ANTONIUS' HANDBOOK

Useful Formulas, Constants, Units and Definitions
Version 2.001

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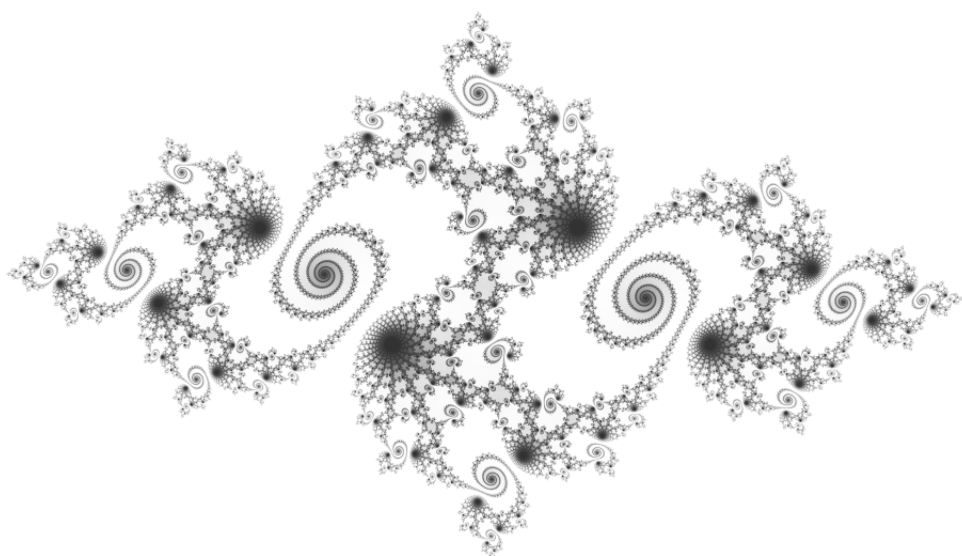

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Preface

This document is a compilation of useful formulas, definitions, constants, and general information used throughout my own schooling as a reference while furthering education. It's purpose is to provide a complete 'encyclopedia' per say of various mathematical and significant ideas used often. The idea and motivation behind it is to be a quick reference providing easily accessible access to necessary information for either double checking or recalling proper formula for use in various situations due to my own shortcomings in matters of memorization. All the material in this document was either directly copied from one of the references listed at the end or derived from scratch. On occasion typos may exist due to human error but will be corrected when discovered.

The version number is updated every time the document is distributed, printed, or referred to. This ensures that there is no two copies with different information and similar version numbers. The latest update date is automatically set to the current date each time the document is edited. Please refrain from distributing this handbook without permission from the original author/compiler. As of version 1.035 this book is formatted for printing.

Courses Covered In This Book

This document encompasses a large portion of formula used throughout specific courses at Michigan state University. The courses which have information pertaining to something in this book are more than just listed below; however, below is a list of classes that the author took whilst compiling the information in this book. All course numbers correspond to Michigan State University courses at the time of adding them.

- AST 207/208/304: Astrophysics I/II/III
- PHY 215: Thermodynamics & Modern Physics
- MTH 310: Abstract Algebra/Number Theory
- PHY 321: Classical Mechanics I
- PHY 410: Thermal & Statistical Physics
- PHY 415: Methods Of Theoretical Physics
- PHY 440: Electronics
- PHY 471/472: Quantum Physics I/II
- PHY 481/482: Electricity and Magnetism I/II
- PHY 492: Introduction to Nuclear Physics

The information in this book is in no way limited to the material used within the courses above. They serve as a simple guideline to what you will find within this document. For more information about this book or details about how to obtain your own copy please visit:

https://msu.edu/~torodean/AHandbook.html

Disclaimer

This book contains formulas, definitions, and theorems that by nature are very precise. Due to this, some of the material in this book was taken directly from other sources such as but not limited to Wolfram Mathworld. This is only such in cases where a change in wording could cause ambiguities or loss of information quality. Following this, all sources used are listed in the references section.

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(Yes, this is a contradiction.)

Contents

Constants and units

0.1: Physical Constants

Constant	Symbol	Value	Units
Speed of light in a vacuum	c	2.99792458×10^8	m/s
Elementary charge	e	$1.602176565(35) \times 10^{-19}$	C
Gravitational constant	G	$6.67384(80) \times 10^{-11}$	$\mathrm{m^3kg^{-1}s^{-2}}$
Avagadro's number	N_a	$6.02214129(27) \times 10^{23}$	mol$\cdot s^{-1}$
Planck constant	h	$6.62606872(52) \times 10^{-34}$	J·s
		4.135668×10^{-15}	eV·s
	hc	1239.84	eV·nm
Reduced planck constant	$\hbar \equiv h/2\pi$	1.05×10^-34	J·s
Permittivity of the vacuum	ϵ_0	8.854×10^{-12}	$\mathrm{C^2N^{-1}m^{-2}}$
Permeability of the vacuum	μ_0	$4\pi \times 10^{-7}$	W·m
Boltzmann constant	k	$1.38064852 \times 10^{-23}$	J/K
		8.61733×10^{-5}	eV/K
Stefan-Boltzmann constant	σ_{SB}	$5.670367(13) \times 10^{-8}$	$\mathrm{W{\cdot}m^{-2}K^{-4}}$
Thomson cross-section	σ_e	6.652×10^{-29}	m^2
The Bohr Magneton	$\mu_B \equiv \frac{e\hbar}{2m}$	5.788×10^{-5}	eV/T
		9.274×10^{-24}	$\mathrm{Am^2}$
Mass of an electron	m_e	$9.10938291(40) \times 10^{-31}$	kg
		510.9989	keV/c^2
Mass of a proton	m_p	$1.6726218 \times 10^{-27}$	kg
		938.27203	MeV/c^2
Mass of a neutron	m_n	$1.6749274 \times 10^{-27}$	kg
		939.56536	MeV/c^2
Unified amu	u	$1.660538782 \times 10^{-27}$	kg
		931.494028	$\mathrm{MeV/c^2}$

0.2: Stellar Data

Spectral Type	T_{eff} (K)	$M/M.$	$L/L.$	$R/R.$	V_{mag}
O5	44,500	60	7.9×10^5	12	-5.7
B5	15,400	5.9	830	3.9	-1.2
A5	8,200	2.0	14	1.7	1.9
F5	6,440	1.4	3.2	1.3	3.4
G5	5,770	0.92	0.79	0.92	4.9
K5	4,350	0.67	0.15	0.72	6.7
M5	3,170	0.21	0.011	0.27	12.3

0.3: Astronomical Constants

Constant	Symbol	Value	Units
Mass of Earth	$M_\oplus$	5.974×10^{24}	kg
Mass of Sun	$M_\odot$	1.989×10^{30}	kg
Mass of Moon	$M_☾$	7.36×10^{22}	kg
Equatorial radius of Earth	$R_\oplus$	6.378×10^{6}	m
Equatorial radius of Sun	$R_\odot$	6.6955×10^{8}	m
Equatorial radius of Moon	$R_☾$	1.737×10^{6}	m
Mean density of Earth		5515	$\text{kg}\cdot\text{m}^{-3}$
Mean density of Sun		1408	$\text{kg}\cdot\text{m}^{-3}$
Mean density of Moon		3346	$\text{kg}\cdot\text{m}^{-3}$
Earth-Moon distance		3.84×10^{8}	m
Earth-Sun distance		1.496×10^{11}	m
Luminosity of Sun	$L_\odot$	3.839×10^{26}	W
Effective temp. of Sun		5778	K
Hubble constant	H_0	70 ± 5	$\text{km}\cdot\text{s}^{-1}\text{Mpc}^{-1}$
Parsec	pc	206264.81	AU
		3.0856776×10^{16}	m
		3.2615638	ly
Astronomical Unit	AU	1.496×10^{11}	m
Light year	ly	9.461×10^{15}	m
1 year on Earth	yr	365.25	days
		3.15576×10^{7}	s

0.4: Solar System

Planet	Symbol	Mass (kg)	Radius (m)	Sun-Distance (km)
Mercury	☿	3.285×10^{23}	2.44×10^{6}	5.791×10^{10}
Venus	♀	4.867×10^{24}	6.052×10^{6}	1.082×10^{11}
Mars	♂	6.39×10^{23}	3.390×10^{6}	2.279×10^{11}
Jupiter	♃	1.898×10^{27}	3.83×10^{11}	7.785×10^{11}
Saturn	♄	5.683×10^{26}	5.8232×10^{7}	1.429×10^{12}
Uranus	⛢	8.681×10^{25}	2.5362×10^{7}	2.871×10^{12}
Neptune	♅	1.024×10^{26}	2.4622×10^{7}	4.498×10^{12}
Pluto	♇	1.309×10^{22}	1.187×10^{6}	5.906×10^{12}

0.5: Unit conversions

The International System of Units (SI) defines seven units of measure as a basic set from which all other SI units can be derived. These are [length](m), [time](s), [mass](kg), [electric current] $\equiv$ [Ampere](A), [temperature](K), [luminous intensity](cd), [amount of substance](mol).

Unit Symbol	Unit	Equivalence
C	[Coulomb]	[Ampere][time]
N	[Newton]	[mass][length][time]$^{-2}$
P	[Pascal]	[mass][length]$^{-1}$[time]$^{-2}$
J	[Joule]	[mass][length]^{2}[time]$^{-2}$
W	[Watt]	[mass][length]^{2}[time]$^{-3}$
		[Ohm][Ampere]^{2}
		[Volt]^{2}[Ohm]$^{-1}$
V	[Volt]	[mass][length]^{2}[time]$^{-3}$[Ampere]$^{-1}$
Wb	[Weber]	[mass][length]^{2}[time]$^{-2}$[Ampere]$^{-1}$
T	[Tesla]	[mass][time]$^{-2}$[Ampere]$^{-1}$
H	[henry]	[mass][length]^{2}[time]$^{-2}$[Ampere]$^{-2}$
Ω	[Ohm]	[mass][length]^{2}[time]$^{-3}$[Ampere]$^{-2}$
F	[Farad]	[mass]$^{-1}$[length]$^{-2}$[time]^{4}[Ampere]^{2}
Hz	[Hertz]	[time]$^{-1}$

0.6: Number Sets ($i \equiv \sqrt{-1}$)

Symbol	Set	Symbol	Set
$\mathbb{R}$	Real numbers	$\emptyset$	$\{\}$
$\mathbb{N} \equiv \mathbb{N}_1$	$\{1,2,3,4,\ldots\}$	$\mathbb{Z}$	$\{\ldots,-2,1,0,1,2,\ldots\}$
$\mathbb{Z}^+ \equiv \mathbb{N}_0$	$\{0,1,2,3,\ldots\}$	$\mathbb{Z}^-$	$\{0,-1,-2,-3,-4,\ldots\}$
$\mathbb{C}$	$\{x+iy \mid x,y \in \mathbb{R}\}$	$\mathbb{Q}$	$\{\frac{x}{y} \mid x,y \in \mathbb{Z}\}$
$\mathbb{I}$	$\{ix \mid x \in \mathbb{R}\}$	$\mathbb{U}$	Universal Set [a]
$\mathbb{A}$	Algebraic Numbers[b]	$\mathbb{T}$	Transcendental Numbers [c]

[a] Definition: The set containing all objects or elements and of which all other sets are subsets.
[b] Any number that is a solution to a polynomial equation with rational coefficients.
[c] Any number that is not an Algebraic Number.

General Mathematics

Definitions

$$\sin(x) = \frac{1}{2i}(e^{ix} - e^{-ix}) \tag{2.0.1}$$
$$\sinh(x) = \frac{1}{2}(e^{x} - e^{-x}) \tag{2.0.2}$$
$$= -i\sin(ix) \tag{2.0.3}$$

$$\cos(x) = \frac{1}{2}(e^{ix} + e^{-ix}) \tag{2.0.4}$$
$$\cosh(x) = \frac{1}{2}(e^{x} + e^{-x}) \tag{2.0.5}$$
$$= \cos(ix) \tag{2.0.6}$$

Curl Theorem: A special case of Stokes' theorem in which $\vec{F}$ is a vector field and M is an oriented, compact embedded 2-manifold with boundary in $\mathbb{R}^3$, and a generalization of Green's theorem from the plane into three-dimensional space. The curl theorem states

$$\int_S (\nabla \times \vec{F}) \cdot d\vec{a} = \int_{\partial S} \vec{F} \cdot d\vec{s} \tag{2.0.7}$$

Green's theorem is a vector identity which is equivalent to the curl theorem

$$\iint\limits_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \oint_{\partial S} P(x,y)dx + Q(x,y)dy \tag{2.0.8}$$

The **divergence theorem** is also known as Gauss's theorem (e.g., Arfken 1985) or the Gauss-Ostrogradsky theorem. Let V be a region in space with boundary ∂V. Then the volume integral of the divergence $\nabla \cdot \vec{F}$ of $\vec{F}$ over V and the surface integral of $\vec{F}$ over the boundary ∂V of V are related by

$$\int_V (\nabla \cdot \vec{F}) dV = \int_{\partial V} \vec{F} \cdot d\vec{a} \tag{2.0.9}$$

The **gradient theorem** (where the integral is a line integral) is

$$\int_a^b (\nabla f) \cdot d\vec{s} = f(b) - f(a) \tag{2.0.10}$$

The **Gamma function** Γ and the **Riemann zeta function** ζ are given by

$$\Gamma(z) \equiv \int_0^\infty t^{z-1} e^{-t} dt \tag{2.0.11}$$
$$\zeta(z) = \sum_{k=1}^\infty \frac{1}{k^z} \implies \zeta'(z) = -\sum_{k=1}^\infty \frac{\ln(k)}{k^z} \tag{2.0.12}$$
$$\zeta(z)\Gamma(z) = \int_0^\infty \frac{u^{z-1}}{e^u - 1} du \tag{2.0.13}$$

The most general case of the binomial theorem is the binomial series identity

$$(x+y)^n = \sum_{i=1}^n \binom{n}{k} x^{n-k} y^k \tag{2.0.14}$$

The **binomial coefficient** is defined as follows, with Pascals Formula implied.

$$ {}_nC_r \equiv \binom{n}{k} \equiv \frac{n!}{(n-k)!k!} \equiv \frac{\Gamma(n+1)}{\Gamma(k+1)\Gamma(n-k+1)} \tag{2.0.15} $$

$$ \binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \tag{2.0.16} $$

The general formula for the power sum of the first n positive integers,

$$ \sum_{k=1}^{n} k^p = \frac{1}{p+1} \sum_{i=1}^{p+1} (-1)^{\delta_{ip}} \binom{p+1}{i} B_{p+1-i} n^i, \tag{2.0.17} $$

where δ_{ip} is the Kronecker delta and B_i is the ith Bernoulli number. The Bernoulli numbers B_n are a sequence of signed rational numbers that can be defined by the exponential generating function

$$ \frac{x}{e^x - 1} \equiv \sum_{n=0}^{\infty} \frac{B_n x^n}{n!}. \tag{2.0.18} $$

The simplest interpretation of the **Kronecker delta** is as the discrete version of the delta function defined by

$$ \delta_{ij} \equiv \begin{cases} 0 & \text{for } i \neq j \\ 1 & \text{for } i = j. \end{cases} \tag{2.0.19} $$

Determining the sums for the first few terms of the power sum give

$$ \sum_{i=0}^{n} i = \frac{1}{2} n(n+1) \tag{2.0.20} $$

$$ \sum_{i=0}^{n} i^2 = \frac{1}{6}(2n^3 + 3n^2 + n) = \frac{1}{6} n(2n+1)(n+1) \tag{2.0.21} $$

$$ \sum_{i=0}^{n} i^3 = \frac{1}{4}(n^4 + 2n^3 + n^2) = \frac{1}{4} n^2 (n+1)^2 \tag{2.0.22} $$

$$ \sum_{i=0}^{n} i^4 = \frac{1}{30}(6n^5 + 15n^4 + 10n^3 - n) \tag{2.0.23} $$

$$ \sum_{i=0}^{n} i^5 = \frac{1}{12}(2n^6 + 6n^5 + 5n^4 - n^2). \tag{2.0.24} $$

A Taylor series is an expansion of a function about a point. A one-dimensional Taylor series of a real function $f(x)$ about the point x=a is given by

$$ f(x) = f(a) + f'(a)(x-a) + \frac{1}{2!} f''(a)(x-a)^2 + \frac{1}{3!} f'''(a)(x-a)^4 + \cdots \tag{2.0.25} $$

Some common series expansions include:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots \tag{2.0.26}$$

$$\ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1} = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \cdots \quad \text{with } |x| < 1 \tag{2.0.27}$$

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \tag{2.0.28}$$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \tag{2.0.29}$$

$$\tan(x) = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots [|x| < \pi/2] \tag{2.0.30}$$

$$\sinh(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots \tag{2.0.31}$$

$$\cosh(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots \tag{2.0.32}$$

$$\tanh(x) = x - \frac{1}{3}x^3 + \frac{2}{15}x^5 - \cdots [|x| < \pi/2] \tag{2.0.33}$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \cdots [|x| < 1] \tag{2.0.34}$$

Dirac Delta Function: The delta function is a generalized function that can be defined as the limit of a class of delta sequences.

$$\delta(x) = \frac{1}{\pi} \lim_{\epsilon \to 0} \frac{\epsilon}{x^2 + \epsilon^2} = \frac{1}{2} \lim_{\epsilon \to 0} \epsilon |x|^{\epsilon - 1} = \lim_{\epsilon \to 0} \frac{1}{\pi x} \sin\left(\frac{x}{\epsilon}\right) = \lim_{\epsilon \to 0^+} \frac{1}{2\sqrt{\pi\epsilon}} e^{-x^2/(4\epsilon)} \tag{2.0.35}$$

The Dirac delta can be thought of as a function on the real line which is zero everywhere except where the arguments of the function are zero, where it is infinite,

$$\delta(x) = \begin{cases} \infty & x = 0 \\ 0 & x \neq 0 \end{cases} \tag{2.0.36}$$

For any $\epsilon > 0$, the delta function has the fundamental property that

$$\int_{-\infty}^{\infty} f(x)\delta(x-a)dx = f(a) \qquad \text{and} \qquad \int_{x-\epsilon}^{x+\epsilon} f(x)\delta(x-a)dx = f(a) \tag{2.0.37}$$

The fundamental equation that defines derivatives of the delta function $\delta(x)$ is

$$\int f(x)\delta^{(n)}(x)dx \equiv -\int \frac{\partial f}{\partial x}\delta^{(n-1)}(x)dx \tag{2.0.38}$$

This implies

$$x^n \delta^{(n)}(x) = (-1)^n n! \delta(x) \tag{2.0.39}$$

A few identities and common expressions using the delta function are

$$\int_{-\infty}^{\infty} f(x)\delta(ax)dx = \frac{1}{|a|}f(0) \tag{2.0.40}$$

$$\int_{-1}^{1} \delta\left(\frac{1}{x}\right)dx = 0 \tag{2.0.41}$$

Coordinate Systems

Cylindrical coordinates

$$r = \sqrt{x^2+y^2} \tag{2.1.1}$$
$$x = r\cos(\theta) \tag{2.1.2}$$
$$y = r\sin(\theta) \tag{2.1.3}$$
$$z = z \tag{2.1.4}$$
$$dV = r \text{ dr d}\theta \text{ dz} \tag{2.1.5}$$

Spherical coordinates

$$r = \sqrt{x^2+y^2+z^2} \tag{2.1.6}$$
$$x = r\sin(\theta)\cos(\phi) \tag{2.1.7}$$
$$y = r\sin(\theta)\sin(\phi) \tag{2.1.8}$$
$$z = r\cos(\theta) \tag{2.1.9}$$
$$dV = r^2\sin(\theta)\text{dr d}\theta \text{ d}\phi \tag{2.1.10}$$

Polar Coordinates

$$r = \sqrt{x^2+y^2} \tag{2.1.11}$$
$$x = r\cos(\theta) \tag{2.1.12}$$
$$y = r\sin(\theta) \tag{2.1.13}$$
$$dA = r \text{ dr d}\theta \tag{2.1.14}$$

Elliptic cylindrical coordinates

$$x = a\cosh(u)\cos(v) \tag{2.1.15}$$
$$y = a\sinh(u) sin(v) \tag{2.1.16}$$
$$z = z \tag{2.1.17}$$
$$dV = a^2[\sinh^2(u)+\cosh^2(v)]\text{du dv dz} \tag{2.1.18}$$

Vector Operations

For any vector $\vec{r} = (r_1, r_2, \ldots, r_n)$ in n-dimensions, the magnitude and unit vector is

$$|\vec{r}| \equiv \sqrt{\vec{r}\cdot\vec{r}} = \sqrt{r_1^2+r_2^2+\cdots+r_n^2} \qquad \hat{r} \equiv \frac{\vec{r}}{|\vec{r}|} \tag{2.2.1}$$

Dot and cross products for 3-dimensional vectors, where θ is the smallest angle between them, $\vec{r} = (r_x, r_y, r_z)$ and $\vec{s} = (s_x, s_y, s_z)$

$$\vec{r}\cdot\vec{s} = |\vec{r}||\vec{s}|\cos(\theta) = r_x s_x + r_y s_y + r_z s_z \tag{2.2.2}$$

$$\vec{r}\times\vec{s} = |\vec{r}||\vec{s}|\sin(\theta) = (r_y s_z - r_z s_y, r_z s_x - r_x s_z, r_x s_y - r_y s_x) = \begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ r_x & r_y & r_z \\ s_x & s_y & s_z \end{vmatrix} \tag{2.2.3}$$

Triple product vector identities

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{B}\cdot(\vec{C}\times\vec{A}) = \vec{C}\cdot(\vec{A}\times\vec{B}) = -\vec{B}\cdot(\vec{A}\times\vec{C}) = -\vec{C}\cdot(\vec{B}\times\vec{A}) = -\vec{A}\cdot(\vec{C}\times\vec{B}) \tag{2.2.4}$$

$$\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B}) \tag{2.2.5}$$

Product rule vector identities

$$\nabla(fg) = f(\nabla g) + g(\nabla f) \tag{2.2.6}$$

$$\nabla(\vec{A}\cdot\vec{B}) = \vec{A}\times(\nabla\times\vec{B}) + \vec{B}\times(\nabla\times\vec{A}) + (\vec{A}\cdot\nabla)\vec{B} + (\vec{B}\cdot\nabla)\vec{A} \tag{2.2.7}$$

$$\nabla\cdot(f\vec{A}) = f(\nabla\cdot\vec{A}) + \vec{A}\cdot(\nabla f) \tag{2.2.8}$$

$$\nabla\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\nabla\times\vec{A}) - \vec{A}\cdot(\nabla\times\vec{B}) \tag{2.2.9}$$

$$\nabla\times(f\vec{A}) = f(\nabla\times\vec{A}) - \vec{A}\times(\nabla f) \tag{2.2.10}$$

$$\nabla\times(\vec{A}\times\vec{B}) = (\vec{B}\cdot\nabla)\vec{A} - (\vec{A}\cdot\nabla)\vec{B} + \vec{A}(\nabla\cdot\vec{B}) - \vec{B}(\nabla\cdot\vec{A}) \tag{2.2.11}$$

Second derivative vector identities

$$\nabla\cdot(\nabla\times\vec{A}) = 0 \qquad \nabla\times(\nabla f) = 0 \qquad \nabla\times(\nabla\times\vec{A}) = \nabla(\nabla\cdot\vec{A}) - \nabla^2\vec{A} \tag{2.2.12}$$

A few other useful identities include

$$\nabla\cdot\frac{\hat{r}}{r^2} = 4\pi\delta^2(\vec{r}) \qquad \nabla\times\frac{\hat{r}}{r^2} = \frac{\vec{r}}{r^3} = 0 \qquad \nabla\cdot\frac{\hat{r}}{r} = \frac{1}{r^2} \tag{2.2.13}$$

Triangles

Let a triangle have side lengths a, b, and c with opposite angles A, B, and C.

