A Look at Mathematical Analytic Continuation

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Abstract

Analytic continuation is a tool in which a function can be extended onto a new domain. It is a process that has been used in physics and mathematics for centuries. In this paper, analytic continuation is defined and a few examples of its use are demonstrated. The examples chosen are such that different properties or unique qualities of the analytical continuation technique can be shown. The first example is of an analytical continuation of $f(x) = \sqrt{x}$. The second is a demonstration of determining interesting results using a form of the Riemann Zeta function. Lastly, Euler numbers E_n are defined and expanded upon such that negative and non-integer values of n can be used.

I. Introduction

It is often the case in mathematics that a function be defined on a specific domain. In such a case, the meaning of the function may not be meaningful or useful outside of such domain. Analytic continuation is the process of taking a domain that complex functions are defined on and expanding them such that the a new equivalent function can be defined on a larger domain. This is most commonly used in the case of a complex analytic function that is defined near a point z_0 by a power series [1]

$$f(z) = \sum_{n=0}^{\infty} a_k (z - z_0)^k. \tag{1}$$

Analytic continuation many possible applications in physics and mathematics. Examples of where it can be used are in the derivation of the quantum Casimir effect [2], calculations regarding three-body quantum Coulomb problems [3], quantum chromodynamic phase diagram calculations [4], and more.

We can define analytic continuation precisely by considering two functions f_1 and f_2 on domains of Ω_1 and Ω_2 respectively. If $f_1 = f_2$ on the non-empty domain $\Omega_1 \cap \Omega_2$, then f_1 is the analytical continuation of f_2 on Ω_1 and f_2 is the analytical continuation of f_1 on Ω_2 [1]. The analytically continued function f_1 on Ω_2 is unique.

II. Simple Example

We can demonstrate the process of analytic continuation by the function $f : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ defined by $f(x) = \sqrt{x}$. Notice that this function is defined only over a positive domain which maps to the positive real numbers. We can take the well defined Taylor polynomial about $x_0 = 1$,

$$g(x) = \sum_{n=0}^{\infty} \frac{(x - x_0)^n}{n!} f^{(n)}(x_0), \tag{2}$$

where $f^{(n)}(x_0)$ denotes the n^{th} derivative of $f(x)$ evaluated at x_0. If we analyze the argument to this summation, we can find the radius of convergence given $|z| = 1$. First, observe the differential term of f. If we look at the first few derivatives evaluated at z_0 we get

$$f^{(0)}(z_0) = 1 \tag{3}$$

$$f^{(1)}(z_0) = \frac{1}{2} \tag{4}$$

$$f^{(2)}(z_0) = -\frac{1}{2^2} \tag{5}$$

$$f^{(3)}(z_0) = \frac{3!!}{2^3} \tag{6}$$

$$f^{(4)}(z_0) = -\frac{5!!}{2^4} \tag{7}$$

$$f^{(5)}(z_0) = \frac{7!!}{2^5}. \tag{8}$$

If we continue this pattern, we can see that the n$^{\text{th}}$ derivative term can be expressed by

$$f^{(n)}(z_0) = (-1)^{n+1}\frac{(2n-3)!!}{2^n}. \tag{9}$$

By plugging this into Eqn. 2, we get

$$g(x) = \sum_{n=0}^{\infty}(-1)^{n+1}\frac{(x-x_0)^n(2n-3)!!}{n!2^n}. \tag{10}$$

