of the system, such as its temperature, pressure, and volume. The goal of statistical mechanics is to determine the probabilities of different microstates leading to a given macrostate.

- **Ensembles**: Statistical mechanics uses ensembles, which are collections of similar systems, to study statistical properties. Common ensembles include the microcanonical ensemble (isolated system with constant energy), canonical ensemble (system in thermal contact with a heat reservoir), and grand canonical ensemble (system with exchange of energy and particles with a heat reservoir).

- **Boltzmann Distribution**: The Boltzmann distribution relates the probabilities of different energy states to their corresponding energies and the system's temperature. It allows us to predict how particles are distributed among different energy levels in a system.

- **Entropy** and Entropy Maximization: Entropy is a fundamental concept in statistical mechanics, representing the measure of a system's disorder or randomness. The second law of thermodynamics states that isolated systems tend to evolve toward states of higher entropy. Statistical mechanics provides a statistical interpretation of entropy and explains the tendency of systems to maximize their entropy over time.

- **Statistical Thermodynamics**: Statistical mechanics connects with thermodynamics, relating the macroscopic thermodynamic properties (e.g., internal energy, temperature, and heat capacity) to the statistical properties of microscopic constituents. This connection allows us to derive thermodynamic quantities from the statistical behavior of particles.

Statistical mechanics has wide-ranging applications in various scientific disciplines, including physics, chemistry, biology, and materials science. It is essential for understanding phase transitions, chemical reactions, and the behavior of matter under different conditions. Additionally, statistical mechanics forms the basis for exploring complex systems, such as gases, liquids, and solids, and plays a crucial role in the development of many modern technologies.

Astrophysics

Astrophysics is a branch of astronomy that deals with the study of celestial objects, phenomena, and the physical processes that govern the universe. It combines principles of physics with observational data to understand the behavior, composition, and evolution of stars, galaxies, planets, and other cosmic entities. Key areas of study in astrophysics include:

- **Stellar Astrophysics**: This field focuses on the properties and life cycles of stars. It examines stellar formation, nuclear fusion processes that power stars, their structure, and the various stages of stellar evolution, including the final fate of stars as supernovae, white dwarfs, neutron stars, or black holes.
- **Galactic Astrophysics**: Galactic astrophysics explores the structure, dynamics, and evolution of galaxies, which are vast collections of stars, gas, dust, and dark matter. It investigates the formation of galaxies, their interactions, and the supermassive black holes at their centers.
- **Cosmology**: Cosmology studies the large-scale properties and evolution of the universe as a whole. It addresses questions about the universe's origin, its expansion, the distribution of galaxies, dark matter, dark energy, and the ultimate fate of the cosmos.
- **Exoplanets and Planetary Systems**: This area examines planets outside our solar system (exoplanets) and investigates planetary systems' formation and dynamics. It aims to find potentially habitable exoplanets and understand the diversity of planetary systems in the universe.
- **High-Energy Astrophysics**: High-energy astrophysics focuses on cosmic phenomena that involve extreme conditions, such as black hole accretion disks, active galactic nuclei, gamma-ray bursts, and cosmic rays. It explores high-energy emissions and their impact on the surrounding space.
- **Astrobiology**: Astrobiology is an interdisciplinary field that combines aspects of astronomy, biology, and chemistry to study the potential for life beyond Earth. It seeks to understand the conditions required for life to exist on other planets and moons in our solar system and beyond.

Astrophysicists use various observational and theoretical tools, such as telescopes, space missions, computer simulations, and mathematical models, to unravel the mysteries of the cosmos. Their research not only expands our knowledge of the universe's workings but also addresses fundamental questions about our place in the cosmos and the potential existence of life elsewhere in the universe.

Electromagnetism

Electromagnetism is a branch of physics that deals with the study of electric and magnetic fields and their interactions with charged particles and currents. It unifies two important phenomena: **electricity** and **magnetism**, which were initially considered separate until the 19th century. Key concepts and principles of electromagnetism include:

- **Electric Fields**: Electric fields are regions around charged particles or objects where electric forces can influence other charged particles. They are represented by vectors, indicating the direction and strength of the force experienced by a test charge placed in the field.
- **Magnetic Fields**: Magnetic fields arise due to moving charges or currents. They also have a vector nature and affect magnetic objects, such as magnets and magnetic materials. Magnetic fields are created in closed loops and can be visualized using field lines.
- **Electromagnetic Waves**: Electromagnetic waves are a form of energy propagation resulting from oscillating electric and magnetic fields. These waves do not require a medium for transmission and travel at the speed of light. Light itself is an electromagnetic wave.
- **Maxwell's Equations**: These are a set of four fundamental equations that form the basis of classical electromagnetism. They describe how electric and magnetic fields are generated by charges and currents and how these fields interact with one another.
- **Electromagnetic Induction**: Electromagnetic induction is the process where a changing magnetic field induces an electric field, leading to the generation of an electric current in a conductor. This phenomenon is the foundation of electric generators and transformers.
- **Electromagnetic Force**: Electric and magnetic forces are fundamental forces in nature. Charged particles experience electric forces due to their electric charges, while moving charges experience magnetic forces in the presence of magnetic fields.

Electromagnetism has extensive applications in modern technology and everyday life. It is the basis for the operation of electrical circuits, power generation, telecommunications, and numerous electrical devices. Moreover, it plays a vital role in the understanding of light and optics, electromagnetic radiation, and the behavior of charged particles in magnetic fields, making it one of the fundamental pillars of modern physics. The study of electromagnetism is crucial for advancing technology and our understanding of the natural world.

