Exploration of Multiple Pendulum Systems

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Abstract

A common problem in any classical mechanics course is the double pendulum system. In this paper, I will examine setting up the Lagrange and exploring the equations of motion for a pendulum with an arbitrary number (n) of rods. Using a small angle approximation, we can also create a system that approximates a rope.

1 Moments of Inertia

When first determining the setup of our pendulum system, we need to decide how applicable we want our system to be. For example, we could setup a pendulum system in which we neglect the mass of each rod, or one in which we have each rod as a variable density as a function of its length. We could even have a system in which our rods are not uniform in shape which would greatly impact the calculations. In our setup we will look at a few cases. To begin, we will start with a 3 dimensional rod of length L, width W, and depth D such that $L > W \geq D$. We will give this rod a uniform mass density ρ. We will allow our pivot point to be about any point in the center of the rod along the length of the rod. Since we are assuming uniform density, we have $\rho = M/V$, where M is the total mass of the rod, and V is the total volume. Since the volume of the rod is clearly just $V = LWD$ and therefore $\rho = \frac{M}{LWD}$. The moment of inertia with respect to a given axis of a solid body with density $\rho(\vec{r})$ is defined by the volume integral [1]

$$I \equiv \int \rho(\vec{r}) r_\perp^2 \, dV. \tag{1}$$

Setting the pivot point of our rod as the origin for our Cartesian coordinate system then allows us to easily evaluate the moment of inertia of the rod. First, let us have the length of our rod fall along the x-axis. Then, since we do not have a specified pivot point, allow the length of the rod towards the negative x axis be ΔL and thus the length in the positive direction would be $L - \Delta L$. From here, our mass is distributed evenly by our density function. Note that with the way we've set up our coordinates, we can say we have the length along our x axis, the width along the z axis, and the depth along the y axis. If we imagine our rod rotating about the z axis, then the distance to any point within our rod from the axis of rotation is given by $r_\perp = \sqrt{x^2}$. Therefore our moment of inertia becomes

$$I = \int_{-\Delta L}^{L-\Delta L} \int_{-D/2}^{D/2} \int_{-W/2}^{W/2} \frac{M}{LWH} (x^2) \mathrm{dzdydx}. \tag{2}$$

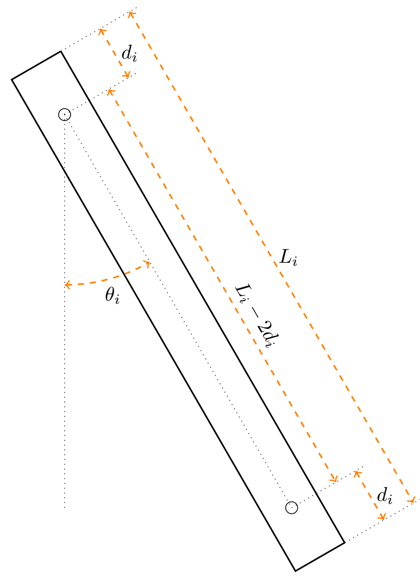

Figure 1: A rod of length L_i with pivot points located at an arbitrary distance d_i from each end. The Rod is at some angle θ_i with respect to the vertical axis.

This can be greatly simplified by noticing symmetry along the y and z axis as well as pulling out constants. Our integral then evaluates as

$$
\begin{aligned}
I &= \frac{4M}{LWD}\int_{-\Delta L}^{L-\Delta L}\int_{0}^{D/2}\int_{0}^{W/2}(x^2)\mathrm{dzdydx} \\
&= \frac{4M}{LWD}\int_{-\Delta L}^{L-\Delta L}\int_{0}^{D/2}\frac{Wx^2}{2}\mathrm{dydx} \\
&= \frac{4M}{LWD}\int_{-\Delta L}^{L-\Delta L}\frac{WDx^2}{4}\mathrm{dx} \\
&= \frac{4M}{LWD}\left[\frac{WD(L-\Delta L)^3}{12}+\frac{WD(\Delta L)^3}{12}\right]\mathrm{dx} \\
&= \frac{M}{3}(L^2-3L\Delta L+3\Delta L^2). \qquad (3)
\end{aligned}
$$

A simple check in the above calculation allows us to compare what we calculated to a few well known moments. For example, the moment of a rod about the end point is $I_{rod,end} = \frac{mL^2}{3}$ and the moment of a rod about the center is $I_{rod,center} = \frac{mL^2}{12}$. These moments correspond to a ΔL value of 0 and $L/2$ respectively. Inserting these values in for ΔL indeed gives us the correct values.

