Exploration of the Quantum Casimir Effect

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Abstract

Named after the Dutch Physicist Hendrik Casimir, The Casimir effect is a physical force that arises from fluctuations in electromagnetic field and explained by quantum field theory. The typical example of this is an apparent attraction created between two very closely placed parallel plates within a vacuum. Due to the nature of the vacuum's quantized field having to do with virtual particles, a force becomes present in the system. This effect creates ideas and explanations for subjects such as zero-point energy and relativistic Van der Waals forces. In this paper I will explore the Casimir effect and some of the astonishing mathematical results that originally come about from quantum field theory that explain it along side an approach that does not reference the zero-point energy from quantum field theory.

1 Introduction And History

The Casimir effect is a small attractive force caused by quantum fluctuations of the electromagnetic field in vacuum (Figure 1). In 1948 the Dutch physicist Hendrick Casimir published a paper predicting this effect [1, 2]. According to Quantum field theory, a vacuum contains particles (photons), the numbers of which are in a continuous state of fluctuation and can be thought of as popping in and out of existence [3]. These particles can cause a force of attraction. Most generally, the quantum Casimir effect is thought about in regards to two closely parallel plates. As the plates are brought together, Casimir realized that between them, only those virtual photons whose wavelengths fit a whole number of times should be counted whilst calculating the vacuum energy [1]. This leads to a decrease in energy density between the plates as they are moved closer which implies that a small force is drawing them together. similarly you can say that due to the smaller space between the plates only smaller exotic particles can exist between them. From this difference in particles outside the plates and those between the plates, a small pressure change can be calculated which creates a force pushing the plates towards one another [3]. This force is the Casimir effect.

In 1996, the small force was measured to within 5% uncertainty to that of the theoretical prediction by Steven Lamoreaux [4]. All bosons make a contribution to the Casimir force, but fermions make a repulsive contribution to the force. All of these particles make a contribution to the force though only that from photons is measurable. The theory states that the lowest energy state of a vacuum (the zero-point energy) is infinite when considering all possible photon modes. The original Casimir force derivation comes about from a situation in which the differences in infinities cancel out which arises from very interesting mathematics. There are inconsistencies and puzzles that arise from the existence of this effect, especially when applying it to the theories of quantum gravity. The solutions to these inconsistencies are however expected to be found within the solution to a theory of quantum gravity [1].

In 2005, Jaffe made it clear that the zero-point fluctuations formulated in quantum field theory was not observable in any laboratory experiments though the vacuum value of the stress tensor (energy density of the vacuum) $\langle T_{\mu v}\rangle \equiv -\varepsilon g_{\mu v}$ even appears in the right hand side of Einstein's equation for gravity in general theory of relativity [6]

$$\frac{1}{2}g_{\mu v}R - R_{\mu v} = 8\pi G(\tilde{\mathrm{T}}_{\mu v} - \varepsilon g_{\mu v}). \qquad (1)$$

Jaffe also demonstrates in his paper that you can calculate the Casimir force without reference to the zero-point energy [6], which suggests that the zero-point energy may simply just be a nice mathematical construct in this situation to arrive at a measurable result. This will be discussed later in more detail.

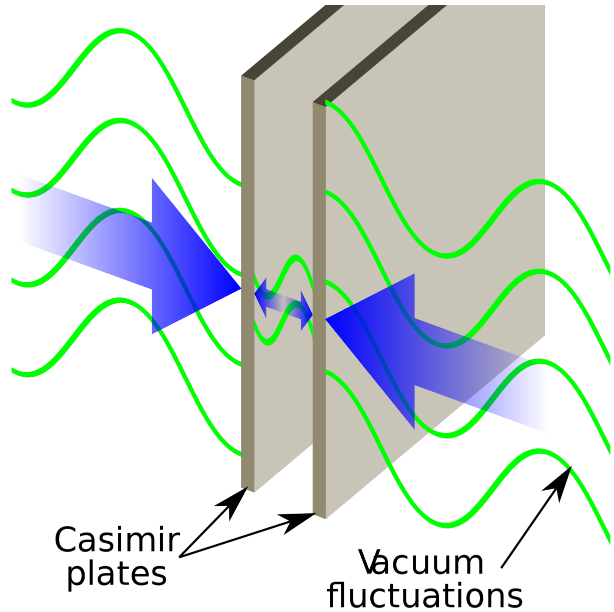

Figure 1: A simple diagram of two parallel plates and a representation of vacuum fluctuations [5].