10.1 Maxwell's Equations

10.1.1 Derivation of The Speed Of Light c In Vacuum

We begin with Maxwell's equations which follow in the MKS system of units [14] as

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \tag{10.1.1}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{10.1.2}$$

$$\nabla \cdot \vec{B} = 0 \tag{10.1.3}$$

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t} \tag{10.1.4}$$

In the above equations, t is time, ϵ_0 is the permittivity of free space, μ_0 is the permeability of free space, $\vec{E}$ is the electric field, $\vec{B}$ is the magnetic field, and $\vec{J}$ is the current density. The definition of current is given by $\vec{I} \equiv \frac{dQ}{dt}\hat{I}$ which allows us to represent the current density as

$$\vec{J} = \frac{d\vec{I}}{da_\perp} = \frac{d}{da_\perp}\left(\frac{dQ}{dt}\right)\hat{I}. \tag{10.1.5}$$

Now, we can take the curl of equation (10.1.2) which gives

$$\nabla \times (\nabla \times \vec{E}) = \nabla \times \left(-\frac{\partial \vec{B}}{\partial t}\right). \tag{10.1.6}$$

The left-hand side of this can be manipulated using the vector identity

$$\nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}. \tag{10.1.7}$$

The right-hand side can also be rearranged since ∇ is not an operation with respect to time which gives

$$\nabla \times (\nabla \times \vec{E}) = \nabla(\nabla \cdot \vec{E}) - \nabla^2 \vec{E} = \nabla \times \left(-\frac{\partial \vec{B}}{\partial t}\right) = -\frac{\partial}{\partial t}(\nabla \times \vec{B}). \tag{10.1.8}$$

We can use equation (10.1.1) to write this as

$$\nabla\left(\frac{\rho}{\epsilon}\right) - \nabla^2 \vec{E} = -\frac{\partial}{\partial t}(\nabla \times \vec{B}). \tag{10.1.9}$$

If we assume we are in a perfect vacuum, then there would contain no matter and thus no charge. Therefore the charge density would be $\rho = 0$. This implies the total charge is zero and thus $\frac{dQ}{dt} = 0$. Hence, we can clearly see from equation (10.1.5) that within a vacuum $\vec{J} = 0$. Using these results as well as (10.1.4), we can write equation (10.1.9) as

$$-\nabla^2 \vec{E} = -\frac{\partial}{\partial t}\left(\epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}\right) \implies \nabla^2 \vec{E} = \epsilon_0 \mu_0 \frac{\partial^2 \vec{E}}{\partial t^2}. \tag{10.1.10}$$

Similarly, the magnetic field can be shown to satisfy the same relationship as the electric field. This result in equation (9) can be recognized as the wave equation which is generally in the form

$$\nabla^2 \vec{\psi} = \frac{1}{v^2}\frac{\partial^2 \vec{\psi}}{\partial t^2}, \tag{10.1.11}$$

where v is the velocity of a wave. Finally, if we think of the electric and magnetic fields $\vec{E}$ as a wave moving through a vacuum together, then we can determine it's velocity by comparing equations (10.1.10) and (10.1.11). This is the speed of an electromagnetic wave (the speed of light) which gives

$$v = c = \frac{1}{\sqrt{\epsilon_0 \mu_0}}. \tag{10.1.12}$$

We can approximate this based on the values of $\epsilon_0 = 8.85418782 \times 10^{-12} s^4 A^2/(m^3 kg)$ and $\mu_0 = 4\pi \times 10^{-7} Wm$ which gives

$$c = \frac{1}{\sqrt{\epsilon_0 \mu_0}} = \boxed{299,792,458 \frac{m}{s}}. \tag{10.1.13}$$

10.2 The Zeeman Effect

Note. This entire section was copied from some notes I had from 2018.

The **Zeeman effect** is an effect that comes about from putting an atom in a magnetic field. The magnetic field causes a magnetic moment to be produced and perturbs the energy levels of the atom causing spectral line shifts to appear when modeled or measured with spectroscopy.

10.2.1 Hydrogen Atom First Order Magnetic Field Energy Corrections

To begin, the unperturbed wave functions of hydrogen is given by

$$H^{(0)} = \underbrace{\frac{\vec{p}^2}{2m}}_{\text{Kinetic term}} - \underbrace{\frac{\hbar c\alpha}{r}}_{\text{Potential term}} . \tag{10.2.1}$$

Consider the atom now in a magnetic field $\vec{B}$. The electrons spin will yield a magnetic moment $\vec{\mu}_S = \frac{-eg_s}{2m}\vec{S}$. The energy interaction of a single magnetic moment $\vec{\mu}$ with a fixed external magnetic field is given by $H^{\mu} = -\mu \cdot \vec{B}$ and so the Hamiltonian change due to this effect is $H_S^{(1)} = -\mu_s \cdot \vec{B} = \frac{eg_s}{2m}\vec{S} \cdot \vec{B}$. There is also a correction due to the angular momentum of the electron. If we assume that the electron is moving in a circle, then this has a magnetic dipole $\mu_L = \frac{-e}{2m}\vec{L}$ and so there exists another correction to the Hamiltonian $H_L^{(1)} = -\mu_L \cdot \vec{B} = \frac{eg_L}{2m}\vec{L} \cdot \vec{B}$. Now, g_s is a bi-products of the Dirac equation has a value of $g_s = s$. Combining these terms together give the total Zeeman effect to the Hamiltonian $H^{(1)}_{magnetic}$,

$$H^{(1)}_{magnetic} \equiv H^{(1)}_{mag} = \frac{e}{2m}(\vec{L} + 2\vec{S}) \cdot \vec{B} = \frac{\mu_B}{\hbar}(\vec{L} + 2\vec{S}) \cdot \vec{B}, \tag{10.2.2}$$