The area of a triangle can be given by

$$A = \sqrt{s(s-a)(s-b)(s-c)} \tag{2.3.1}$$

$$s = (a+b+c)/2 \tag{2.3.2}$$

Law of Cosines:

$$c^2 = a^2 + b^2 - 2ab\cos(C) \tag{2.3.3}$$

Law of Sines:

$$\frac{\sin(A)}{a} = \frac{\sin(B)}{b} = \frac{\sin(C)}{c} \tag{2.3.4}$$

Law of tangents

$$\frac{a-b}{a+b} = \frac{\tan((A-B)/2)}{\tan((A+B)/2)} \tag{2.3.5}$$

Mollweide's Formulas

$$\frac{b-c}{a} = \frac{\sin[(B-C)/2]}{\cos(A/2)} \tag{2.3.6}$$

$$\frac{c-a}{b} = \frac{\sin[(C-A)/2]}{\cos(B/2)} \tag{2.3.7}$$

$$\frac{a-b}{c} = \frac{\sin[(A_B)/2]}{\cos(C/2)} \tag{2.3.8}$$

Trigonometric Identities

Pythagorean identities

$$1 = \sin^2(\theta) + \cos^2(\theta) \tag{2.4.1}$$
$$1 = \sec^2(\theta) - \tan^2(\theta) \tag{2.4.2}$$
$$1 = \csc^2(\theta) - \cot^2(\theta) \tag{2.4.3}$$
$$1 = \cosh^2(\theta) - \sinh^2(\theta) \tag{2.4.4}$$
$$1 = \mathrm{sech}^2(\theta) + \tanh^2(\theta) \tag{2.4.5}$$

Sum-Difference Formulas

$$\sin(\theta \pm \phi) = \sin(\theta)\cos(\phi) \pm \cos(\theta) sin(\phi) \tag{2.4.6}$$
$$\cos(\theta \pm \phi) = \cos(\theta)\cos(\phi) \mp \sin(\theta) sin(\phi) \tag{2.4.7}$$
$$\tan(\theta \pm \phi) = \frac{\tan(\theta) \pm \tan(\phi)}{1 \mp \tan(\theta)\tan(\phi)} \tag{2.4.8}$$

Double Angle formulas

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta) \tag{2.4.9}$$
$$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) \tag{2.4.10}$$
$$= 2\cos^2(\theta) - 1 \tag{2.4.11}$$
$$= 1 - 2\sin^2(\theta) \tag{2.4.12}$$
$$\tan(2\theta) = \frac{2\tan(\theta)}{1 - \tan^2(\theta)} \tag{2.4.13}$$

Power-Reducing/Half Angle Formulas

$$\sin^2(\theta) = \frac{1 - \cos(2\theta)}{2} \tag{2.4.14}$$
$$\cos^2(\theta) = \frac{1 + \cos(2\theta)}{2} \tag{2.4.15}$$
$$\tan^2(\theta) = \frac{1 - \cos(2\theta)}{1 + \cos(2\theta)} \tag{2.4.16}$$

Other relations

$$\sin(-\theta) = -\sin(\theta) \tag{2.4.17}$$
$$\cos(-\theta) = \cos(\theta) \tag{2.4.18}$$
$$\sin(\theta \pm \pi/2) = \pm\cos(\theta) \tag{2.4.19}$$
$$\sin(\theta \pm \pi) = -\sin(\theta) \tag{2.4.20}$$
$$\cos(\theta \pm \pi/2) = \mp\sin(\theta) \tag{2.4.21}$$
$$\cos(\theta \pm \pi) = -\cos(\theta) \tag{2.4.22}$$

Half-angle formulas

$$\sin\left(\frac{\theta}{2}\right) = (-1)^{\theta/(2\pi)}\sqrt{\frac{1 - \cos(\theta)}{2}} \tag{2.4.23}$$
$$\cos\left(\frac{\theta}{2}\right) = (-1)^{(\theta+\pi)/(2\pi)}\sqrt{\frac{1 + \cos(\theta)}{2}} \tag{2.4.24}$$

The Weierstrass substitution makes use of the half-angle formulas

$$\cos(\theta) = \frac{1 - \tan^2(\theta/2)}{1 + \tan^2(\theta/2)} \tag{2.4.25}$$
$$\sin(\theta) = \frac{2\tan(\theta/2)}{1 + \tan^2(\theta/2)} \tag{2.4.26}$$

The half angle identity for tangent.

$$\tan\left(\frac{\theta}{2}\right) = (-1)^{x/\pi}\sqrt{\frac{1 - \cos(\theta)}{1 + \cos(\theta)}} = \frac{\sin(\theta)}{1 + \cos(\theta)} = \frac{1 - \cos(\theta)}{\sin(\theta)} = \frac{\tan(\theta)\sin(\theta)}{\tan(\theta) + \sin(\theta)} \tag{2.4.27}$$

Other identities

$$\cos(\theta)cos(\phi) = \frac{1}{2}[\cos(\theta+\phi)+\cos(\theta-\phi)] \tag{2.4.28}$$

$$\sin(\theta)sin(\phi) = \frac{1}{2}[\cos(\theta-\phi)-\cos(\theta+\phi)] \tag{2.4.29}$$

$$\sin(\theta)cos(\phi) = \frac{1}{2}[\sin(\theta+\phi)+\sin(\theta-\phi)] \tag{2.4.30}$$

$$\cos(\theta)+\cos(\phi) = 2\cos\left(\frac{\theta+\phi}{2}\right)\cos\left(\frac{\theta-\phi}{2}\right) \tag{2.4.31}$$

$$\cos(\theta)-\cos(\phi) = 2\sin\left(\frac{\theta+\phi}{2}\right)\sin\left(\frac{\theta-\phi}{2}\right) \tag{2.4.32}$$

Multiple-angle formulas are given by

$$\sin(nx) = \sum_{k=0}^{n}\binom{n}{k}\cos^k(x)\sin^{n-k}(x)\sin\big((n-k)\pi/2\big) \tag{2.4.33}$$

$$\cos(nx) = \sum_{k=0}^{n}\binom{n}{k}\cos^k(x)\sin^{n-k}(x)\cos\big((n-k)\pi/2\big) \tag{2.4.34}$$

Arbitrary Orthogonal Curvilinear Coordinates

A coordinate system composed of intersecting surfaces. If the intersections are all at right angles, then the curvilinear coordinates are said to form an orthogonal coordinate system. The scale factors are h_i,

$$\vec{a}_i \equiv \frac{\partial\vec{r}}{\partial e_i} = \frac{\partial x}{\partial e_i}\hat{x} + \frac{\partial y}{\partial e_i}\hat{y} + \frac{\partial x}{\partial e_i}\hat{z} = h_i\hat{e}_i = |\vec{a}_i|\hat{e}_i \tag{2.5.1}$$

$$h_i \equiv \left|\frac{\partial\vec{r}}{\partial e_i}\right| = |\vec{a}_i| = \sqrt{\frac{\partial x}{\partial e_i} + \frac{\partial y}{\partial e_i} + \frac{\partial z}{\partial e_i}} \tag{2.5.2}$$

$$\hat{e}_i = \frac{1}{h_1}\frac{\partial\vec{r}}{\partial e_i} = \frac{\vec{a}_i}{|\vec{a}_i|} \tag{2.5.3}$$

The line element $d\vec{s}$ is determined by

$$d\vec{s} \equiv d\vec{x} + d\vec{y} + d\vec{z} \equiv \vec{a}_1 de_1 + \vec{a}_2 de_2 + \vec{a}_3 de_3 \tag{2.5.4}$$

From this, ds^2 is given by

$$ds^2 = d\vec{s}\cdot d\vec{s} = dx^2 + dy^2 + dz^2 = h_1^2 de_1^2 + h_2^2 de_2^2 + h_3^2 de_3^2 \tag{2.5.5}$$

The differential vector and volume elements are therefore

$$d\vec{r} = h_1 du_1\hat{u}_1 + h_2 du_2\hat{u}_2 + h_3 du_3\hat{u}_3 \tag{2.5.6}$$

$$dV = h_1h_2h_3 du_1 du_2 du_3 = \left|\frac{\partial(x,y,z)}{\partial(u_1,u_2,u_3)}\right| du_1 du_2 du_3 \tag{2.5.7}$$

The gradient in arbitrary curvilinear coordinates such that the gradient theorem is preserved:

$$\nabla f = \frac{1}{h_1}\frac{\partial f}{\partial x_1}\hat{x}_1 + \frac{1}{h_2}\frac{\partial f}{\partial x_2}\hat{x}_2 + \frac{1}{h_3}\frac{\partial f}{\partial x_3}\hat{x}_3 \tag{2.5.8}$$

The divergence in arbitrary curvilinear coordinates such that the divergence theorem is preserved:

$$\nabla \cdot \vec{v} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial v_1}{\partial x_1} h_2 h_3 + \frac{\partial v_2}{\partial x_2} h_1 h_3 + \frac{\partial v_3}{\partial x_3} h_1 h_2\right] \tag{2.5.9}$$

The Laplacian for a scalar function ϕ (where the h_i are the scale factors of the coordinate system - Weinberg 1972, p. 109; Arfken 1985, p. 92) is a scalar differential operator defined by

$$\nabla^2 \phi = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_1 h_3}{h_2}\frac{\partial}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial}{\partial u_3}\right)\right]\phi \tag{2.5.10}$$

The form of the Laplacian in several common coordinate systems (cartesian, cylindrical, parabolic, parabolic cylindrical, spherical and oblate spheroidal respectively) are

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \tag{2.5.11}$$

$$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2} \tag{2.5.12}$$

$$\nabla^2 f = \frac{1}{uv(u^2+v^2)}\left[\frac{\partial}{\partial u}\left(uv\frac{\partial f}{\partial u}\right) + \frac{\partial}{\partial v}\left(uv\frac{\partial f}{\partial v}\right)\right] + \frac{1}{v^2 u^2}\frac{\partial^2 f}{\partial \theta^2} \tag{2.5.13}$$

$$\nabla^2 f = \frac{1}{u^2+v^2}\left(\frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2}\right) + \frac{\partial^2 f}{\partial z^2} \tag{2.5.14}$$

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin^2\phi}\frac{\partial^2 f}{\partial \theta^2} + \frac{1}{r^2\sin\phi}\frac{\partial}{\partial \phi}\left(\sin\phi\frac{\partial f}{\partial \phi}\right) \tag{2.5.15}$$

$$\nabla^2 f = \frac{1}{a^2(\zeta^2+\xi^2)}\left[\frac{\partial}{\partial \zeta}\left((1+\zeta^2)\frac{\partial f}{\partial \zeta}\right) + \frac{\partial}{\partial \xi}\left((1-\xi^2)\frac{\partial f}{\partial \xi}\right)\right] + \frac{1}{a^2(1+\zeta^2)(1-\xi^2)}\frac{\partial^2 f}{\partial \phi^2} \tag{2.5.16}$$

The curl can be similarly defined in arbitrary orthogonal curvilinear coordinates as

$$\nabla \times \vec{F} \equiv \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1\hat{e}_1 & h_2\hat{e}_2 & h_3\hat{e}_3 \\ \frac{\partial}{\partial e_1} & \frac{\partial}{\partial e_2} & \frac{\partial}{\partial e_3} \\ h_1 F_1 & h_2 F_2 & h_3 F_3 \end{vmatrix} \tag{2.5.17}$$

$$\begin{aligned} &= \frac{1}{h_2 h_3}\left[\frac{\partial}{\partial u_2}(h_3 F_3) - \frac{\partial}{\partial u_3}(h_2 F_2)\right]\hat{u}_1 + \frac{1}{h_1 h_3}\left[\frac{\partial}{\partial u_3}(h_1 F_1) - \frac{\partial}{\partial u_1}(h_3 F_3)\right]\hat{u}_2 \\ &\quad + \frac{1}{h_1 h_2}\left[\frac{\partial}{\partial u_1}(h_2 F_2) - \frac{\partial}{\partial u_2}(h_1 F_1)\right]\hat{u}_3 \end{aligned} \tag{2.5.18}$$

The **Jacobian** is defined as the determinant of a matrix of partial derivatives [5],

$$\frac{\partial(a,b)}{\partial(c,d)} \equiv \begin{vmatrix} \left(\frac{\partial a}{\partial c}\right)_d & \left(\frac{\partial a}{\partial d}\right)_c \\ \left(\frac{\partial b}{\partial c}\right)_d & \left(\frac{\partial b}{\partial d}\right)_c \end{vmatrix} = \left(\frac{\partial a}{\partial c}\right)_d \left(\frac{\partial b}{\partial d}\right)_c - \left(\frac{\partial a}{\partial d}\right)_c \left(\frac{\partial b}{\partial c}\right)_d . \tag{2.5.19}$$

By the above definition, we can show the relations,

$$\frac{\partial(b,a)}{\partial(c,d)} = -\frac{\partial(a,b)}{\partial(c,d)} \qquad \text{and} \qquad \frac{\partial(a,b)}{\partial(c,d)} = -\frac{\partial(a,b)}{\partial(d,c)} \tag{2.5.20}$$

It then follows directly that

$$\frac{\partial(a,s)}{\partial(c,s)} = \left(\frac{\partial a}{\partial c}\right)_s \qquad \text{and} \qquad \frac{\partial(a,b)}{\partial(a,b)} = 1 \qquad \text{and} \qquad \frac{\partial(a,b)}{\partial(c,d)}\frac{\partial(c,d)}{\partial(s,t)} = \frac{\partial(a,b)}{\partial(s,t)} \tag{2.5.21}$$

Complex Analysis

Complex Numbers

The set of complex numbers is defined such that

$$\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}. \tag{3.1.1}$$

A complex number can be defined by its real part and it's imaginary part

$$i^2 = -1 \iff i = \sqrt{-1} \iff \frac{1}{i} = -i \tag{3.1.2}$$

$$z = x + iy \iff z^* = x - iy \tag{3.1.3}$$

We can express the real and imaginary parts of a complex number in terms of the number and its complex conjugate

$$\Re(z) = \frac{1}{2}(z + z^*) \tag{3.1.4}$$

$$\Im(z) = \frac{1}{2}i(z - z^*) \tag{3.1.5}$$

Just like a two-dimensional vector, a complex number has the magnitude $|z|$ as well as an angle θ with respect to the horizontal axis of the complex plane.

$$|z|^2 = z^* z = x^2 + y^2 = |z|e^{-i\theta}|z|e^{i\theta} \tag{3.1.6}$$

$$\tan(\theta) = \frac{\Im(z)}{\Re(z)} = \frac{y}{x} = \frac{i(z - z^*)}{(z + z^*)} \tag{3.1.7}$$

A complex number can thus be expressed in terms of magnitude and the phase angle

$$z = |z|(\cos(\theta) + i\sin(\theta)) \tag{3.1.8}$$

Euler's Identity/relation

$$e^{i\theta} = cos(\theta) + i\sin(\theta) \tag{3.1.9}$$

With the aid of Eulers identity, we can write any complex number as

$$z = |z|e^{i\theta} \tag{3.1.10}$$

$$z^n = |z|^n e^{in\theta} \tag{3.1.11}$$

A useful property of conjugates is

$$a^* + b^* = (a + b)^* \tag{3.1.12}$$

Powers and roots of a complex number can be determined from the exponential form of a complex number

$$z^n = (re^{i\theta})^n = r^n e^{in\theta} \tag{3.1.13}$$

$$(e^{i\theta})^n = e^{in\theta} = (\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta) \tag{3.1.14}$$

$$z^{1/n} = (re^{i\theta})^{1/n} = r^{1/n}e^{i\theta/n} = \sqrt[n]{r}\left(\cos\frac{\theta}{n} + i\sin\frac{\theta}{n}\right) \tag{3.1.15}$$

Much like in trigonometry, we can define complex numbers using trigonometric identities:

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \qquad \cos z = \frac{e^{iz} + e^{-iz}}{2}, \qquad \sinh z = \frac{e^{z} - e^{-z}}{2}, \qquad \cosh z = \frac{e^{z} + e^{-z}}{2} \tag{3.1.16}$$

The logarithm of a complex number can be manipulated as a normal log with

$$\ln(z) = \ln(re^{i\theta}) = \ln(r) + \ln(e^{i\theta}) = \ln(r) + i\theta \tag{3.1.17}$$

A few Trigonometric identities follow as:

$$\arcsin z = -i\ln(iz \pm \sqrt{1 - z^2}) \tag{3.1.18}$$

$$\arccos z = i\ln(z \pm \sqrt{z^2 - 1}) \tag{3.1.19}$$

$$\arctan z = \frac{1}{2i}\ln\left(\frac{1 + iz}{1 - iz}\right) \tag{3.1.20}$$

Complex Functions

A complex function of z can be expressed in terms of two real functions $u(x, y)$ and $v(x, y)$,

$$f(z) = f(x + iy) = u(x, y) + iv(x, y). \tag{3.2.1}$$

The derivative of $f(z)$ is defined by

$$f'(z) = \frac{df}{dz} = \lim_{\Delta z \to 0} \frac{\Delta f}{\Delta z} = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} = \lim_{\Delta x, \Delta y \to 0} \frac{f(z + \Delta x + i\Delta y) - f(z)}{\Delta x + i\Delta y}. \tag{3.2.2}$$

Definition 2.1: Analytic Function [2]

A function $f(z)$ is **analytic** (or regular or holomorphic or mono-genic) in a region of the complex plane if it has a (unique) derivative at every point of the region. The statement $f(z)$ is analytic at a point $z = a$ means that $f(z)$ has a derivative at every point inside some small circle about $z = a$.

The **Cauchy-Riemann conditions** state that if $f(z) = u(x, y) + iv(x, y)$ is analytic in a region, then in that region

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \text{and} \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}. \tag{3.2.3}$$

From this we also have

$$\frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}, \qquad \text{and} \qquad \frac{\partial f}{\partial y} = \frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}. \tag{3.2.4}$$

Definition 2.2: Regular and Singular Points [2]

A **regular point** of $f(z)$ is a point at which $f(z)$ is analytic. A **singular point** or singularity of $f(z)$ is a point at which $f(z)$ is not analytic. It is called an isolated singular point if $f(z)$ is analytic everywhere else inside some small circle about the singular point.

A few useful theorems [2] include

1. If $f(z)$ is analytic in a region, then it has derivatives of all orders at points inside the region and can be expanded in a Taylor series about any point z_0 inside the region. The power series converges inside the circle about z_0 that extends to the nearest singular point.

2. If $f(z) = u + iv$ is analytic in a region, then u and v satisfy Laplace's equation in the region (that is, u and v are harmonic functions).

3. Any function u (or v) satisfying Laplace's equation in a simply-connected region, is the real or imaginary part of an analytic function $f(z)$.

Given a complex function $f(z) = u(x, y) + iv(x, y)$, we can take the integral along a path ℓ using

$$\int_\ell f(z)dz = \int_\ell f(x + iy)(dx + idy) = \int_\ell u(x, y)dx - v(x, y)dy + i\int_\ell u(x, y)dy + v(x, y)dx. \tag{3.2.5}$$

2.1: Cauchy's Theorem [2]

Let C be a simple closed curve (one which does not cross itself) with a continuously turning tangent except possibly at a finite number of points (that is, we allow a finite number of corners, but otherwise the curve must be smooth). If $f(z)$ is analytic on and inside C, then

$$\oint_C f(z)dz = 0 \tag{3.2.6}$$

2.2: Cauchy's Integral Formula [2]

If $f(z)$ is analytic on and inside a simple closed curve C, the value of $f(z)$ at a point $z = a$ inside C is given by the following contour integral along C:

$$f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}dz \tag{3.2.7}$$

2.3: Laurent's Theorem [2], [17]

Let C_1 and C_2 be two circles with center at $z = z_0$ with radii r_1 and $r_2 < r_1$ respectively. Let $f(z)$ be analytic in the region R between the circles. Then $f(z)$ can be expanded in a series of the form

$$f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots + \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \cdots \tag{3.2.8}$$

$$= \sum_{k=0}^{\infty} a_k(z-z_0)^k + \sum_{k=1}^{\infty} b_k(z-z_0)^{-k}, \tag{3.2.9}$$

convergent in R. Such a series is called a Laurent series. The b series is called the principal part of the Laurent series. The coefficients have the solutions

$$a_k = \frac{1}{2\pi i}\oint_{C_1}\frac{f(z')dz'}{(z'-z_0)^{k+1}}, \qquad \text{and} \qquad b_k = \frac{1}{2\pi i}\oint_{C_2}(z'-z_0)^{k-1}f(z')dz' \tag{3.2.10}$$

Definition 2.3: Poles, Residue and Singularities [2]

1. If all the b's are zero, $f(z)$ is analytic at $z = z_0$, and we call z_0 a **regular point**.
2. If $b_n \neq 0$, but all the b's after b_n are zero, $f(z)$ is said to have a **pole** of order n at $z = z_0$. If $n = 1$, we say that $f(z)$ has a simple pole.
3. If there are an infinite number of b's different from zero, $f(z)$ has an **essential singularity** at $z = z_0$.
4. The coefficient b_1 of $1/(z - z_0)$ is called the **residue** of $f(z)$ at $z = z_0$.

Matrix Algebra

The product C of two matrices A and B is defined (where j is summed over for all possible values of i and k) as (using the Einstein summation convention)

$$c_{ik} = a_{ij}b_{jk} = \sum_{j=1}^{m} a_{ij}b_{jk} \tag{4.0.1}$$

In order for matrix multiplication to be defined, the dimensions of the matrices must satisfy

$$(n \times m)(m \times p) = (n \times p) \tag{4.0.2}$$

where $(a \times b)$ denotes a matrix with a rows and b columns. Writing out the product explicitly,

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{np} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1p} \\ c_{21} & c_{22} & \cdots & c_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{np} \end{bmatrix} \tag{4.0.3}$$

where,

$$c_{11} = a_{11}b_{11} + a_{12}b_{21} + \cdots + a_{1m}b_{m1} \tag{4.0.4}$$
$$c_{12} = a_{11}b_{12} + a_{12}b_{22} + \cdots + a_{1m}b_{m2} \tag{4.0.5}$$
$$c_{1p} = a_{11}b_{1p} + a_{12}b_{2p} + \cdots + a_{1m}b_{mp} \tag{4.0.6}$$
$$c_{21} = a_{21}b_{11} + a_{22}b_{21} + \cdots + a_{2m}b_{m1} \tag{4.0.7}$$
$$c_{22} = a_{21}b_{12} + a_{22}b_{22} + \cdots + a_{2m}b_{m2} \tag{4.0.8}$$
$$c_{2p} = a_{21}b_{1p} + a_{22}b_{2p} + \cdots + a_{2m}b_{mp} \tag{4.0.9}$$
$$c_{n1} = a_{n1}b_{11} + a_{n2}b_{21} + \cdots + a_{nm}b_{m1} \tag{4.0.10}$$
$$c_{n2} = a_{n1}b_{12} + a_{n2}b_{22} + \cdots + a_{nm}b_{m2} \tag{4.0.11}$$
$$c_{np} = a_{n1}b_{1p} + a_{n2}b_{2p} + \cdots + a_{nm}b_{mp} \tag{4.0.12}$$

Matrix multiplication is also distributive. If A and B are $m \times n$ matrices and C and D are $n \times p$ matrices, then

$$A(C + D) = AC + AD \qquad \text{and} \qquad (A + B)C = AC + BC \tag{4.0.13}$$

The trace of an $n \times n$ square matrix A is defined to be

$$\mathrm{Tr}(A) \equiv \sum_{i=1}^{n} a_{ii} \tag{4.0.14}$$

The determinant of an arbitrary 2×2 matrix A is given by

$$\det(M) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21} \tag{4.0.15}$$

The determinant of an arbitrary 3×3 matrix A is given by

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \tag{4.0.16}$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}) \tag{4.0.17}$$

The diagonal space of an $n \times n$ matrix A, denoted $\text{dis}(A)$ is defined as

$$\text{dis}(A) \equiv \left] \begin{matrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{matrix} \right[= \frac{1}{2} \sum_{n=1}^{n} \sum_{k=1}^{n} (a_{kn}a_{nk} - a_{nn}a_{kk}) \tag{4.0.18}$$

For a 3×3 matrix A, the diagonal space can be helpful in finding eigenvalues and is given by

$$\text{dis}(A) \equiv \left] \begin{matrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{matrix} \right[= \frac{1}{2} \sum_{n=1}^{3} \sum_{k=1}^{3} (a_{kn}a_{nk} - a_{nn}a_{kk}) \tag{4.0.19}$$

$$= (a_{12}a_{21} + a_{13}a_{31} + a_{23}a_{32}) - (a_{11}a_{22} + a_{11}a_{33} + a_{22}a_{33}) \tag{4.0.20}$$

The eigenvalues λ_i and eigenvectors $\vec{v}_i$ of a matrix A are given by solving

$$\det(A - \lambda I) = 0 \qquad \text{and} \qquad A\vec{v}_i = \lambda_i \vec{v}_i \qquad \text{or} \qquad (A - \lambda_i I)\vec{v}_i = 0. \tag{4.0.21}$$

By defining the determinant, trace, and diagonal space the way we have above, the eigenvalues of a 3×3 matrix A become the solutions for λ of

$$0 = -\lambda^3 + \text{Tr}(A)\lambda^2 + \text{dis}(A)\lambda + \det(A) \tag{4.0.22}$$

Similarly, the eigenvalues $\lambda_\pm$ of a 2×2 matrix A become the solutions for λ of

$$0 = \lambda^2 - \text{Tr}(A)\lambda + \det(A) \implies \lambda_\pm = \frac{1}{2}\left(\text{Tr}(A) \pm \sqrt{\text{Tr}(A)^2 - 4\det(A)}\right) \tag{4.0.23}$$

$$\implies \lambda_\pm = \frac{1}{2}\left((a_{11} + a_{22}) \pm \sqrt{(a_{11} - a_{22})^2 + 4a_{12}a_{21}}\right) \tag{4.0.24}$$

Properties of Matrices: A matrix has an inverse if and only if it has a non-zero determinant (it is non singular).

$$A^T = A \implies \text{Symmetric} \tag{4.0.25}$$

$$A^T = A^{-1} \implies \text{Anti-Symmetric} \tag{4.0.26}$$

$$A^* = A \implies \text{Real} \tag{4.0.27}$$

$$A^* = -A \implies \text{Imaginary} \tag{4.0.28}$$

$$A^\dagger = A \implies \text{Hermitian} \tag{4.0.29}$$

$$A^T A = AA^T = I \implies A^T = A^{-1} \implies \text{Orthogonal} \tag{4.0.30}$$

$$A^\dagger A = AA^\dagger \implies A^\dagger = A^{-1} \implies \text{Unitary} \tag{4.0.31}$$

$$\det(A) = 0 \implies \text{Singular} \tag{4.0.32}$$

Abstract Algebra and Number Theory

Definition 0.1: Ring

A ring is a triple $(R, \oplus, \odot)$ such that

(i) R is a set.

(ii) $\oplus$ is a function (called ring addition) and $R \times R$ is a subset of the domain of $\oplus$. For $(a, b) \in R \times R$, $a \oplus b$ denotes the image of (a, b) under $\oplus$.

(iii) $\odot$ is a function (called ring multiplication) and $R \times R$ is a subset of the domain of $\odot$. For $(a, b) \in R \times R$, $a \odot b$ (and also ab) denotes the image of (a, b) under $\odot$.

and such that the following eight statements (axioms) hold:

(1) [Closure of addition]: $a + b \in R$ for all $a, b \in R$.

(2) [Associative addition]]: $a + (b + c) = (a + b) + c$ for all $a, b, c \in R$.

(3) [Commutative addition]: $a + b = b + a$ for all $a, b \in R$.

(4) [Additive identity]: There exists an element in R, denoted by 0_R and called 'zero R', such that $a = a + 0_R = a$ and $a = 0_R + a$ for all $a \in R$.

(5) [Additive inverses]: For each $a \in R$ there exists an element in R, denoted by $-a$ and called 'negative a', such that $a + (-a) = 0_R$.

(6) [Closure for multiplication]: $ab \in R$ for all $a, b \in R$.

(7) [Associative multiplication]: $a(bc) = (ab)c$ for all $a, b, c \in R$.

(8) [Distributive laws]: $a(b + c) = ab + ac$ and $(a + b)c = ac + bc$ for all $a, b, c \in R$.

Definition 0.2: Commutative Ring

Let R be a ring. Then R is called commutative if

(9) [Commutative multiplication]: $ab = ba$ for all $a, b \in R$.