To analyze the radius of convergence, we need this in the form of $\sum^{\infty} c_k x^k$. By looking at large k terms we can determine the limit supremum of the highest order term. For $k+1$ we have $c_{k+1} = \frac{(2k-1)!!}{2^{k+1}(k+1)!}$ and $c_k = \frac{(2k-3)!!}{2^k k!}$. In order to easily find the limit supremum, we must show it converges to some value. This will be true if $c_{k+1} < c_k$ for all k. This can be shown directly or by contradiction. If we prove this by contradiction we can assume $\frac{(2k-3)!!}{2^k k!} \leq \frac{(2k-1)!!}{2^{k+1}(k+1)!}$, which leads to $-1 \geq 2$ which is clearly false and thus we see $c_{k+1} < c_k$ for all k. From this, we can find the radius of convergence by taking the limit of the largest $|c_k^{1/k}|$ term (corresponding to $k \to \infty$ term) which gives

$$\lim_{k\to\infty}\left|\frac{(2k-3)!!}{2^k k!}\right|^{1/k} \to 1, \tag{11}$$

which then gives

$$c = \limsup |c_k^{1/k}| = 1. \tag{12}$$

It follows that the radius of convergence is then found by the formula [5]

$$R = \frac{1}{c} = 1. \tag{13}$$

This radius of convergence shows that we can use this expansion g as an analytic continuation of f about $|z| = 1$. This is visual shown in figure 1. This is one demonstration of analytic continuation.

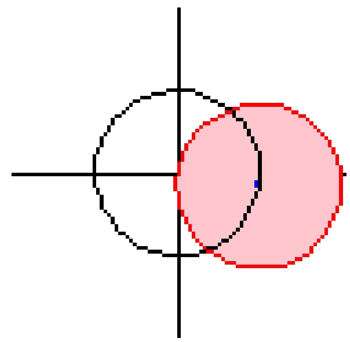

Figure 1: Radius of convergence about $z_0 = 1$.

III. 'Astounding' Mathematical Results.

One of my favorite results I have encountered in my studies follows as

$$\sum_{n=1}^{\infty} n \to -\frac{1}{12}. \tag{14}$$

This result does not explicitly make sense because the sum in equation (14) is a divergent sum. However, due to analytic continuation, some divergent sums can have a finite value. In 1913, this appeared in the work of a very famous mathematician from India, Srinivasa Ramanujan and is an important result for String Theory and other branches of physics. The Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s} = 1 + 2^{-s} + 3^{-s} + 4^{-s} + \cdots \tag{15}$$

is widely studied and used often in physics. In quantum physics, the energy density of a vacuum should be proportional to $\zeta(-3) = 1 + 8 + 27 + 64 + \cdots$, which is a divergent series and thus does not make much sense as an energy density [6]. When we write this using equation (15) and use the process of analytic continuation, this can be written

$$\zeta(-3) = \sum_{n=1}^{\infty} \frac{1}{n^{-3}} = 1 + 2^3 + \cdots \to \frac{1}{120}. \quad (16)$$

The way Ramanujan expresses functions that are divergent such as this (from the Riemann zeta function) is

$$\sum_{k=\alpha}^{x} f(k) \sim \int_{\alpha}^{x} f(t)dt + c + \frac{1}{2} f(x) + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} f^{(2k-1)}(x), \quad (17)$$

[7]. This is a process of analytically continuing these divergent series and coming up with a finite result without any 'magic'. I say magic because there is a process in which one can ignore (in a sense) the divergent nature of a sum and come up with these results as well.

As an example, I will give a 'proof' of equation (14) using this method, which was first shown by Euler around 1735 [8]. Consider the following well defined sum

$$f(x) = 1 + x + x^2 + x^3 + x^4 + \cdots = \frac{1}{1-x}, \quad (18)$$

for $|x| < 1$. Differentiating this gives

$$f'(x) = 1 + 2x + 3x^2 + \cdots = \frac{1}{(1-x)^2}. \quad (19)$$

If we evaluate the result at $x = -1$ we get

$$f'(-1) = 1 - 2 + 3 - 4 + \cdots = \frac{1}{4}. \quad (20)$$

Note that this is troublesome because we defined $f'(x)$ based on a function only valid for when $|x| < 1$. However, for our purposes suppose we can extend our limits and make f(x) differentiable at $x = -1$. Now, if we take $2^{-s}\zeta(s)$ we have

$$\begin{aligned} 2^{-s}\zeta(s) &= 2^{-s} \sum_{n=1}^{\infty} \frac{1}{n^s} = \sum_{n=1}^{\infty} \frac{2^{-s}}{n^s} \\ &= 2^{-s} + 4^{-s} + 6^{-s} + 8^{-s} \cdots . \end{aligned} \quad (21)$$