2 'Astounding' Mathematical Results.

One of my favorite results I have encountered in my studies (which led me to the Casimir effect) follows as

$$\sum_{n=1}^{\infty} n \to -\frac{1}{12}. \qquad (2)$$

This result does not explicitly make sense because the sum in equation (2) is a divergent sum. However, due to a process known as analytic continuation, some divergent sums can have a finite value. In 1913, this appeared in the work of a very famous mathematician from India, Srinivasa Ramanujan and is an important result for String Theory and other branches of physics. The Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = 1 + 2^{-s} + 3^{-s} + 4^{-s} + \cdots \quad (3)$$

is widely studied and used often in physics. In quantum physics, the energy density of a vacuum should be proportional to $\zeta(-3) = 1 + 8 + 27 + 64 + \cdots$, which is a divergent series and thus does not make much sense as an energy density [7]. When we write this using equation (3) and use the process of analytic continuation, this can be written

$$\zeta(-3) = \sum_{n=1}^{\infty} \frac{1}{n^{-3}} = 1 + 2^3 + \cdots \to \frac{1}{120}. \quad (4)$$

The way Ramanujan expresses functions that are divergent such as this (from the Riemann zeta function) is

$$\sum_{k=\alpha}^{x} f(k) \sim \int_{\alpha}^{x} f(t)dt + c + \frac{1}{2} f(x) + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} f^{(2k-1)}(x), \quad (5)$$

[8]. This is a process of analytically continuing these divergent series and coming up with a finite result without any 'magic'. I say magic because there is a process in which one can ignore (in a sense) the divergent nature of a sum and come up with these results as well.

As an example, I will give a 'proof' of equation (2) using this method, which was first shown by Euler around 1735 [9]. Consider the following well defined sum

$$f(x) = 1 + x + x^2 + x^3 + x^4 + \cdots = \frac{1}{1-x}, \quad (6)$$

for $|x| < 1$. Differentiating this gives

$$f'(x) = 1 + 2x + 3x^2 + \cdots = \frac{1}{(1-x)^2}. \quad (7)$$

If we evaluate the result at $x = -1$ we get

$$f'(-1) = 1 - 2 + 3 - 4 + \cdots = \frac{1}{4}. \quad (8)$$

Note that this is troublesome because we defined $f'(x)$ based on a function only valid for when $|x| < 1$. However, for our purposes suppose we can extend our limits and make f(x) differentiable at $x = -1$. Now, if we take $2^{-s}\zeta(s)$ we have

$$\begin{aligned} 2^{-s}\zeta(s) &= 2^{-s} \sum_{n=1}^{\infty} \frac{1}{n^s} = \sum_{n=1}^{\infty} \frac{2^{-s}}{n^s} \\ &= 2^{-s} + 4^{-s} + 6^{-s} + 8^{-s} \cdots . \end{aligned} \quad (9)$$

Now, if we take $g(s) = [1 - 2(2^{-s})]\zeta(s)$ we have

$$\begin{aligned} g(s) &= [1 - 2(2^{-s})]\zeta(s) \\ &= \zeta(s) - 2(2^{-s})\zeta(s) \\ &= 1 + 2^{-s} + 3^{-s} + 4^{-s} + 5^{-s} + 6^{-s} + \cdots \\ &\underline{\quad - 2(\;\; 2^{-s} \qquad + 4^{-s} \qquad + 6^{-s} + \cdots)} \\ &= 1 - 2^{-s} + 3^{-s} - 4^{-s} + 5^{-s} - 6^{-s} + \cdots . \end{aligned} \quad (10)$$

Finally, if we set $s = -1$, we can see that $g(-1) = \zeta(-1) - 2(2)\zeta(-1) = -3\zeta(-1)$ and evaluating this from equation (10) and then using our result from equation (8) gives us

$$\begin{aligned} -3\zeta(-1) &= 1 - 2 + 3 - 4 + \cdots = \frac{1}{4} \\ \implies \zeta(-1) &= -\frac{1}{12}. \end{aligned} \quad (11)$$

Now, notice that plugging in $s = -1$ into the Riemann zeta function gives us the same result from equation (2) and thus

$$\zeta(-1) = -\frac{1}{12} \implies \sum_{n=1}^{\infty} n \to -\frac{1}{12}. \tag{12}$$

This result is very important to obtaining the $24 + 2 = 26$ dimensions in bosonic string theory [10]. It is also a simpler example than that of equation (4) to illustrate.