where $\mu_B = \frac{e\hbar}{2m}$ is the Bohr magneton.
If we are just considering the changes that come about from the Zeeman effect, we can ignore other Hamiltonian perturbations such as the fine structure and hyperfine structure interactions. Assume that $\vec{B} \equiv B\hat{z}$ then the Hamiltonian will only contain z components after performing the dot product giving $H^{(1)}_{mag} = \frac{\mu_B}{\hbar}(L_z + 2S_z)B$. Now, $\vec{J} = \vec{L} + \vec{S} \implies L_z = J_z - S_z$ and so $H^{(1)}_{mag} = \frac{\mu_B}{\hbar}(J_z + S_z)B$. From perturbation theory we can solve for the first order energy corrections to this using

$$\begin{aligned} E_n^{(1)} &= \langle njm_j\ell s|H^{(1)}_{mag}|njm_j\ell s\rangle \\ &= \frac{\mu_B B}{\hbar}\langle njm_j\ell s|(J_z + S_z)|njm_j\ell s\rangle \\ &= \frac{\mu_B B}{\hbar}\left(\hbar m_j + \langle njm_j\ell s|S_z|njm_j\ell s\rangle\right). \end{aligned} \tag{10.2.3}$$

We can determine S_z by projecting $\vec{S}$ onto $\vec{J}$ which gives $\vec{S} = \frac{\vec{J}\cdot\vec{S}}{\vec{J}^2}\vec{J}$ and then using the substitution that

$$\vec{J} \cdot \vec{S} = (\vec{L} + \vec{S}) \cdot \vec{S} = \vec{L} \cdot \vec{S} + \vec{S}^2 = \frac{1}{2}(\vec{J}^2 - \vec{L}^2 - \vec{S}^2) + \vec{S}^2 = \frac{1}{2}(\vec{J}^2 - \vec{L}^2 + \vec{S}^2). \tag{10.2.4}$$

Thus, we have

$$S_z = \frac{(\vec{J}^2 - \vec{L}^2 + \vec{S}^2)}{2\vec{J}^2}J_z, \tag{10.2.5}$$

and so the undetermined matrix element in 10.2.3 becomes

$$\begin{aligned}\langle njm_j\ell s|S_z|njm_j\ell s\rangle &= \langle njm_j\ell s|\frac{(\vec{J}^2-\vec{L}^2+\vec{S}^2)}{2\vec{J}^2}J_z|njm_j\ell s\rangle \\ &= m_j\hbar\frac{j(j+1)-\ell(\ell+1)+s(s+1)}{2j(j+1)}.\end{aligned} \tag{10.2.6}$$

Plugging this back into 10.2.3 gives a general result

$$\boxed{E_n^{(1)} = \mu_B B m_j\left[1+\frac{j(j+1)-\ell(\ell+1)+s(s+1)}{2j(j+1)}\right] = \mu_B B m_j g_L}, \tag{10.2.7}$$

where g_L is the Landé g factor

$$g_L = 1+\frac{j(j+1)-\ell(\ell+1)+s(s+1)}{2j(j+1)}. \tag{10.2.8}$$

Note that for the hydrogen atom such as we have, the spin $s=\frac{1}{2}$ so we can further simplify this to

$$E_n^{(1)} = \mu_B B m_j\left[1+\frac{j(j+1)-\ell(\ell+1)+\frac{3}{4}}{2j(j+1)}\right]. \tag{10.2.9}$$

10.2.2 Example Energy Correction: $2P_{3/2}$

This result in 10.2.7 can be used for an explicit state of the hydrogen atom. For example, consider the $2P_{3/2}$ state. This has $n=2$, $\ell=1$, $s=\frac{1}{2}$, and $j=\frac{3}{2}$. The m_j values can thus be determined by the set $m_j=\{j+k|-j\le m_j\le j \text{ and } k\in\mathbb{Z}\}=\{-\frac{3}{2},-\frac{1}{2},\frac{1}{2},\frac{3}{2}\}$, which tells us there will be 4 different energy splits for this state. Using these values gives us a $g_L=\frac{4}{3}$ and so our energy splits are

$$E^{(1)}_{2P_{3/2}} = \begin{cases} 2\mu_B B & \text{for } m_j=\frac{3}{2} \\ \frac{2\mu_B B}{3} & \text{for } m_j=\frac{1}{2} \\ -\frac{2\mu_B B}{3} & \text{for } m_j=-\frac{1}{2} \\ -2\mu_B B & \text{for } m_j=-\frac{3}{2} \end{cases} \tag{10.2.10}$$

10.2.3 Linearly Changing Magnetic Field

Consider Maxwell's Equations, which are the system of partial differential equations describing classical electromagnetism. $\vec{P}$ is the polarization field, $\vec{D}$ is the electric displacement field, ρ is the charge density, $\vec{E}$ is the electric field, $\vec{B}$ is the magnetic field, and $\vec{J}$ is the current density. In the MKS system of units (where ϵ_0 is the permittivity of free space and μ_0 is the permeability of free space), the equations are written

$$\nabla\cdot\vec{E} = \frac{\rho}{\epsilon_0} \tag{10.2.11}$$

$$\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \tag{10.2.12}$$

$$\nabla\cdot\vec{B} = 0 \tag{10.2.13}$$

$$\nabla\times\vec{B} = \mu_0\vec{J}+\epsilon_0\mu_0\frac{\partial\vec{E}}{\partial t} \tag{10.2.14}$$

It may seem reasonable to look at the Zeeman effect with regards to a linear magnetic field since that would be a simple case. However should we model this as a linear $\vec{B}$ with regard to position or linear $\vec{B}$ with regard

to time? If we allow $\vec{B}_{linear,z} \equiv Bz\hat{z}$, then we can see there is a violation of 10.2.13. This can be easily seen as $\nabla \cdot Bz\hat{z} = B \neq 0$ for any $b \neq 0$. This is an impossible magnetic field.
Consider $\vec{B} = Bt\hat{z}$. The divergence is zero since it has no spacial dependence which satisfies 10.2.13. Similarly, the curl is zero for the same reason. If we assume we are within a vacuum (other than our single hydrogen atom we are modeling), then by 10.2.14, we have $\frac{\partial \vec{E}}{\partial t} = 0$, which implies the electric field is a constant throughout time. By 10.2.12, the electric field must satisfy $\nabla \times \vec{E} = -B\hat{z}$. If we take the curl of both sides of this we get

$$\nabla \times (\nabla \times \vec{E}) = \nabla(\nabla \cdot \vec{E}) - \nabla^2 \vec{E} = -\nabla \times B\hat{z} = 0. \tag{10.2.15}$$