2 Multiple Pendulum System

We will first start by deciding on a proper coordinate system and defining the proper variables. Imagine an arbitrary rod as in Figure 1 with an arbitrary length L_1. With respect to the vertical axis, the rod has some angle θ_1. We can define along the length of the rod, some distance d_1 in which pivot points will be placed that distance from each end of the rod. The distance between the respective pivot points is then $L_1 - 2d_1$. If we allow one of these pivot points, be the origin to our coordinate system, we can use the other pivot point to attach another rod of an arbitrary length L_2 with some angle θ_2 with respect to the vertical axis and some pivot distance d_2. Thus, we have defined a double pendulum. From here, continue to attach n number of rods to the pivot points as previously done with the last rod having a length of L_n, pivot distance of d_n and angle with respect to the axis as θ_n. This is demonstrated in Figure 2.

To begin, we can start by defining our origin as the coordinate $\vec{P}_0 = (0, 0)$ on an xy-Cartesian plane. We will assume that our multiple pendulum system only has 2-degrees of freedom in which it can move. From this, we can define the pivot point at the end of each rod (the pivot not connected to the previous rod/origin) as $\vec{P}_i$, for $0 < i \leq n$. It will also be useful to denote the distance $R_i = L_i - 2d_i$. Since R_i is the length of the section of each respective rod between its two pivot points, we can use it with the angles to find the positions of each pivot point as a function of theta. Note that our system only has angles that depend on time, as we are assuming the lengths of the pivot distances and the rods are fixed.

We can use what we have established so far to determine the first few pivot points as vectors. This will be needed later in determining the coordinate vectors for the center of mass of each rod in our system. Starting with $\vec{P}_1$, we can see from Figure 2 that the pivot point in question forms a right triangle with respect to the horizontal axis and, the vertical axis, and $\vec{P}_0$ and therefore the x position will simply be $R_1 \sin(\theta_1)$ and the y as $R_1 \cos(\theta_1)$. Following this, the next respective pivot point forms a right triangle with $\vec{P}_1$, and the xy-axes. Determining the first few pivot vectors then gives us

$$\begin{aligned}
\vec{P}_0 &= 0\hat{x} - 0\hat{y} \\
\vec{P}_1 &= \vec{P}_0 + R_1 \sin(\theta_1)\hat{x} - R_1 \cos(\theta_1)\hat{y} \\
\vec{P}_2 &= \vec{P}_1 + R_2 \sin(\theta_2)\hat{x} - R_2 \cos(\theta_2)\hat{y} \\
&\vdots \\
\vec{P}_n &= \vec{P}_{n-1} + R_n \sin(\theta_n)\hat{x} - R_n \cos(\theta_n)\hat{y}.
\end{aligned}$$

Because we are defining our angle with respect to the vertical axis and the origin is the pivot point one, we subtract the y components within each $\vec{P}_i$ so that we get their position in the negative direction with respect to the origin when they are below it. If we define $R_0 = 0$ then we can write each term of the above sequence as a sum of the previous terms

$$\vec{P}_k = \sum_{i=0}^{k} R_i \sin(\theta_i)\hat{x} - \sum_{i=0}^{k} R_i \cos(\theta_i)\hat{y}. \tag{4}$$

Therefore, using equation (4), we can determine the position of any pivot point throughout our system.

One way we can represent the evolution of our system is by setting up the Lagrange. By definition, the Lagrange equation is given by the total kinetic energy (KE_{total}) of a system minus the total potential energy (PE_{total})

$$\mathcal{L} = KE_{total} - PE_{total}. \tag{5}$$

By definition, we can take the Kinetic energy to be dependent on the sum of the kinetic energies of each rod. If we observe each rod individually, we can evaluate the kinetic energy of each individual rod (KE_i) as being dependent on the magnitude of velocity about its center of mass and the rotational energy about its center of mass. This gives us

$$KE_i = \frac{1}{2} m_i ||\vec{v}_i||^2 + \frac{1}{2} I_i \omega_i^2, \tag{6}$$

where m_i is the total mass of the rod, $\vec{v}_i$ is the velocity of the center of mass, I_i is the moment of inertia about the center of mass, and ω_i is the angular speed about the center of mass; all of which are with respect to a single rod i. It is important to note that within the coordinate system we set up, each angle is dependent on time, (as we are not assuming the system is in motion) thus we will have $\omega_i = \frac{d}{dt}\theta_i = \dot{\theta}_i$.