3 Casimir Force Derivation.

In Casimir's original paper, he did not use the result in equation (4) explicitly, though in a more recent derivation assuming zeta-regularization, one can see how it is obtained. Let k_x, k_y, and k_z represent the wave numbers in the x, y and z directions respectively. If we allow two plates to be parallel in the $x-y$ plane at a distance a apart, then we can define the cavity between the plates by

$$0 \le x \le \sqrt{A} \tag{13}$$
$$0 \le y \le \sqrt{A} \tag{14}$$
$$0 \le z \le a, \tag{15}$$

where the plates are a square of area A. If we adopt a periodic boundary condition, then we can show

$$k_x = \frac{2\pi n_x}{\sqrt{A}} \implies dn_x = \frac{\sqrt{A}}{2\pi} dk_x \tag{16}$$
$$k_y = \frac{2\pi n_y}{\sqrt{A}} \implies dn_y = \frac{\sqrt{A}}{2\pi} dk_y \tag{17}$$
$$k_z = \frac{n_z \pi}{a}, \tag{18}$$

with $(n_x, n_y, n_z) \in \mathbb{Z}$. The frequency of this wave is $\omega_n = v|\vec{k}| = v\sqrt{k_x^2 + k_y^2 + k_z^2}$. If we assume we are in a vacuum, then the speed of any electromagnetic wave is just c and thus $\omega_{n_z} = c\sqrt{k_x^2 + k_y^2 + k_z^2}$. The vacuum energy is the sum over all possible modes. The zero-point (ground state) energy associated with the n_z^{th} mode is given by $E_{n_z} = \frac{\hbar\omega_{n_z}}{2}$. The energy of all combined modes is then the sum over all n_z or $E = \sum_{n_z=1}^{\infty} \frac{\hbar\omega_{n_z}}{2}$. For simplicity we can allow $n \equiv n_z$. Taking the expectation value of the energy over the entire area of the plates can be done by integrating over all possible values of n_x, n_y and all possible expectation modes which yields

$$\langle E \rangle = \frac{\hbar}{2} \iint \sum_{n=1}^{\infty} \omega_n dn_x dn_y \tag{19}$$
$$= \frac{A\hbar}{8\pi^2} \iint \sum_{n=1}^{\infty} \omega_n dk_x dk_y. \tag{20}$$

This expression is clearly infinite due to the diverging sum. If we use zeta-regulation, we can find a finite energy per unit area by defining a quantity $\langle E(s) \rangle$ which goes to equation (20) when s=0.

$$\frac{\langle E(s) \rangle}{A} = \frac{\hbar}{8\pi^2} \iint \sum_{n=1}^{\infty} \omega_n |\omega_n|^{-s} dk_x dk_y \tag{21}$$
$$= \frac{\hbar}{8\pi^2} \sum_{n=1}^{\infty} \iint \omega_n |\omega_n|^{-s} dk_x dk_y. \tag{22}$$

Simplifying the above expression (Using Mathematica to take the integral over dk_x and dk_y) gives us

$$\frac{\langle E(s) \rangle}{A} = \frac{\hbar c^{1-s} \pi^{2-s}}{2a^{3-s}(3-s)} \sum_{n=1}^{\infty} |n|^{3-s}. \tag{23}$$

This may then be analytically continued to $s = 0$ where it becomes finite when using equation (4).

$$\frac{\langle E \rangle}{A} = \lim_{s\to 0} \frac{\langle E(s) \rangle}{A} = -\frac{\hbar c \pi^2}{6a^3}\zeta(-3). \tag{24}$$

Now, plugging in equation (4) in the above expression gives us

$$\frac{\langle E \rangle}{A} = \frac{-\hbar c \pi^2}{720a^3}. \tag{25}$$

The Casimir force per unit area between two parallel plates within a vacuum is therefore given by $F = -\nabla\langle E \rangle$ which is

$$\frac{F_c}{A} = -\frac{d}{da}\frac{\langle E \rangle}{A} = \frac{-\hbar c \pi^2}{240a^4}. \tag{26}$$

As we can clearly see, this result would not have come about without the use of analytical continuation. In a sense, this is due to nature not containing apparent infinities. Rather, the continuation allowed us to arrive at a finite solution which is experimentally confirmed. The fact that our expression came out negative suggests that the force is an attractive force and due to the presence of $\hbar$, we can see that the force is of a quantum origin. In the original derivation, Casimir computed non-convergent sums using Euler-Maclaurin summation with a regularizing function [2].