Using our assumption of a vacuum and 10.2.11, this becomes

$$\nabla^2 \vec{E} = 0 \implies \frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_y}{\partial y^2} + \frac{\partial^2 E_z}{\partial z^2} = 0 \tag{10.2.16}$$

Consider a one dimensionally changing electric field with spacial dependence and no time dependence. This could potentially be produced in a lab so it is a good example. We can see a trivial solution to this would be $\vec{E}_1 = (ax + by + cz)\hat{x}$, where a, b, c are some constants. By 10.2.12, we know

$$c\hat{y} - b\hat{z} = -B\hat{z} \implies b \equiv B \text{ and } c \equiv 0 \implies \vec{E}_1 = By\hat{x}.. \tag{10.2.17}$$

We can see that a has no requirements and so it can be arbitrary. For simplicity we can make $a = 0$. In fact, upon a closer look, $a = 0$ is the only possible choice in order for 10.2.11 to remain satisfied. So thus from out magnetic field $\vec{B} = Bt\hat{z}$, we require that there also be an electric field $\vec{E}_1 = By\hat{x}$ in order for Maxwell's equations to be satisfied. Let us double check quickly.

$$\text{(10.2.11)}: \ \nabla \cdot \vec{E}_1 = \nabla \cdot By\hat{x} = \frac{\partial}{\partial x} By = 0 \qquad \checkmark \tag{10.2.18}$$

$$\text{(10.2.13)}: \ \nabla \cdot \vec{B} = \nabla \cdot Bt\hat{z} = \frac{\partial}{\partial z} Bt = 0 \qquad \checkmark \tag{10.2.19}$$

$$\text{(10.2.12)}: \ \nabla \times \vec{E}_1 = \nabla \times By\hat{x} = -B\hat{z} = -\frac{\partial \vec{B}}{\partial t} \qquad \checkmark \tag{10.2.20}$$

$$\text{(10.2.14)}: \ \nabla \times \vec{B} = \nabla \times Bt\hat{z} = 0 = \mu\epsilon_0 \frac{\partial \vec{E}}{\partial t} \qquad \checkmark \tag{10.2.21}$$

Does this solution make sense physically? We have a magnetic field changing uniformly through time and it creates an electric field that is linearly increasing in y space in the $\hat{x}$ direction. By analogy we can also find another solution to $\vec{E}$ by assuming $\vec{E}_2 = (ax + by + cz)\hat{y}$ and then by a method analogous to 10.2.17, we get $\vec{E}_2 = -Bx\hat{y}$. And so by superposition we can also have a solution

$$\vec{E}_{1+2} = B(y\hat{x} - x\hat{y}). \tag{10.2.22}$$

This electric field as well as the magnetic field given both satisfy the wave equation as they should. This result shows that with a changing magnetic field, we would see an electric field appear which could itself cause another corrections to the perturbed energies. This is known as the Stark Effect and in our case would introduce the correction

$$H^{(1)}_{elec} = -e\vec{r} \cdot \vec{E} = 0. \tag{10.2.23}$$

If we plot 10.2.22 using Mathematica we can see that it's a radial function around the z axis where our magnetic field is changing, which makes sense for it to be radially dependent. However, if we notice, it is increasing with r, which means it is divergent and thus does not make sense physically (see Figure 1).

