

AUGUST 11, 2004

Default Correlation: From Definition to Proposed Solutions

This paper is different from most papers on default correlation in that we focus on default correlation. Meaning, we do not address asset correlation, equity correlation, rating migration, or any other quantity used as a *proxy* for default correlation. Another difference from other papers is that we do not propose our own solution for incorporating default correlation into portfolio credit analysis. Instead, we evaluate two widely used approaches. Finally, this paper is a lot easier to read than other default correlation papers. We make the maximum use of graphic illustrations, the minimum use of algebra, and no use of calculus. This article should be completely understandable to anyone who knows what a standard deviation is and mostly understandable to someone who does not.

In *Definition and Theory*, we define default correlation, discuss its drivers, and show why we care about it. We show pictorial representations of default probability and default correlation and derive mathematical formulas relating default correlation to default probability. The difficulty of the problem becomes evident when we show that pairwise default correlations are not sufficient to understand the behavior of a credit risky portfolio and introduce “higher orders of default correlation.”

In *Empirical Results and Problems*, we survey the meager work done on historic default correlation. We show that default correlations within well-diversified portfolios vary by the ratings of the credits and also by the time period over which defaults are examined. But we spend most of our time describing the major problems in measuring and even thinking about default correlation. The thorniest problem is that when looking at historical rates of default, it is impossible to distinguish default correlation from changing default probability.

In *Proposed Solutions*, we compare different approaches of incorporating default correlation into portfolio credit analysis. We opine that the CSFB approach makes the most direct use of historical data and is the easier to understand. But we conclude this paper on default correlation by wishing that more work was being done on default probability.¹

¹ The first two sections of this paper owe a lot to our “Default Correlation and Credit Analysis,” *Journal of Fixed Income*, March 1995. One reason for our revisiting this topic is the many papers on default correlation proxies (asset correlation, etc.) that make passing use of the formulas derived in our 1995 article and ignore the article’s more problematic findings. (We also would have appreciated being cited more often as the source of those formulas.) The third section of this paper owes a lot to our “Default Correlation for Cash Flow CDOs” in *CDO Insight*, UBS, June 30, 2004. Another reason for writing this paper is in reaction to the hyper-mathematical approaches to default correlation that build fantastically elaborate superstructures on very shaky empirical foundations.

Definition and Theory

Default correlation measures whether credit risky assets are more likely to default together or separately. For example, default correlation answers the following question:

If ten bonds each have a 10% probability of default, does that mean:

one and only one is definitely going to default?

or

there's a 10% chance *all* of them will default
and a 90% chance *none* of them are going to default?

If the answer is “in between,” where in between?

Default correlation is essential to understanding the risk of credit portfolios. Along with *default probability* and *loss in the event of default*, default correlation determines the credit risk of a portfolio and the economic capital required to support that portfolio. It is important to portfolio managers, investors, CFOs, bankers, rating agencies, and regulators.

Default Correlation Defined

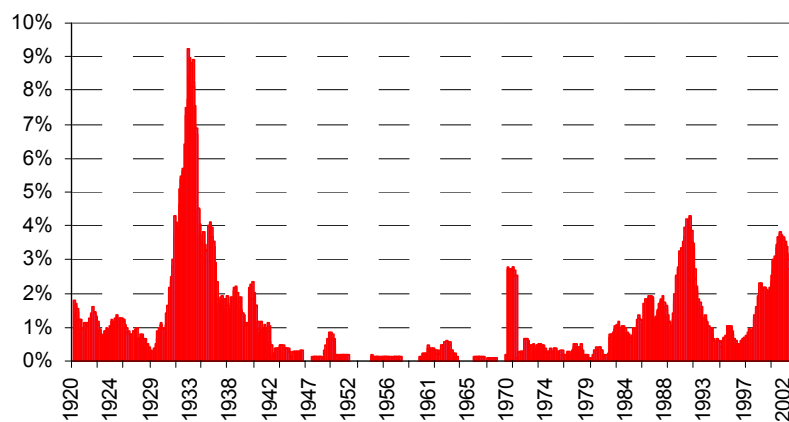
Default correlation is the phenomenon that the likelihood of one obligor defaulting on its debt is affected by whether or not *another* obligor has defaulted on *its* debts. A simple example of this is if one firm is the creditor of another: if Credit A defaults on its obligations to Credit B, we think it is more likely that Credit B will be unable to pay its own obligations. This is an example of *positive* default correlation. The default of one credit makes it *more* likely the other credit will default.

There could also be *negative* default correlation. Suppose that Credit A and Credit B are competitors. If Credit A defaults and goes out of business, it might be the case that Credit B will get Credit A's customers and be able to get price concessions from Credit A's suppliers. If this is true, the default of one credit makes it *less* likely the other credit will default. This would be an example of *negative* correlation.

But default correlation is not normally discussed with respect to the particular business relationship between one credit and another. And the existence of default correlation does not imply that one credit's default directly *causes* the change in another credit's default probability. It is a maxim of statistics that *correlation does not imply causation*. Nor do we think negative default correlation is very common. Primarily, we think *positive* default correlation generally exists among credits because the fortunes of individual companies are linked together via the health of the general economy or the health of broad subsets of the general economy.

Drivers of Default Correlation

The pattern of yearly default rates for U.S. corporations since 1920, shown in Exhibit 1 (next page), is notable for the high concentrations of defaults around 1933, 1991, and 2001. A good number of firms in almost all industries defaulted on their credit obligations in these depressions and recessions. The boom years of the 1950s and 1960s, however, produced very few defaults. To varying degrees, all businesses tend to be affected by the health of the general economy, regardless of their specific characteristics. The phenomena of companies tending to default together or not default together is indicative of positive default correlation.

Exhibit 1. U.S. Corporate Default Rates Since 1920

Source: Moody's Investors Service

But defaults can also be caused by industry-specific events that only affect firms in those particular industries. Despite a favorable overall economy, low oil prices caused 22 companies in the oil industry to default on rated debt between 1982 and 1986. Bad investments or perhaps bad regulation caused 19 thrifts to default in 1989 and 1990. Recently, we experienced the defaults of

numerous dot coms due to the correction of “irrational exuberance.” Again, the phenomena of companies in a particular industry tending to default together or not default together is indicative of positive default correlation.

There are other default-risk relationships among businesses that do not become obvious until they occur. The effect of low oil prices rippled through the Texas economy affecting just about every industry and credit in the state. A spike in the price of silver once negatively affected both film manufacturers and silverware makers. The failure of the South American anchovy harvest in 1972 drove up the price of alternative sources of cattle feed and put both Peruvian fishermen and Midwest cattle ranchers under pressure. These default-producing characteristics hide until, because of the defaults they cause, their presence becomes obvious.

Finally, there are truly company-specific default factors such as the health of a company's founder or the chance a warehouse will be destroyed by fire. These factors do not transfer default contagion to other credits. Recent defaults brought on by corporate fraud are also considered to be company-specific events. For example, the default of Parmalat did not widen the credit default swap premiums of other industrial companies.

Defaults are therefore the result of an unknown and unspecified multi-factor model of default that seems akin to a multi-factor equity-pricing model. Default correlation occurs when, for example, economy-wide or industry-wide default-causing variables assume particular values and cause widespread havoc. Uncorrelated defaults occur when company-specific default-causing variables cause trouble for individual credits.

Why We Care About Default Correlation

Default correlation is very important in understanding and predicting the behavior of credit *portfolios*. It directly affects the risk-return profile of investors in credit risky assets and is therefore important to the creditors and regulators of these investors. Default correlation also has implications for industrial companies that expose themselves to the credit risk of their suppliers and customers through the normal course of business. We will prove these assertions via an example.

Exhibit 2. Extreme Positive Default Correlation

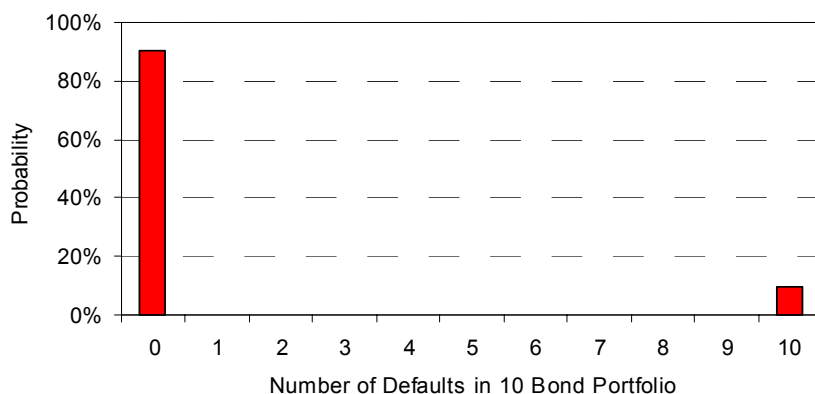
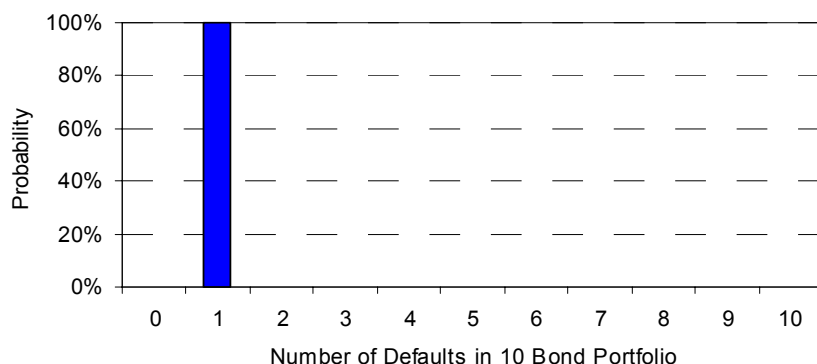


Exhibit 3. Extreme Negative Default Correlation



Suppose we wish to understand the risk of a bond portfolio and we know that each of the 10 bonds in the portfolio has a 10% probability of default over the next five years. What does this tell us about the behavior of the portfolio as a whole? Not much, it turns out, unless we also understand the default correlation among credits in the portfolio.

It could be, for example, that all the bonds in the portfolio always default together. Or to put it another way, if one of the 10 bonds default, they all default. If so, this would be an example of “perfect” *positive* default correlation. Combined with the fact that each bond has a 10% probability

of default, we can make a conclusion about how this portfolio will perform. There is a 10% probability that *all* the bonds in the portfolio will default. And there is a 90% probability that *none* of the bonds will default. Perfect positive default correlation, the fact that all the bonds will either default together or not default at all, combines with the 10% probability of default to produce this extreme distribution, as shown in Exhibit 2 (above).

At the other extreme, it could be the case that bonds in the portfolio *always* default separately. Or to put it another way, if one of the 10 bonds defaults, no other bonds default. This would be an example of “perfect” *negative* default correlation. Combined with the fact that each bond has a 10% probability of default, we can make a conclusion about how this portfolio will perform: there is a 100% probability that *one* and only one bond in the portfolio will default. Perfect negative default correlation, the fact that when one bond defaults no other bonds default, combines with the 10% probability of default to produce this extreme distribution, as shown in Exhibit 3 (above).

