

ANTONIUS'

HANDBOOK

Useful Formulas, Constants, Units and Definitions

Volume I - Mathematical Mansion

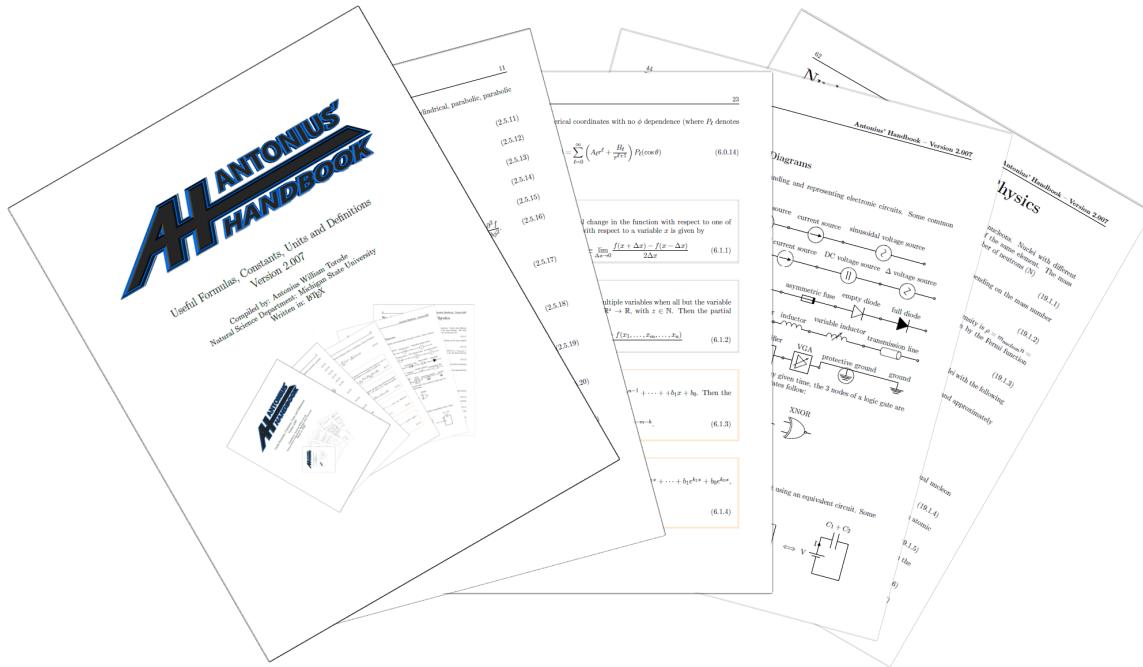
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Preface

This document is a compilation of useful formulas, definitions, constants, and general information used throughout my own schooling as a reference while furthering education. Its purpose is to provide a complete 'encyclopedia' per say of various mathematical and significant ideas used often. The idea and motivation behind it is to be a quick reference providing easily accessible access to necessary information for either double checking or recalling proper formula for use in various situations due to my own shortcomings in matters of memorization. All the material in this document was either directly copied from one of the references listed at the end or derived from scratch. On occasion *typos may exist* due to human error but will be corrected when discovered.

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Courses Covered In This Book

This document encompasses a large portion of formula used throughout specific courses at Michigan state University. The courses which have information pertaining to something in this book are more than just listed below; however, below is a list of classes that the author took whilst compiling the information in this book. All course numbers correspond to Michigan State University courses at the time of adding them.

- | | |
|--|--|
| <ul style="list-style-type: none">• AST 207/208/304/308: Astrophysics I-IV• PHY 215: Thermodynamics & Modern Physics• MTH 310/411: Abstract Algebra I/II• MTH 320: Analysis I• PHY 321: Classical Mechanics I• PHY 410: Thermal & Statistical Physics• PHY 415: Methods Of Theoretical Physics | <ul style="list-style-type: none">• PHY 440: Electronics• MTH 441: Partial Differential Equations• PHY 471/472: Quantum Physics I/II• PHY 481/482: Electricity and Magnetism I/II• PHY 491: Condensed Matter Physics• PHY 492: Introduction to Nuclear Physics• MTH 496: Mathematical Capstone |
|--|--|

The information in this book is in no way limited to the material used within the courses above. They serve as a simple guideline to what you will find within this document. For more information about this book or details about how to obtain your own copy please visit:

<https://msu.edu/~torodean/AHandbook.html>

Disclaimer

This book contains formulas, definitions, and theorems that by nature are very precise. Due to this, some of the material in this book was taken directly from other sources such as but not limited to Wolfram Mathworld. This is only such in cases where a change in wording could cause ambiguities or loss of information quality. Following this, all sources used are listed in the references section.

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Constants and units

1.0.1: Physical Constants

Constant	Symbol	Value	Units
Speed of light in a vacuum	$c \equiv 1/\sqrt{\mu_0\epsilon_0}$	2.99792458×10^8	m/s
Elementary charge	e	$1.602176565(35) \times 10^{-19}$	C
Gravitational constant	G	$6.67384(80) \times 10^{-11}$	$\text{m}^3\text{kg}^{-1}\text{s}^{-2}$
Avagadro's number	N_a	$6.02214129(27) \times 10^{23}$	$\text{mol}\cdot\text{s}^{-1}$
Planck constant	h	$6.62606872(52) \times 10^{-34}$ 4.135668×10^{-15}	J·s eV·s
	hc	1239.84	eV·nm
Reduced planck constant	$\hbar \equiv h/2\pi$	1.05×10^{-34}	J·s
Permittivity of the vacuum	ϵ_0	8.854×10^{-12}	$\text{C}^2\text{N}^{-1}\text{m}^{-2}$
Permeability of the vacuum	μ_0	$4\pi \times 10^{-7}$	N/A^2
Permeability of the vacuum	μ_0	$4\pi \times 10^{-7}$	N/A^2
Boltzmann constant	k_B	$1.38064852 \times 10^{-23}$ 8.61733×10^{-5}	J/K eV/K
Stefan-Boltzmann constant	$\sigma_B \equiv \frac{\pi^2 k_B^4}{60 h^3 c^3}$	$5.670367(13) \times 10^{-8}$	$\text{W}\cdot\text{m}^{-2}\text{K}^{-4}$
Thomson cross-section	σ_e	6.652×10^{-29}	m^2
The Bohr Magneton	$\mu_B \equiv \frac{e\hbar}{2m}$	5.788×10^{-5} 9.274×10^{-24}	eV/T Am ²
Mass of an electron	m_e	$9.10938291(40) \times 10^{-31}$ 510.9989	kg keV/c ²
Mass of a proton	m_p	$1.6726218 \times 10^{-27}$ 938.27203	kg MeV/c ²
Mass of a neutron	m_n	$1.6749274 \times 10^{-27}$ 939.56536	kg MeV/c ²
Unified amu	u	$1.660538782 \times 10^{-27}$ 931.494028	kg MeV/c ²

1.0.2: Stellar Data

Spectral Type	T_{eff} (K)	M/M_\odot	L/L_\odot	R/R_\odot	V_{mag}
O5	44,500	60	7.9×10^5	12	-5.7
B5	15,400	5.9	830	3.9	-1.2
A5	8,200	2.0	14	1.7	1.9
F5	6,440	1.4	3.2	1.3	3.4
G5	5,770	0.92	0.79	0.92	4.9
K5	4,350	0.67	0.15	0.72	6.7
M5	3,170	0.21	0.011	0.27	12.3

1.0.3: Astronomical Constants

Constant	Symbol	Value	Units
Mass of Earth	M_{\oplus}	5.974×10^{24}	kg
Mass of Sun	M_{\odot}	1.989×10^{30}	kg
Mass of Moon	$M_{\mathbb{M}}$	7.36×10^{22}	kg
Equatorial radius of Earth	R_{\oplus}	6.378×10^6	m
Equatorial radius of Sun	R_{\odot}	6.6955×10^8	m
Equatorial radius of Moon	$R_{\mathbb{M}}$	1.737×10^6	m
Mean density of Earth		5515	$\text{kg} \cdot \text{m}^{-3}$
Mean density of Sun		1408	$\text{kg} \cdot \text{m}^{-3}$
Mean density of Moon		3346	$\text{kg} \cdot \text{m}^{-3}$
Earth-Moon distance		3.84×10^8	m
Earth-Sun distance		1.496×10^{11}	m
Luminosity of Sun	L_{\odot}	3.839×10^{26}	W
Effective temp. of Sun		5778	K
Hubble constant	H_0	70 ± 5	$\text{km} \cdot \text{s}^{-1} \text{Mpc}^{-1}$
Parsec	pc	206264.81	AU
		3.0856776×10^{16}	m
		3.2615638	ly
Astronomical Unit	AU	1.496×10^{11}	m
Light year	ly	9.461×10^{15}	m
1 year on Earth	yr	365.25	days
		3.15576×10^7	s

1.0.4: Solar System

Planet	Symbol	Mass (kg)	Radius (m)	Sun-Distance (km)
Mercury	☿	3.285×10^{23}	2.44×10^6	5.791×10^{10}
Venus	♀	4.867×10^{24}	6.052×10^6	1.082×10^{11}
Mars	♂	6.39×10^{23}	3.390×10^6	2.279×10^{11}
Jupiter	♃	1.898×10^{27}	3.83×10^{11}	7.785×10^{11}
Saturn	♄	5.683×10^{26}	5.8232×10^7	1.429×10^{12}
Uranus	♅	8.681×10^{25}	2.5362×10^7	2.871×10^{12}
Neptune	♆	1.024×10^{26}	2.4622×10^7	4.498×10^{12}
Pluto	♇	1.309×10^{22}	1.187×10^6	5.906×10^{12}

1.0.5: Unit conversions

The International System of Units (SI) defines seven units of measure as a basic set from which all other SI units can be derived. These are [length](m), [time](s), [mass](kg), [electric current] \equiv [Ampere](A), [temperature](K), [luminous intensity](cd), [amount of substance](mol).

Unit Symbol	Unit	Equivalence
C	[Coulomb]	[Ampere][time]
N	[Newton]	[mass][length][time] $^{-2}$
P	[Pascal]	[mass][length] $^{-1}$ [time] $^{-2}$
J	[Joule]	[mass][length] 2 [time] $^{-2}$
W	[Watt]	[mass][length] 2 [time] $^{-3}$
		[Ohm][Ampere] 2
		[Volt] 2 [Ohm] $^{-1}$
V	[Volt]	[mass][length] 2 [time] $^{-3}$ [Ampere] $^{-1}$
Wb	[Weber]	[mass][length] 2 [time] $^{-2}$ [Ampere] $^{-1}$
T	[Tesla]	[mass][time] $^{-2}$ [Ampere] $^{-1}$
H	[henry]	[mass][length] 2 [time] $^{-2}$ [Ampere] $^{-2}$
Ω	[Ohm]	[mass][length] 2 [time] $^{-3}$ [Ampere] $^{-2}$
F	[Farad]	[mass] $^{-1}$ [length] $^{-2}$ [time] 4 [Ampere] 2
Hz	[Hertz]	[time] $^{-1}$

1.0.6: Number Sets ($i \equiv \sqrt{-1}$)

Symbol	Set	Symbol	Set
\mathbb{R}	Real numbers	\emptyset	{ }
$\mathbb{N} \equiv \mathbb{N}_1$	$\{1,2,3,4,\dots\}$	\mathbb{Z}	$\{\dots,-2,1,0,1,2,\dots\}$
$\mathbb{Z}^+ \equiv \mathbb{N}_0$	$\{0,1,2,3,\dots\}$	\mathbb{Z}^-	$\{0,-1,-2,-3,-4,\dots\}$
\mathbb{C}	$\{x + iy x, y \in \mathbb{R}\}$	\mathbb{Q}	$\{\frac{x}{y} x, y \in \mathbb{Z}\}$
\mathbb{I}	$\{ix x \in \mathbb{R}\}$	\mathbb{U}	Universal Set ^a
\mathbb{A}	Algebraic Numbers ^b	\mathbb{T}	Transcendental Numbers ^c

^aDefinition: The set containing all objects or elements and of which all other sets are subsets.

^bAny number that is a solution to a polynomial equation with rational coefficients.

^cAny number that is not an Algebraic Number.

1.0.7: Mathematical Notation

\forall	For all	\exists	There exists	\therefore	Because
\in	Is an element of	\notin	Is not an element of	\therefore	Therefore
\implies	Implies	\iff	Bi conditional	\approx	Approximately
\rightarrow	Mapped to	$\not\subseteq$	Is not a subset of	\ll	Much smaller than
\subset	Is a subset of	\subseteq	Is a subset or equal to	\gg	Much greater than
\propto	Is proportional to	\equiv	Is equivalent to	\cup/\cap	Union/Intersection
\perp	Is perpendicular to	\parallel	Is parallel to	$:$ or $ $	Such that

General Mathematics

Trigonometric definitions

$$\sin(x) = \frac{1}{2i}(e^{ix} - e^{-ix}) \quad (2.0.1)$$

$$\sinh(x) = \frac{1}{2}(e^x - e^{-x}) \quad (2.0.2)$$

$$= -i \sin(ix) \quad (2.0.3)$$

$$\cos(x) = \frac{1}{2}(e^{ix} + e^{-ix}) \quad (2.0.4)$$

$$\cosh(x) = \frac{1}{2}(e^x + e^{-x}) \quad (2.0.5)$$

$$= \cos(ix) \quad (2.0.6)$$

Curl Theorem: A special case of Stokes' theorem in which \vec{F} is a vector field and M is an oriented, compact embedded 2-manifold with boundary in \mathbb{R}^3 , and a generalization of Green's theorem from the plane into three-dimensional space. The curl theorem states

$$\int_S (\nabla \times \vec{F}) \cdot d\vec{a} = \int_{\partial S} \vec{F} \cdot d\vec{s} \quad (2.0.7)$$

The differential form of **Lagrange's identity** where $L(u) = \frac{d}{dx}(p \frac{du}{dx}) + qu$ and **Green's formula** which is the integral form of Lagrange's identity [10] are

$$uL(v) - vL(u) = \frac{d}{dx} \left[p \left(u \frac{dv}{dx} - v \frac{du}{dx} \right) \right] \implies \int_a^b [uL(v) - vL(u)] dx = p \left(u \frac{dv}{dx} - v \frac{du}{dx} \right) \Big|_a^b \quad (2.0.8)$$

The two dimensional version of **Green's formula** [10] is

$$\iint_S (v \nabla^2 u - u \nabla^2 v) dA = \oint_{\partial S} (v \nabla u - u \nabla v) \cdot \hat{n} ds \quad (2.0.9)$$

Green's theorem is a vector identity which is equivalent to the curl theorem

$$\iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy = \oint_{\partial S} P(x, y) dx + Q(x, y) dy \quad (2.0.10)$$

The **divergence theorem** is also known as Gauss's theorem (e.g., Arfken 1985) or the Gauss-Ostrogradsky theorem. Let V be a region in space with boundary ∂V . Then the volume integral of the divergence $\nabla \cdot \vec{F}$ of \vec{F} over V and the surface integral of \vec{F} over the boundary ∂V of V are related by

$$\int_V (\nabla \cdot \vec{F}) dV = \int_{\partial V} \vec{F} \cdot d\vec{a} \quad (2.0.11)$$

The **gradient theorem** (where the integral is a line integral) is

$$\int_a^b (\nabla f) \cdot d\vec{s} = f(b) - f(a) \quad (2.0.12)$$

The **Gamma function** Γ and the **Riemann zeta function** ζ are given by

$$\Gamma(z) \equiv \int_0^\infty t^{z-1} e^{-t} dt \quad (2.0.13)$$

$$\zeta(z) = \sum_{k=1}^{\infty} \frac{1}{k^z} \implies \zeta'(z) = - \sum_{k=1}^{\infty} \frac{\ln(k)}{k^z} \quad (2.0.14)$$

$$\zeta(z)\Gamma(z) = \int_0^\infty \frac{u^{z-1}}{e^u - 1} du \quad (2.0.15)$$

The most general case of the **binomial theorem** is the binomial series identity

$$(x + y)^n = \sum_{i=1}^n \binom{n}{k} x^{n-k} y^k \quad (2.0.16)$$

The **binomial coefficient** is defined as follows, with Pascals Formula implied.

$${}_n C_r \equiv \binom{n}{k} \equiv \frac{n!}{(n-k)!k!} \equiv \frac{\Gamma(n+1)}{\Gamma(k+1)\Gamma(n-k+1)} \quad (2.0.17)$$

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \quad (2.0.18)$$

The general formula for the power sum of the first n positive integers,

$$\sum_{k=1}^n k^p = \frac{1}{p+1} \sum_{i=1}^{p+1} (-1)^{\delta_{ip}} \binom{p+1}{i} B_{p+1-i} n^i, \quad (2.0.19)$$

where δ_{ip} is the Kronecker delta and B_i is the i th Bernoulli number. The Bernoulli numbers B_n are a sequence of signed rational numbers that can be defined by the exponential generating function

$$\frac{x}{e^x - 1} \equiv \sum_{n=0}^{\infty} \frac{B_n x^n}{n!}. \quad (2.0.20)$$

The simplest interpretation of the **Kronecker delta** is as the discrete version of the delta function defined by

$$\delta_{ij} \equiv \begin{cases} 0 & \text{for } i \neq j \\ 1 & \text{for } i = j. \end{cases} \quad (2.0.21)$$

Determining the sums for the first few terms of the power sum give

$$\sum_{i=0}^n i = \frac{1}{2} n(n+1) \quad (2.0.22)$$

$$\sum_{i=0}^n i^2 = \frac{1}{6}(2n^3 + 3n^2 + n) = \frac{1}{6}n(2n+1)(n+1) \quad (2.0.23)$$

$$\sum_{i=0}^n i^3 = \frac{1}{4}(n^4 + 2n^3 + n^2) = \frac{1}{4}n^2(n+1)^2 \quad (2.0.24)$$

$$\sum_{i=0}^n i^4 = \frac{1}{30}(6n^5 + 15n^4 + 10n^3 - n) \quad (2.0.25)$$

$$\sum_{i=0}^n i^5 = \frac{1}{12}(2n^6 + 6n^5 + 5n^4 - n^2). \quad (2.0.26)$$

A **Taylor series** is an expansion of a function about a point. A one-dimensional Taylor series of a real function $f(x)$ about the point $x=a$ is given by

$$f(x) = f(a) + f'(a)(x-a) + \frac{1}{2!}f''(a)(x-a)^2 + \frac{1}{3!}f'''(a)(x-a)^3 + \dots \quad (2.0.27)$$

Some common series expansions include:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \dots \quad (2.0.28)$$

$$\ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1} = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \dots \quad \text{with } |x| < 1 \quad (2.0.29)$$

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \quad (2.0.30)$$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots \quad (2.0.31)$$

$$\tan(x) = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \dots [|x| < \pi/2] \quad (2.0.32)$$

$$\sinh(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots \quad (2.0.33)$$

$$\cosh(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots \quad (2.0.34)$$

$$\tanh(x) = x - \frac{1}{3}x^3 + \frac{2}{15}x^5 - \dots [|x| < \pi/2] \quad (2.0.35)$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \dots [|x| < 1] \quad (2.0.36)$$

Dirac Delta Function: The delta function is a generalized function that can be defined as the limit of a class of delta sequences.

$$\delta(x) = \frac{1}{\pi} \lim_{\epsilon \rightarrow 0} \frac{\epsilon}{x^2 + \epsilon^2} = \frac{1}{2} \lim_{\epsilon \rightarrow 0} \epsilon|x|^{\epsilon-1} = \lim_{\epsilon \rightarrow 0} \frac{1}{\pi x} \sin\left(\frac{x}{\epsilon}\right) = \lim_{\epsilon \rightarrow 0^+} \frac{1}{2\sqrt{\pi\epsilon}} e^{-x^2/(4\epsilon)} \quad (2.0.37)$$

The Dirac delta can be thought of as a function on the real line which is zero everywhere except where the arguments of the function are zero, where it is infinite,

$$\delta(x) = \begin{cases} \infty & x = 0 \\ 0 & x \neq 0 \end{cases} \quad (2.0.38)$$

For any $\epsilon > 0$, the delta function has the fundamental property that

$$\int_{-\infty}^{\infty} f(x)\delta(x-a)dx = f(a) \quad \text{and} \quad \int_{x-\epsilon}^{x+\epsilon} f(x)\delta(x-a)dx = f(a) \quad (2.0.39)$$

The fundamental equation that defines derivatives of the delta function $\delta(x)$ is

$$\int f(x)\delta^{(n)}(x)dx \equiv - \int \frac{\partial f}{\partial x} \delta^{(n-1)}(x)dx \quad (2.0.40)$$

This implies

$$x^n \delta^{(n)}(x) = (-1)^n n! \delta(x) \quad (2.0.41)$$

A few identities and common expressions using the delta function are

$$\int_{-\infty}^{\infty} f(x)\delta(ax)dx = \frac{1}{|a|}f(0) \quad (2.0.42)$$

$$\int_{-1}^1 \delta\left(\frac{1}{x}\right)dx = 0 \quad (2.0.43)$$

2.1 Coordinate Systems

Cylindrical coordinates

$$r = \sqrt{x^2 + y^2} \quad (2.1.1)$$

$$x = r \cos(\theta) \quad (2.1.2)$$

$$y = r \sin(\theta) \quad (2.1.3)$$

$$z = z \quad (2.1.4)$$

$$dV = r \, dr \, d\theta \, dz \quad (2.1.5)$$

Spherical coordinates

$$r = \sqrt{x^2 + y^2 + z^2} \quad (2.1.6)$$

$$x = r \sin(\theta) \cos(\phi) \quad (2.1.7)$$

$$y = r \sin(\theta) \sin(\phi) \quad (2.1.8)$$

$$z = r \cos(\theta) \quad (2.1.9)$$

$$dV = r^2 \sin(\theta) \, dr \, d\theta \, d\phi \quad (2.1.10)$$

Polar Coordinates

$$r = \sqrt{x^2 + y^2} \quad (2.1.11)$$

$$x = r \cos(\theta) \quad (2.1.12)$$

$$y = r \sin(\theta) \quad (2.1.13)$$

$$dA = r \, dr \, d\theta \quad (2.1.14)$$

Elliptic cylindrical coordinates

$$x = a \cosh(u) \cos(v) \quad (2.1.15)$$

$$y = a \sinh(u) \sin(v) \quad (2.1.16)$$

$$z = z \quad (2.1.17)$$

$$dV = a^2 [\sinh^2(u) + \cosh^2(v)] du \, dv \, dz \quad (2.1.18)$$

2.2 Vector Operations

For any vector $\vec{r} = (r_1, r_2, \dots, r_n)$ in n -dimensions, the magnitude and unit vector is

$$|\vec{r}| \equiv \sqrt{\vec{r} \cdot \vec{r}} = \sqrt{r_1^2 + r_2^2 + \dots + r_n^2} \quad \hat{r} \equiv \frac{\vec{r}}{|\vec{r}|} \quad (2.2.1)$$

Dot and cross products for 3-dimensional vectors, where θ is the smallest angle between them, $\vec{r} = (r_x, r_y, r_z)$ and $\vec{s} = (s_x, s_y, s_z)$

$$\vec{r} \cdot \vec{s} = |\vec{r}| |\vec{s}| \cos(\theta) = r_x s_x + r_y s_y + r_z s_z \quad (2.2.2)$$

$$\vec{r} \times \vec{s} = |\vec{r}| |\vec{s}| \sin(\theta) = (r_y s_z - r_z s_y, r_z s_x - r_x s_z, r_x s_y - r_y s_x) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ r_x & r_y & r_z \\ s_x & s_y & s_z \end{vmatrix} \quad (2.2.3)$$

Triple product vector identities

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{B} \cdot (\vec{C} \times \vec{A}) = \vec{C} \cdot (\vec{A} \times \vec{B}) = -\vec{B} \cdot (\vec{A} \times \vec{C}) = -\vec{C} \cdot (\vec{B} \times \vec{A}) = -\vec{A} \cdot (\vec{C} \times \vec{B}) \quad (2.2.4)$$

$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B}) \quad (2.2.5)$$

Product rule vector identities

$$\nabla(fg) = f(\nabla g) + g(\nabla f) \quad (2.2.6)$$

$$\nabla(\vec{A} \cdot \vec{B}) = \vec{A} \times (\nabla \times \vec{B}) + \vec{B} \times (\nabla \times \vec{A}) + (\vec{A} \cdot \nabla) \vec{B} + (\vec{B} \cdot \nabla) \vec{A} \quad (2.2.7)$$

$$\nabla \cdot (f\vec{A}) = f(\nabla \cdot \vec{A}) + \vec{A} \cdot (\nabla f) \quad (2.2.8)$$

$$\nabla \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\nabla \times \vec{A}) - \vec{A} \cdot (\nabla \times \vec{B}) \quad (2.2.9)$$

$$\nabla \times (f\vec{A}) = f(\nabla \times \vec{A}) - \vec{A} \times (\nabla f) \quad (2.2.10)$$

$$\nabla \times (\vec{A} \times \vec{B}) = (\vec{B} \cdot \nabla) \vec{A} - (\vec{A} \cdot \nabla) \vec{B} + \vec{A}(\nabla \cdot \vec{B}) - \vec{B}(\nabla \cdot \vec{A}) \quad (2.2.11)$$

Second derivative vector identities

$$\nabla \cdot (\nabla \times \vec{A}) = 0 \quad \nabla \times (\nabla f) = 0 \quad \nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A} \quad (2.2.12)$$

A few other useful identities include

$$\nabla \cdot \frac{\hat{r}}{r^2} = 4\pi\delta^2(\vec{r}) \quad \nabla \times \frac{\hat{r}}{r^2} = \frac{\vec{r}}{r^3} = 0 \quad \nabla \cdot \frac{\hat{r}}{r} = \frac{1}{r^2} \quad (2.2.13)$$

2.3 Triangles

Let a triangle have side lengths a , b , and c with opposite angles A , B , and C .

The area of a triangle can be given by

$$A = \sqrt{s(s-a)(s-b)(s-c)} \quad (2.3.1)$$

$$s = (a+b+c)/2 \quad (2.3.2)$$

Law of Cosines:

$$c^2 = a^2 + b^2 - 2ab \cos(C) \quad (2.3.3)$$

Law of Sines:

$$\frac{\sin(A)}{a} = \frac{\sin(B)}{b} = \frac{\sin(C)}{c} \quad (2.3.4)$$

Law of tangents:

$$\frac{a-b}{a+b} = \frac{\tan((A-B)/2)}{\tan((A+B)/2)} \quad (2.3.5)$$

Mollweide's Formulas:

$$\frac{b-c}{a} = \frac{\sin[(B-C)/2]}{\cos(A/2)} \quad (2.3.6)$$

$$\frac{c-a}{b} = \frac{\sin[(C-A)/2]}{\cos(B/2)} \quad (2.3.7)$$

$$\frac{a-b}{c} = \frac{\sin[(A_B)/2]}{\cos(C/2)} \quad (2.3.8)$$

2.4 Trigonometric Identities

Pythagorean identities:

$$1 = \sin^2(\theta) + \cos^2(\theta) \quad (2.4.1)$$

$$1 = \sec^2(\theta) - \tan^2(\theta) \quad (2.4.2)$$

$$1 = \csc^2(\theta) - \cot^2(\theta) \quad (2.4.3)$$

$$1 = \cosh^2(\theta) - \sinh^2(\theta) \quad (2.4.4)$$

$$1 = \operatorname{sech}^2(\theta) + \tanh^2(\theta) \quad (2.4.5)$$

Sum-Difference Formulas:

$$\sin(\theta \pm \phi) = \sin(\theta)\cos(\phi) \pm \cos(\theta)\sin(\phi) \quad (2.4.6)$$

$$\cos(\theta \pm \phi) = \cos(\theta)\cos(\phi) \mp \sin(\theta)\sin(\phi) \quad (2.4.7)$$

$$\tan(\theta \pm \phi) = \frac{\tan(\theta) \pm \tan(\phi)}{1 \mp \tan(\theta)\tan(\phi)} \quad (2.4.8)$$

Double Angle formulas:

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta) \quad (2.4.9)$$

$$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) \quad (2.4.10)$$

$$= 2\cos^2(\theta) - 1 \quad (2.4.11)$$

$$= 1 - 2\sin^2(\theta) \quad (2.4.12)$$

$$\tan(2\theta) = \frac{2\tan(\theta)}{1 - \tan^2(\theta)} \quad (2.4.13)$$

Power-Reducing/Half Angle Formulas:

$$\sin^2(\theta) = \frac{1 - \cos(2\theta)}{2} \quad (2.4.14)$$

$$\cos^2(\theta) = \frac{1 + \cos(2\theta)}{2} \quad (2.4.15)$$

$$\tan^2(\theta) = \frac{1 - \cos(2\theta)}{1 + \cos(2\theta)} \quad (2.4.16)$$

Phase changes follow:

$$\sin(-\theta) = -\sin(\theta) \quad (2.4.17)$$

$$\cos(-\theta) = \cos(\theta) \quad (2.4.18)$$

$$\sin(\theta \pm \pi/2) = \pm \cos(\theta) \quad (2.4.19)$$

$$\sin(\theta \pm \pi) = -\sin(\theta) \quad (2.4.20)$$

$$\cos(\theta \pm \pi/2) = \mp \sin(\theta) \quad (2.4.21)$$

$$\cos(\theta \pm \pi) = -\cos(\theta) \quad (2.4.22)$$

Half-angle formulas

$$\sin\left(\frac{\theta}{2}\right) = (-1)^{\theta/(2\pi)} \sqrt{\frac{1 - \cos(\theta)}{2}} \quad (2.4.23)$$

$$\cos\left(\frac{\theta}{2}\right) = (-1)^{(\theta+\pi)/(2\pi)} \sqrt{\frac{1 + \cos(\theta)}{2}} \quad (2.4.24)$$

The **Weierstrass substitution** makes use of the half-angle formulas

$$\cos(\theta) = \frac{1 - \tan^2(\theta/2)}{1 + \tan^2(\theta/2)} \quad (2.4.25)$$

$$\sin(\theta) = \frac{2\tan(\theta/2)}{1 + \tan^2(\theta/2)} \quad (2.4.26)$$

Other relations and identities:

$$\cos(x)\cos(y) = \frac{1}{2}[\cos(x-y) + \cos(x+y)] \quad (2.4.27)$$

$$\sin(x)\sin(y) = \frac{1}{2}[\cos(x-y) - \cos(x+y)] \quad (2.4.28)$$

The half angle identity for tangent.

$$\tan\left(\frac{\theta}{2}\right) = (-1)^{x/\pi} \sqrt{\frac{1 - \cos(\theta)}{1 + \cos(\theta)}} = \frac{\sin(\theta)}{1 + \cos(\theta)} = \frac{1 - \cos(\theta)}{\sin(\theta)} = \frac{\tan(\theta)\sin(\theta)}{\tan(\theta) + \sin(\theta)} \quad (2.4.29)$$

Multiple-angle formulas are given by

$$\sin(nx) = \sum_{k=0}^n \binom{n}{k} \cos^k(x) \sin^{n-k}(x) \sin((n-k)\pi/2) \quad (2.4.30)$$

$$\cos(nx) = \sum_{k=0}^n \binom{n}{k} \cos^k(x) \sin^{n-k}(x) \cos((n-k)\pi/2) \quad (2.4.31)$$

Other identities

$$\cos(\theta)\cos(\phi) = \frac{1}{2}[\cos(\theta + \phi) + \cos(\theta - \phi)] \quad (2.4.32)$$

$$\sin(\theta)\sin(\phi) = \frac{1}{2}[\cos(\theta - \phi) - \cos(\theta + \phi)] \quad (2.4.33)$$

$$\sin(\theta)\cos(\phi) = \frac{1}{2}[\sin(\theta + \phi) + \sin(\theta - \phi)] \quad (2.4.34)$$

$$\cos(\theta) + \cos(\phi) = 2 \cos\left(\frac{\theta + \phi}{2}\right) \cos\left(\frac{\theta - \phi}{2}\right) \quad (2.4.35)$$

$$\cos(\theta) - \cos(\phi) = 2 \sin\left(\frac{\theta + \phi}{2}\right) \sin\left(\frac{\theta - \phi}{2}\right) \quad (2.4.36)$$

2.5 Arbitrary Orthogonal Curvilinear Coordinates

A coordinate system composed of intersecting surfaces. If the intersections are all at right angles, then the curvilinear coordinates are said to form an orthogonal coordinate system. The scale factors are h_i ,

$$\vec{a}_i \equiv \frac{\partial \vec{r}}{\partial e_i} = \frac{\partial x}{\partial e_i} \hat{x} + \frac{\partial y}{\partial e_i} \hat{y} + \frac{\partial z}{\partial e_i} \hat{z} = h_i \hat{e}_i = |\vec{a}_i| \hat{e}_i \quad (2.5.1)$$

$$h_i \equiv \left| \frac{\partial \vec{r}}{\partial e_i} \right| = |\vec{a}_i| = \sqrt{\frac{\partial x}{\partial e_i}^2 + \frac{\partial y}{\partial e_i}^2 + \frac{\partial z}{\partial e_i}^2} \quad (2.5.2)$$

$$\hat{e}_i = \frac{1}{h_i} \frac{\partial \vec{r}}{\partial e_i} = \frac{\vec{a}_i}{|\vec{a}_i|} \quad (2.5.3)$$

The line element $d\vec{s}$ is determined by

$$d\vec{s} \equiv d\vec{x} + d\vec{y} + d\vec{z} \equiv \vec{a}_1 de_1 + \vec{a}_2 de_2 + \vec{a}_3 de_3 \quad (2.5.4)$$

From this, ds^2 is given by

$$ds^2 = d\vec{s} \cdot d\vec{s} = dx^2 + dy^2 + dz^2 = h_1^2 de_1^2 + h_2^2 de_2^2 + h_3^2 de_3^2 \quad (2.5.5)$$

The differential vector and volume elements are therefore

$$d\vec{r} = h_1 du_1 \hat{u}_1 + h_2 du_2 \hat{u}_2 + h_3 du_3 \hat{u}_3 \quad (2.5.6)$$

$$dV = h_1 h_2 h_3 du_1 du_2 du_3 = \left| \frac{\partial(x, y, z)}{\partial(u_1, u_2, u_3)} \right| du_1 du_2 du_3 \quad (2.5.7)$$

The gradient in arbitrary curvilinear coordinates such that the gradient theorem is preserved:

$$\nabla f = \frac{1}{h_1} \frac{\partial f}{\partial x_1} \hat{x}_1 + \frac{1}{h_2} \frac{\partial f}{\partial x_2} \hat{x}_2 + \frac{1}{h_3} \frac{\partial f}{\partial x_3} \hat{x}_3 \quad (2.5.8)$$

The divergence in arbitrary curvilinear coordinates such that the divergence theorem is preserved:

$$\nabla \cdot \vec{v} = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial v_1}{\partial x_1} h_2 h_3 + \frac{\partial v_2}{\partial x_2} h_1 h_3 + \frac{\partial v_3}{\partial x_3} h_1 h_2 \right] \quad (2.5.9)$$

The Laplacian for a scalar function ϕ (where the h_i are the scale factors of the coordinate system - Weinberg 1972, p. 109; Arfken 1985, p. 92 [24]) is a scalar differential operator defined by

$$\nabla^2 \phi = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial u_1} \left(\frac{h_2 h_3}{h_1} \frac{\partial}{\partial u_1} \right) + \frac{\partial}{\partial u_2} \left(\frac{h_1 h_3}{h_2} \frac{\partial}{\partial u_2} \right) + \frac{\partial}{\partial u_3} \left(\frac{h_1 h_2}{h_3} \frac{\partial}{\partial u_3} \right) \right] \phi \quad (2.5.10)$$

The form of the Laplacian in several common coordinate systems (cartesian, cylindrical, parabolic, parabolic cylindrical, spherical and oblate spheroidal respectively) are

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \quad (2.5.11)$$

$$\nabla^2 f = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial f}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2} \quad (2.5.12)$$

$$\nabla^2 f = \frac{1}{uv(u^2 + v^2)} \left[\frac{\partial}{\partial u} \left(uv \frac{\partial f}{\partial u} \right) + \frac{\partial}{\partial v} \left(uv \frac{\partial f}{\partial v} \right) \right] + \frac{1}{v^2 u^2} \frac{\partial^2 f}{\partial \theta^2} \quad (2.5.13)$$

$$\nabla^2 f = \frac{1}{u^2 + v^2} \left(\frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2} \right) + \frac{\partial^2 f}{\partial z^2} \quad (2.5.14)$$

$$\nabla^2 f = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial f}{\partial r} \right) + \frac{1}{r^2 \sin^2 \phi} \frac{\partial^2 f}{\partial \theta^2} + \frac{1}{r^2 \sin \phi} \frac{\partial}{\partial \phi} \left(\sin \phi \frac{\partial f}{\partial \phi} \right) \quad (2.5.15)$$

$$\nabla^2 f = \frac{1}{a^2(\zeta^2 + \xi^2)} \left[\frac{\partial}{\partial \zeta} \left((1 + \zeta^2) \frac{\partial f}{\partial \zeta} \right) + \frac{\partial}{\partial \xi} \left((1 - \xi^2) \frac{\partial f}{\partial \xi} \right) \right] + \frac{1}{a^2(1 + \zeta^2)(1 - \xi^2)} \frac{\partial^2 f}{\partial \phi^2}. \quad (2.5.16)$$

The **curl** can be similarly defined in arbitrary orthogonal curvilinear coordinates as

$$\nabla \times \vec{F} \equiv \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \hat{e}_1 & h_2 \hat{e}_2 & h_3 \hat{e}_3 \\ \frac{\partial}{\partial e_1} & \frac{\partial}{\partial e_2} & \frac{\partial}{\partial e_3} \\ h_1 F_1 & h_2 F_2 & h_3 F_3 \end{vmatrix} \quad (2.5.17)$$

$$= \frac{1}{h_2 h_3} \left[\frac{\partial}{\partial u_2} (h_3 F_3) - \frac{\partial}{\partial u_3} (h_2 F_2) \right] \hat{u}_1 + \frac{1}{h_1 h_3} \left[\frac{\partial}{\partial u_3} (h_1 F_1) - \frac{\partial}{\partial u_1} (h_3 F_3) \right] \hat{u}_2 + \frac{1}{h_1 h_2} \left[\frac{\partial}{\partial u_1} (h_2 F_2) - \frac{\partial}{\partial u_2} (h_1 F_1) \right] \hat{u}_3. \quad (2.5.18)$$

The **Jacobian** is defined as the determinant of a matrix of partial derivatives [5],

$$\frac{\partial(a, b)}{\partial(c, d)} \equiv \begin{vmatrix} \left(\frac{\partial a}{\partial c} \right)_d & \left(\frac{\partial a}{\partial d} \right)_c \\ \left(\frac{\partial b}{\partial c} \right)_d & \left(\frac{\partial b}{\partial d} \right)_c \end{vmatrix} = \left(\frac{\partial a}{\partial c} \right)_d \left(\frac{\partial b}{\partial d} \right)_c - \left(\frac{\partial a}{\partial d} \right)_c \left(\frac{\partial b}{\partial c} \right)_d. \quad (2.5.19)$$

By the above definition, we can show the relations,

$$\frac{\partial(b, a)}{\partial(c, d)} = -\frac{\partial(a, b)}{\partial(c, d)} \quad \text{and} \quad \frac{\partial(a, b)}{\partial(c, d)} = -\frac{\partial(a, b)}{\partial(d, c)}. \quad (2.5.20)$$

It then follows directly that

$$\frac{\partial(a, s)}{\partial(c, s)} = \left(\frac{\partial a}{\partial c} \right)_s \quad \text{and} \quad \frac{\partial(a, b)}{\partial(a, b)} = 1 \quad \text{and} \quad \frac{\partial(a, b)}{\partial(c, d)} \frac{\partial(c, d)}{\partial(s, t)} = \frac{\partial(a, b)}{\partial(s, t)}. \quad (2.5.21)$$

Complex Analysis

3.1 Complex Numbers

The set of complex numbers is defined such that

$$\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}. \quad (3.1.1)$$

A complex number can be defined by its real part and it's imaginary part

$$i^2 = -1 \iff i = \sqrt{-1} \iff \frac{1}{i} = -i \quad (3.1.2)$$

$$z = x + iy \iff z^* = x - iy \quad (3.1.3)$$

We can express the real and imaginary parts of a complex number in terms of the number and its complex conjugate

$$\Re(z) = \frac{1}{2}(z + z^*) \quad (3.1.4)$$

$$\Im(z) = \frac{1}{2}i(z - z^*) \quad (3.1.5)$$

Just like a two-dimensional vector, a complex number has the magnitude $|z|$ as well as an angle θ with respect to the horizontal axis of the complex plane.

$$|z|^2 = z^*z = x^2 + y^2 = |z|e^{-i\theta}|z|e^{i\theta} \quad (3.1.6)$$

$$\tan(\theta) = \frac{\Im(z)}{\Re(z)} = \frac{y}{x} = \frac{i(z - z^*)}{(z + z^*)} \quad (3.1.7)$$

A complex number can thus be expressed in terms of magnitude and the phase angle

$$z = |z|(\cos(\theta) + i \sin(\theta)) \quad (3.1.8)$$

Euler's Identity/relation

$$e^{i\theta} = \cos(\theta) + i \sin(\theta) \quad (3.1.9)$$

With the aid of Eulers identity, we can write any complex number as

$$z = |z|e^{i\theta} \quad (3.1.10)$$

$$z^n = |z|^n e^{in\theta} \quad (3.1.11)$$

A useful property of conjugates is

$$a^* + b^* = (a + b)^* \quad (3.1.12)$$

Powers and roots of a complex number can be determined from the exponential form of a complex number

$$z^n = (re^{i\theta})^n = r^n e^{in\theta} \quad (3.1.13)$$

$$(e^{i\theta})^n = e^{in\theta} = (\cos \theta + i \sin \theta)^n = \cos(n\theta) + i \sin(n\theta) \quad (3.1.14)$$

$$z^{1/n} = (re^{i\theta})^{1/n} = r^{1/n} e^{i\theta/n} = \sqrt[n]{r} \left(\cos \frac{\theta}{n} + i \sin \frac{\theta}{n} \right) \quad (3.1.15)$$

Much like in trigonometry, we can define complex numbers using trigonometric identities:

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \quad \cos z = \frac{e^{iz} + e^{-iz}}{2}, \quad \sinh z = \frac{e^z - e^{-z}}{2}, \quad \cosh z = \frac{e^z + e^{-z}}{2} \quad (3.1.16)$$

The logarithm of a complex number can be manipulated as a normal log with

$$\ln(z) = \ln(re^{i\theta}) = \ln(r) + \ln(e^{i\theta}) = \ln(r) + i\theta \quad (3.1.17)$$

A few Trigonometric identities follow as:

$$\arcsin z = -i \ln(iz \pm \sqrt{1 - z^2}) \quad (3.1.18)$$

$$\arccos z = i \ln(z \pm \sqrt{z^2 - 1}) \quad (3.1.19)$$

$$\arctan z = \frac{1}{2i} \ln \left(\frac{1 + iz}{1 - iz} \right) \quad (3.1.20)$$

3.2 Complex Functions

A complex function of z can be expressed in terms of two real functions $u(x, y)$ and $v(x, y)$,

$$f(z) = f(x + iy) = u(x, y) + iv(x, y). \quad (3.2.1)$$

The derivative of $f(z)$ is defined by

$$f'(z) = \frac{df}{dz} = \lim_{\Delta z \rightarrow 0} \frac{\Delta f}{\Delta z} = \lim_{\Delta z \rightarrow 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} = \lim_{\Delta x, \Delta y \rightarrow 0} \frac{f(z + \Delta x + i\Delta y) - f(z)}{\Delta x + i\Delta y}. \quad (3.2.2)$$

Definition 3.2.1: Analytic Function [2]

A function $f(z)$ is **analytic** (or regular or holomorphic or mono-genic) in a region of the complex plane if it has a (unique) derivative at every point of the region. The statement $f(z)$ is analytic at a point $z = a$ means that $f(z)$ has a derivative at every point inside some small circle about $z = a$.

