Useful Formulas, Constants, Units, and Definitions
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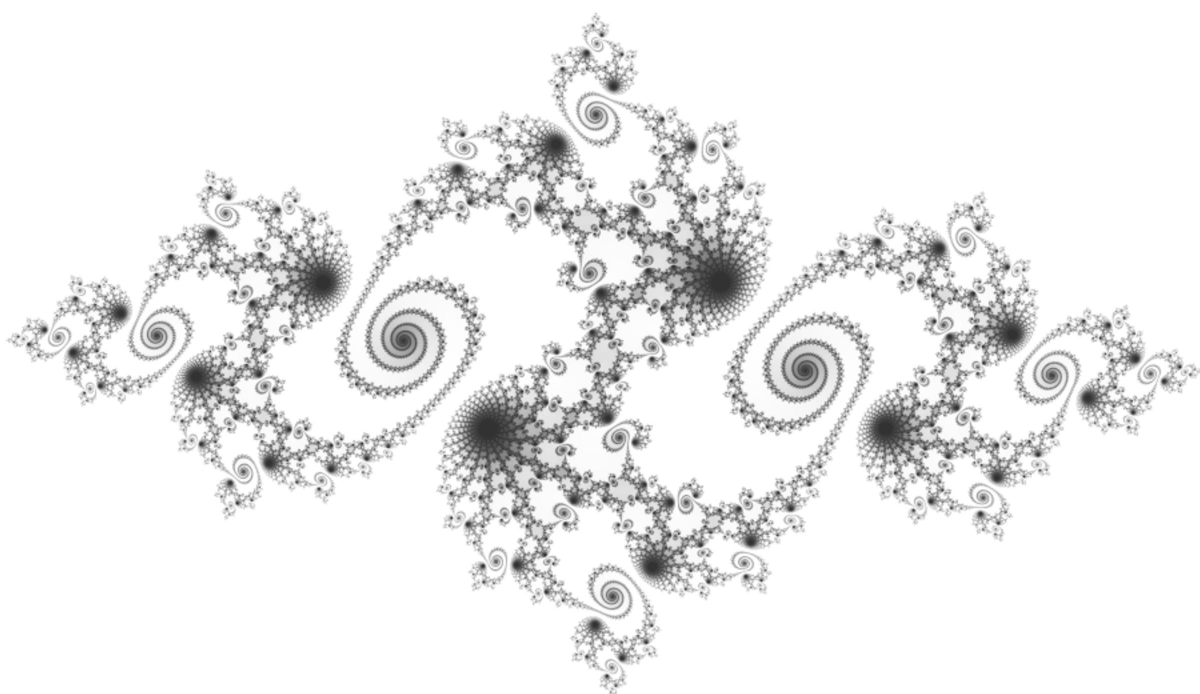


Contents

1 Constants and units

1.1: Physical Constants

Constant	Symbol	Value	Units
Speed of light in a vacuum	c	299,792,458	m/s
		2.99792458×10^{8}	m/s
Elementary charge	e	$1.602176565(35) \times 10^{-19}$	C
Gravitational constant	G	$6.67384(80) \times 10^{-11}$	$\mathrm{m^3kg^{-1}s^{-2}}$
Avagadro's number	N_a	$6.02214129(27) \times 10^{23}$	$\mathrm{mol}\cdot s^{-1}$
Planck constant	h	6.6261×10^{-34}	J·s
		4.135668×10^{-15}	eV·s
	hc	1239.84	eV·nm
Reduced planck constant	$\hbar \equiv h/2\pi$	$1.05 \times 10^{-}34$	J·s
Permittivity of the vacuum	ϵ_0	8.854×10^{-12}	$\mathrm{C^2N^{-1}m^{-2}}$
Permeability of the vacuum	μ_0	$4\pi \times 10^{-7}$	W·m
Boltzmann constant	k	1.381×10^{-23}	$\mathrm{m^2kg\cdot s^{-2}K^{-1}}$
Stefan-Boltzmann constant	σ_{SB}	5.679×10^{-8}	$\mathrm{W\cdot m^{-2}K^{-4}}$
Thomson cross-section	σ_e	6.652×10^{-29}	m^2
Mass of an electron	m_e	$9.10938291(40) \times 10^{-31}$	kg
		510.9989	keV/c^2
Mass of a proton	m_p	$1.6726219 \times 10^{-27}$	kg
		0.938272	GeV/c^2
Mass of a neutron	m_n	m_p	

1.2: Astronomical Constants

Constant	Symbol	Value	Units
Mass of Earth	$M_\oplus$	5.974×10^{24}	kg
Mass of Sun	$M_\odot$	1.989×10^{30}	kg
Mass of Moon	$M_☾$	7.36×10^{22}	kg
Equatorial radius of Earth	$R_\oplus$	6.378×10^6	m
Equatorial radius of Sun	$R_\odot$	6.6955×10^8	m
Equatorial radius of Moon	$R_☾$	1.737×10^6	m
Mean density of Earth		5515	$\mathrm{kg{\cdot}m^{-3}}$
Mean density of Sun		1408	$\mathrm{kg{\cdot}m^{-3}}$
Mean density of Moon		3346	$\mathrm{kg{\cdot}m^{-3}}$
Earth-Moon distance		3.84×10^8	m
Earth-Sun distance		1.496×10^{11}	m
Luminosity of Sun	$L_\odot$	3.839×10^{26}	W
Effective temperature of Sun		5778	K
Hubble constant	H_0	70 ± 5	$\mathrm{km{\cdot}s^{-1}Mpc^{-1}}$
Parsec	pc	206264.81	AU
		3.0856776×10^{16}	m
		3.2615638	ly
Astronomical Unit	AU	1.496×10^{11}	m
Light year	ly	9.461×10^{15}	m
1 year on Earth	yr	365.25	days
		3.15576×10^7	s

1.3: Solar System

Constant	Symbol	Value	Units
Mass of Jupiter	$M_♃$	1.898×10^{27}	kg
Radius of Jupiter	$R_♃$	3.83×10^{11}	m

2 General Mathematics

Definitions

$$\sin(x) = \frac{1}{2}(e^{ix} - e^{-ix}) \tag{2.1}$$
$$\sinh(x) = \frac{1}{2}(e^{x} - e^{-x}) \tag{2.2}$$
$$= -i\sin(ix) \tag{2.3}$$

$$\cos(x) = \frac{1}{2}(e^{ix} + e^{-ix}) \tag{2.4}$$
$$\cosh(x) = \frac{1}{2}(e^{x} + e^{-x}) \tag{2.5}$$
$$= \cos(ix) \tag{2.6}$$

The Gamma function Γ and the Riemann zeta function ζ are given by

$$\Gamma(z) \equiv \int_0^\infty t^{z-1}e^{-t}dt \tag{2.7}$$
$$\zeta(z) = \sum_{k=1}^{\infty} \frac{1}{k^n} \tag{2.8}$$
$$\zeta(z)\Gamma(z) = \int_0^\infty \frac{u^{z-1}}{e^u - 1}du \tag{2.9}$$

The most general case of the binomial theorem is the binomial series identity

$$(x+y)^n = \sum_{i=1}^{n} \binom{n}{k} x^{n-k}y^k \tag{2.10}$$

The binomial coefficient is defined as follows, with Pascals Formula implied.

$${}_nC_r \equiv \binom{n}{k} \equiv \frac{n!}{(n-k)!k!} \equiv \frac{\Gamma(n+1)}{\Gamma(k+1)\Gamma(n-k+1)} \tag{2.11}$$
$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \tag{2.12}$$

A Taylor series is a series expansion of a function about a point. A one-dimensional Taylor series is an expansion of a real function $f(x)$ about a point x=a is given by

$$f(x) = f(a) + f'(a)(x-a) + \frac{1}{2!}f''(a)(z-a)^2 + \frac{1}{3!}f'''(a)(x-a)^4 + \cdots \tag{2.13}$$

Some common series expansions include:

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots \tag{2.14}$$

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \cdots [|x| < 1] \tag{2.15}$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \tag{2.16}$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \tag{2.17}$$

$$\tan(x) = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots [|x| < \pi/2] \tag{2.18}$$

$$\sinh(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots \tag{2.19}$$

$$\cosh(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots \tag{2.20}$$

$$\tanh(x) = x - \frac{1}{3}x^3 + \frac{2}{15}x^5 - \cdots [|x| < \pi/2] \tag{2.21}$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \cdots [|x| < 1] \tag{2.22}$$

2.1 Coordinate Systems

Cylindrical coordinates

$$x = r\cos(\theta) \tag{2.23}$$

$$y = r\sin(\theta) \tag{2.24}$$

$$z = z \tag{2.25}$$

$$dA = r dr d\theta dz \tag{2.26}$$

Spherical coordinates

$$x = r\sin(\theta)\cos(\phi) \tag{2.27}$$

$$y = r\sin(\theta)\sin(\phi) \tag{2.28}$$

$$z = r\cos(\theta) \tag{2.29}$$

$$dV = r^2\sin(\theta) dr d\theta d\phi \tag{2.30}$$

2.2 Vector Operations

For any vector $\vec{r}_n = (r_1, r_2, \ldots, r_n)$ in n-dimensions, the magnitude is

$$||\vec{r}_n|| = \sqrt{r_1^2 + r_2^2 + \cdots + r_n^2} \tag{2.31}$$

Dot and cross products for 3-dimensional vectors $\vec{r} = (r_x, r_y, r_z)$ and $\vec{s} = (s_x, s_y, s_z)$

$$\vec{r} \cdot \vec{s} = ||\vec{r}||||\vec{s}|| \cos(\theta) = r_x s_x + r_y s_y + r_z s_z \tag{2.32}$$

$$\vec{r} \times \vec{s} = (r_y s_z - r_z s_y, r_z s_x - r_x s_z, r_x s_y - r_y s_x) = \det \begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ r_x & r_y & r_z \\ s_x & s_y & s_z \end{vmatrix} \tag{2.33}$$