The difference in the distributions depicted in Exhibits 2 and 3 has profound implications for investors in these portfolios. Remember that in both cases, the default probability of bonds in the portfolio is 10% and the expected number of defaults is one. But one knows with *certainty* the result of the portfolio depicted in Exhibit 3: one and only one bond is going to default. This certainty would be of comfort to a lender to this investor. The lender knows with certainty that nine of

the bonds are going to perform and that par and interest from those nine performing bonds will be available to repay the investor's indebtedness.

The investor in the portfolio depicted in Exhibit 2 has the greatest uncertainty. Ninety percent of the time the portfolio will have no defaults and 10% of the time every bond in the portfolio will default. A lender to an investor with this portfolio has a 10% risk that *no* bonds in the portfolio will perform.

A complete analysis of the risk of these two example portfolios would depend on the distribution of default recoveries. But it is obvious that the portfolio depicted in Exhibit 2 is much more risky than the portfolio depicted in Exhibit 3, even though the default probabilities of bonds in the portfolios are the same. The difference in risk profiles, which is due only to default correlation, has profound implications to investors, lenders, rating agencies, and regulators. Debt backed by the portfolio depicted in Exhibit 2 should bear a higher premium for credit risk and be rated lower. If this is a regulated entity, it should be required to have more capital.

In the next few pages of this section, we will show default probability and default correlation pictorially, present the basic algebra of default correlation, and then delve into the deficiency of pairwise correlations in explaining default distributions.

Picturing Default Probability

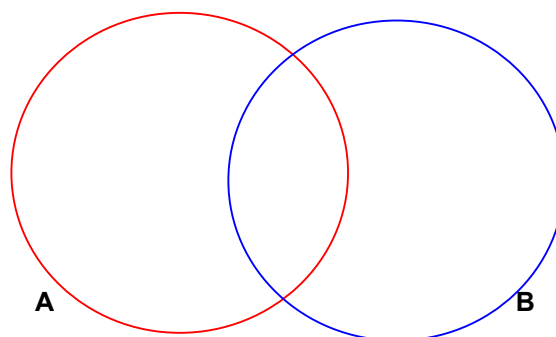
Suppose we have two obligors, Credit A and Credit B, each with 10% default probability. The circles A and B in Exhibit 4 (below) represent the 10% probability that A and B will default, respectively.

There are four possibilities depicted in Exhibit 4:

- (1) both A and B default, as shown by the overlap of circles A and B;
- (2) only A defaults, as shown by circle A that does not overlap with B;
- (3) only B defaults, as shown by circle B that does not overlap with A;
- (4) neither A or B default, as implied by the area outside both circles A and B.

Recall that we defined positive and negative default correlation by how one revises their assessment of the default probability of one credit once one finds out whether another credit has defaulted. If upon the default of one credit you revise the default probability of the second credit *upwards*, you implicitly think there is *positive* default correlation between the two credits. And if

Exhibit 4. Credit A and Credit B Default Probability, Pictorially



upon the default of one credit you revise the default probability of the second credit *downwards*, you implicitly think there is *negative* default correlation between the two credits.

Exhibit 4 is purposely drawn so that knowing whether one credit defaults does not cause us to revise our estimation of the default probability of the other credit. Exhibit 4 pictorially represents no or *zero default correlation* between Credits A and B, neither positive or negative default correlation. In other words, knowing that A has defaulted does not change our assessment of the probability that B will default.

Here's the explanation. Recall that the probability of A defaulting is 10% and the probability of B defaulting is 10%. Suppose A has defaulted. Now, pictorially, we are within the circle labeled A in Exhibit 4. No or zero correlation means that we do not change our estimation of Credit B's default probability just because Credit A has defaulted. We still think there is a 10% probability that B will default. Given that we are within circle A, and circle A represents 10% probability, the probability that B will default must be 10% of circle A or 10% of 10% or 1%. The intersection of circles A and B depicts this 1% probability. This leads to a very simple general formula for calculating the probability that both A and B will default when there is no or zero default correlation.

Recall the phrase in the above paragraph that the overlap of A and B, or the space where both A and B default is "10% of 10% or 1%." What this means mathematically is the probability of both Credits A and B defaulting (the joint probability of default for Credits A and B) is 10% * 10% or 1%. Working from the specific to the general (which we label Equation 1), our notation gives us the following:

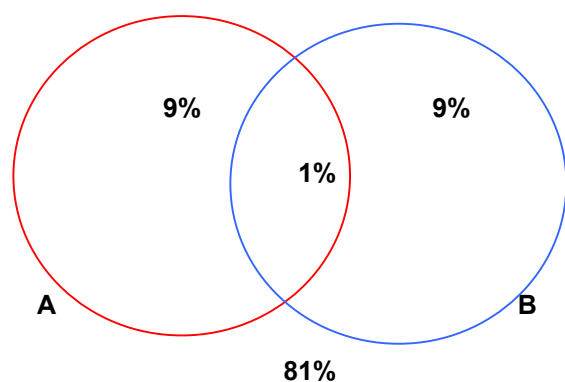
$$10\% \times 10\% = 1\%$$

$$P(A) * P(B) = P(A \text{ and } B) \quad (1)$$

Where:

- $P(A)$ = the probability of Credit A defaulting (10% in our example);
- $P(B)$ = the probability of Credit B defaulting (10% in our example);
- $P(A \text{ and } B)$ = the probability of both Credits A and B defaulting, a.k.a., the *joint probability* of default for Credits A and B (1% in our example).

Exhibit 5. Default Probabilities, Pictorially



This is the general statistical formula for joint default probability assuming zero correlation.

Now that we have calculated the joint probability of A and B defaulting, we can assign probabilities to all the alternatives in Exhibit 4. We do this in Exhibit 5 (left). We assumed that the default probability of Credit A was 10%, which we represent by the circle labeled A in Exhibit 5. We have already determined that the joint probability and Credit A and Credit B defaulting, as represented by the intersection of the circles labeled A and B,

is 1%. Therefore, the probability that Credit A *will* default and Credit B *will not* default, represented by the area within circle A but also outside circle B, is 9%. Likewise, the probability that Credit B will default and Credit A will not default is 9%. The probabilities that *either or both* Credit A and Credit B will default, the area within circles A and B, adds up to 19%. Therefore, the probabilities that *neither* Credit A nor Credit B will default, represented by the area outside circles A and B, is 81%.

These results are also shown in Exhibit 6 (below) throwing some “nots,” “ors,” and “neithers” into the notation. $P(A \text{ not } B)$ means that A defaults and B does not default. $P(A \text{ or } B)$ means that either A or B defaults and includes the possibility that *both* A and B default. “Neither” means neither A or B defaults.

Exhibit 6. Default Probabilities, Notationally

$$\begin{aligned} P(A) &= 10\% \\ P(A \text{ and } B) &= 1\% \\ P(A \text{ not } B) &= P(A) - P(A \text{ and } B) = 10\% - 1\% = 9\% \\ P(A \text{ or } B) &= P(A) + P(B) - P(A \text{ and } B) = 10\% + 10\% - 1\% = 19\% \\ P(\text{neither } A \text{ or } B) &= 100\% - P(A \text{ or } B) = 100\% - 19\% = 81\% \end{aligned}$$

Exhibit 7. No Joint Probability

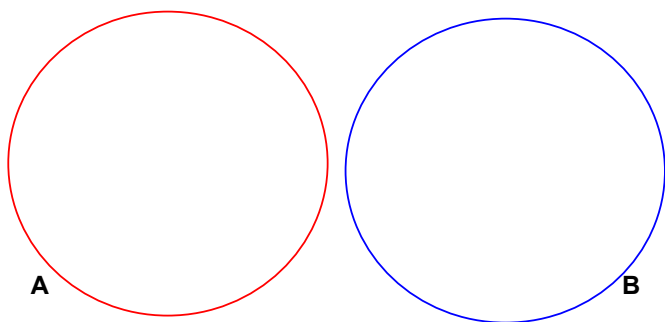
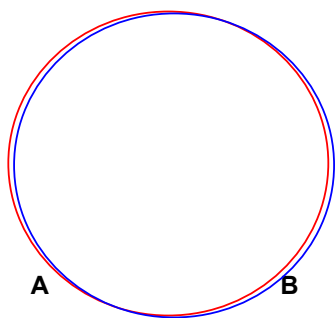


Exhibit 8. Maximum Joint Probability



Picturing Default Correlation

We've pictorially covered scenarios of joint default, single default, and no-default probabilities in our two credit world *assuming zero default correlation*. Exhibit 5, showing moderate overlap of the “default circles” has been our map to these scenarios. There are, of course, other possibilities. There could be no overlap, or 0% joint default probability, between Credit A and Credit B, as depicted in Exhibit 7 (left). Or there could be *complete* overlap as depicted in Exhibit 8 (left). The joint default probability equals 10% because we assume that Credit A and B each have a 10% probability of default and in Exhibit 8 they are depicted as always defaulting together. (Note that we draw the circles in Exhibit 8 a little off-set so you can see that there are two of them. Otherwise, they rest exactly on top of each other.)

One will recall, hopefully, that Exhibit 7 depicts perfect *negative* default correlation since if one credit defaults we know the other will not. Exhibit 8 depicts perfect *positive* default

correlation since if one credit defaults we know the other one will too. Unfortunately, our Equation 1 (above) does not take into account the situations depicted in Exhibits 7 and 8. That formula does not help us calculate joint default probability in either of these circumstances or in any circumstance other than zero default correlation. Which leads us to the next part of this section.

Calculating Default Correlation Mathematically

We are perhaps already further than most introductory statistics courses would go with respect to correlation. But with Venn diagrams under our belt, we can become more precise in understanding default correlation with a little high school algebra. What we are going to do in this section is mathematically define default correlation. Once defined, the equation will allow us to compute default correlation between any two credits given their individual default probabilities and their joint default probability. Then, we will solve the same equation for joint default probability. The reworked equation will allow us to calculate the joint default probability of any two credits given their individual default probabilities and the default correlation between the two credits.

What we would like to have is a mathematical way to express the degree of overlap in the Venn diagrams or the joint default probability of the credits depicted in the Venn diagrams. From no overlap depicted in Exhibit 7 to “moderate” overlap depicted back in Exhibit 5, to complete overlap depicted in Exhibit 8. One way is to refer to the joint probability of default. It’s 0% in Exhibit 7, 1% in Exhibit 5, and 10% in Exhibit 8. All possible degrees of overlap could be described via the continuous scale of joint default probability running from 0% to 10%. However, this measure is tied up with the individual credit’s probability of default. A 1% joint probability of default is very high default correlation if both credits have only a 1% probability of default to begin with. A 1% joint probability of default is very negative default correlation if both credits have a 50.5% probability of default to begin with. We would like a measure of overlap that does not depend on the default probabilities of the credits.