The **Cauchy-Riemann conditions** state that if $f(z) = u(x, y) + iv(x, y)$ is analytic in a region, then in that region

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}. \quad (3.2.3)$$

From this we also have

$$\frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}, \quad \text{and} \quad \frac{\partial f}{\partial y} = \frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}. \quad (3.2.4)$$

Definition 3.2.2: Regular and Singular Points [2]

A **regular point** of $f(z)$ is a point at which $f(z)$ is analytic. A **singular point** or singularity of $f(z)$ is a point at which $f(z)$ is not analytic. It is called an isolated singular point if $f(z)$ is analytic everywhere else inside some small circle about the singular point.

A few useful theorems [2] include

1. If $f(z)$ is analytic in a region, then it has derivatives of all orders at points inside the region and can be expanded in a Taylor series about any point z_0 inside the region. The power series converges inside the circle about z_0 that extends to the nearest singular point.
2. If $f(z) = u + iv$ is analytic in a region, then u and v satisfy Laplace's equation in the region (that is, u and v are harmonic functions).
3. Any function u (or v) satisfying Laplace's equation in a simply-connected region, is the real or imaginary part of an analytic function $f(z)$.

Given a complex function $f(z) = u(x, y) + iv(x, y)$, we can take the integral along a path ℓ using

$$\int_{\ell} f(z) dz = \int_{\ell} f(x + iy)(dx + idy) = \int_{\ell} u(x, y) dx - v(x, y) dy + i \int_{\ell} u(x, y) dy + v(x, y) dx. \quad (3.2.5)$$

3.2.1: Cauchy's Theorem [2]

Let C be a simple closed curve (one which does not cross itself) with a continuously turning tangent except possibly at a finite number of points (that is, we allow a finite number of corners, but otherwise the curve must be smooth). If $f(z)$ is analytic on and inside C , then

$$\oint_C f(z) dz = 0 \quad (3.2.6)$$

3.2.2: Cauchy's Integral Formula [2]

If $f(z)$ is analytic on and inside a simple closed curve C , the value of $f(z)$ at a point $z = a$ inside C is given by the following contour integral along C :

$$f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - a} dz \quad (3.2.7)$$

3.2.3: Laurent's Theorem [2], [24]

Let C_1 and C_2 be two circles with center at $z = z_0$ with radii r_1 and $r_2 < r_1$ respectively. Let $f(z)$ be analytic in the region R between the circles. Then $f(z)$ can be expanded in a series of the form

$$f(z) = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots + \frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + \cdots \quad (3.2.8)$$

$$= \sum_{k=0}^{\infty} a_k (z - z_0)^k + \sum_{k=1}^{\infty} b_k (z - z_0)^{-k}, \quad (3.2.9)$$

convergent in R . Such a series is called a Laurent series. The b series is called the principal part of the Laurent series. The coefficients have the solutions

$$a_k = \frac{1}{2\pi i} \oint_{C_1} \frac{f(z') dz'}{(z' - z_0)^{k+1}}, \quad \text{and} \quad b_k = \frac{1}{2\pi i} \oint_{C_2} (z' - z_0)^{k-1} f(z') dz' \quad (3.2.10)$$

Definition 3.2.3: Poles, Residue and Singularities [2]

1. If all the b 's are zero, $f(z)$ is analytic at $z = z_0$, and we call z_0 a **regular point**.
2. If $b_n \neq 0$, but all the b 's after b_n are zero, $f(z)$ is said to have a **pole** of order n at $z = z_0$. If $n = 1$, we say that $f(z)$ has a simple pole.
3. If there are an infinite number of b 's different from zero, $f(z)$ has an **essential singularity** at $z = z_0$.
4. The coefficient b_1 of $1/(z - z_0)$ is called the **residue** of $f(z)$ at $z = z_0$.

Matrix Algebra

The product C of two matrices A and B is defined (where j is summed over for all possible values of i and k) as (using the Einstein summation convention)

$$c_{ik} = a_{ij}b_{jk} = \sum_{j=1}^m a_{ij}b_{jk} \quad (4.0.1)$$

In order for matrix multiplication to be defined, the dimensions of the matrices must satisfy

$$(n \times m)(m \times p) = (n \times p) \quad (4.0.2)$$

where $(a \times b)$ denotes a matrix with a rows and b columns. Writing out the product explicitly,

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{np} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1p} \\ c_{21} & c_{22} & \cdots & c_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{np} \end{bmatrix} \quad (4.0.3)$$

where,

$$c_{11} = a_{11}b_{11} + a_{12}b_{21} + \cdots + a_{1m}b_{m1} \quad (4.0.4)$$

$$c_{12} = a_{11}b_{12} + a_{12}b_{22} + \cdots + a_{1m}b_{m2} \quad (4.0.5)$$

$$c_{1p} = a_{11}b_{1p} + a_{12}b_{2p} + \cdots + a_{1m}b_{mp} \quad (4.0.6)$$

$$c_{21} = a_{21}b_{11} + a_{22}b_{21} + \cdots + a_{2m}b_{m1} \quad (4.0.7)$$

$$c_{22} = a_{21}b_{12} + a_{22}b_{22} + \cdots + a_{2m}b_{m2} \quad (4.0.8)$$

$$c_{2p} = a_{21}b_{1p} + a_{22}b_{2p} + \cdots + a_{2m}b_{mp} \quad (4.0.9)$$

$$c_{n1} = a_{n1}b_{11} + a_{n2}b_{21} + \cdots + a_{nm}b_{m1} \quad (4.0.10)$$

$$c_{n2} = a_{n1}b_{12} + a_{n2}b_{22} + \cdots + a_{nm}b_{m2} \quad (4.0.11)$$

$$c_{np} = a_{n1}b_{1p} + a_{n2}b_{2p} + \cdots + a_{nm}b_{mp} \quad (4.0.12)$$

Matrix multiplication is also distributive. If A and B are $m \times n$ matrices and C and D are $n \times p$ matrices, then

$$A(C + D) = AC + AD \quad \text{and} \quad (A + B)C = AC + BC \quad (4.0.13)$$

The **trace** of an $n \times n$ square matrix A is defined to be

$$\text{Tr}(A) \equiv \sum_{i=1}^n a_{ii} \quad (4.0.14)$$

The determinant of an arbitrary 2×2 matrix A is given by

$$\det(M) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21} \quad (4.0.15)$$

The determinant of an arbitrary 3×3 matrix A is given by

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \quad (4.0.16)$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}) \quad (4.0.17)$$

The diagonal space of an $n \times n$ matrix A , denoted $\text{dis}(A)$ is defined as

$$\text{dis}(A) \equiv \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} = \frac{1}{2} \sum_{n=1}^n \sum_{k=1}^n (a_{kn}a_{nk} - a_{nn}a_{kk}) \quad (4.0.18)$$

For a 3×3 matrix A , the diagonal space can be helpful in finding eigenvalues and is given by

$$\text{dis}(A) \equiv \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \frac{1}{2} \sum_{n=1}^3 \sum_{k=1}^3 (a_{kn}a_{nk} - a_{nn}a_{kk}) \quad (4.0.19)$$

$$= (a_{12}a_{21} + a_{13}a_{31} + a_{23}a_{32}) - (a_{11}a_{22} + a_{11}a_{33} + a_{22}a_{33}) \quad (4.0.20)$$

The eigenvalues λ_i and eigenvectors \vec{v}_i of a matrix A are given by solving

$$\det(A - \lambda I) = 0 \quad \text{and} \quad A\vec{v}_i = \lambda_i \vec{v}_i \quad \text{or} \quad (A - \lambda_i I)\vec{v}_i = 0. \quad (4.0.21)$$

By defining the determinant, trace, and diagonal space the way we have above, the eigenvalues of a 3×3 matrix A become the solutions for λ of

$$0 = -\lambda^3 + \text{Tr}(A)\lambda^2 + \text{dis}(A)\lambda + \det(A) \quad (4.0.22)$$

Similarly, the eigenvalues λ_{\pm} of a 2×2 matrix A become the solutions for λ of

$$0 = \lambda^2 - \text{Tr}(A)\lambda + \det(A) \implies \lambda_{\pm} = \frac{1}{2} \left(\text{Tr}(A) \pm \sqrt{\text{Tr}(A)^2 - 4\det(A)} \right) \quad (4.0.23)$$

$$\implies \lambda_{\pm} = \frac{1}{2} \left((a_{11} + a_{22}) \pm \sqrt{(a_{11} - a_{22})^2 + 4a_{12}a_{21}} \right) \quad (4.0.24)$$

Properties of Matrices: A matrix has an inverse if and only if it has a non-zero determinant (it is non singular).

$$A^T = A \implies \text{Symmetric} \quad (4.0.25)$$

$$A^T = A^{-1} \implies \text{Anti-Symmetric} \quad (4.0.26)$$

$$A^* = A \implies \text{Real} \quad (4.0.27)$$

$$A^* = -A \implies \text{Imaginary} \quad (4.0.28)$$

$$A^\dagger = A \implies \text{Hermitian} \quad (4.0.29)$$

$$A^T A = A A^T = I \implies A^T = A^{-1} \implies \text{Orthogonal} \quad (4.0.30)$$

$$A^\dagger A = A A^\dagger \implies A^\dagger = A^{-1} \implies \text{Unitary} \quad (4.0.31)$$

$$\det(A) = 0 \implies \text{Singular} \quad (4.0.32)$$

Abstract Algebra and Number Theory

Definition 5.0.1: Ring

A **ring** is a triple (R, \oplus, \odot) such that

- (i) R is a set.
- (ii) \oplus is a function (called ring addition) and $R \times R$ is a subset of the domain of \oplus . For $(a, b) \in R \times R$, $a \oplus b$ denotes the image of (a, b) under \oplus .
- (iii) \odot is a function (called ring multiplication) and $R \times R$ is a subset of the domain of \odot . For $(a, b) \in R \times R$, $a \odot b$ (and also ab) denotes the image of (a, b) under \odot .

and such that the following eight statements (axioms) hold:

- (1) [Closure of addition]: $a + b \in R$ for all $a, b \in R$.
- (2) [Associative addition]: $a + (b + c) = (a + b) + c$ for all $a, b, c \in R$.
- (3) [Commutative addition]: $a + b = b + a$ for all $a, b \in R$.
- (4) [Additive identity]: There exists an element in R , denoted by 0_R and called 'zero R ', such that $a = a + 0_R = a$ and $a = 0_R + a$ for all $a \in R$.
- (5) [Additive inverses]: For each $a \in R$ there exists an element in R , denoted by $-a$ and called 'negative a ', such that $a + (-a) = 0_R$.
- (6) [Closure for multiplication]: $ab \in R$ for all $a, b \in R$.
- (7) [Associative multiplication]: $a(bc) = (ab)c$ for all $a, b, c \in R$.
- (8) [Distributive laws]: $a(b + c) = ab + ac$ and $(a + b)c = ac + bc$ for all $a, b, c \in R$.

Definition 5.0.2: Commutative Ring

Let R be a ring. Then R is called commutative if

- (9) [Commutative multiplication]: $ab = ba$ for all $a, b \in R$.

Definition 5.0.3: Ring With Identity

Let R be a ring. We say that R is a ring with identity if there exists an element, denoted by 1_R and called 'one R ', such that

- (10) [Multiplicative identity]: $a = 1_R \cdot a$ and $a = a \cdot 1_R$ for all $a \in R$.

Definition 5.0.4: Subring

Let (R, \oplus, \odot) be a ring and S a subset of R . Then (S, \oplus, \odot) is called a subring of (R, \oplus, \odot) provided that (S, \oplus, \odot) is a ring.

Theorem 5.0.1: Subring Theorem

Suppose that R is a ring and $S \subseteq R$. Then S is a subring of R if and only if the following four conditions hold:

- (i) $0_R \in S$.
- (ii) S is closed under addition (that is: if $a, b \in S$, then $a + b \in S$).
- (iii) S is closed under multiplication (that is: if $a, b \in S$, then $ab \in S$).
- (iv) S is closed under negatives (that is: if $a \in S$, then $-a \in S$).

Definition 5.0.5: Integral Domain

An ring R is called an integral domain provided that R is commutative, R has identity, $1_R \neq 0_R$ and for any $a, b \in R$, $ab = 0_R \implies a = 0_R$ or $b = 0_R$.

Definition 5.0.6: Injective & Surjective

Let $f : R \rightarrow S$ be a function.

- (a) f is said to be injective provided: $f(a) = f(b) \implies a = b$ for all $a, b \in R$.
- (b) f is said to be surjective provided: for every $y \in S$ there exists $x \in R$ such that $f(x) = y$.
- (c) f is bijective if it is both injective and surjective.

Definition 5.0.7: Equivalence Relation

Let \sim be a relation on a set A (that is a relation from A to A). Then

- (a) \sim is called reflexive if $a \sim a$ for all $a \in A$.
- (b) \sim is called symmetric if $[a \sim b \implies b \sim a]$ for all $a, b \in A$.
- (c) \sim is called transitive if $[a \sim b \text{ and } b \sim c \implies a \sim c]$ for all $a, b, c \in A$.
- (d) \sim is called an equivalence relation if \sim is reflexive, symmetric and transitive.

Definition 5.0.8: Unit

Let R be a ring with identity.

- (a) Let $u \in R$. Then u is called a unit in R if there exists an element in R , denoted by u^{-1} and called ‘ u -inverse’, with $uu^{-1} = 1_R = u^{-1}u$.
- (b) Let $u, v \in R$. Then v is called an (multiplicative) inverse of u if $uv = 1_R = vu$.
- (c) Let $e \in R$. Then e is called an (multiplicative) identity of R , if $ea = a = ae$ for all $a \in R$.

Definition 5.0.9: Common divisor

- (a) Let R be a ring and $a, b, c \in R$. We say that c is a common divisor of a and b in R provided that $c|a$ and $c|b$.
- (b) Let a, b and d be integers. We say that d is a greatest common divisor of a and b in \mathbb{Z} , and we write $d = \gcd(a, b)$ provided that
 - (i) d is a common divisor of a and b in \mathbb{Z} .
 - (ii) If c is a common divisor of a and b in \mathbb{Z} then $c \leq d$.

Definition 5.0.10: Isomorphism and Homomorphism

Let $(R, +, \cdot)$ and (S, \oplus, \odot) be rings and let $f : R \rightarrow S$ be a function.

- (a) f is called a homomorphism from $(R, +, \cdot)$ to (S, \oplus, \odot) if
 - (i) [f respects addition]: $f(a + b) = f(a) \oplus f(b)$, and
 - (ii) [f respects multiplication]: $f(a \cdot b) = f(a) \odot f(b)$
 for all $a, b \in R$.
- (b) f is called an isomorphism from $(R, +, \cdot)$ to (S, \oplus, \odot) , if f is a homomorphism from $(R, +, \cdot)$ to (S, \oplus, \odot) and f is bijective.
- (c) $(R, +, \cdot)$ is called isomorphic to (S, \oplus, \odot) , if there exists an isomorphism from $(R, +, \cdot)$ to (S, \oplus, \odot) .

Definition 5.0.11: Ideal

Let I be a subset of the ring R

- (a) We say that I absorbs R if $ra \in I$ and $ar \in I$ for all $a \in I, r \in R$.
- (b) We say that I is an ideal of R (denoted $I \triangleleft R$) if I is a subring of R and I absorbs R .

Theorem 5.0.2: Ideal Theorem

Let I be a subset of the ring R . Then I is an ideal in R ($I \triangleleft R$) if and only if the following four conditions hold:

- (i) $0_R \in I$.
- (ii) $a + b \in I$ for all $a, b \in I$.
- (iii) $ra \in I$ and $ar \in I$ for all $a \in I$ and $r \in R$.
- (iv) $-a \in I$ for all $a \in I$.

Definition 5.0.12: Principle Ideal

Let R be a ring.

- (a) Let $a \in R$. Then $aR = \{ar : a \in R\}$.
- (b) Suppose R is commutative and $I \subseteq R$. Then I is called a principal ideal in R if $I = aR$ for some $a \in R$. This can be denoted (a) .

Definition 5.0.13: Ideal modulus

Let I be an ideal in the ring R . The relation ' $\equiv (\text{mod } I)$ ' on R is defined by $a \equiv b \pmod{I}$ if $a - b \in I$.

Definition 5.0.14: Cosets

- (a) Let $a \in I$. Then $a + I$ (the coset of I in R containing a) denotes the equivalence class of ' $\equiv \text{mod } I$ ' containing a . so

$$a + I = \{b \in R | a \equiv b \pmod{I}\} = \{b \in R | a - b \in I\}. \quad (5.0.1)$$

- (b) R/I is the set of cosets if I in R/I and the set of equivalence classes of ' $\equiv \text{mod } I$ '. So

$$R/I = \{a + I | a \in R\}. \quad (5.0.2)$$

Definition 5.0.15: Ideal operations

Let I be an ideal in the ring R . Then we define an addition $+$ and multiplication \cdot on R by

$$(a + I) + (b + I) = (a + b) + I \quad \text{and} \quad (a + I) \cdot (b + I) = ab + I$$

for all $a, b \in R$.

Definition 5.0.16: Kernal

Let $f : R \rightarrow S$ be a homomorphism of rings. Then $\ker f$ (the kernel of f) is

$$\ker f = \{a \in R | f(a) = 0_S\}.$$

Definition 5.0.17: natural homomorphism

Let I be an ideal in the ring R . The function

$$\pi : R \rightarrow R/I, r \mapsto r + I$$

is called the natural homomorphism from R to R/I .

Theorem 5.0.3: First Isomorphism Theorem

Let $f : R \rightarrow S$ be a ring homomorphism. Recall that $\text{Im } f = \{f(a) | a \in R\}$. The function

$$\bar{f} : R/\ker f \rightarrow \text{Im } f, \quad (a + \ker f) \mapsto f(a) \tag{5.0.3}$$

is a well-defined ring isomorphism. In particular $R/\ker f$ and $\text{Im } f$ are isomorphic rings.

Mathematical Analysis

Definition 6.0.1: Field

A *field* F can be defined as a commutative ring with identity such that for each $a \in F$ where $a \neq 0$, there is an element $a^{-1} \in F$ such that $aa^{-1} = 1$.

Theorem 6.0.1: Rational Zeros Theorem[19]

Suppose c_0, c_1, \dots, c_n are integers and r is a rational number satisfying the polynomial equation

$$c_nx^n + c_{n-1}x^{n-1} + \dots + c_1x + c_0 = 0 \quad (6.0.1)$$

where $n \geq 1, c_n \neq 0$ and $c_0 \neq 0$. Let $r = \frac{c}{d}$ where c, d are integers having no common factors and $d \neq 0$. Then c divides c_0 and d divides c_n .

In other words, the only rational candidates for solutions to (6.0.1) have the form $\frac{c}{d}$ where c divides c_0 and d divides c_n .

Theorem 6.0.2: Properties of Absolute Values in \mathbb{R}

For all $a, b \in \mathbb{R}$, $|a| \geq 0$, $|ab| = |a| \cdot |b|$, and $|a + b| \leq |a| + |b|$ (The *triangle inequality*).

Definition 6.0.2: Minimum and Maximum [19]

Let S be a non-empty subset of \mathbb{R} .

- (a) If S contains a largest element s_m [that is $s_m \in S$ and $s \leq s_m$ for all $s \in S$], then we call s_m the *maximum* of S and write $s_m = \max S$.
- (b) If S contains a smallest element s_m [that is $s_m \in S$ and $s_m \leq s$ for all $s \in S$], then we call s_m the *minimum* of S and write $s_m = \min S$.

Definition 6.0.3: Bounded Sets [19]

Let S be a non-empty subset of \mathbb{R} .

- (a) If $M \in \mathbb{R}$ satisfies $s \leq M$ for all $s \in S$, then M is called an *upper bound* of S and the set S is said to be bounded above.
- (b) If $m \in \mathbb{R}$ satisfies $m \leq s$ for all $s \in S$, then m is called a *lower bound* of S and the set S is said to be bounded below.
- (c) The set S is said to be bounded if it is bounded above and bounded below. Thus S is bounded if there exist $m, M \in \mathbb{R}$ such that $S \subseteq [m, M]$

Definition 6.0.4: Supremum and Infimum [19]

Let S be a non-empty subset of \mathbb{R} .

- (a) If S is bounded above and S has a least upper bound, then we will call it the *supremum* of S and denote it $\sup S$.
- (b) If S is bounded below and S has a greatest lower bound, then we will call it the *infimum* of S and denote it $\inf S$.

Definition 6.0.5: Completeness Axiom [19]

Every nonempty subset S of \mathbb{R} that is bounded above has a least upper bound. In other words, $\sup S$ exists and is a real number.

Definition 6.0.6: Archimedean Property [19]

If $a > 0$ and $b > 0$, then for some positive integer n , we have $na > b$.

Definition 6.0.7: Denseness of \mathbb{Q} [19]

If $a, b \in \mathbb{R}$ and $a < b$, then there is a rational $r \in \mathbb{Q}$ such that $a < r < b$.

6.1 Sequences and Series

Definition 6.1.1: A Sequence

A sequence is a function $\{n \in \mathbb{Z} : n \geq m\} \rightarrow \mathbb{R}, n \mapsto s_n$.

1. $m = 0$ or $m = 1$ most often.
2. Often denoted as (s_1, s_2, s_3, \dots) , $(s_n)_{n=m}^{\infty}$, $(s_n)_{n \in \mathbb{N}}$, or (s_n) .

Definition 6.1.2: Limits, Converging, and Diverging [19]

A sequence (s_n) of real numbers is said to *converge* to the real number s provided that for each $\epsilon > 0$ there exists a number N such that $n > N \implies |s_n - s| < \epsilon$. If (s_n) converges to s , we will write $\lim_{n \rightarrow \infty} s_n = s$, or $s_n \rightarrow s$. The number s is called the *limit* of the sequence (s_n) . A sequence that does not converge to some real number is said to *diverge*.

Theorem 6.1.1: Sequence and Limit Properties [19]

- i Convergent sequences are bounded.
- ii If the sequence (s_n) converges to s and $k \in \mathbb{R}$, then the sequence (ks_n) converges to ks . That is, $\lim ks_n = k \cdot \lim s_n$.
- iii If the sequence (s_n) converges to s and (t_n) converges to t , then $(s_n + t_n)$ converges to $s + t$. That is, $\lim s_n + t_n = \lim s_n + \lim t_n$.
- iv If the sequence (s_n) converges to s and (t_n) converges to t , then $(s_n t_n)$ converges to st . That is, $\lim s_n \cdot t_n = (\lim s_n)(\lim t_n)$.
- v Suppose (s_n) converges to s and (t_n) converges to t . If $s \neq 0$ and $s_n \neq 0$ for all n , then $(\frac{t_n}{s_n})$ converges to $\frac{t}{s}$.
- vi Let (s_n) and (t_n) be sequences such that $\lim s_n = \pm\infty$ and $\lim t_n > 0$. Then $\lim s_n t_n = \pm\infty$.
- vii For a sequence (s_n) of positive real numbers, we have $\lim s_n = \infty$ if and only if $\lim(\frac{1}{s_n}) = 0$.
- viii All bounded monotonic sequences converge.
- ix If (s_n) is an unbounded increasing or decreasing sequence, then $\lim s_n = \pm\infty$ respectively.
- x If $t \in \mathbb{R}$, then there is a subsequence of (s_n) converging to t if and only if the set $\{n \in \mathbb{N} : |s_n - t| < \epsilon\}$ is infinite for all $\epsilon > 0$.
- xi If the sequence (s_n) is unbounded above or below, it has a subsequence with limit $\pm\infty$ respectively.
- xii Let (s_n) be any sequence. There exists a monotonic subsequence whose limit is $\limsup s_n$, and there exists a monotonic subsequence whose limit is $\liminf s_n$.

Definition 6.1.3: Divergence [19]

Let (s_n) be a sequence.

- i We write $\lim s_n = \infty$ (diverges to $+\infty$) provided for each $M > 0$, there is a number N such that $n > N \implies s_n > M$.
- ii We write $\lim s_n = -\infty$ (diverges to $-\infty$) provided for each $M < 0$, there is a number N such that $n > N \implies s_n < M$.

Definition 6.1.4: Monotonic Sequences [19]

A sequence (s_n) of real numbers is called an *increasing sequence* if $s_n \leq s_{n+1}$ for all n , and (s_n) is called a *decreasing sequence* if $s_n \geq s_{n+1}$ for all n . Note that if (s_n) is increasing, then $s_n \leq s_m$ whenever $n < m$. A sequence that is increasing or decreasing will be called a *monotone sequence* or a *monotonic sequence*.

- (i) Monotonic sequences will always either converge, or diverge to $\pm\infty$.

Theorem 6.1.2: Limits of Supremum and Infimum [19]

Let (s_n) be a sequence in \mathbb{R} .

- (i) If $\lim s_n$ is defined (as a real number or $\pm\infty$), then $\liminf s_n = \lim s_n = \limsup s_n$.
- (ii) If $\liminf s_n = \limsup s_n$, then $\lim s_n$ is defined and $\lim s_n = \liminf s_n = \limsup s_n$.
- (iii) If (s_n) converges to a positive real number s and (t_n) is any sequence, then $\limsup s_n t_n = s \cdot \limsup t_n$.
- (iv) If (s_n) is any sequence of nonzero real numbers then

$$\liminf \left| \frac{s_{n+1}}{s_n} \right| \leq \liminf |s_n|^{1/n} \leq \limsup |s_n|^{1/n} \leq \limsup \left| \frac{s_{n+1}}{s_n} \right|$$

Definition 6.1.5: Cauchy Sequence [19]

A sequence (s_n) of real numbers is called a *Cauchy sequence* if for each $\epsilon > 0$ there exists a number N such that $m, n > N \implies |s_n - s_m| < \epsilon$.

- (i) Cauchy sequences are bounded.
- (ii) A sequence is a convergent sequence if and only if it is a Cauchy sequence.

Definition 6.1.6: Subsequence [19]

Suppose $(s_n)_{n \in \mathbb{N}}$ is a sequence. A *subsequence* of this sequence is a sequence of the form $(t_k)_{k \in \mathbb{N}}$ where for each k there is a positive integer n_k such that $n_1 < n_2 < \dots < n_k < n_{k+1} < \dots$ and $t_k = s_{n_k}$.

- (i) If the sequence (s_n) converges, then every subsequence converges to the same limit.
- (ii) Every sequence (s_n) has a monotonic subsequence.
- (iii) Every bounded sequence has a convergent subsequence (Bolzano-Weierstrass Theorem).

Definition 6.1.7: Subsequential limit [19]

Let (s_n) be a sequence in \mathbb{R} . A *subsequential limit* (occasionally denoted S) is any real number or symbol $\pm\infty$ that is the limit of some subsequence of (s_n) .

- (i) S is non-empty.
- (ii) $\sup S = \limsup s_n$ and $\inf S = \liminf s_n$.
- (iii) $\lim s_n$ exists if and only if S has exactly one element, namely $\lim s_n$.
- (iv) Suppose (t_n) is a sequence in $S \cap R$ and that $t = \lim t_n$. Then t belongs to S .

Definition 6.1.8: Cauchy Critereon [19]

We say a series $\sum a_n$ satisfies the *Cauchy Critereon* if its subsequence (s_n) of partial sums is a Cauchy sequence: for each $\epsilon > 0$ there exists a number N such that $m, n > N$ implies $|s_n - s_m| < \epsilon$.

1. A series converges if and only if it satisfies the Cauchy criterion.

6.2 Convergence/Divergence Tests

6.2.1: Comparison Test [19]

Let $\sum a_n$ be a series where $a_n \geq 0$ for all n .

- (i) If $\sum a_n$ converges and $|b_n| \leq a_n$ for all n , then $\sum b_n$ converges.
- (ii) If $\sum a_n = \infty$ and $b_n \geq a_n$ for all n , then $\sum b_n = \infty$.

6.2.2: Ratio Test [19]

Let $\sum a_n$ be a series of nonzero terms.

- (i) $\sum a_n$ converges absolutely if $\limsup \left| \frac{a_{n+1}}{a_n} \right| < 1$.
- (ii) $\sum a_n$ diverges if $\liminf \left| \frac{a_{n+1}}{a_n} \right| > 1$.
- (iii) Otherwise, $\liminf \left| \frac{a_{n+1}}{a_n} \right| \leq 1 \leq \limsup \left| \frac{a_{n+1}}{a_n} \right|$ and the test gives no information.

6.2.3: Root Test [19]

Let $\sum a_n$ be a series and let $\alpha = \lim \sum |a_n|^{1/n}$.

- (i) The series $\sum a_n$ converges absolutely if $\alpha < 1$.
- (ii) The series $\sum a_n$ diverges if $\alpha > 1$.

6.2.4: Integral Test [19]

- (i) If $\lim_{n \rightarrow \infty} \int_0^n f(x) dx = \infty$ then the series $\sum f(x)$ will diverge.
- (ii) If $\lim_{n \rightarrow \infty} \int_0^n f(x) dx < \infty$ then the series $\sum f(x)$ will converge.

Theorem 6.2.1: Miscellaneous Convergent Properties and Theorems [19]

- (i) If a series $\sum a_n$ converges, then $\lim a_n = 0$.
- (ii) $\sum \frac{1}{n^p}$ converges if and only if $p > 1$.

Theorem 6.2.2: Alternating Series Theorem [19]

If $a_1 \geq a_2 \geq \dots \geq a_n \geq \dots \geq 0$ and $\lim a_n = 0$, then the alternating series $\sum (-1)^{n+1} a_n$ converges. Moreover, the partial sums $s_n = \sum_{k=1}^n (-1)^{k+1} a_k$ satisfy $|s - s_n| \leq a_n$ for all n .

6.3 Functions

Definition 6.3.1: Continuous Function [19]

Let f be a real-valued function whose domain is a subset of \mathbb{R} . The function f is *continuous* at x_0 in $\text{dom}(f)$ if, for every sequence (x_n) in $\text{dom}(f)$ converging to x_0 , we have $\lim_n f(x_n) = f(x_0)$. If f is continuous at each point of a set $S \subseteq \text{dom}(f)$, then f is said to be *continuous* on S . The function f is said to be *continuous* if it is continuous on $\text{dom}(f)$.

Theorem 6.3.1: Continuous Function [19]

Let f be a real-valued function with $\text{dom}(f) \subseteq \mathbb{R}$. Then f is *continuous* at x_0 in $\text{dom}(f)$ if and only if for each $\epsilon > 0$, there exists $\delta > 0$ such that $x \in \text{dom}(f)$ and $|x - x_0| < \delta$ imply $|f(x) - f(x_0)| < \epsilon$.

Definition 6.3.2: Continuous Functions [19]

Let f be a real-valued function whose domain is a subset of \mathbb{R} .

- (i) f is continuous at x_0 in $\text{dom}(f)$ if, for every sequence (x_n) in $\text{dom}(f)$ converging to x_0 , we have $\lim_n f(x_n) = f(x_0)$. If f is continuous at each point of a set $S \subseteq \text{dom}(f)$, then f is said to be *continuous* on S . The function f is said to be *continuous* if it is continuous on $\text{dom}(f)$.
- (ii) f is continuous at x_0 in $\text{dom}(f)$ if and only if for each $\epsilon > 0$ there exists $\delta > 0$ such that $x \in \text{dom}(f)$ and $|x - x_0| < \delta$ imply $|f(x) - f(x_0)| < \epsilon$.
- (iii) If f is continuous at x_0 in $\text{dom}(f)$, then $|f|$ and $kf, k \in \mathbb{R}$, are continuous at x_0 .

Theorem 6.3.2: [19]

If f is continuous at x_0 and g is continuous at $f(x_0)$, then $g \circ f$ is continuous at x_0 .

Theorem 6.3.3: [19]

Let f and g be real-valued functions that are continuous at x_0 in \mathbb{R} . Then

- (i) $f + g$ is continuous at x_0 ;
- (ii) fg is continuous at x_0 ;
- (iii) $\frac{f}{g}$ is continuous at x_0 if $g(x_0) \neq 0$.

Theorem 6.3.4: [19]

Let f be a continuous real-valued function on a closed interval $[a, b]$. Then f is a bounded function. Moreover, f assumes its maximum and minimum values on $[a, b]$; that is, there exist $x_0, y_0 \in [a, b]$ such that $f(x_0) \leq f(x) \leq f(y_0)$ for all $x \in [a, b]$.

Theorem 6.3.5: Intermediate Value Theorem [19]

If f is a continuous real-valued function on an interval I , then f has the intermediate value property on I : Whenever $a, b \in I$, $a < b$ and y lies between $f(a)$ and $f(b)$ [i.e., $f(a) < y < f(b)$ or $f(b) < y < f(a)$], there exists at least one $x \in (a, b)$ such that $f(x) = y$.

- (i) *Corollary.* If f is a continuous real-valued function on an interval I , then the set $f(I) = \{f(x) : x \in I\}$ is also an interval or a single point.

Theorem 6.3.6: [19]

Let f be a continuous strictly increasing function on some interval I . Then $f(I)$ is an interval J by the Intermediate Value Theorem and f^{-1} represents a function with domain J . The function f^{-1} is a continuous strictly increasing function on J .

Theorem 6.3.7: [19]

Let g be a strictly increasing function on an interval J such that $g(J)$ is an interval I . Then g is continuous on J .

Theorem 6.3.8: [19]

Let f be a one-to-one continuous function on an interval I . Then f is strictly increasing [$x_1 < x_2 \implies f(x_1) < f(x_2)$] or strictly decreasing [$x_1 < x_2 \implies f(x_1) > f(x_2)$].

Definition 6.3.3: Uniformly Continuous [19]

Let f be a real-valued function defined on a set $S \subseteq \mathbb{R}$. Then f is *Uniformly Continuous* on S if for each $\epsilon > 0$ there exists $\delta > 0$ such that $x, y \in S$ and $|x - y| < \delta$ imply $|f(x) - f(y)| < \epsilon$. We will say f is uniformly continuous if f is *uniformly continuous* on $\text{dom}(f)$.

1. If f is continuous on a closed interval $[a, b]$, then f is uniformly continuous on $[a, b]$.
2. If f is uniformly continuous on a set S and (s_n) is a Cauchy sequence in S , then $(f(s_n))$ is a Cauchy sequence.
3. A real-valued function f on (a, b) is uniformly continuous on (a, b) if and only if it can be extended to a continuous function \bar{f} on $[a, b]$.
4. Let f be a continuous function on an interval I [I may be bounded or unbounded]. Let I° be the interval obtained by removing from I any endpoints that happen to be in I . If f is differentiable on I° and if f' is bounded on I° , then f is uniformly continuous on I .

Definition 6.3.4: Limits of Functions [19]

- (i) Let $S \subseteq \mathbb{R}$. Let a and L be real numbers or $\pm\infty$, where a is the limit of some sequence in S . We write $\lim_{x \rightarrow a} sf(x) = L$ if f is a function defined on S and for every sequence (x_n) in S with limit a , we have $\lim_{n \rightarrow \infty} f(x_n) = L$.
- (ii) For $a \in \mathbb{R}$ and a function f we write $\lim_{x \rightarrow a} f(x) = L$ provided $\lim_{x \rightarrow a} sf(x) = L$ for some set $S = J \setminus \{a\}$ where J is an open interval containing a . $\lim_{x \rightarrow a} f(x)$ is called the *two-sided* limit of f at a . Note f need not be defined at a and, even if f is defined at a , the value $f(a)$ need not equal $\lim_{x \rightarrow a} f(x)$. In fact, $f(a) = \lim_{x \rightarrow a} f(x)$ if and only if f is defined on an open interval containing a and f is continuous at a .
- (iii) For $a \in \mathbb{R}$ and a function f we write $\lim_{x \rightarrow a^+} f(x) = L$ provided $\lim_{x \rightarrow a} sf(x) = L$ for some open interval $S = (a, b)$. $\lim_{x \rightarrow a^+} f(x)$ is the *right-hand* limit of f at a . Again f need not be defined at a .
- (iv) For $a \in \mathbb{R}$ and a function f we write $\lim_{x \rightarrow a^-} f(x) = L$ provided $\lim_{x \rightarrow a} sf(x) = L$ for some open interval $S = (c, a)$. $\lim_{x \rightarrow a^-} f(x)$ is the *left-hand* limit of f at a .
- (v) For a function f we write $\lim_{x \rightarrow \infty} f(x) = L$ provided $\lim_{x \rightarrow \infty} sf(x) = L$ for some interval $S = (c, \infty)$. Likewise, we write $\lim_{x \rightarrow -\infty} f(x) = L$ provided $\lim_{x \rightarrow -\infty} sf(x) = L$ for some interval $S = (-\infty, b)$.