Definition 0.3: Ring With Identity

Let R be a ring. We say that R is a ring with identity if there exists an element, denoted by 1_R and called 'one R', such that

(10) [Multiplicative identity]: $a = 1_R \cdot a$ and $a = a \cdot 1_R$ for all $a \in R$.

Definition 0.4: Subring

Let $(R, \oplus, \odot)$ be a ring and S a subset of R. Then $(S, \oplus, \odot)$ is called a subring of $(R, \oplus, \odot)$ provided that $(S, \oplus, \odot)$ is a ring.

Theorem 0.1: Subring Theorem

Suppose that R is a ring and $S \subseteq R$. Then S is a subring of R if and only if the following four conditions hold:

(i) $0_R \in S$.

(ii) S is closed under addition (that is: if $a, b \in S$, then $a + b \in S$).

(iii) S is closed under multiplication (that is: if $a, b \in S$, then $ab \in S$).

(iv) S is closed under negatives (that is: if $a \in S$, then $-a \in S$).

Definition 0.5: Integral Domain

An ring R is called an integral domain provided that R is commutative, R has identity, $1_R \neq 0_R$ and for any $a, b \in R$, $ab = 0_R \implies a = 0_R$ or $b = 0_R$.

Definition 0.6: Injective & Surjective

Let $f : R \to S$ be a function.

(a) f is said to be injective provided: $f(a) = f(b) \implies a = b$ for all $a, b \in R$.

(b) f is said to be surjective provided: for every $y \in S$ there exists $x \in R$ such that $f(x) = y$.

(c) f is bijective if it is both injective and surjective.

Definition 0.7: Equivalence Relation

Let $\sim$ be a relation on a set A (that is a relation from A to A). Then

(a) $\sim$ is called reflexive if $a \sim a$ for all $a \in A$.

(b) $\sim$ is called symmetric if $[a \sim b \implies b \sim a]$ for all $a, b \in A$.

(c) $\sim$ is called transitive if $[a \sim b$ and $b \sim c \implies a \sim c]$ for all $a, b, c \in A$.

(d) $\sim$ is called an equivalence relation if $\sim$ is reflexive, symmetric and transitive.

Definition 0.8: Unit

Let R be a ring with identity.

(a) Let $u \in R$. Then u is called a unit in R if there exists an element in R, denoted by u^{-1} and called 'u-inverse', with $uu^{-1} = 1_R = u^{-1}u$.

(b) Let $u, v \in R$. Then v is called an (multiplicative) inverse of u if $uv = 1_R = vu$.

(c) Let $e \in R$. Then e is called an (multiplicative) identity of R, if $ea = a = ae$ for all $a \in R$.

Definition 0.9: Common divisor

(a) Let R be a ring and $a, b, c \in R$. We say that c is a common divisor of a and b in R provided that $c|a$ and $c|b$.

(b) Let a, b and d be integers. We say that d is a greatest common divisor of a and b in $\mathbb{Z}$, and we write $d = \gcd(a, b)$ provided that

   (i) d is a common divisor of a and b in $\mathbb{Z}$.

   (ii) If c is a common divisor of a and b in $\mathbb{Z}$ then $c \leq d$.

Definition 0.10: Isomorphism and Homomorphism

Let $(R, +, \cdot)$ and $(S, \oplus, \odot)$ be rings and let $f : R \to S$ be a function.

(a) f is called a homomorphism from $(R, +, \cdot)$ to $(S, \oplus, \odot)$ if

   (i) [f respects addition]: $f(a + b) = f(a) \oplus f(b)$, and

   (ii) [f respects multiplication]: $f(a \cdot b) = f(a) \odot f(b)$

   for all $a, b \in R$.

(b) f is called an isomorphism from $(R, +, \cdot)$ to $(S, \oplus, \odot)$, if f is a homomorphism from $(R, +, \cdot)$ to $(S, \oplus, \odot)$ and f is bijective.

(c) $(R, +, \cdot)$ is called isomorphic to $(S, \oplus, \odot)$, if there exists an isomorphism from $(R, +, \cdot)$ to $(S, \oplus, \odot)$.

Definition 0.11: Ideal

Let I be a subset of the ring R

(a) We say that I absorbs R if $ra \in I$ and $ar \in I$ for all $a \in I$, $r \in R$.

(b) We say that I is an ideal of R (denoted $I \lhd R$) if I is a subring of R and I absorbs R.

Theorem 0.2: Ideal Theorem

Let I be a subset of the ring R. Then I is an ideal in R ($I \lhd R$) if and only if the following four conditions hold:

(i) $0_R \in I$.

(ii) $a + b \in I$ for all $a, b \in I$.

(iii) $ra \in I$ and $ar \in I$ for all $a \in I$ and $r \in R$.

(iv) $-a \in I$ for all $a \in I$.

Definition 0.12: Principle Ideal

Let R be a ring.

(a) Let $a \in R$. Then $aR = \{ar : a \in R\}$.

(b) Suppose R is commutative and $I \subseteq R$. Then I is called a principal ideal in R if $I = aR$ for some $a \in R$. This can be denoted (a).

Definition 0.13: Ideal modulus

Let I be an ideal in the ring R. The relation '$\equiv \pmod{I}$' on R is defined by $a \equiv b \pmod{I}$ if $a - b \in I$.

Definition 0.14: Cosets

(a) Let $a \in I$. Then $a + I$ (the coset of I in R containing a) denotes the equivalence class of '$\equiv$mod I' containing a. so

$$a + I = \{b \in R | a \equiv b \mod I\} = \{b \in R | a - b \in I\}. \tag{5.0.1}$$

(b) R/I is the set of cosets if I in R/I and the set of equivalence classes of '$\equiv$ mod I'. So

$$R/I = \{a + I | a \in R\}. \tag{5.0.2}$$

Definition 0.15: Ideal operations

Let I be an ideal in the ring R. Then we define an addition $+$ and multiplication $\cdot$ on R by

$$(a + I) + (b + I) = (a + b) + I \quad \text{and} \quad (a + I) \cdot (b + I) = ab + I$$

for all $a, b \in R$.

Definition 0.16: Kernal

Let $f : R \to S$ be a homomorphism of rings. Then $\ker f$ (the kernel of f) is

$$\ker f = \{a \in R | f(a) = 0_R\}.$$

Definition 0.17: natural homomorphism

Let I be an ideal in the ring R. The function

$$\pi : R \to R/I, r \to r + I$$

is called the natural homomorphism from R to R/I.

Theorem 0.3: First Isomorphism Theorem

Let $f : R \to S$ be a ring homomorphism. Recall that $\Im f = \{f(a) | a \in R\}$. The function

$$\bar{f} : R/\ker f \to \Im f, \quad (a + \ker f) \longmapsto f(a) \tag{5.0.3}$$

is a well-defined ring isomorphism. In particular $R/\ker f$ and $\Im f$ are isomorphic rings.

Differential Equations

Definition 0.1: Del Operator

The Del operator with respect to n-dimensions:

$$\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n}\right) = \sum_{i=1}^{n} \vec{e}_i \frac{\partial}{\partial x_i} \tag{6.0.1}$$

The gradient of a 3-dimensional function (cartesian, spherical, and cylindrical coordinates)

$$\begin{aligned}
\text{grad } f = \nabla f &= \frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y} + \frac{\partial f}{\partial z}\hat{z} && (6.0.2)\\
&= \frac{\partial f}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\theta} + \frac{1}{r\sin(\theta)}\frac{\partial f}{\partial \phi}\hat{\phi} && (6.0.3)\\
&= \frac{\partial f}{\partial \rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial f}{\partial \phi}\hat{\phi} + \frac{\partial f}{\partial z}\hat{z} && (6.0.4)
\end{aligned}$$

The curl of a vector is the limit as the volume goes to zero of the ratio of the integral of it's cross product (with respect to the normal) over a closed surface, to the volume enclosed by the surface [11]. The curl of a 3-dimensional function (cartesian, spherical, and cylindrical coordinates)

$$\begin{aligned}
\text{curl } \vec{A} = \nabla \times \vec{A} &= \lim_{V\to 0} \frac{1}{V} \oint_S (\hat{n} \times \vec{A}) da \qquad \text{(definition)} && (6.0.5)\\
&= \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{x} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{y} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{z} && (6.0.6)\\
&= \frac{1}{r\sin(\theta)}\left[\frac{\partial}{\partial \theta}\sin(\theta)A_\phi - \frac{\partial A_\theta}{\partial \phi}\right]\hat{r} + \left[\frac{1}{r\sin(\theta)}\frac{\partial A_r}{\partial \phi} - \frac{1}{r}\frac{\partial}{\partial r}(rA_\phi)\right]\hat{\theta}\\
&\qquad + \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial \theta}\right]\hat{\phi} && (6.0.7)\\
&= \left[\frac{1}{\rho}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right]\hat{\rho} + \left[\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}\right]\hat{\phi} + \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}(\rho A_\phi) - \frac{\partial A_\rho}{\partial \phi}\right]\hat{z} && (6.0.8)
\end{aligned}$$

The divergence of a vector is the limit of its surface integral per unit volume as the volume enclosed by the surface goes to zero [11]. The divergence of a 3-dimensional function (cartesian, spherical, and cylindrical coordinates) is

$$\begin{aligned}
\text{div } \vec{A} = \nabla \cdot \vec{A} &= \lim_{V\to 0} \frac{1}{V} \oint_S (\vec{A} \cdot \hat{n}) da \qquad \text{(definition)} && (6.0.9)\\
&= \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} && (6.0.10)\\
&= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin(\theta)}\frac{\partial}{\partial \theta}\sin(\theta)A_\theta + \frac{1}{r\sin(\theta)}\frac{\partial A_\phi}{\partial \phi} && (6.0.11)\\
&= \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z} && (6.0.12)
\end{aligned}$$

The Laplace Operator (expanded in 3-dimensions below)

$$\Delta = \nabla \cdot \nabla = \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{6.0.13}$$

The general solution to Laplaces equation in spherical coordinates with no ϕ dependence (where P_ℓ denotes the Legendre polynomials) is

$$\nabla^2 V(r,\theta) = 0 \implies V(r,\theta) = \sum_{\ell=0}^{\infty}\left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right)P_\ell(\cos\theta) \tag{6.0.14}$$

Differentiation

Definition 1.1: Derivative of a function

The derivative of a function represents an infinitesimal change in the function with respect to one of its variables. Let $f:\mathbb{R}\to\mathbb{R}$. Then the derivative of f with respect to a variable x is given by

$$\frac{d}{dx}f(x) \equiv f'(x) \equiv \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x} \equiv \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x-\Delta x)}{2\Delta x} \tag{6.1.1}$$

Definition 1.2: Partial Derivative of a function

Partial derivatives are defined as derivatives of a function of multiple variables when all but the variable of interest are held fixed during the differentiation. Let $f:\mathbb{R}^z\to\mathbb{R}$, with $z\in\mathbb{N}$. Then the partial derivative of f with respect to a variable x_m is given by

$$\frac{\partial}{\partial x_m}f(x_1,\dots,x_n) \equiv \lim_{\Delta x\to 0}\frac{f(x_1,\dots,x_m+\Delta x,\dots,x_n)-f(x_1,\dots,x_m,\dots,x_n)}{\Delta x} \tag{6.1.2}$$

1.1: Derivative of an n^{th} degree polynomial.

Let $f:\mathbb{R}\to\mathbb{R}$ such that $f(x)\in\mathbb{P}_n$ of the form $f(x)=b_nx^n+b_{n-1}x^{n-1}+\dots++b_1x+b_0$. Then the m^{th} derivative of $f(x)$ is given by

$$\frac{d^m}{dx^m}f(x) = f^{(m)}(x) = \sum_{k=0}^{n-m}\frac{(n-k)!}{(n-m-k)!}b_{n-k}x^{n-m-k}. \tag{6.1.3}$$

1.2: Derivative of a sum of simple exponential functions

Let $f:\mathbb{R}\to\mathbb{R}$ be a continuous function such that $f(x)=b_ne^{k_nx}+b_{n-a}e^{k_{n-1}x}+\dots+b_1e^{k_1x}+b_0e^{k_0x}$, with b_n and k_n as constants. The m^{th} derivative is then given by

$$\frac{d^m}{dx^m}f(x) = \frac{d^m}{dx^m}\sum_{i=0}^{n}b_ie^{k_ix} = \sum_{i=0}^{n}b_{n-i}k_{n-i}^m e^{k_{n-i}x}. \tag{6.1.4}$$

1.3: Repeating product rule applied to arbitrary functions

Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be continuous and differentiable functions of the variable x. Then the m^{th} derivative of $f(x)g(x)$ with respect to x is

$$\frac{d^m}{dx^m} f(x)g(x) = \sum_{i=0}^{n} \binom{n}{i} \left[\frac{d^i}{dx^i} f(x)\right]\left[\frac{d^{n-i}}{dx^{n-i}} g(x)\right].$$

Legendre differential equation

The Legendre polynomials are normalized solutions to the Legendre differential equation

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \ell(\ell+1)y = 0. \tag{6.2.1}$$

which have the general solution below. For even ℓ, we must take $a_1 = 0$ to obtain a convergent solution, and for odd ℓ, we must take $a_0 = 0$.

$$y = a_0 \left[1 - \frac{\ell(\ell+1)}{2!}x^2 + \frac{\ell(\ell+1)(\ell-2)(\ell+3)}{4!}x^4 - \cdots\right] \tag{6.2.2}$$

$$+ a_1 \left[x - \frac{(\ell-1)(\ell+2)}{3!}x^3 + \frac{(\ell-1)(\ell+2)(\ell-3)(\ell+4)}{5!}x^5 - \cdots\right] \tag{6.2.3}$$

The Legendre polynomial $P_\ell(z)$ can be defined by the contour integral

$$P_\ell(z) = \frac{1}{2\pi i}\oint (1 - 2tz + t^2)^{-1/2} t^{-\ell-1} dt. \tag{6.2.4}$$

The first few Legendre Polynomials follow as:

$$P_0(x) = 1 \tag{6.2.5}$$

$$P_1(x) = x \tag{6.2.6}$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1) \tag{6.2.7}$$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x) \tag{6.2.8}$$

$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3) \tag{6.2.9}$$

$$P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x) \tag{6.2.10}$$

$$\vdots$$

The Rodrigues representation provides a formula for solving for the Legendre Polynomials

$$P_\ell(x) = \frac{1}{2^\ell \ell!}\frac{d^\ell}{dx^\ell}(x^2-1)^\ell \tag{6.2.11}$$

The Legendre polynomials are orthogonal over $(-1, 1)$ with weighting function 1 and satisfy

$$\int_{-1}^{1} P_m(x)P_n(x)dx = \frac{2}{2n+1}\delta_{mn} \qquad \text{and} \qquad \int_{0}^{1} P_m(x)P_n(x)dx = \frac{1}{2n+1}\delta_{mn} \tag{6.2.12}$$

The associated Legendre differential equation is given by

$$\frac{d}{dx}\left[(1-x^2)\frac{dy}{dx}\right] + \left[\ell(\ell-1) - \frac{m^2}{1-x^2}\right] y = 0 \tag{6.2.13}$$

When $m, \ell \in \mathbb{Z}^+$ and $m \leq \ell$, the solutions to the above equation are the **associated Legendre polynomials**,

$$P_\ell^m(x) = (-1)^m (1-x^2)^{m/2} \frac{d^m}{dx^m} P_\ell(x) = \frac{(-1)^m}{2^\ell \ell!} (1-x^2)^{m/2} \frac{d^{\ell+m}}{dx^{\ell+m}} (x^2-1)^\ell \tag{6.2.14}$$

The associated Legendre polynomials for $m < 0$ are defined by

$$P_\ell^{-m}(x) = (-1)^m \frac{(\ell-m)!}{(\ell+m)!} P_\ell^m(x) \tag{6.2.15}$$

The first few associated Legendre Polynomials with $m > 0$ are

$$P_0^0(x) = 1 \tag{6.2.16}$$
$$P_1^0(x) = x \tag{6.2.17}$$
$$P_1^1(x) = -(1-x^2)^{1/2} \tag{6.2.18}$$
$$P_2^0(x) = \frac{1}{2}(3x^2-1) \tag{6.2.19}$$
$$P_2^1(x) = -3x(1-x^2)^{1/2} \tag{6.2.20}$$
$$P_2^2(x) = 3(1-x^2) \tag{6.2.21}$$
$$P_3^0(x) = \frac{1}{2}x(5x^2-3) \tag{6.2.22}$$
$$P_3^1(x) = \frac{3}{2}(1-5x^2)(1-x^2)^{1/2} \tag{6.2.23}$$
$$P_3^2(x) = 15x(1-x^2) \tag{6.2.24}$$
$$P_3^3(x) = -15(1-x^2)^{3/2} \tag{6.2.25}$$
$$\vdots$$

The first few associated Legendre Polynomials with $m < 0$ are

$$P_1^{-1}(x) = \frac{1}{2}(1-x^2)^{1/2} \tag{6.2.26}$$
$$P_2^{-1}(x) = \frac{1}{2}x(1-x^2)^{1/2} \tag{6.2.27}$$
$$P_2^{-2}(x) = \frac{1}{8}(1-x^2) \tag{6.2.28}$$
$$P_3^{-1}(x) = \frac{1}{8}(5x^2-1)(1-x^2)^{1/2} \tag{6.2.29}$$
$$P_3^{-2}(x) = \frac{1}{8}x(1-x^2) \tag{6.2.30}$$
$$P_3^{-3}(x) = \frac{1}{48}(1-x^2)^{3/2} \tag{6.2.31}$$
$$P_4^{-1}(x) = \frac{1}{8}(7x^3-3x)(1-x^2)^{1/2} \tag{6.2.32}$$
$$\vdots$$

The associated Legendre polynomials are orthogonal over $[-1, 1]$ such that

$$\int_{-1}^{1} P_\ell^m(x) P_{\ell'}^m(x) dx = \frac{2}{2\ell+1} \frac{(\ell+1)!}{(\ell-1)!} \delta_{\ell\ell'} \tag{6.2.33}$$

$$\int_{-1}^{1} P_\ell^m(x) P_\ell^{m'}(x) \frac{dx}{1-x^2} = \frac{(\ell+m)!}{m(\ell-m)!} \delta_{mm'} \tag{6.2.34}$$

The derivative about the origin for an associated Legendre polynomial is given by

$$\left[\frac{dP_\ell^m(x)}{dx}\right]_{x=0} = \frac{2^{m+1} \sin\left[\frac{1}{2}\pi(\ell+m)\right] \Gamma\left(\frac{1}{2}\ell + \frac{1}{2}m + 1\right)}{\pi^{1/2} \Gamma\left(\frac{1}{2}\ell - \frac{1}{2}m + \frac{1}{2}\right)} \tag{6.2.35}$$

Laguerre differential equation

The general associated Laguerre differential equation is defined by,

$$xy''(x) + (k+1-x)y'(x) + ny(x) = 0. \tag{6.3.1}$$

A solution to the above differential equation is any generalized Laguerre polynomial $L_n^k(x)$. The Rodrigues representation for the associated Laguerre polynomials is

$$L_n^k = \frac{e^x x^{-k}}{n!}\frac{d^n}{dx^n}(e^{-x}x^{n+k}) = (-1)^k \frac{d^k}{dx^k}[L_{n+k}(x)] = \sum_{m=0}^{n} \frac{(-1)^m (n+k)! x^m}{(n-m)!(k+m)!m!}. \tag{6.3.2}$$

An alternate definitions of the Laguerre polynomials is given as

$$L_n^k(x) = \frac{1}{n!}\sum_{i=0}^{n} \frac{n!}{i!}\binom{k+n}{n-i}(-x)^i. \tag{6.3.3}$$

The associated Laguerre polynomials are orthogonal over $[0,\infty)$ in the following way,

$$\int_0^\infty e^{-x}x^k L_n^k(x) L_m^k(x)dx = \frac{(n+k)!}{n!}\delta_{mn}. \tag{6.3.4}$$

They also satisfy,

$$\int_0^\infty e^{-x}x^{k+1}[L_n^k(x)]^2 dx = \frac{(n+k)!}{n!}(2n+k+1). \tag{6.3.5}$$

The first few Associated Laguerre polynomials are

$$L_0^k(x) = 1 \tag{6.3.6}$$

$$L_1^k(x) = -x+k+1 \tag{6.3.7}$$

$$L_2^k(x) = \frac{1}{2}\left[x^2 - 2(k+2)x + (k+1)(k+2)\right] \tag{6.3.8}$$

$$L_3^k(x) = \frac{1}{6}\left[-x^3 + 3(k+3)x^2 - 3(k+2)(k+3)x + (k+1)(k+2)(k+3)\right] \tag{6.3.9}$$

$$\vdots \tag{6.3.10}$$

A special case of the Associated Laguerre polynomials occurs when $k=0$ which can be defined by a sum, the Rodrigues representation, or the contour integral (respectively)

$$L_n(x) = \sum_{k=0}^{n} \frac{(-1)^k}{k!}\binom{n}{k}x^k \equiv \frac{e^x}{n!}\frac{d^n}{dx^n}(x^n e^{-x}) \equiv \frac{1}{2\pi i}\oint \frac{e^{-zt/(1-t)}}{(1-t)t^{n+1}}dt \tag{6.3.11}$$

Second-order Homogeneous

$$\ddot{x} + Ax = 0 \implies x(t) = C_1 e^{i\sqrt{A}t} + C_2 e^{-i\sqrt{A}t} \tag{6.4.1}$$
$$\implies x(t) = C_1 \cos(\sqrt{A}t) + C_2 \sin(\sqrt{A}t) \tag{6.4.2}$$
$$\ddot{x} - Ax = 0 \implies x(t) = C_1 e^{\sqrt{A}t} + C_2 e^{-\sqrt{A}t} \tag{6.4.3}$$
$$\implies x(t) = C_1 \sinh(\sqrt{A}t) + C_2 \cosh(\sqrt{A}t) \tag{6.4.4}$$

4.1: $\ddot{x} + A\dot{x} + Bx = 0$

Given any differential equation of the form $\ddot{x} + A\dot{x} + Bx = 0$, a general solution of the following form exists:

$$x(t) = C_1 \exp\left[-\frac{1}{2}t(\sqrt{A^2 - 4B} + A)\right] + C_2 \exp\left[\frac{1}{2}t(\sqrt{A^2 - 4B} - A)\right]. \tag{6.4.5}$$

Following this, three special cases arise

(i) $A^2 > 4B \implies$

$$x(t) = C_1 \exp\left[\frac{-At}{2}\right] \cosh\left(\frac{t\sqrt{A^2 - 4B}}{2}\right) + C_2 \exp\left[\frac{-At}{2}\right] \sinh\left(\frac{t\sqrt{A^2 - 4B}}{2}\right) \tag{6.4.6}$$

(ii) $A^2 < 4B \implies$

$$x(t) = C_1 \exp\left[\frac{-At}{2}\right] \cos\left(\frac{t\sqrt{4B - A^2}}{2}\right) + C_2 \exp\left[\frac{-At}{2}\right] i \sin\left(\frac{t\sqrt{4B - A^2}}{2}\right) \tag{6.4.7}$$

(iii) $A^2 = 4B \implies$

$$x(t) = C_1 \exp\left[\frac{-At}{2}\right] \tag{6.4.8}$$

Second-order Linear Ordinary

$$\ddot{x} + Ax = B \implies x(t) = \frac{B}{A} + C_1 e^{i\sqrt{A}t} + C_2 e^{-i\sqrt{A}t} \tag{6.5.1}$$

$$\implies x(t) = \frac{B}{A} + C_1 \cos(\sqrt{A}t) + C_2 \sin(\sqrt{A}t) \tag{6.5.2}$$

$$\ddot{x} - Ax = B \implies x(t) = -\frac{B}{A} + C_1 e^{\sqrt{A}t} + C_2 e^{-\sqrt{A}t} \tag{6.5.3}$$

$$\implies x(t) = -\frac{B}{A} + C_1 \sinh(\sqrt{A}t) + C_2 \cosh(\sqrt{A}t) \tag{6.5.4}$$

$$\ddot{x} + x = t(A - t) \implies x(t) = C_1 \cos(t) + C_2 \sin(t) - t^2 + At + 2 \tag{6.5.5}$$

$$\ddot{x} + A\dot{x} + Bx = t \implies x(t) = C_1 \exp\left[-\frac{1}{2}t(\sqrt{A^2 - 4B} + A)\right] \tag{6.5.6}$$

$$+ C_2 \exp\left[\frac{1}{2}t(\sqrt{A^2 - 4B} - A)\right] - \frac{A}{B^2} + \frac{t}{B} \tag{6.5.7}$$

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f_0 e^{i\omega t} \implies x(t) = \frac{f_0 e^{i\omega t}}{\omega_0^2 - \omega^2 + 2\beta i\omega} \tag{6.5.8}$$

$$\implies x(t) = A\cos(\omega t - \delta) + A_{tr} e^{-\beta t} \cos(\omega_1 t - \delta_{tr}) \tag{6.5.9}$$

$$\delta = \arctan\left(\frac{2\beta\omega}{\omega_0^2 - \omega^2}\right) \tag{6.5.10}$$

$$\implies x(t) = A\cos(\omega t - \delta) + e^{-\beta t}[B_1 \cos(\omega_1 t) + B_2 \sin(\omega_1 t)] \tag{6.5.11}$$

$$\ddot{x} + 2\beta\dot{x} + x = te^{-\alpha t} \implies x(t) = C_1 e^{-\alpha t} + C_2 t e^{-\alpha t} + C_3 e^{-\beta t} \sin(\omega_1 t) + C_4 e^{-\beta t} \cos(\omega_1 t) \tag{6.5.12}$$

$$\omega_1^2 = 1 - \beta^2 \tag{6.5.13}$$

Higher Order Differential Equations

6.1: Particular solution to a sum of exponential functions

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous and differential function and let C_n, a_n, b_n and k_n be constant for all n. Given a differential equation of the form

$$\frac{d^n}{dt^n} f(t) + \frac{d^{n-1}}{dt^{n-1}} C_{n-1} f(t) + \cdots + \frac{d}{dt} C_1 f(t) + C_0 f(t) = \sum_{i=0}^{\ell} a_i e^{k_i t},$$

A solution of the following form exists:

$$f(t) = \sum_{i=0}^{\ell} b_i e^{k_i t} = \sum_{i=0}^{\ell} \frac{a_i e^{k_i t}}{k_{\ell-i}^n + C_{n-1} k_{\ell-i}^{n-1} + \cdots + C_0}$$

Frobenius Method

Consider a second-order ordinary differential equation

$$y'' + P(x)y' + Q(x)y = 0. \tag{6.7.1}$$

If P(x) and Q(x) remain finite at $x = x_0$, then x_0 is called an ordinary point. If either P(x) or Q(x) diverges as $x \to x_0$, then x_0 is called a singular point. If either P(x) or Q(x) diverges as $x \to x_0$ but $(x - x_0)P(x)$ and $(x - x_0)^2 Q(x)$ remain finite as $x \to x_0$, then $x = x_0$ is called a **regular singular point** (or nonessential singularity)[17].