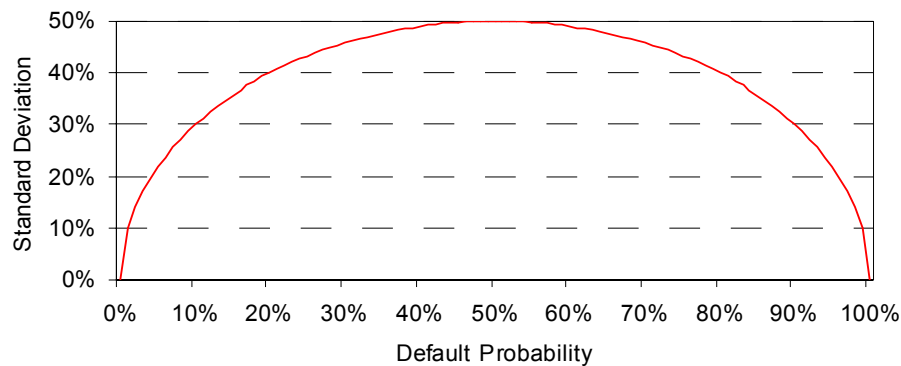
This is exactly what default correlation, a number running from -1 to $+1$, does. Default correlation is defined mathematically as:

$$\text{Default Correlation(A and B)} = \frac{\text{Covariance(A,B)}}{\text{Standard Deviation(A)} * \text{Standard Deviation(B)}} \quad (2)$$

What we are going to do now is to delve more into the right side of Equation (2) and better define default correlation between Credits A and B.

In the denominator of the formula, *standard deviation* is a measure of how much A can vary. A, in this case, is whether or not Credit A defaults. What this means intuitively is how certain or uncertain we are that A will default. We are very certain about whether A will default if A’s default probability is 0% or 100%. Then we know with certainty whether or not A is going to default. At 50% default probability of default, we are most uncertain whether A is going to default.

The term for an event like default, where either the event happens or does not happen, and there is no in between, is *binomial*. And the mathematical formula for the standard deviation of a

Exhibit 9. Standard Deviation and Default Probability

binomial variable is:

$$\text{Standard Deviation}(A) = \{P(A) * [1-P(A)]\}^{1/2} \quad (3)$$

In the example we have been working with, where the default probability of A is 10%, or $P(A) = 10\%$, the standard deviation of A is:

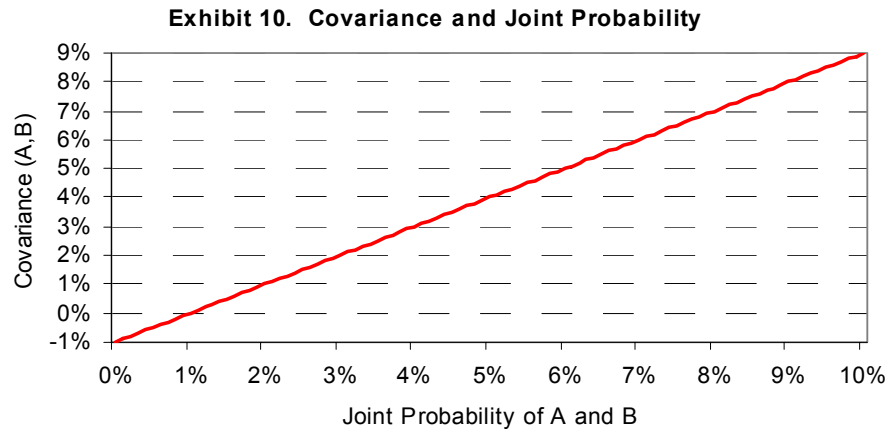
$$\begin{aligned} \text{Standard Deviation}(A) &= (P(A) * [1-P(A)])^{1/2} \\ &= (10\% * 90\%)^{1/2} \\ &= (9\%)^{1/2} \\ &= 30\% \end{aligned}$$

All the possible standard deviations of a binomial event, where the probability varies from 0% to 100%, are shown in Exhibit 9 (above). Above 10% probability on the X axis we can see that the standard deviation is indeed 30%. The chart also illustrates the statements we made before likening standard deviation to the uncertainty of whether or not the credit is going to default. At 0% and 100% default probability, where we are completely certain what is going to happen, standard deviation is 0%. At 50% default probability, where we are least certain whether the credit is going to default, standard deviation is at its highest.

The *covariance* of A and B is a measure of how far the actual joint probability of A and B is from the joint probability that would obtain if there was zero default correlation. Mathematically, this is simply *actual joint probability of A and B* minus the *joint probability of A and B assuming zero correlation*. Recall from Equation 1 above that the joint probability of A and B assuming zero correlation is $P(A) * P(B)$. Therefore the covariance² between A and B is:

$$\text{Covariance}(A, B) = P(A \text{ and } B) - P(A) * P(B) \quad (4)$$

² Covariance is more formally defined as the $\text{Expectation}(A*B) - \text{Expectation}(A) * \text{Expectation}(B)$. When we define default as 1 and no default as 0, Equation 4 is the result.



In our example, from our work around Exhibit 5, we worked out that the joint probability of default, assuming zero default correlation, is 1%. From Exhibit 7, we know that given perfect negative default correlation, the actual joint probability can be as small as 0%. From Exhibit 8, we know that given perfect positive default correlation, the actual joint probability can be as high as 10%. Exhibit 10 (above) depicts the relationship between joint default probability and covariance graphically.

Recall that a few pages back in Equation 2 we presented this formula for default correlation.

$$\text{Correlation(A and B)} = \frac{\text{Covariance(A,B)}}{\text{Standard Deviation(A)} * \text{Standard Deviation(B)}} \quad (2)$$

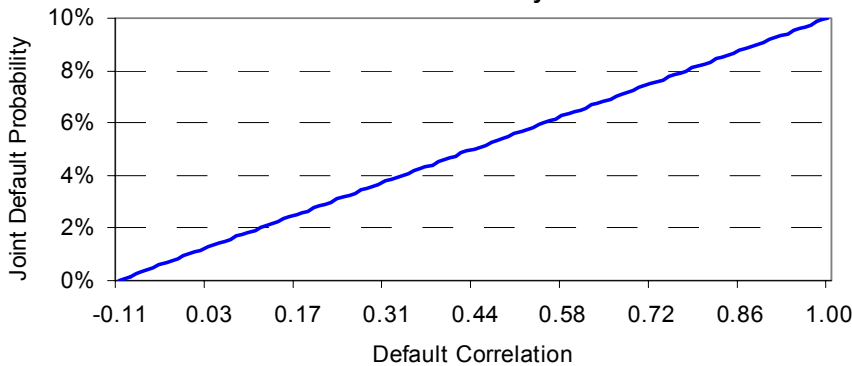
In Equations 3 and 4, we refined the numerator and denominator of Equation 2. Substituting Equations 3 and 4 into Equation 2 we get:

$$\text{Correlation(A and B)} = \frac{P(A \text{ and } B) - P(A) * P(B)}{\{P(A) * [1 - P(A)]\}^{1/2} * \{P(B) * [1 - P(B)]\}^{1/2}} \quad (5)$$

Now, finally, we can define mathematically the default correlation we saw visually in Exhibits 5, 7, and 8. In Exhibit 5, the joint default probability of A and B, $P(A \text{ and } B)$, was 1%, simply because we wanted to show the case where the default probability of one credit does not depend on whether another credit had defaulted. The product of A's and B's default probabilities, $P(A) * P(B)$, is 10% * 10%, or 1%. Moving to the denominator of Equation 5, the product of A's and B's standard deviations, $\{P(A) * [1 - P(A)]\}^{1/2} * \{P(B) * [1 - P(B)]\}^{1/2}$ is 9%. Putting this all together, we get:

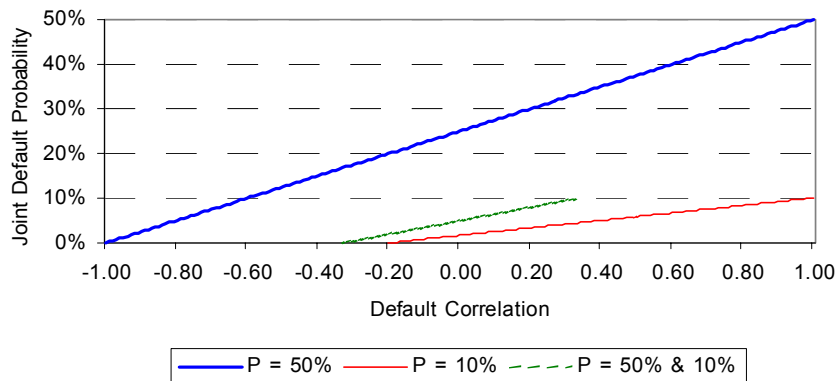
$$\text{Correlation(A and B)} = \frac{1\% - 1\%}{9\%} = 0.00$$

Exhibit 11. Default Correlation and Joint Default Probability



Similarly, for Exhibit 7, where joint default probability is 0%, default correlation is -0.11. And in Exhibit 8, where joint default probability is 10%, default correlation is +1.00. In our example, as joint default probability moves from 0% to 10%, default correlation increases linearly from -0.11 to +1.00, as shown in Exhibit 11 (left).

Exhibit 12. Default Correlation, Joint Default Probability, and Underlying Default Probability



Theoretically, correlation can range from -1.00 to +1.00. But for binomial events like default, the range of possible default correlations is dictated by the default probabilities of the two credits. With 10% probability of default for both credits, the possible range of default correlation is reduced to the range from -0.11 to +1.00. If both credits do not have the same default probability, they cannot have +1.00 default correlation.

Only if the default probability of both credits were 50% would it be mathematically possible for default correlation to range fully from -1.00 to +1.00.

Exhibit 12 (above) shows the relationship between default correlation and joint default probability when the individual default probability of both credits is 50%, when the individual default probability of both credits is 10%, and when the individual default probabilities of credits are 10% and 50%, respectively.

Note that as described, default correlation in the case where the default probability of both credits is 50% ranges from +1.00 to -1.00. Also, note the slope of the two lines. The same increase in default correlation has a bigger effect on the joint probability of default when individual default probabilities are 50% than when individual default probabilities are 10%.

We will see later that Equation 5 allows us to calculate historic default correlations from empirical default data. But right now, we rearrange Equation 5 to solve for the joint probability of default so we can calculate the joint default probability of A and B given their individual probabilities of default and their default correlation.

$$P(A \text{ and } B) =$$

$$\text{Correlation}(A \text{ and } B) * \{P(A) * [1 - P(A)]\}^{1/2} * \{P(B) * [1 - P(B)]\}^{1/2} + P(A) * P(B) \quad (6)$$

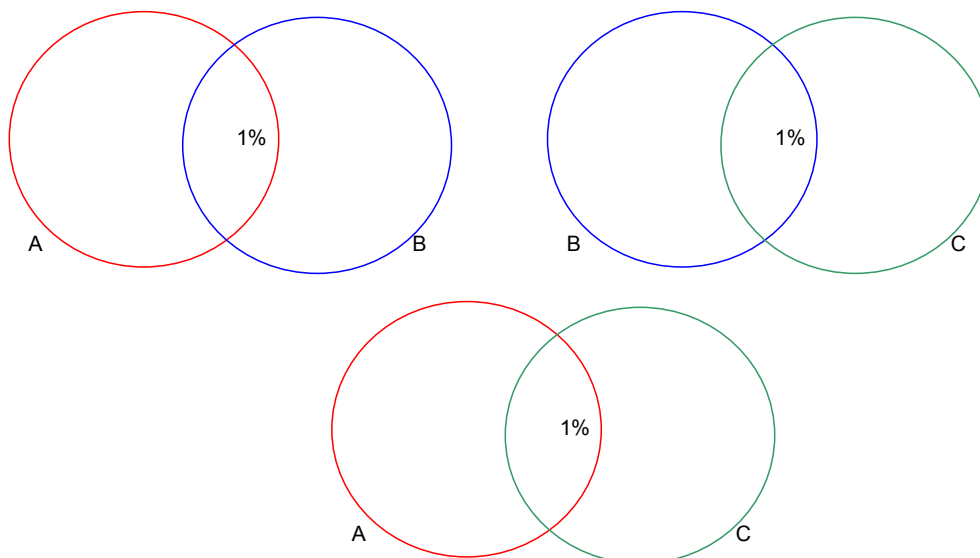
Default Correlation in a Ménage à Trois

As this point, readers who are familiar with the concept of correlation for continuous variables like stock returns or interest rates are apt to find some surprises. We have already seen, in Exhibit 11, how the range of default correlation can be restricted. But perhaps you are used to looking at portfolio risk in the context of Harry Markowitz's portfolio theory and variance-covariance matrices.³ In that framework, if you know the standard deviation of each variable, and the correlation of each pair of variables, you can explain the behavior of the entire portfolio. Not so with a binomial variable like default. We illustrate the difference in this section.