Theorem 6.3.9: Limits of Functions [19]

Let f_1 and f_2 be functions for which the limits $L_1 = \lim_{x \rightarrow a} sf_1(x)$ and $L_2 = \lim_{x \rightarrow a} sf_2(x)$ exist and are finite. Then

- (i) $\lim_{x \rightarrow a} s(f_1 + f_2)(x)$ exists and equals $L_1 + L_2$
- (ii) $\lim_{x \rightarrow a} s(f_1 f_2)(x)$ exists and equals $L_1 L_2$
- (iii) $\lim_{x \rightarrow a} s(f_1/f_2)(x)$ exists and equals L_1/L_2 provided $L_2 \neq 0$ and $f_2(x) \neq 0$ for $x \in S$.

Theorem 6.3.10: Limits of Functions [19]

Let f be a function for which the limit $L = \lim_{x \rightarrow a} sf(x)$ exists and is finite. If g is a function defined on $\{f(x) : x \in S\} \cup \{L\}$ that is continuous at L , then $\lim_{x \rightarrow a} sg \circ f(x)$ exists and equals $g(L)$.

Theorem 6.3.11: Limits of Functions [19]

Let f be a function defined on a subset S of \mathbb{R} , let a be a real number that is the limit of some sequence in S , and let L be a real number, then $\lim_{x \rightarrow a} sf(x) = L$ if and only if for each $\epsilon > 0$ there exists $\delta > 0$ such that $x \in S$ and $|x - a| < \delta$ imply $|f(x) - L| < \epsilon$.

1. Let f be a function defined on $J \setminus \{a\}$ for some open interval J containing a , and let L be a real number. Then $\lim_{x \rightarrow a} f(x) = L$ if and only if for each $\epsilon > 0$ there exists $\delta > 0$ such that $0 < |x - a| < \delta \implies |f(x) - L| < \epsilon$.
2. Let f be a function defined on some interval (a, b) , and let L be a real number. Then $\lim_{x \rightarrow a^+} f(x) = L$ if and only if for each $\epsilon > 0$ there exists $\delta > 0$ such that $0 < x - a < \delta \implies |f(x) - L| < \epsilon$.

Theorem 6.3.12: [19]

Let f be a function defined on $J \setminus \{a\}$ for some open interval J containing a . Then $\lim_{x \rightarrow a} f(x)$ exists if and only if the limits $\lim_{x \rightarrow a^+} f(x)$ and $\lim_{x \rightarrow a^-} f(x)$ both exist and are equal, in which case all three limits are equal.

6.4 Sequences and Series of Functions

Theorem 6.4.1: Convergence of Power Series [19]

For a power series $\sum a_n x^n$, we say $\beta = \limsup |a_n|^{1/n}$ and the radius of convergence $R = \frac{1}{\beta}$. For $\beta = 0$, we set $R = \infty$ and for $\beta = \infty$, we set $R = 0$. The power series will converge for $|x| < R$ and diverge for $|x| > R$.

Definition 6.4.1: Pointwise Convergence [19]

Let (f_n) be a sequence of real-valued functions defined on a set $S \subseteq R$. The sequence (f_n) converges pointwise [i.e., at each point] to a function f defined on S if $\lim_{n \rightarrow \infty} f_n(x) = f(x)$ for all $x \in S$.

Definition 6.4.2: [19]

Let (f_n) be a sequence of real-valued functions defined on a set $S \subseteq R$. The sequence (f_n) converges uniformly on S to a function f defined on S if for each $\epsilon > 0$ there exists a number N such that $|f_n(x) - f(x)| < \epsilon$ for all $x \in S$ and all $n > N$. We write $\lim f_n = f$ uniformly on S or $f_n \rightarrow f$ uniformly on S .

Theorem 6.4.2: Uniform Limits of Continuous Functions [19]

The uniform limit of continuous functions is continuous. More precisely, let (f_n) be a sequence of functions on a set $S \subseteq R$, suppose $f_n \rightarrow f$ uniformly on S , and suppose $S = \text{dom}(f)$. If each f_n is continuous at $x_0 \in S$, then f is continuous at x_0 . [So if each f_n is continuous on S , then f is continuous on S .]

Theorem 6.4.3: [19]

Let (f_n) be a sequence of continuous functions on $[a, b]$, and suppose $f_n \rightarrow f$ uniformly on $[a, b]$. Then $\lim_{n \rightarrow \infty} \int_a^b f_n(x) dx = \int_a^b f(x) dx$

Definition 6.4.3: Uniformly Cauchy [19]

A sequence (f_n) of functions defined on a set $S \subseteq R$ is uniformly Cauchy on S if for each $\epsilon > 0$ there exists a number N such that $|f_n(x) - f_m(x)| < \epsilon$ for all $x \in S$ and all $m, n > N$.

Theorem 6.4.4: [19]

Let (f_n) be a sequence of functions defined and uniformly Cauchy on a set $S \subseteq R$. Then there exists a function f on S such that $f_n \rightarrow f$ uniformly on S .

Theorem 6.4.5: [19]

Consider a series $\sum_{k=0}^{\infty} g_k$ of functions on a set $S \subseteq R$. Suppose each g_k is continuous on S and the series converges uniformly on S . Then the series $\sum_{k=0}^{\infty} g_k$ represents a continuous function on S .

Theorem 6.4.6: [19]

If a series $\sum_{k=0}^{\infty} g_k$ of functions satisfies the Cauchy criterion uniformly on a set S , then the series converges uniformly on S .

Definition 6.4.4: Weierstrass M-test [19]

Let (M_k) be a sequence of nonnegative real numbers where $\sum M_k < \infty$. If $|g_k(x)| \leq M_k$ for all x in a set S , then $\sum g_k$ converges uniformly on S .

Definition 6.4.5: Uniformly Converging Power Series [19]

Let $\sum a_n x^n$ be a power series with radius of convergence $R > 0$ [possibly $R = \infty$]. If $0 < R_1 < R$, then the power series converges uniformly on $[-R_1, R_1]$ to a continuous function.

Theorem 6.4.7: [19]

If the power series $\sum_{n=0}^{\infty} a_n x^n$ has radius of convergence R , then the power series $\sum_{n=0}^{\infty} n a_n x^{n-1}$ and $\sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$ also have radius of convergence R .

Theorem 6.4.8: [19]

Suppose $f(x) = \sum_{n=0}^{\infty} a_n x^n$ has radius of convergence $R > 0$. Then $\int_0^x f(t) dt = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$ for $|x| < R$

Theorem 6.4.9: [19]

Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ have radius of convergence $R > 0$. Then f is differentiable on $(-R, R)$ and $f'(x) = \sum_{n=0}^{\infty} n a_n x^{n-1}$ for $|x| < R$.

Theorem 6.4.10: Abels Theorem [19]

Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ be a power series with finite positive radius of convergence R . If the series converges at $x = R$, then f is continuous at $x = R$. If the series converges at $x = -R$, then f is continuous at $x = -R$.

6.5 Derivatives

Definition 6.5.1: [19]

Let f be a real-valued function defined on an open interval containing a point a . We say f is differentiable at a , or f has a derivative at a , if the limit $\lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$ exists and is finite. We will write $f'(a)$ for the derivative of f at a :

$$f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} = \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{f(x + h) - f(x - h)}{2h}, \quad (6.5.1)$$

whenever this limit exists and is finite.

Theorem 6.5.1: [19]

If f is differentiable at a point a , then f is continuous at a .

Theorem 6.5.2: Differentiation formulas [19]

Let f and g be functions that are differentiable at the point a . Each of the functions cf (c a constant), $f + g$, fg and f/g is also differentiable at a , except f/g if $g(a) = 0$ since f/g is not defined at a in this case. The formulas are

1. $(cf)'(a) = cf'(a);$
2. $(f + g)'(a) = f'(a) + g'(a);$
3. (product rule) $(fg)'(a) = f(a)g'(a) + f'(a)g(a);$
4. (quotient rule) $(f/g)'(a) = [g(a)f'(a) - f(a)g'(a)]/g^2(a)$ if $g(a) \neq 0$.

Theorem 6.5.3: Chain Rule [19]

If f is differentiable at a and g is differentiable at $f(a)$, then the composite function $g \circ f$ is differentiable at a and we have $(g \circ f)'(a) = g'(f(a))f'(a)$.

Theorem 6.5.4: [19]

If f is defined on an open interval containing x_0 , if f assumes its maximum or minimum at x_0 , and if f is differentiable at x_0 , then $f'(x_0) = 0$.

Theorem 6.5.5: Rolle's Theorem [19]

Let f be a continuous function on $[a, b]$ that is differentiable on (a, b) and satisfies $f(a) = f(b)$. There exists [at least one] $x \in (a, b)$ such that $f'(x) = 0$.

Theorem 6.5.6: Mean Value Theorem [19]

Let f be a continuous function on $[a, b]$ that is differentiable on (a, b) . Then there exists [at least one] $x \in (a, b)$ such that $f'(a) = \frac{f(x) - f(a)}{x - a}$.

1. Let f be a differentiable function on (a, b) such that $f'(x) = 0$ for all $x \in (a, b)$. Then f is a constant function on (a, b) .
2. Let f and g be differentiable functions on (a, b) such that $f' = g'$ on (a, b) . Then there exists a constant c such that $f(x) = g(x) + c$ for all $x \in (a, b)$.

Definition 6.5.2: Increasing and Decreasing Functions [19]

Let f be a real-valued function defined on an interval I .

- (i) We say f is *strictly increasing* on I if $(x_1, x_2 \in I \text{ and } x_1 < x_2) \implies f(x_1) < f(x_2)$.
- (ii) We say f is *strictly decreasing* on I if $(x_1, x_2 \in I \text{ and } x_1 < x_2) \implies f(x_1) > f(x_2)$.
- (iii) We say f is *increasing* on I if $(x_1, x_2 \in I \text{ and } x_1 < x_2) \implies f(x_1) \leq f(x_2)$.
- (iv) We say f is *decreasing* on I if $(x_1, x_2 \in I \text{ and } x_1 < x_2) \implies f(x_1) \geq f(x_2)$.

Theorem 6.5.7: [19]

Let f be a differentiable function on an interval (a, b) . Then

- (i) f is *strictly increasing* if $f'(x) > 0$ for all $x \in (a, b)$.
- (ii) f is *strictly decreasing* if $f'(x) < 0$ for all $x \in (a, b)$.
- (iii) f is *increasing* if $f'(x) \geq 0$ for all $x \in (a, b)$.
- (iv) f is *decreasing* if $f'(x) \leq 0$ for all $x \in (a, b)$.

Theorem 6.5.8: Intermediate Value Theorem for Derivatives [19]

Let f be a differentiable function on (a, b) . If $a < x_1 < x_2 < b$, and if c lies between $f'(x_1)$ and $f'(x_2)$, there exists [at least one] x in (x_1, x_2) such that $f'(x) = c$.

Theorem 6.5.9: [19]

Let f be a one-to-one continuous function on an open interval I , and let $J = f(I)$. If f is differentiable at $x_0 \in I$ and if $f'(x_0) \neq 0$, then f^{-1} is differentiable at $y_0 = f(x_0)$ and $(f^{-1})'(y_0) = \frac{1}{f'(x_0)}$.

Theorem 6.5.10: Generalized Mean Value Theorem [19]

Let f and g be continuous functions on $[a, b]$ that are differentiable on (a, b) . Then there exists [at least one] $x \in (a, b)$ such that $f'(x)[g(b) - g(a)] = g'(x)[f(b) - f(a)]$.

Definition 6.5.3: L'Hospital's Rule [19]

Let s signify $a, a^+, a^-,$ or $\pm\infty$ where $a \in R$, and suppose f and g are differentiable functions for which the limit $\lim_{x \rightarrow s} \frac{f'(x)}{g'(x)} = L$ exists. If $\lim_{x \rightarrow s} f(x) = \lim_{x \rightarrow s} g(x) = 0$ or $\lim_{x \rightarrow s} |g(x)| = \infty$ then $\lim_{x \rightarrow s} \frac{f(x)}{g(x)} = L$.

Definition 6.5.4: Taylor Series [19]

Let f be a function defined on some open interval containing c . If f possesses derivatives of all orders at c , then the series $\sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x - c)^k$ is called the *Taylor series* for f about c . For $n \geq 1$, remainder $R_n(x)$ is defined by $R_n(x) = f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(c)}{k!}(x - c)^k$. is defined by

Theorem 6.5.11: Taylor's Theorem [19]

Let f be defined on (a, b) where $a < c < b$. Here we allow $a = -\infty$ or $b = \infty$. Suppose the n th derivative $f(n)$ exists on (a, b) . Then for each $x \neq c$ in (a, b) there is some y between c and x such that $R_n(x) = \frac{f^{(n)}(y)}{n!}(x - c)^n$.

Theorem 6.5.12: [19]

Let f be defined on (a, b) where $a < c < b$, and suppose the n th derivative $f(n)$ exists and is continuous on (a, b) . Then for $x \in (a, b)$ we have $R_n(x) = \int_c^x \frac{(x-t)^{n-1}}{(n-1)!} f^{(n)}(t) dt$.

Theorem 6.5.13: Binomial Series Theorem [19]

If $\alpha \in \mathbb{R}$ and $|x| < 1$, then $(1 + x)^\alpha = 1 + \sum_{k=1}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!} x^k$.

6.6 Groups

Definition 6.6.1: Group [12]

A *group* (G, \circ) is a set G together with the law of composition $(a, b) \mapsto a \circ b$ that satisfies the following axioms:

- (i) The law of composition is associate, and so $(a \circ b) \circ c = a \circ (b \circ c)$ for all $a, b, c \in G$.
- (ii) There exists an element $e \in G$, called the identity element, such that for any element $a \in G$, $e \circ a = a \circ e = a$.
- (iii) For each element $a \in G$, there exists an inverse element $a^{-1} \in G$ such that $a \circ a^{-1} = a^{-1} \circ a = e$.

Definition 6.6.2: Abelian Group [12]

A group G with the property that $a \circ b = b \circ a$ for all $a, b \in G$ is called *abelian* or *commutative*. Groups not satisfying this property are said to be *nonabelian* or *noncommutative*.

Definition 6.6.3: General Linear Group (2×2) [12]

The set of all $n \times n$ matrices is denoted by $\mathbb{M}_n(\mathbb{R})$. The set of invertible matrices, denoted $GL_n(\mathbb{R})$ form a group called the *general linear group*.

Theorem 6.6.1: Group Properties [12]

Let G be a group.

1. The identity element $e \in G$ is unique. There exists only one element e such that $eg = ge = e$ for all $g \in G$.
2. If $g \in G$, then the inverse of g denoted g^{-1} is unique.
3. If $a, b \in G$, then $(ab)^{-1} = b^{-1}a^{-1}$.
4. For any $a \in G$, $(a^{-1})^{-1} = a$.
5. If $a, b \in G$, then the equations $ax = b$ and $xa = b$ have unique solutions.
6. If $a, b, c \in G$, then $ba = ca \implies b = c$ and $ab = ac \implies b = c$.
7. If $a, b \in G$, then $a^m a^n = a^{m+n}$, $(a^m)^n = a^{mn}$ and $(ab)^n = (b^{-1}a^{-1})^{-n}$ for all $m, n \in \mathbb{Z}$.

Definition 6.6.4: Subgroup

Let G be a group. We say for some set S such that $S \subseteq G$, that S is a subgroup of G , denoted $S \trianglelefteq G$ if S is a group on its own when restricted to the operation of G .

Theorem 6.6.2: Subgroup Theorem [12]

A subset S of G is a subgroup if and only if it satisfies the following conditions.

1. The identity $e \in G$ is contained in S .
2. If $a, b \in S$, then $ab \in S$.
3. if $a \in S$, then $a^{-1} \in S$.

Similarly, Let G be a group and $S \subseteq G$. Then $S \trianglelefteq G \iff (S \neq \emptyset, \text{ and } a, b \in S \implies ab^{-1} \in S)$.

Differential Equations

Definition 7.0.1: Del Operator

The **Del** operator with respect to n -dimensions:

$$\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n} \right) = \sum_{i=1}^n \vec{e}_i \frac{\partial}{\partial x_i} \quad (7.0.1)$$

Definition 7.0.2: Laplace Operator

The Laplace Operator (expanded in 3-dimensions below)

$$\Delta = \nabla \cdot \nabla = \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \quad (7.0.2)$$

Definition 7.0.3: Gradient

The **gradient** of a 3-dimensional function (cartesian, spherical, and cylindrical coordinates)

$$\text{grad } f = \nabla f = \frac{\partial f}{\partial x} \hat{x} + \frac{\partial f}{\partial y} \hat{y} + \frac{\partial f}{\partial z} \hat{z} = \frac{\partial f}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial f}{\partial \theta} \hat{\theta} + \frac{1}{r \sin(\theta)} \frac{\partial f}{\partial \phi} \hat{\phi} = \frac{\partial f}{\partial \rho} \hat{\rho} + \frac{1}{\rho} \frac{\partial f}{\partial \phi} \hat{\phi} + \frac{\partial f}{\partial z} \hat{z} \quad (7.0.3)$$

Definition 7.0.4: Curl

The **curl** of a vector is the limit as the volume goes to zero of the ratio of the integral of it's cross product (with respect to the normal) over a closed surface, to the volume enclosed by the surface [16]. The curl of a 3-dimensional function (cartesian, spherical, and cylindrical coordinates)

$$\text{curl } \vec{A} = \nabla \times \vec{A} = \lim_{V \rightarrow 0} \frac{1}{V} \oint_S (\hat{n} \times \vec{A}) da \quad (\text{definition}) \quad (7.0.4)$$

$$= \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) \hat{x} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) \hat{y} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \hat{z} \quad (7.0.5)$$

$$= \frac{1}{r \sin(\theta)} \left[\frac{\partial}{\partial \theta} \sin(\theta) A_\phi - \frac{\partial A_\theta}{\partial \phi} \right] \hat{r} + \left[\frac{1}{r \sin(\theta)} \frac{\partial A_r}{\partial \phi} - \frac{1}{r} \frac{\partial}{\partial r} (r A_\phi) \right] \hat{\theta} \\ + \frac{1}{r} \left[\frac{\partial}{\partial r} (r A_\theta) - \frac{\partial A_r}{\partial \theta} \right] \hat{\phi} \quad (7.0.6)$$

$$= \left[\frac{1}{\rho} \frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z} \right] \hat{\rho} + \left[\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho} \right] \hat{\phi} + \frac{1}{\rho} \left[\frac{\partial}{\partial \rho} (\rho A_\phi) - \frac{\partial A_\rho}{\partial \phi} \right] \hat{z} \quad (7.0.7)$$

Definition 7.0.5: Divergence

The **divergence** of a vector is the limit of its surface integral per unit volume as the volume enclosed by the surface goes to zero [16]. The divergence of a 3-dimensional function (cartesian, spherical, and cylindrical coordinates) is

$$\operatorname{div} \vec{A} = \nabla \cdot \vec{A} = \lim_{V \rightarrow 0} \frac{1}{V} \oint_S (\vec{A} \cdot \hat{n}) da \quad (\text{definition}) \quad (7.0.8)$$

$$= \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \quad (7.0.9)$$

$$= \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 A_r) + \frac{1}{r \sin(\theta)} \frac{\partial}{\partial \theta} \sin(\theta) A_\theta + \frac{1}{r \sin(\theta)} \frac{\partial A_\phi}{\partial \phi} \quad (7.0.10)$$

$$= \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho A_\rho) + \frac{1}{\rho} \frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z} \quad (7.0.11)$$

Theorem 7.0.1: Solving Laplaces equation

The general solution to Laplaces equation in spherical coordinates with no ϕ dependence (where P_ℓ denotes the Legendre polynomials) is

$$\nabla^2 V(r, \theta) = 0 \implies V(r, \theta) = \sum_{\ell=0}^{\infty} \left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}} \right) P_\ell(\cos \theta) \quad (7.0.12)$$

7.1 Differentiation

Definition 7.1.1: Derivative of a function

The derivative of a function represents an infinitesimal change in the function with respect to one of its variables. Let $f : \mathbb{R} \rightarrow \mathbb{R}$. Then the derivative of f with respect to a variable x is given by

$$\frac{d}{dx} f(x) \equiv f'(x) \equiv \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \equiv \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x - \Delta x)}{2\Delta x} \quad (7.1.1)$$

Definition 7.1.2: Partial Derivative of a function

Partial derivatives are defined as derivatives of a function of multiple variables when all but the variable of interest are held fixed during the differentiation. Let $f : \mathbb{R}^z \rightarrow \mathbb{R}$, with $z \in \mathbb{N}$. Then the partial derivative of f with respect to a variable x_m is given by

$$\frac{\partial}{\partial x_m} f(x_1, \dots, x_n) \equiv \lim_{\Delta x \rightarrow 0} \frac{f(x_1, \dots, x_m + \Delta x, \dots, x_n) - f(x_1, \dots, x_m, \dots, x_n)}{\Delta x} \quad (7.1.2)$$

7.1.1: Derivative of an n^{th} degree polynomial.

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ such that $f(x) \in \mathbb{P}_n$ of the form $f(x) = b_n x^n + b_{n-1} x^{n-1} + \cdots + b_1 x + b_0$. Then the m^{th} derivative of $f(x)$ is given by

$$\frac{d^m}{dx^m} f(x) = f^{(m)}(x) = \sum_{k=0}^{n-m} \frac{(n-k)!}{(n-m-k)!} b_{n-k} x^{n-m-k}. \quad (7.1.3)$$

7.1.2: Derivative of a sum of simple exponential functions

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function such that $f(x) = b_n e^{k_n x} + b_{n-1} e^{k_{n-1} x} + \cdots + b_1 e^{k_1 x} + b_0 e^{k_0 x}$, with b_n and k_n as constants. The m^{th} derivative is then given by

$$\frac{d^m}{dx^m} f(x) = \frac{d^m}{dx^m} \sum_{i=0}^n b_i e^{k_i x} = \sum_{i=0}^n b_{n-i} k_{n-i}^m e^{k_{n-i} x}. \quad (7.1.4)$$

REVIEW OF DIFFERENTIATION

Rules

1. Constant: $\frac{d}{dx} c = 0$

2. Constant Multiple: $\frac{d}{dx} cf(x) = c f'(x)$

3. Sum: $\frac{d}{dx} [f(x) \pm g(x)] = f'(x) \pm g'(x)$

4. Product: $\frac{d}{dx} f(x)g(x) = f(x)g'(x) + g(x)f'(x)$

5. Quotient: $\frac{d}{dx} \frac{f(x)}{g(x)} = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$

6. Chain: $\frac{d}{dx} f(g(x)) = f'(g(x))g'(x)$

7. Power: $\frac{d}{dx} x^n = nx^{n-1}$

8. Power: $\frac{d}{dx} [g(x)]^n = n[g(x)]^{n-1} g'(x)$

Functions

Trigonometric:

9. $\frac{d}{dx} \sin x = \cos x$

10. $\frac{d}{dx} \cos x = -\sin x$

11. $\frac{d}{dx} \tan x = \sec^2 x$

12. $\frac{d}{dx} \cot x = -\csc^2 x$

13. $\frac{d}{dx} \sec x = \sec x \tan x$

14. $\frac{d}{dx} \csc x = -\csc x \cot x$

Inverse trigonometric:

15. $\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$

16. $\frac{d}{dx} \cos^{-1} x = -\frac{1}{\sqrt{1-x^2}}$

17. $\frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2}$

18. $\frac{d}{dx} \cot^{-1} x = -\frac{1}{1+x^2}$

19. $\frac{d}{dx} \sec^{-1} x = \frac{1}{|x|\sqrt{x^2-1}}$

20. $\frac{d}{dx} \csc^{-1} x = -\frac{1}{|x|\sqrt{x^2-1}}$

Hyperbolic:

21. $\frac{d}{dx} \sinh x = \cosh x$

22. $\frac{d}{dx} \cosh x = \sinh x$

23. $\frac{d}{dx} \tanh x = \operatorname{sech}^2 x$

24. $\frac{d}{dx} \coth x = -\operatorname{csch}^2 x$

25. $\frac{d}{dx} \operatorname{sech} x = -\operatorname{sech} x \tanh x$

26. $\frac{d}{dx} \operatorname{csch} x = -\operatorname{csch} x \coth x$

Inverse hyperbolic:

27. $\frac{d}{dx} \sinh^{-1} x = \frac{1}{\sqrt{x^2+1}}$

28. $\frac{d}{dx} \cosh^{-1} x = \frac{1}{\sqrt{x^2-1}}$

29. $\frac{d}{dx} \tanh^{-1} x = \frac{1}{1-x^2}$

30. $\frac{d}{dx} \coth^{-1} x = \frac{1}{1-x^2}$

31. $\frac{d}{dx} \operatorname{sech}^{-1} x = -\frac{1}{x\sqrt{1-x^2}}$

32. $\frac{d}{dx} \operatorname{csch}^{-1} x = -\frac{1}{|x|\sqrt{x^2+1}}$

Exponential:

33. $\frac{d}{dx} e^x = e^x$

34. $\frac{d}{dx} a^x = a^x (\ln a)$

Logarithmic:

35. $\frac{d}{dx} \ln|x| = \frac{1}{x}$

36. $\frac{d}{dx} \log_a x = \frac{1}{x(\ln a)}$

⁰This "Review of Differentiation" is taken directly from Dennis G. Zill - A First Course in Differential Equations, 10th Ed. When time permits it will be re-created in an original format.

7.2.1: Repeating product rule applied to arbitrary functions

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ and $g : \mathbb{R} \rightarrow \mathbb{R}$ be continuous and differentiable functions of the variable x . Then the m^{th} derivative of $f(x)g(x)$ with respect to x is

$$\frac{d^n}{dx^n} f(x)g(x) = \sum_{i=0}^n \binom{n}{i} \left[\frac{d^i}{dx^i} f(x) \right] \left[\frac{d^{n-i}}{dx^{n-i}} g(x) \right].$$

7.3 Legendre differential equation

The **Legendre polynomials** are normalized solutions to the Legendre differential equation

$$(1 - x^2) \frac{d^2y}{dx^2} - 2x \frac{dy}{dx} + \ell(\ell + 1)y = 0. \quad (7.3.1)$$

which have the general solution below. For even ℓ , we must take $a_1 = 0$ to obtain a convergent solution, and for odd ℓ , we must take $a_0 = 0$.

$$y = a_0 \left[1 - \frac{\ell(\ell + 1)}{2!} x^2 + \frac{\ell(\ell + 1)(\ell - 2)(\ell + 3)}{4!} x^4 - \dots \right] \quad (7.3.2)$$

$$+ a_1 \left[x - \frac{(\ell - 1)(\ell + 2)}{3!} x^3 + \frac{(\ell - 1)(\ell + 2)(\ell - 3)(\ell + 4)}{5!} x^5 - \dots \right] \quad (7.3.3)$$

The Legendre polynomial $P_\ell(z)$ can be defined by the contour integral

$$P_\ell(z) = \frac{1}{2\pi i} \oint (1 - 2tz + t^2)^{-1/2} t^{-\ell-1} dt. \quad (7.3.4)$$

The first few Legendre Polynomials follow as:

$P_0(x) = 1 \quad (7.3.5)$	$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3) \quad (7.3.9)$
$P_1(x) = x \quad (7.3.6)$	$P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x) \quad (7.3.10)$
$P_2(x) = \frac{1}{2}(3x^2 - 1) \quad (7.3.7)$	\vdots
$P_3(x) = \frac{1}{2}(5x^3 - 3x) \quad (7.3.8)$	

The Rodrigues representation provides a formula for solving for the Legendre Polynomials

$$P_\ell(x) = \frac{1}{2^\ell \ell!} \frac{d^\ell}{dx^\ell} (x^2 - 1)^\ell \quad (7.3.11)$$

The Legendre polynomials are orthogonal over $(-1, 1)$ with weighting function 1 and satisfy

$$\int_{-1}^1 P_m(x) P_n(x) dx = \frac{2}{2n + 1} \delta_{mn} \quad \text{and} \quad \int_0^1 P_m(x) P_n(x) dx = \frac{1}{2n + 1} \delta_{mn} \quad (7.3.12)$$

The associated Legendre differential equation is given by

$$\frac{d}{dx} \left[(1 - x^2) \frac{dy}{dx} \right] + \left[\ell(\ell - 1) - \frac{m^2}{1 - x^2} \right] y = 0 \quad (7.3.13)$$

When $m, \ell \in \mathbb{Z}^+$ and $m \leq \ell$, the solutions to the above equation are the **associated Legendre polynomials**,

$$P_\ell^m(x) = (-1)^m (1-x^2)^{m/2} \frac{d^m}{dx^m} P_\ell(x) = \frac{(-1)^m}{2^\ell \ell!} (1-x^2)^{m/2} \frac{d^{\ell+m}}{dx^{\ell+m}} (x^2 - 1)^\ell \quad (7.3.14)$$

The associated Legendre polynomials for $m < 0$ are defined by

$$P_\ell^{-m}(x) = (-1)^m \frac{(\ell-m)!}{(\ell+m)!} P_\ell^m(x) \quad (7.3.15)$$

The first few associated Legendre Polynomials with $m > 0$ are

$$P_0^0(x) = 1 \quad (7.3.16)$$

$$P_1^0(x) = x \quad (7.3.17)$$

$$P_1^1(x) = -(1-x^2)^{1/2} \quad (7.3.18)$$

$$P_2^0(x) = \frac{1}{2}(3x^2 - 1) \quad (7.3.19)$$

$$P_2^1(x) = -3x(1-x^2)^{1/2} \quad (7.3.20)$$

$$P_2^2(x) = 3(1-x^2) \quad (7.3.21)$$

$$P_3^0(x) = \frac{1}{2}x(5x^2 - 3) \quad (7.3.22)$$

$$P_3^1(x) = \frac{3}{2}(1-5x^2)(1-x^2)^{1/2} \quad (7.3.23)$$

$$P_3^2(x) = 15x(1-x^2) \quad (7.3.24)$$

$$P_3^3(x) = -15(1-x^2)^{3/2} \quad (7.3.25)$$

⋮

The first few associated Legendre Polynomials with $m < 0$ are

$$P_1^{-1}(x) = \frac{1}{2}(1-x^2)^{1/2} \quad (7.3.26)$$

$$P_2^{-1}(x) = \frac{1}{2}x(1-x^2)^{1/2} \quad (7.3.27)$$

$$P_2^{-2}(x) = \frac{1}{8}(1-x^2) \quad (7.3.28)$$

$$P_3^{-1}(x) = \frac{1}{8}(5x^2 - 1)(1-x^2)^{1/2} \quad (7.3.29)$$

$$P_3^{-2}(x) = \frac{1}{8}x(1-x^2) \quad (7.3.30)$$

$$P_3^{-3}(x) = \frac{1}{48}(1-x^2)^{3/2} \quad (7.3.31)$$

$$P_4^{-1}(x) = \frac{1}{8}(7x^3 - 3x)(1-x^2)^{1/2} \quad (7.3.32)$$

⋮

The associated Legendre polynomials are orthogonal over $[-1, 1]$ such that

$$\int_{-1}^1 P_\ell^m(x) P_{\ell'}^m(x) dx = \frac{2}{2\ell+1} \frac{(\ell+1)!}{(\ell-1)!} \delta_{\ell\ell'} \quad (7.3.33)$$

$$\int_{-1}^1 P_\ell^m(x) P_{\ell'}^{m'}(x) \frac{dx}{1-x^2} = \frac{(\ell+m)!}{m(\ell-m)!} \delta_{mm'} \quad (7.3.34)$$

The derivative about the origin for an associated Legendre polynomial is given by

$$\left[\frac{dP_\ell^m(x)}{dx} \right]_{x=0} = \frac{2^{m+1} \sin [\frac{1}{2}\pi(\ell+m)] \Gamma (\frac{1}{2}\ell + \frac{1}{2}m + 1)}{\pi^{1/2} \Gamma (\frac{1}{2}\ell - \frac{1}{2}m + \frac{1}{2})} \quad (7.3.35)$$

7.4 Laguerre differential equation

The general associated Laguerre differential equation is defined by,

$$xy''(x) + (k+1-x)y'(x) + ny(x) = 0. \quad (7.4.1)$$

A solution to the above differential equation is any generalized **Laguerre polynomial** $L_n^k(x)$. The Rodrigues representation for the associated Laguerre polynomials is

$$L_n^k = \frac{e^x x^{-k}}{n!} \frac{d^n}{dx^n}(e^{-x} x^{n+k}) = (-1)^k \frac{d^k}{dx^k}[L_{n+k}(x)] = \sum_{m=0}^n \frac{(-1)^m (n+k)! x^m}{(n-m)! (k+m)! m!}. \quad (7.4.2)$$

An alternate definition of the Laguerre polynomials is given as

$$L_n^k(x) = \frac{1}{n!} \sum_{i=0}^n \frac{n!}{i!} \binom{k+n}{n-i} (-x)^i. \quad (7.4.3)$$

The associated Laguerre polynomials are orthogonal over $[0, \infty)$ in the following way,

$$\int_0^\infty e^{-x} x^k L_n^k(x) L_m^k(x) dx = \frac{(n+k)!}{n!} \delta_{mn}. \quad (7.4.4)$$

They also satisfy,

$$\int_0^\infty e^{-x} x^{k+1} [L_n^k(x)]^2 dx = \frac{(n+k)!}{n!} (2n+k+1). \quad (7.4.5)$$

The first few Associated Laguerre polynomials are

$$L_0^k(x) = 1 \quad (7.4.6)$$

$$L_1^k(x) = -x + k + 1 \quad (7.4.7)$$

$$L_2^k(x) = \frac{1}{2} [x^2 - 2(k+2)x + (k+1)(k+2)] \quad (7.4.8)$$

$$L_3^k(x) = \frac{1}{6} [-x^3 + 3(k+3)x^2 - 3(k+2)(k+3)x + (k+1)(k+2)(k+3)] \quad (7.4.9)$$

$$\vdots \quad (7.4.10)$$

A special case of the Associated Laguerre polynomials occurs when $k = 0$ which can be defined by a sum, the Rodrigues representation, or the contour integral (respectively)

$$L_n(x) = \sum_{k=0}^n \frac{(-1)^k}{k!} \binom{n}{k} x^k \equiv \frac{e^x}{n!} \frac{d^n}{dx^n}(x^n e^{-x}) \equiv \frac{1}{2\pi i} \oint \frac{e^{-zt/(1-t)}}{(1-t)t^{n+1}} dt \quad (7.4.11)$$

7.5 Second-order Homogeneous

$$\ddot{x} = 0 \implies x(t) = C_1 x + C_2 \quad (7.5.1)$$

$$\ddot{x} + Ax = 0 \implies x(t) = C_1 e^{i\sqrt{A}t} + C_2 e^{-i\sqrt{A}t} \quad (7.5.2)$$

$$\implies x(t) = C_1 \cos(\sqrt{A}t) + C_2 \sin(\sqrt{A}t) \quad (7.5.3)$$

$$\ddot{x} - Ax = 0 \implies x(t) = C_1 e^{\sqrt{A}t} + C_2 e^{-\sqrt{A}t} \quad (7.5.4)$$

$$\implies x(t) = C_1 \sinh(\sqrt{A}t) + C_2 \cosh(\sqrt{A}t) \quad (7.5.5)$$

$$\ddot{x} \pm Ax^2 = 0 \implies x(t) = C_1 \pm \frac{C_2}{A} \log(At \mp C_1) \quad (7.5.6)$$

$$\ddot{x} \pm \frac{A}{x}\dot{x} = 0 \implies x(t) = \frac{C_1}{1 \mp A} t^{1 \mp A} + C_2 \quad (7.5.7)$$

7.5.1: $\ddot{x} + A\dot{x} + Bx = 0$

Given any differential equation of the form $\ddot{x} + A\dot{x} + Bx = 0$, a general solution of the following form exists:

$$x(t) = C_1 \exp \left[-\frac{1}{2}t(\sqrt{A^2 - 4B} + A) \right] + C_2 \exp \left[\frac{1}{2}t(\sqrt{A^2 - 4B} - A) \right]. \quad (7.5.8)$$

Following this, three special cases arise

$$(i) \quad A^2 > 4B \implies$$

$$x(t) = C_1 \exp \left[\frac{-At}{2} \right] \cosh \left(\frac{t\sqrt{A^2 - 4B}}{2} \right) + C_2 \exp \left[\frac{-At}{2} \right] \sinh \left(\frac{t\sqrt{A^2 - 4B}}{2} \right) \quad (7.5.9)$$

$$(ii) \quad A^2 < 4B \implies$$

$$x(t) = C_1 \exp \left[\frac{-At}{2} \right] \cos \left(\frac{t\sqrt{4B - A^2}}{2} \right) + C_2 \exp \left[\frac{-At}{2} \right] i \sin \left(\frac{t\sqrt{4B - A^2}}{2} \right) \quad (7.5.10)$$

$$(iii) \quad A^2 = 4B \implies$$

$$x(t) = C_1 \exp \left[\frac{-At}{2} \right] \quad (7.5.11)$$

7.6 Second-order Linear Ordinary

$$\ddot{x} + Ax = B \implies x(t) = \frac{B}{A} + C_1 e^{i\sqrt{A}t} + C_2 e^{-i\sqrt{A}t} \quad (7.6.1)$$

$$\implies x(t) = \frac{B}{A} + C_1 \cos(\sqrt{A}t) + C_2 \sin(\sqrt{A}t) \quad (7.6.2)$$

$$\ddot{x} - Ax = B \implies x(t) = -\frac{B}{A} + C_1 e^{\sqrt{A}t} + C_2 e^{-\sqrt{A}t} \quad (7.6.3)$$

$$\implies x(t) = -\frac{B}{A} + C_1 \sinh(\sqrt{A}t) + C_2 \cosh(\sqrt{A}t) \quad (7.6.4)$$

$$\ddot{x} + x = t(A - t) \implies x(t) = C_1 \cos(t) + C_2 \sin(t) - t^2 + At + 2 \quad (7.6.5)$$

$$\ddot{x} + A\dot{x} + Bx = t \implies x(t) = C_1 \exp\left[-\frac{1}{2}t(\sqrt{A^2 - 4B} + A)\right] \quad (7.6.6)$$

$$+ C_2 \exp\left[\frac{1}{2}t(\sqrt{A^2 - 4B} - A)\right] - \frac{A}{B^2} + \frac{t}{B} \quad (7.6.7)$$

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f_0 e^{i\omega t} \implies x(t) = \frac{f_0 e^{i\omega t}}{\omega_0^2 - \omega^2 + 2\beta i\omega} \quad (7.6.8)$$

$$\implies x(t) = A \cos(\omega t - \delta) + A_{tr} e^{-\beta t} \cos(\omega_1 t - \delta_{tr}) \quad (7.6.9)$$

$$\delta = \arctan\left(\frac{2\beta\omega}{\omega_0^2 - \omega^2}\right) \quad (7.6.10)$$

$$\implies x(t) = A \cos(\omega t - \delta) + e^{-\beta t} [B_1 \cos(\omega_1 t) + B_2 \sin(\omega_1 t)] \quad (7.6.11)$$

$$\ddot{x} + 2\beta\dot{x} + x = te^{-\alpha t} \implies x(t) = C_1 e^{-\alpha t} + C_2 t e^{-\alpha t} + C_3 e^{-\beta t} \sin(\omega_1 t) + C_4 e^{-\beta t} \cos(\omega_1 t) \quad (7.6.12)$$

$$\omega_1^2 = 1 - \beta^2 \quad (7.6.13)$$

7.7 Higher Order Differential Equations

7.7.1: Particular solution to a sum of exponential functions

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous and differentiable function and let C_n, a_n, b_n and k_n be constant for all n . Given a differential equation of the form

$$\frac{d^n}{dt^n} f(t) + \frac{d^{n-1}}{dt^{n-1}} C_{n-1} f(t) + \cdots + \frac{d}{dt} C_1 f(t) + C_0 f(t) = \sum_{i=0}^{\ell} a_i e^{k_i t},$$

A solution of the following form exists:

$$f(t) = \sum_{i=0}^{\ell} b_i e^{k_i t} = \sum_{i=0}^{\ell} \frac{a_i e^{k_i t}}{k_{\ell-i}^n + C_{n-1} k_{\ell-i}^{n-1} + \cdots + C_0}$$

7.8 Frobenius Method

Consider a second-order ordinary differential equation

$$y'' + P(x)y' + Q(x)y = 0. \quad (7.8.1)$$

If $P(x)$ and $Q(x)$ remain finite at $x = x_0$, then x_0 is called an ordinary point. If either $P(x)$ or $Q(x)$ diverges as $x \rightarrow x_0$, then x_0 is called a singular point. If either $P(x)$ or $Q(x)$ diverges as $x \rightarrow x_0$ but $(x - x_0)P(x)$ and $(x - x_0)^2Q(x)$ remain finite as $x \rightarrow x_0$, then $x = x_0$ is called a **regular singular point** (or nonessential singularity)[24].