If $x = 0$ is a regular singular point of the ordinary differential equation, $y''(x) + P(x)y'(x) + Q(x)y(x) = 0$, solutions may be found by the Frobenius method or by expansion in a Laurent series. In the Frobenius method, assume a solution of the form

$$y(x) = x^{\alpha} \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+\alpha}. \tag{6.7.2}$$

Taking the first and second derivative of this with respect to x yield

$$y'(x) = \sum_{n=0}^{\infty} (n+\alpha) a_n x^{n+\alpha-1}, \qquad \text{and} \qquad y''(x) = \sum_{n=0}^{\infty} (n+\alpha)(n+\alpha-1) a_n x^{n+\alpha-2}. \tag{6.7.3}$$

If we allow $xP(x) = p_0 + p_1 x + p_2 x^2 + \cdots$ and $x^2 Q(x) = q_0 + q_1 x + q_2 x^2 + \cdots$, then we can consolidate coefficients, take the limit as $x \to 0$ and arise at an **Indicial equation** to solve for possible α values

$$0 = \alpha(\alpha - 1) + \alpha p_0 + q_0 \quad \text{with} \quad \begin{cases} p_0 = \lim\limits_{x\to 0} xP(x) \\ q_0 = \lim\limits_{x\to 0} x^2 Q(x) \end{cases} \tag{6.7.4}$$

7.1: Fuchs's Theorem [17]

At least one power series solution will be obtained when applying the Frobenius method if the expansion point is an ordinary, or regular, singular point. The number of roots is given by the roots of the indicial equation.

When the roots of the indicial equation are the same (or sometimes when they differ by an integer), there will be a Frobenious series solution $S_1(x)$ and another solution (where $S_2(x)$ is another Frobenious series) of the form

$$y(x) = S_1(x)\ln(x) + S_2(x) \implies y(x) = \ln(x) \sum_{n=0}^{\infty} a_n x^{n+\alpha} + \sum_{n=0}^{\infty} b_n x^{n+\beta}. \tag{6.7.5}$$

Integrals

Basic indefinite integrals (c =constant)

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\arctan\left(\frac{x}{a}\right)+c \tag{7.0.1}$$

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin\left(\frac{x}{|a|}\right)+c = \arctan\left(\frac{x}{\sqrt{a^2-x^2}}\right)+c \tag{7.0.2}$$

$$\int \frac{dx}{x+x^2} = \ln\left(\frac{x}{1+x}\right)+c \tag{7.0.3}$$

$$\int \frac{dx}{\sqrt{x^2-1}} = \operatorname{arccosh}(x)+c \tag{7.0.4}$$

$$\int \frac{dx}{x\sqrt{x^2-1}} = \arccos\left(\frac{1}{x}\right)+c \tag{7.0.5}$$

$$\int \frac{dx}{(a^2+x^2)^{3/2}} = \frac{x}{a^2\sqrt{a^2+x^2}}+c \tag{7.0.6}$$

$$\int \frac{xdx}{(a^2+x^2)^{3/2}} = -\frac{1}{\sqrt{a^2+x^2}}+c \tag{7.0.7}$$

$$\int \frac{dx}{1-x^2} = \operatorname{arctanh}(x)+c \tag{7.0.8}$$

$$\int \frac{dx}{\sqrt{a^2+x^2}} = \operatorname{arcsinh}\left(\frac{x}{a}\right)+c = \ln\left|x+\sqrt{a^2+x^2}\right|+c \tag{7.0.9}$$

$$\int \frac{xdx}{1+x^2} = \frac{1}{2}\ln(1+x^2)+c \tag{7.0.10}$$

$$\int \frac{xdx}{\sqrt{1+x^2}} = \sqrt{1+x^2}+c \tag{7.0.11}$$

$$\int \frac{\sqrt{x}dx}{\sqrt{1-x}} = \arcsin(\sqrt{x})-\sqrt{x(1-x)}+c \tag{7.0.12}$$

$$\int \frac{x^2}{a^2+x^2}dx = \frac{-x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)+c \tag{7.0.13}$$

$$\int \frac{1}{(a^2+x^2)^2}dx = \frac{x}{2a^2(x^2+a^2)}+\frac{1}{2a^3}\arctan\left(\frac{x}{a}\right)+c \tag{7.0.14}$$

$$\int \frac{x^2}{(a^2+x^2)^2}dx = \frac{-x}{2(x^2+a^2)}+\frac{1}{2a}\arctan\left(\frac{x}{a}\right)+c \tag{7.0.15}$$

$$\int \ln(x) = x\ln(x)-x+c \tag{7.0.16}$$

Exponential integrals

$$\int_{-\infty}^{\infty} \frac{e^{-iax}}{(1+x^2)}dx = \pi e^{-|a|} \tag{7.0.17}$$

Trigonometric integrals

$$\int \tan(x)dx = -\ln(\cos(x)) + c \tag{7.0.18}$$

$$\int \tanh(x)dx = \ln(\cosh(x)) + c \tag{7.0.19}$$

$$\int \sin^2(x)dx = \frac{1}{2}\big(x - \sin(x)\cos(x)\big) + c = \frac{1}{4}\big(2x - \sin(2x)\big) + c \tag{7.0.20}$$

$$\int \cos^2(x)dx = \frac{1}{2}\big(x + \sin(x)\cos(x)\big) + c = \frac{1}{4}\big(2x + \sin(2x)\big) + c \tag{7.0.21}$$

$$\int \sin^2(x)\cos(x)dx = \frac{1}{3}\sin^3(x) + c \tag{7.0.22}$$

$$\int \cos^2(x)\sin(x)dx = -\frac{1}{3}\cos^3(x) + c \tag{7.0.23}$$

$$\int \sin^3(x)dx = -\frac{1}{3}\cos(x)\big(\sin^2(x) + 2\big) + c \tag{7.0.24}$$

$$\int x\sin^2(x)dx = \frac{1}{4}\big(x^2 - x\sin(2x) - \frac{1}{2}\cos(2x)\big) + c \tag{7.0.25}$$

$$\int x^2\sin^2(x)dx = \frac{x^3}{6} - \left(\frac{x^2}{4} - \frac{1}{8}\right)\sin(2x) - \frac{x}{4}\cos(2x) + c \tag{7.0.26}$$

$$\int x^n\sin(ax)dx = -\frac{x^n}{a}\cos(ax) + \frac{n}{a}\int x^{n-1}\cos(ax)dx \tag{7.0.27}$$

$$\int x^n\cos(ax)dx = \frac{x^n}{a}\sin(ax) - \frac{n}{a}\int x^{n-1}\sin(ax)dx \tag{7.0.28}$$

The Wallis Cosine Formula

$$\int_0^{\pi/2}\cos^n(x)dx = \int_0^{\pi/2}\sin^n(x)dx = \frac{(n-1)!!}{n!!}\begin{cases}\pi/2 & \text{for } n = 2, 4, \ldots \\ 1 & \text{for } n = 3, 5, \ldots\end{cases} \tag{7.0.29}$$

Gaussian Integrals

The integral of an arbitrary Gaussian function is

$$\int x^n e^{\beta x} dx = e^{\beta x} \sum_{k=0}^{n} (-1)^k \frac{n! x^{n-k}}{(n-k)! \beta^{k+1}} + c \tag{7.1.1}$$

Some general Gaussian integrals evaluate as

$$\int_{-\infty}^{\infty} e^{-\alpha x^2} dx = \sqrt{\frac{\pi}{\alpha}} \tag{7.1.2}$$

$$I_n = \int x^n e^{-x/\alpha} dx \tag{7.1.3}$$

$$I_0 = -\alpha e^{-x/\alpha} \tag{7.1.4}$$

$$I_1 = -(\alpha^2 + \alpha x) e^{-x/\alpha} \tag{7.1.5}$$

$$I_2 = -(2\alpha^3 + 2\alpha^2 x + \alpha x^2) e^{-x/\alpha} \tag{7.1.6}$$

$$I_{n+1} = \alpha^2 \frac{\partial I_n}{\partial \alpha} \tag{7.1.7}$$

$$\int_0^{\infty} e^{-x/\alpha} dx = \alpha \tag{7.1.8}$$

$$\int_0^{\infty} x e^{-x/\alpha} dx = \alpha^2 \tag{7.1.9}$$

$$\int_0^{\infty} x^2 e^{-x/\alpha} dx = 2\alpha^3 \tag{7.1.10}$$

$$\int_0^{\infty} x^n e^{-x/\alpha} dx = n! \alpha^{n+1} \tag{7.1.11}$$

The integral of an arbitrary Gaussian function with an n-dimensional linear term (with $n \in \mathbb{Z}$) is

$$\int_0^{\infty} x^{2n} e^{-\alpha x^2} dx = \sqrt{\frac{\pi}{\alpha}} \frac{(2n-1)!!}{2^{n+1} \alpha^n} \implies \int_{-\infty}^{\infty} x^{2n} e^{-\alpha x^2} dx = \sqrt{\frac{\pi}{\alpha}} \frac{(2n-1)!!}{(2\alpha)^n} \tag{7.1.12}$$

$$\int_0^{\infty} x^{2n+1} e^{-\alpha x^2} dx = \frac{n!}{2a^{n+1}} \implies \int_{-\infty}^{\infty} x^{2n+1} e^{-\alpha x^2} dx = 0 \tag{7.1.13}$$

Therefore a general solution is

$$\int_0^{\infty} x^n e^{-\alpha x^2} dx = \begin{cases} \dfrac{(n-1)!!}{2^{n/2+1} a^{n/2}} \sqrt{\dfrac{\pi}{\alpha}} & \text{for } n \text{ even} \\ \dfrac{[\frac{1}{2}(n-1)]!}{2a^{(n+1)/2}} & \text{for } n \text{ odd} \end{cases} \tag{7.1.14}$$

The below form of a gaussian integral evaluates to zero when n is odd due to the function being odd, but when n is even, the more general integral has the following closed form

$$\int_{-\infty}^{\infty} x^n e^{-\alpha x^2 + \beta x} = \sqrt{\frac{\pi}{\alpha}} e^{\beta^2/(4\alpha)} \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} (2k-1)!! (2a)^{k-n} \beta^{n-2k} \tag{7.1.15}$$

Fourier Series

The computation of the (usual) Fourier series is based on the integral identities

$$\int_{-\pi}^{\pi} \sin(mx)\sin(nx)dx = \pi\delta_{mn} \tag{8.0.1}$$

$$\int_{-\pi}^{\pi} \cos(mx)\cos(nx)dx = \pi\delta_{mn} \tag{8.0.2}$$

$$\int_{-\pi}^{\pi} \sin(mx)\cos(nx)dx = 0 \tag{8.0.3}$$

$$\int_{-\pi}^{\pi} \sin(mx)dx = 0 \tag{8.0.4}$$

$$\int_{-\pi}^{\pi} \cos(mx)dx = 0 \tag{8.0.5}$$

$$\delta_{mn} = \frac{1}{2\pi i}\oint_{\gamma} z^{m-n-1}dz \tag{8.0.6}$$

Using the method for a generalized Fourier series, the usual Fourier series involving sines and cosines is obtained by taking $f_1(x) = cosx$ and $f_2(x) = sinx$. Since these functions form a complete orthogonal system over $[-\pi, \pi]$, the Fourier series of a function $f(x)$ is given by (with $n \in \mathbb{N}$)

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx) \tag{8.0.7}$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx \tag{8.0.8}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx \tag{8.0.9}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx \tag{8.0.10}$$

The notion of a Fourier series can also be extended to complex coefficients.

$$f(x) = \sum_{n=-\infty}^{\infty} A_n e^{inx} \quad \text{with} \quad A_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx \tag{8.0.11}$$

For a function f(x) periodic on an interval [-L,L] instead of [-pi,pi], a simple change of variables can be used to transform the interval of integration from [-pi,pi] to [-L,L]. Let

$$x \equiv \frac{\pi x'}{L} \Longleftrightarrow x' \equiv \frac{Lx}{\pi} \implies dx = \frac{\pi dx'}{L} \tag{8.0.12}$$

Astronomy, Optics and Telescopes

A parsec is defined so

$$1 \text{ parsec} = \frac{1 \text{ AU}}{\tan(1")} \approx \frac{1 \text{ AU}}{1"} \tag{9.0.1}$$

The Flux (F) of a star relates to it's luminosity (L) and distance (d) via

$$F = \frac{L}{4\pi R^2} = \sigma_{SB} T_{eff}^4 \tag{9.0.2}$$

The flux received by a telescope at distance d is then

$$F(d) = \frac{L}{4\pi d^2} = \sigma T_{eff}^4 \left(\frac{R}{d}\right)^2 \tag{9.0.3}$$

The ratio of two magnitudes using different filters from a single star gives a rough estimation of the stars color.

$$B - V = m_B - m_V = -2.5 \log_{10}\left(\frac{F_B}{F_V}\right) \tag{9.0.4}$$

$$\frac{F_B}{F_V} = 10^{-(M_B - M_V)/2.5} \tag{9.0.5}$$

We define the distance modulus (DM) as the difference in apparent magnitude (m) between a given star and the absolute magnitude (M) it would have if it were at 10 pc.

$$DM \equiv m - m(10 \text{ pc}) \equiv m - M \tag{9.0.6}$$

$$M \equiv m - DM \tag{9.0.7}$$

The full form of intensity as a function of angle from the beam axis is

$$I = I_0 \left[\frac{\sin(\pi D/\lambda \sin(\theta))}{\sin(\pi d/\lambda \sin(\theta))}\right]^2 \tag{9.0.8}$$

Snell's Law:

$$\frac{\sin(\theta_1)}{\sin(\theta_2)} = \frac{v_1}{v_2} = \frac{\lambda_1}{\lambda_2} = \frac{n_2}{n_1} \tag{9.0.9}$$

Celestial Orbits

Suppose we have a exoplanet system with a planet p and a star s. The vector from the star to the planet is $\vec{r}_{sp} = \vec{r}_p - \vec{r}_s$, and the force that the star exerts on the planet is ($\vec{r}_n$ is the vector from the origin to n)

$$\vec{F}_{sp} = -\frac{GM_pM_s}{|\vec{r}_{sp}|^3}\vec{r}_{sp} \tag{9.1.1}$$

If we put the origin at the center of mass ($\vec{R}$ is the vector from the origin to the center of mass)

$$\vec{R} = \frac{M_s\vec{r}_s + M_p\vec{r}_p}{M_s + M_p} \tag{9.1.2}$$

Then the star and planets have positions

$$\vec{x}_s = \vec{r}_s - \vec{R} = -\frac{M_p}{M_p + M_s}\vec{r}_{sp} \tag{9.1.3}$$

$$\vec{x}_p = \vec{r}_p - \vec{R} = -\frac{M_s}{M_p + M_s}\vec{r}_{sp} \tag{9.1.4}$$

And thus accelerations

$$\frac{d^2\vec{x}_s}{dt^2} = -\frac{M_p}{M_p + M_s}\frac{d^2\vec{r}_{sp}}{dt^2} \tag{9.1.5}$$

$$\frac{d^2\vec{x}_p}{dt^2} = -\frac{M_s}{M_p + M_s}\frac{d^2\vec{r}_{sp}}{dt^2} \tag{9.1.6}$$

Substituting the acceleration into the equation of motion for the planet,

$$M_p\frac{d^2\vec{x}_p}{dt^2} = \vec{F}_{sp} \tag{9.1.7}$$

Then we can get the reduced equation of motion as

$$\frac{d^2\vec{r}_{sp}}{dt^2} = -G\frac{M_s + M_p}{|\vec{r}_{sp}|^3}\vec{r}_{sp} \tag{9.1.8}$$

Keplar's Third law: The solution to this is an elliptical orbit with the center-of-force at one focus of the ellipse. The period (T) depends on the semi-major axis (a)

$$T^2 = \frac{4\pi^2}{G(M_s + M_p)}a^3 \tag{9.1.9}$$

$$a^3 = \frac{G(M_s + M_p)}{4\pi^2}T^2 \tag{9.1.10}$$

If the orbit is circular, so that $|\vec{r}_sp = a$ is constant, then the orbital speed of the star is

$$v_s = \frac{2\pi a M_p}{T(M_p + M_s)} = \sqrt{\frac{GM_p^2}{a(M_p + M_s)}} \tag{9.1.11}$$

For a particle in a circular orbit, $v = r\Omega\hat{\theta}$; using Keplers law (r is the distance from the center of mass), we have

$$L = mr^2\Omega = m\sqrt{GMr}. \quad (9.1.12)$$

The orbital angular momentum of the two-body system is

$$L = \frac{M_1 M_2}{M_1 + M_2} a^2 \Omega \quad (9.1.13)$$
$$= \frac{M_1 M_2}{M_1 + M_2} \sqrt{G(M_1 + M_2)a}. \quad (9.1.14)$$

The angular momentum of a sphere is

$$L = \frac{8\pi}{15}\rho\Omega R^5 = \frac{2}{5}MR^2\Omega. \quad (9.1.15)$$

The equations of motion in a rotating frame are

$$\frac{d^2\vec{r}'}{dt^2} = \frac{1}{m}\vec{F}_{rot} \quad (9.1.16)$$
$$= \frac{1}{m}\vec{F} + \underbrace{r\Omega^2\hat{r}}_{\text{centrifugal}} + \underbrace{2\Omega(v_\theta\hat{r} - v_r\hat{\theta})}_{\text{coriolis}}. \quad (9.1.17)$$

Particles within a sphere of radius R_H are dominated by the gravitational attraction of M_2; R_H (the Hill radius) is

$$R_H \approx a\left[\frac{M_2}{3(M_1 + M_2)}\right]^{1/3}. \quad (9.1.18)$$

Celestial & Stellar Atmospheres

The equation of hydrostatic equilibrium

$$\frac{dP}{dr} = -\rho g. \quad (9.2.1)$$

The ideal gas law can be written (with m=mass of 1 mole of our gas) as

$$P = \left(\frac{mN/N_A}{V}\right)\frac{kN_A}{m}T \equiv \rho\frac{kN_A}{m}T. \quad (9.2.2)$$

Combining the above two equations and assuming T=constant then yields a relation between pressure and height as

$$\frac{dP}{P} = -\frac{mg}{N_A kT}dz. \quad (9.2.3)$$

This then gives a pressure Dependant on height as

$$P(z) = P_0 \exp\left[-\frac{mgz}{N_A kT}\right]. \quad (9.2.4)$$

In addition to the Coriolis acceleration from the Earth rotation, horizontal pressure gradients will also produce an acceleration

$$-\frac{1}{\rho}\nabla P. \quad (9.2.5)$$

The equation for force and acceleration along r$\hat{r}$ is therefore

$$\underbrace{\frac{v^2}{r}}_{\text{centripital}} + \underbrace{2v\Omega\sin(\lambda)}_{\text{coriolis}} - \underbrace{\frac{1}{\rho}\frac{dP}{dr}}_{\text{pressure}} = 0. \quad (9.2.6)$$

If matter is in thermal equilibrium, then populations of a two different states of a given atom are given by Boltzmanns formula,

$$\frac{n_i}{n_j} = \frac{g_i}{g_j} exp\left(\frac{E_j - E_i}{kT}\right). \quad (9.2.7)$$

Hydrostatic equilibrium (where m is the mass within a sphere of radius r, P is the pressure, and ρ is the mass density) gives two equations of stellar structure,

$$\frac{dm}{dr} = 4\pi r^2\rho \quad \text{and} \quad \frac{dP}{dr} = -\rho\frac{Gm}{r^2} \quad (9.2.8)$$

From the virial theorem, the average pressure and density are

$$\bar{\rho} = \frac{GM}{4\pi R^3} \quad \text{and} \quad \bar{P} \propto \frac{GM^2}{R^4} \quad (9.2.9)$$

The optical depth for an outward-directed ray is

$$\tau_\mu = \int_z^\infty \rho\kappa_\mu dz' \implies \frac{d\tau}{dz} = -\rho\kappa \quad (9.2.10)$$

From this, an estimate of the photospheric pressure can be determined for a gray atmosphere in LTE[1] by,

$$\frac{dP}{d\tau} = -\left(\frac{d\tau}{dz}\right)^{-1}\rho g = \frac{g}{\kappa}. \quad (9.2.11)$$

[1] Local Thermodynamical Equilibrium

From hydrostatic equilibrium and taking $\rho = constant$ (where μm_u is the average mass of a particle in the plasma), the central pressure and temperature are given by

$$T_c = \frac{GM\mu m_u}{2Rk_B} \tag{9.2.12}$$

$$P_c = \frac{3GM^2}{8\pi R^4} \tag{9.2.13}$$

Bringing a small amount of mass dm from infinity onto a sphere of mass m and radius r gives a potential change of

$$d\Omega = -\frac{Gm}{r}dm. \tag{9.2.14}$$

For a constant density, we have $r = R(m/M)^{1/3}$ and so

$$\Omega = -\frac{3GM^2}{5R}. \tag{9.2.15}$$

Using this, the mean temperature and pressure for a constant density sphere is

$$\bar{T} = \frac{GM\mu m_u}{5Rk_B} \tag{9.2.16}$$

$$\bar{P} = \frac{3GM^2}{20\pi R^4}. \tag{9.2.17}$$

The free fall time it would take for a star to collapse if all internal pressures were removed is

$$\tau_{ff} = \frac{\pi}{\sqrt{GM}}\left(\frac{R}{2}\right)^{3/2} = \left(\frac{3}{32\pi}\right)^{1/2}\frac{1}{\sqrt{G\bar{\rho}}}. \tag{9.2.18}$$

The dynamical timescale of the star is defined from the proportionality constant of the free fall time

$$t_{dyn} \equiv \frac{1}{\sqrt{G\bar{\rho}}}. \tag{9.2.19}$$

Any change in pressure is communicated through a star by sound waves which travel at the speed

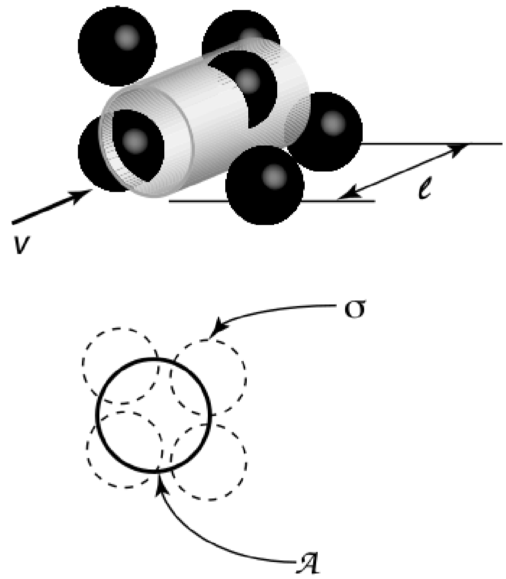
$$c_s = \left(\gamma\frac{P}{\rho}\right)^{1/2} = \left(\gamma\frac{k_BT}{\mu m_u}\right)^{1/2}. \tag{9.2.20}$$

The time it takes for a sound wave to travel a distance R is then

$$\tau_{sc} = \frac{R}{c_s} = \sqrt{\frac{3R^3}{GM}} = \left(\frac{3}{2\sqrt{\pi}}\right)\frac{1}{\sqrt{G\bar{\rho}}}. \tag{9.2.21}$$

The **Kelvin-Helmholtz timescale** is the time it would take the sun to radiate all of it's gravitational energy away with it's current luminosity $L_\odot$,

$$t_{KH} \approx \frac{GM_\odot^2}{R_\odot L_\odot} \approx 3\times 10^7 yr. \tag{9.2.22}$$



As displayed by the image above[2], the probability of a particle making it through a density of obstacles n with cross section σ is

$$\mathcal{P} = \frac{n(\mathcal{A}\ell)\sigma}{\mathcal{A}} = n\sigma\ell. \tag{9.2.23}$$

The **mean free path** is defined to be the length at which $\mathcal{P} \to 1$ which is when the particle will suffer a collision:

$$\ell = \frac{1}{n\sigma}. \tag{9.2.24}$$

Convection

The temperature gradient in a star ($\kappa = opacity$) is

$$\frac{dT}{dr} = -\frac{3\rho\kappa}{4acT^3}\frac{L(r)}{4\pi r^2}. \tag{9.3.1}$$

[2]"Schematic of a particle incident on a group of particles." [4].