Instead of the two-credit world we have been focused on, let's suppose we have three credits, A, B, and C, each with a 10% probability of default. Let us suppose that the default correlation between each pair of credits is 0.00. As we have discussed before, around Exhibit 5, this means that the joint probability between each pair of credits is 1%. We illustrate this situation in Exhibit 13 (below).

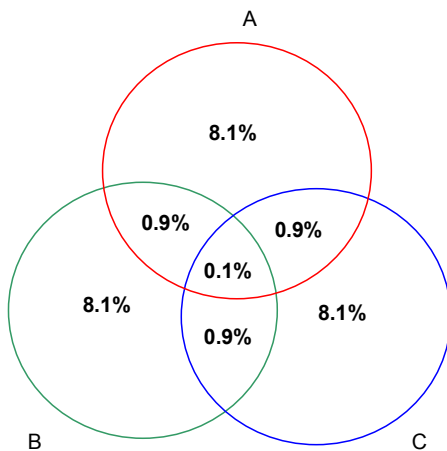
Now we are eager to understand the behavior of all three credits together. We seem to have a lot of information: each credit's default probability and the default correlation between each pair of credits. What does this tell us about how defaults will occur among all three credits? Not much, it turns out. Many people would jump to the conclusion that if the pairs AB, BC, and AC all are zero

Exhibit 13. Pairwise Joint Default Correlation



³ H.M. Markowitz, "Portfolio Selection," *Journal of Finance*, March 1952. But, really any introductory finance textbook.

Exhibit 14. Zero Pairwise and Zero Triplet Default Correlation



default correlated, the default correlations between the *pair* AB and the single credit C, or the *pair* BC and the single credit A, or the *pair* AC and B must also all be zero default correlated. Since the zero default correlation joint default probability of any pair is 10% * 10% or 1%, the zero default correlation triplet joint probability of default is 1% * 10% or 0.1%. In general, the triple joint default probability assuming zero pairwise and zero triplet default correlation is:

$$\begin{aligned} P(A \text{ and } B \text{ and } C) &= P(A) * P(B) * P(C) \\ &= 10\% * 10\% * 10\% \\ &= 0.1\% \end{aligned}$$

Once we know that $P(A \text{ and } B \text{ and } C) = 0.1\%$, we can figure out that $P(A \text{ and } B \text{ not } C)$ is 0.9%, and that $P(A \text{ not } B \text{ not } C)$ is 8.1%. This is illustrated pictorially in Exhibit 14 (above). Exhibit 15 (below) shows the probabilities of all possible default outcomes under the heading “0.00 Triplet Default Correlation.”

But there is no reason why just because *pairs* of credits have zero default correlation that the

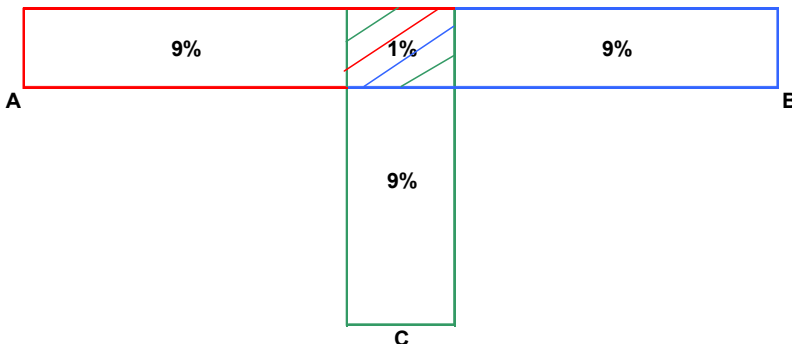
Exhibit 15. Default Probabilities Under 0.00 Pairwise Default Correlation and Various Triplet Default Correlation

Number of Defaults	-0.03 Triplet Default Correlation	0.00 Triplet Default Correlation	0.30 Triplet Default Correlation
0	73.0%	72.9%	72.0%
1	24.0%	24.3%	27.0%
2	3.0%	2.7%	0.0%
3	0.0%	0.1%	1.0%

default correlation between a *pair* and a *third* credit must also be zero. Exhibit 16 (bottom, left) and 17 (next page) show the extremes of possible correlation. (Note the switch from circles to rectangles and ovals in these Venn diagrams to show the overlapping probabilities clearly.)

In Exhibit 16, whenever two credits default, the third credit joins them in default and there is no situation where only *two* credits default. Exhibit 15 shows the probabilities of all possible default outcomes under the heading “0.30 Triplet Default Correlation.” There is a 1% probability that all three credits default, 0% probability that two credits default, 27% probability that one credit will default and 72% probability that no credits will default.

Exhibit 16. Zero Pairwise and Positive Triplet Default Correlation



This sounds like positive default correlation: if you know that any two credits have defaulted, your estimate of the default probability of the third credit increases from 10% to 100%. We can solve for the triplet default correlation by treating the default of A and B as *one* event and comparing that event to the default of C. Using Equation 5, and substitution in AB for A and C for B we have:

$$\begin{aligned}
 \text{Correlation(AB and C)} &= \frac{P(\text{AB and C}) - P(\text{AB}) * P(\text{C})}{\{P(\text{AB}) * [1 - P(\text{AB})]\}^{1/2} * \{P(\text{C}) * [1 - P(\text{C})]\}^{1/2}} \\
 &= \frac{1\% - 1\% * 10\%}{\{1\% * [1 - 1\%]\}^{1/2} * \{10\% * [1 - 10\%]\}^{1/2}} \\
 &= \frac{0.90\%}{2.98\%} \\
 &= 0.30
 \end{aligned}$$

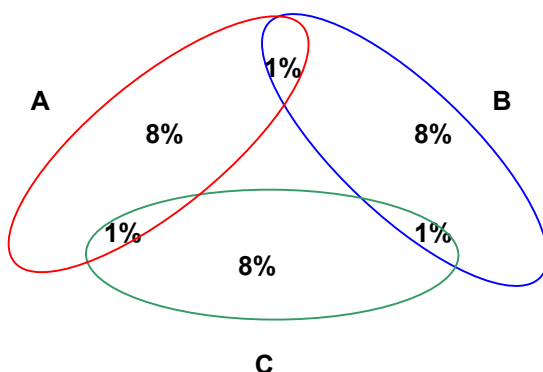
But this triplet default correlation of 0.30 occurs while all *pairwise* default correlations are zero.

In Exhibit 16, in contrast, there is no situation where all *three* credits default. In this case, if you know that two credits have defaulted, your estimate of the default probability of the third credit decreases from 10% to 0%. This sounds like negative default correlation. In this situation, the triplet default correlation is -0.03. Exhibit 17 shows the probabilities of all possible default outcomes under the heading “-0.03 Triplet Default Correlation.” There is a 0% probability that all three credits default, 3% probability that two credits default, 24% probability that one credit will default and 73% probability that no credits will default.

Note that the expected number of defaults in each triplet correlation scenario is the same. In the zero triplet correlation scenario, the expected number of defaults is $24.3\% * 1 + 2.7\% * 2 + 0.1\% * 3$ or 0.3. In the positive triplet correlation scenario, the expected number of defaults in the

portfolio is $27\% * 1 + 1\% * 3$ or 0.3. In the negative triplet correlation scenario, the expected number of defaults in the portfolio is $24\% * 1 + 3\% * 2$ or also 0.3.

Exhibit 17. Zero Pairwise and Negative Triplet Default Correlation



Note also that the probability of any two credits defaulting at the same time in any of the triplet default scenarios is 1%. In the -0.03 triplet correlation scenario, the 3.0% probability of two defaults divides into a 1% probability of any pair of credits defaulting. In the positive triplet correlation scenario, the probability of *all* three credits defaulting at the same time is 1%. Which means that the probability of each possible pair of credits defaulting is

Exhibit 18. Range of Triplet Default Correlation

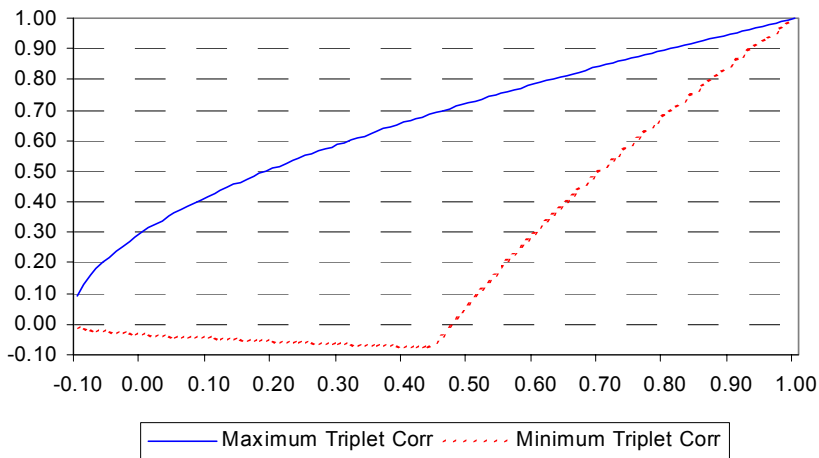


Exhibit 19. Default Probabilities with Minimum Triplet Default Correlation

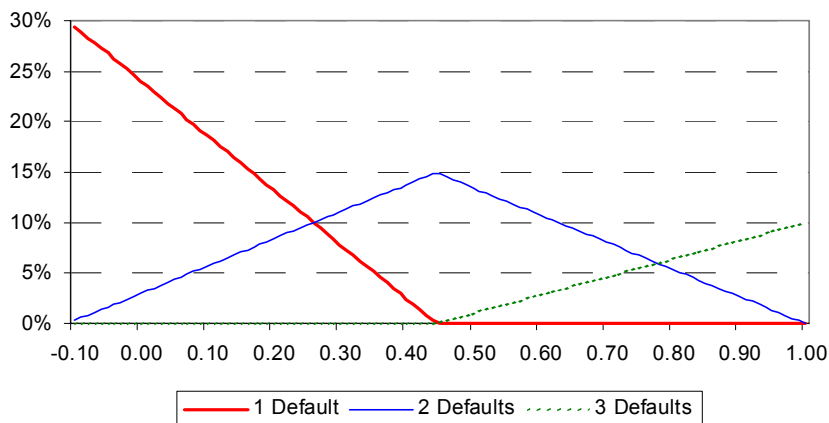
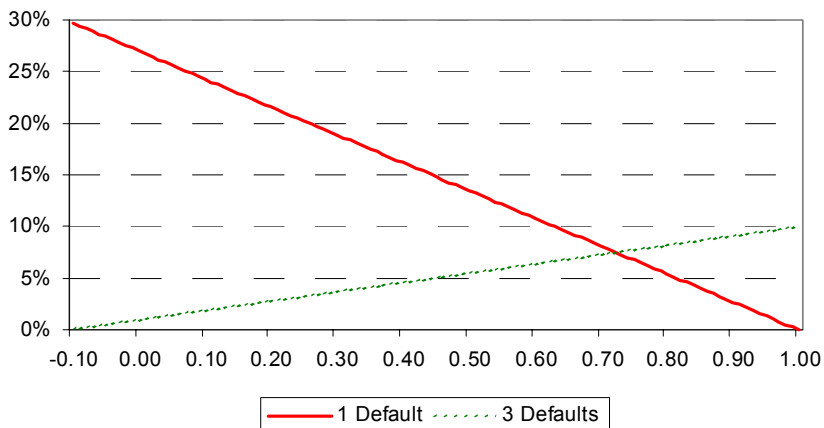