If $x = 0$ is a regular singular point of the ordinary differential equation, $y''(x) + P(x)y'(x) + Q(x)y(x) = 0$, solutions may be found by the Frobenius method or by expansion in a Laurent series. In the Frobenius method, assume a solution of the form

$$y(x) = x^\alpha \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+\alpha}. \quad (7.8.2)$$

Taking the first and second derivative of this with respect to x yield

$$y'(x) = \sum_{n=0}^{\infty} (n + \alpha) a_n x^{n+\alpha-1}, \quad \text{and} \quad y''(x) = \sum_{n=0}^{\infty} (n + \alpha)(n + \alpha - 1) a_n x^{n+\alpha-2}. \quad (7.8.3)$$

If we allow $xP(x) = p_0 + p_1x + p_2x^2 + \dots$ and $x^2Q(x) = q_0 + q_1x + q_2x^2 + \dots$, then we can consolidate coefficients, take the limit as $x \rightarrow 0$ and arise at an **Indicial equation** to solve for possible α values

$$0 = \alpha(\alpha - 1) + \alpha p_0 + q_0 \quad \text{with} \quad \begin{cases} p_0 = \lim_{x \rightarrow 0} xP(x) \\ q_0 = \lim_{x \rightarrow 0} x^2Q(x) \end{cases} \quad (7.8.4)$$

7.8.1: Fuchs's Theorem [24]

At least one power series solution will be obtained when applying the Frobenius method if the expansion point is an ordinary, or regular, singular point. The number of roots is given by the roots of the indicial equation.

When the roots of the indicial equation are the same (or sometimes when they differ by an integer), there will be a Frobenius series solution $S_1(x)$ and another solution (where $S_2(x)$ is another Frobenius series) of the form

$$y(x) = S_1(x) \ln(x) + S_2(x) \implies y(x) = \ln(x) \sum_{n=0}^{\infty} a_n x^{n+\alpha} + \sum_{n=0}^{\infty} b_n x^{n+\beta}. \quad (7.8.5)$$

Integrals

Basic indefinite integrals. Add c to each result for completion ($c = \text{constant}$).

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \arctan\left(\frac{x}{a}\right) \quad (8.0.1)$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin\left(\frac{x}{|a|}\right) \quad (8.0.2)$$

$$= \arctan\left(\frac{x}{\sqrt{a^2 - x^2}}\right) \quad (8.0.3)$$

$$\int \frac{dx}{x + x^2} = \ln\left(\frac{x}{1+x}\right) \quad (8.0.4)$$

$$\int \frac{dx}{\sqrt{x^2 - 1}} = \operatorname{arccosh}(x) \quad (8.0.5)$$

$$\int \frac{dx}{x\sqrt{x^2 - 1}} = \arccos\left(\frac{1}{x}\right) \quad (8.0.6)$$

$$\int \frac{dx}{(a^2 + x^2)^{3/2}} = \frac{x}{a^2\sqrt{a^2 + x^2}} \quad (8.0.7)$$

$$\int \frac{xdx}{(a^2 + x^2)^{3/2}} = -\frac{1}{\sqrt{a^2 + x^2}} \quad (8.0.8)$$

$$\begin{aligned} \int \frac{dx}{\sqrt{a^2 + x^2}} &= \operatorname{arcsinh}\left(\frac{x}{a}\right) \\ &= \ln|x + \sqrt{a^2 + x^2}| \end{aligned} \quad (8.0.9) \quad (8.0.10)$$

$$\int \frac{xdx}{1 + x^2} = \frac{1}{2} \ln(1 + x^2) \quad (8.0.11)$$

$$\int \frac{dx}{1 - x^2} = \operatorname{arctanh}(x) \quad (8.0.12)$$

$$\int \frac{xdx}{\sqrt{1 + x^2}} = \sqrt{1 + x^2} \quad (8.0.13)$$

$$\int \frac{\sqrt{x}dx}{\sqrt{1-x}} = \arcsin(\sqrt{x}) - \sqrt{x(1-x)} \quad (8.0.14)$$

$$\int \frac{x^2}{a^2 + x^2} dx = \frac{a^2}{2} \sin^{-1}\left(\frac{x}{a}\right) - \frac{x}{2} \sqrt{a^2 - x^2} \quad (8.0.15)$$

$$\int \frac{dx}{(a^2 + x^2)^2} = \frac{x}{2a^2(x^2 + a^2)} + \frac{1}{2a^3} \arctan\left(\frac{x}{a}\right) \quad (8.0.16)$$

$$\int \frac{x^2 dx}{(a^2 + x^2)^2} = \frac{-x}{2(x^2 + a^2)} + \frac{1}{2a} \arctan\left(\frac{x}{a}\right) \quad (8.0.17)$$

$$\int \ln(x) = x \ln(x) - x \quad (8.0.18)$$

By use of the product rule $\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$, we can integrate both sides to arrive at a useful formula for solving various integrals known as **integration by parts**

$$\int f(x)g'(x) = f(x)g(x) - \int g(x)f'(x) \quad \text{or} \quad \int u dv = uv - \int v du \quad (8.0.19)$$

Generalizing this result gives us integration by parts as

$$\int f^{(n)}(x)g(x)dx = g(x)f^{(n-1)}(x) - g'(x)f^{(n-2)}(x) + g''(x)f^{(n-3)}(x) - \dots + (-1)^n \int f(x)g^{(n)}(x)dx \quad (8.0.20)$$

Some numerical results (with ζ being the usual zeta function) include

$$\int_{-\infty}^{\infty} \frac{x^2 e^x}{(e^x + 1)^2} dx = \frac{\pi^2}{3} \quad (8.0.21)$$

$$\int_0^{\infty} \frac{x}{e^{ax} + 1} dx = \frac{\pi^2}{12a^2} \quad (8.0.22)$$

$$\int_0^{\infty} \frac{\sqrt{x}dx}{e^x - 1} = \frac{\sqrt{\pi}}{2} \zeta\left(\frac{3}{2}\right) \approx 1.306\sqrt{\pi} \quad (8.0.23)$$

BRIEF TABLE OF INTEGRALS

1. $\int u^n du = \frac{u^{n+1}}{n+1} + C, n \neq -1$
2. $\int \frac{1}{u} du = \ln|u| + C$
3. $\int e^u du = e^u + C$
4. $\int a^u du = \frac{1}{\ln a} a^u + C$
5. $\int \sin u du = -\cos u + C$
6. $\int \cos u du = \sin u + C$
7. $\int \sec^2 u du = \tan u + C$
8. $\int \csc^2 u du = -\cot u + C$
9. $\int \sec u \tan u du = \sec u + C$
10. $\int \csc u \cot u du = -\csc u + C$
11. $\int \tan u du = -\ln|\cos u| + C$
12. $\int \cot u du = \ln|\sin u| + C$
13. $\int \sec u du = \ln|\sec u + \tan u| + C$
14. $\int \csc u du = \ln|\csc u - \cot u| + C$
15. $\int u \sin u du = \sin u - u \cos u + C$
16. $\int u \cos u du = \cos u + u \sin u + C$
17. $\int \sin^2 u du = \frac{1}{2}u - \frac{1}{4}\sin 2u + C$
18. $\int \cos^2 u du = \frac{1}{2}u + \frac{1}{4}\sin 2u + C$
19. $\int \tan^2 u du = \tan u - u + C$
20. $\int \cot^2 u du = -\cot u - u + C$
21. $\int \sin^3 u du = -\frac{1}{3}(2 + \sin^2 u)\cos u + C$
22. $\int \cos^3 u du = \frac{1}{3}(2 + \cos^2 u)\sin u + C$
23. $\int \tan^3 u du = \frac{1}{2}\tan^2 u + \ln|\cos u| + C$
24. $\int \cot^3 u du = -\frac{1}{2}\cot^2 u - \ln|\sin u| + C$
25. $\int \sec^3 u du = \frac{1}{2}\sec u \tan u + \frac{1}{2}\ln|\sec u + \tan u| + C$
26. $\int \csc^3 u du = -\frac{1}{2}\csc u \cot u + \frac{1}{2}\ln|\csc u - \cot u| + C$
27. $\int \sin au \cos bu du = \frac{\sin(a-b)u}{2(a-b)} - \frac{\sin(a+b)u}{2(a+b)} + C$
28. $\int \cos au \cos bu du = \frac{\sin(a-b)u}{2(a-b)} + \frac{\sin(a+b)u}{2(a+b)} + C$
29. $\int e^{au} \sin bu du = \frac{e^{au}}{a^2 + b^2}(a \sin bu - b \cos bu) + C$
30. $\int e^{au} \cos bu du = \frac{e^{au}}{a^2 + b^2}(a \cos bu + b \sin bu) + C$
31. $\int \sinh u du = \cosh u + C$
32. $\int \cosh u du = \sinh u + C$
33. $\int \operatorname{sech}^2 u du = \tanh u + C$
34. $\int \operatorname{csch}^2 u du = -\coth u + C$
35. $\int \tanh u du = \ln(\cosh u) + C$
36. $\int \coth u du = \ln|\sinh u| + C$
37. $\int \ln u du = u \ln u - u + C$
38. $\int u \ln u du = \frac{1}{2}u^2 \ln u - \frac{1}{4}u^2 + C$
39. $\int \frac{1}{\sqrt{a^2 - u^2}} du = \sin^{-1} \frac{u}{a} + C$
40. $\int \frac{1}{\sqrt{a^2 + u^2}} du = \ln \left| u + \sqrt{a^2 + u^2} \right| + C$
41. $\int \sqrt{a^2 - u^2} du = \frac{u}{2} \sqrt{a^2 - u^2} + \frac{a^2}{2} \sin^{-1} \frac{u}{a} + C$
42. $\int \sqrt{a^2 + u^2} du = \frac{u}{2} \sqrt{a^2 + u^2} + \frac{a^2}{2} \ln \left| u + \sqrt{a^2 + u^2} \right| + C$
43. $\int \frac{1}{a^2 + u^2} du = \frac{1}{a} \tan^{-1} \frac{u}{a} + C$
44. $\int \frac{1}{a^2 - u^2} du = \frac{1}{2a} \ln \left| \frac{a+u}{a-u} \right| + C$

⁰This "Brief Table of Integrals" is taken directly from Dennis G. Zill - A First Course in Differential Equations, 10th Ed. When time permits it will be re-created in an original format.

Exponential integrals

$$\int_{-\infty}^{\infty} \frac{e^{-iax}}{(1+x^2)} dx = \pi e^{-|a|} \quad (8.1.1)$$

8.1.1: Trigonometric Integrals

$$\int \tan(x) dx = -\ln(\cos(x)) + c \quad (8.1.2)$$

$$\int \tanh(x) dx = \ln(\cosh(x)) + c \quad (8.1.3)$$

$$\int \sin^2(x) dx = \frac{1}{2}(x - \sin(x)\cos(x)) + c = \frac{1}{4}(2x - \sin(2x)) + c \quad (8.1.4)$$

$$\int \cos^2(x) dx = \frac{1}{2}(x + \sin(x)\cos(x)) + c = \frac{1}{4}(2x + \sin(2x)) + c \quad (8.1.5)$$

$$\int \sin^2(x) \cos(x) dx = \frac{1}{3} \sin^3(x) + c \quad (8.1.6)$$

$$\int \cos^2(x) \sin(x) dx = -\frac{1}{3} \cos^3(x) + c \quad (8.1.7)$$

$$\int \sin^3(x) dx = -\frac{1}{3} \cos(x)(\sin^2(x) + 2) + c \quad (8.1.8)$$

$$\int x \sin^2(x) dx = \frac{1}{4}(x^2 - x \sin(2x) - \frac{1}{2} \cos(2x)) + c \quad (8.1.9)$$

$$\int x^2 \sin^2(x) dx = \frac{x^3}{6} - \left(\frac{x^2}{4} - \frac{1}{8} \right) \sin(2x) - \frac{x}{4} \cos(2x) + c \quad (8.1.10)$$

$$\int x^n \sin(ax) dx = -\frac{x^n}{a} \cos(ax) + \frac{n}{a} \int x^{n-1} \cos(ax) dx \quad (8.1.11)$$

$$\int x^n \cos(ax) dx = \frac{x^n}{a} \sin(ax) - \frac{n}{a} \int x^{n-1} \sin(ax) dx \quad (8.1.12)$$

Definition 8.1.1: The Wallis Cosine Formula

$$\int_0^{\pi/2} \cos^n(x) dx = \int_0^{\pi/2} \sin^n(x) dx = \frac{(n-1)!!}{n!!} \begin{cases} \pi/2 & \text{for } n = 2, 4, \dots \\ 1 & \text{for } n = 3, 5, \dots \end{cases} \quad (8.1.13)$$

8.2 Gaussian Integrals

The integral of an arbitrary Gaussian function is

$$\int x^n e^{\beta x} dx = e^{\beta x} \sum_{k=0}^n (-1)^k \frac{n! x^{n-k}}{(n-k)! \beta^{k+1}} + c \quad (8.2.1)$$

Some general Gaussian integrals evaluate as

$$\int_{-\infty}^{\infty} e^{-\alpha x^2} dx = \sqrt{\frac{\pi}{\alpha}} \quad (8.2.2)$$

$$I_n = \int x^n e^{-x/\alpha} dx \quad (8.2.3) \quad \int_0^{\infty} e^{-x/\alpha} dx = \alpha \quad (8.2.8)$$

$$I_0 = -\alpha e^{-x/\alpha} \quad (8.2.4) \quad \int_0^{\infty} x e^{-x/\alpha} dx = \alpha^2 \quad (8.2.9)$$

$$I_1 = -(\alpha^2 + \alpha x) e^{-x/\alpha} \quad (8.2.5) \quad \int_0^{\infty} x^2 e^{-x/\alpha} dx = 2\alpha^3 \quad (8.2.10)$$

$$I_2 = -(2\alpha^3 + 2\alpha^2 x + \alpha x^2) e^{-x/\alpha} \quad (8.2.6) \quad \int_0^{\infty} x^n e^{-x/\alpha} dx = n! \alpha^{n+1} \quad (8.2.11)$$

$$I_{n+1} = \alpha^2 \frac{\partial I_n}{\partial \alpha} \quad (8.2.7)$$

The integral of an arbitrary Gaussian function with an n-dimensional linear term (with $n \in \mathbb{Z}$) is

$$\int_0^{\infty} x^{2n} e^{-\alpha x^2} dx = \sqrt{\frac{\pi}{\alpha}} \frac{(2n-1)!!}{2^{n+1} \alpha^n} \implies \int_{-\infty}^{\infty} x^{2n} e^{-\alpha x^2} dx = \sqrt{\frac{\pi}{\alpha}} \frac{(2n-1)!!}{(2\alpha)^n} \quad (8.2.12)$$

$$\int_0^{\infty} x^{2n+1} e^{-\alpha x^2} dx = \frac{n!}{2a^{n+1}} \implies \int_{-\infty}^{\infty} x^{2n+1} e^{-\alpha x^2} dx = 0 \quad (8.2.13)$$

Therefore a general solution is

$$\int_0^{\infty} x^n e^{-\alpha x^2} dx = \begin{cases} \frac{(n-1)!!}{2^{n/2+1} a^{n/2}} \sqrt{\frac{\pi}{\alpha}} & \text{for } n \text{ even} \\ \frac{[\frac{1}{2}(n-1)]!}{2a^{(n+1)/2}} & \text{for } n \text{ odd} \end{cases} \quad (8.2.14)$$

The below form of a gaussian integral evaluates to zero when n is odd due to the function being odd, but when n is even, the more general integral has the following closed form

$$\int_{-\infty}^{\infty} x^n e^{-\alpha x^2 + \beta x} dx = \sqrt{\frac{\pi}{\alpha}} e^{\beta^2/(4\alpha)} \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} (2k-1)!! (2a)^{k-n} \beta^{n-2k} \quad (8.2.15)$$

Definition 8.2.1: Saddle Point Approximation

One can use the **saddle point approximation** to approximate integrals of a certain form (where x_0 is the critical point of $f'(x) = 0$ - specifically the minimum)

$$I = \int_{-\infty}^{\infty} g(x) e^{-f(x)} dx \approx g(x_0) e^{-f(x_0)} \sqrt{\frac{2\pi}{f''(x_0)}} \quad (8.2.16)$$

Fourier Series

The computation of the (usual) Fourier series is based on the integral identities

$$\int_{-\pi}^{\pi} \sin(mx) \sin(nx) dx = \pi \delta_{mn} \quad (9.0.1)$$

$$\int_{-\pi}^{\pi} \cos(mx) \cos(nx) dx = \pi \delta_{mn} \quad (9.0.2)$$

$$\int_{-\pi}^{\pi} \sin(mx) \cos(nx) dx = 0 \quad (9.0.3)$$

$$\int_{-\pi}^{\pi} \sin(mx) dx = 0 \quad (9.0.4)$$

$$\int_{-\pi}^{\pi} \cos(mx) dx = 0 \quad (9.0.5)$$

$$\delta_{mn} = \frac{1}{2\pi i} \oint_{\gamma} z^{m-n-1} dz \quad (9.0.6)$$

Using the method for a generalized Fourier series, the usual Fourier series involving sines and cosines is obtained by taking $f_1(x) = \cos x$ and $f_2(x) = \sin x$. Since these functions form a complete orthogonal system over $[-\pi, \pi]$, the Fourier series of a function $f(x)$ is given by (with $n \in \mathbb{N}$)

$$f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx) \quad (9.0.7)$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx \quad (9.0.8)$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx \quad (9.0.9)$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx \quad (9.0.10)$$

The notion of a Fourier series can also be extended to complex coefficients.

$$f(x) = \sum_{n=-\infty}^{\infty} A_n e^{inx} \quad \text{with} \quad A_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx \quad (9.0.11)$$

For a function $f(x)$ periodic on an interval $[-L, L]$ instead of $[-\pi, \pi]$, a simple change of variables can be used to transform the interval of integration from $[-\pi, \pi]$ to $[-L, L]$. Let

$$x \equiv \frac{\pi x'}{L} \iff x' \equiv \frac{Lx}{\pi} \implies dx = \frac{\pi dx'}{L} \quad (9.0.12)$$

We can define the **Fourier transform** of $f(x)$ [10] by

$$F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{i\omega t} dx \iff \frac{f(x^+) + f(x^-)}{2} = \int_{-\infty}^{\infty} F(\omega) e^{-i\omega t} d\omega. \quad (9.0.13)$$

The **Fourier transform** of a **Gaussian** [10] is given by

$$F(\omega) = \frac{1}{\sqrt{4\pi\beta}} e^{-\omega^2/4\beta} \iff f(x) = \int_{-\infty}^{\infty} e^{-\beta x^2} e^{-i\omega t} d\omega \quad (9.0.14)$$

$$F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\alpha\omega^2} e^{i\omega t} dx \iff f(x) = \sqrt{\frac{\pi}{\alpha}} e^{-x^2/4\alpha}. \quad (9.0.15)$$

The **Convolution theorem** states

$$\int_{-\infty}^{\infty} F(\omega) G(\omega) e^{-i\omega t} d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(\bar{x}) f(x - \bar{x}) d\bar{x} = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\omega) g(x - \omega) d\omega \quad (9.0.16)$$

Statistics

A probability distribution: Given a Poisson process, the probability of obtaining exactly m successes in n trials is given by the limit of a binomial distribution

$$\mathcal{P}_n(m; p) = \binom{n}{m} p^m (1-p)^{n-m} \quad (10.0.1)$$

Letting the sample size n become large, the distribution then approaches the Poisson Distribution

$$\mathcal{P}(m, \lambda) = \frac{\lambda^m}{m!} e^{-\lambda} \quad (10.0.2)$$

The mean number of events is

$$\langle m \rangle = \sum_{m=0}^{\infty} m \frac{\lambda^m}{m!} e^{-\lambda} = \lambda \quad (10.0.3)$$

And the standard deviation is

$$\sigma = \sqrt{\lambda} \quad (10.0.4)$$

The normal, or Gaussian distribution

$$\mathcal{P}(x; \mu, \sigma) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right] \quad (10.0.5)$$

$$\mathcal{P}(a \leq x \leq b) = \int_a^b \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right] \quad (10.0.6)$$

If the mean is not equal to zero, a more general distribution known as the noncentral chi-squared distribution results. In particular, if x_i are independent variates with a normal distribution having means μ_i and variances σ_i^2 for $i = 1, \dots, n$, then

$$\chi^2 \equiv \sum_{i=1}^n \frac{(x_i - \mu_i)^2}{\sigma_i^2}. \quad (10.0.7)$$

Given some function $f(x_1, x_2, \dots, x_n)$, the error of a calculation with each respective variable being denoted by σ_i , can be determined by

$$\sigma_f^2 = \left(\frac{\partial f}{\partial x_1}\right)^2 \sigma_{x_1}^2 + \left(\frac{\partial f}{\partial x_2}\right)^2 \sigma_{x_2}^2 + \dots + \left(\frac{\partial f}{\partial x_n}\right)^2 \sigma_{x_n}^2 = \sum_{i=1}^n \left(\frac{\partial f}{\partial x_i}\right)^2 \sigma_{x_i}^2 \quad (10.0.8)$$

Astronomy, Optics and Telescopes

A **parsec** is defined so

$$1 \text{ parsec} = \frac{1 \text{ AU}}{\tan(1'')} \approx \frac{1 \text{ AU}}{1''} \quad (11.0.1)$$

The **flux** (F) of a star relates to it's luminosity (L) and distance (d) via

$$F = \frac{L}{4\pi R^2} = \sigma_{SB} T_{eff}^4 \quad (11.0.2)$$

The flux received by a telescope at distance d is then

$$F(d) = \frac{L}{4\pi d^2} = \sigma T_{eff}^4 \left(\frac{R}{d} \right)^2 \quad (11.0.3)$$

We can relate apparent magnitude m and absolute magnitude M by the distance of an object

$$10^{(m-M)/2.5} = \frac{F(10\text{pc})}{F(d)} = \left(\frac{d}{10\text{pc}} \right)^2. \quad (11.0.4)$$

The ratio of two magnitudes using different filters from a single star gives a rough estimation of the stars color.

$$B - V = m_B - m_V = -2.5 \log_{10} \left(\frac{F_B}{F_V} \right) \quad (11.0.5)$$

$$\frac{F_B}{F_V} = 10^{-(M_B - M_V)/2.5} \quad (11.0.6)$$

We define the **distance modulus** (DM) as the difference in apparent magnitude (m) between a given star and the absolute magnitude (M) it would have if it were at 10 pc.

$$DM \equiv m - m(10 \text{ pc}) \equiv m - M \quad (11.0.7)$$

$$M \equiv m - DM \quad (11.0.8)$$

The full form of intensity as a function of angle from the beam axis is

$$I = I_0 \left[\frac{\sin(\pi D/\lambda \sin(\theta))}{\sin(\pi d/\lambda \sin(\theta))} \right]^2 \quad (11.0.9)$$

Snell's Law:

$$\frac{\sin(\theta_1)}{\sin(\theta_2)} = \frac{v_1}{v_2} = \frac{\lambda_1}{\lambda_2} = \frac{n_2}{n_1} \quad (11.0.10)$$

11.1 Celestial Orbits

Suppose we have a exoplanet system with a planet p and a star s . The vector from the star to the planet is $\vec{r}_{sp} = \vec{r}_p - \vec{r}_s$, and the force that the star exerts on the planet is (\vec{r}_n is the vector from the origin to n)

$$\vec{F}_{sp} = -\frac{GM_p M_s}{|\vec{r}_{sp}|^3} \vec{r}_{sp} \quad (11.1.1)$$

If we put the origin at the center of mass (\vec{R} is the vector from the origin to the center of mass)

$$\vec{R} = \frac{M_s \vec{r}_s + M_p \vec{r}_p}{M_s + M_p} \quad (11.1.2)$$

Then the star and planets have positions

$$\vec{x}_s = \vec{r}_s - \vec{R} = -\frac{M_p}{M_p + M_s} \vec{r}_{sp} \quad (11.1.3)$$

$$\vec{x}_p = \vec{r}_p - \vec{R} = -\frac{M_s}{M_p + M_s} \vec{r}_{sp} \quad (11.1.4)$$

And thus accelerations

$$\frac{d^2 \vec{x}_s}{dt^2} = -\frac{M_p}{M_p + M_s} \frac{d^2 \vec{r}_{sp}}{dt^2} \quad (11.1.5)$$

$$\frac{d^2 \vec{x}_p}{dt^2} = -\frac{M_s}{M_p + M_s} \frac{d^2 \vec{r}_{sp}}{dt^2} \quad (11.1.6)$$

Substituting the acceleration into the equation of motion for the planet,

$$M_p \frac{d^2 \vec{x}_p}{dt^2} = \vec{F}_{sp} \quad (11.1.7)$$

Then we can get the reduced equation of motion as

$$\frac{d^2 \vec{r}_{sp}}{dt^2} = -G \frac{M_s + M_p}{|\vec{r}_{sp}|^3} \vec{r}_{sp} \quad (11.1.8)$$

Keplar's Third law: The solution to this is an elliptical orbit with the center-of-force at one focus of the ellipse. The period (T) depends on the semi-major axis (a)

$$T^2 = \frac{4\pi^2}{G(M_s + M_p)} a^3 \quad (11.1.9)$$

$$a^3 = \frac{G(M_s + M_p)}{4\pi^2} T^2 \quad (11.1.10)$$

If the orbit is circular, so that $|\vec{r}_s p| = a$ is constant, then the orbital speed of the star is

$$v_s = \frac{2\pi a M_p}{T(M_p + M_s)} = \sqrt{\frac{GM_p^2}{a(M_p + M_s)}} \quad (11.1.11)$$

For a particle in a circular orbit, $v = r\Omega\hat{\theta}$; using Kepler's law (r is the distance from the center of mass), we have

$$L = mr^2\Omega = m\sqrt{GMr}. \quad (11.1.12)$$

The orbital angular momentum of the two-body system is

$$L = \frac{M_1 M_2}{M_1 + M_2} a^2 \Omega \quad (11.1.13)$$

$$= \frac{M_1 M_2}{M_1 + M_2} \sqrt{G(M_1 + M_2)a}. \quad (11.1.14)$$

The angular momentum of a sphere is

$$L = \frac{8\pi}{15} \rho \Omega R^5 = \frac{2}{5} MR^2 \Omega. \quad (11.1.15)$$

The equations of motion in a rotating frame are

$$\frac{d^2\vec{r}'}{dt^2} = \frac{1}{m} \vec{F}_{rot} \quad (11.1.16)$$

$$= \frac{1}{m} \vec{F} + \underbrace{\frac{r\Omega^2}{m} \hat{r}}_{\text{centrifugal}} + \underbrace{2\Omega(v_\theta \hat{r} - v_r \hat{\theta})}_{\text{coriolis}}. \quad (11.1.17)$$

Particles within a sphere of radius R_H are dominated by the gravitational attraction of M_2 ; R_H (the Hill radius) is

$$R_H \approx a \left[\frac{M_2}{3(M_1 + M_2)} \right]^{1/3}. \quad (11.1.18)$$

11.2 Celestial & Stellar Atmospheres

The equation of hydrostatic equilibrium

$$\frac{dP}{dr} = -\rho g. \quad (11.2.1)$$

The ideal gas law can be written (with m =mass of 1 mole of our gas) as

$$P = \left(\frac{mN/N_A}{V} \right) \frac{kN_A}{m} T \equiv \rho \frac{kN_A}{m} T. \quad (11.2.2)$$

Combining the above two equations and assuming $T=\text{constant}$ then yields a relation between pressure and height as

$$\frac{dP}{P} = -\frac{mg}{N_A k T} dz. \quad (11.2.3)$$

This then gives a pressure dependant on height as

$$P(z) = P_0 \exp \left[-\frac{mgz}{N_A k T} \right]. \quad (11.2.4)$$

In addition to the Coriolis acceleration from the Earth rotation, horizontal pressure gradients will also produce an acceleration

$$-\frac{1}{\rho} \nabla P. \quad (11.2.5)$$

The equation for force and acceleration along $r\hat{r}$ is therefore

$$\underbrace{\frac{v^2}{r}}_{\text{centripital}} + \underbrace{2v\Omega \sin(\lambda)}_{\text{coriolis}} - \underbrace{\frac{1}{\rho} \frac{dP}{dr}}_{\text{pressure}} = 0. \quad (11.2.6)$$

If matter is in thermal equilibrium, then populations of a two different states of a given atom are given by Boltzmann's formula,

$$\frac{n_i}{n_j} = \frac{g_i}{g_j} \exp \left(\frac{E_j - E_i}{kT} \right). \quad (11.2.7)$$

Hydrostatic equilibrium (where m is the mass within a sphere of radius r , P is the pressure, and ρ is the mass density) gives two equations of stellar structure,

$$\frac{dm}{dr} = 4\pi r^2 \rho \quad \text{and} \quad \frac{dP}{dr} = -\rho \frac{Gm}{r^2} \quad (11.2.8)$$

From the virial theorem, the average pressure and density are

$$\bar{\rho} = \frac{GM}{4\pi R^3} \quad \text{and} \quad \bar{P} \propto \frac{GM^2}{R^4} \quad (11.2.9)$$

The optical depth for an outward-directed ray is

$$\tau_\mu = \int_z^\infty \rho \kappa_\mu dz' \implies \frac{d\tau}{dz} = -\rho \kappa \quad (11.2.10)$$

From this, an estimate of the photospheric pressure can be determined for a gray atmosphere in LTE¹ by,

$$\frac{dP}{d\tau} = - \left(\frac{d\tau}{dz} \right)^{-1} \rho g = \frac{g}{\kappa}. \quad (11.2.11)$$

From hydrostatic equilibrium and taking $\rho = \text{constant}$ (where μm_u is the average mass of a particle in the plasma), the central pressure and temperature are given by

$$T_c = \frac{GM\mu m_u}{2Rk_B} \quad (11.2.12)$$

$$P_c = \frac{3GM^2}{8\pi R^4} \quad (11.2.13)$$

Bringing a small amount of mass dm from infinity onto a sphere of mass m and radius r gives a potential change of

$$d\Omega = - \frac{Gm}{r} dm. \quad (11.2.14)$$

For a constant density, we have $r = R(m/M)^{1/3}$ and so

$$\Omega = - \frac{3GM^2}{5R}. \quad (11.2.15)$$

Using this, the mean temperature and pressure for a constant density sphere is

$$\bar{T} = \frac{GM\mu m_u}{5Rk_B} \quad (11.2.16)$$

$$\bar{P} = \frac{3GM^2}{20\pi R^4}. \quad (11.2.17)$$

The free fall time it would take for a star to collapse if all internal pressures were removed is

$$\tau_{ff} = \frac{\pi}{\sqrt{GM}} \left(\frac{R}{2} \right)^{3/2} = \left(\frac{3}{32\pi} \right)^{1/2} \frac{1}{\sqrt{G\rho}}. \quad (11.2.18)$$

The **dynamical timescale** of the star is defined from the proportionality constant of the free fall time

$$t_{dyn} \equiv \frac{1}{\sqrt{G\rho}}. \quad (11.2.19)$$

Any change in pressure is communicated through a star by sound waves which travel at the speed

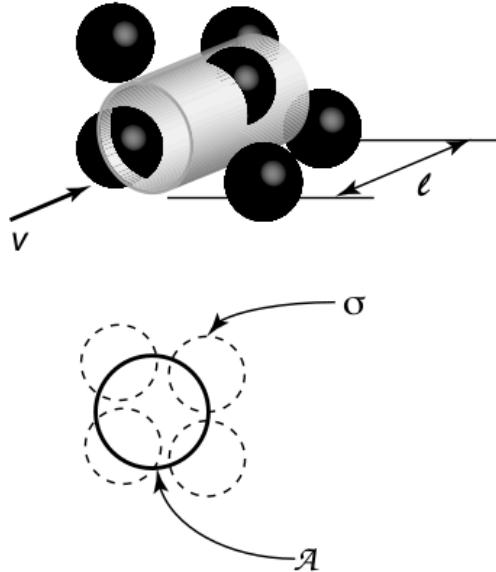
$$c_s = \left(\gamma \frac{P}{\rho} \right)^{1/2} = \left(\gamma \frac{k_B T}{\mu m_u} \right)^{1/2}. \quad (11.2.20)$$

The time it takes for a sound wave to travel a distance R is then

$$\tau_{sc} = \frac{R}{c_s} = \sqrt{\frac{3R^3}{GM}} = \left(\frac{3}{2\sqrt{\pi}} \right) \frac{1}{\sqrt{G\rho}}. \quad (11.2.21)$$

The **Kelvin-Helmholtz timescale** is the time it would take the sun to radiate all of its gravitational energy away with its current luminosity L_\odot ,

$$t_{KH} \approx \frac{GM_\odot^2}{R_\odot L_\odot} \approx 3 \times 10^7 \text{ yr}. \quad (11.2.22)$$



As displayed by the image above², the probability of a particle making it through a density of obstacles n with cross section σ is

$$\mathcal{P} = \frac{n(\mathcal{A}\ell)\sigma}{\mathcal{A}} = n\sigma\ell. \quad (11.2.23)$$

The **mean free path** is defined to be the length at which $\mathcal{P} \rightarrow 1$ which is when the particle will suffer a collision:

$$\ell = \frac{1}{n\sigma}. \quad (11.2.24)$$

¹Local Thermodynamical Equilibrium

²"Schematic of a particle incident on a group of particles." [4].

11.3 Convection

The temperature gradient in a star ($\kappa = \text{opacity}$) is

$$\frac{dT}{dr} = -\frac{3\rho\kappa}{4acT^3} \frac{L(r)}{4\pi r^2}. \quad (11.3.1)$$

From the first law of thermodynamics,

$$dQ = dU - \frac{P}{\rho^2} d\rho \quad (11.3.2)$$

$$dU = \left(\frac{\partial U}{\partial T}\right)_\rho dT + \left(\frac{\partial U}{\partial \rho}\right)_T d\rho \quad (11.3.3)$$

$$dQ = \left(\frac{\partial U}{\partial T}\right)_\rho dT + \left[\left(\frac{\partial U}{\partial \rho}\right)_T - \frac{P}{\rho^2}\right] d\rho. \quad (11.3.4)$$

While holding density fixed, the heat needed to raise the temperature of one kilogram of fluid is then

$$C_\rho \equiv \left(\frac{\partial Q}{\partial T}\right)_\rho = \left(\frac{\partial U}{\partial T}\right)_\rho. \quad (11.3.5)$$

From this, heat transfer can be expressed as a function of temperature and pressure

$$dQ = \left[C_\rho + \frac{P}{\rho T}\right] dT - \frac{1}{\rho} dP \quad (11.3.6)$$

$$= \left[C_\rho + \frac{k_B}{\mu m_u}\right] dT - \frac{1}{\rho} dP. \quad (11.3.7)$$

Hence, while holding pressure fixed, the heat needed to raise the temperature of one kilogram of fluid is

$$C_P = \left(\frac{\partial Q}{\partial T}\right)_P = C_\rho + \frac{k_B}{\mu m_u}. \quad (11.3.8)$$

For a plasma of ions and electrons,

$$C_\rho = \frac{3k_B}{2\mu m_u} = \frac{3}{5} C_P. \quad (11.3.9)$$

Thus the ratio of specific heats for an ideal gas is

$$\gamma = \frac{C_P}{C_\rho} = \frac{5}{3}. \quad (11.3.10)$$

During adiabatic motion, no heat exchange occurs and so $TdS = dQ = 0$ which leads to

$$T = T_0 \left(\frac{P}{P_0}\right)^{(\gamma-1)/\gamma}. \quad (11.3.11)$$

The temperature change with pressure in an adiabatically stratified gas is given by

$$\frac{P}{T} \left(\frac{\partial T}{\partial P}\right)_S = \left(\frac{\partial \ln T}{\partial \ln P}\right)_S = \frac{\gamma-1}{\gamma}. \quad (11.3.12)$$

For stable convection, we must have

$$\left(\frac{\partial V}{\partial S}\right)_P \frac{dS}{dr} = \frac{T}{C_P} \left(\frac{\partial V}{\partial T}\right)_P \frac{dT}{dr} > 0. \quad (11.3.13)$$

The stability requirements for convection can also be derived in terms of local gradients of temperature and pressure. The fluid is unstable to convection if

$$\frac{P}{P_{rad}} \frac{\kappa}{16\pi Gc} \frac{L(r)}{m(r)} > \left(\frac{\partial \ln T}{\partial \ln P}\right)_S = \frac{\gamma-1}{\gamma}. \quad (11.3.14)$$

11.4 Main Sequence Stars

For an enclosed mass we have the following relations for radiative and convective regions respectively

$$\frac{dT}{dr} = -\frac{L}{4\pi r^2} \frac{3\rho\kappa}{4acT^3} \quad (11.4.1)$$

$$\frac{dT}{dr} = \frac{T}{P} \left(\frac{\partial \ln T}{\partial \ln P}\right)_S \frac{dP}{dr}. \quad (11.4.2)$$

The fourth order equation of stellar structure (with ϵ being the heating rate per unit mass) is

$$\frac{dL}{dr} = 4\pi r^2 \rho \epsilon. \quad (11.4.3)$$

11.5 Galaxies

The Friedmann-Lemaître-Robertson-Walker (FLRW) Metric which relates space and time and is an exact solution of Einsteins field equations in general relativity using the assumptions of homogeneous and isotropic space ($a(t)$ is the expansion parameter).

$$c^2 ds^2 = -c^2 dt^2 + a(t)^2 \left(\frac{dr^2}{1 - (r/r_0)^2} + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \right) = -c^2 dt^2 + d\vec{\Sigma}^2 \quad (11.5.1)$$

From this, we can determine a flux-luminosity relation that is dependent upon the expansion parameter.

$$F_\nu(\nu) = \frac{L(\nu a^{-1})a}{4\pi r(a)^2}. \quad (11.5.2)$$

Friedman's Equation is given by

$$H^2 - \frac{8\pi}{3}G\rho = -r_0^{-2}a^{-2} \implies \Omega - 1 = (r_0 Ha)^{-2} \quad (11.5.4)$$

Using this we can come to the following result:

$$\int_0^1 [\Omega_{m,0}a^{-3} + \Omega_{r,0}a^{-4} + \Omega_{\nu,0} + (1 - \Omega_0)a^{-2}]^{-1/2} da = (\Omega_0 - 1)^{-1/2} \sin^{-1}[H_0 r(\Omega_0 - 1)^{1/2}] \quad (11.5.5)$$

A black object emites a Planck spectrum. The energy emmited per second per m^2 per steradian per hertz follows, whereas a grey body emits $\epsilon B_\nu(T)$ (where ϵ is the emissivity).

$$B_\nu(T) = \frac{2h\nu^3}{c^2 e^{\frac{h\nu}{kT}} - c^2} \quad (11.5.6)$$

Wien's Law is given by

$$\lambda_{max}T = 2.9 \text{mm}\cdot\text{K}. \quad (11.5.7)$$

The Rayleigh-Jeans approximation says that if $h\nu \ll kT$ (the energy of a photon is much smaller

We can define a density parameter for each type of matter as

$$\Omega_i = \frac{8\pi G \rho_i}{3H^2}. \quad (11.5.3)$$

than the characteristic energy for a specific temperature) then

$$B_\nu = \frac{2kT\nu^2}{c^2} \quad (11.5.8)$$

The history of the background radiation is inversely proportional to the expansion parameter.

$$T(a) = \frac{2.7\text{K}}{a} \quad (11.5.9)$$

Classical Mechanics

Newton's Second Law in Cartesian coordinates and 2D Polar coordinates

$$\vec{F} = m\vec{a} = m\ddot{\vec{r}} \iff \begin{cases} F_x = m\ddot{x} \\ F_y = m\ddot{y} \\ F_z = m\ddot{z} \end{cases} \quad (12.0.1)$$

$$\iff \begin{cases} F_r = m(\ddot{r} - r\dot{\phi}^2) \\ F_\phi = m(r\ddot{\phi} + 2\dot{r}\dot{\phi}) \end{cases} \quad (12.0.2)$$