4 Implications From The Zero-Point Derivation.

The Casimir effect extends quantum field theory to allow for negative energy densities with respect to the ordinary vacuum energy. It has been suggested by numerous physicists such as Stephen Hawking, Kip Thorne, and many more that such a thing will allow the possibilities of stabilizing traversable wormholes [11]. Miguel Alcubierre, creator of the Alcubierre Drive has also suggested using the Casimir effect to obtain negative energy required for his designs [11]. In many cases, this effect has been shown to have possible applications in propulsion drives for space craft. It also has possible application in nanotechnology which has been suggested by some [11]. Due to the small scale that this force is observed on, this would make sense that it could present possible applications in nanotechnology. For instance, the force of attraction could be used as an architecture for moving components on a microscopic scale or something much more complex.

5 Caimir Force Without Referencing Zero-Point Energy.

As mentioned earlier, Jaffe argues that the Casimir force can be constructed without considering zero-point fluctuations of quantized electromagnetic field and is a result from the material of the plates and not resulting from zero-point energies. If we use the Drude model of metals, then the metal/conductor properties are characterized by a plasma frequency ω_p and a skin depth δ. The original result does not depend on anything other than the distance of the plates and fundamental constants. However, this result assumed that the plates were perfect conductors which do not exist in reality. The skin depth of a material is a measure of how far electromagnetic waves penetrate through a material and thus can cause a relationship between the waves within the plates to those outside.

Jaffe argues that both ω_p and δ are dependent on the fine structure constant α. He then argues that the perfect conductor approximation is good for sufficiently large α which in the case of the Casimir measurement scales for experimental verification are satisfied by the physical value of $\alpha \approx 1/137$ which is why the original derivation is supported by experimental results. Similarly, he also argues that the Casimir force vanishes as $\alpha \to 0$.

6 Conclusion

I have shown that while referencing zero-point energy one can derive the Casimir force using zeta-function regularization, however, it can also be calculated without reference to the zero-point energy which suggests that it may not be related to the energies that are suggested to come about from quantum field theory but instead the fine structure constant and properties of materials. It is fascinating to note that the same experimentally observed result can be obtained through a simple method using zeta-function regularization and ignoring divergences which may possibly suggest that this is a useful mathematical construct that could potentially have many real world applications. Much like the early use of imaginary numbers, which appeared to have no physical application, it may prove to be a useful method of mathematical manipulation that could lead us to new unique breakthroughs much like in the case of the Casimir force.

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References

[1] Gibbs, Philip. ”What Is the Casimir Effect?” The Casimir Effect. University of California, 1997. Web. 16 Oct. 2016.

[2] H.B.G. Casimir, Proc. Kon. Ned. Akad. Wetensch. B51, 793 (1948)

[3] Casimir Effect & Black Holes - Sixty Symbols. Prod. Brady Haran. Perf. Mike Merrifield Ph.D. Youtube. University of Nottingham, 20 Mar. 2014. Web.

[4] S. Lamoreaux, Phys. Rev. Lett. 78, 5 (1996)

[5] Image from `https://en.wikipedia.org/wiki/Casimir_effect`

[6] Jaffe. R. L. ”Casimir effect and the quantum vacuum” 10.1103/PhysRevD.72.021301. 2005.

[7] Berman, David. ”Infinity or -1/12?” Plus.maths.org. +Plus Magazine, 18 Feb. 2014. Web. 16 Oct. 2016.

[8] Library of Congress Cataloging in Publication Data Ramanujan Aiyangar, Srinivasa, 1887-l 920. Ramanujans notebooks. Mathematics-Collected works. 1. Berndt, Bruce C., 1939- II. Title. QA3. R33. 1985. 510. 84-20201

[9] John Baez on the number 24. The Rankin Lectures 2008. Youtube. John Baez. University of California. 16 May. 2012. Web.

[10] Sum of Natural Numbers (second proof and extra footage). Prod. Brady Haran. Perf. Ed Copeland Ph.D and Tony Padilla Ph.D. Youtube. University of Nottingham, 11 Jan. 2015. Wb.

[11] Casimir effect. (n.d.). Retrieved October 16, 2016, from `https://en.wikipedia.org/wiki/Casimir_effect#cite_note-45`