From the first law of thermodynamics,

$$dQ = dU - \frac{P}{\rho^2}d\rho \tag{9.3.2}$$

$$dU = \left(\frac{\partial U}{\partial T}\right)_\rho dT + \left(\frac{\partial U}{\partial \rho}\right)_T d\rho \tag{9.3.3}$$

$$dQ = \left(\frac{\partial U}{\partial T}\right)_\rho dT + \left[\left(\frac{\partial U}{\partial \rho}\right)_T - \frac{P}{\rho^2}\right] d\rho. \tag{9.3.4}$$

While holding density fixed, the heat needed to raise the temperature of one kilogram of fluid is then

$$C_\rho \equiv \left(\frac{\partial Q}{\partial T}\right)_\rho = \left(\frac{\partial U}{\partial T}\right)_\rho. \tag{9.3.5}$$

From this, heat transfer can be expressed as a function of temperature and pressure

$$dQ = \left[C_\rho + \frac{P}{\rho T}\right] dT - \frac{1}{\rho}dP \tag{9.3.6}$$

$$= \left[C_\rho + \frac{k_B}{\mu m_u}\right] dT - \frac{1}{\rho}dP. \tag{9.3.7}$$

Hence, while holding pressure fixed, the heat needed to raise the temperature of one kilogram of fluid is

$$C_P = \left(\frac{\partial Q}{\partial T}\right)_P = C_\rho + \frac{k_B}{\mu m_u}. \tag{9.3.8}$$

For a plasma of ions and electrons,

$$C_\rho = \frac{3k_B}{2\mu m_u} = \frac{3}{5}C_P. \tag{9.3.9}$$

Thus the ratio of specific heats for an ideal gas is

$$\gamma = \frac{C_P}{C_\rho} = \frac{5}{3}. \tag{9.3.10}$$

During adiabatic motion, no heat exchange occurs and so $TdS = dQ = 0$ which leads to

$$T = T_0\left(\frac{P}{P_0}\right)^{(\gamma-1)/\gamma}. \tag{9.3.11}$$

The temperature change with pressure in an adiabatically stratified gas is given by

$$\frac{P}{T}\left(\frac{\partial T}{\partial P}\right)_S = \left(\frac{\partial \ln T}{\partial \ln P}\right)_S = \frac{\gamma - 1}{\gamma}. \tag{9.3.12}$$

For stable convection, we must have

$$\left(\frac{\partial V}{\partial S}\right)_P \frac{dS}{dr} = \frac{T}{C_P}\left(\frac{\partial V}{\partial T}\right)_P \frac{dS}{dr} > 0. \tag{9.3.13}$$

The stability requirements for convection can also be derived in terms of local gradients of temperature and pressure. The fluid is unstable to convection if

$$\frac{P}{P_{rad}}\frac{\kappa}{16\pi Gc}\frac{L(r)}{m(r)} > \left(\frac{\partial \ln T}{\partial \ln P}\right)_S = \frac{\gamma - 1}{\gamma}. \tag{9.3.14}$$

Main Sequence Stars

For an enclosed mass we have the following relations for radiative and convective regions respectively

$$\frac{dT}{dr} = -\frac{L}{4\pi r^2}\frac{3\rho\kappa}{4acT^3} \tag{9.4.1}$$

$$\frac{dT}{dr} = \frac{T}{P}\left(\frac{\partial \ln T}{\partial \ln P}\right)_S \frac{dP}{dr}. \tag{9.4.2}$$

The fourth order equation of stellar structure (with ϵ being the heating rate per unit mass) is

$$\frac{dL}{dr} = 4\pi r^2 \rho\epsilon. \tag{9.4.3}$$

Classical Mechanics

Newtons Second Law in Cartesian coordinates and 2D Polar coordinates

$$\vec{F} = m\vec{a} = m\ddot{\boldsymbol{r}} \iff \begin{cases} F_x = m\ddot{x} \\ F_y = m\ddot{y} \\ F_z = m\ddot{z} \end{cases} \tag{10.0.1}$$

$$\iff \begin{cases} F_r = m(\ddot{r} - r\dot{\phi}^2) \\ F_\phi = m(r\ddot{\phi} + 2\dot{r}\dot{\phi}) \end{cases} \tag{10.0.2}$$

Conservation of energy

$$E = \text{constant} = KE + PE \tag{10.0.3}$$

$$= \frac{1}{2}m|\vec{v}|^2 + mgh \tag{10.0.4}$$

Equation of motion for a rocket

$$m\dot{v} = -\dot{m}v_{ex} + F^{external} \tag{10.0.5}$$

The center of mass of several particles with a total mass M is

$$\vec{R} = \frac{1}{M}\sum_{\alpha=1}^{n} m_\alpha \vec{r}_\alpha = \frac{m_1\vec{r}_1 + \cdots m_n\vec{r}_n}{M} \tag{10.0.6}$$

$$\vec{R} = \frac{1}{M}\int \vec{r}dm = \frac{1}{M}\int \rho\vec{r}dV \tag{10.0.7}$$

The mass of an object is defined by the density multiplied by the volume.

$$M \equiv \rho V \equiv \iiint_Q \rho(x, y, z)dV \tag{10.0.8}$$

The moment of inertia with respect to a given axis of a solid body with density $\rho(r)$, where $r_\perp$ is the perpendicular distance from the axis of rotation, is defined by the volume integral

$$I \equiv \int \rho(\boldsymbol{r})r_\perp^2 dV \equiv \iiint_Q \rho(x, y, z)||\boldsymbol{r}||^2 dV \tag{10.0.9}$$

Angular momentum

$$\vec{L} = \vec{r} \times \vec{p} = I\vec{\omega} = I\dot{\theta} \tag{10.0.10}$$

The net external torque is given by

$$\vec{\tau}_{ext} = \vec{r} \times \vec{F} = \frac{d\vec{L}}{dt} \tag{10.0.11}$$

The change in kinetic energy as it moves from point a to point b is

$$\Delta K \equiv K_2 - K_1 = \int_a^b \vec{F} \cdot d\vec{r} \equiv W(a \to b) \tag{10.0.12}$$

$$K = \frac{1}{2}mv^2 = \frac{1}{2}I\omega^2 = \frac{1}{2}I\dot{\theta}^2 \tag{10.0.13}$$

A force $\vec{F}$ on a particle is **conservative** if (i) it depends only on the particles position, $\vec{F} = \vec{F}(\vec{r})$ and (ii) $\nabla \times \vec{F} = 0$. If $\vec{F}$ is conservative we can define a corresponding **potential energy** so that

$$U(\boldsymbol{r}) = -W(\boldsymbol{r}_0 \to \boldsymbol{r}) \equiv \int_{\boldsymbol{r}_0}^{\boldsymbol{r}} \boldsymbol{F}(\boldsymbol{r}') \cdot d\boldsymbol{r}' \tag{10.0.14}$$

$$\vec{F} = -\nabla\vec{U} \tag{10.0.15}$$

Hooke's Law states that the force needed to extend or compress a spring by some distance is proportional to that distance.

$$F = -kx \iff U = \text{constant} + \frac{1}{2}kx^2 \tag{10.0.16}$$

Simple harmonic motion

$$\ddot{x} = -\omega^2 x \iff A\cos(\omega t - \delta) \tag{10.0.17}$$

Damped oscillations: If the oscillator is subject to a damping force $-bv$, the

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0 \text{ and } \beta < \omega_0$$

$$\iff x(t) = Ae^{-\beta t}\cos(\omega_1 t - \delta) \tag{10.0.18}$$

$$\beta = \frac{b}{2m}, \quad \omega_0 = \sqrt{\frac{k}{m}}, \quad \omega_1 = \sqrt{\omega_0^2 - \beta^2} \tag{10.0.19}$$

If the oscillator is also subject to a sinusoidal driving force $F(t) = mf_0\cos(\omega t)$, the long-term motion has the form

$$x(t) = A\cos(\omega t - \delta) \tag{10.0.20}$$

$$A^2 = \frac{f_0^2}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2} \tag{10.0.21}$$

It is always possible to write a sum of sinusoidal functions as a single sinusoid the form

$$f(\theta) = A\cos(\theta) + B\sin(\theta) \Longleftrightarrow f(\theta) = C\cos(\theta+\delta) \tag{10.0.22}$$

$$\delta = \arctan(-B/A) \qquad \text{and} \qquad C = \pm\sqrt{A^2+B^2} \tag{10.0.23}$$

$$f(\theta) = A\cos(\theta) + B\sin(\theta) \Longleftrightarrow f(\theta) = \operatorname{sgn}(A)\sqrt{A^2+B^2}\cos\big(\theta + \arctan(-B/A)\big) \tag{10.0.24}$$

Any periodic function with period τ can be written as (A Fourier series with $n \geq 1$)

$$f(t) = \sum_{n=0}^{\infty}[a_n\cos(n\omega t) + b_n\sin(n\omega t)] \tag{10.0.25}$$

$$a_n = \frac{2}{\tau}\int_{-\tau/2}^{\tau/2} f(t)\cos(n\omega t)dt \tag{10.0.26}$$

$$b_n = \frac{2}{\tau}\int_{-\tau/2}^{\tau/2} f(t)\sin(n\omega t)dt \tag{10.0.27}$$

$$a_0 = \frac{1}{\tau}\int_{-\tau/2}^{\tau/2} f(t)dt \tag{10.0.28}$$

It is sometimes useful to express the above Fourier series as an exponential

$$f(t) = \sum_{n=-\infty}^{\infty} A_n e^{in\omega t} \tag{10.0.29}$$

$$A_n = \frac{1}{\tau}\int_{-\tau/2}^{\tau/2} f(t)e^{-in\omega t}dt \tag{10.0.30}$$

It is important to know $A_n = A^*_{-n}$ so we can write $A_n = \Re(A_n) + i\Im(A_n)$. An important relationship between A_n, a_n and b_n then follows as,

$$a_n = 2\Re(A_n) \quad \text{and} \quad b_n = -2\Im(A_n) \tag{10.0.31}$$

The root-mean square displacement is a good measure of the average response of the oscillator and is given by parseval's theorem

$$x_{rms} = \sqrt{\frac{1}{\tau}\int_0^{\tau} x^2 dt} \tag{10.0.32}$$

$$= \sqrt{A_0^2 + \frac{1}{2}\sum_{n=1}^{\infty} A_n^2} \tag{10.0.33}$$

The non-relativistic Lagrangian $\mathcal{L}$ for a conservative system can be defined in terms of the kinetic energy and potential energy of a system as

$$\mathcal{L} = KE - PE \tag{10.0.34}$$

An integral of the form

$$S = \int_{x_1}^{x_2} f[y(x), y'(x), x]dx \tag{10.0.35}$$

taken along a path $y = y(x)$ is stationary with respect to variations of that path if and only if $y(x)$ satisfies the Euler-Lagrange Equation

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0. \tag{10.0.36}$$

If there are n dependent variables in the original integral, there are n Euler-Langrange equations. For instance, an integral of the form

$$S = \int_{u_1}^{u_2} f[x(u), y(u), x'(u), y'(u), u]du \tag{10.0.37}$$

with two dependent variables [$x(u)$ and $y(u)$], is stationary with respect to variations of $x(u)$ and $y(u)$ if and only if these two functions satisfy the two equations

$$\frac{\partial f}{\partial x} = \frac{d}{du}\frac{\partial f}{\partial x'} \quad \text{and} \quad \frac{\partial f}{\partial y} = \frac{d}{du}\frac{\partial f}{\partial y'} \tag{10.0.38}$$

For any holonomic system, Newtons second law is equivalent to the n Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} \tag{10.0.39}$$

The ith generalized momentum p_i is defined to be the derivative

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \tag{10.0.40}$$

If $\partial\mathcal{L}/\partial t = 0$ then $\mathcal{H}$ is conserved; if the coordinates $q_1, \ldots, q_n$ are natural, $\mathcal{H}$ is just the energy of the system. The Hamiltonian $\mathcal{H}$ is defined as

$$\mathcal{H} = \sum_{i=1}^{n} p_i\dot{q}_i - \mathcal{L} \tag{10.0.41}$$

The time evolution of a system is given by Hamilton's equations

$$\dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i} \quad \text{and} \quad \dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i} \tag{10.0.42}$$

The Lagrangian for a charge q in an electromagnetic field is

$$\mathcal{L}(\boldsymbol{r}, \dot{\boldsymbol{r}}, t) = \frac{1}{2}m\dot{\boldsymbol{r}}^2 - q(V - \dot{\boldsymbol{r}}\cdot\boldsymbol{A}) \tag{10.0.43}$$

Electricity and Magnetism

Maxwell's Equations: The system of partial differential equations describing classical electromagnetism. $\vec{P}$ is the polarization field, $\vec{D}$ is the electric displacement field, ρ is the charge density, $\vec{E}$ is the electric field, $\vec{B}$ is the magnetic field, and $\vec{J}$ is the current density. In the so-called cgs system of units, the Maxwell equations are given by

$$\nabla \cdot \vec{E} = 4\pi\rho \tag{11.0.1}$$
$$\nabla \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} \tag{11.0.2}$$
$$\nabla \cdot \vec{B} = 0 \tag{11.0.3}$$
$$\nabla \times \vec{B} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial \vec{E}}{\partial t} \tag{11.0.4}$$

In the MKS system of units (where ϵ_0 is the permittivity of free space and μ_0 is the permeability of free space), the equations are written

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \tag{11.0.5}$$
$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{11.0.6}$$
$$\nabla \cdot \vec{B} = 0 \tag{11.0.7}$$
$$\nabla \times \vec{B} = \mu_0\vec{J} + \epsilon_0\mu_0\frac{\partial \vec{E}}{\partial t} \tag{11.0.8}$$

In the special case of a steady state, known as **electrostatics**, with stationary charges and currents,

$$\nabla \times \vec{E} = 0 \implies \oint \vec{E} \cdot d\vec{\ell} = 0 \tag{11.0.9}$$

If we consider both bound and free charges (where the free charges are the charges we place within a system), we have

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} = \frac{\rho_{bound} + \rho_{free}}{\epsilon_0} = \frac{-\nabla \cdot \vec{P}}{\epsilon_0} + \frac{\rho_{free}}{\epsilon_0} \tag{11.0.10}$$
$$\implies \nabla \cdot \vec{D} = \rho_{free} \implies \oint_S \vec{D} \cdot d\vec{a} = Q_{free} = \int \rho_{free} d\tau'. \tag{11.0.11}$$

The dipole moment is defined by

$$\vec{p} \equiv \sum_i q_i \vec{r}_i \qquad\qquad \vec{p} \equiv \int_V \rho(\vec{r}')\vec{r}' d\tau' \tag{11.0.12}$$

The polarization field of a linearly polarized dielectric is characterized by its dipole moment per unit volume and can be defined by the susceptibility constant χ_e and the dielectric constant ϵ_R,

$$\vec{P} = \lim \frac{\Delta\vec{p}}{\Delta v} = \frac{1}{\Delta v}\sum_i \vec{p}_i \equiv \epsilon_0\chi_e\vec{E} = \frac{\chi_e}{1+\chi_e}\vec{D} = \frac{\chi_e}{\epsilon_R}\vec{D} \quad \longrightarrow \quad \begin{cases} \chi_e \to 0 & \implies \vec{P} \text{ for a vacuum} \\ \chi_e \to \infty & \implies \vec{P} \text{ for a metal} \end{cases} \tag{11.0.13}$$

From this, the bound charge densities for both the surface and volume are defined by

$$\rho_B = -\nabla \cdot \vec{P} \qquad \text{and} \qquad \sigma_B = \vec{P} \cdot \hat{n}. \tag{11.0.14}$$

The electric displacement field is defined such that

$$\vec{D} = \epsilon_0\vec{E} + \vec{P} \implies \vec{D} = \epsilon_0(1+\chi_e)\vec{E} = \epsilon_0\epsilon_R\vec{E} \iff \vec{P} = \epsilon_0(1-\epsilon_R)\vec{E}. \tag{11.0.15}$$

Coulomb's Law: The force on a test charge Q due to a single point charge q with the separation between them being $|\vec{\mathfrak{r}}|$ (note: $\vec{\mathfrak{r}} = \vec{r} - \vec{r}'$ is the separation vector from the location of q - denoted $\vec{r}'$ - to the location of Q - denoted $\vec{r}$)) is given by

$$\vec{F} = \frac{1}{4\pi\epsilon_0}\frac{qQ}{|\vec{\mathfrak{r}}|^2}\hat{\mathfrak{r}} \tag{11.0.16}$$

Given stationary charges and currents, the electric field $\vec{E}(\vec{r})$ can be written as

$$\begin{aligned}
\vec{E}(\vec{r}) &= \frac{1}{4\pi\epsilon_0}\int \frac{dq}{|\vec{\mathfrak{r}}|^2}\hat{\mathfrak{r}} && (11.0.17)\\
&\equiv \frac{1}{4\pi\epsilon_0}\iiint\limits_V \frac{\rho(\vec{r}')}{|\vec{\mathfrak{r}}|^2}\hat{\mathfrak{r}}d\tau' \qquad \text{(volume charge)} && (11.0.18)\\
&\equiv \frac{1}{4\pi\epsilon_0}\iint\limits_A \frac{\sigma(\vec{r}')}{|\vec{\mathfrak{r}}|^2}\hat{\mathfrak{r}}da' \qquad \text{(area charge)} && (11.0.19)\\
&\equiv \frac{1}{4\pi\epsilon_0}\int_l \frac{\lambda(\vec{r}')}{|\vec{\mathfrak{r}}|^2}\hat{\mathfrak{r}}d\ell' \qquad \text{(line charge)} && (11.0.20)
\end{aligned}$$

Gauss's Law: The electric flux Φ_E through a surface S enclosing any volume is proportional to the total charge enclosed within the volume. This is an alternate form of one of Maxwell's equations.

$$\Phi_E = \oiint\limits_S \vec{E}\cdot d\vec{a} = \oiint\limits_S (\vec{E}\cdot\hat{n})da = \frac{Q_{enc}}{\epsilon_0} \tag{11.0.21}$$

An electric potential V is a continuous function and is defined as

$$V(\vec{r}) \equiv -\int_{\mathcal{O}}^{r} \vec{E}(\vec{r}')\cdot d\vec{\ell}' \equiv \frac{1}{4\pi\epsilon_0}\iiint\limits_V \frac{\rho(\vec{r}')}{|\vec{\mathfrak{r}}|}d\tau' \tag{11.0.22}$$

Using this and the fundamental theorem for gradients, we have

$$\int_a^b (\nabla V)\cdot d\vec{\ell} = -\int_a^b \vec{E}\cdot d\vec{\ell} \implies \vec{E} = -\nabla V \tag{11.0.23}$$

Poisson's equation can be used to determine the charge density of a function from the electric potential.

$$\nabla^2 V(\vec{r}) = -\frac{\rho(\vec{r}')}{\epsilon_0} \tag{11.0.24}$$

Using a special case of Poisson's equation when $\rho = 0$, we can derive the multi-pole expansion.

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\sum_i \frac{q_i}{|\vec{\mathfrak{r}}_i|}\frac{1}{4\pi\epsilon_0}\sum_i \frac{q_i}{|\vec{r} - \vec{r}_i'|} \tag{11.0.25}$$

From **the multi-pole expansion**, we can approximate the potential as

$$V(r,\theta) \approx \underbrace{\frac{Q_{tot}}{4\pi\epsilon_0 r}}_{monopole} + \underbrace{\frac{\vec{p}\cdot\hat{r}}{4\pi\epsilon_0 r^2}}_{dipole} = \frac{Q_{tot}}{4\pi\epsilon_0 r} + \frac{1}{4\pi\epsilon_0 r^2}\int_V \rho(\vec{r}')\vec{r}'\cdot\hat{r}d\tau' \tag{11.0.26}$$

The work on a system due to an electric field is given by

$$W_{sys} \equiv \sum_{j=1}^{m} W_j = \sum_{j=1}^{m} \left(\sum_{k=1}^{j-1} \frac{q_j q_k}{4\pi\epsilon_0 r_{jk}} \right) \equiv \frac{1}{2} \int \rho(\vec{r}) V(\vec{r}) d\tau \implies W_{sys} = \frac{\epsilon_0}{2} \int E^2 d\tau \tag{11.0.27}$$

The energy stored due to an electric field and magnetic field is given by

$$U = \frac{1}{2} \int \left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2 \right) d\tau \tag{11.0.28}$$

The Lorentz force law: The magnetic force on a charge q, moving with velocity $\vec{v}$ due to a magnetic field $\vec{B}$ and an electric field $\vec{E}$ is

$$\vec{F} = q[\vec{E} + (\vec{v} \times \vec{B})] \tag{11.0.29}$$

A line charge λ traveling down a wire at speed v constitutes a **current** $\vec{I} = \lambda \vec{v}$ [6]. The magnetic force on a segment of current-carrying wire is

$$\vec{F}_{mag} = \int (\vec{v} \times \vec{B}) dq = \int (\vec{v} \times \vec{B}) \lambda d\ell = \int (\vec{I} \times \vec{B}) d\ell = \int I (d\vec{\ell} \times \vec{B}). \tag{11.0.30}$$

When charge (q) flows over a surface or through a volume, we describe it by the surface current density $\vec{K}$ and the volume current density $\vec{J}$ respectively. By definition (where N is number of charge carriers for unit volume with some velocity $\vec{v}$), these are

$$\vec{K} \equiv \frac{d\vec{I}}{d\ell_\perp} \qquad \text{and} \qquad \vec{J} \equiv \frac{d\vec{I}}{da_\perp} = \vec{J}_{bound} + \vec{J}_{free} \equiv \sum_i N_i q_i \vec{v}_i. \tag{11.0.31}$$

The total current through a surface can be defined

$$I = \int_S (\vec{J} \cdot \hat{n}) da = \frac{dQ}{dt} \tag{11.0.32}$$

From the surface and volume currents, we can express the magnetic force as

$$\vec{F}_{mag} \equiv \int (\vec{v} \times \vec{B}) \sigma da = \int (\vec{K} \times \vec{B}) da \tag{11.0.33}$$

$$\vec{F}_{mag} \equiv \int (\vec{v} \times \vec{B}) \rho d\tau = \int (\vec{J} \times \vec{B}) d\tau. \tag{11.0.34}$$

The **continuity equation** is a precise mathematical statement of local charge conservation.

$$\nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t}, \qquad \text{Magneto-statics} \implies \nabla \cdot \vec{J} = 0 \tag{11.0.35}$$

The **Biot-Savart law** gives the magnetic field of a steady state line current

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{I} \times \hat{\mathfrak{r}}}{|\vec{\mathfrak{r}}|^2} d\ell' = \frac{\mu_0}{4\pi} I \int \frac{d\vec{\ell'} \times \hat{\mathfrak{r}}}{|\vec{\mathfrak{r}}|^2}. \tag{11.0.36}$$

When dealing with surface and volume currents, the Biot-Savart law becomes

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{K}(\vec{r'}) \times \hat{\mathfrak{r}}}{|\vec{\mathfrak{r}}|^2} da' \qquad \text{and} \qquad \vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r'}) \times \hat{\mathfrak{r}}}{|\vec{\mathfrak{r}}|^2} d\tau' \tag{11.0.37}$$

Ampere's Law: For a straight line current, the integral of $\vec{B}$ around an Amperien path centered at a wire is related to the total enclosed current by

$$\oint \vec{B} \cdot d\vec{\ell} = \mu_0 I_{enc} = \mu_0 \int \vec{J} \cdot d\vec{a} = \int (\nabla \times \vec{B}) \cdot d\vec{a} \tag{11.0.38}$$

From Maxwell's equation $\nabla \cdot \vec{B} = 0$, we can define the vector potential $\vec{A}$ such that $\nabla \cdot (\nabla \times \vec{A}) = 0 \implies \vec{B} = \nabla \times \vec{A}$. From this, we can also define the gauge freedom such that $\nabla \cdot (\nabla \vec{A}) = 0$ which gives us

$$\nabla^2 \vec{A} = -\mu_0 \vec{J} \implies \vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r'}) d\tau'}{|\vec{\mathfrak{r}}|} \equiv \frac{\mu_0}{4\pi} \int \frac{\vec{K}(\vec{r'}) da'}{|\vec{\mathfrak{r}}|} \tag{11.0.39}$$

We can define the **magnetic dipole moment** $\vec{m}$ (which is a measurable quantity) and the magnetization $\vec{M}$ of a material in terms of a line current or surface current density to be

$$\vec{m} \equiv I \int_S d\vec{a} = I\vec{a}, \quad \text{or} \quad \vec{m} \equiv \frac{1}{2} \int_V (\vec{r} \times \vec{J}) dV = \int_V \vec{M} dV = \frac{I}{2} \oint_C \vec{r} \times d\vec{\ell}, \tag{11.0.40}$$

$$\text{with} \quad \vec{M} = \frac{d\vec{m}}{dV} \equiv \lim_{\Delta v \to 0} \frac{1}{\Delta v} \sum_i \vec{m}_i \tag{11.0.41}$$

From this, we can do a multi-pole expansion of the vector potential. The mono-pole term evaluates to zero and the dipole term becomes useful in many cases and can be written as follows by use of Stokes theorem.

$$\vec{A}(\vec{r}) = \frac{\mu_0 I}{4\pi} \oint \frac{d\vec{\ell}}{|\vec{\mathfrak{r}}|} = \frac{\mu_0 I}{4\pi} \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \oint (r')^n P_n(\cos\alpha) d\vec{\ell} \implies \vec{A}_{dip}(\vec{r}) = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \hat{r}}{r^2} \tag{11.0.42}$$

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int_S \frac{\vec{K}_B(\vec{r'})}{|\vec{\mathfrak{r}}|} da' + \frac{\mu_0}{4\pi} \int_V \frac{\vec{J}_B(\vec{r'})}{|\vec{\mathfrak{r}}|} d\tau' \tag{11.0.43}$$

Using the solution form of the Biot-Savart law, we can write an expression for the vector potential

$$\vec{A}(\vec{r}) = \frac{1}{4\pi} \int \frac{\vec{B}(\vec{r'}) \times \hat{\mathfrak{r}}}{|\vec{\mathfrak{r}}|^2} d\tau' \tag{11.0.44}$$

From this, if we place $\vec{m}$ at the origin pointing in the $\hat{z}$ direction, the magnetic field of a perfect dipole is calculated as follows.

$$\vec{B}_{dip}(\vec{r}) = \underbrace{\frac{\mu_0}{4\pi r^3} [3(\vec{m} \cdot \hat{r})\hat{r} - \vec{m}] + \frac{2\mu_0}{3} \vec{m} \delta^3(\vec{r})}_{\text{The true field of a magnetic dipole.}^1} \equiv \frac{\mu_0 m}{4\pi r^3} \left(2\cos\theta \hat{r} + \sin\theta \hat{\theta}\right). \tag{11.0.45}$$

We can define the potential of a bound volume current $\vec{J}_B$ and bound surface current $\vec{K}_B$ as

$$\vec{J}_B = \nabla \times \vec{M} \qquad \text{and} \qquad \vec{K}_B = \vec{M} \times \hat{n} \tag{11.0.46}$$

From Ampere's Law we can define the **magnetizing field** $\vec{H}$ and thus have,

$$\frac{1}{\mu_0} (\nabla \times \vec{B}) = \vec{J}_f + (\nabla \times \vec{M}) \implies \nabla \times \left(\frac{\vec{B}}{\mu_0} - \vec{M}\right) = \nabla \times \vec{H} = \vec{J}_f \tag{11.0.47}$$

$$\implies \oint \vec{H} \cdot d\vec{\ell} = I_{free} = \int \vec{J}_{free} \cdot d\vec{a} \qquad \text{and} \qquad \int (\nabla \times \vec{H}) \cdot d\vec{a} = \int \vec{J} \cdot d\vec{a} \tag{11.0.48}$$