Conservation of energy

$$E = \text{constant} = KE + PE \quad (12.0.3)$$

$$= \frac{1}{2}m|\vec{v}|^2 + mgh \quad (12.0.4)$$

Equation of motion for a rocket

$$\dot{m}\vec{v} = -\dot{m}\vec{v}_{ex} + \vec{F}^{external} \quad (12.0.5)$$

The center of mass of several particles with a total mass M is

$$\vec{R} = \frac{1}{M} \sum_{\alpha=1}^n m_\alpha \vec{r}_\alpha = \frac{m_1 \vec{r}_1 + \dots + m_n \vec{r}_n}{M} \quad (12.0.6)$$

$$\vec{R} = \frac{1}{M} \int \vec{r} dm = \frac{1}{M} \int \rho \vec{r} dV \quad (12.0.7)$$

The mass of an object is defined by the density multiplied by the volume.

$$M \equiv \rho V \equiv \iiint_Q \rho(x, y, z) dV \quad (12.0.8)$$

The moment of inertia with respect to a given axis of a solid body with density $\rho(r)$, where r_\perp is the perpendicular distance from the axis of rotation, is defined by the volume integral

$$I \equiv \int \rho(\vec{r}) r_\perp^2 dV \equiv \iiint_Q \rho(x, y, z) ||\vec{r}||^2 dV \quad (12.0.9)$$

Angular momentum

$$\vec{L} = \vec{r} \times \vec{p} = I\vec{\omega} = I\dot{\theta} \quad (12.0.10)$$

The net external torque is given by

$$\vec{\tau}_{ext} = \vec{r} \times \vec{F} = \frac{d\vec{L}}{dt} \quad (12.0.11)$$

The change in kinetic energy as it moves from point a to point b is

$$\Delta K \equiv K_2 - K_1 = \int_a^b \vec{F} \cdot d\vec{r} \equiv W(a \rightarrow b) \quad (12.0.12)$$

$$K = \frac{1}{2}mv^2 = \frac{1}{2}I\omega^2 = \frac{1}{2}I\dot{\theta}^2 \quad (12.0.13)$$

A force \vec{F} on a particle is **conservative** if (i) it depends only on the particles position, $\vec{F} = \vec{F}(\vec{r})$ and (ii) $\nabla \times \vec{F} = 0$. If \vec{F} is conservative we can define a corresponding **potential energy** so that

$$U(\vec{r}) = -W(\vec{r}_0 \rightarrow \vec{r}) \equiv \int_{\vec{r}_0}^{\vec{r}} \vec{F}(\vec{r}') \cdot d\vec{r}' \quad (12.0.14)$$

$$\vec{F} = -\nabla U \quad (12.0.15)$$

Hooke's Law states that the force needed to extend or compress a spring by some distance is proportional to that distance.

$$F = -kx \iff U = \text{constant} + \frac{1}{2}kx^2 \quad (12.0.16)$$

Simple harmonic motion

$$\ddot{x} = -\omega^2 x \iff A \cos(\omega t - \delta) \quad (12.0.17)$$

Damped oscillations: If the oscillator is subject to a damping force $-bv$, the

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0 \text{ and } \beta < \omega_0 \iff x(t) = A e^{-\beta t} \cos(\omega_1 t - \delta) \quad (12.0.18)$$

$$\beta = \frac{b}{2m}, \quad \omega_0 = \sqrt{\frac{k}{m}}, \quad \omega_1 = \sqrt{\omega_0^2 - \beta^2} \quad (12.0.19)$$

If the oscillator is also subject to a sinusoidal driving force $F(t) = m f_0 \cos(\omega t)$, the long-term motion has the form

$$x(t) = A \cos(\omega t - \delta) \quad (12.0.20)$$

$$A^2 = \frac{f_0^2}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2} \quad (12.0.21)$$

It is always possible to write a sum of sinusoidal functions as a single sinusoid the form

$$f(\theta) = A \cos(\theta) + B \sin(\theta) \iff f(\theta) = C \cos(\theta + \delta) \quad (12.0.22)$$

$$\delta = \arctan(-B/A) \quad \text{and} \quad C = \pm \sqrt{A^2 + B^2} \quad (12.0.23)$$

$$f(\theta) = A \cos(\theta) + B \sin(\theta) \iff f(\theta) = \operatorname{sgn}(A) \sqrt{A^2 + B^2} \cos(\theta + \arctan(-B/A)) \quad (12.0.24)$$

Any periodic function with period τ can be written as (A Fourier series with $n \geq 1$)

$$f(t) = \sum_{n=0}^{\infty} [a_n \cos(n\omega t) + b_n \sin(n\omega t)] \quad (12.0.25)$$

$$a_n = \frac{2}{\tau} \int_{-\tau/2}^{\tau/2} f(t) \cos(n\omega t) dt \quad (12.0.26)$$

$$b_n = \frac{2}{\tau} \int_{-\tau/2}^{\tau/2} f(t) \sin(n\omega t) dt \quad (12.0.27)$$

$$a_0 = \frac{1}{\tau} \int_{-\tau/2}^{\tau/2} f(t) dt \quad (12.0.28)$$

It is sometimes useful to express the above Fourier series as an exponential

$$f(t) = \sum_{n=-\infty}^{\infty} A_n e^{in\omega t} \quad (12.0.29)$$

$$A_n = \frac{1}{\tau} \int_{-\tau/2}^{\tau/2} f(t) e^{-in\omega t} dt \quad (12.0.30)$$

It is important to know $A_n = A_{-n}^*$ so we can write $A_n = \Re(A_n) + i\Im(A_n)$. An important relationship between A_n , a_n and b_n then follows as,

$$a_n = 2\Re(A_n) \quad \text{and} \quad b_n = -2\Im(A_n) \quad (12.0.31)$$

The root-mean square displacement is a good measure of the average response of the oscillator and is given by **parseval's theorem**

$$x_{rms} = \sqrt{\frac{1}{\tau} \int_0^\tau x^2 dt} \quad (12.0.32)$$

$$= \sqrt{A_0^2 + \frac{1}{2} \sum_{n=1}^{\infty} A_n^2} \quad (12.0.33)$$

The non-relativistic Lagrangian \mathcal{L} for a conservative system can be defined in terms of the kinetic energy and potential energy of a system as

$$\mathcal{L} = KE - PE \quad (12.0.34)$$

An integral of the form

$$S = \int_{x_1}^{x_2} f[y(x), y'(x), x] dx \quad (12.0.35)$$

taken along a path $y = y(x)$ is stationary with respect to variations of that path if and only if $y(x)$ satisfies the Euler-Lagrange Equation

$$\frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} = 0. \quad (12.0.36)$$

If there are n dependent variables in the original integral, there are n Euler-Langrange equations. For instance, an integral of the form

$$S = \int_{u_1}^{u_2} f[x(u), y(u), x'(u), y'(u), u] du \quad (12.0.37)$$

with two dependent variables $[x(u)$ and $y(u)$], is stationary with respect to variations of $x(u)$ and $y(u)$ if and only if these two functions satisfy the two equations

$$\frac{\partial f}{\partial x} = \frac{d}{du} \frac{\partial f}{\partial x'} \quad \text{and} \quad \frac{\partial f}{\partial y} = \frac{d}{du} \frac{\partial f}{\partial y'} \quad (12.0.38)$$

For any holonomic system, Newtons second law is equivalent to the n Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \quad (12.0.39)$$

The i th generalized momentum p_i is defined to be the derivative

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \quad (12.0.40)$$

If $\partial \mathcal{L}/\partial t = 0$ then \mathcal{H} is conserved; if the coordinates q_1, \dots, q_n are natural, \mathcal{H} is just the energy of the system. The Hamiltonian \mathcal{H} is defined as

$$\mathcal{H} = \sum_{i=1}^n p_i \dot{q}_i - \mathcal{L} \quad (12.0.41)$$

The time evolution of a system is given by **Hamilton's equations**

$$\dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i} \quad \text{and} \quad \dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i} \quad (12.0.42)$$

The Lagrangian for a charge q in an electromagnetic field is

$$\mathcal{L}(\mathbf{r}, \dot{\mathbf{r}}, t) = \frac{1}{2} m \dot{\mathbf{r}}^2 - q(V - \dot{\mathbf{r}} \cdot \mathbf{A}) \quad (12.0.43)$$

Electricity and Magnetism

Maxwell's Equations: The system of partial differential equations describing classical electromagnetism. \vec{P} is the polarization field, \vec{D} is the electric displacement field, ρ is the charge density, \vec{E} is the electric field, \vec{B} is the magnetic field, and \vec{J} is the current density. In the so-called cgs system of units, the Maxwell equations are given by

$$\nabla \cdot \vec{E} = 4\pi\rho \quad (13.0.1)$$

$$\nabla \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t} \quad (13.0.2)$$

$$\nabla \cdot \vec{B} = 0 \quad (13.0.3)$$

$$\nabla \times \vec{B} = \frac{4\pi}{c} \vec{J} + \frac{1}{c} \frac{\partial \vec{E}}{\partial t} \quad (13.0.4)$$

In the MKS system of units (where ϵ_0 is the permittivity of free space and μ_0 is the permeability of free space), the equations are written

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \quad (13.0.5)$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad (13.0.6)$$

$$\nabla \cdot \vec{B} = 0 \quad (13.0.7)$$

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t} \quad (13.0.8)$$

From the field tensor and dual tensors, Maxwell's equations (where J^μ is the current density 4-vector) are given by

$$\frac{\partial F^{\mu\nu}}{\partial x^\nu} = \mu_0 J^\mu, \quad \frac{\partial G^{\mu\nu}}{\partial x^\nu} = 0 \quad \text{with} \quad J^\mu = (c\rho, J_x, J_y, J_z). \quad (13.0.9)$$

From Maxwell's equations, electric and magnetic fields can be shown to satisfy the wave equation in a vacuum allowing us to derive a speed for both fields which is equivalent to the speed of light (electromagnetic waves) in a vacuum.

$$\nabla^2 \vec{E} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} \quad \text{and} \quad \nabla^2 \vec{B} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2} \implies c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}. \quad (13.0.10)$$

In the special case of a steady state, known as **electrostatics**, with stationary charges and currents,

$$\nabla \times \vec{E} = 0 \implies \oint \vec{E} \cdot d\ell = 0 \quad (13.0.11)$$

The dipole moment is defined by

$$\vec{p} \equiv \sum_i q_i \vec{r}_i \quad (13.0.12)$$

$$\vec{p} \equiv \int_V \rho(\vec{r}') \vec{r}' d\tau' \quad (13.0.13)$$

If we consider both bound and free charges (where the free charges are the charges we place within a system), we have

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} = \frac{\rho_{\text{bound}} + \rho_{\text{free}}}{\epsilon_0} \quad (13.0.14)$$

$$= \frac{-\nabla \cdot \vec{P}}{\epsilon_0} + \frac{\rho_{\text{free}}}{\epsilon_0} \quad (13.0.15)$$

$$\implies \nabla \cdot \vec{D} = \rho_{\text{free}} \quad (13.0.16)$$

$$\implies \oint_S \vec{D} \cdot d\vec{a} = Q_{\text{free}} = \int_S \rho_{\text{free}} d\tau'. \quad (13.0.17)$$

The **polarization field** of a linearly polarized dielectric is characterized by its dipole moment per unit volume and can be defined by the susceptibility constant χ_e and the dielectric constant ϵ_R ,

$$\vec{P} = \lim \frac{\Delta \vec{p}}{\Delta v} = \frac{1}{\Delta v} \sum_i \vec{p}_i \equiv \epsilon_0 \chi_e \vec{E} = \frac{\chi_e}{1 + \chi_e} \vec{D} = \frac{\chi_e}{\epsilon_R} \vec{D} \quad \rightarrow \quad \begin{cases} \chi_e \rightarrow 0 & \implies \vec{P} \text{ for a vacuum} \\ \chi_e \rightarrow \infty & \implies \vec{P} \text{ for a metal} \end{cases} \quad (13.0.18)$$

From this, the bound charge densities for both the surface and volume are defined by

$$\rho_B = -\nabla \cdot \vec{P} \quad \text{and} \quad \sigma_B = \vec{P} \cdot \hat{n}. \quad (13.0.19)$$

The electric displacement field is defined such that

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P} \implies \vec{D} = \epsilon_0(1 + \chi_e) \vec{E} = \epsilon_0 \epsilon_R \vec{E} \iff \vec{P} = \epsilon_0(1 - \epsilon_R) \vec{E}. \quad (13.0.20)$$

Coulomb's Law: The force on a test charge Q due to a single point charge q with the separation between them being $|\vec{r}|$ (note: $\vec{r} = \vec{r} - \vec{r}'$ is the separation vector from the location of q - denoted \vec{r}' - to the location of Q - denoted \vec{r}) is given by

$$\vec{F} = \frac{1}{4\pi\epsilon_0} \frac{qQ}{|\vec{r}|^2} \hat{r} \quad (13.0.21)$$

Given stationary charges and currents, the electric field $\vec{E}(\vec{r})$ can be written as

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{dq}{|\vec{r}'|^2} \hat{r} \quad (13.0.22)$$

$$\equiv \frac{1}{4\pi\epsilon_0} \iiint_V \frac{\rho(\vec{r}')}{|\vec{r}'|^2} \hat{r} d\tau' \quad (\text{volume}) \quad (13.0.23)$$

$$\equiv \frac{1}{4\pi\epsilon_0} \iint_A \frac{\sigma(\vec{r}')}{|\vec{r}'|^2} \hat{r} da' \quad (\text{area}) \quad (13.0.24)$$

$$\equiv \frac{1}{4\pi\epsilon_0} \int_l \frac{\lambda(\vec{r}')}{|\vec{r}'|^2} \hat{r} dl' \quad (\text{line}) \quad (13.0.25)$$

Gauss's Law: The electric flux Φ_E through a surface S enclosing any volume is proportional to the total charge enclosed within the volume. This is an alternate form of one of Maxwell's equations.

$$\Phi_E = \iint_S \vec{E} \cdot d\vec{a} = \iint_S (\vec{E} \cdot \hat{n}) da = \frac{Q_{enc}}{\epsilon_0} \quad (13.0.26)$$

An electric potential V is a continuous function and is defined as

$$V(\vec{r}) \equiv - \int_{\mathcal{O}}^r \vec{E}(\vec{r}') \cdot d\vec{l}' \quad (13.0.27)$$

$$\equiv \frac{1}{4\pi\epsilon_0} \iiint_V \frac{\rho(\vec{r}')}{|\vec{r}'|} d\tau' \quad (13.0.28)$$

Using this and the fundamental theorem for gradients, we have

$$\int_a^b (\nabla V) \cdot d\vec{l}' = - \int_a^b \vec{E} \cdot d\vec{l}' \quad (13.0.29)$$

$$\implies \vec{E} = -\nabla V \quad (13.0.30)$$

Poisson's equation can be used to determine the charge density of a function from the electric potential.

$$\nabla^2 V(\vec{r}) = -\frac{\rho(\vec{r}')}{\epsilon_0} \quad (13.0.31)$$

Using a special case of Poisson's equation when $\rho = 0$, we can derive the multi-pole expansion.

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \sum_i \frac{q_i}{|\vec{r}_i|} \frac{1}{4\pi\epsilon_0} \sum_i \frac{q_i}{|\vec{r} - \vec{r}_i|} \quad (13.0.32)$$

From the **multi-pole expansion**, we can approximate the potential as

$$V(r, \theta) \approx \underbrace{\frac{Q_{tot}}{4\pi\epsilon_0 r}}_{\text{monopole}} + \underbrace{\frac{\vec{p} \cdot \hat{r}}{4\pi\epsilon_0 r^2}}_{\text{dipole}} \quad (13.0.33)$$

$$= \frac{Q_{tot}}{4\pi\epsilon_0 r} + \frac{1}{4\pi\epsilon_0 r^2} \int_V \rho(\vec{r}') \vec{r}' \cdot \hat{r} d\tau' \quad (13.0.34)$$

The work on a system due to an electric field is given by

$$W_{sys} \equiv \sum_{j=1}^m W_j = \sum_{j=1}^m \left(\sum_{k=1}^{j-1} \frac{q_j q_k}{4\pi\epsilon_0 r_{jk}} \right) \equiv \frac{1}{2} \int \rho(\vec{r}) V(\vec{r}) d\tau \implies W_{sys} = \frac{\epsilon_0}{2} \int E^2 d\tau \quad (13.0.35)$$

The energy stored due to an electric field and magnetic field is given by

$$U = \frac{1}{2} \int \left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2 \right) d\tau \quad (13.0.36)$$

The Lorentz force law: The magnetic force on a charge q , moving with velocity \vec{v} due to a magnetic field \vec{B} and an electric field \vec{E} is

$$\vec{F} = q[\vec{E} + (\vec{v} \times \vec{B})] \quad (13.0.37)$$

A line charge λ traveling down a wire at speed v constitutes a **current** $\vec{I} = \lambda\vec{v}$ [7]. The magnetic force on a segment of current-carrying wire is

$$\vec{F}_{mag} = \int (\vec{v} \times \vec{B}) dq = \int (\vec{v} \times \vec{B}) \lambda d\ell = \int (\vec{I} \times \vec{B}) d\ell = \int I(d\vec{\ell} \times \vec{B}). \quad (13.0.38)$$

When charge (q) flows over a surface or through a volume, we describe it by the surface current density \vec{K} and the volume current density \vec{J} respectively. By definition (where N is number of charge carriers for unit volume with some velocity \vec{v}), these are

$$\vec{K} \equiv \frac{d\vec{I}}{d\ell_{\perp}} \quad \text{and} \quad \vec{J} \equiv \frac{d\vec{I}}{da_{\perp}} = \vec{J}_{bound} + \vec{J}_{free} \equiv \sum_i N_i q_i \vec{v}_i. \quad (13.0.39)$$

The total **current** through a surface can be defined

$$I = \int_S (\vec{J} \cdot \hat{n}) da = \frac{dQ}{dt} \quad (13.0.40)$$

From the surface and volume currents, we can express the magnetic force as

$$\vec{F}_{mag} \equiv \int (\vec{v} \times \vec{B}) \sigma da = \int (\vec{K} \times \vec{B}) da \quad (13.0.41)$$

$$\vec{F}_{mag} \equiv \int (\vec{v} \times \vec{B}) \rho d\tau = \int (\vec{J} \times \vec{B}) d\tau. \quad (13.0.42)$$

The **Biot-Savart law** gives the magnetic field of a steady state line current

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{I} \times \hat{\mathbf{r}}}{|\vec{r}|^2} d\ell' \quad (13.0.43)$$

$$= \frac{\mu_0}{4\pi} I \int \frac{d\vec{\ell}' \times \hat{\mathbf{r}}}{|\vec{r}|^2}. \quad (13.0.44)$$

Magneto-statics is defined when

$$\nabla \cdot \vec{J} = 0 \quad (13.0.45)$$

The **continuity equation** is a precise mathematical statement of local charge conservation.

$$\nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t} \quad (13.0.46)$$

When dealing with surface and volume currents, the Biot-Savart law becomes

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{K}(\vec{r}') \times \hat{\mathbf{r}}}{|\vec{r}'|^2} da' \quad (13.0.47)$$

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}') \times \hat{\mathbf{r}}}{|\vec{r}'|^2} d\tau' \quad (13.0.48)$$

Ampere's Law: For a straight line current, the integral of \vec{B} around an Amperien path centered at a wire is related to the total enclosed current by

$$\oint \vec{B} \cdot d\vec{\ell} = \mu_0 I_{enc} = \mu_0 \int \vec{J} \cdot d\vec{a} \quad (13.0.49)$$

$$= \int (\nabla \times \vec{B}) \cdot d\vec{a} \quad (13.0.50)$$

From Maxwell's equation $\nabla \cdot \vec{B} = 0$, we can define the vector potential \vec{A} such that $\nabla \cdot (\nabla \times \vec{A}) = 0 \implies \vec{B} = \nabla \times \vec{A}$. From this, we can also define the gauge freedom such that $\nabla \cdot (\nabla \vec{A}) = 0$ which gives us

$$\nabla^2 \vec{A} = -\mu_0 \vec{J} \implies \vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}') d\tau'}{|\vec{r}'|} \equiv \frac{\mu_0}{4\pi} \int \frac{\vec{K}(\vec{r}') da'}{|\vec{r}'|} \quad (13.0.51)$$

We can define the **magnetic dipole moment** \vec{m} (which is a measurable quantity) and the magnetization \vec{M} of a material in terms of a line current or surface current density to be

$$\vec{m} \equiv I \int_S d\vec{a} = I\vec{a}, \quad \text{or} \quad \vec{m} \equiv \frac{1}{2} \int_V (\vec{r} \times \vec{J}) dV = \int_V \vec{M} dV = \frac{I}{2} \oint_C \vec{r} \times d\vec{\ell}, \quad (13.0.52)$$

$$\text{with} \quad \vec{M} = \frac{d\vec{m}}{dV} \equiv \lim_{\Delta v \rightarrow 0} \frac{1}{\Delta v} \sum_i \vec{m}_i \quad (13.0.53)$$

From this, we can do a multi-pole expansion of the vector potential. The mono-pole term evaluates to zero and the dipole term becomes useful in many cases and can be written as follows by use of Stokes theorem.

$$\vec{A}(\vec{r}) = \frac{\mu_0 I}{4\pi} \oint \frac{d\vec{\ell}}{|\vec{r}|} = \frac{\mu_0 I}{4\pi} \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \oint (\vec{r}')^n P_n(\cos \alpha) d\vec{\ell} \implies \vec{A}_{dip}(\vec{r}) = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \hat{r}}{r^2} \quad (13.0.54)$$

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int_S \frac{\vec{K}_B(\vec{r}')}{|\vec{r}|} da' + \frac{\mu_0}{4\pi} \int_V \frac{\vec{J}_B(\vec{r}')}{|\vec{r}|} d\tau' \quad (13.0.55)$$

Using the solution form of the Biot-Savart law, we can write an expression for the vector potential

$$\vec{A}(\vec{r}) = \frac{1}{4\pi} \int \frac{\vec{B}(\vec{r}') \times \hat{r}}{|\vec{r}|^2} d\tau' \quad (13.0.56)$$

From this, if we place \vec{m} at the origin pointing in the \hat{z} direction, the magnetic field of a perfect dipole is calculated as follows.

$$\vec{B}_{dip}(\vec{r}) = \underbrace{\frac{\mu_0}{4\pi r^3} [3(\vec{m} \cdot \hat{r})\hat{r} - \vec{m}]}_{\text{The true field of a magnetic dipole.}^1} + \frac{2\mu_0}{3} \vec{m} \delta^3(\vec{r}) \equiv \frac{\mu_0 m}{4\pi r^3} (2 \cos \theta \hat{r} + \sin \theta \hat{\theta}). \quad (13.0.57)$$

We can define the potential of a bound volume current \vec{J}_B and bound surface current \vec{K}_B as

$$\vec{J}_B = \nabla \times \vec{M} \quad \text{and} \quad \vec{K}_B = \vec{M} \times \hat{n} \quad (13.0.58)$$

From Ampere's Law we can define the **magnetizing field** \vec{H} and thus have,

$$\frac{1}{\mu_0} (\nabla \times \vec{B}) = \vec{J}_f + (\nabla \times \vec{M}) \implies \nabla \times \left(\frac{\vec{B}}{\mu_0} - \vec{M} \right) = \nabla \times \vec{H} = \vec{J}_f \quad (13.0.59)$$

$$\implies \oint \vec{H} \cdot d\vec{\ell} = I_{free} = \int \vec{J}_{free} \cdot d\vec{a} \quad \text{and} \quad \int (\nabla \times \vec{H}) \cdot d\vec{a} = \int \vec{J} \cdot d\vec{a} \quad (13.0.60)$$

The **magnetic susceptibility** χ_m is a dimensionless quantity that is dependent on the substance. For a linear media, we have the relation

$$\vec{M} = \chi_m \vec{H} \implies \vec{B} = \mu_0 (1 + \chi_m) \vec{H} \quad \text{with} \quad \begin{cases} \chi_m > 0, & \text{paramagnetic} \\ \chi_m < 0, & \text{diamagnetic} \\ \chi_m = 0, & \text{vacuum} \end{cases} \quad (13.0.61)$$

The force on a magnetic dipole due to a varying magnetic field is

$$F = \nabla(\vec{m} \cdot \vec{B}) \quad (13.0.62)$$

Kirchhoff's Laws apply for electric circuits and are derived from the static equations $\nabla \cdot \vec{J} = 0$ and $\nabla \times \vec{E} = 0$ which become

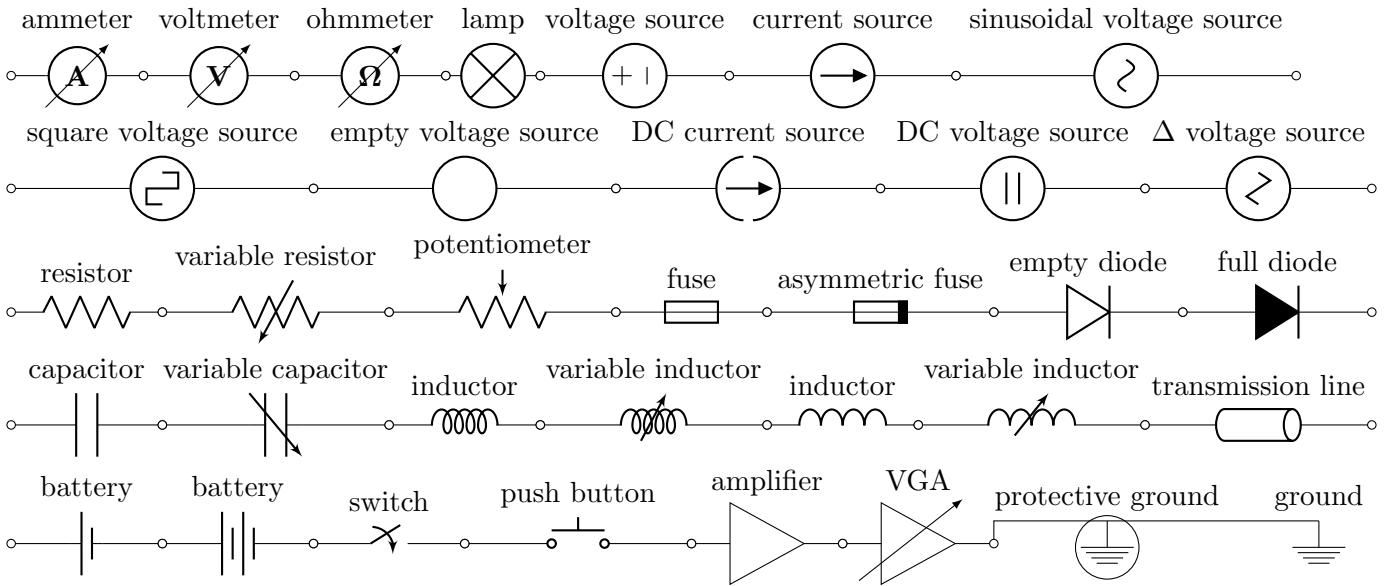
$$\sum I_i = 0 \text{ at a branch point} \quad \sum V_i = 0 \text{ around a loop} \quad (13.0.63)$$

¹The delta-function is responsible for the hyperfine splitting in atomic spectra[7]

Electronics

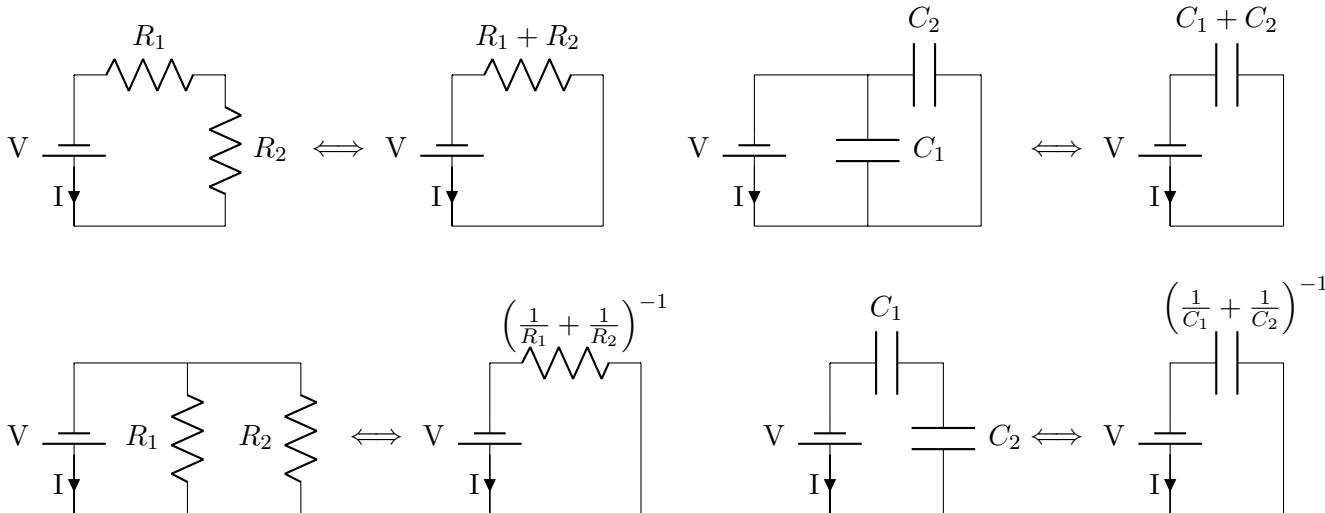
14.1 Electronic Symbols & Circuit Diagrams

Circuit diagrams are a major part of understanding and representing electronic circuits. Some common **circuit diagram symbols** follow:

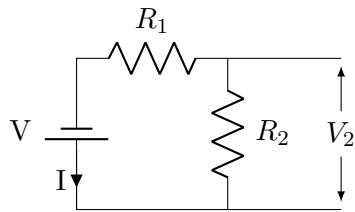


14.2 Equivalent Circuits

When dealing with circuit diagrams, it is often helpful to simplify a circuit using an equivalent circuit. Some basic **circuit equivalences** follow:



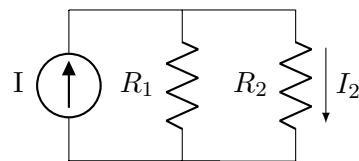
A voltage divider:



$$V_2 = \frac{VR_2}{R_1 + R_2} \quad (14.2.1)$$

$$V = I_1(R_1 + R_2) = I_2(R_1 + R_2) \quad (14.2.2)$$

A current divider:



$$I_2 = \frac{IR_1}{R_1 + R_2} \quad (14.2.3)$$

Electrical **impedance** is "the total opposition to alternating current by an electric circuit." [6]

$$Z_{\text{series}} = \sum_i Z_i \quad \text{and} \quad Z_{\text{parallel}} = \left[\sum_i \frac{1}{Z_i} \right]^{-1}. \quad (14.2.4)$$

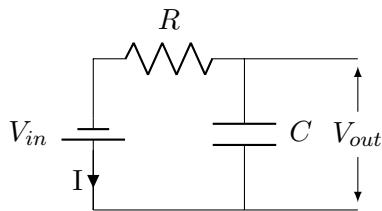
The electrical impedance for a resistor, capacitor, and inductor is given by

$$Z_{\text{resistor}} = R \quad \text{and} \quad Z_{\text{capacitor}} = \frac{1}{i\omega C} \quad \text{and} \quad Z_{\text{inductor}} = i\omega L. \quad (14.2.5)$$

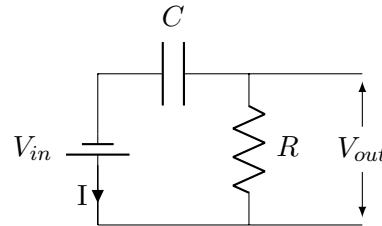
The complex form of **Ohm's Law** can be written in terms of **phasors** (denoted with a tilde overhead, i.e \tilde{X}): $\tilde{V} \equiv \tilde{I}Z$. For cases with an alternating power source, we can use the phasors to find the voltage and current at a given time

$$V(t) \equiv \Re[\tilde{V}e^{i\omega t}] \quad \text{and} \quad I(t) \equiv \Re[\tilde{I}e^{i\omega t}] \quad (14.2.6)$$

A low pass filter:

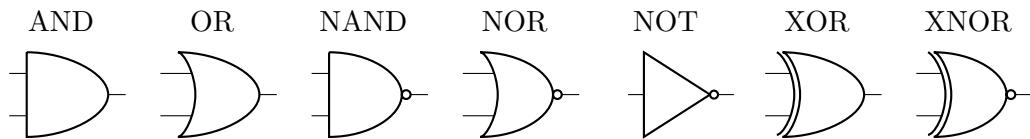


A high pass filter:



14.3 Logic, Gates, and Truth Tables

An elementary building block of a circuit is a **logic gate**. At any given time, the 3 nodes of a logic gate are either true (1) or false (0). The common notation for the logic gates follow:



The logic gates generally have two inputs (A and B) as well as an output which is used to determine an operation on the inputs and perform 'logic' to determine the appropriate output. The truth tables (using notation "A B | OUT") for the above logic gates is given as

AND		OR		NAND		NOR		NOT		XOR		XNOR	
0	0	0	0	0	1	0	0	1	0	1	0	0	0
0	1	0	0	1	1	0	1	1	0	1	0	1	0
1	0	0	1	0	1	1	0	1	0	0	1	1	0
1	1	1	1	1	0	1	1	0	1	1	0	1	1

Logic circuits can be combined by writing them into expressions using the output OUT as a function of the inputs A, B, C, etc...

$$A + B \equiv A \vee B \equiv A \text{ or } B \quad A \cdot B \equiv A \wedge B \equiv A \text{ and } B \quad \bar{A} \equiv \neg A \equiv \text{ NOT } A \quad (14.3.1)$$

Special Relativity

Relativistic time dilation and length contraction (where Δt_0 and Δl_0 are the proper time and length)

$$\Delta t = \frac{\Delta t_0}{\sqrt{1 - \beta^2}} = \gamma \Delta t_0 \quad (15.0.1)$$

$$\Delta l = \Delta l_0 \sqrt{1 - \beta^2} = \frac{\Delta l_0}{\gamma} \quad (15.0.2)$$

$$\beta = \frac{v}{c} \quad (15.0.3)$$

$$\gamma = \frac{1}{\sqrt{1 - \beta^2}} \quad (15.0.4)$$

Lorentz Transformations for space and time coordinates for a frame moving at a constant velocity v in the \hat{x} direction.

$$\bar{x} = \gamma(x - vt) \quad (15.0.5)$$

$$\bar{y} = y \quad \text{and} \quad \bar{z} = z \quad (15.0.6)$$

$$\bar{t} = \gamma(t - vx/c^2) \quad (15.0.7)$$

The relativistic velocity transformation is.

$$\bar{u}_x = \frac{u_x - v}{1 - vu_x/c^2} \iff u_x = \frac{\bar{u}_x + v}{1 + v\bar{u}_x/c^2} \quad (15.0.8)$$

The rest energy of a particle

$$E_0 = mc^2 \quad (15.0.9)$$

The Lorentz transformation for momentum and energy is.

$$\bar{p}_x = \gamma(p_x - vE/c^2) \quad (15.0.10)$$

$$\bar{p}_y = p_y \quad \text{and} \quad \bar{p}_z = p_z \quad (15.0.11)$$

$$\bar{E} = \gamma(E - vp_x) \quad (15.0.12)$$

Relativistic mass and momentum (where m is the rest mass of an object measured in its rest frame).

$$E = \gamma mc^2 = cp^0 \quad (15.0.13)$$

$$p = \gamma mv \quad (15.0.14)$$

Combining the above equations give

$$\frac{E}{p} = \frac{c^2}{v} \implies E = \frac{pc^2}{v} \quad (15.0.15)$$

Mass-energy equivalence and kinetic energy (K_E).

$$E^2 = (mc^2)^2 + (pc)^2 \quad (15.0.16)$$

$$E = K_E + E_0 \quad (15.0.17)$$

$$K_E = (\gamma - 1)mc^2 \quad (15.0.18)$$

Combining the above equations gives

$$p = \frac{1}{c} \sqrt{K_E^2 + 2K_E E_0} \quad (15.0.19)$$

Invariant dot product in $c=1$ notation

$$A \cdot B = (E, \vec{p}) \cdot (U, \vec{q}) = EU - \vec{p} \cdot \vec{q} \quad (15.0.20)$$

Relativistic frequency and wavelength shifts

$$f = f_0 \sqrt{\frac{c \pm v}{c \mp v}} \iff \pm v = \frac{f^2 - f_0^2}{f^2 + f_0^2} \quad (15.0.21)$$

$$\lambda = \lambda_0 \sqrt{\frac{c \mp v}{c \pm v}} \iff \mp v = \frac{\lambda^2 - \lambda_0^2}{\lambda^2 + \lambda_0^2} \quad (15.0.22)$$

Space-time equivalence (same in all reference frames)

$$S \equiv (c\Delta t)^2 - (\Delta x)^2 \equiv E^2 - (pc)^2 \quad (15.0.23)$$

Assuming a frame moving with a constant velocity to another, we can relate the accelerations observed between two frames by

$$\bar{a}_x = a_x \left(1 - \frac{u_x}{c}\beta\right)^{-3} (1 - \beta^2)^{3/2} \quad (15.0.24)$$

$$\implies \bar{a}_x \approx a_x \left(1 + 3\beta \frac{u_x}{c} - \frac{3}{2}\beta^2\right) \quad (15.0.25)$$

15.0.1 Einstein Notation

The Lorentz components can be defined by $X^0 \equiv ct$, $X^1 \equiv x$, $X^2 \equiv y$, and $X^3 \equiv z$, from which the Lorentz transformations follow as

$$\bar{X}^0 = \gamma(X^0 - \beta X^1) \quad (15.0.26)$$

$$\bar{X}^1 = \gamma(X^1 - \beta X^0) \quad (15.0.27)$$

$$\bar{X}^2 = X^2 \quad (15.0.28)$$

$$\bar{X}^3 = X^3 \quad (15.0.29)$$

In relativity, it is useful to work with what is known as **4-vectors**. We can summarize the above transformations with matrix notation by

$$\begin{pmatrix} \bar{X}^0 \\ \bar{X}^1 \\ \bar{X}^2 \\ \bar{X}^3 \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} X^0 \\ X^1 \\ X^2 \\ X^3 \end{pmatrix} \quad (15.0.30)$$

This notation can be compacted as

$$\bar{X}^\mu = \sum_{\nu=0}^3 \Lambda_\nu^\mu X^\nu \equiv \Lambda_\nu^\mu X^\nu \quad (15.0.31)$$

The displacement is a contra-variant 4-vector

$$(\Delta \bar{X})^\mu = \Lambda_\nu^\mu (\Delta X)^\nu \quad (15.0.32)$$

The dot product between 4-vectors are defined by

$$a_\mu b^\mu \equiv -a^0 b^0 + a^1 b^1 + a^2 b^2 + a^3 b^3 \quad (15.0.33)$$

With respect to the Lorentz transformations, the following products are invariant

$$X_\mu \bar{X}^\mu \equiv \bar{X}_\mu \bar{X}^\mu \quad (15.0.34)$$

Similarly, the **space-time interval** is invariant and defined as follows. The spacial and time separation of an event is d and Δt respectively.

$$I \equiv (\Delta X)_\mu (\Delta X)^\mu \equiv (\Delta \bar{X})_\mu (\Delta \bar{X})^\mu \quad (15.0.35)$$

$$= -(c\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 \quad (15.0.36)$$

$$= d^2 - c\Delta t^2 \quad (15.0.37)$$

A 4-velocity can be transformed as follows which is similarly invariant in all frames (the proper time is τ).

$$\bar{u}^\mu = \frac{(\Delta \bar{X})^\mu}{\Delta \tau} = \Lambda_\nu^\mu (\Delta x)^\mu \quad (15.0.38)$$

We define η^μ as the 4-velocity

$$\eta^\mu = \frac{dX^\mu}{d\tau} \quad (15.0.39)$$

From which it follows that the 4-momentum (where m is the rest mass of an object measured in its rest frame) is

$$p^\mu = m\eta^\mu = m \frac{dX^\mu}{d\tau} \quad (15.0.40)$$

$$\bar{p}^\mu = \Lambda_\nu^\mu p^\nu \quad (15.0.41)$$

Given a velocity in any direction \vec{v} while treating $\vec{\beta} = \beta_x \hat{x} + \beta_y \hat{y} + \beta_z \hat{z}$, the transformation matrix element becomes

$$\Lambda_\nu^\mu = \begin{pmatrix} \gamma & -\gamma\beta_x & -\gamma\beta_y & -\gamma\beta_z \\ -\gamma\beta_x & \frac{\beta^2 + (\gamma-1)\beta_x^2}{\beta^2} & \frac{(\gamma-1)\beta_x\beta_y}{\beta^2} & \frac{(\gamma-1)\beta_x\beta_z}{\beta^2} \\ -\gamma\beta_y & \frac{(\gamma-1)\beta_x\beta_y}{\beta^2} & \frac{\beta^2 + (\gamma-1)\beta_y^2}{\beta^2} & \frac{(\gamma-1)\beta_y\beta_z}{\beta^2} \\ -\gamma\beta_z & \frac{(\gamma-1)\beta_x\beta_z}{\beta^2} & \frac{(\gamma-1)\beta_y\beta_z}{\beta^2} & \frac{\beta^2 + (\gamma-1)\beta_z^2}{\beta^2} \end{pmatrix} \quad (15.0.42)$$

It is often useful to refer to the **rapidity** of a particle defined as $\phi = \cosh^{-1} \gamma$. From this,

$$\beta = \tanh \phi \quad (15.0.43)$$

$$\beta\gamma = \sinh \phi \quad (15.0.44)$$

15.0.2 Electromagnetic Fields

The **Minkowski Force** K^μ is defined and related to the non-relativistic force by

$$K^\mu \equiv \frac{dp^\mu}{d\tau} \quad (15.0.45)$$

$$\vec{K} = \frac{d\vec{p}}{d\tau} = \frac{d\vec{p}}{dt} \frac{dt}{d\tau} = \gamma \vec{F} \quad (15.0.46)$$

The electric field Lorentz transformations are

$$\bar{E}_x = E_x \quad (15.0.47)$$

$$\bar{E}_y = \gamma(E_y - vB_z) \quad (15.0.48)$$

$$\bar{E}_z = \gamma(E_z + vB_y) \quad (15.0.49)$$

The magnetic field Lorentz transformations are

$$\bar{B}_x = B_x \quad (15.0.50)$$

$$\bar{B}_y = \gamma(B_y + vE_z/c^2) \quad (15.0.51)$$

$$\bar{B}_z = \gamma(B_z - vE_y/c^2) \quad (15.0.52)$$

The **field tensor** $F^{\mu\nu}$ can be written

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix} \quad (15.0.53)$$

The **dual tensor** $G^{\mu\nu}$ can be written

$$G^{\mu\nu} = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z/c & E_y/c \\ -B_y & E_z/c & 0 & -E_x/c \\ -B_z & -E_y/c & E_x/c & 0 \end{pmatrix} \quad (15.0.54)$$

Thermodynamics

Useful constants: the specific heat of water is c

$$c = 4186 \text{ J/(kg}\cdot\text{K)} \quad (16.0.1)$$

$$1 \text{ cal} = 4.186 \text{ J} \quad (16.0.2)$$

Temperature relationships.

$${}^{\circ}\text{F} = \frac{9}{5} {}^{\circ}\text{C} + 32 \quad (16.0.3)$$

$${}^{\circ}\text{C} = \frac{5}{9}({}^{\circ}\text{F} - 32) \quad (16.0.4)$$

$${}^{\circ}\text{K} = {}^{\circ}\text{C} + 273.15 \quad (16.0.5)$$

The heat required to raise the temperature of a mass m by ΔT is

$$Q = cm\Delta T \quad (16.0.6)$$

The temperature of an object determines the radiated power of the object, which is given by the **Stefan-Boltzmann equation**

$$P_{\text{radiated}} = \sigma\epsilon AT^4 \quad (16.0.7)$$

$$\sigma = 5.67 \times 10^{-8} \text{ W/K}^4\text{m}^2 \quad (16.0.8)$$

$$\epsilon = \text{emissivity, and } 0 \leq \epsilon \leq 1 \quad (16.0.9)$$

The work done on a system in going from initial volume (V_i) to a final volume (V_f) is

$$W = \int dW = \int_{V_i}^{V_f} pdV. \quad (16.0.10)$$

The first law of thermodynamics, with internal energy (dU), heat transferred (dQ), pressure (P) and volume (dV)

$$\Delta E_{\text{internal}} = Q - W \quad (16.0.11)$$

$$dU = dQ - PdV \quad (16.0.12)$$

different processes include

- (i) An adiabatic process is one where $dQ = 0$.
- (ii) In a constant-volume process, $W = 0$.
- (iii) In a closed-loop process, $Q = W$.
- (iv) In an adiabatic free expansion, $Q = W = \Delta E_{\text{internal}} = 0$.