Vector identities

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{B}\cdot(\vec{C}\times\vec{A}) = \vec{C}\cdot(\vec{A}\times\vec{B}) \tag{2.34}$$
$$\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B}) \tag{2.35}$$

2.3 Complex numbers

The set of complex numbers is defined such that

$$\mathbb{C} = \{a+bi : a,b\in\mathbb{R}\}. \tag{2.36}$$

A complex number can be defined by its real part and it's imaginary part

$$i^2 = -1 \iff i = \sqrt{-1} \iff \frac{1}{i} = -i \tag{2.37}$$
$$z = x+iy \iff z^* = x-iy \tag{2.38}$$

We can express the real and imaginary parts of a complex number in terms of the number and its complex conjugate

$$\mathfrak{R}(z) = \frac{1}{2}(z+z^*) \tag{2.39}$$
$$\mathfrak{I}(z) = \frac{1}{2}i(z-z^*) \tag{2.40}$$

Just like a two-dimensional vector, a complex number has the magnitude $|z|$ as well as an angle θ with respect to the horizontal axis of the complex plane.

$$|z|^2 = z^*z = x^2+y^2 = |z|e^{-i\theta}|z|e^{i\theta} \tag{2.41}$$
$$\tan(\theta) = \frac{\mathfrak{I}(z)}{\mathfrak{R}(z)} = \frac{i(z-z^*)}{(z+z^*)} \tag{2.42}$$

A complex number can thus be expressed in terms of magnitude and the phase angle

$$z = |z|(\cos(\theta)+i\sin(\theta)) \tag{2.43}$$

Euler's Identity/relation

$$e^{i\theta} = cos(\theta) + i\sin(\theta) \tag{2.44}$$

With the aid of Eulers identity, we can write any complex number as

$$z = |z|e^{i\theta} \tag{2.45}$$
$$z^n = |z|^n e^{in\theta} \tag{2.46}$$

2.4 Triangles

Let a triangle have side lengths a, b, and c with opposite angles A, B, and C.

The area of a triangle can be given by

$$A = \sqrt{s(s-a)(s-b)(s-c)} \tag{2.47}$$
$$s = (a+b+c)/2 \tag{2.48}$$

Law of Cosines:

$$c^2 = a^2+b^2-2ab\cos(C) \tag{2.49}$$

Law of Sines:

$$\frac{\sin(A)}{a} = \frac{\sin(B)}{b} = \frac{\sin(C)}{c} \tag{2.50}$$

Law of tangents

$$\frac{a-b}{a+b} = \frac{\tan((A-B)/2)}{\tan((A+B)/2)} \tag{2.51}$$

Mollweide's Formulas

$$\frac{b-c}{a} = \frac{\sin[(B-C)/2]}{\cos(A/2)} \tag{2.52}$$

$$\frac{c-a}{b} = \frac{\sin[(C-A)/2]}{\cos(B/2)} \tag{2.53}$$

$$\frac{a-b}{c} = \frac{\sin[(A_B)/2]}{\cos(C/2)} \tag{2.54}$$

2.5 Matrix Algebra

The product C of two matrices A and B is defined (where j is summed over for all possible values of i and k) as (using the Einstein summation convention)

$$c_{ik} = a_{ij}b_{jk} = \sum_{j=1}^{m} a_{ij}b_{jk} \tag{2.55}$$

In order for matrix multiplication to be defined, the dimensions of the matrices must satisfy

$$(n \times m)(m \times p) = (n \times p) \tag{2.56}$$

where $(a \times b)$ denotes a matrix with a rows and b columns. Writing out the product explicitly,

$$\begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1p} \\ c_{21} & c_{22} & \cdots & c_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{np} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{np} \end{bmatrix} \tag{2.57}$$

where,

$$c_{11} = a_{11}b_{11} + a_{12}b_{21} + \cdots + a_{1m}b_{m1} \tag{2.58}$$

$$c_{12} = a_{11}b_{12} + a_{12}b_{22} + \cdots + a_{1m}b_{m2} \tag{2.59}$$

$$c_{1p} = a_{11}b_{1p} + a_{12}b_{2p} + \cdots + a_{1m}b_{mp} \tag{2.60}$$

$$c_{21} = a_{21}b_{11} + a_{22}b_{21} + \cdots + a_{2m}b_{m1} \tag{2.61}$$

$$c_{22} = a_{21}b_{12} + a_{22}b_{22} + \cdots + a_{2m}b_{m2} \tag{2.62}$$

$$c_{2p} = a_{21}b_{1p} + a_{22}b_{2p} + \cdots + a_{2m}b_{mp} \tag{2.63}$$

$$c_{n1} = a_{n1}b_{11} + a_{n2}b_{21} + \cdots + a_{nm}b_{m1} \tag{2.64}$$

$$c_{n2} = a_{n1}b_{12} + a_{n2}b_{22} + \cdots + a_{nm}b_{m2} \tag{2.65}$$

$$c_{np} = a_{n1}b_{1p} + a_{n2}b_{2p} + \cdots + a_{nm}b_{mp} \tag{2.66}$$

Matrix multiplication is also distributive. If A and B are $m \times n$ matrices and C and D are $n \times p$ matrices, then

$$A(C+D) = AC + AD \tag{2.67}$$

$$(A+B)C = AC + BC \tag{2.68}$$

2.6 Trigonometric Identities

Pythagorean identities

$$1 = \sin^2(\theta) + \cos^2(\theta) \tag{2.69}$$
$$1 = \sec^2(\theta) - \tan^2(\theta) \tag{2.70}$$
$$1 = \csc^2(\theta) - \cot^2(\theta) \tag{2.71}$$
$$1 = \cosh^2(\theta) - \sinh^2(\theta) \tag{2.72}$$
$$1 = \mathrm{sech}^2(\theta) + \tanh^2(\theta) \tag{2.73}$$

Sum-Difference Formulas

$$\sin(\theta \pm \phi) = \sin(\theta)\cos(\phi) \pm \cos(\theta) sin(\phi) \tag{2.74}$$
$$\cos(\theta \pm \phi) = \cos(\theta)\cos(\phi) \mp \sin(\theta) sin(\phi) \tag{2.75}$$
$$\tan(\theta \pm \phi) = \frac{\tan(\theta) \pm \tan(\phi)}{1 \mp \tan(\theta)\tan(\phi)} \tag{2.76}$$

Double Angle formulas

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta) \tag{2.77}$$
$$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) \tag{2.78}$$
$$= 2\cos^2(\theta) - 1 \tag{2.79}$$
$$= 1 - 2\sin^2(\theta) \tag{2.80}$$
$$\tan(2\theta) = \frac{2\tan(\theta)}{1 - \tan^2(\theta)} \tag{2.81}$$

Power-Reducing/Half Angle Formulas

$$\sin^2(\theta) = \frac{1 - \cos(2\theta)}{2} \tag{2.82}$$
$$\cos^2(\theta) = \frac{1 + \cos(2\theta)}{2} \tag{2.83}$$
$$\tan^2(\theta) = \frac{1 - \cos(2\theta)}{1 + \cos(2\theta)} \tag{2.84}$$

Other relations

$$\sin(-\theta) = -\sin(\theta) \tag{2.85}$$
$$\cos(-\theta) = \cos(\theta) \tag{2.86}$$
$$\sin(\theta \pm \pi/2) = \pm\cos(\theta) \tag{2.87}$$
$$\sin(\theta \pm \pi) = -\sin(\theta) \tag{2.88}$$
$$\cos(\theta \pm \pi/2) = \mp\sin(\theta) \tag{2.89}$$
$$\cos(\theta \pm \pi) = -\cos(\theta) \tag{2.90}$$

Half-angle formulas

$$\sin\left(\frac{\theta}{2}\right) = (-1)^{\theta/(2\pi)}\sqrt{\frac{1 - \cos(\theta)}{2}} \tag{2.91}$$
$$\cos\left(\frac{\theta}{2}\right) = (-1)^{(\theta+\pi)/(2\pi)}\sqrt{\frac{1 + \cos(\theta)}{2}} \tag{2.92}$$

The Weierstrass substitution makes use of the half-angle formulas

$$\cos(\theta) = \frac{1 - \tan^2(\theta/2)}{1 + \tan^2(\theta/2)} \tag{2.93}$$
$$\sin(\theta) = \frac{2\tan(\theta/2)}{1 + \tan^2(\theta/2)} \tag{2.94}$$

The half angle identity for tangent.