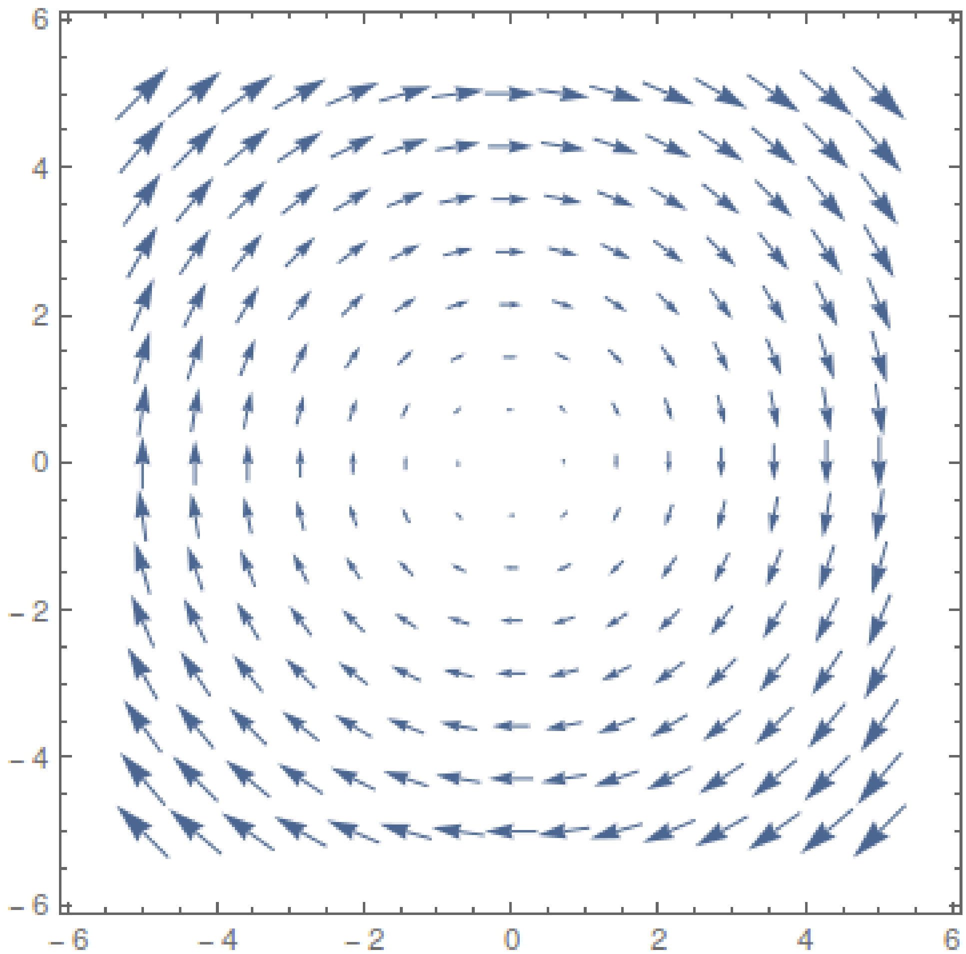

Figure 1. A plot of $\vec{E}_{1+2}$ using B=1 in the x-y plane.

From this example we can see that we cannot just guess a solution to 10.2.16. Instead we can find a general solution and impose boundary conditions in order to find an electric field that will physically work and make sense in our situation.

Hypothesis. I suspect the possible inconsistency above comes about from assuming we are in a vacuum if not, perhaps a mistake in a calculation somewhere.

10.2.4 Landé g Analysis

Consider the Lande é g factor derived previously in 10.2.8 with a spin value of $s = \frac{1}{2}$,

$$g_L = \frac{12j(j+1) - 4\ell(\ell+1) + 3}{8j(j+1)}. \tag{10.2.24}$$

For our system, the j value is given by $j = \ell \otimes s = \ell \pm \frac{1}{2}$. Thus, we can say

$$g_L = \begin{cases} \frac{2+2l}{1+2\ell} & \text{for } j = l + \frac{1}{2} \\ \frac{2l}{1+2\ell} & \text{for } j = l - \frac{1}{2} \end{cases}. \tag{10.2.25}$$

Now, $m_j = \{j + k| - j \leq m_j \leq j \text{ and } k \in \mathbb{Z}\}$ which can be expressed in terms of ℓ as

$$m_j = \{\ell \pm \frac{1}{2} + k| - \ell \leq m_j \pm \frac{1}{2} \leq \ell \pm 1 \text{ and } k \in \mathbb{Z}\}. \tag{10.2.26}$$

Using these two forms of m_j and g_L, we can write out energy corrections in terms of a single variable as

$$E_n^{(1)} = \mu_B B \begin{cases} \frac{2+2l}{1+2\ell}(\ell + \frac{1}{2} + k) & \text{for } -\ell \leq \ell + 1 + k \leq \ell + 1 \text{ and } k \in \mathbb{Z} \\ \frac{2l}{1+2\ell}(\ell - \frac{1}{2} + k) & \text{for } -\ell \leq \ell - 1 + k \leq \ell - 1 \text{ and } k \in \mathbb{Z} \end{cases}. \tag{10.2.27}$$

Notice that $k < 0$ for both cases due to how we defined m_j. We can make k positive by flipping the sign of it and notice that then k must range from 0 to $2\ell + 1$ and $2\ell - 1$ depending on the case, and so this becomes

$$\boxed{E_n^{(1)} = \mu_B B \begin{cases} \frac{2+2l}{1+2\ell}(\ell + \frac{1}{2} - k) & \text{for } 0 \leq k \leq 2\ell + 1 \text{ and } j = \ell + \frac{1}{2} \\ \frac{2l}{1+2\ell}(\ell - \frac{1}{2} - k) & \text{for } 0 \leq k \leq 2\ell - 1 \text{ and } j = \ell - \frac{1}{2} \end{cases}}. \tag{10.2.28}$$

By plugging in $\ell = 1$ we get the same energy values as earlier in 10.2.10. Now we have a general formula for the corrected energy levels that we can plot for any ℓ value. for spin $s = \frac{1}{2}$ hydrogen.

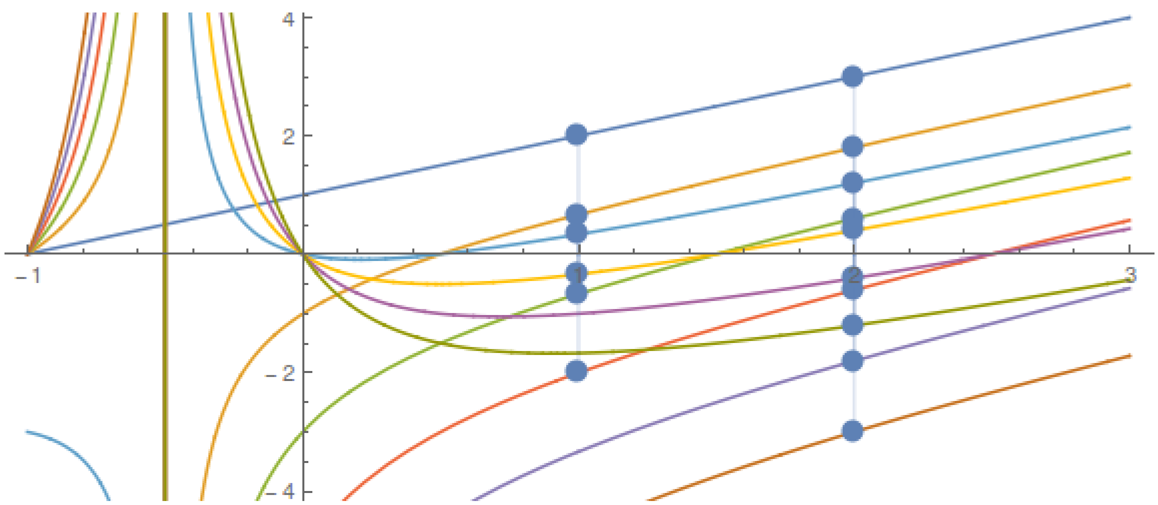

Figure 1. Relative hydrogen Energy corrections for $\ell = 1$ and $\ell = 2$ with units of $\mu_B = B = 1$ modeled using the derived energy correction formula over all real values.