The efficiency of a system is defined by

$$Eff = \frac{W_{\text{cycle}}}{Q_{\text{in}}} = 1 - \frac{Q_c}{Q_h} < 100\% \quad (16.0.13)$$

If heat is added to an object, its change in temperature (with C =heat capacity of the object) is given by

$$\Delta T = \frac{Q}{C} \quad (16.0.14)$$

If heat is added to an object with mass m, its change in temperature (with c =specific heat of the object) is given by

$$\Delta T = \frac{Q}{cm} \quad (16.0.15)$$

The ideal gas law: For an ideal gas of n particles in a volume V at pressure P and temperature T, the equation of state is

$$PV = nN_A kT \equiv nRT \quad (16.0.16)$$

With a constant number of moles we get from the ideal gas law the following relation:

$$\frac{P_1 V_1}{T_1} = \frac{P_2 V_2}{T_2} \quad (16.0.17)$$

Dalton's law - The total pressure exerted by a mixture of gases is equal to the sum of the partial pressures pf the gases in the mixture.

$$P_{\text{total}} = P_1 + P_2 + P_3 + \dots + P_n \quad (16.0.18)$$

The work done by an ideal gas at constant temperature is

$$W = nRT \ln \left(\frac{V_f}{V_i} \right) \quad (16.0.19)$$

The average kinetic energy of an ideal gas

$$K_{\text{ave}} = \frac{1}{N} \sum_{i=1}^N K_i \quad (16.0.20)$$

$$= \frac{1}{N} \sum_{i=1}^N \frac{1}{2}mv_i^2 \quad (16.0.21)$$

$$= \frac{1}{2}mv_{rms}^2 \quad (16.0.22)$$

The root-mean-square speed of gas molecules is

$$v_{rms} = \sqrt{\frac{1}{N} \sum_{i=1}^N v_i^2} = \sqrt{\frac{3RT}{m}} \quad (16.0.23)$$

For an adiabatic process (with C_V =specific heat at constant volume, C_P =specific heat at constant pres-

sure), we have

$$dE_{internal} = -PdV = nC_VdT \quad (16.0.24)$$

$$PV^\gamma = \text{constant} \quad (16.0.25)$$

$$\gamma = \frac{C_P}{C_V} \quad (16.0.26)$$

$$P_f V_f^\gamma = P_i V_i^\gamma \quad (16.0.27)$$

$$T_f V_f^{\gamma-1} = T_i V_i^{\gamma-1} \quad (16.0.28)$$

In classical thermodynamics the entropy S is defined by The fundamental temperature τ is defined by the relation

$$\frac{1}{T} = \left(\frac{\partial S}{\partial U} \right)_N. \quad (16.0.29)$$

Thermal & Statistical Physics

17.1 States of a Model System

The multiplicity function for a system of N magnets with a spin excess $2s = N_\uparrow - N_\downarrow$ is

$$g(N, s) = \frac{N!}{(\frac{N}{2} + s)! (\frac{N}{2} - s)!} = \frac{N!}{N_\uparrow! N_\downarrow!}. \quad (17.1.1)$$

It is often useful to evaluate $g(N, s)$ within a logarithm in which the **Stirling approximation** becomes useful.

$$N! \approx N^N \sqrt{2\pi N} \exp\left(-N + \frac{1}{12N} + \dots\right). \quad (17.1.2)$$

It is often useful to take the logarithm of this which gives

$$\log N! \cong \frac{\log 2\pi}{2} + \left(N + \frac{1}{2}\right) \log N - N. \quad (17.1.3)$$

In the limit $s/N \ll 1$, with $N \gg 1$, we have the Gaussian approximation

$$g(N, s) \cong g(N, 0) \exp\left(\frac{-2s^2}{N}\right) \quad (17.1.4)$$

$$g(N, 0) \cong 2^N \sqrt{\frac{2}{\pi N}}. \quad (17.1.5)$$

The exact value of $g(N, 0)$ is given by

$$g(N, 0) = \frac{N!}{(N/2)!(N/2)!}. \quad (17.1.6)$$

The average value, or mean value, of a function $f(s)$ taken over a probability distribution $P(s)$ is defined as

$$\langle f \rangle = \sum_s f(s) P(s), \quad (17.1.7)$$

$$1 = \sum_s P(s). \quad (17.1.8)$$

The binomial distribution has the property

$$\sum_s g(N, s) = 2^N. \quad (17.1.9)$$

If all states of the model spin system are equally likely, the average value of s^2 is

$$\langle s^2 \rangle = \frac{\int_{-\infty}^{\infty} s^2 g(N, s) ds}{\int_{-\infty}^{\infty} g(N, s) ds} = \frac{N}{4} \quad (17.1.10)$$

The energy interaction of a single magnetic moment \vec{m} with a fixed external magnetic field \vec{B} is

$$U = -\vec{m} \cdot \vec{B}. \quad (17.1.11)$$

For a model system of N elementary magnets, each with two allowed orientations in a uniform magnetic field \vec{B} , the total potential energy U is

$$U = \sum_{i=0}^N U_i = -\vec{B} \cdot \sum_{i=0}^N m_i \quad (17.1.12)$$

$$= -2smB = -MB. \quad (17.1.13)$$

17.2 Entropy And Temperature

If $P(s)$ is the probability that a system is in the state X , the average value of a quantity X is

$$\langle X \rangle = \sum_s X(s) P(s). \quad (17.2.1)$$

The number of combined systems 1 and 2 (with $s = s_1 + s_2$) is

$$g(s) = \sum_s g_1(s_1) g_2(s - s_1). \quad (17.2.2)$$

The relation $s = k_B \sigma$ connects the conventional entropy S with the fundamental entropy σ . The **entropy** $\sigma(N, U)$ is given by

$$\sigma(N, U) = \log g(N, U). \quad (17.2.3)$$

The fundamental temperature τ is defined by the relation

$$\frac{1}{\tau} = \left(\frac{\partial \sigma}{\partial U} \right)_{N,V}. \quad (17.2.4)$$

17.3 Boltzmann Distribution and Helmholtz Free Energy

The **partition function** Z is

$$Z \equiv \sum_s \exp\left(-\frac{\epsilon_s}{\tau}\right). \quad (17.3.1)$$

The probability of finding a system of N particles in a state s of energy ϵ_s when the system is in thermal contact with a large reservoir at temperature τ is

$$P(\epsilon_s) = \frac{1}{Z} \exp\left(-\frac{\epsilon_s}{\tau}\right). \quad (17.3.2)$$

The pressure is given by

$$P = -\left(\frac{\partial U}{\partial V}\right)_\sigma = \tau \left(\frac{\partial \sigma}{\partial V}\right)_U. \quad (17.3.3)$$

The **Helmholtz Free Energy** is a minimum in equilibrium for a system held at constant τ, V and is defined as

$$F \equiv U - \tau\sigma = -\tau \log(Z) \quad (17.3.4)$$

$$dF = -\sigma d\tau - pdV + \mu dN. \quad (17.3.5)$$

From this we have

$$\sigma = -\left(\frac{\partial F}{\partial \tau}\right)_V \quad (17.3.6)$$

$$P = -\left(\frac{\partial F}{\partial V}\right)_\tau \quad (17.3.7)$$

$$\mu = \left(\frac{\partial F}{\partial N}\right)_{\tau,V}. \quad (17.3.8)$$

For an ideal monotonic gas of N atoms of spin zero with $n = N/V \ll n_Q$,

$$Z_n = \frac{Z_1^N}{N!} = \frac{(n_Q V)^N}{N!}, \quad (17.3.9)$$

The quantum concentration n_Q is defined by

$$n_Q \equiv \left(\frac{M\tau}{2\pi\hbar^2}\right)^{3/2}. \quad (17.3.10)$$

Furthermore, for an ideal gas we have

$$PV = N\tau \implies P = n\tau \quad (17.3.11)$$

$$\sigma = N \left[\log\left(\frac{n_Q}{n}\right) + \frac{5}{2} \right] \quad (17.3.12)$$

$$C_V = \frac{3}{2}N \quad \text{and} \quad C_p = \frac{5}{2}N. \quad (17.3.13)$$

The thermal average energy of an atom in a box is

$$U = \langle \epsilon \rangle = \tau^2 \frac{\partial \log(Z_1)}{\partial \tau}. \quad (17.3.14)$$

For a system of fixed volume in thermal contact with a reservoir, the mean square fluctuation in energy of the system is

$$\langle (\epsilon - \langle \epsilon \rangle)^2 \rangle = \tau^2 \left(\frac{\partial U}{\partial \tau} \right)_V \quad (17.3.15)$$

17.4 Thermal Radiation and Planck Distribution

The **Planck distribution function** for the thermal average number of photons in a cavity mode of frequency ω is

$$\langle s \rangle = \frac{1}{\exp(\hbar\omega/\tau) - 1}. \quad (17.4.1)$$

The **Stefan-Boltzmann law** for the radiant energy density in a cavity at temperature τ is

$$\frac{U}{V} = \frac{\pi^2 \tau^4}{15\hbar^3 c^3} \implies \frac{\sigma}{V} = \frac{4\pi^2 \tau^3}{45\hbar^3 c^3}. \quad (17.4.2)$$

The **Planck radiation law** for the energy per unit volume per unit range of frequency is

$$u_\omega = \frac{\hbar}{\pi^2 c^3} \frac{\omega^3}{\exp(\hbar\omega/\tau) - 1}. \quad (17.4.3)$$

The flux density of radiant energy J_ν and the Stefan-Boltzmann constant are

$$J_\nu = \sigma_B T^4 \quad (17.4.4)$$

$$\sigma_B = \frac{\pi^2 k_B^4}{60\hbar^3 c^3}. \quad (17.4.5)$$

The Debye low temperature limit of the heat capacity of a dielectric solid (where ϑ is the Debye temperature) is, in conventional units,

$$C_V = \frac{12\pi^4 N k_B}{5} \left(\frac{T}{\vartheta}\right)^3 \quad (17.4.6)$$

$$\vartheta = \frac{\hbar c}{k_B} \left(\frac{6\pi^2 N}{V}\right)^{1/3}. \quad (17.4.7)$$

The **Debye concentration** is

$$n_D = n_{max} = \left(\frac{6N}{\pi}\right)^{1/3} \quad (17.4.8)$$

17.5 Chemical Potential and Gibbs Distribution

The **chemical potential** is defined as follows. Two systems are in diffusive equilibrium if $\mu_1 = \mu_2$.

$$\mu(\tau, V, N) \equiv \left(\frac{\partial F}{\partial N} \right)_{\tau, V} \quad (17.5.1)$$

$$\mu = \left(\frac{\partial U}{\partial N} \right)_{\sigma, V} = -\tau \left(\frac{\partial \sigma}{\partial N} \right)_{U, V}. \quad (17.5.2)$$

The chemical potential is made up of two parts, external and internal. The external part is the potential energy of a particle in an external field of force. The internal part is of thermal origin; for an ideal **monatomic gas**

$$\mu(int) = \tau \log \left(\frac{n}{n_Q} \right). \quad (17.5.3)$$

The **Gibbs factor** gives the probability that a system at chemical potential μ and temperature τ will have N particles and be in a quantum state s of energy ϵ_s .

$$P(N, \epsilon_s) = \frac{1}{\mathcal{Z}} \exp \left[\frac{N\mu - \epsilon_s}{\tau} \right]. \quad (17.5.4)$$

The **Gibbs sum** is taken over all states for all numbers of particles.

$$\mathcal{Z} \equiv \sum_{ASN} \exp \left[\frac{N\mu - \epsilon_{s(N)}}{\tau} \right]. \quad (17.5.5)$$

The **absolute activity** λ is defined by

$$\lambda \equiv \exp \left[\frac{\mu}{\tau} \right]. \quad (17.5.6)$$

The thermal average number of particles is

$$\langle N \rangle = \lambda \frac{\partial}{\partial \lambda} \log(\mathcal{Z}). \quad (17.5.7)$$

For a system in diffusive contact with a reservoir, the number of particles is not constant and

$$\langle N \rangle = \frac{\tau}{\mathcal{Z}} \left(\frac{\partial \mathcal{Z}}{\partial \mu} \right)_{\tau, V} \quad (17.5.8)$$

$$\langle N^2 \rangle = \frac{\tau^2}{\mathcal{Z}} \left(\frac{\partial^2 \mathcal{Z}}{\partial \mu^2} \right). \quad (17.5.9)$$

17.6 Ideal Gas

The Fermi-Dirac (+) and Bose-Einstein (-) distribution functions are

$$f(\epsilon) = \frac{1}{\exp[(\epsilon - \mu)/\tau] \pm 1} \quad (17.6.1)$$

Occupancy of an orbital in the classical limit of $f(\epsilon) \ll 1$

$$f(\epsilon) = \lambda \exp(-\epsilon/\tau). \quad (17.6.2)$$

Given N , we can determine λ in the classical limit as

$$\lambda = \frac{N}{\sum \exp(-\epsilon_n/\tau)} = \frac{N}{n_Q V}. \quad (17.6.3)$$

The energy of a free particle orbital of quantum number n in a cube of volume V is

$$\epsilon_n = \frac{1}{2M} \left(\frac{n\pi\hbar}{V^{1/3}} \right)^2. \quad (17.6.4)$$

A useful transformation from the summation to the integral follows as

$$\sum_n e^{-\epsilon_n/\tau} = \frac{\pi}{2} \int n^2 e^{-\epsilon_n/\tau} dn. \quad (17.6.5)$$

Some important relationships are

$$F = \int \mu dN = N\tau[\log(n/n_Q) - 1] \quad (17.6.6)$$

$$P = - \left(\frac{\partial F}{\partial V} \right)_{\tau, N} = \frac{N\tau}{V}. \quad (17.6.7)$$

For an ideal gas, the **entropy** (with σ_1 beign a constant independent of τ and V)is

$$\sigma = C_v \log(\tau) + N \log(V) + \sigma_1. \quad (17.6.8)$$

The average pressure in a system in thermal contact with a heat reservoir is given by

$$P = - \frac{1}{Z} \sum_s \left(\frac{\partial \epsilon_s}{\partial V} \right)_N e^{-\epsilon_s/\tau}. \quad (17.6.9)$$

For a gas of free particles

$$\left(\frac{\partial \epsilon_s}{\partial V} \right)_N = - \frac{2\epsilon_s}{3V}. \quad (17.6.10)$$

The Gibbs sum for an ideal gas with identical atoms and the probability that there are N atoms in the gas in a volume V in diffusive contact with a reservoir is

$$\mathcal{Z} = \exp(\lambda n_Q V) \quad (17.6.11)$$

$$P(N) = \frac{\langle N \rangle^N}{N!} e^{-\langle N \rangle}. \quad (17.6.12)$$

17.7 Heat and Work

Heat is the transfer of energy by thermal contact with a reservoir. In a reversible process

$$dQ = \tau d\sigma. \quad (17.7.1)$$

The **Carnot Energy** conversion efficiency η_C is the upper limit to the ratio W/Q_h of the work generated to the heat added:

$$\eta_C = \left(\frac{W}{Q_h} \right)_{rev} = \frac{(\tau_h - \tau_l)}{\tau_h} = \frac{(T_h - T_l)}{T_h}. \quad (17.7.2)$$

The **Carnot coefficient** of refrigerator performance is the upper limit of Q_l/W of the heat extracted to the work consumed:

$$\gamma_C = \left(\frac{Q_l}{W} \right)_{rev} = \frac{\tau_l}{\tau_h - \tau_l} = \frac{T_l}{T_h - T_l}. \quad (17.7.3)$$

The effective work performed on a system at constant temperature and pressure in a reversible process is equal to the change in the Gibbs free energy:

$$G \equiv U - \tau\sigma + pV. \quad (17.7.4)$$

The chemical work performed on a system in the reversible transfer of dN particles to the system is

$$W_\mu = \mu dN. \quad (17.7.5)$$

The change in the free energy density of a superconductor of type I caused by an external magnetic field is $B^2/2\mu_0$ in SI units and $B^28\pi$ in CGS units.

17.8 Gibbs Free Energy and Chemical Reactions

From the **Gibbs free energy**, we have

$$dG = \mu dN - \sigma d\tau + V dp. \quad (17.8.1)$$

From this we can determine the following relations:

$$\mu = \left(\frac{\partial G}{\partial N} \right)_{\tau,p} \quad (17.8.2)$$

$$-\sigma = \left(\frac{\partial G}{\partial \tau} \right)_{N,p} \quad (17.8.3)$$

$$V = \left(\frac{\partial G}{\partial p} \right)_{N,\tau}. \quad (17.8.4)$$

The Gibbs Free energy is related to chemical potential via

$$G(\tau, p, V) = N\mu(\tau, p). \quad (17.8.5)$$

The **Law of mass action** for a chemical reaction is

$$K(\tau) = \prod_j n_j^{\mu_j} \quad (17.8.6)$$

The **Maxwell Relations** are

$$\left(\frac{\partial V}{\partial \tau} \right)_{p,N} = - \left(\frac{\partial \sigma}{\partial p} \right)_{\tau,N} \quad (17.8.7)$$

$$\left(\frac{\partial V}{\partial N} \right)_p = \left(\frac{\partial \mu}{\partial p} \right)_N \quad (17.8.8)$$

$$\left(\frac{\partial \mu}{\partial \tau} \right)_N = - \left(\frac{\partial \sigma}{\partial N} \right)_\tau \quad (17.8.9)$$

$$(17.8.10)$$

The **acidity** or alkalinity of a solution in terms of pH is defined as

$$pH \equiv -\log_{10}[H^+]. \quad (17.8.11)$$

The **reaction quotient** for the chemical reaction $\alpha A + \beta B \rightleftharpoons \gamma C + \delta D$ is

$$Q = \frac{[C]^\gamma [D]^\delta}{[A]^\alpha [B]^\beta}. \quad (17.8.12)$$

17.9 Phase Transformations

Enthalpy is defined as

$$H \equiv U - pV. \quad (17.9.1)$$

The coexistence curve in the $p - \tau$ plane between two phases must satisfy the **Clausius-Clapeyron equation** (L is the latent heat $L = H_1 - H_2$)

$$\frac{dp}{d\tau} = \frac{L}{\tau \Delta v}. \quad (17.9.2)$$

The **van der Waals** equation of state is

$$(p + N^2 a/V^2)(V - Nb) = N\tau. \quad (17.9.3)$$

The critical points for a van der Waals gas are defined as

$$\tau_c = \frac{8a}{27b} \quad (17.9.4)$$

$$p_c = \frac{a}{27b^2} \quad (17.9.5)$$

$$V_c = 3Nb. \quad (17.9.6)$$

17.9.1 Kinetic Theory

The **Maxwell distribution** describes the probability that an atom has a velocity v in dv ,

$$P(v)dv = 4\pi \left(\frac{M}{2\pi\tau}\right)^{3/2} v^2 e^{-\frac{Mv^2}{2\tau}} dv \quad (17.9.7)$$

The root mean square velocity is given by

$$v_{rms} = \sqrt{\langle v^2 \rangle} = \sqrt{\frac{3\tau}{M}}. \quad (17.9.8)$$

The most probable value of the speed is

$$v_{mp} = \sqrt{\frac{2\tau}{M}}. \quad (17.9.9)$$

The mean speed is

$$\bar{c} = \langle v \rangle = \int_0^\infty v P(v) dv = \sqrt{\frac{8\tau}{\pi M}} \quad (17.9.10)$$

Diffusion is described by the mean free path ℓ and the mean speed $\bar{c} \equiv \langle v \rangle$,

$$\vec{J}_n = -D \nabla n, \quad D = \frac{1}{3} \bar{c} \ell \quad (17.9.11)$$

Thermal conductivity is described by the specific heat per unit volume \bar{C}_V ,

$$\vec{J}_\mu = -K \nabla \tau, \quad K = D \bar{C}_V = \frac{1}{3} \bar{C}_V \bar{c} \ell \quad (17.9.12)$$

The coefficient of **viscosity** is given in terms of the mass density (where d is the molecular diameter) as

$$\eta = D\rho = \frac{1}{3} \rho \bar{c} \ell = \frac{M \bar{c}}{3\pi d^2} \quad (17.9.13)$$

The electrical conductivity of a Fermi gas is given in terms of the relaxation time τ_c ,

$$\sigma = \frac{nq^2 \tau_c}{m} \quad (17.9.14)$$

The electric current density is

$$\vec{J}_q = \sigma \vec{E} \quad (17.9.15)$$

The **Wiedemann-Franz ratio** holds for a classical gas of particles of charge q ,

$$\frac{K}{\tau\sigma} = \frac{3}{2q^2}, \quad \text{or} \quad \frac{K}{T\sigma} = \frac{3k_B^2}{2q^2}. \quad (17.9.16)$$

The **Boltzmann transport equation** in the relaxation time approximation is

$$\frac{\partial f}{\partial t} + \alpha \nabla_V f + v \nabla_r f = -\frac{f - f_0}{\tau_c} \quad (17.9.17)$$

17.9.1: Table from chapter 14 of Kittel and Kroemer's "Thermal Physics" [17].

Effect	Flux of Particle Property	Gradient	Coefficient	Law	Name of Law	Expression for Coefficient
Diffusion	Particle Number	∇n	Diffusivity, D	$\mathbf{J}_n = -D \nabla n$	Fick's Law	$D = \frac{\lambda \langle v \rangle}{3}$
Electrical Conductivity	Charge	$\nabla \phi$	Conductivity, σ	$\mathbf{J}_q = -\sigma \nabla \phi$	Ohm's Law	$\sigma = \frac{nq^2 \lambda}{M \langle v \rangle}$
Thermal Conductivity	Energy	$\nabla \tau$	Thermal Conductivity, κ	$\mathbf{J}_U = -\kappa \nabla \tau$	Fourier's Law	$\kappa = \frac{\bar{C}_V \lambda \langle v \rangle}{3}$
Viscosity	Transverse Momentum	$\frac{dv_x}{dz}$	Viscosity, η	$\mathbf{J}_{p_x} = -\eta \frac{dv_x}{dz}$	Newtonian Viscosity	$\eta = \frac{M \langle v \rangle}{3\pi d^2}$

Elementary Quantum Physics

Wien's Displacement Law

$$\lambda_{MAX}T = 2.898 \times 10^{-3} m * K \quad (18.0.1)$$

Total Power Stefan-Boltzmann Law

$$R(T) = \int_0^{\infty} I(\lambda, T) d\lambda = \epsilon \sigma T^4 \quad (18.0.2)$$

ϵ = emmisivity (unitless) $(18.0.3)$

$$\sigma = 5.67 \times 10^{-8} \frac{w}{m^2 k^4} \quad (18.0.4)$$

Max Planck's Radiation Law:

$$I(\lambda, T) = \frac{2\pi c^2 h}{\lambda^5} \frac{1}{e^{hc/\lambda kT} - 1} \quad (18.0.5)$$

The kinetic energy of an emitted photoelectron is
(Where ϕ = binding energy of electron to metal surface or the work function)

$$KE = hv - \phi \quad (18.0.6)$$

$$E_{photon} = KE_{electron} + \phi \quad (18.0.7)$$

$$KE_{electrons} = 0 \text{ (at threshold)} \quad (18.0.8)$$

Rutherford Scattering Formula: Any particle hitting an area σ around the nucleus will be scattered through an angle of θ or greater.

$$b = (r_{min}/2) \cot(\theta/2) \quad (18.0.9)$$

$$r_{min} = \frac{Z_1 Z_2 e^2}{4\pi\epsilon_0 K} \quad (18.0.10)$$

$$\sigma = \pi b^2 = \text{cross sectional area} \quad (18.0.11)$$

$$\frac{e^2}{4\pi\epsilon_0} = 1.44 \times 10^{-9} \text{ eV}\cdot\text{m} \quad (18.0.12)$$

A common unit of σ is one barn.

$$\text{barn (unit)} = 10^{-28} m^2 = 100 fm^2 \quad (18.0.13)$$

$$\frac{\# \text{ atoms}}{\text{area}} = \frac{\text{atoms}}{\text{volume}} \times \text{thickness} \quad (18.0.14)$$

$$n = \left(N_A \frac{\text{atoms}}{\text{mole}} \right) \left(\frac{1 \text{ mole}}{A \text{ gm}} \right) \left(\rho \frac{\text{gm}}{\text{cm}^3} \right) \quad (18.0.15)$$

$$= \frac{\rho N_A}{A} \quad (18.0.16)$$

The **Compton effect** describes the photon wavelength λ' after a photon of wavelength λ scatters off an electron.

$$\lambda' = \lambda + \frac{h}{m_e c} (1 - \cos(\theta)) \quad (18.0.17)$$

The Compton wavelength of an electron is

$$\lambda_e = \frac{h}{m_e c} = 2.426 \times 10^{-12} m \quad (18.0.18)$$

Heisenberg Uncertainty relation

$$\Delta x \Delta p_x \geq \frac{1}{2} \hbar \quad (18.0.19)$$

The **de Broglie wavelength** is defined as

$$\lambda = \frac{h}{p} = \frac{h}{mv\gamma} = \frac{h}{mv} \sqrt{1 - \frac{v^2}{c^2}} \quad (18.0.20)$$

$$= \frac{hc}{\sqrt{K_E^2 + 2K_E E_0}} \quad (18.0.21)$$

Rutherford Scattering.

$$K = \frac{1}{4\pi\epsilon_0} \frac{Z_1 Z_2 e^2}{R_{min}} \quad (18.0.22)$$

$$R_{min} = \frac{1}{4\pi\epsilon_0} \frac{Z_1 Z_2 e^2}{K} \quad (18.0.23)$$

Z's are the atomic masses of the particles within the interaction and R_{min} is the minimum distance they reach (from center to center), and e is

$$e = 1.602177 \times 10^{-19} C \quad (18.0.24)$$

$$\epsilon \approx 8.854 \times 10^{-12} F/m \quad (18.0.25)$$

The Rutherford Scattering Formula

$$N(\theta) = \frac{N_i n t}{16 r^2} (R_{min})^2 \frac{1}{\sin^4(\theta/2)} \quad (18.0.26)$$

Centripetal force due to coulomb attraction

$$\frac{1}{4\pi\epsilon_0} \frac{e^2}{r^2} = ma_c = m \frac{v^2}{r} \quad (18.0.27)$$

$$\Rightarrow v^2 = \frac{1}{4\pi\epsilon_0} \frac{e^2}{mr} \quad (18.0.28)$$

$$\Rightarrow r = 4\pi\epsilon_0 \frac{n^2 \hbar^2}{me^2} \quad (18.0.29)$$

Energy levels

$$E = KE + PE \quad (18.0.30)$$

$$= -\frac{1}{2} \frac{e^2}{4\pi\epsilon_0 r} \quad (18.0.31)$$

$$\Rightarrow E = \frac{-E_0}{n^2}, \quad (18.0.32)$$

$$\text{where } E_0 = \alpha^2 mc^2 / 2 = 13.6 \text{ eV.} \quad (18.0.33)$$

Energy of emitted radiation

$$E = E_n - E_m \quad (18.0.34)$$

$$= E_0 \left(\frac{1}{m^2} - \frac{1}{n^2} \right) \quad (18.0.35)$$

Note. Using the Planck formula in the above equation leads to the Rydberg formula.

The **Rydberg formula:** Wavelength of the spectral lines in Hydrogen:

$$\frac{1}{\lambda} = R_H \left(\frac{1}{n_f^2} - \frac{1}{n_i^2} \right) \quad (18.0.36)$$

$$n \in \mathbb{N} = 1, 2, 3, 4, 5, \dots \quad (18.0.37)$$

Note. ZnS (Zinc Sulfide) emits a faint flash of light when struck by an α -ray.

L quantized

$$L = mvr = n\hbar \quad (18.0.38)$$

Stationary state orbits

$$r = a_0 n^2 \quad (18.0.39)$$

$$a_0 = \text{Bohr Radius} \quad (18.0.40)$$

Stationary state energies

$$E_n = -Z^2 \frac{E_0}{n^2} \quad (18.0.41)$$

Uncertainty relation of energy and the measurement of time.

$$\Delta E \cdot \Delta t \geq \frac{1}{2}\hbar \quad (18.0.42)$$

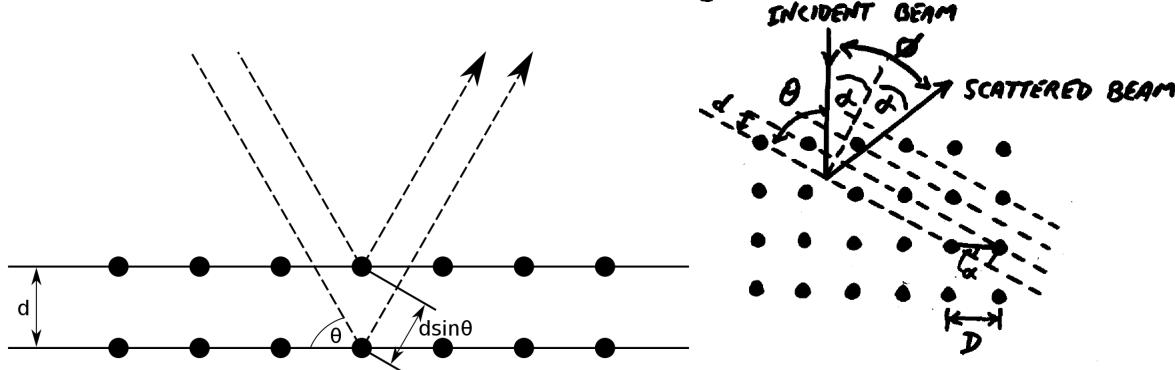
Bragg's Law: When scattering off of crystal structures, the wavelengths will peak at specific angles determined by the diagrams below

$$n\lambda = 2d \sin(\theta) = 2D \cos(\alpha) = 2D \sin(\alpha) \cos(\alpha) = D \sin(2\alpha) = D \sin(\phi) \quad (18.0.43)$$

$$d = D \sin(\alpha) \quad (18.0.44)$$

$$\phi = 2\alpha \quad (18.0.45)$$

$$\theta = 90^\circ - \alpha \quad (18.0.46)$$



The potential the electron moves in

$$V(r) = \frac{-e^2}{(4\pi\epsilon_0 r)} \quad (18.0.47)$$

The angular momentum of an electron in the atom

$$L = mvr = \hbar\sqrt{\ell(\ell+1)} \quad \text{and} \quad L_z = m_\ell\hbar \quad (18.0.48)$$

An electron orbiting around a nucleus has magnetic moment $\vec{\mu}$

$$\vec{\mu} = IA\hat{n} = \frac{-e}{(2\pi r/v)}(\pi r^2)\hat{n} = \frac{-erv}{2}\hat{n} = \frac{-e}{2m}\vec{L} \quad (18.0.49)$$

$$\mu_z = \frac{-e}{2m}L_z = \frac{-e}{2m}m_\ell\hbar = -m_\ell\mu_B \quad (18.0.50)$$

In an external magnetic field, B , the magnetic dipole feels a torque $\vec{\tau}$ and has a potential energy U_B

$$\vec{\tau} = \vec{\mu} \times \vec{B} \quad (18.0.51)$$

$$V_B = -\vec{\mu} \cdot \vec{B} = \frac{-e}{2m}\vec{L} \cdot \vec{B} \implies V_{Bz} = \mu_B m_\ell B_z \quad (18.0.52)$$

Quantum Mechanics

Plane waves with electromagnetic wave frequencies and wavelengths

$$\psi(x, t) = A \cos[2\pi(x - ct)/\lambda] \quad (19.0.1)$$

$$c = f\lambda \iff f = \frac{c}{\lambda} \iff \lambda = \frac{c}{f} \quad (19.0.2)$$

$$T = 1/f \quad (19.0.3)$$

$$\psi(x, t) = A \cos(kx - \omega t) \quad (19.0.4)$$

$$k = 2\pi/\lambda \quad (19.0.5)$$

$$\omega = 2\pi f = 2\pi/T \quad (19.0.6)$$

The energy in a photon (packet of light)

$$E = hf = \frac{hc}{\lambda} = \hbar\omega \quad (19.0.7)$$

$$dE = -\frac{hc}{\lambda^2} d\lambda = -\frac{E^2}{hc} d\lambda \quad (19.0.8)$$

$$|\Delta\lambda| = hc \frac{\Delta E}{E^2} \quad (19.0.9)$$

The wave equation

$$\frac{1}{v^2} \frac{\partial^2 \psi}{\partial t^2} = \nabla^2 \psi \quad (19.0.10)$$

A periodic wave can be constructed from a sum of plane waves

$$\psi(x, t) = \sum_{i=1}^n A_i \cos(k_i x_i - \omega_i t) \quad (19.0.11)$$

The **cubit** is defined as

$$|\psi\rangle = c_1|1\rangle + c_0|0\rangle \quad (19.0.12)$$

$$|\psi\rangle = c_{11}|11\rangle + c_{01}|01\rangle + c_{10}|10\rangle + c_{00}|00\rangle \quad (19.0.13)$$

⋮

The quantum mechanical **expectation value** of an observable \hat{X} in a normalized state ψ is found by integrating over the entire space ψ^* times the result obtained when the corresponding operator acts on ψ .

$$\langle \psi | \hat{X} | \psi \rangle = \int_{-\infty}^{\infty} \psi^* \hat{X} \psi dx \quad (19.0.14)$$

The state of the system is given by a wavefunction $\psi(\vec{r}, t)$. The probability density is the square modulus of the amplitude

$$P(\vec{r})d\vec{r} = |\psi(\vec{r})|^2 d\vec{r} \quad (19.0.15)$$

The probability of a particle being between x_1 and x_2 given a normalized wave function $\psi(x, t)$ is

$$P_{x \in x_1:x_2}(t) = \int_{x_1}^{x_2} |\psi(x, t)|^2 dx = \int_{x_1}^{x_2} \psi^*(x, t) \psi(x, t) dx \quad (19.0.16)$$

The normalization of a wave function implies the probability over all space is 1.

$$P_{x \in -\infty:\infty}(t) = \int_{-\infty}^{\infty} |\psi(x, t)|^2 dx = 1 \quad (19.0.17)$$

The **Schrödinger Equation** is a partial differential equation that describes how the wavefunction of a physical system evolves over time. The (non-relativistic) Schrödinger Equation for a particle moving in a 3-dimensional potential energy field $V(\vec{r})$ is

$$\hat{E}\psi(\vec{r}, t) = \frac{-\hbar^2}{2m} \nabla^2 \psi(\vec{r}, t) + V(\vec{r})\psi(\vec{r}, t) \equiv \hat{H}\psi(\vec{r}, t) \quad (19.0.18)$$

The general solution to the time-dependent **Schrödinger Equation** is

$$\psi(\vec{r}, t) = \sum c_n \psi_n(\vec{r}) e^{-iE_n t/\hbar}. \quad (19.0.19)$$

The Dirac equation: the generalization of the time dependent Schrödinger equation for the relativistically correct relationship between energy and momentum. It leads to negative energy states and antiparticles.

$$\left[\gamma^0 mc^2 + \sum_{i=1}^3 \gamma^i \hat{p}_i c \right] \psi(\vec{r}, t) = i\hbar \frac{\partial}{\partial t} \psi(\vec{r}, t) \quad (19.0.20)$$

Each observable corresponds to a linear operator. A linear operator is something that acts on a state and gives another state. The Hamiltonian operator is defined as the operator \hat{H} such the energy E of a system with wavefunction ψ is an eigenvalue of $\hat{H}\psi$ or $\hat{H}\psi = E\psi$.

$$\hat{H} = \hat{K} + V(\hat{r}) = \frac{\hat{p}^2}{2m} + V(\hat{r}) = \frac{-\hbar^2}{2m} \nabla^2 + V(\vec{r}) \quad (19.0.21)$$

The Energy operator

$$i\hbar \frac{\partial \psi}{\partial t} = i\hbar \frac{\partial}{\partial t} A e^{i(kx - \omega t)} = i\hbar(-i\omega)\psi = \hbar\omega\psi = E\psi \implies \hat{E} = i\hbar \frac{\partial}{\partial t} \quad (19.0.22)$$

$$\langle \psi | E | \psi \rangle = \int_{-\infty}^{\infty} \psi^* \hat{E} \psi dx = i\hbar \int_{-\infty}^{\infty} \psi^* \frac{\partial \psi}{\partial t} dx \quad (19.0.23)$$