Now, if we take $g(s) = [1 - 2(2^{-s})]\zeta(s)$ we have

$$\begin{aligned} g(s) &= [1 - 2(2^{-s})]\zeta(s) \\ &= \zeta(s) - 2(2^{-s})\zeta(s) \\ &= 1 + 2^{-s} + 3^{-s} + 4^{-s} + 5^{-s} + 6^{-s} + \cdots \\ &\quad \underline{-\ 2(\ 2^{-s} \qquad + 4^{-s} \qquad + 6^{-s} + \cdots)} \\ &= 1 - 2^{-s} + 3^{-s} - 4^{-s} + 5^{-s} - 6^{-s} + \cdots . \end{aligned} \quad (22)$$

Finally, if we set $s = -1$, we can see that $g(-1) = \zeta(-1) - 2(2)\zeta(-1) = -3\zeta(-1)$ and evaluating this from equation (22) and then using our result from equation (20) gives us

$$\begin{aligned} -3\zeta(-1) = 1 - 2 + 3 - 4 + \cdots = \frac{1}{4} \\ \implies \zeta(-1) = -\frac{1}{12}. \end{aligned} \quad (23)$$

Now, notice that plugging in $s = -1$ into the Riemann zeta function gives us the same result from equation (14) and thus

$$\zeta(-1) = -\frac{1}{12} \implies \sum_{n=1}^{\infty} n \to -\frac{1}{12}. \quad (24)$$

This result is very important to obtaining the $24 + 2 = 26$ dimensions in bosonic string theory [**?**]. It is also a simpler example than that of equation (16) to illustrate. The modern way to show this without using the 'magic' mentioned before would be to start with the well defined Riemann zeta function [9] which can be expressed in the form

$$\zeta(s) = \frac{1}{1 - 2^{1-s}} \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \sum_{k=0}^{n} \binom{n}{k} \frac{(k+1)^{-s}}{(-1)^k}. \quad (25)$$

By substituting $s = -1$ this becomes

$$\zeta(-1) = -\frac{1}{3} \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \sum_{k=0}^{n} \binom{n}{k} \frac{(k+1)}{(-1)^k}. \quad (26)$$

Now, it is not obvious what this sum evaluates to but one can define the function

$$D(m) = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \sum_{k=0}^{n} \binom{n}{k} \frac{k^m}{(-1)^k}. \quad (27)$$

For negative values of s this allows us to easily foil out Eqn. 25 in terms of $D(m)$ for some value of m dependent on the order of k. We can solve for the needed values of $D(m)$ by applying the Kronecker delta [9] which give

$$D(0) = \sum_{n=0}^{\infty} \frac{\delta_{0,n}}{2^{n+1}} = \frac{1}{2} \tag{28}$$

$$D(1) = -\sum_{n=0}^{\infty} n \frac{\delta_{1,n}}{2^{n+1}} = -\frac{1}{4}. \tag{29}$$

By applying these we get

$$\zeta(-1) = -\frac{1}{3} \sum_{n=0}^{\infty} \frac{\delta_{0,n} - n\delta_{1,n}}{2^{n+1}} \tag{30}$$

$$= -\frac{1}{3} \left(\frac{1}{2} - \frac{1}{2^2} \right) \tag{31}$$

$$= -\frac{1}{12} \tag{32}$$

This process can be applied in a similar manner to determine $\zeta(-3)$. First, the Kronecker delta identities that are needed are

$$D(2) = \sum_{n=0}^{\infty} n^2 \frac{\delta_{0,n}}{2^{n+1}} = 0 \tag{33}$$

$$D(3) = \sum_{n=0}^{\infty} \frac{\delta_{2,n}}{2^{n+1}} = \frac{1}{8}. \tag{34}$$