Quantum Mechanics

Quantum mechanics is a fundamental branch of physics that describes the behavior of matter and energy at the smallest scales, such as subatomic particles and photons. It provides a unique and revolutionary framework for understanding the peculiar and counterintuitive behavior of particles at the quantum level. Key concepts and principles of quantum mechanics include:

- **Wave-Particle Duality**: One of the central tenets of quantum mechanics is the wave-particle duality. It states that particles, such as electrons and photons, exhibit both particle-like and wave-like characteristics. They can be described by wave functions, which represent probabilities of finding a particle at different locations.
- **Quantization of Energy**: Quantum mechanics introduced the concept of quantized energy levels, where energy levels of particles are restricted to discrete values rather than continuous values. This is exemplified in the energy levels of electrons in an atom, resulting in the discrete emission and absorption of photons.
- **Uncertainty Principle**: The Heisenberg uncertainty principle states that it is impossible to simultaneously know both the position and momentum of a particle with absolute precision. The more accurately one quantity is known, the less precisely the other can be determined. This fundamental limitation is inherent in quantum mechanics.
- **Quantum Superposition**: Quantum systems can exist in a state of superposition, where they are in multiple states simultaneously. For example, an electron can exist in a superposition of spin-up and spin-down states until measured, at which point it collapses into one definite state.
- **Quantum Entanglement**: Quantum entanglement is a phenomenon where the properties of two or more particles become correlated in such a way that the state of one particle is directly related to the state of another, regardless of distance. This has profound implications for quantum information and potential applications in quantum computing.

  Quantum Mechanics and Measurement: The process of measurement in quantum mechanics is non-deterministic. Upon measurement, the system's wave function collapses to a specific state corresponding to the observed measurement outcome, introducing inherent randomness into quantum events.

Quantum mechanics has revolutionized our understanding of the subatomic world and is the foundation for modern technologies such as transistors, lasers, and MRI machines. While it has proven to be highly successful in describing the behavior of particles on a small scale, it also challenges our classical intuition and raises profound philosophical questions about the nature of reality and our perception of the universe.

Relativity

Relativity refers to two groundbreaking theories developed by Albert Einstein: **Special Relativity** and **General Relativity**. Both theories revolutionized our understanding of the universe and how it functions, especially at high speeds and in the presence of strong gravitational fields.

- **Special Relativity**: Introduced in 1905, Special Relativity deals with the behavior of objects moving at constant velocities, especially near the speed of light. It is based on two postulates: the principle of relativity (laws of physics are the same for all inertial observers) and the constancy of the speed of light in a vacuum. Key principles of Special Relativity include:
  - **Time Dilation**: Moving clocks appear to run slower compared to stationary clocks from the perspective of an observer at rest.
  - **Length Contraction**: Moving objects appear shorter in the direction of motion relative to a stationary observer.
  - **Mass-Energy Equivalence**: $E = mc^2$, where "E" is energy, "m" is mass, and "c" is the speed of light. This famous equation shows that mass and energy are interchangeable.
- **General Relativity**: Formulated in 1915, General Relativity is a theory of gravity that describes the curvature of spacetime caused by the presence of mass and energy. Unlike Newtonian gravity, which considers gravity as an attractive force between masses, General Relativity attributes gravity to the curvature of spacetime caused by massive objects. Key principles of General Relativity include:
  - **Curved Spacetime**: Massive objects like stars and planets curve the fabric of spacetime around them, and other objects move along the curved paths in response to this curvature.
  - **Gravitational Time Dilation**: Clocks in stronger gravitational fields (e.g., near massive objects) run slower compared to clocks in weaker fields.
  - **Gravitational Waves**: General Relativity predicts the existence of gravitational waves, ripples in spacetime caused by violent cosmic events.

Both Special and General Relativity have been confirmed through numerous experiments and observations, and they have far-reaching implications for our understanding of the cosmos. Special Relativity's effects become significant at high speeds, near the speed of light, while General Relativity is essential for describing gravity and the behavior of massive objects, such as stars, galaxies, and black holes. These theories have not only revolutionized fundamental physics but have also influenced technology, astronomy, and our perception of the nature of space, time, and the universe.

12.1 Special Relativity

12.1.1 Time Dilation and Length Contraction

To begin, we start with Einsteins postulates which are given as

1. The laws of physics must always hold in all inertial reference frames.
2. The speed of light in a vacuum c is the same in all reference frames.

From this, we can explore the implications and effects of these assumptions. First, consider a frame[1] of reference $\mathcal{F}'$. Suppose we set up two mirrors in this frame that are facing each other and have a beam of light traveling back and forth between them. Let these mirrors be aligned such that the beam of light is traveling vertically and assume we are in a vacuum. If we set $t = 0$ as the point when the light beam hits the bottom mirror, then the time it takes for the light to travel to the top mirror can be determined simply by the distance formula

$$v = \frac{\Delta x}{\Delta t} \implies \Delta t = \frac{\Delta x}{v}. \tag{12.1.1}$$