$$\tan\left(\frac{\theta}{2}\right) = (-1)^{x/\pi}\sqrt{\frac{1 - \cos(\theta)}{1 + \cos(\theta)}} = \frac{\sin(\theta)}{1 + \cos(\theta)} = \frac{1 - \cos(\theta)}{\sin(\theta)} = \frac{\tan(\theta)\sin(\theta)}{\tan(\theta) + \sin(\theta)} \tag{2.95}$$

Other identities

$$\cos(\theta) cos(\phi) = \frac{1}{2}[\cos(\theta + \phi) + \cos(\theta - \phi)] \tag{2.96}$$

$$\sin(\theta) sin(\phi) = \frac{1}{2}[\cos(\theta - \phi) - \cos(\theta + \phi)] \tag{2.97}$$

$$\sin(\theta) cos(\phi) = \frac{1}{2}[\sin(\theta + \phi) + \sin(\theta - \phi)] \tag{2.98}$$

$$\cos(\theta) + \cos(\phi) = 2\cos\left(\frac{\theta + \phi}{2}\right)\cos\left(\frac{\theta - \phi}{2}\right) \tag{2.99}$$

$$\cos(\theta) - \cos(\phi) = 2\sin\left(\frac{\theta + \phi}{2}\right)\sin\left(\frac{\theta - \phi}{2}\right) \tag{2.100}$$

Multiple-angle formulas are given by

$$\sin(nx) = \sum_{k=0}^{n} \binom{n}{k} \cos^k(x)\sin^{n-k}(x)\sin\big((n-k)\pi/2\big) \tag{2.101}$$

$$\cos(nx) = \sum_{k=0}^{n} \binom{n}{k} \cos^k(x)\sin^{n-k}(x)\cos\big((n-k)\pi/2\big) \tag{2.102}$$

3 Differential Equations

Definition 3.1: Del Operator

The Del operator with respect to n-dimensions:

$$\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n}\right) = \sum_{i=1}^{n} \vec{e}_i \frac{\partial}{\partial x_i} \tag{3.1}$$

Gradient of a 3-dimensional function (cartesian, spherical, and cylindrical coordinates)

$$\text{grad } f = \nabla f = \frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y} + \frac{\partial f}{\partial z}\hat{z} \tag{3.2}$$

$$= \frac{\partial f}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\theta} + \frac{1}{r\sin(\theta)}\frac{\partial f}{\partial \phi}\hat{\phi} \tag{3.3}$$

$$= \frac{\partial f}{\partial \rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial f}{\partial \phi}\hat{\phi} + \frac{\partial f}{\partial z}\hat{z} \tag{3.4}$$

Curl of a 3-dimensional function (cartesian, spherical, and cylindrical coordinates)

$$\text{curl } \vec{A} = \nabla \times \vec{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{x} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{y} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{z} \tag{3.5}$$

$$= \frac{1}{r\sin(\theta)}\left[\frac{\partial}{\partial \theta}\sin(\theta)A_\phi - \frac{\partial A_\theta}{\partial \phi}\right]\hat{r} + \left[\frac{1}{r\sin(\theta)}\frac{\partial A_r}{\partial \phi} - \frac{1}{r}\frac{\partial}{\partial r}(rA_\phi)\right]\hat{\theta} + \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial \theta}\right]\hat{\phi} \tag{3.6}$$

$$= \left[\frac{1}{\rho}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right]\hat{\rho} + \left[\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}\right]\hat{\phi} + \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}(\rho A_\phi) - \frac{\partial A_\rho}{\partial \phi}\right]\hat{z} \tag{3.7}$$

Divergence of a 3-dimensional function (cartesian, spherical, and cylindrical coordinates)

$$\text{div } \vec{A} = \nabla \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \tag{3.8}$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin(\theta)}\frac{\partial}{\partial \theta}\sin(\theta)A_\theta + \frac{1}{r\sin(\theta)}\frac{\partial A_\phi}{\partial \phi} \tag{3.9}$$

$$= \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z} \tag{3.10}$$

The Laplace Operator

$$\Delta = \nabla \cdot \nabla = \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{3.11}$$

3.1 Second-order Homogeneous

$$\ddot{x} + Ax = 0 \implies x(t) = C_1 e^{i\sqrt{A}t} + C_2 e^{-i\sqrt{A}t} \tag{3.12}$$

$$\implies x(t) = C_1 \cos(\sqrt{A}t) + C_2 \sin(\sqrt{A}t) \tag{3.13}$$

$$\ddot{x} - Ax = 0 \implies x(t) = C_1 e^{\sqrt{A}t} + C_2 e^{-\sqrt{A}t} \tag{3.14}$$

$$\implies x(t) = C_1 \sinh(\sqrt{A}t) + C_2 \cosh(\sqrt{A}t) \tag{3.15}$$

$$\ddot{x} + A\dot{x} + Bx = 0 \implies x(t) = C_1 \mathrm{exp}\left[-\frac{1}{2}t(\sqrt{A^2 - 4B} + A)\right] \tag{3.16}$$

$$+ C_2 \mathrm{exp}\left[\frac{1}{2}t(\sqrt{A^2 - 4B} - A)\right] \tag{3.17}$$

3.2 Second-order Linear Ordinary

$$\ddot{x} + Ax = B \implies x(t) = \frac{B}{A} + C_1 e^{i\sqrt{A}t} + C_2 e^{-i\sqrt{A}t} \tag{3.18}$$

$$\implies x(t) = \frac{B}{A} + C_1 \cos(\sqrt{A}t) + C_2 \sin(\sqrt{A}t) \tag{3.19}$$

$$\ddot{x} - Ax = B \implies x(t) = -\frac{B}{A} + C_1 e^{\sqrt{A}t} + C_2 e^{-\sqrt{A}t} \tag{3.20}$$

$$\implies x(t) = -\frac{B}{A} + C_1 \sinh(\sqrt{A}t) + C_2 \cosh(\sqrt{A}t) \tag{3.21}$$

$$\ddot{x} + x = t(A - t) \implies x(t) = C_1 \cos(t) + C_2 \sin(t) - t^2 + At + 2 \tag{3.22}$$

$$\ddot{x} + A\dot{x} + Bx = t \implies x(t) = C_1 \mathrm{exp}\left[-\frac{1}{2}t(\sqrt{A^2 - 4B} + A)\right] \tag{3.23}$$

$$+ C_2 \mathrm{exp}\left[\frac{1}{2}t(\sqrt{A^2 - 4B} - A)\right] - \frac{A}{B^2} + \frac{t}{B} \tag{3.24}$$

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f_0 e^{i\omega t} \implies x(t) = \frac{f_0 e^{i\omega t}}{\omega_0^2 - \omega^2 + 2\beta i\omega} \tag{3.25}$$

$$\implies x(t) = A\cos(\omega t - \delta) + A_{tr} e^{-\beta t} \cos(\omega_1 t - \delta_{tr}) \tag{3.26}$$

$$\delta = \arctan\left(\frac{2\beta\omega}{\omega_0^2 - \omega^2}\right) \tag{3.27}$$

$$\implies x(t) = A\cos(\omega t - \delta) + e^{-\beta t}[B_1 \cos(\omega_1 t) + B_2 \sin(\omega_1 t)] \tag{3.28}$$

$$\ddot{x} + 2\beta\dot{x} + x = te^{-\alpha t} \implies x(t) = C_1 e^{-\alpha t} + C_2 t e^{-\alpha t} + C_3 e^{-\beta t} \sin(\omega_1 t) + C_4 e^{-\beta t} \cos(\omega_1 t) \tag{3.29}$$

$$\omega_1^2 = 1 - \beta^2 \tag{3.30}$$

4 Integrals

Basic indefinite integrals (c =constant)

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\arctan\left(\frac{x}{a}\right)+c \tag{4.1}$$

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin\left(\frac{x}{|a|}\right)+c = \arctan\left(\frac{x}{\sqrt{a^2-x^2}}\right)+c \tag{4.2}$$

$$\int \frac{dx}{x+x^2} = \ln\left(\frac{x}{1+x}\right)+c \tag{4.3}$$

$$\int \frac{dx}{\sqrt{x^2-1}} = \operatorname{arccosh}(x)+c \tag{4.4}$$

$$\int \frac{dx}{x\sqrt{x^2-1}} = \arccos\left(\frac{1}{x}\right)+c \tag{4.5}$$

$$\int \frac{dx}{(a^2+x^2)^{3/2}} = \frac{x}{a^2\sqrt{a^2+x^2}}+c \tag{4.6}$$

$$\int \frac{xdx}{(a^2+x^2)^{3/2}} = -\frac{1}{\sqrt{a^2+x^2}}+c \tag{4.7}$$

$$\int \frac{dx}{1-x^2} = \operatorname{arctanh}(x)+c \tag{4.8}$$

$$\int \frac{dx}{\sqrt{a^2+x^2}} = \operatorname{arcsinh}\left(\frac{x}{a}\right)+c = \ln\left|x+\sqrt{a^2+x^2}\right|+c \tag{4.9}$$

$$\int \frac{xdx}{1+x^2} = \frac{1}{2}\ln(1+x^2)+c \tag{4.10}$$

$$\int \frac{xdx}{\sqrt{1+x^2}} = \sqrt{1+x^2}+c \tag{4.11}$$

$$\int \frac{\sqrt{x}dx}{\sqrt{1-x}} = \arcsin(\sqrt{x})-\sqrt{x(1-x)}+c \tag{4.12}$$

