Deriving The Speed Of Light In a Vacuum

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We begin with Maxwell's equations which follow in the MKS system of units [1] as

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \tag{1}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{2}$$

$$\nabla \cdot \vec{B} = 0 \tag{3}$$

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t} \tag{4}$$

In the above equations, t is time, ϵ_0 is the permittivity of free space, μ_0 is the permeability of free space, $\vec{E}$ is the electric field, $\vec{B}$ is the magnetic field, and $\vec{J}$ is the current density. The definition of current is given by $\vec{I} \equiv \frac{dQ}{dt}\hat{I}$ which allows us to represent the current density as

$$\vec{J} = \frac{d\vec{I}}{da_\perp} = \frac{d}{da_\perp}\left(\frac{dQ}{dt}\right)\hat{I}. \tag{5}$$

Now, we can take the curl of equation (2) which gives

$$\nabla \times (\nabla \times \vec{E}) = \nabla \times \left(-\frac{\partial \vec{B}}{\partial t}\right). \tag{6}$$

The left-hand side of this can be manipulated using the vector identity $\nabla\times(\nabla\times\vec{A}) = \nabla(\nabla\cdot\vec{A}) - \nabla^2\vec{A}$. The right-hand side can also be rearranged since ∇ is not an operation with respect to time which gives

$$\nabla \times (\nabla \times \vec{E}) = \nabla(\nabla \cdot \vec{E}) - \nabla^2 \vec{E} = \nabla \times \left(-\frac{\partial \vec{B}}{\partial t}\right) = -\frac{\partial}{\partial t}(\nabla \times \vec{B}). \tag{7}$$

We can use equation (1) to write this as

$$\nabla\left(\frac{\rho}{\epsilon}\right) - \nabla^2 \vec{E} = -\frac{\partial}{\partial t}(\nabla \times \vec{B}). \tag{8}$$

If we assume we are in a perfect vacuum, then there would contain no matter and thus no charge. Therefore the charge density would be $\rho = 0$. This implies the total charge is zero and thus $\frac{dQ}{dt} = 0$. Hence, we can clearly see from equation (5) that within a vacuum $\vec{J} = 0$. Using these results as well as (4), we can write equation (8) as

$$-\nabla^2 \vec{E} = -\frac{\partial}{\partial t}\left(\epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}\right) \implies \nabla^2 \vec{E} = \epsilon_0 \mu_0 \frac{\partial^2 \vec{E}}{\partial t^2}. \tag{9}$$

This result in equation (9) can be recognized as the wave equation which is generally in the form

$$\nabla^2 \vec{\psi} = \frac{1}{v^2} \frac{\partial^2 \vec{\psi}}{\partial t^2}, \tag{10}$$

where v is the velocity of a wave. Finally, if we think of the electric field $\vec{E}$ as a wave moving through a vacuum, then we can determine it's velocity by comparing equations (9) and (10). This is the speed of an electromagnetic wave (the speed of light) which gives

$$v = c = \frac{1}{\sqrt{\epsilon_0 \mu_0}}. \tag{11}$$

We can approximate this based on the values of $\epsilon_0 = 8.85418782 \times 10^{-12} s^4 A^2/(m^3 kg)$ and $\mu_0 = 4\pi \times 10^{-7} Wm$ which gives

$$c = \frac{1}{\sqrt{\epsilon_0 \mu_0}} \approx \frac{1}{\sqrt{[8.85418782 \times 10^{-12} s^4 A^2/(m^3 kg)][4\pi \times 10^{-7} Wm]}} \approx \boxed{299,792,458 \frac{m}{s}}. \tag{12}$$

References

[1] "Wolfram MathWorld: The Web's Most Extensive Mathematics Resource." Wolfram MathWorld. N.p., n.d. Web.