[1] The delta-function is responsible for the hyperfine splitting in atomic spectra[6]

The **magnetic susceptibility** χ_m is a dimensionless quantity that is dependent on the substance. For a linear media, we have the relation

$$\vec{M} = \chi_m \vec{H} \implies \vec{B} = \mu_0(1+\chi_m)\vec{H} \qquad \text{with} \qquad \begin{cases} \chi_m > 0, & \text{paramagnetic} \\ \chi_m < 0, & \text{diamagnetic} \\ \chi_m = 0, & \text{vacuum} \end{cases} \tag{11.0.49}$$

The force on a magnetic dipole due to a varying magnetic field is

$$F = \nabla(\vec{m} \cdot \vec{B}) \tag{11.0.50}$$

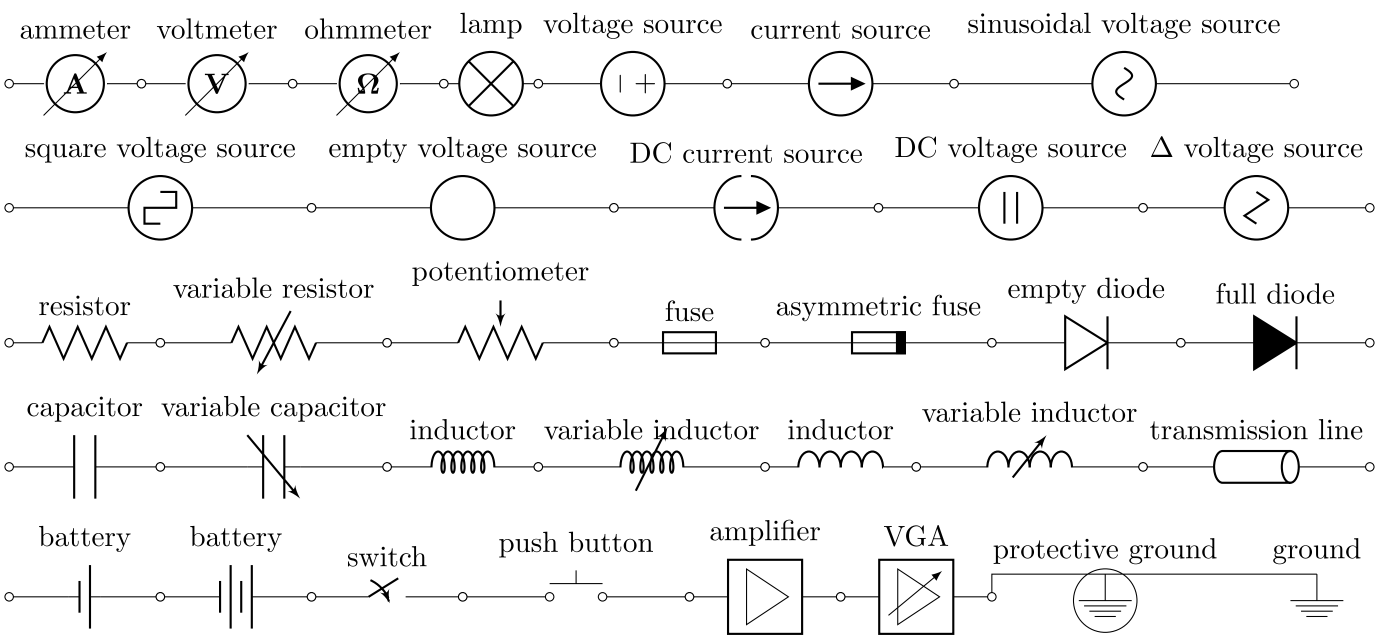
Kirchhoff's Laws apply for electric circuits and are derived from the static equations $\nabla \cdot \vec{J} = 0$ and $\nabla \times \vec{E} = 0$ which become

$$\sum I_i = 0 \text{ at a branch point} \qquad \sum V_i = 0 \text{ around a loop} \tag{11.0.51}$$

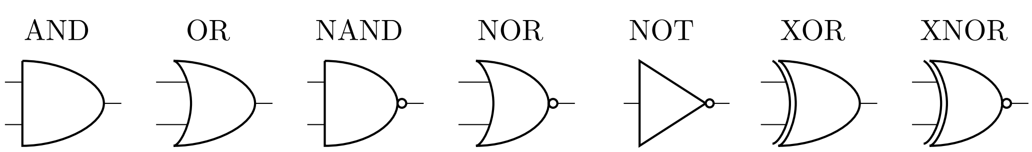
Electronics

Electronic Symbols & Circuit Diagrams

Circuit diagrams are a major part of understanding and representing electronic circuits. Some common **circuit diagram symbols** follow:

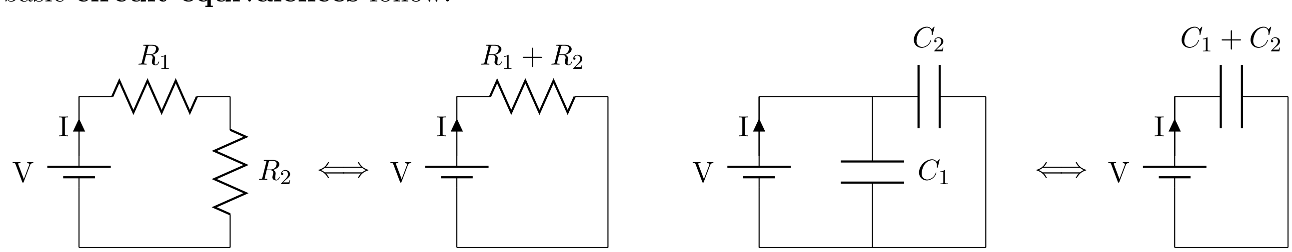


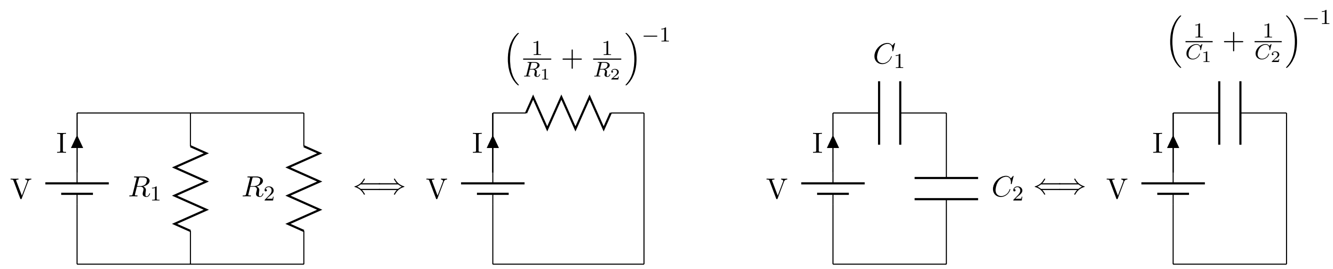
An elementary building block of a circuit is a **logic gate**. At any given time, the 3 nodes of a logic gate are either true (1) or false (0). The common notation for the logic gates follow:



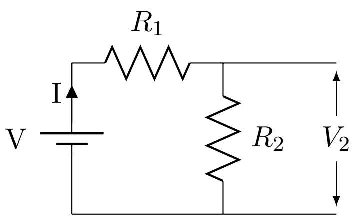
Equivalent Circuits

When dealing with circuit diagrams, it is often helpful to simplify a circuit using an equivalent circuit. Some basic **circuit equivalences** follow:





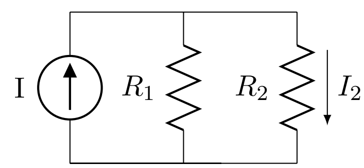
A **voltage divider**:



$$V_2 = \frac{VR_2}{R_1 + R_2} \tag{12.2.1}$$

$$V = I_1(R_1 + R_2) = I_2(R_1 + R_2) \tag{12.2.2}$$

A **Current divider**:



$$I_2 = \frac{IR_1}{R_1 + R_2} \tag{12.2.3}$$

Special Relativity

Relativistic time dilation and length contraction.

$$\Delta t = \frac{\Delta t_o}{\sqrt{1-\beta^2}} = \gamma \Delta t_0 \tag{13.0.1}$$

$$\Delta l = \Delta l_0 \sqrt{1-\beta^2} = \frac{\Delta l_0}{\gamma} \tag{13.0.2}$$

$$\beta = \frac{v}{c} \tag{13.0.3}$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} \tag{13.0.4}$$

Lorentz Transformations for space and time coordinates.

$$x' = \gamma(x - vt) \tag{13.0.5}$$

$$y' = y \tag{13.0.6}$$

$$z' = z \tag{13.0.7}$$

$$t' = \gamma(t - vx/c^2) \tag{13.0.8}$$

The relativistic velocity transformation is.

$$u' = \frac{u - v}{1 - vu/c^2} \tag{13.0.9}$$

$$u = \frac{u' + v}{1 + vu'/c^2} \tag{13.0.10}$$

The rest energy of a particle

$$E_0 = mc^2 \tag{13.0.11}$$

the lorentz transformation for momentum and energy is.

$$p'_x = \gamma(p_x - vE/c^2) \tag{13.0.12}$$

$$p'_y = p_y \tag{13.0.13}$$

$$p'_z = p_z \tag{13.0.14}$$

$$E' = \gamma(E - vp_x) \tag{13.0.15}$$

Relativistic mass and momentum.

$$E = \gamma mc^2 \tag{13.0.16}$$

$$p = \gamma mv \tag{13.0.17}$$

Combining the above equations give

$$\frac{E}{p} = \frac{c^2}{v} \implies E = \frac{pc^2}{v} \tag{13.0.18}$$

Mass-energy equivalence and kinetic energy (K_E).

$$E^2 = (mc^2)^2 + (pc)^2 \tag{13.0.19}$$

$$E = K_E + E_0 \tag{13.0.20}$$

$$K_E = (\gamma - 1)mc^2 \tag{13.0.21}$$

Combining the above equations gives

$$p = \frac{1}{c}\sqrt{K_E^2 + 2K_E E_0} \tag{13.0.22}$$

Invariant dot product in c=1 notation

$$A \cdot B = (E, \vec{p}) \cdot (U, \vec{q}) = EU - \vec{p} \cdot \vec{q} \tag{13.0.23}$$

Relativistic frequency and wavelength shifts

$$f = f_0 \sqrt{\frac{c \pm v}{c \mp v}} \Longleftrightarrow \pm v = \frac{f^2 - f_0^2}{f^2 + f_0^2} \tag{13.0.24}$$

$$\lambda = \lambda_0 \sqrt{\frac{c \mp v}{c \pm v}} \Longleftrightarrow \mp v = \frac{\lambda^2 - \lambda_0^2}{\lambda^2 + \lambda_0^2} \tag{13.0.25}$$

Space-time equivalence (same in all reference frames)

$$S \equiv (c\Delta t)^2 - (\Delta x)^2 \equiv E^2 - (pc)^2 \tag{13.0.26}$$

Statistics

A probability distribution: Given a Poisson process, the probability of obtaining exactly m successes in n trials is given by the limit of a binomial distribution

$$\mathcal{P}_n(m;p) = \binom{n}{m} p^m (1-p)^{n-m} \tag{14.0.1}$$

Letting the sample size n become large, the distribution then approaches the Poisson Distribution

$$\mathcal{P}(m,\lambda) = \frac{\lambda^m}{m!} e^{-\lambda} \tag{14.0.2}$$

The mean number of events is

$$\langle m \rangle = \sum_{m=0}^{\infty} m \frac{\lambda^m}{m!} e^{-\lambda} = \lambda \tag{14.0.3}$$

And the standard deviation is

$$\sigma = \sqrt{\lambda} \tag{14.0.4}$$

The normal, or Gaussian distribution

$$\mathcal{P}(x;\mu,\sigma) = \frac{1}{\sqrt{2\pi}\sigma} exp\Bigg[-\frac{(x-\mu)^2}{2\sigma^2} \Bigg] \tag{14.0.5}$$

$$\mathcal{P}(a \leq x \leq b) = \int_a^b \frac{1}{\sqrt{2\pi}\sigma} exp\Bigg[-\frac{(x-\mu)^2}{2\sigma^2} \Bigg] \tag{14.0.6}$$

If the mean is not equal to zero, a more general distribution known as the noncentral chi-squared distribution results. In particular, if x_i are independent variates with a normal distribution having means μ_i and variances σ_i^2 for $i = 1, ..., n$, then

$$\chi^2 \equiv \sum_{i=1}^{n} \frac{(x_i - \mu_i)^2}{\sigma_i^2}. \tag{14.0.7}$$

Given some function $f(x_1, x_2, \ldots, x_n)$, the error of a calculation with each respective variable being denoted by σ_i, can be determined by

$$\sigma_f^2 = \left(\frac{\partial f}{\partial x_1}\right)^2 \sigma_{x_1}^2 + \left(\frac{\partial f}{\partial x_2}\right)^2 \sigma_{x_2}^2 + \cdots + \left(\frac{\partial f}{\partial x_n}\right)^2 \sigma_{x_n}^2 = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i}\right)^2 \sigma_{x_i}^2 \tag{14.0.8}$$

Thermodynamics

Useful constants: the specific heat of water is c

$$c = 4186 \text{ J/(kg·K)}. \tag{15.0.1}$$
$$1 \text{ cal} = 4.186 \text{ J} \tag{15.0.2}$$

Temperature relationships.

$$^{\circ}F = \frac{9}{5}^{\circ}C + 32 \tag{15.0.3}$$
$$^{\circ}C = \frac{5}{9}(^{\circ}F - 32) \tag{15.0.4}$$
$$^{\circ}K =^{\circ} C + 273.15 \tag{15.0.5}$$

The heat required to raise the temperature of a mass m by ΔT is

$$Q = cm\Delta T \tag{15.0.6}$$

he temperature of an object determines the radiated power of the object, which is given by the **Stefan-Boltzmann equation**

$$P_{radiated} = \sigma\epsilon A T^4 \tag{15.0.7}$$
$$\sigma = 5.67 \times 10^{-8} \text{ W}/K^4 m^2 \tag{15.0.8}$$
$$\epsilon = \text{emissivity, and } 0 \leq \epsilon \leq 1 \tag{15.0.9}$$

The work done on a system in going from initial volume (V_i) to a final volume (V_f) is

$$W = \int dW = \int_{V_i}^{V_f} pdV. \tag{15.0.10}$$

The first law of thermodynamics, with internal energy (dU), heat transferred (dQ), pressure (P) and volume (dV)

$$\Delta E_{internal} = Q - W \tag{15.0.11}$$
$$dU = dQ - PdV \tag{15.0.12}$$

different processes include

(i) An adiabatic process is one where $dQ = 0$.

(ii) In a constant-volume process, $W = 0$.

(iii) In a closed-loop process, $Q = W$.

(iv) In an adiabatic free expansion, $Q = W = \Delta E_{internal} = 0$.

The efficiency of a system is defined by

$$Eff = \frac{W_{cycle}}{Q_{in}} = 1 - \frac{Q_c}{Q_h} < 100\% \tag{15.0.13}$$

If heat is added to an object, its change in temperature (with C =heat capacity of the object) is given by

$$\Delta T = \frac{Q}{C} \tag{15.0.14}$$

If heat is added to an object with mass m, its change in temperature (with c =specific heat of the object) is given by

$$\Delta T = \frac{Q}{cm} \tag{15.0.15}$$

The ideal gas law: For an ideal gas of n particles in a volume V at pressure and temperature P and T, the equation of state is

$$PV = nN_A kT \equiv nRT \tag{15.0.16}$$

With a constant number of moles we get from the ideal gas law the following relation:

$$\frac{P_1 V_1}{T_1} = \frac{P_2 V_2}{T_2} \tag{15.0.17}$$

Dalton's law - The total pressure exerted by a mixture of gases is equal to the sum of the partial pressures pf the gases in the mixture.

$$P_{total} = P_1 + P_2 + P_3 + \cdots + P_n \tag{15.0.18}$$

The work done by an ideal gas at constant temperature is

$$W = nRT \ln\left(\frac{V_f}{V_i}\right) \tag{15.0.19}$$

The average kinetic energy of an ideal gas

$$K_{ave} = \frac{1}{N}\sum_{i=1}^{N} K_i \tag{15.0.20}$$
$$= \frac{1}{N}\sum_{i=1}^{N} \frac{1}{2} m v_i^2 \tag{15.0.21}$$
$$= \frac{1}{2} m v_{rms}^2 \tag{15.0.22}$$

The root-mean-square speed of gas molecules is

$$v_{rms} = \sqrt{\frac{1}{N}\sum_{i=1}^{N} v_i^2} = \sqrt{\frac{3RT}{m}} \tag{15.0.23}$$

For an adiabatic process (with C_V=specific heat at constant volume, C_P=specific heat at constant pressure), we have

$$dE_{internal} = -PdV = nC_V dT \tag{15.0.24}$$

$$PV^\gamma = \text{constant} \tag{15.0.25}$$

$$\gamma = \frac{C_P}{C_V} \tag{15.0.26}$$

$$P_f V_f^\gamma = P_i V_i^\gamma \tag{15.0.27}$$

$$T_f V_f^{\gamma-1} = T_i V_i^{\gamma-1} \tag{15.0.28}$$

In classical thermodynamics the entropy S is defined by The fundamental temperature τ is defined by the relation

$$\frac{1}{T} = \left(\frac{\partial S}{\partial U}\right)_N . \tag{15.0.29}$$

Thermal & Statistical Physics

States of a Model System

The multiplicity function for a system of N magnets with a spin excess $2s = N_\uparrow - N_\downarrow$ is

$$g(N,s) = \frac{N!}{(\frac{N}{2}+s)!(\frac{N}{2}-s)!} = \frac{N!}{N_\uparrow! N_\downarrow!}. \quad (16.1.1)$$

It is often useful to evaluate $g(N,s)$ within a logarithm in which the **Stirling approximation** becomes useful.

$$N! \approx N^N \sqrt{2\pi N} \exp\left(-N + \frac{1}{12N} + \cdots\right). \quad (16.1.2)$$

It is often useful to take the logarithm of this which gives

$$\log N! \cong \frac{\log 2\pi}{2} + \left(N + \frac{1}{2}\right) \log N - . \quad (16.1.3)$$

In the limit $s/N << 1$, with $N >> 1$, we have the Gaussian approximation

$$g(N,s) \cong g(N,0) \exp\left(\frac{-2s^2}{N}\right) \quad (16.1.4)$$

$$g(N,0) \simeq 2^N \sqrt{\frac{2}{\pi N}}. \quad (16.1.5)$$

The exact value of $g(N,0)$ is given by

$$g(N,0) = \frac{N!}{(N/2)!(N/2)!}. \quad (16.1.6)$$

The average value, or mean value, of a function f(s) taken over a probability distribution P(s) is defined as

$$\langle f \rangle = \sum_s f(s)P(s), \quad (16.1.7)$$

$$1 = \sum_s P(s). \quad (16.1.8)$$

The binomial distribution has the property

$$\sum_s g(N,s) = 2^N. \quad (16.1.9)$$

If all states of the model spin system are equally likely, the average value of s^2 is

$$\langle s^2 \rangle = \frac{\int_{-\infty}^{\infty} s^2 g(N,s)ds}{\int_{-\infty}^{\infty} g(N,s)ds} = \frac{N}{4} \quad (16.1.10)$$

The energy interaction of a single magnetic moment $\vec{m}$ with a fixed external magnetic field $\vec{B}$ is

$$U = -\vec{m} \cdot \vec{B}. \quad (16.1.11)$$

For a model system of N elementary magnets, each with two allowed orientations in a uniform magnetic field $\vec{B}$, the total potential energy U is

$$U = \sum_{i=0}^{N} U_i = -\vec{B} \cdot \sum_{i=0}^{N} m_i \quad (16.1.12)$$

$$= -2smB = -MB. \quad (16.1.13)$$

Entropy And Temperature

If $P(s)$ is the probability that a system is in the state X, the average value of a quantity X is

$$\langle X \rangle = \sum_s X(s)P(s). \quad (16.2.1)$$

The number of combined systems 1 and 2 (with $s = s_1 + s_2$) is

$$g(s) = \sum_s g_1(s_1)g_2(s - s_1). \quad (16.2.2)$$

The relation $s = k_B\sigma$ connects the conventional entropy S with the fundamental entropy σ. The **entropy** $\sigma(N,U)$ is given by

$$\sigma(N,U) = \log g(N,U). \quad (16.2.3)$$

The fundamental temperature τ is defined by the relation

$$\frac{1}{\tau} = \left(\frac{\partial\sigma}{\partial U}\right)_{N,V}. \quad (16.2.4)$$

Boltzmann Distribution & Helmholtz Free Energy

The partition function Z is

$$Z \equiv \sum_s \exp\left(-\frac{\epsilon_s}{\tau}\right). \tag{16.3.1}$$

The probability of finding a system of N particles in a state s of energy ϵ_s when the system is in thermal contact with a large reservoir at temperature τ is

$$P(\epsilon_s) = \frac{1}{Z}\exp\left(-\frac{\epsilon_s}{\tau}\right). \tag{16.3.2}$$

The pressure is given by

$$P = -\left(\frac{\partial U}{\partial V}\right)_\sigma = \tau\left(\frac{\partial \sigma}{\partial V}\right)_U. \tag{16.3.3}$$

The **Helmholtz Free Energy** is a minimum in equilibrium for a system held at constant τ, V and is defined as

$$F \equiv U - \tau\sigma. \tag{16.3.4}$$

From this we have

$$\sigma = -\left(\frac{\partial F}{\partial \tau}\right)_V \tag{16.3.5}$$

$$P = -\left(\frac{\partial F}{\partial V}\right)_\tau. \tag{16.3.6}$$

For an ideal monotonic gas of N atoms of spin zero with $n = N/V << n_Q$,

$$Z_n = \frac{Z_1^N}{N!} = \frac{(n_Q V)^N}{N!}, \tag{16.3.7}$$

The quantum concentration n_Q is defined by

$$n_Q \equiv \left(\frac{M\tau}{2\pi\hbar^2}\right)^{3/2}. \tag{16.3.8}$$

Furthermore, we have

$$PV = N\tau \tag{16.3.9}$$

$$\sigma = N\left[\log\left(\frac{n_Q}{n}\right) + \frac{5}{2}\right] \tag{16.3.10}$$

$$C_V = \frac{3}{2}N. \tag{16.3.11}$$

The thermal average energy of an atom a a box is

$$U = \langle\epsilon\rangle = \tau^2\frac{\partial \log(Z_1)}{\partial\tau}. \tag{16.3.12}$$

For a system of fixed volume in thermal contact with a reservoir, the mean square fluctuation in energy of the system is

$$\langle(\epsilon - \langle\epsilon\rangle)^2\rangle = \tau^2\left(\frac{\partial U}{\partial\tau}\right)_V \tag{16.3.13}$$

Thermal Radiation & Planck Distribution

The Planck distribution function for the thermal average number of photons in a cavity mode of frequency ω is

$$\langle s\rangle = \frac{1}{\exp(\hbar\omega/t) - 1}. \tag{16.4.1}$$

The Stefan-Boltzmann law for the radiant energy density in a cavity at temperature τ is

$$\frac{U}{V} = \frac{\pi^2}{15\hbar^3 c^3}\tau^4. \tag{16.4.2}$$

The Planck radiation law for the energy per unit volume per unit range of frequency is

$$u_\omega = \frac{\hbar}{\pi^2 c^3}\frac{\omega^3}{\exp(\hbar\omega/t) - 1}. \tag{16.4.3}$$

The flux density of radiant energy J_ν and the Stefan-Boltzmann constant are

$$J_\nu = \sigma_B T^4 \tag{16.4.4}$$

$$\sigma_B = \frac{\pi^2 k_B^4}{60\hbar^3 c^3}. \tag{16.4.5}$$

The Debye low temperature limit of the heat capacity of a dielectric solid (where ϑ is the Debye temperature) is, in conventional units,

$$C_V = \frac{12\pi^4 N k_B}{5}\left(\frac{T}{\vartheta}\right)^3 \tag{16.4.6}$$

$$\vartheta = \frac{\hbar c}{k_B}\left(\frac{6\pi^2 N}{V}\right)^{1/3}. \tag{16.4.7}$$

Elementary Quantum Physics

Wien's Displacement Law

$$\lambda_{MAX} T = 2.898 \times 10^{-3} m * K \tag{17.0.1}$$

Total Power Stefan-Boltzmann Law

$$R(T) = \int_0^\infty I(\lambda, T) d\lambda = \epsilon \sigma T^4 \tag{17.0.2}$$

$$\epsilon = \text{emmisivity (unitless)} \tag{17.0.3}$$

$$\sigma = 5.67 \times 10^{-8} \frac{w}{m^2 k^4} \tag{17.0.4}$$

Max Planck's Radiation Law:

$$I(\lambda, T) = \frac{2\pi c^2 h}{\lambda^5} \frac{1}{e^{hc/\lambda kT} - 1} \tag{17.0.5}$$

The kinetic energy of an emitted photoelectron is (Where ϕ = binding energy of electron to metal surface or the work function)

$$KE = hv - \phi \tag{17.0.6}$$

$$E_{photon} = KE_{electron} + \phi \tag{17.0.7}$$

$$KE_{electrons} = 0 \text{ (at threshold)} \tag{17.0.8}$$

Ruthford Scattering Formula:Any particle hitting an area σ around the nucleus will be scattered through an angle of θ or greater.

$$b = (r_{min}/2) \cot(\theta/2) \tag{17.0.9}$$

$$r_{min} = \frac{Z_1 Z_2 e^2}{4\pi\epsilon_0 K} \tag{17.0.10}$$

$$\sigma = \pi b^2 = \text{ cross sectional area} \tag{17.0.11}$$

$$\frac{e^2}{4\pi\epsilon_0} = 1.44 \times 10^{-9} \text{eV}\cdot\text{m} \tag{17.0.12}$$

A common unit of σ is one barn.

$$\text{barn (unit)} = 10^{-28} m^2 = 100 fm^2 \tag{17.0.13}$$

$$\frac{\#\text{ atoms}}{\text{area}} = \frac{\text{atoms}}{\text{volume}} \times \text{thickness} \tag{17.0.14}$$

$$n = \left(N_A \frac{atoms}{mole}\right)\left(\frac{1}{A}\frac{mole}{gm}\right)\left(\rho \frac{gm}{cm^3}\right) \tag{17.0.15}$$

$$= \frac{\rho N_A}{A} \tag{17.0.16}$$

The Compton effect describes the photon wavelength λ' after a photon of wavelength λ scatters off an electron.

$$\lambda' = \lambda + \frac{h}{m_e c}(1 - cos(\theta)) \tag{17.0.17}$$

The Compton wavelength of an electron is

$$\lambda_e = \frac{h}{m_e c} = 2.426 \times 10^{-12} m \tag{17.0.18}$$

Heisenberg Uncertainty relation

$$\Delta x \Delta p_x \geq \frac{1}{2}\hbar \tag{17.0.19}$$

The de Broglie wavelength is defined as

$$\lambda = \frac{h}{p} = \frac{h}{mv\gamma} = \frac{h}{mv}\sqrt{1 - \frac{v^2}{c^2}} \tag{17.0.20}$$

$$= \frac{hc}{\sqrt{K_E^2 + 2K_E E_0}} \tag{17.0.21}$$

Rutherford Scattering.

$$K = \frac{1}{4\pi\epsilon_0}\frac{Z_1 Z_2 e^2}{R_{min}} \tag{17.0.22}$$

$$R_{min} = \frac{1}{4\pi\epsilon_0}\frac{Z_1 Z_2 e^2}{K} \tag{17.0.23}$$

Z's are the atomic masses of the particles within the interaction and R_{min} is the minimum distance they reach (from center to center), and e is

$$e = 1.602177 \times 10^{-19} C \tag{17.0.24}$$

$$\epsilon \approx 8.854 \times 10^{-12} F/m \tag{17.0.25}$$

The Rutherford Scattering Formula

$$N(\theta) = \frac{N_i n t}{16 r^2}(R_{min})^2 \frac{1}{\sin^4(\theta/2)} \tag{17.0.26}$$

Centripetal force due to coulomb attraction

$$\frac{1}{4\pi\epsilon_0}\frac{e^2}{r^2} = m a_c = m\frac{v^2}{r} \tag{17.0.27}$$

$$\Longrightarrow v^2 = \frac{1}{4\pi\epsilon_0}\frac{e^2}{mr} \tag{17.0.28}$$

$$\Longrightarrow r = 4\pi\epsilon_0 \frac{n^2\hbar^2}{me^2} \tag{17.0.29}$$

Energy levels

$$E = KE + PE \tag{17.0.30}$$
$$= -\frac{1}{2}\frac{e^2}{4\pi\epsilon_0 r} \tag{17.0.31}$$
$$\implies E = \frac{-E_0}{n^2}, \tag{17.0.32}$$
$$\text{where } E_0 = \alpha^2 mc^2/2 = 13.6 \text{ eV}. \tag{17.0.33}$$

Energy of emitted radiation

$$E = E_n - E_m \tag{17.0.34}$$
$$= E_0\left(\frac{1}{m^2} - \frac{1}{n^2}\right) \tag{17.0.35}$$

Note. Using the Planck formula in the above equation leads to the Rydberg formula.