$$\int \ln(x) = x\ln(x)-x+c \tag{4.13}$$

Trigonometric integrals

$$\int \tan(x)dx = -\ln(\cos(x)) + c \tag{4.14}$$

$$\int \tanh(x)dx = \ln(\cosh(x)) + c \tag{4.15}$$

$$\int \sin^2(x)dx = \frac{1}{2}\big(x - \sin(x)\cos(x)\big) + c = \frac{1}{4}\big(2x - \sin(2x)\big) + c \tag{4.16}$$

$$\int \cos^2(x)dx = \frac{1}{2}\big(x + \sin(x)\cos(x)\big) + c = \frac{1}{4}\big(2x + \sin(2x)\big) + c \tag{4.17}$$

$$\int \sin^2(x)\cos(x)dx = \frac{1}{3}\sin^3(x) + c \tag{4.18}$$

$$\int \cos^2(x)\sin(x)dx = -\frac{1}{3}\cos^3(x) + c \tag{4.19}$$

$$\int \sin^3(x)dx = -\frac{1}{3}\cos(x)\big(\sin^2(x) + 2\big) + c \tag{4.20}$$

$$\int x\sin^2(x)dx = \frac{1}{4}\big(x^2 - x\sin(2x) - \frac{1}{2}\cos(2x)\big) + c \tag{4.21}$$

$$\int x^2\sin^2(x)dx = \frac{x^3}{6} - \left(\frac{x^2}{4} - \frac{1}{8}\right)\sin(2x) - \frac{x}{4}\cos(2x) + c \tag{4.22}$$

The Wallis Cosine Formula

$$\int_0^{\pi/2} \cos^n(x)dx = \int_0^{\pi/2} \sin^n(x)dx = \frac{(n-1)!!}{n!!}\begin{cases}\pi/2 & \text{for } n = 2, 4, \dots \\ 1 & \text{for } n = 3, 5, \dots\end{cases} \tag{4.23}$$

The integral of an arbitrary Gaussian function is

$$\int x^n e^{\beta x}dx = e^{\beta x}\sum_{k=0}^{n}(-1)^k\frac{n!x^{n-k}}{(n-k)!\beta^{k+1}} + c \tag{4.24}$$

Some general Gaussian integrals evaluate as

$$\int_{-\infty}^{\infty} e^{-\alpha x^2}dx = \sqrt{\frac{\pi}{\alpha}} \tag{4.25}$$

$$I_n = \int x^n e^{-x/\alpha}dx \tag{4.26}$$

$$I_0 = -\alpha e^{-x/\alpha} \tag{4.27}$$

$$I_1 = -(\alpha^2 + \alpha x)e^{-x/\alpha} \tag{4.28}$$

$$I_2 = -(2\alpha^3 + 2\alpha^2 x + \alpha x^2)e^{-x/\alpha} \tag{4.29}$$

$$I_{n+1} = \alpha^2\frac{\partial I_n}{\partial\alpha} \tag{4.30}$$

$$\int_0^{\infty} e^{-x/\alpha}dx = \alpha \tag{4.31}$$

$$\int_0^{\infty} xe^{-x/\alpha}dx = \alpha^2 \tag{4.32}$$

$$\int_0^{\infty} x^2e^{-x/\alpha}dx = 2\alpha^3 \tag{4.33}$$

$$\int_0^{\infty} x^ne^{-x/\alpha}dx = n!\alpha^{n+1} \tag{4.34}$$

The integral of an arbitrary Gaussian function with an n-dimensional linear term (with $n \in \mathbb{Z}$) is

$$\int_0^\infty x^{2n} e^{-\alpha x^2} dx = \sqrt{\frac{\pi}{\alpha}} \frac{(2n-1)!!}{2^{n+1}\alpha^n} \implies \int_{-\infty}^\infty x^{2n} e^{-\alpha x^2} dx = \sqrt{\frac{\pi}{\alpha}} \frac{(2n-1)!!}{(2\alpha)^n} \tag{4.35}$$

$$\int_0^\infty x^{2n+1} e^{-\alpha x^2} dx = \frac{n!}{2a^{n+1}} \implies \int_{-\infty}^\infty x^{2n+1} e^{-\alpha x^2} dx = 0 \tag{4.36}$$

Therefore a general solution is

$$\int_0^\infty x^n e^{-\alpha x^2} dx = \begin{cases} \dfrac{(n-1)!!}{2^{n/2+1} a^{n/2}} \sqrt{\dfrac{\pi}{\alpha}} & \text{for } n \text{ even} \\ \dfrac{[\frac{1}{2}(n-1)]!}{2a^{(n+1)/2}} & \text{for } n \text{ odd} \end{cases} \tag{4.37}$$

The below form of a gaussian integral evaluates to zero when n is odd due to the function being odd, but when n is even, the more general integral has the following closed form

$$\int_{-\infty}^\infty x^n e^{-\alpha x^2 + \beta x} = \sqrt{\frac{\pi}{\alpha}} e^{\beta^2/(4\alpha)} \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} (2k-1)!!(2a)^{k-n} \beta^{n-2k} \tag{4.38}$$

5 Fourier Series

The computation of the (usual) Fourier series is based on the integral identities

$$\int_{-\pi}^{\pi} \sin(mx)\sin(nx)dx = \pi\delta_{mn} \tag{5.1}$$

$$\int_{-\pi}^{\pi} \cos(mx)\cos(nx)dx = \pi\delta_{mn} \tag{5.2}$$

$$\int_{-\pi}^{\pi} \sin(mx)\cos(nx)dx = 0 \tag{5.3}$$

$$\int_{-\pi}^{\pi} \sin(mx)dx = 0 \tag{5.4}$$

$$\int_{-\pi}^{\pi} \cos(mx)dx = 0 \tag{5.5}$$

$$\delta_{mn} = \frac{1}{2\pi i}\oint_{\gamma} z^{m-n-1}dz \tag{5.6}$$

Using the method for a generalized Fourier series, the usual Fourier series involving sines and cosines is obtained by taking $f_1(x) = cosx$ and $f_2(x) = sinx$. Since these functions form a complete orthogonal system over $[-\pi, \pi]$, the Fourier series of a function $f(x)$ is given by (with $n \in \mathbb{N}$)

$$f(x) = \frac{1}{2}a_0 + \sum_{i=1}^{\infty} a_n \cos(nx) + \sum_{i=1}^{\infty} b_n \sin(nx) \tag{5.7}$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx \tag{5.8}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx \tag{5.9}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx \tag{5.10}$$

For a function f(x) periodic on an interval [-L,L] instead of [-pi,pi], a simple change of variables can be used to transform the interval of integration from [-pi,pi] to [-L,L]. Let

$$x \equiv \frac{\pi x'}{L} \implies x' = \frac{Lx}{\pi} \tag{5.11}$$

$$dx = \frac{\pi dx'}{L} \tag{5.12}$$

6 Statistics

A probability distribution: Given a Poisson process, the probability of obtaining exactly m successes in n trials is given by the limit of a binomial distribution

$$\mathcal{P}_n(m;p) = \binom{n}{m} p^m (1-p)^{n-m} \tag{6.1}$$

Letting the sample size n become large, the distribution then approaches the Poisson Distribution

$$\mathcal{P}(m,\lambda) = \frac{\lambda^m}{m!} e^{-\lambda} \tag{6.2}$$

The mean number of events is

$$\langle m \rangle = \sum_{m=0}^{\infty} m \frac{\lambda^m}{m!} e^{-\lambda} = \lambda \tag{6.3}$$

And the standard deviation is

$$\sigma = \sqrt{\lambda} \tag{6.4}$$

The normal, or Gaussian distribution

$$\mathcal{P}(x;\mu,\sigma) = \frac{1}{\sqrt{2\pi}\sigma} exp\Big[-\frac{(x-\mu)^2}{2\sigma^2}\Big] \tag{6.5}$$

$$\mathcal{P}(a \leq x \leq b) = \int_a^b \frac{1}{\sqrt{2\pi}\sigma} exp\Big[-\frac{(x-\mu)^2}{2\sigma^2}\Big] \tag{6.6}$$

If the mean is not equal to zero, a more general distribution known as the noncentral chi-squared distribution results. In particular, if x_i are independent variates with a normal distribution having means μ_i and variances σ_i^2 for $i = 1, ..., n$, then

$$\chi^2 \equiv \sum_{i=1}^{n} \frac{(x_i - \mu_i)^2}{\sigma_i^2}. \tag{6.7}$$

Given some function $f(x_1, x_2, \ldots, x_n)$, the error of a calculation with each respective variable being denoted by σ_i, can be determined by

$$\sigma_f^2 = \left(\frac{\partial f}{\partial x_1}\right)^2 \sigma_{x_1}^2 + \left(\frac{\partial f}{\partial x_2}\right)^2 \sigma_{x_2}^2 + \cdots + \left(\frac{\partial f}{\partial x_n}\right)^2 \sigma_{x_n}^2 = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i}\right)^2 \sigma_{x_i}^2 \tag{6.8}$$

7 Astronomy, Optics and Telescopes

A parsec is defined so

$$1 \text{ parsec} = \frac{1 \text{ AU}}{\tan(1")} \approx \frac{1 \text{ AU}}{1"} \tag{7.1}$$