If we maintain our Cartesian coordinate system, we can represent the vector for each center of mass in terms of some vector $\vec{C}_i = x_i\hat{x} + y_i\hat{y}$, where x_i is the x coordinate of the center of mass and y_i is the y coordinate of the center of mass. From this, taking the time derivative gives us the velocity of the center of mass $\vec{v}_i = \frac{d}{dt}\vec{C}_i = \frac{d}{dt}x_i\hat{x} + \frac{d}{dt}y_i\hat{y} = \dot{x}_i\hat{x} + \dot{y}_i\hat{y}$. Then, the magnitude of each center of mass velocity is given by

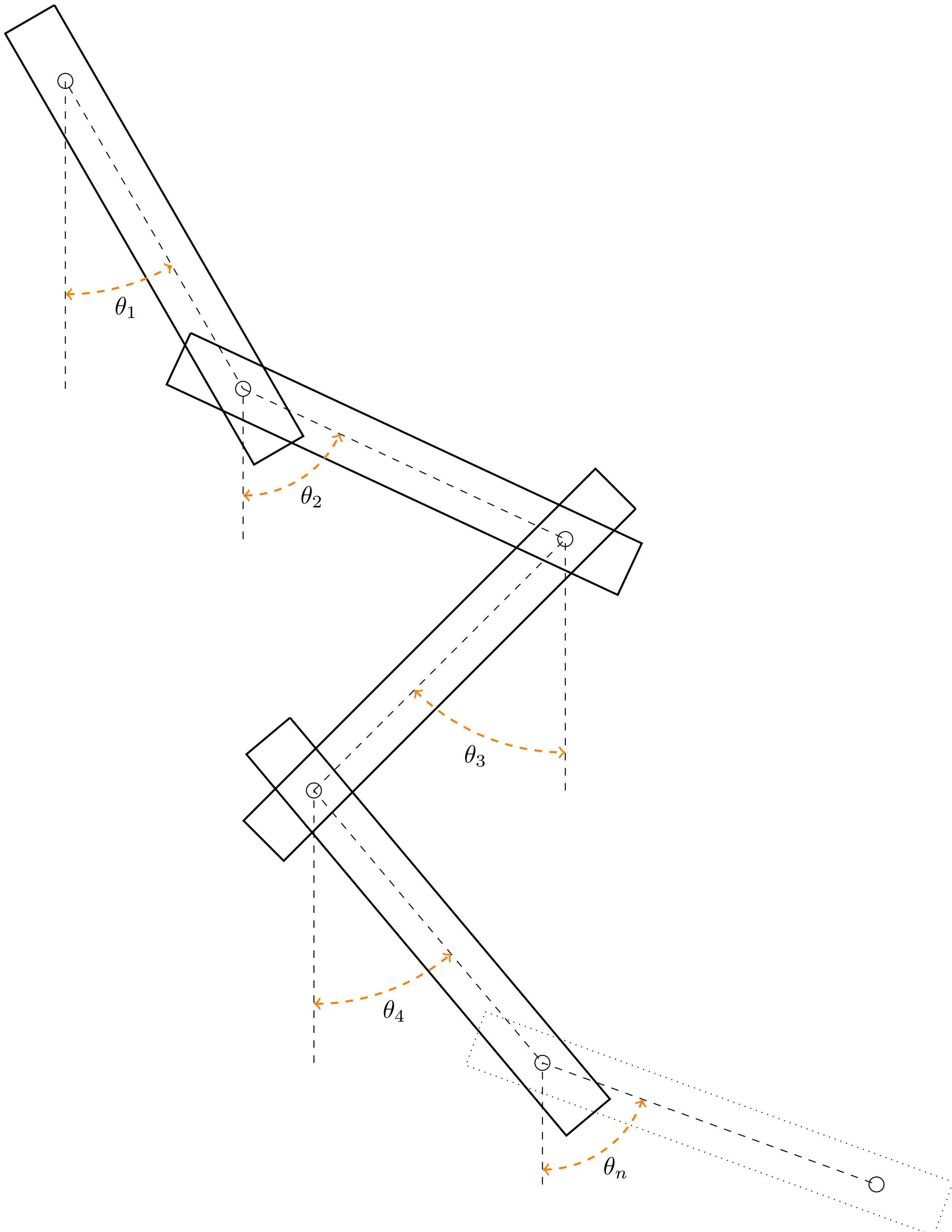

Figure 2: A representation of a pendulum with an arbitrary (n) number of rods. Each rod is tilted at some angle θ_n with respect to the vertical axis and connected to the previous rod at some pivot point.