The Rydberg formula: Wavelength of the spectral lines in Hydrogen:

$$\frac{1}{\lambda} = R_H\left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right) \tag{17.0.36}$$
$$n \in \mathbb{N} = 1, 2, 3, 4, 5, \ldots \tag{17.0.37}$$

Note. ZnS (Zinc Sulfide) emits a faint flash of light when struck by an α−ray.

L quantized

$$L = mvr = n\hbar \tag{17.0.38}$$

Stationary state orbits

$$r = a_0 n^2 \tag{17.0.39}$$
$$a_0 = \text{Bohr Radius} \tag{17.0.40}$$

Stationary state energies

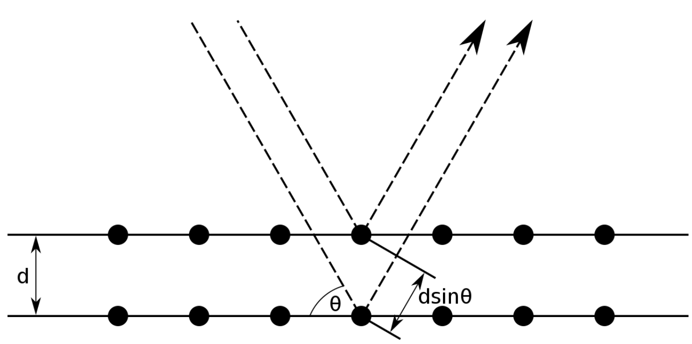
$$E_n = -Z^2\frac{E_0}{n^2} \tag{17.0.41}$$

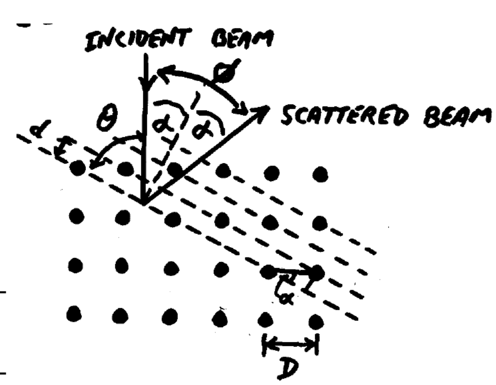
Uncertainty relation of energy and the measurement of time.

$$\Delta E \cdot \Delta t \geq \frac{1}{2}\hbar \tag{17.0.42}$$

Bragg's Law: When scattering off of crystal structures, the wavelengths will peak at specific angles determined by the diagrams below

$$n\lambda = 2d\sin(\theta) = 2d\cos(\alpha) = 2D\sin(\alpha)\cos(\alpha) = D\sin(2\alpha) = D\sin(\phi) \tag{17.0.43}$$
$$d = Dsin(\alpha) \tag{17.0.44}$$
$$\phi = 2\alpha \tag{17.0.45}$$
$$\theta = 90^\circ - \alpha \tag{17.0.46}$$





The potential the electron moves in

$$V(r) = \frac{-e^2}{(4\pi\epsilon_0 r)} \tag{17.0.47}$$

The angular momentum of an electron in the atom

$$L = mvr = \hbar\sqrt{\ell(\ell+1)} \tag{17.0.48}$$

$$L_z = m_\ell \hbar \tag{17.0.49}$$

An electron orbiting around a nucleus has magnetic moment $\vec{\mu}$

$$\vec{\mu} = IA\hat{n} = \frac{-e}{(2\pi r/v)}(\pi r^2)\hat{n} = \frac{-erv}{2}\hat{n} = \frac{-e}{2m}\vec{L} \tag{17.0.50}$$

$$\mu_z = \frac{-e}{2m}L_z = \frac{-e}{2m}m_l\hbar = -m_\ell \mu_B \tag{17.0.51}$$

In an external magnetic field, B, the magnetic dipole feels a torque $\vec{\tau}$ and has a potential energy U_B

$$\vec{\tau} = \vec{\mu} \times \vec{B} \tag{17.0.52}$$

$$V_B = -\vec{\mu} \cdot \vec{B} = \frac{-e}{2m}\vec{L} \cdot \vec{B} \implies V_{Bz} = \mu_B m_\ell B_z \tag{17.0.53}$$

Quantum Mechanics

Plane waves with electromagnetic wave frequencies and wavelengths

$$\psi(x,t) = A\cos[2\pi(x-ct)/\lambda] \tag{18.0.1}$$
$$c = f\lambda \Longleftrightarrow f = \frac{c}{\lambda} \Longleftrightarrow \lambda = \frac{c}{f} \tag{18.0.2}$$
$$T = 1/f \tag{18.0.3}$$
$$\psi(x,t) = A\cos(kx - \omega t) \tag{18.0.4}$$
$$k = 2\pi/\lambda \tag{18.0.5}$$
$$\omega = 2\pi f = 2\pi/T \tag{18.0.6}$$

The energy in a photon (packet of light)

$$E = hf = \frac{hc}{\lambda} = \hbar\omega \tag{18.0.7}$$
$$dE = -\frac{hc}{\lambda^2}d\lambda = -\frac{E^2}{hc}d\lambda \tag{18.0.8}$$
$$|\Delta\lambda| = hc\frac{\Delta E}{E^2} \tag{18.0.9}$$

The wave equation

$$\frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2} = \nabla^2\psi \tag{18.0.10}$$

A periodic wave can be constructed from a sum of plane waves

$$\psi(x,t) = \sum_{i=1}^{n} A_i\cos(k_i x_i - \omega_i t) \tag{18.0.11}$$

The cubit is defined as

$$|\psi\rangle = c_1|1\rangle + c_0|0\rangle \tag{18.0.12}$$
$$|\psi\rangle = c_{11}|11\rangle + c_{01}|01\rangle + c_{10}|10\rangle + c_{00}|00\rangle \tag{18.0.13}$$
$$\vdots$$

The quantum mechanical **expectation value** of an observable $\hat{X}$ in a normalized state ψ is found by integrating over the entire space ψ^* times the result obtained when the corresponding operator acts on ψ.

$$\langle\psi|\hat{X}|\psi\rangle = \int_{-\infty}^{\infty}\psi^*\hat{X}\psi dx \tag{18.0.14}$$

The state of the system is given by a wavefunction $\psi(\vec{r},t)$. The probability density is the square modulus of the amplitude

$$P(\vec{r})d\vec{r} = |\psi(\vec{r})|^2 d\vec{r} \tag{18.0.15}$$

The probability of a particle being between x_1 and x_2 given a normalized wave function $\psi(x,t)$ is

$$P_{x\in x_1:x_2}(t) = \int_{x_1}^{x_2}|\psi(x,t)|^2 dx = \int_{x_1}^{x_2}\psi^*(x,t)\psi(x,t)dx \tag{18.0.16}$$

The normalization of a wave function implies the probability over all space is 1.

$$P_{x\in-\infty:\infty}(t) = \int_{-\infty}^{\infty}|\psi(x,t)|^2 dx = 1 \tag{18.0.17}$$

The **Schrödingr Equation** is a partial differential equation that describes how the wavefunction of a physical system evolves over time. The (non-relativistic) Schrödingr Equation for a particle moving in a 3-dimensional potential energy field $V(\vec{r})$ is

$$\hat{E}\psi(\vec{r},t) = \frac{-\hbar^2}{2m}\nabla^2\psi(\vec{r},t) + V(\vec{r})\psi(\vec{r},t) \equiv \hat{H}\psi(\vec{r},t) \tag{18.0.18}$$

The Dirac equation: the generalization of the time dependent Schrödinger equation for the relativistically correct relationship between energy and momentum. It leads to negative energy states and antiparticles.

$$\left[\gamma^0 mc^2 + \sum_{i=1}^{3}\gamma^i\hat{p}_i c\right]\psi(\vec{r},t) = i\hbar\frac{\partial}{\partial t}\psi(\vec{r},t) \tag{18.0.19}$$

Each observable corresponds to a linear operator. A linear operator is something that acts on a state and gives another state. The Hamiltonian operator is defined as the operator $\hat{H}$ such the energy E of a system with wavefunction ψ is an eigenvalue of $\hat{H}\psi$ or $\hat{H}\psi = E\psi$.

$$\hat{H} = \hat{K} + V(\hat{r}) = \frac{\hat{p}^2}{2m} + V(\hat{r}) = \frac{-\hbar^2}{2m}\nabla^2 + V(\vec{r}) \tag{18.0.20}$$

The Energy operator

$$i\hbar\frac{\partial\psi}{\partial t} = i\hbar\frac{\partial}{\partial t}Ae^{i(kx-\omega t)} = i\hbar(-i\omega)\psi = \hbar\omega\psi = E\psi \implies \hat{E} = i\hbar\frac{\partial}{\partial t} \tag{18.0.21}$$

$$\langle\psi|E|\psi\rangle = \int_{-\infty}^{\infty}\psi^*\hat{E}\psi dx = i\hbar\int_{-\infty}^{\infty}\psi^*\frac{\partial\psi}{\partial t}dx \tag{18.0.22}$$

The operator for a particles kinetic energy is

$$\hat{K} = \frac{-\hbar^2}{2m}\nabla^2 \tag{18.0.23}$$

$$\hat{K}\psi = \frac{1}{2m}\hat{p}^2\psi = \frac{1}{2m}\left(-i\hbar\frac{\partial}{\partial x}\right)^2\psi = \frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi \tag{18.0.24}$$

$$\langle\psi|\hat{K}|\psi\rangle = \int_{-\infty}^{\infty}\psi^*\hat{K}\psi dx = \int_{-\infty}^{\infty}\psi^*\frac{1}{2m}\hat{p}^2\psi dx = \frac{-\hbar^2}{2m}\int_{-\infty}^{\infty}\psi^*\frac{\partial^2}{\partial x^2}\psi dx \tag{18.0.25}$$

The momentum operator

$$\hat{p} \equiv -i\hbar\nabla = \frac{\hbar}{i}\nabla \tag{18.0.26}$$

$$\frac{\partial\psi}{\partial x} = \frac{\partial}{\partial x}Ae^{i(kx-\omega t)} = ik\psi = \frac{ip}{\hbar}\psi \implies \hat{p} = -i\hbar\frac{\partial}{\partial x} \tag{18.0.27}$$

$$\langle\psi|\hat{p}|\psi\rangle = \int_{-\infty}^{\infty}\psi^*\hat{p}\psi dx = -i\hbar\int_{-\infty}^{\infty}\psi^*\frac{\partial\psi}{\partial x}dx \tag{18.0.28}$$

The wave function solution for a particle confined to an infinite potential well with walls at $x = 0$ and $x = a$ is as follows, with the corresponding energy eigenvalues (with $n \in \mathbb{N}$)

$$\psi(x) = \begin{cases}\sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right) & 0 \le x \le a \\ 0 & \text{otherwise}\end{cases} \tag{18.0.29}$$

$$E_n = \frac{\hbar^2\pi^2}{2ma^2}n^2 \tag{18.0.30}$$

The solution to The Schrödingr Equation for a finite potential well with the potential

$$V(x) = \begin{cases} \infty & \text{for } x < 0 \\ 0 & \text{for } 0 \le x \le a \\ V_1 & \text{for } x > a \end{cases} \tag{18.0.31}$$

with $E > V_1$ is

$$\psi(x) = \begin{cases} 0 & \text{for } x < 0 \\ D\sin(kx) & \text{for } 0 \le x \le a \\ F\cos(k'x) + G\sin(k'x) & \text{for } x > a \end{cases} \tag{18.0.32}$$

$$\text{with } k' = \sqrt{k^2 - \frac{2mV_1}{\hbar^2}} \tag{18.0.33}$$

with $E < V_1$ is

$$\psi(x) = \begin{cases} 0 & \text{for } x < 0 \\ D\sin(kx) & \text{for } 0 \le x \le a \\ Fe^{-\gamma x} & \text{for } x > a \end{cases} \tag{18.0.34}$$

$$\text{with } \gamma^2 = \frac{2m(V_1 - E)}{\hbar^2} = \frac{2mV_1}{\hbar^2} - k^2 \tag{18.0.35}$$

The solutions for a finite barrier (with the probability of reflection as $R = |B|^2/|A|^2$ and the probability of transmission as $T = |F|^2/|A|^2$) are

$$\psi_1 = Ae^{ikx} + Be^{-ikx} \qquad \text{(incident + reflected)} \tag{18.0.36}$$

$$\psi_2 = Ce^{ik'x} + De^{-ik'x} \qquad \text{(intermediate)} \tag{18.0.37}$$

$$\psi_3 = Fe^{ikx} \qquad \text{(transmitted)} \tag{18.0.38}$$

$$k = \sqrt{2mE/\hbar^2} \tag{18.0.39}$$

$$k' = \sqrt{2m(E - U_0)/\hbar^2} \tag{18.0.40}$$

For any two Hermitian operators A and B,

$$\Delta A \Delta B \ge \frac{1}{2}|\langle i[A, B]\rangle| \tag{18.0.41}$$

Atomic quantum numbers

$$n = \text{ Principle Quantum Number } [n \in \mathbb{N}] \tag{18.0.42}$$

$$\ell = \text{ Orbital Angular Momentum Quantum Number } [\ell \in \mathbb{N} \cup \{0\}, \ell < n] \tag{18.0.43}$$

$$m_\ell = \text{ Magnetic Quantum Number } [m_\ell \in [-\ell, \ell], m_\ell \in \mathbb{Z}] \tag{18.0.44}$$

For a simple harmonic oscillator, the energies and eigenfunctions are given by,

$$V(x) = \frac{1}{2}kx^2 = \frac{1}{2}m\omega^2 x^2 \implies k = m\omega^2 \iff \omega = \sqrt{\frac{k}{m}} \tag{18.0.45}$$

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega = \left(n + \frac{1}{2}\right)\hbar\sqrt{\frac{k}{m}} = \left(n + \frac{1}{2}\right)\frac{\hbar}{x}\sqrt{\frac{2V(x)}{m}} \tag{18.0.46}$$

$$H|n\rangle = E_n|n\rangle, \qquad H = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right), \qquad [a, a^\dagger] = 1 \tag{18.0.47}$$

$$|n\rangle = \frac{(a^\dagger)^n|0\rangle}{\sqrt{n!}}, \qquad a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle, \qquad a|n\rangle = \sqrt{n}|n-1\rangle \tag{18.0.48}$$

Each operator $\hat{Y}$ has a set of eigenvalues y which are the possible values you can get on doing a measurement of Y. Each eigenvalues y is associated with an eigenstate $\phi_y(x)$ which is the state for which the values of Y is exactly y with no uncertainty. You can find the eigenstates and eigenvalues of an operator by

$$\hat{Y}\phi_y(x) = y\phi_y(x). \tag{18.0.49}$$

The eigenstates of any operator $\hat{Y}$ form a complete orthonormal basis of states so we can write any state $\psi(x)$ in terms of them.

$$\psi(x) \equiv \sum_y A_y \phi_y(x) \quad \text{or} \quad \psi(x) \equiv \int A(y)\phi_y(x)dy. \tag{18.0.50}$$

To solve for the coefficients in the above expression we can use Fouriers trick, or

$$A(y) = \int \phi_y^*(x)\psi(x)dx \tag{18.0.51}$$

If you are within operator space you can find the expectation value of an operator by

$$\langle \hat{y} \rangle = \int y|A(y)|^2 dy. \tag{18.0.52}$$

The commutation relation is a relationship between two operators and given by

$$[x, y] \equiv xy - yx \implies [x, y] = -[y, x] \tag{18.0.53}$$

$$[x, y] = [y, x] = 0 \implies x \text{ and } y \text{ commute} \tag{18.0.54}$$

$$[xy, z] = x[y, z] + [x, z]y \qquad \text{and} \qquad [x, yz] = y[x, z] + [x, y]z.adv \tag{18.0.55}$$

Position and momentum are related via commutation by the following:

$$[x, p_x] = [y, p_y] = [z, p_z] = i\hbar \tag{18.0.56}$$

$$[x, p_y] = [x, p_z] = [y, p_x] = [y, p_z] = [z, p_x] = [z, p_y] = 0. \tag{18.0.57}$$

The angular momentum operators are related via commutation by the following:

$$[L_x, L_y] = i\hbar L_z \qquad \text{and} \qquad [L_y, L_z] = i\hbar L_x \qquad \text{and} \qquad [L_z, L_x] = i\hbar L_y \tag{18.0.58}$$

$$[L_x, \vec{r}] = i\hbar(z - y) \qquad \text{and} \qquad [L_x, \vec{L}] = i\hbar(L_z - L_y) \qquad \text{and} \qquad [L_x, \vec{p}] = i\hbar(p_z - p_y) \tag{18.0.59}$$

$$[\vec{L}^2, L_z] = 0 \tag{18.0.60}$$

Nuclear and High Energy Physics

Nuclear Physics

The atomic nucleus consists of protons and neutrons collectively called nucleons. Nuclei with different number of neutrons but with the same number of protons are isotopes of the same element. The mass number of an isotope os the sum of the number of protons (Z) and the number of neutrons (N)

$$A = Z + N. \tag{19.1.1}$$

Nuclei are approximately spherical in shape, with the radius of the sphere depending on the mass number ($R_0 = 1.12$ fm)

$$R(A) = R_0 A^{1/3}. \tag{19.1.2}$$

The nucleon density in the interior of a nucleus is $n = 0.17\ \text{fm}^{-3}$, and the mass density is $\rho = m_{nucleon} n = 2.8 \times 10^{17} \text{kg/m}^3$. The dependence of the density on the radial coordinate is given by the Fermi function ($a = 0.54$ fm):

$$n(r) = \frac{n_0}{1 + e^{(r-R(A))/a}}. \tag{19.1.3}$$

The Nuclear (or "strong") force is what binds the protons and neutrons together into nuclei with the following properties:

(i) Within the nucleus, it is about 100 times stronger than the electromagnetic force and approximately 10^{38} times stronger than gravity.

(ii) It is charge-independent.

(iii) It is spin-dependent.

In any nuclear reaction, the following quantities are conserved:

Nuclean Number (A), electric charge, total energy and total momentum.

The mass of a nucleus with Z protons and N neutrons is smaller than the sum of the masses of the individual nucleons, and the binding energy is defined as the mass difference times c^2:

$$B(N, Z) = Zm(0, 1)c^2 + N m_n c^2 - m(N, Z)c^2. \tag{19.1.4}$$

The mass excess of a nucleus is defined as the difference between the mass of a nucleus expressed in atomic mass units and the mass number:

$$\text{mass excess} = m(N, Z) - (A)(1u). \tag{19.1.5}$$

The binding energies of different isotopes can be reproduced well by the Bethe-Weizsächer formula from the liquid-drop model, as the sum of volume, surface, Coulomb, asymmetry, and pairing contributions:

$$B(N, Z) = B_v(N, Z) + B_s(N, Z) + B_c(N, Z) + B_a(N, Z) + B_p(N, Z) \tag{19.1.6}$$

$$= a_v A - a_s A^{2/3} - a_c Z^2 A^{-1/3} - a_a \left(Z - \frac{1}{2} A \right)^2 A^{-1} + a_p \big((-1)^Z + (-1)^N \big) A^{-1/2}. \tag{19.1.7}$$

Dividing this expression by the mass number gives the binding energy per nucleon:

$$\frac{B(N,Z)}{A} = a_v - a_s A^{-1/3} - a_c \frac{Z^2}{A^{4/3}} - a_a \left(\frac{Z}{A} - \frac{1}{2}\right)^2 + a_p \frac{(-1)^Z + (-1)^N}{A^{3/2}}. \tag{19.1.8}$$

Several successful fits have been published for the empirical mass formula above. Using the values obtained by Bertulani and Schechter (2002), we have

$$a_v = 15.85 \text{ MeV},\ a_s = 18.34 \text{ MeV},\ a_c = 0.71 \text{ MeV},\ a_a = 92.86 \text{ MeV},\ a_p = 11.46 \text{ MeV}.$$

The Fermi gas model proposes a quantum gas of nucleons that can move freely inside the nucleus but are confined by the nuclear surface. The density of states in the Fermi gas model is

$$dN(E) = \frac{1}{\pi^2 a^3 \hbar^3}\sqrt{\frac{m^3 E}{2}}dE. \tag{19.1.9}$$

The Fermi energy is

$$E_F = \frac{\hbar^2}{2m}\sqrt[3]{\frac{9}{4}\pi^4 n_0^2} = 38 \text{ MeV}. \tag{19.1.10}$$

In a nuclear reaction, the difference between final and initial kinetic energies is called the Q-value:

$$Q = \Delta KE = -\Delta M c^2 \qquad \begin{cases} Q > 0 \implies \text{Exothermic} \\ Q < 0 \implies \text{Endothermic} \end{cases} \tag{19.1.11}$$

Nuclear Decay

Nuclear decays follow an exponential decay law. The decay constant λ, half life $t_{1/2}$, and mean lifetime τ are related:

$$N(t) = N_0 e^{-\lambda t} \tag{19.2.1}$$

$$t_{1/2} = \frac{\ln(2)}{\lambda} = \tau \ln(2). \tag{19.2.2}$$

In alpha (α) decay, a heavier nucleus (Nuc) emits a helium-4 nucleus:

$${}^{A}_{Z}\text{Nuc} \to {}^{4}_{2}\text{He} + {}^{A-4}_{Z-2}\text{Nuc'} \tag{19.2.3}$$

In a β^- decay, an electron and an anti-neutrino are emitted:

$${}^{A}_{Z}\text{Nuc} \to {}^{A}_{Z+1}\text{Nuc'} + e^- + \bar{v}_e \tag{19.2.4}$$

In a β^+ decay can proceed via a positron emission:

$${}^{A}_{Z}\text{Nuc} \to {}^{A}_{Z-1}\text{Nuc'} + e^+ + v_e \tag{19.2.5}$$

This type of decay can also occur via electron capture

$$e^- + {}^{A}_{Z}\text{Nuc} \to {}^{A}_{Z-1}\text{Nuc'} + v_e \tag{19.2.6}$$

A gamma decay is an emission of a high-energy photon from an excited nucleus, a process that does not transmute the nucleus:

$${}^{A}_{Z}\text{Nuc} \to {}^{A}_{Z}\text{Nuc'} + \gamma \tag{19.2.7}$$

Elementary Particle Physics

Substructure is probed using scattering experiments. The scattering cross section is defined as

$$\sigma = \frac{\text{\# of reactions per scattering center/s}}{\text{\# of impinging particles/s/m}^2}. \tag{19.3.1}$$

The scattering cross section has the physical dimension of area and is measured in the unit barn (b) or millibarn (mb):

$$1 \text{ b} = 10^{-28} \text{ m}^2 \qquad \text{and} \qquad 1 \text{ mb} = 10^{-31} \text{ m}^2. \tag{19.3.2}$$

The classical Rutherford cross section for scattering from a pointlike target by the Coulomb interaction is

$$\frac{d\sigma}{d\Omega} = \left(\frac{kZ_P Z_t e^2}{4K}\right)^2 \frac{1}{\sin^4(\theta/2)}. \tag{19.3.3}$$

For scattering of a plane wave off a point source, the scattering wave function is

$$\psi_{total}(\vec{r}) = \psi_i(\vec{r}) + \psi_f(\vec{r}) = N\left(e^{ikz} + f(\theta)\frac{e^{ikr}}{r}\right) \qquad \text{and} \qquad \frac{d\sigma}{d\Omega} = |f(\theta)|^2. \tag{19.3.4}$$

The form factor is the Fourier transform of the density distribution and measures the deviation of the scattering cross section from the Rutherford cross-section of a point-like target:

$$F^2(\Delta p) = \left|\frac{1}{e}\int \rho(\vec{r})e^{i\Delta\vec{p}\cdot\vec{r}/\hbar}dV\right|^2 \qquad \text{and} \qquad \frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega}\right)_{point} \cdot F^2(\Delta p). \tag{19.3.5}$$

- Elementary fermions have spin $\frac{1}{2}\hbar$ and include the six quarks (up, down, strange, charm, bottom, and top), the electron, muon, and tau leptons, and the electron-, muon-, and tau-neutrinos. Each of these 12 fermions has an antiparticle. Quarks all have a non-integer charge of $-\frac{1}{3}e$ or $+\frac{2}{3}e$ and cannot be observed in isolation.