The operator for a particles kinetic energy is

$$\hat{K} = \frac{-\hbar^2}{2m} \nabla^2 \quad (19.0.24)$$

$$\hat{K}\psi = \frac{1}{2m} \hat{p}^2 \psi = \frac{1}{2m} \left(-i\hbar \frac{\partial}{\partial x} \right)^2 \psi = \frac{-\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \psi \quad (19.0.25)$$

$$\langle \psi | \hat{K} | \psi \rangle = \int_{-\infty}^{\infty} \psi^* \hat{K} \psi dx = \int_{-\infty}^{\infty} \psi^* \frac{1}{2m} \hat{p}^2 \psi dx = \frac{-\hbar^2}{2m} \int_{-\infty}^{\infty} \psi^* \frac{\partial^2}{\partial x^2} \psi dx \quad (19.0.26)$$

The momentum operator

$$\hat{p} \equiv -i\hbar \nabla = \frac{\hbar}{i} \nabla \quad (19.0.27)$$

$$\frac{\partial \psi}{\partial x} = \frac{\partial}{\partial x} A e^{i(kx - \omega t)} = ik\psi = \frac{ip}{\hbar}\psi \implies \hat{p} = -i\hbar \frac{\partial}{\partial x} \quad (19.0.28)$$

$$\langle \psi | \hat{p} | \psi \rangle = \int_{-\infty}^{\infty} \psi^* \hat{p} \psi dx = -i\hbar \int_{-\infty}^{\infty} \psi^* \frac{\partial \psi}{\partial x} dx \quad (19.0.29)$$

The wave function solution for a particle confined to an infinite potential well with walls at $x = 0$ and $x = a$ is as follows, with the corresponding energy eigenvalues (with $n \in \mathbb{N}$)

$$\psi(x) = \begin{cases} \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right) & 0 \leq x \leq a \\ 0 & \text{otherwise} \end{cases} \quad \text{and } E_n = \frac{\hbar^2 \pi^2}{2ma^2} n^2 \quad (19.0.30)$$

The solution to The Schrödinger Equation for a finite potential well with the potential

$$V(x) = \begin{cases} \infty & \text{for } x < 0 \\ 0 & \text{for } 0 \leq x \leq a \\ V_1 & \text{for } x > a \end{cases} \quad (19.0.31)$$

with $E > V_1$ is

$$\psi(x) = \begin{cases} 0 & \text{for } x < 0 \\ D \sin(kx) & \text{for } 0 \leq x \leq a, \\ F \cos(k'x) + G \sin(k'x) & \text{for } x > a \end{cases} \quad \text{with } k' = \sqrt{k^2 - \frac{2mV_1}{\hbar^2}} \quad (19.0.32)$$

with $E < V_1$ is

$$\psi(x) = \begin{cases} 0 & \text{for } x < 0 \\ D \sin(kx) & \text{for } 0 \leq x \leq a, \\ F e^{-\gamma x} & \text{for } x > a \end{cases} \quad \text{with } \gamma^2 = \frac{2m(V_1 - E)}{\hbar^2} = \frac{2mV_1}{\hbar^2} - k^2 \quad (19.0.33)$$

The solutions for a finite barrier (with the probability of reflection as $R = |B|^2/|A|^2$ and the probability of transmission as $T = |F|^2/|A|^2$) are

$$\psi_1 = A e^{ikx} + B e^{-ikx} \quad (\text{incident + reflected}) \quad (19.0.34)$$

$$\psi_2 = C e^{ik'x} + D e^{-ik'x} \quad (\text{intermediate}) \quad (19.0.35)$$

$$\psi_3 = F e^{ikx} \quad (\text{transmitted}) \quad (19.0.36)$$

$$k = \sqrt{2mE/\hbar^2} \quad (19.0.37)$$

$$k' = \sqrt{2m(E - U_0)/\hbar^2} \quad (19.0.38)$$

For any two Hermitian operators A and B ,

$$\Delta A \Delta B \geq \frac{1}{2} |\langle i[A, B] \rangle| \quad (19.0.39)$$

Atomic quantum numbers

$$n = \text{Principle Quantum Number } [n \in \mathbb{N}] \quad (19.0.40)$$

$$\ell = \text{Orbital Angular Momentum Quantum Number } [\ell \in \mathbb{N} \cup \{0\}, \ell < n] \quad (19.0.41)$$

$$m_\ell = \text{Magnetic Quantum Number } [m_\ell \in [-\ell, \ell], m_\ell \in \mathbb{Z}] \quad (19.0.42)$$

19.1 Simple Harmonic Oscillator

For a simple harmonic oscillator, the potential energy is given by

$$V(x) = \frac{1}{2} kx^2 = \frac{1}{2} m\omega^2 x^2 \implies k = m\omega^2 \iff \omega = \sqrt{\frac{k}{m}} \quad (19.1.1)$$

The energies and wave functions are then given by

$$\psi_n = \frac{1}{\sqrt{n!}} (a_+)^n \psi_0 \quad \psi_0(x) = \left(\frac{m\omega}{\pi\hbar} \right)^{1/4} e^{-\frac{m\omega}{2\hbar} x^2} \quad (19.1.2)$$

$$E_n = \left(n + \frac{1}{2} \right) \hbar\omega = \left(n + \frac{1}{2} \right) \hbar \sqrt{\frac{k}{m}} = \left(n + \frac{1}{2} \right) \frac{\hbar}{x} \sqrt{\frac{2V(x)}{m}} \quad (19.1.3)$$

The eigenfunctions, raising, and lowering operators are related to the eigenstates/vectors by

$$a_{\pm} = \frac{1}{\sqrt{2\hbar m\omega}}(\mp ip + m\omega x), \quad H|n\rangle = E_n|n\rangle, \quad H = \hbar\omega \left(a^{\dagger}a + \frac{1}{2}\right), \quad [a, a^{\dagger}] = 1 \quad (19.1.4)$$

$$|n\rangle = \frac{(a^{\dagger})^n|0\rangle}{\sqrt{n!}}, \quad a^{\dagger}|n\rangle \equiv a_+|n\rangle = \sqrt{n+1}|n+1\rangle, \quad a|n\rangle \equiv a_-|n\rangle = \sqrt{n}|n-1\rangle \quad (19.1.5)$$

For the harmonic oscillator, x and p can be expressed in terms of the raising and lowering operators

$$x = \sqrt{\frac{\hbar}{2m\omega}}(a_+ + a_-) \quad p = i\sqrt{\frac{\hbar m\omega}{2}}(a_+ - a_-). \quad (19.1.6)$$

19.2 Spherical Harmonics

The normalized angular wave functions are called **spherical harmonics** and are given by

$$Y_{\ell}^m(\theta, \phi) = \epsilon \sqrt{\frac{(2\ell+1)}{4\pi} \frac{(\ell-|m|)!}{(\ell+|m|)!}} e^{im\phi} P_{\ell}^m(\cos\theta), \quad \text{with} \quad \epsilon = \begin{cases} (-1)^m & m \geq 0 \\ 1 & m < 0 \end{cases}. \quad (19.2.1)$$

The **spherical harmonics** are automatically orthogonal so,

$$\int_0^{2\pi} \int_0^{\pi} [Y_{\ell}^m(\theta, \phi)]^* [Y_{\ell'}^{m'}(\theta, \phi)] \sin\theta d\theta d\phi = \delta_{\ell\ell'} \delta_{mm'} \quad (19.2.2)$$

The normalized **hydrogen wave functions** containing the quantum numbers n, m , and ℓ are

$$\Psi_{n\ell m}(r, \theta, \phi) = \sqrt{\left(\frac{2}{na}\right)^3 \frac{(n-\ell-1)!}{2n[(n+\ell)!]^3}} e^{-r/na} \left(\frac{2r}{na}\right)^{\ell} \left[L_{n-\ell-1}^{2\ell+1}(2r/na)\right] Y_{\ell}^m(\theta, \phi). \quad (19.2.3)$$

The ground state wave functions and energy of Hydrogen is

$$\Psi_{100}(r, \theta, \phi) = \frac{1}{\sqrt{\pi a^3}} e^{-r/a} \quad \text{and} \quad E_1 = - \left[\frac{m}{2\hbar^2} \left(\frac{e^2}{4\pi\epsilon_0} \right)^2 \right] = -13.6\text{eV} \quad (19.2.4)$$

Each operator \hat{Y} has a set of eigenvalues y which are the possible values you can get on doing a measurement of Y . Each eigenvalues y is associated with an eigenstate $\phi_y(x)$ which is the state for which the values of Y is exactly y with no uncertainty. You can find the eigenstates and eigenvalues of an operator by

$$\hat{Y}\phi_y(x) = y\phi_y(x). \quad (19.2.5)$$

The eigenstates of any operator \hat{Y} form a complete orthonormal basis of states so we can write any state $\psi(x)$ in terms of them.

$$\psi(x) \equiv \sum_y A_y \phi_y(x) \quad \text{or} \quad \psi(x) \equiv \int A(y) \phi_y(x) dy. \quad (19.2.6)$$

To solve for the coefficients in the above expression we can use Fouriers trick, or

$$A(y) = \int \phi_y^*(x) \psi(x) dx \quad (19.2.7)$$

If you are within operator space you can find the expectation value of an operator by

$$\langle \hat{y} \rangle = \int y |A(y)|^2 dy. \quad (19.2.8)$$

The commutation relation is a relationship between two operators and given by

$$[x, y] \equiv xy - yx \implies [x, y] = -[y, x] \quad (19.2.9)$$

$$[x, y] = [y, x] = 0 \implies x \text{ and } y \text{ commute} \quad (19.2.10)$$

$$[xy, z] = x[y, z] + [x, z]y \quad \text{and} \quad [x, yz] = y[x, z] + [x, y]z. \quad (19.2.11)$$

Position and momentum are related via commutation by the following:

$$[x, p_x] = [y, p_y] = [z, p_z] = i\hbar \quad (19.2.12)$$

$$[x, p_y] = [x, p_z] = [y, p_x] = [y, p_z] = [z, p_x] = [z, p_y] = 0. \quad (19.2.13)$$

19.3 Angular Momentum and Spin

The angular momentum operators are related via commutation by the following:

$$[L_x, L_y] = i\hbar L_z \quad \text{and} \quad [L_y, L_z] = i\hbar L_x \quad \text{and} \quad [L_z, L_x] = i\hbar L_y \quad (19.3.1)$$

$$[L_x, \vec{r}] = i\hbar(z - y) \quad \text{and} \quad [L_x, \vec{L}] = i\hbar(L_z - L_y) \quad \text{and} \quad [L_x, \vec{p}] = i\hbar(p_z - p_y) \quad (19.3.2)$$

$$[\vec{L}^2, L_{\pm}] = 0 \quad \text{and} \quad [\vec{L}^2, L_z] = 0 \quad \text{and} \quad [L_z, L_{\pm}] = \pm\hbar L_{\pm} \quad (19.3.3)$$

The angular momentum operators as well as the raising and lowering operators are given by

$$\vec{L} = \frac{\hbar}{i} \left(\hat{\phi} \frac{\partial}{\partial \theta} - \frac{\hat{\theta}}{\sin \theta} \frac{\partial}{\partial \phi} \right) \quad \text{and} \quad L_z = \frac{\hbar}{i} \frac{\partial}{\partial \phi} \quad (19.3.4)$$

$$L_x = \frac{L_+ + L_-}{2} = \frac{\hbar}{i} \left(-\sin \phi \frac{\partial}{\partial \theta} - \cot \theta \cos \phi \frac{\partial}{\partial \phi} \right) \quad (19.3.5)$$

$$L_y = \frac{L_+ - L_-}{2i} = \frac{\hbar}{i} \left(\cos \phi \frac{\partial}{\partial \theta} - \cot \theta \sin \phi \frac{\partial}{\partial \phi} \right) \quad (19.3.6)$$

$$L_{\pm} = L_x \pm iL_y = \pm\hbar e^{\pm i\phi} \left(\frac{\partial}{\partial \theta} \pm i \cot(\theta) \frac{\partial}{\partial \phi} \right) \quad (19.3.7)$$

$$\vec{L}^2 = L_x^2 + L_y^2 + L_z^2 = -\hbar^2 \left[\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right] \quad (19.3.8)$$

The angular momentum operators satisfy

$$\vec{L}^2 |\ell, m\rangle = \hbar^2 \ell(\ell + 1) |\ell, m\rangle \quad \text{and} \quad L_z |\ell, m\rangle = \hbar m |\ell, m\rangle \quad (19.3.9)$$

$$L_{\pm} |\ell, m\rangle = \hbar \sqrt{(\ell \mp m)(\ell \pm m + 1)} |\ell, m \pm 1\rangle \quad (19.3.10)$$

The fundamental commutation relations for spin are

$$[S_x, S_y] = i\hbar S_z \quad \text{and} \quad [S_y, S_z] = i\hbar S_x \quad \text{and} \quad [S_z, S_x] = i\hbar S_y. \quad (19.3.11)$$

The general state of a spin-1/2 particle can be expressed as a two element column matrix (or spinor):

$$\chi = \begin{pmatrix} a \\ b \end{pmatrix} = a \begin{pmatrix} 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \end{pmatrix} = a\chi_+ + b\chi_-. \quad (19.3.12)$$

The spin matrices are given by

$$\vec{S}_x = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \vec{S}_y = \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \vec{S}_z = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (19.3.13)$$

$$\vec{S}_+ = \hbar \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \vec{S}_- = \hbar \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \vec{S}^2 = \frac{3\hbar^2}{4} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (19.3.14)$$

The **Pauli Spin Matrices** are then given by

$$\sigma_x \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (19.3.15)$$

The combined state $|s, m\rangle$ with total spin s and z-component m will be some linear combination of the composite states $|s_1, m_1\rangle$ and $|s_2, m_2\rangle$ and depends on the Clebsch-Gordan Coefficients $C_{m_1 m_2 m}^{s_1 s_2 s}$ (page 93):

$$|s, m\rangle = \sum_{m=m_1+m_2} C_{m_1 m_2 m}^{s_1 s_2 s} |s_1, m_1\rangle |s_2, m_2\rangle, \quad |s_1, m_1\rangle |s_2, m_2\rangle = \sum_s C_{m_1 m_2 m}^{s_1 s_2 s} |s, m\rangle \quad (19.3.16)$$

19.3.1: Hund's Rules

To find the state with the lowest energy configuration,

1. Choose the largest S.
2. Choose the Largest L (consistent with the Pauli Exclusion Principle).
3. If the shell is less than one half filled, choose the smallest J. If the shell is greater than one half filled, choose the largest J. If the shell is half filled, then $L = 0 \implies J = S$.

From the **Time-independent perturbation theory** at an approximation gives the correction energy (1^{st} and 2^{nd} order given) and wave function (1^{st} order) as

$$|\psi_n^{(1)}\rangle = \sum_{\ell \neq n} |\psi_\ell^{(0)}\rangle \frac{\langle \psi_\ell^{(0)} | H^1 | \psi_n^{(0)} \rangle}{E_n^{(0)} - E_\ell^{(0)}} \quad E_n^{(1)} = \langle \psi_n^{(0)} | H^1 | \psi_n^{(0)} \rangle \quad E_n^{(2)} = \sum_{m \neq n} \frac{|\langle \psi_m^{(0)} | H^1 | \psi_n^{(0)} \rangle|^2}{E_n^{(0)} - E_m^{(0)}} \quad (19.3.17)$$

For a two state system and a time dependent Hamiltonian, we can derive a relationship for the probability coefficients of the system by solving Shrödinger's equation which is used with $H'_{ij}(t) = \langle \psi_i | H(t) | \psi_j \rangle$ (and with $\omega = \frac{E_2 - E_1}{\hbar}$) which gives

$$\frac{dc_1(t)}{dt} = -\frac{i}{\hbar} H'_{12}(t) e^{-i\omega t} c_2(t) \quad \frac{dc_2(t)}{dt} = -\frac{i}{\hbar} H'_{21}(t) e^{i\omega t} c_1(t). \quad (19.3.18)$$

For a two state system the coefficients of the system up to second order given an initial state of $c_a^{(0)}(t) = a$ and $c_b^{(0)}(t) = b$, with $\omega_0 = (E_b - E_a)/\hbar$ are

$$c_a^{(2)}(t) = a - \frac{ib}{\hbar} \int_0^t H'_{ab}(t') e^{-i\omega_0 t'} dt' - \frac{a}{\hbar^2} \int_0^t H'_{ab} e^{-i\omega_0 t'} \left[\int_0^{t'} H'_{ba}(t'') e^{i\omega_0 t''} dt'' \right] dt' \quad (19.3.19)$$

$$c_b^{(2)}(t) = b - \frac{ia}{\hbar} \int_0^t H'_{ba}(t') e^{i\omega_0 t'} dt' - \frac{b}{\hbar^2} \int_0^t H'_{ba} e^{i\omega_0 t'} \left[\int_0^{t'} H'_{ab}(t'') e^{-i\omega_0 t''} dt'' \right] dt' \quad (19.3.20)$$

The **variational principle** will give an upper bound for the ground state energy for a system described by the Hamiltonian H . This is often useful when you are unable to solve Schrödinger's equation.

$$E_{gs} \leq \langle \psi | H | \psi \rangle \quad (19.3.21)$$

The **Rabi-Flopping frequency** is defined as

$$\omega_r \equiv \frac{1}{2} \sqrt{(\omega - \omega_0)^2 + (|V|/\hbar)^2} \quad (19.3.22)$$

The **transition rate** for stimulated emission from state b to state a, under the influence of incoherent, unpolarized light incident from all directions (where \mathfrak{p} is the matrix element of the electric dipole moment between the two states and $\rho(\omega_0)$ is the energy density in the fields with $\omega_0 = (E_b - E_a)/\hbar$ [9] is

$$R_{b \rightarrow a} = \frac{\pi}{3\epsilon_0 \hbar^2} |\mathfrak{p}|^2 \rho(\omega_0) \quad (19.3.23)$$

The **spontaneous emission** rate is

$$A = \frac{\omega_0^3 |\mathfrak{p}|^2}{3\pi \epsilon_0 \hbar c^3} \quad (19.3.24)$$

19.4 Scattering

The impact parameter b is given by

$$b = R \sin \left(\frac{\pi}{2} - \frac{\theta}{2} \right) = R \cos \left(\frac{\theta}{2} \right). \quad (19.4.1)$$

Particles incident in the area $d\sigma$ scatter into the solid angle $d\Omega$,

$$d\sigma = D(\theta) d\Omega = \underbrace{\frac{b}{\sin(\theta)} \left| \frac{db}{d\theta} \right|}_{D(\theta)} \underbrace{\sin(\theta) d\theta d\phi}_{d\Omega} = b d\theta d\phi \implies D(\theta) = \frac{d\sigma}{d\Omega} = |f(\theta)|^2 \quad (19.4.2)$$

In the case of **hard sphere scattering**,

$$\frac{db}{d\theta} = -\frac{1}{2} R \sin \left(\frac{\theta}{2} \right) \implies D(\theta) = \frac{R \cos(\theta/2)}{\sin(\theta)} \left(\frac{\sin(\theta/2)}{2} \right) = \frac{R^2}{4} \quad (19.4.3)$$

In the quantum theory of scattering, for an incident plane wave $\psi = e^{ikz}$ traveling in the z direction and for large r we can find solutions to the Schrödinger equation (with $k \equiv \frac{\sqrt{2mE}}{\hbar}$ and a_ℓ which is called the partial wave amplitude) of the general form

$$\psi(r, \theta) = A \left(e^{ikz} + k \sum_{\ell=0}^{\infty} i^{\ell+1} (2\ell+1) a_\ell h_\ell^{(1)}(kr) P_\ell(\cos \theta) \right) \approx A \left(e^{ikz} + f(\theta) \frac{e^{ikr}}{r} \right) \quad (19.4.4)$$

The differential cross section can be written as

$$D(\theta) = |f(\theta)|^2 = \left| \sum_{\ell=0}^{\infty} (2\ell+1) a_\ell P_\ell(\cos \theta) \right|^2 = \sum_{\ell} \sum_{\ell'} (2\ell+1)(2\ell'+1) a_\ell^* a_{\ell'} P_\ell(\cos \theta) P_{\ell'}(\cos \theta) \quad (19.4.5)$$

The total cross section is then given by

$$\sigma = 4\pi \sum_{\ell=0}^{\infty} (2\ell+1) |a_\ell|^2 \quad (19.4.6)$$

For a Hydrogen atom, while taking into account the relativistic effects and the spin-orbit coupling, we acquire the energy for the fine structure component given as

$$E_{n,j} = E_n \left[1 + \frac{Z^2 \alpha^2}{3n} \left(\frac{1}{j+1/2} - \frac{3}{4n} \right) \right] \quad (19.4.7)$$

For particles within a magnetic field \vec{B} the **Zeeman Effect** occurs which creates an energy correction (with g_j known as the Landé g factor)

$$E_{zeeman} = m_j g_j \mu_B B \quad \text{with} \quad g_j = 1 + \frac{j(j+1) + s(s+1) - \ell(\ell-1)}{2j(j+1)} \quad (19.4.8)$$

Nuclear and High Energy Physics

20.1 Nuclear Physics

The atomic nucleus consists of protons and neutrons collectively called nucleons. Nuclei with different number of neutrons but with the same number of protons are isotopes of the same element. The mass number of an isotope is the sum of the number of protons (Z) and the number of neutrons (N)

$$A = Z + N. \quad (20.1.1)$$

Nuclei are approximately spherical in shape, with the radius of the sphere depending on the mass number ($R_0 = 1.12 \text{ fm}$)

$$R(A) = R_0 A^{1/3}. \quad (20.1.2)$$

The nucleon density in the interior of a nucleus is $n = 0.17 \text{ fm}^{-3}$, and the mass density is $\rho = m_{\text{nucleon}} n = 2.8 \times 10^{17} \text{ kg/m}^3$. The dependence of the density on the radial coordinate is given by the Fermi function ($a = 0.54 \text{ fm}$):

$$n(r) = \frac{n_0}{1 + e^{(r - R(A))/a}}. \quad (20.1.3)$$

The Nuclear (or “strong”) force is what binds the protons and neutrons together into nuclei with the following properties:

- (i) Within the nucleus, it is about 100 times stronger than the electromagnetic force and approximately 10^{38} times stronger than gravity.
- (ii) It is charge-independent.
- (iii) It is spin-dependent.

In any nuclear reaction, the following quantities are conserved:

Nuclear Number (A), electric charge, total energy and total momentum.

The mass of a nucleus with Z protons and N neutrons is smaller than the sum of the individual nucleon masses, and the binding energy is defined as the mass difference times c^2 :

$$B(N, Z) = Zm(0, 1)c^2 + Nm_n c^2 - m(N, Z)c^2. \quad (20.1.4)$$

The **mass excess** of a nucleus is defined as the difference between the mass of a nucleus expressed in atomic mass units and the mass number:

$$\text{mass excess} = m(N, Z) - (A)(1u). \quad (20.1.5)$$

The binding energies of different isotopes can be reproduced well by the Bethe-Weizsächer formula from the liquid-drop model, as the sum of volume, surface, Coulomb, asymmetry, and pairing contributions:

$$B(N, Z) = B_v(N, Z) + B_s(N, Z) + B_c(N, Z) + B_a(N, Z) + B_p(N, Z) \quad (20.1.6)$$

$$= a_v A - a_s A^{2/3} - a_c Z^2 A^{-1/3} - a_a \left(Z - \frac{1}{2} A \right)^2 A^{-1} + a_p ((-1)^Z + (-1)^N) A^{-1/2}. \quad (20.1.7)$$

Dividing this expression by the mass number gives the binding energy per nucleon:

$$\frac{B(N, Z)}{A} = a_v - a_s A^{-1/3} - a_c \frac{Z^2}{A^{4/3}} - a_a \left(\frac{Z}{A} - \frac{1}{2} \right)^2 + a_p \frac{(-1)^Z + (-1)^N}{A^{3/2}}. \quad (20.1.8)$$

Several successful fits have been published for the empirical mass formula above. Using the values obtained by Bertulani and Schechter (2002), we have

$$a_v = 15.85 \text{ MeV}, a_s = 18.34 \text{ MeV}, a_c = 0.71 \text{ MeV}, a_a = 92.86 \text{ MeV}, a_p = 11.46 \text{ MeV}.$$

The Fermi gas model proposes a quantum gas of nucleons that can move freely inside the nucleus but are confined by the nuclear surface. The density of states in the Fermi gas model is

$$dN(E) = \frac{1}{\pi^2 a^3 \hbar^3} \sqrt{\frac{m^3 E}{2}} dE. \quad (20.1.9)$$

The **Fermi energy** is

$$E_F = \frac{\hbar^2}{2m} \sqrt{\frac{9}{4} \pi^4 n_0^2} = 38 \text{ MeV}. \quad (20.1.10)$$

In a nuclear reaction, the difference between final and initial kinetic energies is called the *Q*-value:

$$Q = \Delta KE = -\Delta Mc^2 \quad \begin{cases} Q > 0 \implies \text{Exothermic} \\ Q < 0 \implies \text{Endothermic} \end{cases} \quad (20.1.11)$$

20.2 Nuclear Decay

Nuclear decays follow an exponential decay law. The decay constant λ , half life $t_{1/2}$, and mean lifetime τ are related:

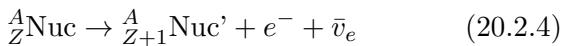
$$N(t) = N_0 e^{-\lambda t} \quad (20.2.1)$$

$$t_{1/2} = \frac{\ln(2)}{\lambda} = \tau \ln(2). \quad (20.2.2)$$

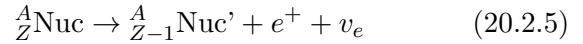
In alpha (α) decay, a heavier nucleus (Nuc) emits a helium-4 nucleus:



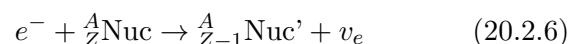
In a β^- decay, an electron and an anti-neutrino are emitted:



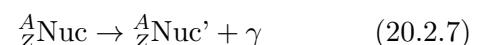
In a β^+ decay can proceed via a positron emission:



This type of decay can also occur via electron capture



A **gamma decay** is an emission of a high-energy photon from an excited nucleus, a process that does not transmute the nucleus:



20.3 Elementary Particle Physics

Substructure is probed using scattering experiments. The scattering cross section is defined as

$$\sigma = \frac{\# \text{ of reactions per scattering center/s}}{\# \text{ of impinging particles/s/m}^2}. \quad (20.3.1)$$

The scattering cross section has the physical dimension of area and is measured in the unit barn (b) or millibarn (mb):

$$1 \text{ b} = 10^{-28} \text{ m}^2 \quad \text{and} \quad 1 \text{ mb} = 10^{-31} \text{ m}^2. \quad (20.3.2)$$

The classical **Rutherford cross section** for scattering from a pointlike target by the Coulomb interaction is

$$\frac{d\sigma}{d\Omega} = \left(\frac{kZ_P Z_t e^2}{4K} \right)^2 \frac{1}{\sin^4(\theta/2)}. \quad (20.3.3)$$

For scattering of a plane wave off a point source, the scattering wave function is

$$\psi_{total}(\vec{r}) = \psi_i(\vec{r}) + \psi_f(\vec{r}) = N \left(e^{ikz} + f(\theta) \frac{e^{ikr}}{r} \right) \quad \text{and} \quad \frac{d\sigma}{d\Omega} = |f(\theta)|^2. \quad (20.3.4)$$

The form factor is the Fourier transform of the density distribution and measures the deviation of the scattering cross section from the Rutherford cross-section of a point-like target:

$$F^2(\Delta p) = \left| \frac{1}{e} \int \rho(\vec{r}) e^{i\Delta\vec{p}\cdot\vec{r}/\hbar} dV \right|^2 \quad \text{and} \quad \frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega} \right)_{point} \cdot F^2(\Delta p). \quad (20.3.5)$$

- Elementary fermions have spin $\frac{1}{2}\hbar$ and include the six quarks (up, down, strange, charm, bottom, and top), the electron, muon, and tau leptons, and the electron-, muon-, and tau-neutrinos. Each of these 12 fermions has an antiparticle. Quarks all have a non-integer charge of $-\frac{1}{3}e$ or $+\frac{2}{3}e$ and cannot be observed in isolation.
- Elementary bosons are the mediators of the interactions between the fermions. They are the photon (electromagnetic), the W and Z bosons (electroweak), the gluon (strong), and the graviton (gravitational). The graviton is yet to be found experimentally. Gluons can also interact with other gluons.
- Elementary quarks and antiquarks can combine to form color singlets, which are particles that can be observed in isolation. A quark and an antiquark can form a meson (pion, kaon, etc.). Three quarks can form a baryon (proton, neutron, delta baryon, etc.). The only stable baryon is the proton, and none of the mesons is stable. Lifetimes of the unstable particles vary from 10^{-23} seconds to 15 minutes.
- The quark-gluon plasma phase transition of the early universe can be probed in the laboratory with relativistic heavy ion collisions. The primordial fraction of 23% helium in the universe can be explained from the neutron-proton mass difference, which fixes the ratio of proton and neutron numbers to $n_n/n_p = e^{(m_n - m_p)c^2/k_B T}$.

Single and double nucleon separation energies (S_{n_1} and S_{n_2}) are given by

$$S_{n_1} = B(N, Z) - B(N - 1, Z) \quad \text{and} \quad S_{n_2} = B(N, Z) - B(N - 2, Z) \quad (20.3.6)$$

$$S_{p_1} = B(N, Z) - B(N, Z - 1) \quad \text{and} \quad S_{p_2} = B(N, Z) - B(N, Z - 2). \quad (20.3.7)$$

The neutron pairing gap as related to the separation energies is given by

$$\Delta_n = \frac{(-1)^N}{2} [S_{n_1}(N, Z) - S_{n_1}(N - 1, Z)]. \quad (20.3.8)$$

Advanced Physics

21.1 Quantum Chromodynamics

The classical Lagrangian density for n non-interacting quarks with masses m_i is

$$\mathcal{L}_{\text{quarks}} = \sum_i^n q_i^{-a} (i\partial - m_i)_{ab} a_i^b. \quad (21.1.1)$$

The **Quantum Chromodynamic Lagrangian**, with the color fields tensor $G_\alpha^{\mu c} = \partial^\mu G_\alpha^c - \partial^c G_\alpha^\mu - g f^{\alpha\beta\gamma} G_\beta^\mu G_\gamma^c$, the four potential of the gluon fields ($\alpha = 1, \dots, 8$) G_α^μ , the 3×3 Gell-Mann matrices - generators of the SU(3) color group t_α , the structure constants of the SU(3) color group $f^{\alpha\beta\gamma}$, the Dirac spinor of the quark field (i represents color) ψ_i , and units where $g = \sqrt{4\pi\alpha_s}$ ($\hbar = c = 1$) [18].

$$\mathcal{L}_{\text{QCD}} = \sum_q (\psi_q^* i\gamma^\mu [\delta_{ij} \partial_\mu + ig(G_\mu^\alpha t_\alpha)_{ij}] \psi_{qj} - m_q \psi_{qi}^* \psi_{qi}) - \frac{1}{4} G_{\mu\nu}^\alpha G_\alpha^{\mu\nu} \quad (21.1.2)$$

$$= \mathcal{L}_{\text{quarks}} + \mathcal{L}_{\text{ghost}} - \frac{1}{2\lambda} (\partial^\mu A_\mu^\alpha)^2 - \frac{1}{4} G_{\mu\nu}^\alpha G_\alpha^{\mu\nu}. \quad (21.1.3)$$

The **Quantum Electrodynamics Lagrange**, with the EM field tensor $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$, the four potential of the photon field A^μ , the Dirac 4x4 matrices γ^μ , the Dirac four-spinor of the electron field ψ_e , and using units $e = \sqrt{4\pi\alpha}$, $1/\alpha \approx 137$, and $\hbar = c = 1$ [18].

$$\mathcal{L}_{\text{QED}} = \psi_e^* i\gamma^\mu [\partial_\mu + ieA_\mu] \psi_e - m_e \psi_e^* \psi_e - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \quad (21.1.4)$$

From the Euler-Lagrange equation of motion for a field, we can get both the Dirac equation for the electron and Maxwell equations for the EM fields in the Lorentz gauge.

$$\underbrace{(i\gamma^\mu \partial_\mu - me)\psi_e}_{\text{Dirac}} = e\gamma^\mu A_\mu \psi_e \quad \underbrace{\partial^\mu \partial_\mu A^\mu}_{\text{Maxwell}} = e\psi_e^* \gamma^\mu \psi_e \quad (21.1.5)$$

The Lagrange equation for a multiple pendulum system with n number of rods, where the i^{th} rod has a length L_i , and mass m_i .

$$\mathcal{L} = \frac{1}{2} \sum_{i=1}^n m_i (\dot{x}_i^2 + \dot{y}_i^2) + \frac{1}{6} \sum_{i=1}^n m_i L_i^2 \dot{\theta}_i^2 - g \sum_{i=1}^n m_i y_i. \quad (21.1.6)$$

34. CLEBSCH-GORDAN COEFFICIENTS, SPHERICAL HARMONICS, AND d FUNCTIONS

Note: A square-root sign is to be understood over *every* coefficient, e.g., for $-8/15$ read $-\sqrt{8/15}$.

$$1/2 \times 1/2 \begin{matrix} 1 \\ +1/2+1/2 \\ 1 \\ 0 \\ 0 \end{matrix}$$

$$\begin{matrix} 1/2 & -1/2 & 1/2 & 1/2 & 1 \\ -1/2 & +1/2 & 1/2 & -1/2 & -1 \end{matrix}$$

$$\begin{matrix} 1/2 \\ -1/2-1/2 \\ 1 \end{matrix}$$

$$Y_1^0 = \sqrt{\frac{3}{4\pi}} \cos \theta$$

$$2 \times 1/2 \begin{matrix} 5/2 \\ +5/2 \\ 5/2 \\ 3/2 \\ 1 \end{matrix}$$

$$\begin{matrix} +2 & +1/2 \\ +1 & +1/2 \end{matrix}$$

$$\begin{matrix} 1/5 & 4/5 \\ 4/5 & -1/5 \end{matrix}$$

$$\begin{matrix} 5/2 & 3/2 \\ +1/2 & +1/2 \end{matrix}$$

J	J	...
M	M	...
m_1	m_2	
m_1	m_2	Coefficients
\vdots	\vdots	
\vdots	\vdots	

$$1 \times 1/2 \begin{matrix} 3/2 \\ +3/2 \\ 3/2 & 1/2 \\ +1 & +1/2 \\ 1 & +1/2+1/2 \end{matrix}$$

$$\begin{matrix} +1 & -1/2 \\ 1/3 & 2/3 \\ 2/3 & -1/3 \end{matrix}$$

$$\begin{matrix} 0 & -1/2 \\ -1 & +1/2 \end{matrix}$$

$$\begin{matrix} 2/3 & 1/3 \\ 1/3 & -2/3 \end{matrix}$$

$$\begin{matrix} 3/2 \\ -3/2 \end{matrix}$$

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}} \sin \theta e^{i\phi}$$

$$Y_2^0 = \sqrt{\frac{5}{4\pi}} \left(\frac{3}{2} \cos^2 \theta - \frac{1}{2} \right)$$

$$Y_2^1 = -\sqrt{\frac{15}{8\pi}} \sin \theta \cos \theta e^{i\phi}$$

$$Y_2^2 = \frac{1}{4} \sqrt{\frac{15}{2\pi}} \sin^2 \theta e^{2i\phi}$$

$$2 \times 1/2 \begin{matrix} 5/2 \\ +5/2 \\ 5/2 & 3/2 \\ +1 & +3/2 \\ 1 & +1/2 \end{matrix}$$

$$\begin{matrix} +1 & -1/2 \\ 2/5 & 3/5 \\ 3/5 & -2/5 \end{matrix}$$

$$\begin{matrix} 5/2 & 3/2 \\ -1/2 & -1/2 \end{matrix}$$

$$2 \times 1 \begin{matrix} 3 \\ +3 \\ 3 & 2 \\ +2 & +1 \\ 1 & +2 & +2 \end{matrix}$$

$$\begin{matrix} +2 & 0 \\ 1/3 & 2/3 \end{matrix}$$

$$\begin{matrix} 3 & 2 \\ +1 & +1 \\ +1 & +1 \end{matrix}$$

$$\begin{matrix} 1 & -1/2 \\ -1 & +1/2 \end{matrix}$$

$$3/2 \times 1 \begin{matrix} 5/2 \\ +5/2 \\ 5/2 & 3/2 \\ +1 & +3/2 \\ 1 & +1/2 \end{matrix}$$

$$\begin{matrix} +3/2 & 0 \\ 2/5 & 3/5 \end{matrix}$$

$$\begin{matrix} 5/2 & 3/2 & 1/2 \\ +1/2 & +1/2 & +1/2 \end{matrix}$$

$$1 \times 1 \begin{matrix} 2 \\ +2 \\ 2 & 1 \\ +1 & +1 \end{matrix}$$

$$\begin{matrix} +2 & -1 \\ 1/5 & 1/3 & 3/5 \end{matrix}$$

$$\begin{matrix} 3 & 2 & 1 \\ +1 & 0 \\ 0 & +1 \end{matrix}$$

$$\begin{matrix} 2/5 & -1/2 \\ 1/10 & \end{matrix}$$

$$3/2 \times 1/2 \begin{matrix} 2 \\ +2 \\ 2 & 1 \\ +1 & +1 \end{matrix}$$

$$\begin{matrix} +3/2 & 0 \\ 2/5 & 3/5 \end{matrix}$$

$$\begin{matrix} 5/2 & 3/2 & 1/2 \\ +1/2 & +1/2 & +1/2 \end{matrix}$$

$$Y_\ell^{-m} = (-1)^m Y_\ell^m$$

$$d_{m,0}^\ell = \sqrt{\frac{4\pi}{2\ell+1}} Y_\ell^m e^{-im\phi}$$

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 JM \rangle$$

$$= (-1)^{J-j_1-j_2} \langle j_2 j_1 m_2 m_1 | j_2 j_1 JM \rangle$$

$$d_{m',m}^j = (-1)^{m-m'} d_{m,m'}^j = d_{-m,-m'}^j$$

$$d_{0,0}^1 = \cos \theta \quad d_{1/2,1/2}^{1/2} = \cos \frac{\theta}{2} \quad d_{1,1}^1 = \frac{1+\cos \theta}{2}$$

$$2 \times 3/2 \begin{matrix} 7/2 \\ +7/2 \\ 7/2 & 5/2 \\ +2 & +3/2 \\ 1 & +5/2 & +5/2 \end{matrix}$$

$$\begin{matrix} +2 & +1/2 \\ 3/7 & 4/7 \\ 4/7 & -3/7 \end{matrix}$$

$$\begin{matrix} 5/2 & 3/2 \\ +3/2 & +3/2 \\ +3/2 & +3/2 \end{matrix}$$

$$3/2 \times 3/2 \begin{matrix} 3 \\ +3 \\ 3 & 2 \\ +2 & +2 \end{matrix}$$

$$\begin{matrix} +3/2 & +1/2 \\ 1/2 & 1/2 \\ 1/2 & -1/2 \end{matrix}$$

$$\begin{matrix} 3 & 2 & 1 \\ +1 & +1 & +1 \end{matrix}$$

$$2 \times 2 \begin{matrix} 4 \\ +4 \\ 4 & 3 \\ +2 & +1 \end{matrix}$$

$$\begin{matrix} +2 & -1 \\ 1/7 & 16/35 & 2/5 \\ 4/7 & 1/35 & -2/5 \\ 0 & +3/2 & 2/7-18/35 \end{matrix}$$

$$\begin{matrix} 5/2 & 3/2 & 1/2 \\ 7/2 & 5/2 & 1/2 \\ 7/2 & 5/2 & 1/2 \end{matrix}$$

$$d_{0,0}^1 = \cos \theta \quad d_{1/2,-1/2}^{1/2} = -\sin \frac{\theta}{2} \quad d_{1,0}^1 = -\frac{\sin \theta}{\sqrt{2}}$$

$$d_{1/2,-1/2}^{1/2} = -\sin \frac{\theta}{2} \quad d_{1,0}^1 = -\frac{\sin \theta}{\sqrt{2}} \quad d_{1,-1}^1 = \frac{1-\cos \theta}{2}$$