$||\vec{v}_i|| = \sqrt{\dot{x}_i^2 + \dot{y}_i^2}$. If we adapt equation (3) to our system using $\Delta L = d_i$, we can see the moment of inertia of each rod becomes $I_i = \frac{m_i}{3}(L_i^2 - 3L_i d_i + 3d_i^2)$. Combining what we have, our kinetic energy for each rod now becomes

$$KE_i = \frac{1}{2}m_i(\dot{x}_i^2 + \dot{y}_i^2) + \frac{1}{6}m_i(L_i^2 - 3L_i d_i + 3d_i^2)\dot{\theta}_i^2. \tag{7}$$

Hence, if we sum each term of this together, we will get the total kinetic energy for n rods as

$$KE_{total} = \frac{1}{2}\sum_{i=1}^{n} m_i(\dot{x}_i^2 + \dot{y}_i^2) + \frac{1}{6}\sum_{i=1}^{n} m_i(L_i^2 - 3L_i d_i + 3d_i^2)\dot{\theta}_i^2. \tag{8}$$

Next, determining x_i and y_i can be done in terms of each pivot point which we determined earlier. This can be done in more than one way. We can either start at the previous pivot point and add the coordinates to half of the R_i length or we can start at each end pivot point and subtract coordinates. If we subtract, the coordinates for $\vec{P}_i$ we will use will have the same i value as the center of mass vector we are trying to solve for. From this, we can say each x_i value will be the end pivot point x coordinate minus $\frac{1}{2}R_i \sin(\theta_i)$ (half the length to the previous pivot point) and similarly, the y_i coordinate will be the y coordinate of the end pivot point plus $\frac{1}{2}R_i \cos(\theta_i)$. This gives us the center of mass position of any rod i as

$$\vec{C}_i = \vec{P}_i - \frac{1}{2}R_i \sin(\theta_i)\hat{x} + \frac{1}{2}R_i \cos(\theta_i)\hat{y}. \tag{9}$$

If we then insert our formula for $\vec{P}_n$ we determined in (4), we have

$$\vec{C}_k = x_k\hat{x} + y_k\hat{y} \tag{10}$$

$$x_k = \sum_{i=0}^{k} R_i \sin(\theta_i)\hat{x} - \frac{1}{2}R_k \sin(\theta_k)\hat{x} \tag{11}$$

$$y_k = -\sum_{i=0}^{k} R_i \cos(\theta_i)\hat{y} + \frac{1}{2}R_k \cos(\theta_k)\hat{y}. \tag{12}$$

Then, taking the time derivative of both of these gives us

$$\frac{d}{dt}x_k = \dot{x}_k = \sum_{i=0}^{k} R_i \cos(\theta_i)\dot{\theta}_i\hat{x} - \frac{1}{2}R_k \cos(\theta_k)\dot{\theta}_k\hat{x} \tag{13}$$

$$\frac{d}{dt}y_k = \dot{y}_k = \sum_{i=0}^{k} R_i \sin(\theta_i)\dot{\theta}_i\hat{y} - \frac{1}{2}R_k \sin(\theta_k)\dot{\theta}_k\hat{y}. \tag{14}$$