The Flux (F) of a star relates to it's luminosity (L) and distance (d) via

$$F = \frac{L}{4\pi d^2} \tag{7.2}$$

The ratio of two magnitudes using different filters from a single star gives a rough estimation of the stars color.

$$B - V = m_B - m_V = -2.5 \log_{10}\left(\frac{F_B}{F_V}\right) \tag{7.3}$$

$$\frac{F_B}{F_V} = 10^{-(M_B - M_V)/2.5} \tag{7.4}$$

We define the distance modulus (DM) as the difference in apparent magnitude (m) between a given star and the absolute magnitude (M) it would have if it were at 10 pc.

$$DM \equiv m - m(10 \text{ pc}) \equiv m - M \tag{7.5}$$

$$M \equiv m - DM \tag{7.6}$$

The full form of intensity as a function of angle from the beam axis is

$$I = I_0 \left[\frac{\sin(\pi D/\lambda \sin(\theta))}{\sin(\pi d/\lambda \sin(\theta))}\right]^2 \tag{7.7}$$

Snell's Law:

$$\frac{\sin(\theta_1)}{\sin(\theta_2)} = \frac{v_1}{v_2} = \frac{\lambda_1}{\lambda_2} = \frac{n_2}{n_1} \tag{7.8}$$

7.1 Planetary Orbits

Suppose we have a exoplanet system with a planet p and a star s. The vector from the star to the planet is $\vec{r}_{sp} = \vec{r}_p - \vec{r}_s$, and the force that the star exerts on the planet is ($\vec{r}_n$ is the vector from the origin to n)

$$\vec{F}_{sp} = -\frac{GM_pM_s}{|\vec{r}_{sp}|^3}\vec{r}_{sp} \tag{7.9}$$

If we put the origin at the center of mass ($\vec{R}$ is the vector from the origin to the center of mass)

$$\vec{R} = \frac{M_s\vec{r}_s + M_p\vec{r}_p}{M_s + M_p} \tag{7.10}$$

Then the star and planets have positions

$$\vec{x}_s = \vec{r}_s - \vec{R} = -\frac{M_p}{M_p + M_s}\vec{r}_{sp} \tag{7.11}$$

$$\vec{x}_p = \vec{r}_p - \vec{R} = -\frac{M_s}{M_p + M_s}\vec{r}_{sp} \tag{7.12}$$

And thus accelerations

$$\frac{d^2\vec{x}_s}{dt^2} = -\frac{M_p}{M_p + M_s}\frac{d^2\vec{r}_{sp}}{dt^2} \tag{7.13}$$

$$\frac{d^2\vec{x}_p}{dt^2} = -\frac{M_s}{M_p + M_s}\frac{d^2\vec{r}_{sp}}{dt^2} \tag{7.14}$$

Substituting the acceleration into the equation of motion for the planet,

$$M_p\frac{d^2\vec{x}_p}{dt^2} = \vec{F}_{sp} \tag{7.15}$$

Then we can get the reduced equation of motion as

$$\frac{d^2\vec{r}_{sp}}{dt^2} = -G\frac{M_s + M_p}{|\vec{r}_{sp}|^3}\vec{r}_{sp} \tag{7.16}$$

Keplar's Third law: The solution to this is an elliptical orbit with the center-of-force at one focus of the ellipse. The period (T) depends on the semi-major axis (a)

$$T^2 = \frac{4\pi^2}{G(M_s + M_p)}a^3 \tag{7.17}$$

$$a^3 = \frac{G(M_s + M_p)}{4\pi^2}T^2 \tag{7.18}$$

If the orbit is circular, so that $|\vec{r}_s p = a$ is constant, then the orbital speed of the star is

$$v_s = \frac{2\pi a M_p}{T(M_p + M_s)} = \sqrt{\frac{GM_p^2}{a(M_p + M_s)}} \tag{7.19}$$

8 Classical Mechanics

Newtons Second Law in Cartesian coordinates

$$\vec{F} = m\vec{a} = m\ddot{\boldsymbol{r}} \Longleftrightarrow \begin{cases} F_x = m\ddot{x} \\ F_y = m\ddot{y} \\ F_z = m\ddot{z} \end{cases} \tag{8.1}$$

Newtons Second Law in 2D polar coordinates

$$\vec{F} = m\vec{a} \Longleftrightarrow \begin{cases} F_r = m(\ddot{r} - r\dot{\phi}^2) \\ F_\phi = m(r\ddot{\phi} + 2\dot{r}\dot{\phi}) \end{cases} \tag{8.2}$$

Newtons Second Law in cylindrical polar coordinates

$$\vec{F} = m\vec{a} \Longleftrightarrow \begin{cases} F_r = m(\ddot{\rho} - \rho\dot{\phi}^2) \\ F_\phi = m(\rho\ddot{\phi} + 2\dot{\rho}\dot{\phi}) \\ F_z = m\ddot{x} \end{cases} \tag{8.3}$$

Conservation of energy

$$E = \text{constant} = KE + PE = \frac{1}{2}m||\vec{v}||^2 + mgh \tag{8.4}$$

The Lorentz Force on a charged particle.

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B}) \tag{8.5}$$

Equation of motion for a rocket

$$m\dot{v} = -\dot{m}v_{ex} + F^{external} \tag{8.6}$$

The center of mass of several particles with a total mass M is

$$\vec{R} = \frac{1}{M}\sum_{\alpha=1}^{n} m_\alpha \vec{r}_\alpha = \frac{m_1\vec{r}_1 + \cdots m_n\vec{r}_n}{M} \tag{8.7}$$

$$\vec{R} = \frac{1}{M}\int \vec{r}dm = \frac{1}{M}\int \rho\vec{r}dV \tag{8.8}$$

The mass of an object is defined by the density multiplied by the volume.

$$M \equiv \rho V \equiv \iiint_Q \rho(x,y,z)dV \tag{8.9}$$

The moment of inertia with respect to a given axis of a solid body with density $\rho(r)$, where $r_\perp$ is the perpendicular distance from the axis of rotation, is defined by the volume integral

$$I \equiv \int \rho(\boldsymbol{r})r_\perp^2 dV \equiv \iiint_Q \rho(x,y,z)||\boldsymbol{r}||^2 dV \tag{8.10}$$

Angular momentum

$$\vec{L} = \vec{r} \times \vec{p} = I\vec{\omega} = I\dot{\theta} \tag{8.11}$$

The net external torque is given by

$$\vec{\tau}_{ext} = \vec{r} \times \vec{F} = \frac{d\vec{L}}{dt} \tag{8.12}$$

The change in kinetic energy as it moves from point a to point b is

$$\Delta K \equiv K_2 - K_1 = \int_a^b \vec{F} \cdot d\vec{r} \equiv W(a \to b) \tag{8.13}$$

$$K = \frac{1}{2}mv^2 = \frac{1}{2}I\omega^2 = \frac{1}{2}I\dot{\theta}^2 \tag{8.14}$$

A force $\vec{F}$ on a particle is **conservative** if (i) it depends only on the particles position, $\vec{F} = \vec{F}(\vec{r})$ and (ii) $\nabla \times \vec{F} = 0$. If $\vec{F}$ is conservative we can define a corresponding **potential energy** so that

$$U(\boldsymbol{r}) = -W(\boldsymbol{r}_0 \to \boldsymbol{r}) \equiv \int_{\boldsymbol{r}_0}^{\boldsymbol{r}} \boldsymbol{F}(\boldsymbol{r}') \cdot d\boldsymbol{r}' \tag{8.15}$$

$$\vec{F} = -\nabla\vec{U} \tag{8.16}$$

Hooke's Law

$$F = -kx \Longleftrightarrow U = \text{constant} + \frac{1}{2}kx^2 \tag{8.17}$$

Simple harmonic motion

$$\ddot{x} = -\omega^2 x \Longleftrightarrow A\cos(\omega t - \delta) \tag{8.18}$$

Damped oscillations: If the oscillator is subject to a damping force $-bv$, the

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0 \text{ and } \beta < \omega_0 \Longleftrightarrow x(t) = Ae^{-\beta t}\cos(\omega_1 t - \delta) \tag{8.19}$$

$$\beta = \frac{b}{2m}, \quad \omega_0 = \sqrt{\frac{k}{m}}, \quad \omega_1 = \sqrt{\omega_0^2 - \beta^2} \tag{8.20}$$

If the oscillator is also subject to a sinusoidal driving force $F(t) = mf_0\cos(\omega t)$, the long-term motion has the form

$$x(t) = A\cos(\omega t - \delta) \tag{8.21}$$

$$A^2 = \frac{f_0^2}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2} \tag{8.22}$$

It is always possible to write a sum of sinusoidal functions as a single sinusoid the form

$$f(\theta) = A\cos(\theta) + B\sin(\theta) \Longleftrightarrow f(\theta) = C\cos(\theta + \delta) \tag{8.23}$$

$$\delta = \arctan(-B/A) \tag{8.24}$$

$$C = \pm\sqrt{A^2 + B^2} \tag{8.25}$$

$$f(\theta) = A\cos(\theta) + B\sin(\theta) \Longleftrightarrow f(\theta) = \operatorname{sgn}(A)\sqrt{A^2 + B^2}\cos\big(\theta + \arctan(-B/A)\big) \tag{8.26}$$