We use $s = -3$ in Eqn. 25 along with these identities which gives

$$\zeta(-3) = -\frac{1}{15} \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \sum_{k=0}^{n} \binom{n}{k} \frac{(k+1)^3}{(-1)^k} \tag{35}$$

$$= -\frac{1}{15} \sum_{n=0}^{\infty} \frac{\delta_{0,n} - 3n\delta_{1,n} + \delta_{2,n}}{2^{n+1}} \tag{36}$$

$$= -\frac{1}{15} \left(\frac{1}{2} - \frac{3}{2^2} + \frac{1}{2^3} \right) \tag{37}$$

$$= \frac{1}{120}. \tag{38}$$

Thus we have shown $\zeta(-3) = \frac{1}{120}$ as mentioned previously. These two results show how we are able to get results extended outside of the normal domain of the defined functions. It's important to understand that the Riemann Zeta function is defined over the complex plane $\mathbb{C}$. which is what allows us to perform this process.

IV. Euler Polynomials and the Zeta Function

The Euler numbers E_n can be defined by the well known function [10]

$$\frac{2}{e^t + 1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}. \tag{39}$$

We can see from this that $n \in \mathbb{N}$ and is not defined for $n < 0$. The first few Euler numbers are thus

$$E_0 = 1, \qquad E_1 = -\frac{1}{2}, \qquad E_3 = \frac{1}{4}, \tag{40}$$

$$E_5 = -\frac{1}{2}, \qquad E_7 = \frac{17}{8}, \qquad E_9 = -\frac{31}{2}, \tag{41}$$

where $E_{2k} = 0$ for all $k \in \mathbb{N}$. The Euler polynomials $E_n(x)$ are similar to the Euler numbers and defined by the generating function [10]

$$F(x,t) = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!} = \left(\frac{2}{e^t + 1} \right) e^{xt}. \tag{42}$$

Following this, a relationship between the Euler polynomials and the Euler numbers can be determined [10]. For n, one has

$$E_k(x) = \sum_{n=0}^{\infty} \binom{n}{k} E_k x^{n-k}. \tag{43}$$

The Hurwitz zeta function can be defined as

$$\zeta(s,x) = 2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(n-x)^s}, \tag{44}$$

where $s \in \mathbb{C}$, $Re(s) > 0$ and $x \in \mathbb{R}$ with $0 \leq x < 1$. The Hurwitz zeta function turns into the Euler zeta function $\zeta(s)$ for $x = 0$ and indexed starting from 1 instead of 0..

It can be shown that the Hurwitz zeta function is related to the Euler zeta function by $E_n(x) = \zeta(-n, x)$ and that the Euler numbers E_n can be related to the Euler zeta function by analytic continuation as $E_s \mapsto E(s) = \zeta(-s)$ [10]. This is an analytic continuation of E_n because it now defined for all values it was before (meaning all E_n agree with values of $E(n)$ for $n \geq 1$) but also extended into a complex plane as defined within the Euler zeta functions. From this, we can find values of the Euler numbers in cases that were previously undefined. An example is for negative values of n. If we now use this new definition that we mapped E_n to, we can see that $E(-1) = \zeta(1) = -2\log(2)$, which is a well defined real number. As seen from this, analytic continuation has expanded these Euler numbers outside of the domain they were originally defined. Another such example of what this has done can be seen when taking the limit as n goes to infinity. From this we have

$$\lim_{n\to-\infty} E_n = \lim_{n\to\infty} \zeta(-n) = -2. \tag{45}$$

This shows that the Euler numbers converge to a value in the negative extended domain which is not something we could have gathered from the non-continued function.

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