Since we are in a vacuum, the speed of our light is simply c and thus we have $\Delta t' = \frac{\Delta x'}{c}$.
Now consider an inertial frame $\mathcal{F}$ such that the first frame is moving horizontally relative to it. The light will remain traveling between these two mirrors, only the observer in frame $\mathcal{F}$ will see the light traveling both horizontally and vertically (since $\mathcal{F}'$ is moving horizontally as seen in $\mathcal{F}$). Say the speed that $\mathcal{F}'$ is moving as seen from $\mathcal{F}$ is v. Assuming no acceleration, since light must travel at a constant speed in our vacuum, the distance the light travels from the bottom mirror to the top mirror as observed in $\mathcal{F}$ is longer than that of $\mathcal{F}'$. The vertical distance between the two plates is still $\Delta x'$, but now there is also a horizontal distance. Suppose it takes Δt for the light to hit the top mirror in $\mathcal{F}$, then the horizontal distance that the light traveled is given by $\Delta x_h = v\Delta t$ and so the total distance the light traveled in $\mathcal{F}$ is

$$\Delta x = \sqrt{(\Delta x')^2 + (\Delta x_h)^2} = \sqrt{(\Delta x')^2 + (v\Delta t)^2}. \tag{12.1.2}$$

Now, by definition, $\Delta x = c\Delta t$ and so if we combine this with (12.1.1) and (12.1.2) we have the equation for time dilation which is

$$(c\Delta t)^2 = (c\Delta t')^2 + (v\Delta t)^2 \implies \boxed{\Delta t = \frac{\Delta t'}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}}. \tag{12.1.3}$$

Now, we can take (12.1.1) and use it to relate $\Delta x'$ and Δx. First, notice that $\Delta x' = c\Delta t'$ and $\Delta x = c\Delta t$. Solving for c and setting these equal to each other gives

$$\frac{\Delta x'}{\Delta t'} = \frac{\Delta x}{\Delta t} \implies \frac{\Delta x'}{\Delta x} = \frac{\Delta t}{\Delta t'}. \tag{12.1.4}$$

Combining this with (12.1.3) will then give us the relationship for length contraction between reference frames which is

$$\frac{\Delta x'}{\Delta x} = \frac{\Delta t}{\Delta t'} \implies \Delta x = \boxed{\Delta x' \sqrt{1 - \left(\frac{v}{c}\right)^2}}. \tag{12.1.5}$$

The factor of $\sqrt{1 - \left(\frac{v}{c}\right)^2}$ appears often within relativity and is generally denoted γ^{-1}. Therefore these equations can also be written as $\Delta t = \gamma \Delta t'$ and $\Delta x = \frac{\Delta x'}{\gamma}$.

12.1.2 Lorentz Coordinate Transformations

Consider again two frames, $\mathcal{F}$ and $\mathcal{F}'$. Suppose we have $\mathcal{F}'$ moving at some speed v as observed in the $\mathcal{F}$ frame in the $\hat{x}$ direction. Suppose there is a bar moving along with frame $\mathcal{F}'$ with length L. After some time, the bar will have moved a distance of vt and thus the new location will be at $x = L + vt \implies L = x - vt$.

[1] Note that $\mathcal{F}'$ is the "proper frame".

Now, by the length contraction we derived previously, this length can be related to the length in it's frame by $L' = \frac{L}{\gamma}$ and so we will have $L' = \gamma(x - vt)$. Now, say at $t' = 0$ that $x' = x$, and so we can set the position of the end of the bar at $x' = L'$ and thus since in the prime frame the position of the bar does not change it remains that $x' = L'$ after some time t and so we have $x' = \gamma(x - vt)$. Notice that if there is no velocity ($v = 0$), this becomes $x' = x$ and so our case, the respective y and z components become $y = y'$ and $z = z'$. These are all of the position transformations contained within the Lorentz coordinate Transformations.

$$\boxed{x' = \gamma(x - vt) \qquad\qquad y = y' \qquad\qquad z = z'} \tag{12.1.6}$$

We can now take the above position change in the x direction and derive the Lorentz transformation for time. Consider some change in position while moving at constant velocity in the prime frame

$$\Delta x' = x'_f - x'_i = \gamma(x_f - vt_f) - \gamma(x_i - vt_i) = \gamma(\Delta x - v\Delta t). \tag{12.1.7}$$

Dividing this by $\Delta t'$ and applying (12.1.4) yields

$$\frac{\Delta x'}{\Delta t'} = \frac{\gamma}{\Delta t'}(\Delta x - v\Delta t) = \frac{\Delta x}{\Delta t} \implies \Delta t' = \gamma\left(\Delta t - \frac{v\Delta t^2}{\Delta x}\right). \tag{12.1.8}$$

Since $\frac{Deltax}{\Delta t} = c$ (from our derivations of time dilation), we can square this and use some manipulation to get $\frac{Deltax^2}{\Delta t^2} = c^2 \implies v\Delta t^2 = \frac{v\Delta x^2}{c^2}$. Plugging this into (12.1.8) gives us

$$\Delta t' = \gamma\left(\Delta t - \frac{v\Delta x}{c^2}\right) \implies t'_f - t'_i = \gamma\left(t_f - t_i - \frac{v(x_f - x_i)}{c^2}\right) \tag{12.1.9}$$

$$\implies t'_f - t'_i = \gamma\left(t_f - \frac{vx_f}{c^2}\right) - \gamma\left(t_i - \frac{vx_i}{c^2}\right). \tag{12.1.10}$$

Examining this result allows us to clearly see a solution to any arbitrary time transformation that will be consistent with the above formulations and that result is the Lorentz time transformation. This is given by

$$\boxed{t' = \gamma\left(t - \frac{vx}{c^2}\right)}. \tag{12.1.11}$$

Quantum Field Theory

Quantum Field Theory (QFT) is a theoretical framework that combines the principles of quantum mechanics with special relativity to describe the behavior of particles as quantized fields. It is one of the fundamental theories in modern theoretical physics and forms the basis for understanding particle interactions at both the microscopic and cosmological scales. Key aspects and principles of Quantum Field Theory include:

- **Quantized Fields**: In QFT, particles are described not as individual discrete entities but as quantized fields that pervade all of space and time. Each particle type corresponds to a specific field, and particles are represented as excitations or quanta of these fields.
- **Lagrangian Formalism**: QFT employs a Lagrangian formalism to construct the equations of motion and interactions for the fields. The Lagrangian describes the dynamics and symmetries of the system and allows for the derivation of fundamental equations, such as the equations of motion and scattering amplitudes.
- **Creation and Annihilation Operators**: QFT introduces creation and annihilation operators to describe particle creation and annihilation processes. These operators act on the field states to generate or destroy particles and provide a mathematical framework for quantized states.
- **Feynman Diagrams**: Feynman diagrams are a visual tool used in QFT to represent particle interactions and scattering processes. They provide a pictorial representation of complex particle interactions and are instrumental in calculating scattering amplitudes.
- **Renormalization**: Similar to Quantum Electrodynamics (QED), QFT encounters infinities in certain calculations. Renormalization is a method to remove these infinities and obtain finite, meaningful predictions.
- **Gauge Theories**: QFT includes gauge theories, which describe interactions involving force-carrying particles (gauge bosons). Examples of gauge theories include Quantum Electrodynamics (QED) and Quantum Chromodynamics (QCD).
- **Quantum Electrodynamics** (QED): QED is a specific example of Quantum Field Theory that describes the electromagnetic force and its interactions with charged particles through quantized electromagnetic fields and photons.

Quantum Field Theory has proven to be highly successful and is an essential framework for describing and understanding the behavior of elementary particles and their interactions. It is the foundation for the Standard Model of particle physics, which provides a comprehensive description of the known elementary particles and their interactions, except for gravity. Additionally, QFT plays a crucial role in studying fundamental questions about the nature of matter, energy, and the universe at the most fundamental level.

Quantum Electrodynamics

Quantum Electrodynamics (QED) is a quantum field theory that describes the electromagnetic force and its interactions with charged particles, such as electrons and photons. It is considered one of the most successful and accurate scientific theories ever developed and is a cornerstone of modern particle physics. Key aspects and principles of Quantum Electrodynamics include:

- **Quantization of Electromagnetic Fields**: QED treats the electromagnetic field as a quantum field, where photons are the quantized particles representing the discrete packets of electromagnetic energy.
- **Feynman Diagrams**: Feynman diagrams are a powerful tool used in QED to visualize and calculate particle interactions. They depict the exchange of photons between charged particles, enabling precise predictions of scattering processes and particle interactions.
- **Renormalization**: QED encounters infinities in certain calculations due to the self-interactions of charged particles with their own electromagnetic fields. Renormalization is a technique used to remove these infinities, allowing meaningful and accurate predictions.
- **Quantum Loops**: In QED, particles can interact with each other through quantum loops, where virtual particles (e.g., virtual photons) briefly pop into existence and influence the interactions between charged particles.
- **Gauge Invariance**: QED exhibits gauge invariance, meaning that different mathematical representations of the theory lead to physically equivalent results. This property ensures that observable quantities are independent of the choice of mathematical description.
- **Electron Self-Energy**: QED accounts for the electron's self-energy, which arises from its interaction with its own electromagnetic field. This self-energy correction leads to subtle shifts in electron properties, such as its mass and magnetic moment.

Quantum Electrodynamics has been extensively tested through precise experiments and is one of the most accurate and well-validated theories in physics. It successfully explains a wide range of phenomena, including electromagnetic interactions between charged particles, the behavior of light and photons, and the fine structure of atomic spectra. Moreover, QED has been instrumental in the development of other quantum field theories, such as Quantum Chromodynamics (QCD) and the electroweak theory, which describe the strong and weak nuclear forces.

Quantum Chromodynamics

Quantum Chromodynamics (QCD) is a fundamental theory in particle physics that describes the strong nuclear force, which is responsible for holding quarks together to form protons, neutrons, and other hadrons. QCD is a quantum field theory and is an essential component of the Standard Model, which describes the known elementary particles and their interactions. Key aspects and principles of Quantum Chromodynamics include:

- **Quarks** and **Gluons**: QCD postulates the existence of quarks, which are elementary particles that come in six flavors (up, down, charm, strange, top, and bottom). Quarks are bound together by exchanging gluons, which are the force-carrying particles of the strong nuclear force.
- **Color Charge**: In QCD, quarks carry a property called "color charge," which is analogous to electric charge in electromagnetism but comes in three types: red, green, and blue (plus their corresponding anticolors). Gluons, which also carry color charge, mediate the strong force between quarks.
- **Asymptotic Freedom and Confinement**: QCD exhibits two remarkable phenomena. At high energies or short distances, the strong force weakens, a property known as "asymptotic freedom." This allows for precise calculations in perturbation theory. However, at low energies or long distances, the force becomes strong, and quarks are confined within hadrons, making isolated quarks inaccessible in nature.
- **Lattice QCD**: Because QCD becomes strongly coupled at low energies, direct calculations become challenging. Lattice QCD is a numerical approach that uses a discrete space-time lattice to perform non-perturbative calculations of QCD phenomena.
- **Hadronization** and **Hadron** Structure: QCD governs the process of hadronization, where quarks and gluons combine to form color-neutral hadrons (e.g., mesons and baryons). QCD also provides insights into the structure and properties of hadrons, such as their masses and decay rates.
- **Strong Interactions** in Particle Colliders: QCD is crucial in understanding the strong interactions observed in high-energy particle colliders, where quarks and gluons are produced in energetic collisions. The study of these interactions helps test the predictions of QCD and explore new physics.

Quantum Chromodynamics is a fundamental theory in the Standard Model, working in conjunction with Quantum Electrodynamics (QED) and the Electroweak Theory to describe the behavior of elementary particles and their interactions. The study of QCD has led to significant advances in our understanding of the strong force, the nature of matter, and the dynamics of quarks and gluons within hadrons. QCD is also a subject of ongoing research and remains a key area of exploration in particle physics.

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Index