35. CLEBSCH-GORDAN COEFFICIENTS, SPHERICAL HARMONICS, AND d FUNCTIONS

Note: A square-root sign is to be understood over *every* coefficient, *e.g.*, for $-8/15$ read $-\sqrt{8/15}$.

Notation:

J	J	...
M	M	...

m_1	m_2	
m_1	m_2	Coefficients
.	.	
.	.	
.	.	

1/2 × 1/2

	1	
+1/2	+1/2	1

	1	0	
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2

	1	
−1/2	−1/2	1

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$$

$$Y_2^0 = \sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

$$Y_2^1 = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\phi}$$

$$Y_2^2 = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{2i\phi}$$

2 × 1/2

	5/2	
+5/2		1

	5/2	3/2	
+3/2	+2	+1/2	1

	5/2	3/2	
+3/2	+1	+1/2	1

	5/2	3/2	
+3/2	0	+1/2	1

	5/2	3/2	
+3/2	−1	+1/2	1

	5/2	3/2	
+3/2	−2	+1/2	1

	1/5	4/5	5/2	3/2
+2	−1/2	4/5	−1/5	+1/2
+1	+1/2	4/5	−1/5	+1/2

	2/5	3/5	5/2	3/2
+1	−1/2	3/5	−2/5	+1/2
0	+1/2	3/5	−2/5	−1/2

	3/5	2/5	5/2	3/2
0	−1/2	2/5	−3/5	−1/2
−1	+1/2	2/5	−3/5	−3/2

	4/5	1/5	5/2
−1	−1/2	1/5	−4/5
−2	+1/2	1/5	−4/5

	5/2	
−2	−1/2	1

3/2 × 1/2

	2	
+2		1

	2	1	
+3/2	+1/2	+1	1

	2	1	
+3/2	−1/2	+1	1

	2	1	
+3/2	−3/2	+1	1

	2	1	
+1/2	+1/2	+1	1

	2	1	
+1/2	−1/2	+1	1

	2	1	
−1/2	−1/2	+1	1

	1/4	3/4	2	1
+3/2	−1/2	3/4	−1/4	2
+1/2	+1/2	3/4	−1/4	0

	1/2	1/2	2	1
+1/2	−1/2	1/2	−1/2	2
−1/2	+1/2	1/2	−1/2	−1

	3/4	1/4	2
−1/2	−1/2	1/4	−3/4
−3/2	+1/2	1/4	−3/4

	2	
−3/2	−1/2	1

2 × 1

	3	
+3		1

	3	2	
+2	+2	+2	1

	3	2	
+2	0	+1	1
+1	+1	+1	1

	1/3	2/3	3	2	1
+2	0	+1	1	+1	+1
+1	+1	+1	1	+1	+1

	1/3	2/3	3	2	1
+2	−1	+1	1	+1	+1
+1	0	+1	1	+1	+1
0	+1	+1	1	+1	+1

	1/15	1/3	3/5
+2	−1	+1	1
+1	0	+1	1
0	+1	+1	1

	8/15	1/6	−3/10
+2	−1	0	1
+1	0	0	1
0	+1	0	1

	2/5	−1/2	1/10
+2	−1	0	1
+1	0	0	1
0	+1	0	1

3/2 × 1

	5/2	
+5/2		1

	5/2	3/2	
+3/2	+1	+3/2	1

	5/2	3/2	
+3/2	0	+3/2	1

	5/2	3/2	
+3/2	−1	+3/2	1

	5/2	3/2	
+3/2	−2	+3/2	1

	2/5	3/5	5/2	3/2	1/2
+3/2	0	+1/2	1	+1/2	+1/2
+1/2	+1	+1/2	1	+1/2	+1/2

	1/10	2/5	1/2
+3/2	−1	+1/2	1
+1/2	0	+1/2	1
−1/2	+1	+1/2	1

	3/5	1/15	−1/3
+1/2	0	+1/2	1
−1/2	+1	+1/2	1

	3/10	−8/15	1/6
+1/2	−1	+1/2	1
−1/2	0	+1/2	1
−3/2	+1	+1/2	1

	5/2	3/2	1/2
−1/2	−1/2	−1/2	−1/2

	3/10	8/15	1/6
+1/2	−1	3/10	8/15	1/6
−1/2	0	3/5	−1/15	−1/3
−3/2	+1	1/10	−2/5	1/2

	5/2	3/2	
−1/2	−1	3/5	−1/5	−1/3
−3/2	0	1/5	−1/5	−1/3

	3/5	2/5	5/2
−1/2	−1	3/5	−2/5	1/2
−3/2	0	2/5	−3/5	−5/2

	5/2	
−3/2	−1	1

1 × 1

	2	
+2		1

	2	1	
+1	+1	+1	1

	2	1	
+1	0	+1	1
0	+1	+1	1

	1/2	1/2	2	1	0
+1	0	+1	1	+1	+1
0	+1	+1	1	+1	+1

	1/2	1/2	2	1	0
+1	−1	+1	1	+1	+1
0	0	+1	1	+1	+1
−1	+1	+1	1	+1	+1

	1/6	1/2	1/3	2	1
+1	−1	+1	1	+1	+1
0	0	+1	1	+1	+1
−1	+1	+1	1	+1	+1

	2/3	0	−1/3	2	1
+1	−1	0	1	+1	+1
0	0	0	1	+1	+1
−1	+1	0	1	+1	+1

	1/6	−1/2	1/3	2	1
+1	−1	+1	1	+1	+1
0	0	+1	1	+1	+1
−1	+1	+1	1	+1	+1

	1/2	1/2	2
0	−1	1	1	+1	+1
−1	0	1	1	+1	+1

	1/2	−1/2	2
0	−1	1	1	+1	+1
−1	0	1	1	+1	+1

	2	
−1	−1	1

	3	2	1
0	0	0	0

	1/5	1/2	3/10	3	2	1
+1	−1	0	1	0	0	0
0	0	0	1	0	0	0
−1	+1	0	1	0	0	0

	3/5	0	−2/5	3	2	1
+1	−1	0	1	0	0	0
0	0	0	1	0	0	0
−1	+1	0	1	0	0	0

	1/5	−1/2	3/10	3	2	1
+1	−1	0	1	0	0	0
0	0	0	1	0	0	0
−1	+1	0	1	0	0	0

	2/5	1/2	1/10	3	2
0	−1	−1	1	−1	−1
−1	0	−1	1	−1	−1
−2	+1	−1	1	−1	−1

	8/15	−1/6	−3/10	3	2
0	−1	−1	1	−1	−1
−1	0	−1	1	−1	−1
−2	+1	−1	1	−1	−1

	1/15	−1/3	3/5	3	2
0	−1	−1	1	−1	−1
−1	0	−1	1	−1	−1
−2	+1	−1	1	−1	−1

	2/3	1/3	3
−1	−1	2	1	−2	−2
−2	0	2	1	−2	−2

	1/3	−2/3	3
−1	−1	2	1	−2	−2
−2	0	2	1	−2	−2

	3
−2	−1	2	1	−2	−2

$$Y_\ell^{-m} = (-1)^m Y_\ell^{m*}$$

$$d^\ell_{m,0} = \sqrt{\frac{4\pi}{2\ell+1}}\, Y_\ell^m\, e^{-im\phi}$$

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 J M\rangle = (-1)^{J-j_1-j_2}\langle j_2 j_1 m_2 m_1 | j_2 j_1 J M\rangle$$