- Elementary bosons are the mediators of the interactions between the fermions. They are the photon (electromagnetic), the W and Z bosons (electroweak), the gluon (strong), and the graviton (gravitational). The graviton is yet to be found experimentally. Gluons can also interact with other gluons.

- Elementary quarks and antiquarks can combine to form color singlets, which are particles that can be observed in isolation. A quark and an antiquark can form a meson (pion, kaon, etx.). Three quarks can form a baryon (proton, neutron, delta baryon, etc.). The only stable baryon is the proton, and none of the mesons is stable. Lifetimes of the unstable particles vary from 10^{-23} seconds to 15 minutes.

- The quark-gluon plasma phase transition of the early universe can be probed in the laboratory with relativistic heavy ion collisions. The primordial fraction of 23% helium in the universe can be explained from the neutron-proton mass difference, which fixes the ratio of proton and neutron numbers to $n_n/n_p = e^{(m_n - m_p)c^2/k_BT}$.

Advanced Physics

Quantum Chromodynamics

The classical Lagrangian density for n non-interacting quarks with masses m_i is

$$\mathcal{L}_{\text{quarks}} = \sum_i^n q_i^{-a}(i\eth - m_i)_{ab} a_i^b. \tag{20.1.1}$$

The Quantum Chromodynamic Lagrangian is given by

$$\mathcal{L}_{\text{QCD}} = -\frac{1}{4}F_{\mu\nu}^A F_A^{\mu\nu} + \sum_i^n q_i^{-a}(i\eth - m_i)_{ab} a_i^b - \frac{1}{2\lambda}(\partial^\mu A_\mu^A)^2 + \mathcal{L}_{\text{ghost}}. \tag{20.1.2}$$

The Lagrange equation for a multiple pendulum system with n number of rods, where the i^{th} rod has a length L_i, and mass m_i.

$$\mathcal{L} = \frac{1}{2}\sum_{i=1}^n m_i(\dot{x}_i^2 + \dot{y}_i^2) + \frac{1}{6}\sum_{i=1}^n m_i L_i^2 \dot{\theta}_i^2 - g\sum_{i=1}^n m_i y_i. \tag{20.1.3}$$

The normalized hydrogen wave functions are:

$$\Psi_{n\ell m}(r,\theta,\phi) = \sqrt{\left(\frac{2}{na}\right)^3 \frac{(n-\ell-1)!}{2n[(n+\ell)!]^3}} e^{-r/na}\left(\frac{2r}{na}\right)^\ell \left[L_{n-\ell-1}^{2\ell+1}(2r/na)\right] Y_\ell^m(\theta,\phi) \tag{20.1.4}$$

Table of Isotopic Masses and Natural Abundances

This table lists the mass and percent natural abundance for the stable nuclides. The mass of the longest lived isotope is given for elements without a stable nuclide. Nuclides marked with an asterisk (*) in the abundance column indicate that it is not present in nature or that a meaningful natural abundance cannot be given. The isotopic mass data is from G. Audi, A. H. Wapstra *Nucl. Phys A.* **1993**, *565*, 1-65 and G. Audi, A. H. Wapstra *Nucl. Phys A.* **1995**, *595*, 409-480. The percent natural abundance data is from the 1997 report of the IUPAC Subcommittee for Isotopic Abundance Measurements by K.J.R. Rosman, P.D.P. Taylor *Pure Appl. Chem.* **1999**, *71*, 1593-1607.

Z	Name	Symbol	Mass of Atom (u)	% Abundance
1	Hydrogen	$^{1}\mathrm{H}$	1.007825	99.9885
	Deuterium	$^{2}\mathrm{H}$	2.014102	0.0115
	Tritium	$^{3}\mathrm{H}$	3.016049	*
2	Helium	$^{3}\mathrm{He}$	3.016029	0.000137
		$^{4}\mathrm{He}$	4.002603	99.999863
3	Lithium	$^{6}\mathrm{Li}$	6.015122	7.59
		$^{7}\mathrm{Li}$	7.016004	92.41
4	Beryllium	$^{9}\mathrm{Be}$	9.012182	100
5	Boron	$^{10}\mathrm{B}$	10.012937	19.9
		$^{11}\mathrm{B}$	11.009305	80.1
6	Carbon	$^{12}\mathrm{C}$	12.000000	98.93
		$^{13}\mathrm{C}$	13.003355	1.07
		$^{14}\mathrm{C}$	14.003242	*
7	Nitrogen	$^{14}\mathrm{N}$	14.003074	99.632
		$^{15}\mathrm{N}$	15.000109	0.368
8	Oxygen	$^{16}\mathrm{O}$	15.994915	99.757
		$^{17}\mathrm{O}$	16.999132	0.038
		$^{18}\mathrm{O}$	17.999160	0.205
9	Fluorine	$^{19}\mathrm{F}$	18.998403	100
10	Neon	$^{20}\mathrm{Ne}$	19.992440	90.48
		$^{21}\mathrm{Ne}$	20.993847	0.27
		$^{22}\mathrm{Ne}$	21.991386	9.25
11	Sodium	$^{23}\mathrm{Na}$	22.989770	100
12	Magnesium	$^{24}\mathrm{Mg}$	23.985042	78.99
		$^{25}\mathrm{Mg}$	24.985837	10.00
		$^{26}\mathrm{Mg}$	25.982593	11.01
13	Aluminum	$^{27}\mathrm{Al}$	26.981538	100
14	Silicon	$^{28}\mathrm{Si}$	27.976927	92.2297
		$^{29}\mathrm{Si}$	28.976495	4.6832
		$^{30}\mathrm{Si}$	29.973770	3.0872
15	Phosphorus	$^{31}\mathrm{P}$	30.973762	100
16	Sulphur	$^{32}\mathrm{S}$	31.972071	94.93
		$^{33}\mathrm{S}$	32.971458	0.76
		$^{34}\mathrm{S}$	33.967867	4.29
		$^{36}\mathrm{S}$	35.967081	0.02
17	Chlorine	$^{35}\mathrm{Cl}$	34.968853	75.78
		$^{37}\mathrm{Cl}$	36.965903	24.22
18	Argon	$^{36}\mathrm{Ar}$	35.967546	0.3365
		$^{38}\mathrm{Ar}$	37.962732	0.0632
		$^{40}\mathrm{Ar}$	39.962383	99.6003
19	Potassium	$^{39}\mathrm{K}$	38.963707	93.2581
		$^{40}\mathrm{K}$	39.963999	0.0117
		$^{41}\mathrm{K}$	40.961826	6.7302
20	Calcium	$^{40}\mathrm{Ca}$	39.962591	96.941
		$^{42}\mathrm{Ca}$	41.958618	0.647
		$^{43}\mathrm{Ca}$	42.958767	0.135
		$^{44}\mathrm{Ca}$	43.955481	2.086
		$^{46}\mathrm{Ca}$	45.953693	0.004
		$^{48}\mathrm{Ca}$	47.952534	0.187
21	Scandium	$^{45}\mathrm{Sc}$	44.955910	100
22	Titanium	$^{46}\mathrm{Ti}$	45.952629	8.25
		$^{47}\mathrm{Ti}$	46.951764	7.44
		$^{48}\mathrm{Ti}$	47.947947	73.72
		$^{49}\mathrm{Ti}$	48.947871	5.41
		$^{50}\mathrm{Ti}$	49.944792	5.18
23	Vanadium	$^{50}\mathrm{V}$	49.947163	0.250
		$^{51}\mathrm{V}$	50.943964	99.750
24	Chromium	$^{50}\mathrm{Cr}$	49.946050	4.345
		$^{52}\mathrm{Cr}$	51.940512	83.789
		$^{53}\mathrm{Cr}$	52.940654	9.501
		$^{54}\mathrm{Cr}$	53.938885	2.365
25	Manganese	$^{55}\mathrm{Mn}$	54.938050	100
26	Iron	$^{54}\mathrm{Fe}$	53.939615	5.845
		$^{56}\mathrm{Fe}$	55.934942	91.754

Z	Name	Symbol	Mass of Atom (u)	% Abundance
		^{57}Fe	56.935399	2.119
		^{58}Fe	57.933280	0.282
27	Cobalt	^{59}Co	58.933200	100
28	Nickel	^{58}Ni	57.935348	68.0769
		^{60}Ni	59.930791	26.2231
		^{61}Ni	60.931060	1.1399
		^{62}Ni	61.928349	3.6345
		^{64}Ni	63.927970	0.9256
29	Copper	^{63}Cu	62.929601	69.17
		^{65}Cu	64.927794	30.83
30	Zinc	^{64}Zn	63.929147	48.63
		^{66}Zn	65.926037	27.90
		^{67}Zn	66.927131	4.10
		^{68}Zn	67.924848	18.75
		^{70}Zn	69.925325	0.62
31	Gallium	^{69}Ga	68.925581	60.108
		^{71}Ga	70.924705	39.892
32	Germanium	^{70}Ge	69.924250	20.84
		^{72}Ge	71.922076	27.54
		^{73}Ge	72.923459	7.73
		^{74}Ge	73.921178	36.28
		^{76}Ge	75.921403	7.61
33	Arsenic	^{75}As	74.921596	100
34	Selenium	^{74}Se	73.922477	0.89
		^{76}Se	75.919214	9.37
		^{77}Se	76.919915	7.63
		^{78}Se	77.917310	23.77
		^{80}Se	79.916522	49.61
		^{82}Se	81.916700	8.73
35	Bromine	^{79}Br	78.918338	50.69
		^{81}Br	80.916291	49.31
36	Krypton	^{78}Kr	77.920386	0.35
		^{80}Kr	79.916378	2.28
		^{82}Kr	81.913485	11.58
		^{83}Kr	82.914136	11.49
		^{84}Kr	83.911507	57.00
		^{86}Kr	85.910610	17.30
37	Rubidium	^{85}Rb	84.911789	72.17
		^{87}Rb	86.909183	27.83

Z	Name	Symbol	Mass of Atom (u)	% Abundance
38	Strontium	^{84}Sr	83.913425	0.56
		^{86}Sr	85.909262	9.86
		^{87}Sr	86.908879	7.00
		^{88}Sr	87.905614	82.58
39	Yttrium	^{89}Y	88.905848	100
40	Zirconium	^{90}Zr	89.904704	51.45
		^{91}Zr	90.905645	11.22
		^{92}Zr	91.905040	17.15
		^{94}Zr	93.906316	17.38
		^{96}Zr	95.908276	2.80
41	Niobium	^{93}Nb	92.906378	100
42	Molybdenum	^{92}Mo	91.906810	14.84
		^{94}Mo	93.905088	9.25
		^{95}Mo	94.905841	15.92
		^{96}Mo	95.904679	16.68
		^{97}Mo	96.906021	9.55
		^{98}Mo	97.905408	24.13
		^{100}Mo	99.907477	9.63
43	Technetium	^{98}Tc	97.907216	*
44	Ruthenium	^{96}Ru	95.907598	5.54
		^{98}Ru	97.905287	1.87
		^{99}Ru	98.905939	12.76
		^{100}Ru	99.904220	12.60
		^{101}Ru	100.905582	17.06
		^{102}Ru	101.904350	31.55
		^{104}Ru	103.905430	18.62
45	Rhodium	^{103}Rh	102.905504	100
46	Palladium	^{102}Pd	101.905608	1.02
		^{104}Pd	103.904035	11.14
		^{105}Pd	104.905084	22.33
		^{106}Pd	105.903483	27.33
		^{108}Pd	107.903894	26.46
		^{110}Pd	109.905152	11.72
47	Silver	^{107}Ag	106.905093	51.839
		^{109}Ag	108.904756	48.161
48	Cadmium	^{106}Cd	105.906458	1.25
		^{108}Cd	107.904183	0.89
		^{110}Cd	109.903006	12.49
		^{111}Cd	110.904182	12.80

Z	Name	Symbol	Mass of Atom (u)	% Abundance
		^{112}Cd	111.902757	24.13
		^{113}Cd	112.904401	12.22
		^{114}Cd	113.903358	28.73
		^{116}Cd	115.904755	7.49
49	Indium	^{113}In	112.904061	4.29
		^{115}In	114.903878	95.71
50	Tin	^{112}Sn	111.904821	0.97
		^{114}Sn	113.902782	0.66
		^{115}Sn	114.903346	0.34
		^{116}Sn	115.901744	14.54
		^{117}Sn	116.902954	7.68
		^{118}Sn	117.901606	24.22
		^{119}Sn	118.903309	8.59
		^{120}Sn	119.902197	32.58
		^{122}Sn	121.903440	4.63
		^{124}Sn	123.905275	5.79
51	Antimony	^{121}Sb	120.903818	57.21
		^{123}Sb	122.904216	42.79
52	Tellurium	^{120}Te	119.904020	0.09
		^{122}Te	121.903047	2.55
		^{123}Te	122.904273	0.89
		^{124}Te	123.902819	4.74
		^{125}Te	124.904425	7.07
		^{126}Te	125.903306	18.84
		^{128}Te	127.904461	31.74
		^{130}Te	129.906223	34.08
53	Iodine	^{127}I	126.904468	100
54	Xenon	^{124}Xe	123.905896	0.09
		^{126}Xe	125.904269	0.09
		^{128}Xe	127.903530	1.92
		^{129}Xe	128.904779	26.44
		^{130}Xe	129.903508	4.08
		^{131}Xe	130.905082	21.18
		^{132}Xe	131.904154	26.89
		^{134}Xe	133.905395	10.44
		^{136}Xe	135.907220	8.87
55	Cesium	^{133}Cs	132.905447	100
56	Barium	^{130}Ba	129.906310	0.106
		^{132}Ba	131.905056	0.101
		^{134}Ba	133.904503	2.417
		^{135}Ba	134.905683	6.592
		^{136}Ba	135.904570	7.854

Z	Name	Symbol	Mass of Atom (u)	% Abundance
		^{137}Ba	136.905821	11.232
		^{138}Ba	137.905241	71.698
57	Lanthanum	^{138}La	137.907107	0.090
		^{139}La	138.906348	99.910
58	Cerium	^{136}Ce	135.907144	0.185
		^{138}Ce	137.905986	0.251
		^{140}Ce	139.905434	88.450
		^{142}Ce	141.909240	11.114
59	Praseodymium	^{141}Pr	140.907648	100
60	Neodymium	^{142}Nd	141.907719	27.2
		^{143}Nd	142.909810	12.2
		^{144}Nd	143.910083	23.8
		^{145}Nd	144.912569	8.3
		^{146}Nd	145.913112	17.2
		^{148}Nd	147.916889	5.7
		^{150}Nd	149.920887	5.6
61	Promethium	^{145}Pm	144.912744	*
62	Samarium	^{144}Sm	143.911995	3.07
		^{147}Sm	146.914893	14.99
		^{148}Sm	147.914818	11.24
		^{149}Sm	148.917180	13.82
		^{150}Sm	149.917271	7.38
		^{152}Sm	151.919728	26.75
		^{154}Sm	153.922205	22.75
63	Europium	^{151}Eu	150.919846	47.81
		^{153}Eu	152.921226	52.19
64	Gadolinium	^{152}Gd	151.919788	0.20
		^{154}Gd	153.920862	2.18
		^{155}Gd	154.922619	14.80
		^{156}Gd	155.922120	20.47
		^{157}Gd	156.923957	15.65
		^{158}Gd	157.924101	24.84
		^{160}Gd	159.927051	21.86
65	Terbium	^{159}Tb	158.925343	100
66	Dysprosium	^{156}Dy	155.924278	0.06
		^{158}Dy	157.924405	0.10
		^{160}Dy	159.925194	2.34
		^{161}Dy	160.926930	18.91
		^{162}Dy	161.926795	25.51
		^{163}Dy	162.928728	24.90

Z	Name	Symbol	Mass of Atom (u)	% Abundance
		^{164}Dy	163.929171	28.18
67	Holmium	^{165}Ho	164.930319	100
68	Erbium	^{162}Er	161.928775	0.14
		^{164}Er	163.929197	1.61
		^{166}Er	165.930290	33.61
		^{167}Er	166.932045	22.93
		^{168}Er	167.932368	26.78
		^{170}Er	169.935460	14.93
69	Thulium	^{169}Tm	168.934211	100
70	Ytterbium	^{168}Yb	167.933894	0.13
		^{170}Yb	169.934759	3.04
		^{171}Yb	170.936322	14.28
		^{172}Yb	171.936378	21.83
		^{173}Yb	172.938207	16.13
		^{174}Yb	173.938858	31.83
		^{176}Yb	175.942568	12.76
71	Lutetium	^{175}Lu	174.940768	97.41
		^{176}Lu	175.942682	2.59
72	Hafnium	^{174}Hf	173.940040	0.16
		^{176}Hf	175.941402	5.26
		^{177}Hf	176.943220	18.60
		^{178}Hf	177.943698	27.28
		^{179}Hf	178.945815	13.62
		^{180}Hf	179.946549	35.08
73	Tantalum	^{180}Ta	179.947466	0.012
		^{181}Ta	180.947996	99.988
74	Tungsten	^{180}W	179.946706	0.12
		^{182}W	181.948206	26.50
		^{183}W	182.950224	14.31
		^{184}W	183.950933	30.64
		^{186}W	185.954362	28.43
75	Rhenium	^{185}Re	184.952956	37.40
		^{187}Re	186.955751	62.60
76	Osmium	^{184}Os	183.952491	0.02
		^{186}Os	185.953838	1.59
		^{187}Os	186.955748	1.96
		^{188}Os	187.955836	13.24
		^{189}Os	188.958145	16.15
		^{190}Os	189.958445	26.26
		^{192}Os	191.961479	40.78

Z	Name	Symbol	Mass of Atom (u)	% Abundance
77	Iridium	^{191}Ir	190.960591	37.3
		^{193}Ir	192.962924	62.7
78	Platinum	^{190}Pt	189.959930	0.014
		^{192}Pt	191.961035	0.782
		^{194}Pt	193.962664	32.967
		^{195}Pt	194.964774	33.832
		^{196}Pt	195.964935	25.242
		^{198}Pt	197.967876	7.163
79	Gold	^{197}Au	196.966552	100
80	Mercury	^{196}Hg	195.965815	0.15
		^{198}Hg	197.966752	9.97
		^{199}Hg	198.968262	16.87
		^{200}Hg	199.968309	23.10
		^{201}Hg	200.970285	13.18
		^{202}Hg	201.970626	29.86
		^{204}Hg	203.973476	6.87
81	Thallium	^{203}Tl	202.972329	29.524
		^{205}Tl	204.974412	70.476
82	Lead	^{204}Pb	203.973029	1.4
		^{206}Pb	205.974449	24.1
		^{207}Pb	206.975881	22.1
		^{208}Pb	207.976636	52.4
83	Bismuth	^{209}Bi	208.980383	100
84	Polonium	^{209}Po	208.982416	*
85	Astatine	^{210}At	209.987131	*
86	Radon	^{222}Rn	222.017570	*
87	Francium	^{223}Fr	223.019731	*
88	Radium	^{226}Ra	226.025403	*
89	Actinium	^{227}Ac	227.027747	*
90	Thorium	^{232}Th	232.038050	100
91	Protactinium	^{231}Pa	231.035879	100
92	Uranium	^{234}U	234.040946	0.0055
		^{235}U	235.043923	0.7200
		^{238}U	238.050783	99.2745

Z	Name	Symbol	Mass of Atom (u)	% Abundance
93	Neptunium	^{237}Np	237.048167	*
94	Plutonium	^{244}Pu	244.064198	*
95	Americium	^{243}Am	243.061373	*
96	Curium	^{247}Cm	247.070347	*
97	Berkelium	^{247}Bk	247.070299	*
98	Californium	^{251}Cf	251.079580	*
99	Einsteinium	^{252}Es	252.082972	*
100	Fermium	^{257}Fm	257.095099	*
101	Mendelevium	^{258}Md	258.098425	*
102	Nobelium	^{259}No	259.101024	*
103	Lawrencium	^{262}Lr	262.109692	*
104	Rutherfordium	^{263}Rf	263.118313	*
105	Dubnium	^{262}Db	262.011437	*
106	Seaborgium	^{266}Sg	266.012238	*
107	Bohrium	^{264}Bh	264.012496	*
108	Hassium	^{269}Hs	269.001341	*
109	Meitnerium	^{268}Mt	268.001388	*
110	Ununnilium	^{272}Uun	272.001463	*
111	Unununium	^{272}Uuu	272.001535	*
112	Ununbium	^{277}Uub	(277)	*
114	Ununquadium	^{289}Uuq	(289)	*
116	Ununhexium	^{289}Uuh	(289)	*
118	Ununoctium	^{293}Uuo	(293)	*

Periodic Table of the Elements

Atomic Number
Symbol
Name
Atomic Mass

1 IA 1A	2 IIA 2A	3 IIIB 3B	4 IVB 4B	5 VB 5B	6 VIB 6B	7 VIIB 7B	8	9 VIII 8	10	11 IB 1B	12 IIB 2B	13 IIIA 3A	14 IVA 4A	15 VA 5A	16 VIA 6A	17 VIIA 7A	18 VIIIA 8A
1 H Hydrogen 1.008																	2 He Helium 4.003
3 Li Lithium 6.941	4 Be Beryllium 9.012											5 B Boron 10.811	6 C Carbon 12.011	7 N Nitrogen 14.007	8 O Oxygen 15.999	9 F Fluorine 18.998	10 Ne Neon 20.180
11 Na Sodium 22.990	12 Mg Magnesium 24.305											13 Al Aluminum 26.982	14 Si Silicon 28.086	15 P Phosphorus 30.974	16 S Sulfur 32.066	17 Cl Chlorine 35.453	18 Ar Argon 39.948
19 K Potassium 39.098	20 Ca Calcium 40.078	21 Sc Scandium 44.956	22 Ti Titanium 47.867	23 V Vanadium 50.942	24 Cr Chromium 51.996	25 Mn Manganese 54.938	26 Fe Iron 55.845	27 Co Cobalt 58.933	28 Ni Nickel 58.693	29 Cu Copper 63.546	30 Zn Zinc 65.38	31 Ga Gallium 69.723	32 Ge Germanium 72.631	33 As Arsenic 74.922	34 Se Selenium 78.971	35 Br Bromine 79.904	36 Kr Krypton 84.798
37 Rb Rubidium 84.468	38 Sr Strontium 87.62	39 Y Yttrium 88.906	40 Zr Zirconium 91.224	41 Nb Niobium 92.906	42 Mo Molybdenum 95.95	43 Tc Technetium 98.907	44 Ru Ruthenium 101.07	45 Rh Rhodium 102.906	46 Pd Palladium 106.42	47 Ag Silver 107.868	48 Cd Cadmium 112.411	49 In Indium 114.818	50 Sn Tin 118.711	51 Sb Antimony 121.760	52 Te Tellurium 127.6	53 I Iodine 126.904	54 Xe Xenon 131.294
55 Cs Cesium 132.905	56 Ba Barium 137.328	57-71	72 Hf Hafnium 178.49	73 Ta Tantalum 180.948	74 W Tungsten 183.84	75 Re Rhenium 186.207	76 Os Osmium 190.23	77 Ir Iridium 192.217	78 Pt Platinum 195.085	79 Au Gold 196.967	80 Hg Mercury 200.592	81 Tl Thallium 204.383	82 Pb Lead 207.2	83 Bi Bismuth 208.980	84 Po Polonium [208.982]	85 At Astatine 209.987	86 Rn Radon 222.018
87 Fr Francium 223.020	88 Ra Radium 226.025	89-103	104 Rf Rutherfordium [261]	105 Db Dubnium [262]	106 Sg Seaborgium [266]	107 Bh Bohrium [264]	108 Hs Hassium [269]	109 Mt Meitnerium [268]	110 Ds Darmstadtium [269]	111 Rg Roentgenium [272]	112 Cn Copernicium [277]	113 Uut Ununtrium unknown	114 Fl Flerovium [289]	115 Uup Ununpentium unknown	116 Lv Livermorium [298]	117 Uus Ununseptium unknown	118 Uuo Ununoctium unknown

Lanthanide Series	57 La Lanthanum 138.905	58 Ce Cerium 140.116	59 Pr Praseodymium 140.908	60 Nd Neodymium 144.243	61 Pm Promethium 144.913	62 Sm Samarium 150.36	63 Eu Europium 151.964	64 Gd Gadolinium 157.25	65 Tb Terbium 158.925	66 Dy Dysprosium 162.500	67 Ho Holmium 164.930	68 Er Erbium 167.259	69 Tm Thulium 168.934	70 Yb Ytterbium 173.055	71 Lu Lutetium 174.967
Actinide Series	89 Ac Actinium 227.028	90 Th Thorium 232.038	91 Pa Protactinium 231.036	92 U Uranium 238.029	93 Np Neptunium 237.048	94 Pu Plutonium 244.064	95 Am Americium 243.061	96 Cm Curium 247.070	97 Bk Berkelium 247.070	98 Cf Californium 251.080	99 Es Einsteinium [254]	100 Fm Fermium 257.095	101 Md Mendelevium 258.1	102 No Nobelium 259.101	103 Lr Lawrencium [262]

Alkali Metal | Alkaline Earth | Transition Metal | Basic Metal | Semimetal | Nonmetal | Halogen | Noble Gas | Lanthanide | Actinide

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