Exhibit 20. Default Probabilities With Maximum Triplet Default Correlation



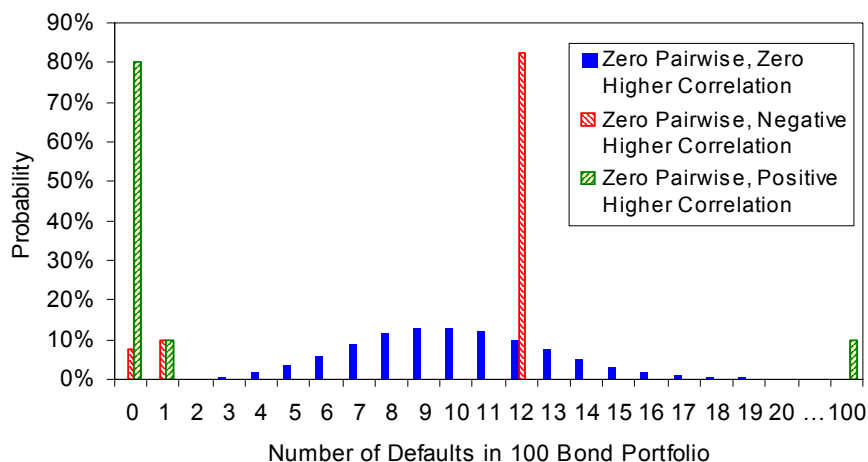
also 1%. In the zero triplet correlation scenario, the 2.7% probability of two defaults divides into a 0.9% probability of any pair of credits defaulting. Also in the 0.00 triplet correlation scenario, the probability of *all* three credits defaulting at the same time is 0.1%. Which adds another 0.1% of probability and brings the total probability of any pair of credits defaulting to 1.0%.

So in all three triplet-correlation scenarios, the defaults of pairs AB, BC, and AC each have a 1% chance of occurring. This is proof that pairwise default correlation is 0.00. But the sad truth is that knowing pairwise default correlations does not tell you everything you would like to know about the behavior of this three credit portfolio. This makes default correlation computationally very difficult.

Pairwise and Triplet Default Correlation

In Exhibit 18 (left) we show the range of triplet default correlation for the whole range of pairwise default correlation, given that the default probability of each of the three credits is 10%. That is, for any point on the x axis giving a possible pairwise default correlation, we show the *minimum* triplet default correlation can be and the *maximum* triplet default correlation can be.

The effects of varying triplet default correlation are shown in Exhibits 19 and 20 (left). In Exhibit 19, we show the probabilities of

Exhibit 21. Default Probabilities in a 100 Credit Portfolio

one, two, and three defaults given triplet default correlation is as *low* as it can be (given pairwise default correlation). In Exhibit 20, we show the probabilities of one, two, and three defaults given triplet default correlation is as *high* as it can be (given pairwise default correlation).

Comparing Exhibits 19 and 20, the probability of extreme default results is greater with maximum triplet default correlation than it is in the minimum triplet default correlation case. In Exhibit 19, two defaults is the most probable outcome under a wide range of pairwise default correlations. In Exhibit 20, two defaults *never* occur. This harkens back to Exhibits 2 and 3, where there was a wide range of default results with positive default correlation and a narrow range of default results with negative default correlation. Given pairwise default correlation, low triplet default correlation works to create a stable number of defaults and high triplet default correlation works to create a wide range in the number of defaults.

Group Default Correlation

We wanted to make sure that higher orders of default correlation were also important for large portfolios. So we consider a 100-credit portfolio where each credit has a 10% probability of default. We computed the probabilities of zero to 100 credits defaulting under three correlation scenarios:

- zero pairwise default correlation and zero higher correlations;
- zero pairwise default correlation and maximum negative higher correlations;
- zero pairwise default correlation and maximum positive higher correlations.

The results are shown in Exhibit 21 (above) and show the extreme distribution of the positive higher correlation portfolio and the very stable distribution of the negative higher correlation portfolio relative to the zero higher correlation portfolio. Again, our conclusion is that pairwise default correlations do not give us all the information we need to understand the behavior of a portfolio.

Intuitively, increasing higher-level default correlation seems logical. Assuming that positive pairwise default correlation exists, the first default in the portfolio will cause us to revise our estimation of the default probability of remaining credits in the portfolio upwards. It seems logical that if a *second* credit defaults, we would want to again revise our estimation of the default probabilities of

remaining credits upwards. This is the effect of higher order positive default correlation. As more and more credits default, we think it more likely that remaining credits will also default.

This ends our more theoretical discussion of default correlation. In the next part of this paper, we explore the determination of historical default correlations, and the problems inherent in empirical default correlation.

Empirical Results and Problems

With enough data, and one very strong assumption that we will discuss in detail later, we can calculate *historic* default correlations. We go back to Equation 5 (page 10):

$$\text{Correlation(A and B)} = \frac{P(\text{A and B}) - P(\text{A}) * P(\text{B})}{\{P(\text{A}) * [1 - P(\text{A})]\}^{1/2} * \{P(\text{B}) * [1 - P(\text{B})]\}^{1/2}} \quad (5)$$

To compute, say, the default correlation of two B-rated companies over one year, we set P(A) and P(B) equal to the historic average one-year default rate for B-rated companies. The remaining variable on the right side of Equation 5 is the joint probability of default, P(A and B). We compute P(A and B) by first counting the number of companies rated B at the beginning of a year that subsequently defaulted over that particular year. We then calculate all possible *pairs* of such defaulting B-rated companies. If X is the number of B-rated companies defaulting in a year, the possible pairs are:

$$\frac{X * (X - 1)}{2}$$

We next calculate all possible pairs of B-rated companies, whether or not they defaulted, using the same formula, $[Y * (Y - 1)]/2$, where Y is the number of B-rated companies *available* to default. The joint default probability of B-rated companies in a particular year is:

$$\frac{[X * (X - 1)] / 2}{[Y * (Y - 1)] / 2} \quad (6)$$

The average of this statistic is taken over available years in the dataset to determine P(A and B). Now, having all the terms on the right hand of Equation 5, we can solve for the default correlation between two B-rated credits A and B. In a similar manner, it is possible to calculate default correlations over longer periods and between groups of credits of different ratings, for example the default correlation between Aa and Ba credits over five years.

In our 1995 journal article on default correlation, we computed default correlations between all combinations of Moody's rating categories for time periods from one to ten years. The data we used included 24 years of default data covering the years 1970 through 1993, including industrial companies, utilities, financial institutions, and sovereign issuers. We reproduce this in Exhibit 22 (next page).

Exhibit 22. Historic Default Correlations

	Aaa	Aa	A	Baa	Ba	B
One-Year Default Correlations * 100						
Aaa	0					
Aa	0	0				
A	0	0	0			
Baa	0	0	0	0		
Ba	0	0	0	0	2	
B	0	1	0	1	4	7
Two-Year Default Correlations * 100						
Aaa	0					
Aa	0	0				
A	0	0	0			
Baa	0	0	0	0		
Ba	0	1	1	1	6	
B	0	1	1	2	10	16
Three-Year Default Correlations * 100						
Aaa	0					
Aa	0	0				
A	0	1	1			
Baa	0	0	0	0		
Ba	0	2	2	1	9	
B	0	2	3	3	17	22
Four-Year Default Correlations * 100						
Aaa	0					
Aa	0	0				
A	0	1	1			
Baa	0	1	1	0		
Ba	0	2	3	3	13	
B	0	2	4	5	22	27
Five-Year Default Correlations * 100						
Aaa	0					
Aa	0	0				
A	0	1	1			
Baa	0	1	1	0		
Ba	0	3	4	3	15	
B	0	4	6	7	25	29
Six-Year Default Correlations * 100						
Aaa	0					
Aa	1	1				
A	1	1	1			
Baa	0	1	1	0		
Ba	1	3	4	3	15	
B	1	4	7	7	25	29
Seven-Year Default Correlations * 100						
Aaa	0					
Aa	1	0				
A	1	1	2			
Baa	0	1	1	0		
Ba	2	2	4	3	13	
B	3	3	9	8	24	30
Eight-Year Default Correlations * 100						
Aaa	1					
Aa	1	0				
A	1	1	2			
Baa	1	1	1	0		
Ba	3	3	5	2	10	
B	6	5	11	7	23	37
Nine-Year Default Correlations * 100						
Aaa	1					
Aa	1	0				
A	2	1	2			
Baa	2	1	1	0		
Ba	4	3	5	2	8	
B	8	6	12	6	20	39
Ten-Year Default Correlations * 100						
Aaa	1					
Aa	2	1				
A	2	2	2			
Baa	2	1	1	0		
Ba	4	3	4	2	8	
B	9	8	9	6	17	38

Our conclusions from studying the results in Exhibit 22 are:

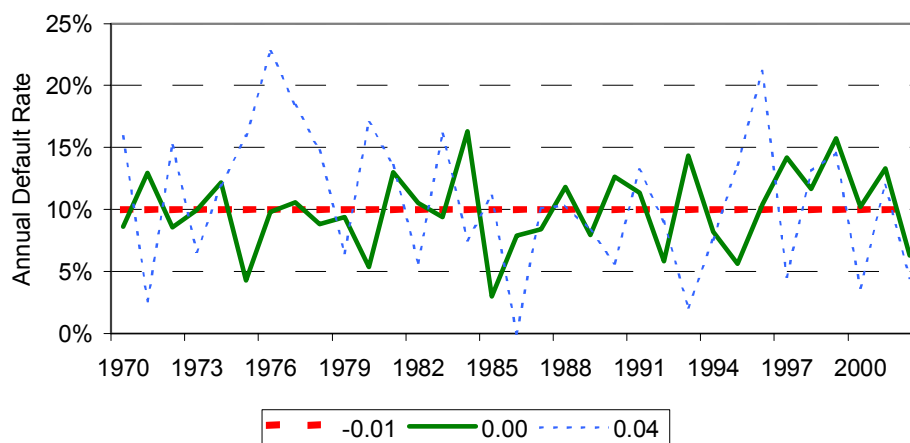
- default correlations increase as ratings decrease;
- default correlations initially increase with time and then decrease with time.