$$2 \times 2 \begin{matrix} 4 \\ +4 \\ 4 & 3 \\ +2 & +1 \end{matrix}$$

$$\begin{matrix} +2 & 0 \\ 3/14 & 1/2 & 2/7 \\ +1 & +1 \\ 0 & +2 \end{matrix}$$

$$\begin{matrix} 4 & 3 & 2 & 1 \\ +1 & +1 & +1 & +1 \end{matrix}$$

$$d_{0,0}^1 = \cos \theta \quad d_{1/2,-1/2}^{1/2} = -\sin \frac{\theta}{2} \quad d_{1,0}^1 = -\frac{\sin \theta}{\sqrt{2}}$$

$$d_{1/2,-1/2}^{1/2} = -\sin \frac{\theta}{2} \quad d_{1,0}^1 = -\frac{\sin \theta}{\sqrt{2}} \quad d_{1,-1}^1 = \frac{1-\cos \theta}{2}$$

$$d_{3/2,3/2}^{3/2} = \frac{1+\cos \theta}{2} \cos \frac{\theta}{2}$$

$$d_{3/2,1/2}^{3/2} = -\sqrt{3} \frac{1+\cos \theta}{2} \sin \frac{\theta}{2}$$

$$d_{3/2,-1/2}^{3/2} = \sqrt{3} \frac{1-\cos \theta}{2} \cos \frac{\theta}{2}$$

$$d_{3/2,-3/2}^{3/2} = -\frac{1-\cos \theta}{2} \sin \frac{\theta}{2}$$

$$d_{1/2,1/2}^{3/2} = \frac{3\cos \theta - 1}{2} \cos \frac{\theta}{2}$$

$$d_{1/2,-1/2}^{3/2} = -\frac{3\cos \theta + 1}{2} \sin \frac{\theta}{2}$$

$$d_{2,2}^2 = \left(\frac{1+\cos \theta}{2} \right)^2 \quad d_{2,1}^2 = -\frac{1+\cos \theta}{2} \sin \theta \quad d_{2,0}^2 = \frac{\sqrt{6}}{4} \sin^2 \theta \quad d_{2,-1}^2 = -\frac{1-\cos \theta}{2} \sin \theta \quad d_{2,-2}^2 = \left(\frac{1-\cos \theta}{2} \right)^2$$

$$d_{1,1}^2 = \frac{1+\cos \theta}{2} (2 \cos \theta - 1) \quad d_{1,0}^2 = -\sqrt{\frac{3}{2}} \sin \theta \cos \theta \quad d_{1,-1}^2 = \frac{1-\cos \theta}{2} (2 \cos \theta + 1) \quad d_{0,0}^2 = \left(\frac{3}{2} \cos^2 \theta - \frac{1}{2} \right)$$

$$d_{3/2,3/2}^{3/2} = \frac{1+\cos \theta}{2} \cos \frac{\theta}{2}$$

$$d_{3/2,1/2}^{3/2} = -\sqrt{3} \frac{1+\cos \theta}{2} \sin \frac{\theta}{2}$$

$$d_{3/2,-1/2}^{3/2} = \sqrt{3} \frac{1-\cos \theta}{2} \cos \frac{\theta}{2}$$

$$d_{3/2,-3/2}^{3/2} = -\frac{1-\cos \theta}{2} \sin \frac{\theta}{2}$$

$$d_{1/2,1/2}^{3/2} = \frac{3\cos \theta - 1}{2} \cos \frac{\theta}{2}$$

$$d_{1/2,-1/2}^{3/2} = -\frac{3\cos \theta + 1}{2} \sin \frac{\theta}{2}$$

$$d_{3/2,3/2}^{3/2} = \frac{1+\cos \theta}{2} \cos \frac{\theta}{2}$$

$$d_{3/2,1/2}^{3/2} = -\sqrt{3} \frac{1+\cos \theta}{2} \sin \frac{\theta}{2}$$

$$d_{3/2,-1/2}^{3/2} = \sqrt{3} \frac{1-\cos \theta}{2} \cos \frac{\theta}{2}$$

$$d_{3/2,-3/2}^{3/2} = -\frac{1-\cos \theta}{2} \sin \frac{\theta}{2}$$

$$d_{1/2,1/2}^{3/2} = \frac{3\cos \theta - 1}{2} \cos \frac{\theta}{2}$$

$$d_{1/2,-1/2}^{3/2} = -\frac{3\cos \theta + 1}{2} \sin \frac{\theta}{2}$$

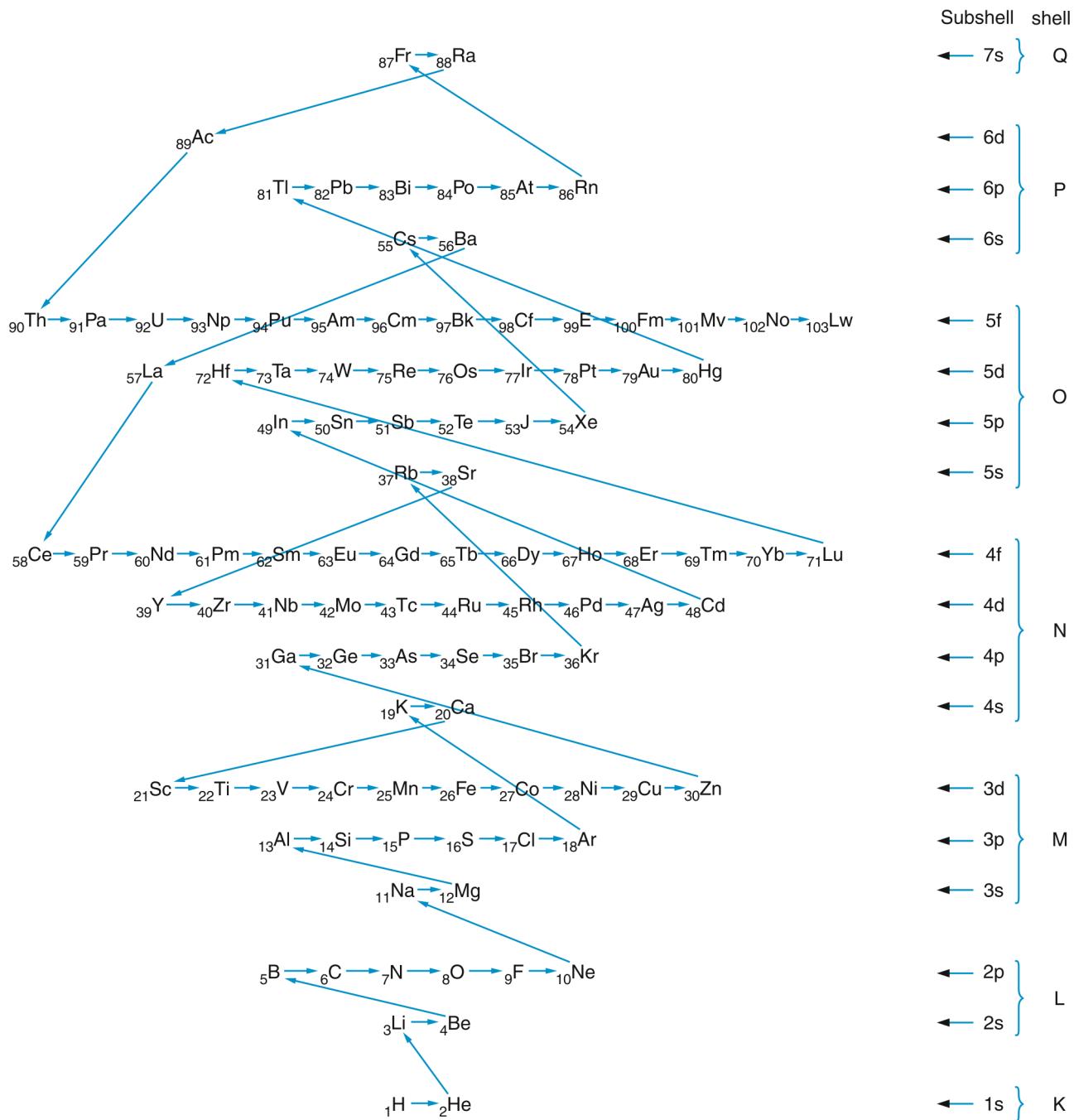
Figure 34.1: The sign convention is that of Wigner (*Group Theory*, Academic Press, New York, 1959), also used by Condon and Shortley (*The Theory of Atomic Spectra*, Cambridge Univ. Press, New York, 1953), Rose (*Elementary Theory of Angular Momentum*, Wiley, New York, 1957), and Cohen (*Tables of the Clebsch-Gordan Coefficients*, North American Rockwell Science Center, Thousand Oaks, Calif., 1974). The coefficients here have been calculated using computer programs written independently by Cohen and at LBNL.

22.2 Ground State Electron Configurations Of Elements

Shell			K	L		M			O	Shell			K	L		M			N			O		
Z		Element	1s	2s	2p	3s	3p	3d	4s	Z		Element	1s	2s	2p	3s	3p	3d	4s	4p	4d	5s	5p	
1	H	Hydrogen	1							28	Ni	Nickel	2	2	6	2	6	8	2					
2	He	Helium	2							29	Cu	Copper	2	2	6	2	6	10	1					
3	Li	Lithium	2	1						30	Zn	Zink	2	2	6	2	6	10	2					
4	Be	Beryllium	2	2						31	Ga	Gallium	2	2	6	2	6	10	2	1				
5	B	Boron	2	2	1					32	Ge	Germanium	2	2	6	2	6	10	2	2				
6	C	Carbon	2	2	2					33	As	Arsenic	2	2	6	2	6	10	2	3				
7	N	Nitrogen	2	2	3					34	Se	Selenium	2	2	6	2	6	10	2	4				
8	O	Oxygen	2	2	4					35	Br	Bromium	2	2	6	2	6	10	2	5				
9	F	Fluorine	2	2	5					36	Kr	Krypton	2	2	6	2	6	10	2	6				
10	Ne	Neon	2	2	6					37	Rb	Rubidium	2	2	6	2	6	10	2	6		1		
11	Na	Sodium	2	2	6	1				38	Sr	Strontium	2	2	6	2	6	10	2	6		2		
12	Mg	Magnesium	2	2	6	2				39	Y	Yttrium	2	2	6	2	6	10	2	6	1	2		
13	Al	Aluminum	2	2	6	2	1			40	Zr	Zirconium	2	2	6	2	6	10	2	6	2	2		
14	Si	Silicon	2	2	6	2	2			41	Nb	Niobium	2	2	6	2	6	10	2	6	4	1		
15	P	Phosphorus	2	2	6	2	3			42	Mo	Molybdenum	2	2	6	2	6	10	2	6	5	1		
16	S	Sulfur	2	2	6	2	4			43	Tc	Technetium	2	2	6	2	6	10	2	6	6	1		
17	Cl	Chlorine	2	2	6	2	5			44	Ru	Ruthenium	2	2	6	2	6	10	2	6	7	1		
18	Ar	Argon	2	2	6	2	6			45	Rh	Rhodium	2	2	6	2	6	10	2	6	8	1		
19	K	Potassium	2	2	6	2	6	1		46	Pd	Palladium	2	2	6	2	6	10	2	6	10			
20	Ca	Calcium	2	2	6	2	6	2		47	Ag	Silver	2	2	6	2	6	10	2	6	10	1		
21	Sc	Scandium	2	2	6	2	6	1	2	48	Cd	Cadmium	2	2	6	2	6	10	2	6	10	2		
22	Ti	Titanium	2	2	6	2	6	2	2	49	In	Indium	2	2	6	2	6	10	2	6	10	2	1	
23	V	Vanadium	2	2	6	2	6	3	2	50	Sn	Tin	2	2	6	2	6	10	2	6	10	2	2	
24	Cr	Chromium	2	2	6	2	6	5	1	51	Sb	Antimony	2	2	6	2	6	10	2	6	10	2	3	
25	Mn	Manganese	2	2	6	2	6	5	2	52	Te	Tellurium	2	2	6	2	6	10	2	6	10	2	4	
26	Fe	Iron	2	2	6	2	6	6	2	53	I	Iodine	2	2	6	2	6	10	2	6	10	2	5	
27	Co	Cobalt	2	2	6	2	6	7	2	54	Xe	Xenon	2	2	6	2	6	10	2	6	10	2	6	
Shell			N	O			P	Shell			N	O			P	Q								
Z		Element	4f	5s	5p	5d	5f	6s	Z		Element	4f	5s	5p	5d	5f	6s	6p	6d	7s				
55	Cs	Cesium		2	6			1	80	Hg	Mercury	14	2	6	10		2							
56	Ba	Barium		2	6			2	81	Tl	Thallium	14	2	6	10		2	1						
57	La	Lanthanum		2	6	1		2	82	Pb	Lead	14	2	6	10		2	2						
58	Ce	Cerium	2	2	6			2	83	Bi	Bismuth	14	2	6	10		2	3						
59	Pr	Praseodymium	3	2	6			2	84	Po	Polonium	14	2	6	10		2	4						
60	Nd	Neodymium	4	2	6			2	85	At	Astatine	14	2	6	10		2	5						
61	Pm	Promethium	5	2	6			2	86	Rn	Radon	14	2	6	10		2	6						
62	Sm	Samarium	6	2	6			2	87	Fr	Francium	14	2	6	10		2	6	1					
63	Eu	Europium	7	2	6			2	88	Ra	Radium	14	2	6	10		2	6	2					
64	Gd	Gadolinium	7	2	6	1		2	89	Ac	Actinium	14	2	6	10		2	6	1	2				
65	Tb	Terbium	9	2	6			2	90	Th	Thorium	14	2	6	10		2	6	2	2				
66	Dy	Dysprosium	10	2	6			2	91	Pa	Protactinium	14	2	6	10	2	2	6	1	2				
67	Ho	Holmium	11	2	6			2	92	U	Uranium	14	2	6	10	3	2	6	1	2				
68	Er	Erbium	12	2	6			2	93	Np	Neptunium	14	2	6	10	5	2	6	2	2				
69	Tm	Thulium	13	2	6			2	94	Pu	Plutonium	14	2	6	10	6	2	6	2	2				
70	Yb	Ytterbium	14	2	6			2	95	Am	Americium	14	2	6	10	7	2	6	2	2				
71	Lu	Lutetium	14	2	6	1		2	96	Cm	Curium	14	2	6	10	7	2	6	1	2				
72	Hf	Hafnium	14	2	6	2		2	97	Bk	Berkelium	14	2	6	10	8	2	6	1	2				
73	Ta	Tantalum	14	2	6	3		2	98	Cf	Californium	14	2	6	10	10	2	6	2	2				
74	W	Tungsten	14	2	6	4		2	99	Es	Einsteinium	14	2	6	10	11	2	6	2	2				
75	Re	Rhenium	14	2	6	5		2	100	Fm	Fermium	14	2	6	10	12	2	6	2	2				
76	Os	Osmium	14	2	6	6		2	101	Md	Mendelevium	14	2	6	10	13	2	6	2	2				
77	Ir	Iridium	14	2	6	7		2	102	No	Nobelium	14	2	6	10	14	2	6	2	2				
78	Pt	Platinum	14	2	6	9	1	103	Lr	Lawrencium	14	2	6	10	14	2	6	1	2					
79	Au	Gold	14	2	6	10	1	104	Rf	Rutherfordium	14	2	6	10	14	2	6	2	2					

These tables show the "Electron configuration in the ground states of the chemical elements" and are taken directly from table 6.2 in [8].

22.3 Building Up The Electron Shells Of Elements



This plot shows the process of "building up the electron shells of all chemical elements" and is taken directly from Fig 6.17 in [8].

Table of Isotopic Masses and Natural Abundances

This table lists the mass and percent natural abundance for the stable nuclides. The mass of the longest lived isotope is given for elements without a stable nuclide. Nuclides marked with an asterisk (*) in the abundance column indicate that it is not present in nature or that a meaningful natural abundance cannot be given. The isotopic mass data is from G. Audi, A. H. Wapstra *Nucl. Phys. A.* **1993**, 565, 1-65 and G. Audi, A. H. Wapstra *Nucl. Phys. A.* **1995**, 595, 409-480. The percent natural abundance data is from the 1997 report of the IUPAC Subcommittee for Isotopic Abundance Measurements by K.J.R. Rosman, P.D.P. Taylor *Pure Appl. Chem.* **1999**, 71, 1593-1607.

Z	Name	Symbol	Mass of Atom (u)	% Abundance	Z	Name	Symbol	Mass of Atom (u)	% Abundance
1	Hydrogen	¹ H	1.007825	99.9885	15	Phosphorus	³¹ P	30.973762	100
	Deuterium	² H	2.014102	0.0115					
	Tritium	³ H	3.016049	*					
2	Helium	³ He	3.016029	0.000137	16	Sulphur	³² S	31.972071	94.93
		⁴ He	4.002603	99.999863					
3	Lithium	⁶ Li	6.015122	7.59			³³ S	32.971458	0.76
		⁷ Li	7.016004	92.41	17	Chlorine	³⁴ S	33.967867	4.29
4	Beryllium	⁹ Be	9.012182	100					
		¹⁰ B	10.012937	19.9	18	Argon	³⁶ Ar	35.967546	0.3365
		¹¹ B	11.009305	80.1					
5	Boron	¹² B	12.000000	98.93	19	Potassium	³⁷ Cl	36.965903	24.22
		¹³ B	13.003355	1.07					
		¹⁴ B	14.003242	*					
6	Carbon	¹² C	12.000000	98.93	20	Calcium	³⁸ Ar	37.962732	0.0632
		¹³ C	13.003355	1.07					
		¹⁴ C	14.003242	*					
7	Nitrogen	¹⁴ N	14.003074	99.632	21	Scandium	⁴⁰ Ar	39.962383	99.6003
		¹⁵ N	15.000109	0.368					
		¹⁶ N	16.999132	0.038					
8	Oxygen	¹⁶ O	15.994915	99.757	22	Titanium	⁴¹ K	40.961826	6.7302
		¹⁷ O	16.999132	0.038					
		¹⁸ O	17.999160	0.205					
9	Fluorine	¹⁹ F	18.998403	100	23	Vanadium	⁴⁰ Ca	39.962591	96.941
		²⁰ F	19.998403	100					
		²¹ F	20.998403	100					
10	Neon	²⁰ Ne	19.992440	90.48	24	Chromium	⁴² Ca	38.963707	93.2581
		²¹ Ne	20.993847	0.27					
		²² Ne	21.991386	9.25					
11	Sodium	²³ Na	22.989770	100	25	Manganese	⁴³ Ca	36.963999	0.0117
		²⁴ Na	23.985042	100					
		²⁵ Na	24.985837	10.00					
12	Magnesium	²⁴ Mg	23.985042	78.99	26	Iron	⁴⁴ Ca	36.964051	0.0117
		²⁵ Mg	24.985837	10.00					
		²⁶ Mg	25.982593	11.01					
13	Aluminum	²⁷ Al	26.981538	100	27	Iron	⁴⁵ Ca	36.964479	5.18
		²⁸ Al	27.976927	92.2297					
		²⁹ Al	28.976495	4.6832					
14	Silicon	²⁸ Si	27.976927	92.2297	28	Iron	⁴⁶ Ca	36.964942	91.754
		²⁹ Si	28.976495	4.6832					
		³⁰ Si	29.973770	3.0872					

Z	Name	Symbol	Mass of Atom (u)	% Abundance	Z	Name	Symbol	Mass of Atom (u)	% Abundance
		⁵⁷ Fe	56.935399	2.119			⁸⁴ Sr	83.913425	0.56
		⁵⁸ Fe	57.933280	0.282			⁸⁶ Sr	85.909262	9.86
27	Cobalt	⁵⁹ Co	58.933200	100			⁸⁷ Sr	86.908879	7.00
28	Nickel	⁵⁸ Ni	57.935348	68.0769			⁸⁸ Sr	87.905614	82.58
		⁶⁰ Ni	59.930791	26.2231			⁸⁹ Y	88.905848	100
		⁶¹ Ni	60.931060	1.1399			⁹⁰ Zr	89.904704	51.45
		⁶² Ni	61.928349	3.6345			⁹¹ Zr	90.905645	11.22
29	Copper	⁶⁴ Ni	63.927970	0.9256			⁹² Zr	91.905040	17.15
		⁶³ Cu	62.929601	69.17			⁹⁴ Zr	93.906316	17.38
		⁶⁵ Cu	64.927794	30.83			⁹⁶ Zr	95.908276	2.80
30	Zinc	⁶⁴ Zn	63.929147	48.63			⁹³ Nb	92.906378	100
		⁶⁶ Zn	65.926037	27.90			⁹² Mo	91.906810	14.84
		⁶⁷ Zn	66.927131	4.10			⁹⁴ Mo	93.905088	9.25
		⁶⁸ Zn	67.924848	18.75			⁹⁵ Mo	94.905841	15.92
		⁷⁰ Zn	69.925325	0.62			⁹⁶ Mo	95.904679	16.68
31	Gallium	⁶⁹ Ga	68.925581	60.108			⁹⁷ Mo	96.906021	9.55
		⁷¹ Ga	70.924705	39.892			⁹⁸ Mo	97.905408	24.13
							¹⁰⁰ Mo	99.907477	9.63
32	Germanium	⁷⁰ Ge	69.924250	20.84			⁹⁸ Tc	97.907216	*
		⁷² Ge	71.922076	27.54			⁹⁶ Ru	95.907598	5.54
		⁷³ Ge	72.923459	7.73			⁹⁸ Ru	97.905287	1.87
		⁷⁴ Ge	73.921178	36.28			⁹⁹ Ru	98.905939	12.76
		⁷⁶ Ge	75.921403	7.61			¹⁰⁰ Ru	99.904220	12.60
33	Arsenic	⁷⁵ As	74.921596	100			¹⁰¹ Ru	100.905582	17.06
							¹⁰² Ru	101.904350	31.55
34	Selenium	⁷⁴ Se	73.922477	0.89			¹⁰⁴ Ru	103.905430	18.62
		⁷⁶ Se	75.919214	9.37			¹⁰³ Rh	102.905504	100
		⁷⁷ Se	76.919915	7.63			¹⁰² Pd	101.905608	1.02
		⁷⁸ Se	77.917310	23.77			¹⁰⁴ Pd	103.904035	11.14
		⁸⁰ Se	79.916522	49.61			¹⁰⁵ Pd	104.905084	22.33
		⁸² Se	81.916700	8.73			¹⁰⁶ Pd	105.903483	27.33
35	Bromine	⁷⁹ Br	78.918338	50.69			¹⁰⁸ Pd	107.903894	26.46
		⁸¹ Br	80.916291	49.31			¹¹⁰ Pd	109.905152	11.72
36	Krypton	⁷⁸ Kr	77.920386	0.35			¹⁰⁷ Ag	106.905093	51.839
		⁸⁰ Kr	79.916378	2.28			¹⁰⁹ Ag	108.904756	48.161
		⁸² Kr	81.913485	11.58			¹⁰⁶ Cd	105.906458	1.25
		⁸³ Kr	82.914136	11.49			¹⁰⁸ Cd	107.904183	0.89
		⁸⁴ Kr	83.911507	57.00			¹¹⁰ Cd	109.903006	12.49
37	Rubidium	⁸⁶ Kr	85.910610	17.30			¹¹¹ Cd	110.904182	12.80
		⁸⁵ Rb	84.911789	72.17					
		⁸⁷ Rb	86.909183	27.83					

Z	Name	Symbol	Mass of Atom (u)	% Abundance	Z	Name	Symbol	Mass of Atom (u)	% Abundance
49	Indium	¹¹² Cd	111.902757	24.13	57	Lanthanum	¹³⁷ Ba	136.905821	11.232
		¹¹³ Cd	112.904401	12.22			¹³⁸ Ba	137.905241	71.698
		¹¹⁴ Cd	113.903358	28.73			¹³⁸ La	137.907107	0.090
		¹¹⁶ Cd	115.904755	7.49			¹³⁹ La	138.906348	99.910
		¹¹³ In	112.904061	4.29			¹³⁶ Ce	135.907144	0.185
		¹¹⁵ In	114.903878	95.71			¹³⁸ Ce	137.905986	0.251
		¹¹² Sn	111.904821	0.97			¹⁴⁰ Ce	139.905434	88.450
		¹¹⁴ Sn	113.902782	0.66			¹⁴² Ce	141.909240	11.114
		¹¹⁵ Sn	114.903346	0.34			¹⁴¹ Pr	140.907648	100
		¹¹⁶ Sn	115.901744	14.54			¹⁴² Nd	141.907719	27.2
50	Tin	¹¹⁷ Sn	116.902954	7.68	60	Neodymium	¹⁴³ Nd	142.909810	12.2
		¹¹⁸ Sn	117.901606	24.22			¹⁴⁴ Nd	143.910083	23.8
		¹¹⁹ Sn	118.903309	8.59			¹⁴⁵ Nd	144.912569	8.3
		¹²⁰ Sn	119.902197	32.58			¹⁴⁶ Nd	145.913112	17.2
		¹²² Sn	121.903440	4.63			¹⁴⁸ Nd	147.916889	5.7
		¹²⁴ Sn	123.905275	5.79			¹⁵⁰ Nd	149.920887	5.6
		¹²¹ Sb	120.903818	57.21			¹⁴⁵ Pm	144.912744	*
		¹²³ Sb	122.904216	42.79			¹⁴⁴ Sm	143.911995	3.07
		¹²⁰ Te	119.904020	0.09			¹⁴⁷ Sm	146.914893	14.99
		¹²² Te	121.903047	2.55			¹⁴⁸ Sm	147.914818	11.24
51	Antimony	¹²³ Te	122.904273	0.89			¹⁴⁹ Sm	148.917180	13.82
		¹²⁴ Te	123.902819	4.74			¹⁵⁰ Sm	149.917271	7.38
		¹²⁵ Te	124.904425	7.07			¹⁵² Sm	151.919728	26.75
		¹²⁶ Te	125.903306	18.84			¹⁵⁴ Sm	153.922205	22.75
		¹²⁸ Te	127.904461	31.74			¹⁵¹ Eu	150.919846	47.81
		¹³⁰ Te	129.906223	34.08			¹⁵³ Eu	152.921226	52.19
		¹²⁷ I	126.904468	100			¹⁵² Gd	151.919788	0.20
		¹²⁹ I	128.904470	0.09			¹⁵⁴ Gd	153.920862	2.18
		¹³¹ I	130.904472	21.18			¹⁵⁵ Gd	154.922619	14.80
		¹³³ I	132.904474	40.80			¹⁵⁶ Gd	155.922120	20.47
52	Tellurium	¹³¹ Xe	130.905082	21.18			¹⁵⁷ Gd	156.923957	15.65
		¹³² Xe	131.904154	26.89			¹⁵⁸ Gd	157.924101	24.84
		¹³⁴ Xe	133.905395	10.44			¹⁶⁰ Gd	159.927051	21.86
		¹³⁶ Xe	135.907220	8.87			¹⁵⁹ Tb	158.925343	100
		¹³³ Cs	132.905447	100			¹⁵⁶ Dy	155.924278	0.06
		¹³⁵ Cs	134.905449	0.09			¹⁵⁸ Dy	157.924405	0.10
		¹³⁷ Cs	136.905451	2.417			¹⁶⁰ Dy	159.925194	2.34
		¹³⁹ Cs	138.905453	6.592			¹⁶¹ Dy	160.926930	18.91
		¹⁴¹ Cs	140.905455	7.854			¹⁶² Dy	161.926795	25.51
		¹⁴³ Cs	142.905457	100			¹⁶³ Dy	162.928728	24.90
53	Iodine	¹³⁵ I	126.904468	100			¹⁴⁴ Sm	143.911995	3.07
		¹³⁷ I	128.904470	0.09			¹⁴⁷ Sm	146.914893	14.99
		¹³⁹ I	130.904472	21.18			¹⁴⁸ Sm	147.914818	11.24
		¹⁴¹ I	132.904474	40.80			¹⁴⁹ Sm	148.917180	13.82
		¹⁴³ I	134.904476	26.89			¹⁵⁰ Sm	149.917271	7.38
		¹⁴⁵ I	136.904478	10.44			¹⁵² Sm	151.919728	26.75
		¹⁴⁷ I	138.904480	8.87			¹⁵⁴ Sm	153.922205	22.75
		¹⁴⁹ I	140.904482	0.09			¹⁵¹ Eu	150.919846	47.81
		¹⁵¹ I	142.904484	21.18			¹⁵³ Eu	152.921226	52.19
		¹⁵³ I	144.904486	40.80			¹⁵⁵ Eu	155.924278	0.06
54	Xenon	¹²⁴ Xe	123.905896	0.09			¹⁵⁷ Eu	157.924405	0.10
		¹²⁶ Xe	125.904269	0.09			¹⁶⁰ Eu	159.925194	2.34
		¹²⁸ Xe	127.903530	1.92			¹⁶¹ Eu	160.926930	18.91
		¹²⁹ Xe	128.904779	26.44			¹⁶² Eu	161.926795	25.51
		¹³⁰ Xe	129.903508	4.08			¹⁶³ Eu	162.928728	24.90
		¹³¹ Xe	130.905082	21.18			¹⁴⁴ Gd	151.919788	0.20
		¹³² Xe	131.904154	26.89			¹⁵⁴ Gd	153.920862	2.18
		¹³⁴ Xe	133.905395	10.44			¹⁵⁵ Gd	154.922619	14.80
		¹³⁶ Xe	135.907220	8.87			¹⁵⁶ Gd	155.922120	20.47
		¹³⁸ Xe	137.907222	0.09			¹⁵⁷ Gd	156.923957	15.65
55	Cesium	¹³³ Cs	132.905447	100			¹⁵⁸ Gd	157.924101	24.84
		¹³⁵ Cs	134.905449	0.09			¹⁶⁰ Gd	159.927051	21.86
		¹³⁷ Cs	136.905451	2.417			¹⁵⁹ Tb	158.925343	100
		¹³⁹ Cs	138.905453	6.592			¹⁵⁶ Dy	155.924278	0.06
		¹⁴¹ Cs	140.905455	7.854			¹⁵⁸ Dy	157.924405	0.10
		¹⁴³ Cs	142.905457	100			¹⁶⁰ Dy	159.925194	2.34
		¹⁴⁵ Cs	144.905459	0.09			¹⁶¹ Dy	160.926930	18.91
		¹⁴⁷ Cs	146.905461	21.18			¹⁶² Dy	161.926795	25.51
		¹⁴⁹ Cs	148.905463	40.80			¹⁶³ Dy	162.928728	24.90
		¹⁵¹ Cs	150.905465	26.89			¹⁴⁴ Sm	143.911995	3.07
56	Barium	¹³⁰ Ba	129.906310	0.106			¹⁴⁷ Sm	146.914893	14.99
		¹³² Ba	131.905056	0.101			¹⁴⁸ Sm	147.914818	11.24
		¹³⁴ Ba	133.904503	2.417			¹⁴⁹ Sm	148.917180	13.82
		¹³⁵ Ba	134.905683	6.592			¹⁵⁰ Sm	149.917271	7.38
		¹³⁶ Ba	135.904570	7.854			¹⁵² Sm	151.919728	26.75
		¹³⁸ Ba	136.904572	0.09			¹⁵³ Sm	152.921226	52.19
		¹⁴⁰ Ba	138.904574	21.18			¹⁵⁴ Sm	154.922619	14.80
		¹⁴² Ba	140.904576	40.80			¹⁵⁶ Sm	155.922120	20.47
		¹⁴⁴ Ba	142.904578	26.89			¹⁵⁷ Sm	156.923957	15.65
		¹⁴⁶ Ba	144.904580	10.44			¹⁵⁸ Sm	157.924101	24.84

Z	Name	Symbol	Mass of Atom (u)	% Abundance	Z	Name	Symbol	Mass of Atom (u)	% Abundance
		¹⁶⁴ Dy	163.929171	28.18					
67	Holmium	¹⁶⁵ Ho	164.930319	100	77	Iridium	¹⁹¹ Ir	190.960591	37.3
68	Erbium	¹⁶² Er	161.928775	0.14	78	Platinum	¹⁹⁰ Pt	189.959930	0.014
		¹⁶⁴ Er	163.929197	1.61			¹⁹² Pt	191.961035	0.782
		¹⁶⁶ Er	165.930290	33.61			¹⁹⁴ Pt	193.962664	32.967
		¹⁶⁷ Er	166.932045	22.93			¹⁹⁵ Pt	194.964774	33.832
		¹⁶⁸ Er	167.932368	26.78			¹⁹⁶ Pt	195.964935	25.242
		¹⁷⁰ Er	169.935460	14.93			¹⁹⁸ Pt	197.967876	7.163
69	Thulium	¹⁶⁹ Tm	168.934211	100	79	Gold	¹⁹⁷ Au	196.966552	100
70	Ytterbium	¹⁶⁸ Yb	167.933894	0.13	80	Mercury	¹⁹⁶ Hg	195.965815	0.15
		¹⁷⁰ Yb	169.934759	3.04			¹⁹⁸ Hg	197.966752	9.97
		¹⁷¹ Yb	170.936322	14.28			¹⁹⁹ Hg	198.968262	16.87
		¹⁷² Yb	171.936378	21.83			²⁰⁰ Hg	199.968309	23.10
		¹⁷³ Yb	172.938207	16.13			²⁰¹ Hg	200.970285	13.18
		¹⁷⁴ Yb	173.938858	31.83			²⁰² Hg	201.970626	29.86
		¹⁷⁶ Yb	175.942568	12.76			²⁰⁴ Hg	203.973476	6.87
71	Lutetium	¹⁷⁵ Lu	174.940768	97.41	81	Thallium	²⁰³ Tl	202.972329	29.524
		¹⁷⁶ Lu	175.942682	2.59			²⁰⁵ Tl	204.974412	70.476
72	Hafnium	¹⁷⁴ Hf	173.940040	0.16	82	Lead	²⁰⁴ Pb	203.973029	1.4
		¹⁷⁶ Hf	175.941402	5.26			²⁰⁶ Pb	205.974449	24.1
		¹⁷⁷ Hf	176.943220	18.60			²⁰⁷ Pb	206.975881	22.1
		¹⁷⁸ Hf	177.943698	27.28			²⁰⁸ Pb	207.976636	52.4
		¹⁷⁹ Hf	178.945815	13.62					
		¹⁸⁰ Hf	179.946549	35.08	83	Bismuth	²⁰⁹ Bi	208.980383	100
73	Tantalum	¹⁸⁰ Ta	179.947466	0.012	84	Polonium	²⁰⁹ Po	208.982416	*
		¹⁸¹ Ta	180.947996	99.988	85	Astatine	²¹⁰ At	209.987131	*
74	Tungsten	¹⁸⁰ W	179.946706	0.12	86	Radon	²²² Rn	222.017570	*
		¹⁸² W	181.948206	26.50	87	Francium	²²³ Fr	223.019731	*
		¹⁸³ W	182.950224	14.31	88	Radium	²²⁶ Ra	226.025403	*
		¹⁸⁴ W	183.950933	30.64	89	Actinium	²²⁷ Ac	227.027747	*
		¹⁸⁶ W	185.954362	28.43					
75	Rhenium	¹⁸⁵ Re	184.952956	37.40					
		¹⁸⁷ Re	186.955751	62.60					
76	Osmium	¹⁸⁴ Os	183.952491	0.02	90	Thorium	²³² Th	232.038050	100
		¹⁸⁶ Os	185.953838	1.59	91	Protactinium	²³¹ Pa	231.035879	100
		¹⁸⁷ Os	186.955748	1.96	92	Uranium	²³⁴ U	234.040946	0.0055
		¹⁸⁸ Os	187.955836	13.24			²³⁵ U	235.043923	0.7200
		¹⁸⁹ Os	188.958145	16.15			²³⁸ U	238.050783	99.2745
		¹⁹⁰ Os	189.958445	26.26					
		¹⁹² Os	191.961479	40.78					

Z	Name	Symbol	Mass of Atom (u)	% Abundance
93	Neptunium	^{237}Np	237.048167	*
94	Plutonium	^{244}Pu	244.064198	*
95	Americium	^{243}Am	243.061373	*
96	Curium	^{247}Cm	247.070347	*
97	Berkelium	^{247}Bk	247.070299	*
98	Californium	^{251}Cf	251.079580	*
99	Einsteinium	^{252}Es	252.082972	*
100	Fermium	^{257}Fm	257.095099	*
101	Mendelevium	^{258}Md	258.098425	*
102	Nobelium	^{259}No	259.101024	*
103	Lawrencium	^{262}Lr	262.109692	*
104	Rutherfordium	^{263}Rf	263.118313	*
105	Dubnium	^{262}Db	262.011437	*
106	Seaborgium	^{266}Sg	266.012238	*
107	Bohrium	^{264}Bh	264.012496	*
108	Hassium	^{269}Hs	269.001341	*
109	Meitnerium	^{268}Mt	268.001388	*
110	Ununnilium	^{272}Uun	272.001463	*
111	Unununium	^{272}Uuu	272.001535	*
112	Ununbium	^{277}Uub	(277)	*
114	Ununquadium	^{289}Uuq	(289)	*
116	Ununhexium	^{289}Uuh	(289)	*
118	Ununoctium	^{293}Uuo	(293)	*

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References

- [1] Bauer, W., and Gary D. Westfall. University Physics with Modern Physics. 2nd ed. Vol. 2. New York, NY: McGraw-Hill Companies, 2014. Print.
- [2] Boas, Mary L. Mathematical Methods in the Physical Sciences. 3rd ed. New York: John Wiley & Sons, 1984. Print.
- [3] Brown, Edward, comp. "Planets And Telescopes." (2015): n. pag. Print. Michigan State University Department of Astronomy and Physics
- [4] Brown, Edward, comp. Astronomy 304 class handouts (2015): n. pag. Print. Michigan State University Department of Astronomy and Physics
- [5] Brown, Edward, comp. "Stellar Astrophysics." (2015): n. pag. Print. Michigan State University Department of Astronomy and Physics
- [6] <http://www.dictionary.com>
- [7] Griffiths, David J. Introduction to Electrodynamics. 4th ed. N.p.: Pearson India Education Services Pvt, 2015. Print.
- [8] Demtröder, Wolfgang. Atoms, Molecules and Photons an Introduction to Atomic, Molecular, and Quantum-Physics: with 677 Figures and 42 Tables. Springer, 2010.
- [9] Griffiths, David J. Introduction to Quantum Mechanics. 2nd ed. Harlow: Pearson, 2014. Print.
- [10] Haberman, R. "Applied Partial Differential Equations: with Fourier series and Boundary Value Problems." 4th ed. Upper Saddle River, New Jersey. 2004. Print.
- [11] Helmenstine, Todd. Periodic Table of the Elements. sciencenotes.org. 2015.
- [12] Judson T. W. "Abstract Algebra: Theory and Applications." Stephen F. Austin State University. August 5, 2017.
- [13] Linnemann, Jim, Dr. Modern Physics Lecture Notes. 2016. PHY 215 lecture notes. Department of Physics and Astronomy, Michigan State University.
- [14] Meierfrankenfeld, Ulrich. MTH 310 Lecture Notes Based on Hungerford, Abstract Algebra. N.p.: Department of Mathematics, Michigan State U, 2015. Print.
- [15] Plonus, Martin. Electronics and Communications for Scientists and Engineers. San Diego: Academic, 2001. Print.
- [16] Reitz, John Richard. Foundations of Electromagnetic Theory. 4th ed. N.p.: Addison-Wesley, 1992. Print.
- [17] Kittel, Charles, and Herbert Kroemer. Thermal Physics. San Francisco: W.H. Freeman, 1980. Print.
- [18] Nazarewicz, Witek. "MSU PHY802 Lecture Slides." Survey of Nuclear Physics 802/492 Spring 2017. Michigan State University, n.d. Web. 31 Mar. 2017.
- [19] Ross, Kenneth A. Elementary Analysis: The Theory of Calculus 2nd edition.
- [20] Ryden, Barbara Sue., and Bradley M. Peterson. Foundations of Astrophysics. San Francisco: Addison-Wesley, 2010. Print.
- [21] "Table of Isotopic Masses and Natural Abundances." (n.d.): n. pag. NC State University. Web. "https://www.ncsu.edu/chemistry/msf/pdf/IsotopicMass_NaturalAbundance.pdf".
- [22] Taylor, John R. Classical Mechanics. Sausalito, CA: U Science, 2005. Print.
- [23] "Wolfram|Alpha: Computational Knowledge Engine." N.p., n.d. Web. 2016 x
- [24] "Wolfram MathWorld: The Web's Most Extensive Mathematics Resource." Wolfram MathWorld. N.p., n.d. Web.
- [25] Zee, Dennis G. A first Course in Differential Equations with Modeling Applications." Edition 10. Book.

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