Finally, squaring each one of these components to determine the magnitude gives

$$\dot{x}_k^2 = \left[\sum_{i=0}^{k} R_i \cos(\theta_i)\dot{\theta}_i\hat{x} - \frac{1}{2}R_k \cos(\theta_k)\dot{\theta}_k\hat{x}\right]^2 \tag{15}$$

$$= \left[\sum_{i=0}^{k} R_i \cos(\theta_i)\dot{\theta}_i\right]^2 - R_k \cos(\theta_k)\dot{\theta}_k \sum_{i=0}^{k} R_i \cos(\theta_i)\dot{\theta}_i + \frac{1}{4}R_k^2 \cos^2(\theta_k)\dot{\theta}_k^2, \tag{16}$$

and

$$\dot{y}_k^2 = \left[\sum_{i=0}^{k} R_i \sin(\theta_i)\dot{\theta}_i - \frac{1}{2}R_k \sin(\theta_k)\dot{\theta}_k\right]^2 \tag{17}$$

$$= \left[\sum_{i=0}^{k} R_i \sin(\theta_i)\dot{\theta}_i\right]^2 - R_k \sin(\theta_k)\dot{\theta}_k \sum_{i=0}^{k} R_i \sin(\theta_i)\dot{\theta}_i + \frac{1}{4}R_k^2 \sin^2(\theta_k)\dot{\theta}_k^2. \tag{18}$$

From here, summing (16) and (18) gives us what we need to finish our kinetic energy equation

$$\dot{x}_k^2 + \dot{y}_k^2 = \left[\sum_{i=0}^{k} R_i \cos(\theta_i)\dot{\theta}_i\right]^2 + \left[\sum_{i=0}^{k} R_i \sin(\theta_i)\dot{\theta}_i\right]^2 - R_k \sin(\theta_k)\dot{\theta}_k \sum_{i=0}^{k} R_i \sin(\theta_i)\dot{\theta}_i$$
$$- R_k \cos(\theta_k)\dot{\theta}_k \sum_{i=0}^{k} R_i \cos(\theta_i)\dot{\theta}_i + \frac{1}{4} R_k^2 \dot{\theta}_k^2. \tag{19}$$

Plugging in equation (19) to our kinetic energy equation in (8) then gives us a complete kinetic energy in terms of the angles of each rod and our initial constants. From here, we can set up our potential energy more simple. By definition, the potential energy of each rod can be represented by $PE_i = m_i g h_i$, where g is the force of gravity and h_i is the height of the center of mass. We have already defined y_i as the vertical distance of each rods center of mass with respect to our origin. This gives us our respective potential energies of a particular rod as $PE_i = m_i g y_i$. Then, our total potential energy is the sum of all potential energies within our system. For n many rods, this becomes

$$PE_{total} = g \sum_{i=1}^{n} m_i y_i. \tag{20}$$

Combining equations (20) and (8), we can then define our Lagrange as

$$\mathcal{L} = \frac{1}{2} \sum_{i=1}^{n} m_i (\dot{x}_i^2 + \dot{y}_i^2) + \frac{1}{6} \sum_{i=1}^{n} m_i (L_i^2 - 3L_i d_i + 3d_i^2)\dot{\theta}_i^2 - g \sum_{i=1}^{n} m_i y_i \tag{21}$$
$$= \sum_{i=1}^{n} \left[\frac{1}{2} m_i (\dot{x}_i^2 + \dot{y}_i^2) + \frac{1}{6} m_i (L_i^2 - 3L_i d_i + 3d_i^2)\dot{\theta}_i^2 - g m_i y_i\right]. \tag{22}$$

Combining equations (22), (19) and (12) then gives us a complete Lagrangian for our system as

$$\mathcal{L} = \sum_{i=1}^{n} \left[\frac{1}{2} m_i \left[\left(\sum_{k=0}^{i} R_k \cos(\theta_k)\dot{\theta}_k\right)^2 + \left(\sum_{k=0}^{i} R_k \sin(\theta_k)\dot{\theta}_k\right)^2 - R_i \sin(\theta_i)\dot{\theta}_i \sum_{k=0}^{i} R_k \sin(\theta_k)\dot{\theta}_k\right.\right.$$
$$\left. - R_i \cos(\theta_i)\dot{\theta}_i \sum_{k=0}^{i} R_k \cos(\theta_k)\dot{\theta}_k + \frac{1}{4} R_i^2 \dot{\theta}_i^2\right] + \frac{1}{6} m_i (L_i^2 - 3L_i d_i + 3d_i^2)\dot{\theta}_i^2$$
$$\left. + g m_i \left(\sum_{k=0}^{i} R_k \cos(\theta_k)\hat{y} - \frac{1}{2} R_i \cos(\theta_i)\hat{y}\right)\right]. \tag{23}$$

References

[1] "Wolfram MathWorld: The Web's Most Extensive Mathematics Resource." Wolfram MathWorld. N.p., n.d. Web.