We guess that default correlations increase as ratings decrease because lower-rated companies are relatively more susceptible to problems in the *general economy* while higher-rated companies are relatively more susceptible to *company-specific* problems. Low-rated companies, being closer to default already, are more likely to be pushed into default because of an economic downturn. As economic conditions affect all low-rated credits simultaneously, defaults among these credits are likely to be correlated. In contrast, defaults of highly-rated companies, besides being rare, are typically the result of company-specific problems. As these problems are by definition isolated to individual credits, they do not produce default correlation.

With respect to default correlation increasing and then decreasing with the time period studied, we note that default correlations peak at five and six year periods for rating pairs Baa/Ba, Baa/B, Ba/Ba, and Ba/B. However, default correlations peak at nine years for rating pairs A/B and B/B.

We note that over arbitrarily short time periods, defaults are necessarily uncorrelated. Imagine a database whose column headings are the names of credits and whose rows represent time intervals, perhaps four-year intervals. The entry in a particular cell is 1 if the credit defaulted in that time interval and 0 otherwise. Ones in the same row indicate that credits defaulted together in that time interval and indicate the presence of positive default correlation. But if a shorter time period is used, fewer ones will appear in the

Exhibit 23. Simulated Annual Default Rates Under Different Default Correlations



same row, lowering perceived default correlation. At some arbitrarily short period of time, no more than one 1 will appear in a row and there will be no evidence of positive default correlation.

The decrease in default correlation that occurs in most rating categories over longer time periods may be caused by the relationship of the time period being studied to the average business cycle. If the time period studied covers the entire ebb and flow of the business cycle, defaults caused by general economic conditions average out over the period, thus lowering default correlation. We think that default correlation is maximized when the time period tested most closely approximates the length of an economic recession or expansion.

Just as the pairwise default correlations in Exhibit 22 can be calculated, so to can higher order default correlations. But rather than demonstrate this, we instead turn to a discussion of the reliability of empirically-observed default correlations.

Problems with Historical Default Correlations

Implicit in the previous section on empirical default correlation is the idea that wide swings in default rates are indicative of positive default correlation while small swings or steady default rates are indicative of low or even negative default correlation. We illustrate this concept explicitly in Exhibit 23 (above).

Exhibit 23 depicts annual default rates for three 100-credit portfolios assuming 10% default probability per year for each credit and pairwise default correlations of -0.01, 0.00, and 0.04, respectively. The thin line, steady at exactly 10%, is produced with perfect negative default correlation, in this case -0.01. The thick line that ranges between 3% and 16% was produced with 0.00 default correlation. Finally, the most volatile series, the dotted line, which varies between 0% and 23%, was produced with default correlation of 0.04. This shows that a little bit of default correlation can cause great swings in experienced defaults.

However, the default rates of the most volatile series in Exhibit 23 could have been produced by *varying default probability* instead of default correlation. Suppose that over the time period

shown in Exhibit 23, annual default probability averaged 10%, but varied from year to year. For example, maybe in 1976 the default probability of credits in the portfolio was 22% and in 1986 it was 1%. In this case, high and low experienced default rates are caused by varying default probability, not positive default correlation. In any particular year, given that year's specific default probability, default correlation could be zero.

For another perspective on our inability to distinguish varying default probability from default correlation, consider Exhibit 1 again. We said the variability in annual corporate default rates since 1920 was evidence of default correlation. Our implicit assumption was that the long-term average of the series, 1%, was the year-in and year-out annual default probability. Of course, we don't directly observe default probability, we only observe default results. But it seems logical that credit analysts in 1934 and 1952 would have had vastly different expectations of future defaults. Put another way, their respective estimates of U.S. corporate default probability would have been very different.

The assumption in calculating default correlation is that default probability is constant for each rating class. This turns out to be unsupportable. Varying default probability, a simple and plausible alternative explanation of fluctuating default rates, puts into question all our work deriving empirical default correlations in the previous part of this section. In fact, it puts into question all consideration of default correlation. We cannot be sure whether the variability in default rates from year to year or over longer periods is due to default correlation or changing default probability.

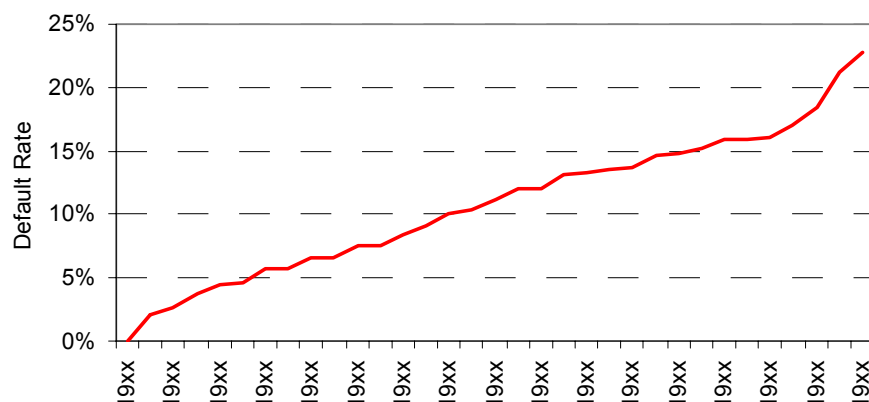
In fact, pragmatic scrutiny of credit ratings and the credit rating process suggests to us that ratings are more *relative* than *absolute* measures of default probability and that default probabilities for different rating categories change year-to-year. It is a hard enough job to arrange credits in an industry in relative order of credit quality. It seems to us very difficult to assess credit quality against an absolute measure like default probability and then calibrate this measure across different industries. In fact, the rating agencies themselves say that ratings are relative measures of credit quality.⁴

If ratings are relative measures of credit quality, or if for any reason the probabilities of default for different rating categories change over time, this would mean that the historically-derived default correlations presented in Exhibit 22 are based on an inaccurate assumption and overstate true default correlation. But more importantly, default correlation is just not the right way to look at or think about experienced default rates.

Another perspective on the idea of varying default rates is shown in Exhibit 24 (next page). Here we have rearranged the annual default rates of the positively correlated series in Exhibit 23 so that the default rates are in strict order from lowest to highest. In the calculation of default correlation, assuming a constant 10% annual default probability, the *order* of default rates does not make a difference. This series would still have default correlation of 0.04.

On average, it is true that the annual default rate is 10%. But looking at this time series, some simple rules to explain and predict default rates present themselves. First, "defaults this year will

⁴ Jerome Fons, Richard Cantor, and Christopher Mahoney, *Understanding Moody's Corporate Bond Ratings and Rating Process*, Moody's Investors Service, May 2002.

Exhibit 24. Time Correlated Default Rates

be what they were last year.” Second, “defaults this year will be what they were last year *plus* the change in default rate between last year and the previous year.” Yet, our method of calculating default correlation would not pick up the “memory,” or *time series correlation*, of default rates. This suggests another type of correlation, along the dimension of time, which also seems important to our understanding of defaults.

The indistinctiveness of default correlation and changing default probability will drive our conclusions as we assess different default correlation methodologies in the next section of this paper.

Proposed Solutions

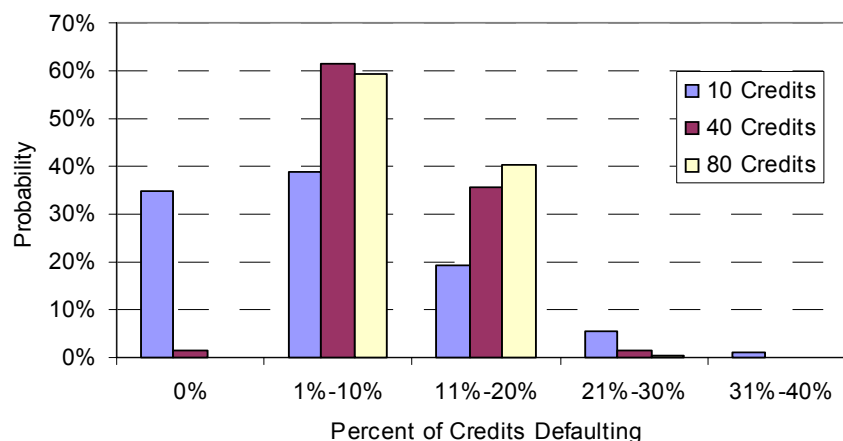
Given the importance of default correlation in evaluating credit portfolios, it is not surprising that a lot of effort has been given to incorporating default correlation into the analysis of credit risky portfolios. The goal of various approaches is to create default probability distributions that accurately depict the effect of default correlation upon a credit portfolio. From the default distribution and assumptions about loss in the event of default, one can determine required economic capital against a credit portfolio or the credit risk of a collateralized debt obligation (CDO) tranche.

The default correlation solutions we will highlight in the next several pages take very different approaches to the problem. We find no single approach completely satisfying, but certain solutions have strengths in certain applications. We present this survey of default correlation methodologies to help our readers understand the comparative advantages and disadvantages of different methodologies and develop an appreciative, but skeptical, view to them all.

Single Name and Industry Limits

The effect of default correlation is not a new discovery, despite the new technologies brought to bear on the challenge. Often, the issue of default correlation is discussed and expressed in terms of portfolio *diversity*. Banks and other fixed income investors have an incentive to create low default correlated, or diverse, portfolios. Investors want loss distributions that are more stable rather than distributions that experience wide swings. For example, from the point of view of capital adequacy, a commercial bank with a portfolio more like Exhibit 3 will require less capital than a bank with a loan portfolio more like Exhibit 2. This is because the potential for large credit losses is lower in a

Exhibit 25. Illustration of Law of Large Numbers



less positively default-correlated portfolio. Ideally, a bank would prefer the stable default distribution of the extremely negatively default correlated portfolio in Exhibit 3. With that portfolio, future defaults and required capital are known with certainty.

One tool that has been used through the ages to manage default correlation and create less volatile default losses is *exposure or concentration limits*. Recall that we discussed industry-specific factors as a cause of defaults and default correlation and cited examples in the oil, thrift, and dotcom industries. The rationale behind industry exposure limits in credit portfolios is that credits within a particular industry are more default-correlated than credits in different industries. Beside industry limits, credit portfolios might have risk limits on obligors from specific countries.

Credit portfolios also have *single name limits*. Technically, this has nothing to do with default correlation but with *portfolio diversity* or the “*law of large numbers*.” Simply put, the more individual credits there are in the portfolio, the more likely it is that the portfolio’s actual credit results will equal the theoretical expectation.⁵ Exhibit 25 (above) shows the probability of defaults in three portfolios comprised of 10, 40, and 80 credits, respectively. We assume a 10% default probability for each credit in the respective portfolios and zero default correlation. The exhibit shows the probability of 0% of the credits defaulting, as well as the probability of defaults in the ranges of 1-10%, 11-20%, 21-30%, and 31-40% of credits in the respective portfolios. In the 10-credit portfolio, the range of portfolio defaults is from 0% to 40% of credits in that portfolio. In the 40-credit portfolio, the range of portfolio defaults is from 0% to 30%. Finally, in the 80-credit portfolio, almost all probability is encompassed between 1% and 20% of the portfolio defaulting. This demonstrates that the more credits added to a portfolio, the more stable its potential outcomes become.