Any periodic function with period τ can be written as (A Fourier series)

$$f(t) = \sum_{n=0}^{\infty}[a_n \cos(n\omega t) + b_n \sin(n\omega t)] \tag{8.27}$$

$$a_n = \frac{2}{\tau}\int_{-\tau/2}^{\tau/2} f(t)\cos(n\omega t)dt \qquad [n \geq 1] \tag{8.28}$$

$$b_n = \frac{2}{\tau}\int_{-\tau/2}^{\tau/2} f(t)\sin(n\omega t)dt \qquad [n \geq 1] \tag{8.29}$$

$$a_0 = \frac{1}{\tau}\int_{-\tau/2}^{\tau/2} f(t)dt \tag{8.30}$$

It is sometimes useful to express the above Fourier series as an exponential

$$f(t) = \sum_{n=-\infty}^{\infty} A_n e^{in\omega t} \qquad \text{with} \qquad A_n = \frac{1}{\tau}\int_{-\tau/2}^{\tau/2} f(t)e^{-in\omega t}dt \tag{8.31}$$

It is important to know $A_n = A^*_{-n}$ so we can write $A_n = \Re(A_n) + i\Im(A_n)$. An important relationship between A_n, a_n and b_n then follows as,

$$a_n = 2\Re(A_n) \qquad \text{and} \qquad b_n = -2\Im(A_n) \tag{8.32}$$

The root-mean square displacement is a good measure of the average response of the oscillator and is given by parseval's theorem

$$x_{rms} = \sqrt{\frac{1}{\tau}\int_0^{\tau} x^2 dt} = \sqrt{A_0^2 + \frac{1}{2}\sum_{n=1}^{\infty} A_n^2} \tag{8.33}$$

The non-relativistic Lagrangian $\mathcal{L}$ for a conservative system can be defined in terms of the kinetic energy and potential energy of a system as

$$\mathcal{L} = KE - PE \tag{8.34}$$

An integral of the form

$$S = \int_{x_1}^{x_2} f[y(x), y'(x), x]dx \tag{8.35}$$

taken along a path $y = y(x)$ is stationary with respect to variations of that path if and only if $y(x)$ satisfies the Euler-Lagrange Equation

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0. \tag{8.36}$$

If there are n dependent variables in the original integral, there are n Euler-Langrange equations. For instance, an integral of the form

$$S = \int_{u_1}^{u_2} f[x(u), y(u), x'(u), y'(u), u]du \tag{8.37}$$

with two dependent variables [$x(u)$ and $y(u)$], is stationary with respect to variations of $x(u)$ and $y(u)$ if and only if these two functions satisfy the two equations

$$\frac{\partial f}{\partial x} = \frac{d}{du}\frac{\partial f}{\partial x'} \qquad \text{and} \qquad \frac{\partial f}{\partial y} = \frac{d}{du}\frac{\partial f}{\partial y'} \tag{8.38}$$

For any holonomic system, Newtons second law is equivalent to the n Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} \tag{8.39}$$

The ith generalized momentum p_i is defined to be the derivative

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \tag{8.40}$$

If $\partial\mathcal{L}/\partial t = 0$ then $\mathcal{H}$ is conserved; if the coordinates $q_1, \ldots, q_n$ are natural, $\mathcal{H}$ is just the energy of the system. The Hamiltonian $\mathcal{H}$ is defined as

$$\mathcal{H} = \sum_{i=1}^{n} p_i \dot{q}_i - \mathcal{L} \tag{8.41}$$

The time evolution of a system is given by Hamilton's equations

$$\dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i} \qquad \text{and} \qquad \dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i} \tag{8.42}$$

The Lagrangian for a charge q in an electromagnetic field is

$$\mathcal{L}(\boldsymbol{r}, \dot{\boldsymbol{r}}, t) = \frac{1}{2}m\dot{\boldsymbol{r}}^2 - q(V - \dot{\boldsymbol{r}} \cdot \boldsymbol{A}) \tag{8.43}$$

9 Special Relativity

Relativistic time dilation and length contraction.

$$\Delta t = \frac{\Delta t_o}{\sqrt{1-\beta^2}} = \gamma \Delta t_0 \tag{9.1}$$

$$\Delta l = \Delta l_0 \sqrt{1-\beta^2} = \frac{\Delta l_0}{\gamma} \tag{9.2}$$

$$\beta = \frac{v}{c} \tag{9.3}$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} \tag{9.4}$$

Lorentz Transformations for space and time coordinates.

$$x' = \gamma(x - vt) \tag{9.5}$$

$$y' = y \tag{9.6}$$

$$z' = z \tag{9.7}$$

$$t' = \gamma(t - vx/c^2) \tag{9.8}$$

The relativistic velocity transformation is.

$$u' = \frac{u - v}{1 - vu/c^2} \tag{9.9}$$

$$u = \frac{u' + v}{a + vu'/c^2} \tag{9.10}$$

The rest energy of a particle

$$E_0 = mc^2 \tag{9.11}$$

the lorentz transformation for momentum and energy is.

$$p'_x = \gamma(p_x - vE/c^2) \tag{9.12}$$

$$p'_y = p_y \tag{9.13}$$

$$p'_z = p_z \tag{9.14}$$

$$E' = \gamma(E - vp_x) \tag{9.15}$$

Relativistic mass and momentum.

$$E = \gamma mc^2 \tag{9.16}$$

$$p = \gamma mv \tag{9.17}$$

Combining the above equations give

$$\frac{E}{p} = \frac{c^2}{v} \implies E = \frac{pc^2}{v} \tag{9.18}$$

Mass-energy equivalence

$$E^2 = (mc^2)^2 + (pc)^2 \tag{9.19}$$

$$E = K_E + E_0 \tag{9.20}$$

$$K_E = \text{ Kinetic Energy}$$

Combining the above equations gives

$$p = \frac{1}{c}\sqrt{K_E^2 + 2K_E E_0} \tag{9.21}$$

Invariant dot product in c=1 notation

$$A \cdot B = (E, \vec{p}) \cdot (U, \vec{q}) = EU - \vec{p} \cdot \vec{q} \tag{9.22}$$

Relativistic frequency and wavelength shifts

$$f = f_0 \sqrt{\frac{c \pm v}{c \mp v}} \tag{9.23}$$

$$\lambda = \lambda_0 \sqrt{\frac{c \mp v}{c \pm v}} \tag{9.24}$$

Space-time equivalence (same in all reference frames)

$$S \equiv (c\Delta t)^2 - (\Delta x)^2 \equiv E^2 - (pc)^2 \tag{9.25}$$

10 Thermodynamics

Useful constants: the specific heat of water is c

$$c = 4186 \text{ J/(kg·K)}. \tag{10.1}$$
$$1 \text{ cal} = 4.186 \text{ J} \tag{10.2}$$

Temperature relationships.

$$^\circ F = \frac{9^\circ}{5} C + 32 \tag{10.3}$$
$$^\circ C = \frac{5}{9}(^\circ F - 32) \tag{10.4}$$
$$^\circ K =^\circ C + 273.15 \tag{10.5}$$

The heat required to raise the temperature of a mass m by ΔT is

$$Q = cm\Delta T \tag{10.6}$$

he temperature of an object determines the radiated power of the object, which is given by the **Stefan-Boltzmann equation**

$$P_{radiated} = \sigma\epsilon A T^4 \tag{10.7}$$
$$\sigma = 5.67 \times 10^{-8} \text{ W}/K^4 m^2 \tag{10.8}$$
$$\epsilon = \text{emissivity, and } 0 \leq \epsilon \leq 1 \tag{10.9}$$

The work done on a system in going from initial volume (V_i) to a final volume (V_f) is

$$W = \int dW = \int_{V_i}^{V_f} pdV. \tag{10.10}$$

The first law of thermodynamics

$$\Delta E_{internal} = Q - W \tag{10.11}$$

different processes include

(i) An adiabatic process is one in which $Q = 0$.

(ii) In a constant-volume process, $W = 0$.

(iii) In a closed-loop process, $Q = W$.

(iv) In an adiabatic free expansion, $Q = W = \Delta E_{internal} = 0$.