The problem with single name and industry limits is that there is no way to determine optimum levels. Diversity is good, but how much is enough? This is a relevant question because diversity is not free. Single name and industry limits require that more individual credits in more industries be

⁵ This is true as long as the portfolio does not have complete positive correlation, which is to say it is almost always true.

included in the portfolio. The cost of reviewing and monitoring credits is expensive. Also, the search for a more diverse portfolio might lead to one that has lower credit quality, resulting in more defaults and greater default severity. Meanwhile, as more and more credits are added to the portfolio, the diversification benefit of adding still more credits diminishes, i.e., the benefit of going from 50 to 60 credits is far less than the benefit of going from 10 to 20. At some point, the cost of diversity exceeds the benefits of diversity. Also, the trade off between single name, industry, and other limits is unquantified. Is it better to have relatively lower single name limits and relatively higher industry limits or the opposite?

Rating agencies began to consider portfolio diversity explicitly when they rated CDOs in the mid-1980s. These are special purpose investment vehicles that issue various seniorities of debt backed by a portfolio of corporate debt. Back then, the rating agencies controlled default correlation risk in CDO portfolios by setting strict limits on industry and single name concentrations. The rating agencies simply refused to rate any CDO that did not comply with their standards. However, the rating agencies also did not give credit to a CDO for having additional industry or single name diversity. The issue of the costs and benefits of industry and single name limits took on more prominence as the CDO market grew in the late 1980s.

Exhibit 26. Moody's Diversity Score Calculation

Number of Credits in Same Industry	Diversity Points
1	1.0
2	1.5
3	2.0
4	2.33
5	2.67
6	3.0
7	3.25
8	3.50
9	3.75
10	4.0
11	4.2
12	4.4
13	4.6
14	4.8
15	5.0

Moody's Diversity Score

In 1989, Moody's struggled to handle default correlation and diversity in the rating of CDOs and developed some simple ad hoc rules.⁶ To handle un-diversifiable default correlation due to general economic conditions, Moody's stressed historic default rates. For example, Moody's first corporate bond default study, also completed in 1989, had just calculated the historic 10-year B2 default rate to be 29.3%. However, when assessing B2 bonds in a CDO portfolio, the rating agency assumed a 37.9% 10-year default rate, reflecting the average historic default rate *plus* two standard deviations based on the historic volatility of the ten-year default rate.

To assess single name and industry concentrations in a CDO portfolio, Moody's developed a single index measurement it christened "Diversity Score." The measure explicitly quantified the trade-off between industry diversity and single name diversity in CDO portfolios.

Moody's divided the economy into 32 industries. As shown in Exhibit 26 (left), the first name in any industry earned a CDO one diversity point, the next two in the same industry earned the CDO 1/2 a point each, the next three 1/3 a point each, the next four 1/4 a point each, and finally the next five after that, 1/5 a diversity point each. The CDO's Diversity Score was the sum of all the points accumulated in each of the industries represented in the CDO portfolio.

⁶ Douglas Lucas, *Rating Cash Flow Transactions Backed by Corporate Debt*, Moody's Investors Service, September, 1989.

folio. So if a CDO had three credits in each of 10 industries, by Exhibit 26 the CDO earned 2.0 diversity points in each of the 10 industries for a Diversity Score of 20. Another ad hoc formulaic adjustment was made to adjust for uneven par amounts in the CDO portfolio.

The idea behind the Diversity Score was that a large number of default-correlated credits behave like a smaller number of uncorrelated credits. In modeling the default distribution of a CDO collateral portfolio, a portfolio with 80 correlated credits with a Diversity Score of 40 would be evaluated as if contained only 40 uncorrelated credits. Intuitively, this is the same as saying that credits in the portfolio always default in pairs. Exhibit 25 provides a graphical representation. Instead of the relatively narrow default distribution of the 80-credit portfolio, the CDO collateral portfolio would be considered to have the wider default distribution of the 40-credit portfolio in that exhibit. The wider distribution of defaults would require the CDO to have more credit enhancement to issue its debts, all other things equal.

Moody's Diversity Score became an obligatory concept in CDOs because it was part of Moody's CDO rating methodology. However, the Diversity Score also obtained recognition outside the area of CDOs when applied to other credit risk portfolios. Its appeal was that it explicitly quantified trade-offs between default correlation caused by the general economy, default correlation caused by industry factors, and the effects of single name diversity. But while these trade-offs were explicitly quantified, they lacked any theoretical or empirical justification. Their appropriateness relied on an intuitive grasp of very unintuitive questions, e.g., is a portfolio of 30 names in 30 industries as diversified as a portfolio of 60 names in 10 industries? Should CDOs of these portfolios be required to have the same credit enhancement, all other things equal? For these reasons, some quantification of default correlation was required.

CSFB's Changing Default Probability Model

Analysts at CSFB made good use of the insight discussed in "Empirical Results and Problems" that there is no objective distinction between changing default probability and default correlation. Their method of incorporating default correlation into credit modeling was to change default probabilities and assume zero default correlation.⁷ An illustration will help make their approach clearer.

Let's assume we have a 10-credit portfolio comprised of high risk loans that we believe have a 10% annual probability of default. Our belief in the loan's default probability springs from the fact that over the last 20 years, loans like these have defaulted at an average annual rate of 10%. However, it turns out that there is great variability in their annual default rate, as shown in Exhibit 27 (next page).

As we have said, we don't know whether volatility of annual default rates stems from default correlation or from changes in default probability from one year to the next. The CSFB approach assumes that each annual default rate reflects that year's default probability for these types of loans. Given that assumption, what are the "probabilities of annual loan default probabilities?"

⁷ The widely distributed and annotated approach was first documented in Tom Wilde, *CreditRisk+: A Credit Risk Management Framework*, Credit Suisse First Boston, 1997, available at www.csfb.com/creditrisk/.

Exhibit 27. Hypothetical Annual Default Rates of High Risk Loans

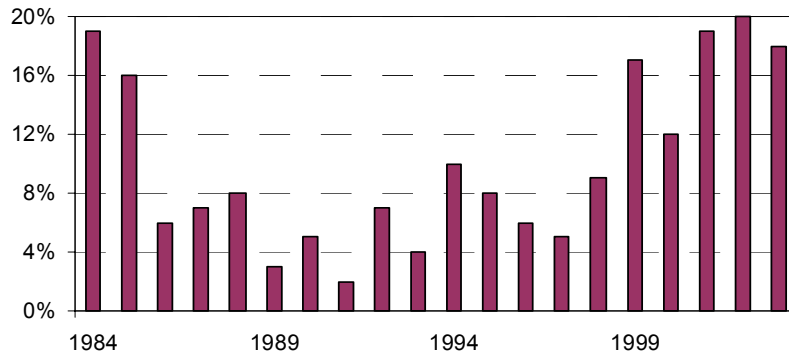


Exhibit 28. Historic Probability of Annual Default Rates

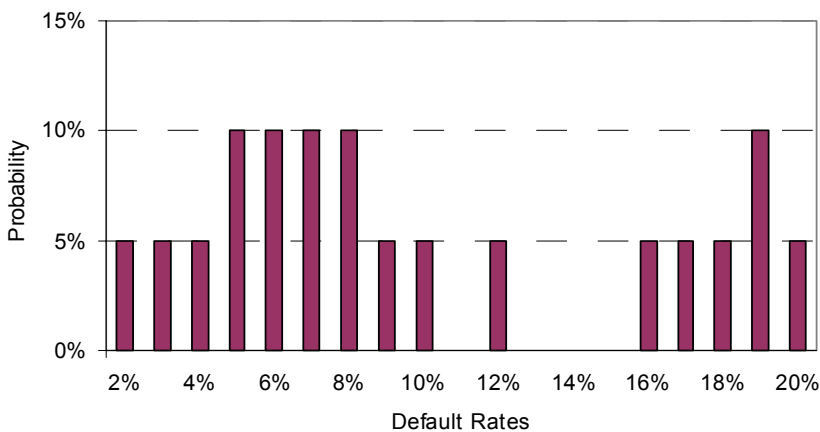


Exhibit 29. Defaults Given 2% Default Probability

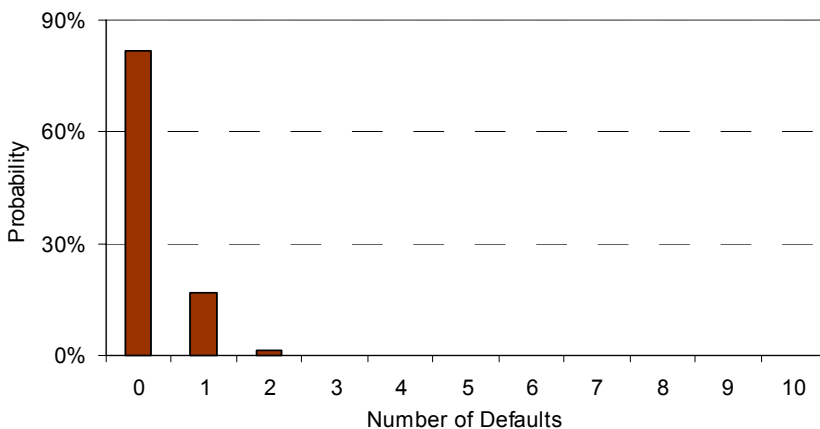


Exhibit 28 (middle, left) reassembles the default rates of Exhibit 27 into a graph showing the historic likelihood of any specific annual default rate. For example, as shown in Exhibit 28, there is a 5% probability of a 2% default probability, a 5% probability of a 3% default probability, and so on.

We would like to determine the default probability distribution of a 10-credit loan portfolio. From year to year, we believe the default probability of these credits ranges from 2% to 20%. When the probability of default is 2%, the probability of different numbers of loan defaults is as shown in Exhibit 29. The default probability distribution in Exhibit 29 (bottom, left) is the result of the 2% default probability, 10 credits in the portfolio, and the assumption of zero default correlation among the credits. As we pointed out in discussing Exhibit 25, if there were more credits in the portfolio, the distribution would be tighter. In contrast to the assumption in Exhibit 29, Exhibit 30 (next page) shows the default probability distribution given that annual loan default probability is 20%. Naturally, the probability of more defaults in the 10-credit loan portfolio is higher.