If heat is added to an object, its change in temperature (with C =heat capacity of the object) is given by

$$\Delta T = \frac{Q}{C} \tag{10.12}$$

If heat is added to an object with mass m, its change in temperature (with c =specific heat of the object) is given by

$$\Delta T = \frac{Q}{cm} \tag{10.13}$$

The ideal gas law

$$PV = nRT \tag{10.14}$$
$$R = 1.38106504(24) \times 10^{-23} \text{ J/K} \tag{10.15}$$

With a constant number of moles we get from the ideal gas law the following relation:

$$\frac{P_1V_1}{T_1} = \frac{P_2V_2}{T_2} \tag{10.16}$$

Dalton's law - The total pressure exerted by a mixture of gases is equal to the sum of the partial pressures pf the gases in the mixture.

$$P_{total} = P_1 + P_2 + P_3 + \cdots + P_n \tag{10.17}$$

The work done by an ideal gas at constant temperature is

$$W = nRT \ln\left(\frac{V_f}{V_i}\right) \tag{10.18}$$

The average kinetic energy of an ideal gas

$$K_{ave} = \frac{1}{N}\sum_{i=1}^{N} K_i \tag{10.19}$$
$$= \frac{1}{N}\sum_{i=1}^{N} \frac{1}{2} m v_i^2 \tag{10.20}$$
$$= \frac{1}{2} m v_{rms}^2 \tag{10.21}$$

The root-mean-square speed of gas molecules is

$$v_{rms} = \sqrt{\frac{1}{N}\sum_{i=1}^{N} v_i^2} = \sqrt{\frac{3RT}{m}} \tag{10.22}$$

For an adiabatic process (with C_V=specific heat at constant volume, C_P=specific heat at constant pressure), we have

$$dE_{internal} = -PdV = nC_V dT \tag{10.23}$$

$$PV^\gamma = \text{constant} \tag{10.24}$$

$$\gamma = \frac{C_P}{C_V} \tag{10.25}$$

$$P_f V_f^\gamma = P_i V_i^\gamma \tag{10.26}$$

$$T_f V_f^{\gamma-1} = T_i V_i^{\gamma-1} \tag{10.27}$$

11 Quantum Mechanics

Electromagnetic wave frequency and wavelength

$$c = \nu\lambda \implies \nu = \frac{c}{\lambda} \implies \lambda = \frac{c}{\nu} \tag{11.1}$$

$$\nu = \text{frequency} \tag{11.2}$$

The energy in a photon (packet of light)

$$E = h\nu = \frac{hc}{\lambda} = \hbar\omega \tag{11.3}$$

$$dE = -\frac{hc}{\lambda^2}d\lambda = -\frac{E^2}{hc}d\lambda \implies |\Delta\lambda| = hc\frac{\Delta E}{E^2} \tag{11.4}$$

Useful units for the proportionality factor (Planck's constant) are

$$hc = 1240eV * nm \tag{11.5}$$

$$\hbar = \frac{h}{2\pi} = 1.05 \times 10^-34J * s \tag{11.6}$$

Wien's Displacement Law

$$\lambda_{MAX}T = 2.898 \times 10^{-3}m * K \tag{11.7}$$

Total Power Stefan-Boltzmann Law

$$R(T) = \int_0^\infty I(\lambda, T)d\lambda = \epsilon\sigma T^4 \tag{11.8}$$

$$\epsilon = \text{emmisivity (unitless)} \tag{11.9}$$

$$\sigma = 5.67 \times 10^{-8}\frac{w}{m^2k^4} \tag{11.10}$$

Max Planck's Radiation Law:

$$I(\lambda, T) = \frac{2\pi c^2 h}{\lambda^5}\frac{1}{e^{hc/\lambda kT} - 1} \tag{11.11}$$

The kinetic energy of an emitted photoelectron is (Where ϕ = binding energy of electron to metal surface or the work function)

$$KE = hv - \phi \tag{11.12}$$

$$E_{photon} = KE_{electron} + \phi \tag{11.13}$$

$$KE_{electrons} = 0 \text{ (at threshold)} \tag{11.14}$$

Ruthford Scattering Formula:Any particle hitting an area σ around the nucleus will be scattered through an angle of θ or greater.

$$b = (r_{min}/2)\cot(\theta/2) \tag{11.15}$$

$$r_{min} = \frac{Z_1 Z_2 e^2}{4\pi\epsilon_0 K} \tag{11.16}$$

$$\sigma = \pi b^2 = \text{ cross sectional area} \tag{11.17}$$

$$\frac{e^2}{4\pi\epsilon_0} = 1.44 \times 10^{-9}\text{eV}\cdot\text{m} \tag{11.18}$$

A common unit of σ is one barn.

$$\text{barn (unit)} = 10^{-28} m^2 = 100 fm^2 \tag{11.19}$$

Number of atoms per area = (atoms/volume)*thickness

$$n = \left(N_A \frac{atoms}{mole} \right) \left(\frac{1}{A} \frac{mole}{gm} \right) \left(\rho \frac{gm}{cm^3} \right) = \frac{\rho N_A}{A} \tag{11.20}$$

The Compton effect describes the photon wavelength λ' after a photon of wavelength λ scatters off an electron.

$$\lambda' = \lambda + \frac{h}{m_e c}(1 - cos(\theta)) \tag{11.21}$$

The Compton wavelength of an electron is

$$\lambda_e = \frac{h}{m_e c} = 2.426 \times 10^{-12} m \tag{11.22}$$

Heisenberg Uncertainty relation

$$\Delta x \cdot \Delta p_x \geq \frac{1}{2}\hbar \tag{11.23}$$

The de Broglie wavelength is defined as

$$\lambda = \frac{h}{p} = \frac{h}{mv\gamma} = \frac{h}{mv}\sqrt{1 - \frac{v^2}{c^2}} = \frac{hc}{\sqrt{K_E^2 + 2K_E E_0}} \tag{11.24}$$

Rutherford Scattering.

$$K = \frac{1}{4\pi\epsilon_0} \frac{Z_1 Z_2 e^2}{R_{min}} \Longleftrightarrow R_{min} = \frac{1}{4\pi\epsilon_0} \frac{Z_1 Z_2 e^2}{K} \tag{11.25}$$

Z's are the atomic masses of the particles within the interaction and R_{min} is the minimum distance they reach (from center to center), and e is

$$e = 1.602177 \times 10^{-19} C \tag{11.26}$$

$$\epsilon \approx 8.854 \times 10^{-12} F/m \tag{11.27}$$

The Rutherford Scattering Formula

$$N(\theta) = \frac{N_i n t}{16 r^2} (R_{min})^2 \frac{1}{\sin^4(\theta/2)} \tag{11.28}$$

Centripetal force due to coulomb attraction

$$\frac{1}{4\pi\epsilon_0} \frac{e^2}{r^2} = m a_c = m\frac{v^2}{r} \implies v^2 = \frac{1}{4\pi\epsilon_0} \frac{e^2}{mr} \tag{11.29}$$

$$\implies r = 4\pi\epsilon_0 \frac{n^2 \hbar^2}{m e^2} \tag{11.30}$$

Energy levels

$$E = KE + PE = -\frac{1}{2}\frac{e^2}{4\pi\epsilon_0 r} \implies E = \frac{-E_0}{n^2}, \tag{11.31}$$
$$\text{where } E_0 = \alpha^2 mc^2/2 = 13.6 \text{ eV}. \tag{11.32}$$

Energy of emitted radiation

$$E = E_n - E_m = E_0\left(\frac{1}{m^2} - \frac{1}{n^2}\right) \tag{11.33}$$

Note. Using the Planck formula in the above equation leads to the Rydberg formula.

The Rydberg formula: Wavelength of the spectral lines in Hydrogen:

$$\frac{1}{\lambda} = R_H\left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right) \tag{11.34}$$
$$n \in \mathbb{N} = 1, 2, 3, 4, 5, \ldots \tag{11.35}$$
$$R_H = \frac{E_0}{hc} = \frac{13.6eV}{1240eV \cdot nm} \tag{11.36}$$
$$= 10{,}967{,}760m^{-1} \tag{11.37}$$
$$= 1.096776 \times 10^7 m^{-1} \text{ (Rydberg's constant)} \tag{11.38}$$

Note. ZnS (Zinc Sulfide) emits a faint flash of light when struck by an $\alpha-$ray.

L quantized

$$L = mvr = n\hbar \tag{11.39}$$

Stationary state orbits

$$r = a_0 n^2 \tag{11.40}$$
$$a_0 = \text{Bohr Radius} \tag{11.41}$$

Stationary state energies

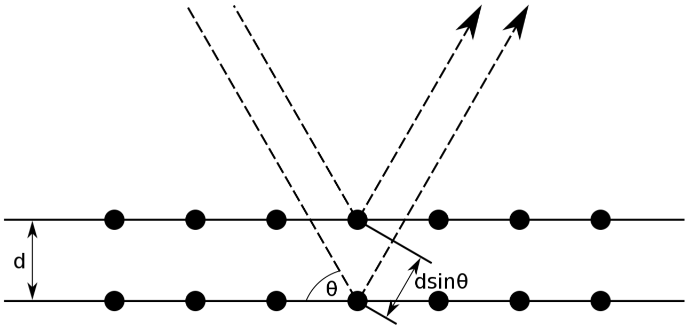
$$E_n = -Z^2\frac{E_0}{n^2} \tag{11.42}$$

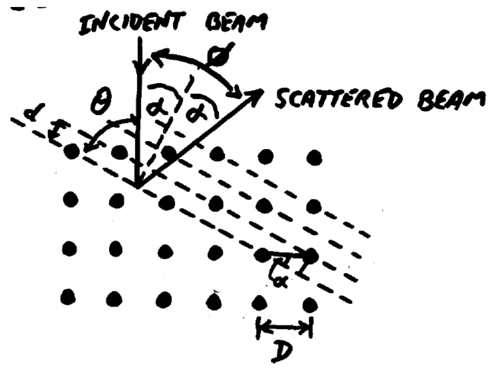
Uncertainty relation of energy and the measurement of time.

$$\Delta E \cdot \Delta t \geq \frac{1}{2}\hbar \tag{11.43}$$

Bragg's Law: When scattering off of crystal structures, the wavelengths will peak at specific angles determined by the diagrams below

$$n\lambda = 2d\sin(\theta) = 2d\cos(\alpha) = 2D\sin(\alpha)\cos(\alpha) = D\sin(2\alpha) = D\sin(\phi) \tag{11.44}$$
$$d = Dsin(\alpha) \tag{11.45}$$
$$\phi = 2\alpha \tag{11.46}$$
$$\theta = 90^\circ - \alpha \tag{11.47}$$





11.1 Matter-Waves

A plane wave

$$\psi(x,t) = A\cos[2\pi(x-ct)/\lambda] \tag{11.48}$$
$$f = c/\lambda \tag{11.49}$$
$$T = 1/f \tag{11.50}$$

$$\psi(x,t) = A\cos(kx-\omega t) \tag{11.51}$$
$$k = 2\pi/\lambda \tag{11.52}$$
$$\omega = 2\pi f = 2\pi/T \tag{11.53}$$

A periodic wave can be constructed from a sum of plane waves

$$\psi(x,t) = \sum_{i=1}^{n} A_i \cos(k_i x_i - \omega_i t) \tag{11.54}$$

Fourier Transform: A wave packet can be constructed as a continuous sum of plane waves

$$\psi(x,t) = \int A(k)\cos(kx-\omega t)dk \tag{11.55}$$

The wave equation

$$\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = \nabla^2\psi \tag{11.56}$$

The wave function solution for a particle confined to an infinite potential well with walls at $x=0$ and $x=a$ is as follows, with the corresponding energy eigenvalues

$$\psi(x) = \begin{cases} 0 & x<0 \\ \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right) \text{ with } n\in\mathbb{N} & 0\le x\le a \\ 0 & x>a \end{cases} \tag{11.57}$$

$$E_n = \frac{\hbar^2\pi^2}{2ma^2}n^2 \tag{11.58}$$

The solution for a finite potential well

$$U(x) = \begin{cases} \infty & \text{for } x<0 \\ 0 & \text{for } 0\le x\le a \\ U_1 & \text{for } x>a \end{cases} \tag{11.59}$$

Similarly, with $E > U_1$ is

$$\psi(x) = \begin{cases} 0 & \text{for } x < 0 \\ D\sin(kx) & \text{for } 0 \leq x \leq a \\ F\cos(k'x) + G\sin(k'x) & \text{for } x > a \end{cases} \tag{11.60}$$

$$\text{with } k' = \sqrt{k^2 - \frac{2mU_1}{\hbar^2}} \tag{11.61}$$

with $E < U_1$ is

$$\psi(x) = \begin{cases} 0 & \text{for } x < 0 \\ D\sin(kx) & \text{for } 0 \leq x \leq a \\ Fe^{-\gamma x} & \text{for } x > a \end{cases} \tag{11.62}$$

$$\text{with } \gamma^2 = \frac{2m(U_1 - E)}{\hbar^2} = \frac{2mU_1}{\hbar^2} - k^2 \tag{11.63}$$

For a simple harmonic oscillator

$$U(x) = \frac{1}{2}kx^2 = \frac{1}{2}m\omega_0^2 x^2 \implies k = m\omega_0^2 \Longleftrightarrow \omega_0 = \sqrt{\frac{k}{m}} \tag{11.64}$$

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega_0 = \left(n + \frac{1}{2}\right)\hbar\sqrt{\frac{k}{m}} = \left(n + \frac{1}{2}\right)\frac{\hbar}{x}\sqrt{\frac{2U(x)}{m}} \tag{11.65}$$

The quantum mechanical expectation value of a quantity is found by integrating over the entire space ψ^* times the result obtained when the corresponding operator acts on ψ. The position expectation value for any function is

$$\langle f(x) \rangle = \int_{-\infty}^{\infty} \psi^* f(x) \psi dx \tag{11.66}$$

The momentum operator

$$\hat{p} = -i\hbar\nabla \tag{11.67}$$

$$\frac{\partial \psi}{\partial x} = \frac{\partial}{\partial x} Ae^{i(kx - \omega t)} = ik\psi = \frac{ip}{\hbar}\psi \implies \hat{p} = -i\hbar\frac{\partial}{\partial x} \tag{11.68}$$

$$\langle p \rangle = \int_{-\infty}^{\infty} \psi^* \hat{p} \psi dx = -i\hbar \int_{-\infty}^{\infty} \psi^* \frac{\partial \psi}{\partial x} dx \tag{11.69}$$

The operator for a particles kinetic energy is

$$\hat{K} = \frac{-\hbar^2}{2m}\nabla^2 \tag{11.70}$$

$$\hat{K}\psi = \frac{1}{2m}\hat{p}^2\psi = \frac{1}{2m}\left(- i\hbar\frac{\partial}{\partial x}\right)^2 \psi = \frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi \tag{11.71}$$

$$\langle K \rangle = \int_{-\infty}^{\infty} \psi^* \hat{K} \psi dx = \int_{-\infty}^{\infty} \psi^* \frac{1}{2m}\hat{p}^2 \psi dx = \frac{-\hbar^2}{2m} \int_{-\infty}^{\infty} \psi^* \frac{\partial^2}{\partial x^2} \psi dx \tag{11.72}$$

The Hamiltonian operator

$$\hat{H} = \hat{K} + U(\hat{r}) = \frac{\hat{p}^2}{2m} + U(\hat{r}) = \frac{-\hbar^2}{2m}\nabla^2 + U(\vec{r}) \tag{11.73}$$

The Energy operator

$$i\hbar\frac{\partial\psi}{\partial t} = i\hbar\frac{\partial}{\partial t}Ae^{i(kx-\omega t)} = i\hbar(-i\omega)\psi = \hbar\omega\psi = E\psi \implies \hat{E} = i\hbar\frac{\partial}{\partial t} \tag{11.74}$$

$$\langle E\rangle = \int_{-\infty}^{\infty}\psi^*\hat{E}\psi dx = i\hbar\int_{-\infty}^{\infty}\psi^*\frac{\partial\psi}{\partial x}dx \tag{11.75}$$

The Schrödingr Equation (non-relativistic) for a particle moving in a 3-dimensional potential energy field $U(\vec{r})$ is

$$\hat{E}\psi = \hat{H}\psi \tag{11.76}$$

$$\hat{E}\psi(\vec{r},t) = \frac{1}{2m}\hat{p}^2\psi(\vec{r},t) + U(\vec{r})\psi(\vec{r},t) \tag{11.77}$$

$$\hat{E}\psi = \frac{-\hbar}{2m}\nabla^2\psi + U(\vec{r})\psi \tag{11.78}$$

The probability of a particle being between x_1 and x_2 is

$$P_{x\in x_1:x_2}(t) = \int_{x_1}^{x_2}|\psi(x,t)|^2dx = \int_{x_1}^{x_2}\psi^*(x,t)\psi(x,t)dx \tag{11.79}$$

The normalization of a wave function

$$P_{x\in x_1:x_2}(t) = 1 \tag{11.80}$$

The cubit is defined as

$$|\psi\rangle = c_1|1\rangle + c_0|0\rangle \tag{11.81}$$

$$|\psi\rangle = c_{11}|11\rangle + c_{01}|01\rangle + c_{10}|10\rangle + c_{00}|00\rangle \tag{11.82}$$

$$\vdots \tag{11.83}$$

The Dirac equation: the generalization of the time dependent Schrödinger equation for the relativistically correct relationship between energy and momentum. It leads to negative energy states and antiparticles.

$$\left[\gamma^0mc^2 + \sum_{i=1}^{3}\gamma^i\hat{p}_ic\right]\psi(\vec{r},t) = i\hbar\frac{\partial}{\partial t}\psi(\vec{r},t) \tag{11.84}$$

Atomic quantum numbers

$$n = \text{ Principle Quantum Number } [n\in\mathbb{N}] \tag{11.85}$$

$$\ell = \text{ Orbital Angular Momentum Quantum Number } [\ell\in\mathbb{N}\cup\{0\}] \tag{11.86}$$

$$m_\ell = \text{ Magnetic Quantum Number } m_\ell \in [(-\ell,\ell)] \tag{11.87}$$

The potential the electron moves in

$$U(\vec{r}) = \frac{-e^2}{(4\pi\epsilon_0 r)} \tag{11.88}$$

The angular momentum of an electron in the atom

$$L = mvr = \hbar\sqrt{\ell(\ell+1)} \tag{11.89}$$

$$L_z = m_\ell \hbar \tag{11.90}$$

An electron orbiting around a nucleus has magnetic moment $\vec{\mu}$

$$\vec{\mu} = IA\hat{n} = \frac{-e}{(w\pi r/v)}(\pi r^2)\hat{n} = \frac{-erv}{2}\hat{n} = \frac{-e}{2m}\vec{L} \tag{11.91}$$

$$\mu_z = \frac{-e}{2m}L_z = \frac{-e}{2m}m_l\hbar = -m_\ell \mu_B \tag{11.92}$$

$$\mu_B = \frac{e\hbar}{2m} = 5.788 \times 10^{-5} \text{ eV/T [The Bohr Magneton]} \tag{11.93}$$

In an external magnetic field, B, the magnetic dipole feels a torque $\vec{\tau}$ and has a potential energy U_B

$$\vec{\tau} = \vec{\mu} \times \vec{B} \tag{11.94}$$

$$U_B = -\vec{\mu} \cdot \vec{B} \tag{11.95}$$

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