Given that annual default probability is distributed as shown in Exhibit 28, what is the default probability distribution allowing

Exhibit 30. Defaults Given 20% Default Probability

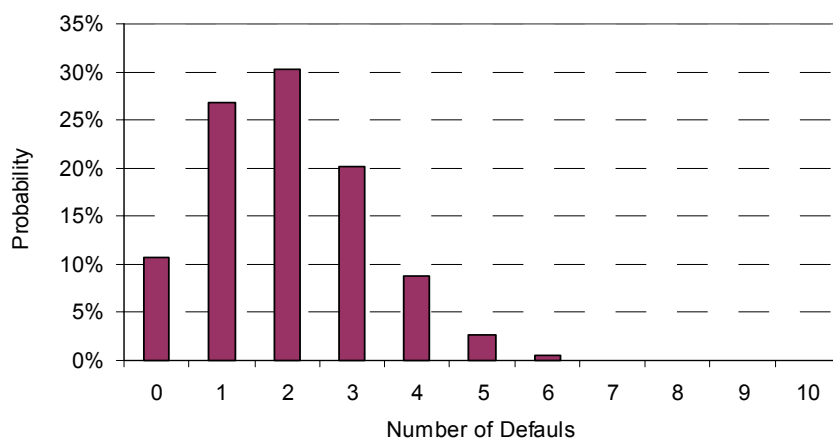
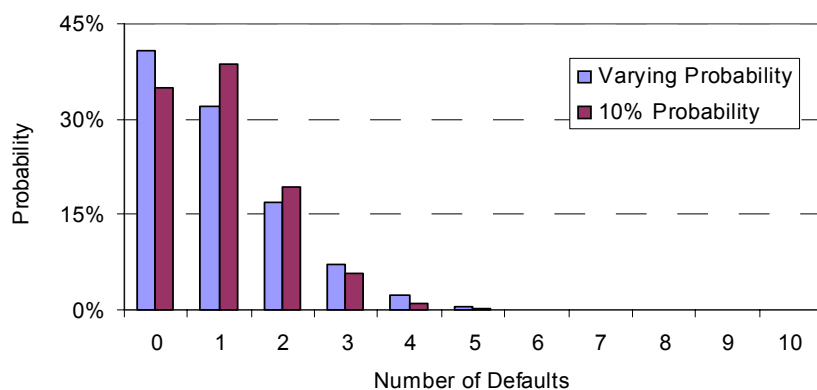


Exhibit 31. Static vs. Varying Default Probability



annual default probability to vary? This is shown in Exhibit 31 (left). For comparison, we show the default probability distribution assuming a constant 10% annual default rate. Varying default probability creates a wider default distribution than assuming a static 10% default probability. This is exactly what we said was the effect of default correlation.

The analysts at CSFB confronted the problem of default probability and default correlation in a unique manner. They avoided altogether the calculation problems associated with default correlation. However, their method is best suited for credits of homogenous quality where a long default history of similar credits is available. The method is ill suited to heterogeneous

credits or credits where a long history of default behavior is not available. Likewise, it does not differentiate between individual credits in a portfolio. All credits in the portfolio are assumed to have the same probability of default and the same default correlation with other credits. To take into account a portfolio comprised of debt with different default probabilities and different default correlations, another approach is necessary.

Merton-KMV Default Probability Model

Any Dickensian green eye-shade, quill-wielding credit analyst from the 19th century might observe that a company whose assets are less than its liabilities is insolvent. But that would not help win a Nobel Prize for Economics. To win such an accolade, one must apply to this insight the physics formula that quantifies the diffusion of heat through a solid. But since we are not in the running for a Nobel Prize of any kind, we will explain the Merton-KMV approach to estimating default probability in less mathematical terms. Consider three quantities:

1. the market value of the firm;
2. the volatility of the market value of the firm;
3. the firm's liabilities.

A firm is insolvent when its market value is less than its liabilities. Creditors will refuse to refinance its debt or advance working capital, the company will run out of cash, and it will default. A credit is less apt to experience this dire scenario (1) the greater its market value, (2) the less volatile its market value, and (3) the smaller its liabilities. Implementing Robert Merton's theoretical framework, the partners of KMV developed a model that estimates the default probabilities of corporate credits. In essence, the model employs the following formula:

$$\frac{\text{The Market Value of the Firm} - \text{The Firm's Liabilities}}{\text{The Volatility of the Market Value of the Firm}}$$

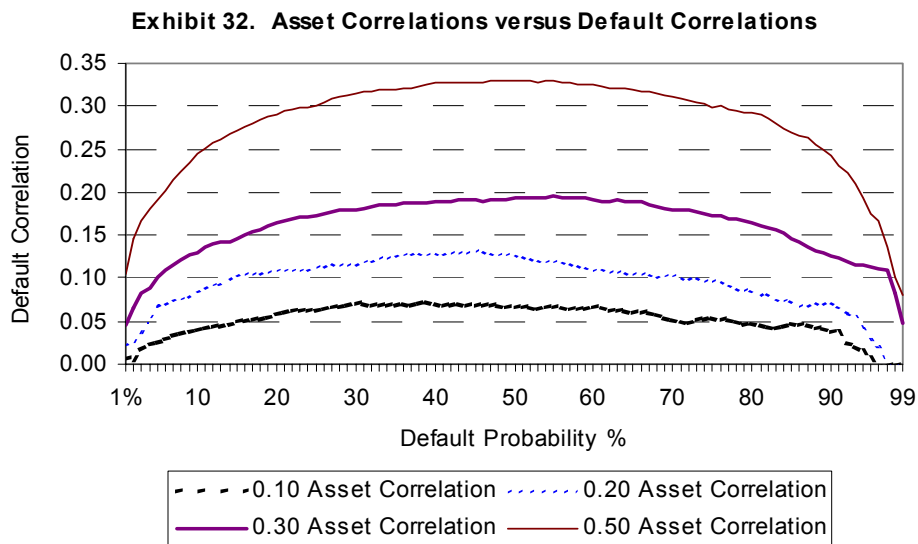
In the formula above, the numerator is the difference in the market value of the firm and the amount of the firm's liabilities. Dividing this dollar amount by the volatility (i.e., standard deviation) of market value standardizes the formula into a number that is comparable between firms. Assuming like distributions, the probability of one credit experiencing a two standard deviation decline in market value is the same as any other credit experiencing a two standard deviation decline in market value.

There are a myriad of practical obstacles to building this model to estimate relative default probability. For example, how to quantify the three variables? How to take into account that not only must the firm be technically insolvent, there must also be a *default trigger*, such as the need to refinance debt or the need for working capital, to precipitate an actual default? KMV made pragmatic decisions to increase the model's ability to rank the relative default probability of credits. For example, they found that weighting long-term debt less than short-term debt in the calculation of a firm's liabilities increased the predictive accuracy of the model. They found that the size of the firm affected the probability of default (larger firms default less often) and chose to include this variable as an adjustment to the volatility of the firm's market value. They made other adjustments to calibrate modeled default probabilities to historical default rates. In the end, the practical implementation of the theoretical model lost some of the latter's mechanistic and objective beauty, but produced a good means of assessing relative credit quality.

A bonus of KMV's default prediction model is its insight into default correlation.⁸ Remember that the default of a firm occurs when its market value sinks below the level of its liabilities. (The firm's liabilities are assumed to remain static in the model.) If one can correlate the volatility of one firm's market value to the volatility of another firm's market value, one discovers something about their default correlation.

For example, suppose the market value of Credit A and Credit B always move in the same direction; i.e., both firms' market values increase or both firms' market values decrease. This means that as the firms' market values change, those market values either both fall closer to the level of each firm's liabilities or both rise up away from the level of each firm's liabilities. If the relationship between the two firm's market values is understood, one can calculate the probability

⁸ Peter Crosbie and Jeffrey Bohn, *Modeling Default Risk*, Moody's KMV, 2002.



that the market values of *both* firms will decline below the level of their liabilities causing *both* credits to default.

This approach to default correlation has several wonderful properties. First, since a firm's market value is a continuous variable, it does not have all the problems, referred to in "Definition and Theory," handling the binomial default variable. Just as for stock returns, pairwise correlations of firm market values are sufficient to calculate a portfolio's default distribution.

When market values are simulated, the KMV model keeps track of individual credits and when defaults occur. The default results of multiple simulations of firm market value can be used to create a default probability distribution. Finally, the KMV model is credit specific. Individual credits can have different default probabilities and pairwise market value correlations can be specific to each pair of credits in the portfolio.

The KMV approach to default correlation is so entrenched that when most practitioners speak of "default correlation" they are really referring to "market value correlation." But the relationship between the two is not straightforward. As Exhibit 32 (above) shows, default correlation (on the vertical axis of the exhibit) is a function of both market value correlation and the two firm's individual default probabilities. But at any default probability, a particular market value correlation produces a much smaller default correlation between the two credits. And given any particular market value correlation, default correlation is highest at 50% individual credit default probability and lowest at 1% or 99% individual credit default probability.

Of all the variables in the KMV Merton default model, the most critical in arriving at default probability distributions is the correlation of firms' potential market values.

Modeling Firm Market Values

There are at least three methods of predicting a firms' market value volatility and the correlation of one firm's market value to another's. KMV's first method, since discarded, used an econometric model of firm value. Each firm's market value was modeled as a unique function of 18 or so

economic and financial variables, including GNP, the level of interest rates, and industry health factors. The steps in determining correlated defaults were to:

- (1) simulate a fluctuation in the econometric variables;
- (2) calculate a new market value for each of the firms from the values of the econometric variables;
- (3) determine which credits' market values had declined below their liabilities and thus defaulted;
- (4) repeat many times to form a default probability distribution.

The problem with this approach was that it depended on how well the econometric model captured firms' potential market value changes. The variables in the econometric model itself limited how credits could be market value- and thus default-correlated. With 18 econometric variables, there are only 18 ways credits can be correlated. Surprising connections between credits (such as our anchovy harvest example) cannot be captured in such a model.

For whatever reasons, KMV abandoned the econometric method and moved to a model of future firm market values that depends on *historic* relationships. KMV now simply looks at the correlation of two firms' *past* market values (which it determines as part of its default probability modeling) to predict a joint distribution of the two firm's *future* market values. This raises the question of whether these market values, which are optimized to help produce good default probability estimates in a statistical model, are also good at estimating default correlation. Competitors of KMV have argued that historic equity price correlations, which are much easier to obtain, are just as good as historic market value correlations in predicting future firm market value correlations. Equity price correlations are certainly more available.

But the larger question is whether the past relationship of two firms' market values (as calculated by KMV) or two firms' equity prices, is so indicative of their future relationship that default correlations can be determined. It seems to us that the relationship between two firms' market values is unstable and might completely change in the future, especially if one or both of them become more at risk to default.

Judging Default Correlation Methodologies

Ultimately, there is no objective way to measure the goodness of a default correlation model. First, when comparing the model's estimated default probability distribution to actual default results, we never know whether the model is wrong or whether the actual result is wrong. By a "wrong" actual result, we mean an atypical result from nature, i.e., a 100-year flood or six-sigma event.

Second, if we could determine that a default probability distribution is wrong, we can never be sure why. The distribution could be off because either the default correlations or default probabilities were wrong. We covered the inability to separate the two factors in "Empirical Results and Problems." Without the ability to objectively measure the performance of default correlation models, one has to rely on an intuitive view of their reasonableness.

Moody's Diversity Score is a good way to express and enforce diversity requirements upon a portfolio manager. It is convenient to have a single measurement that trades off industry diversity and single name diversity. Our view is that the formula over-rewards industry diversity relative to single name diversity. For example, we would rather have a portfolio of 10 firms in the same

industry rather than four firms in four different industries. Moody's Diversity Score would rank these portfolios as equally diverse. The desire of some CDO investors to place a single name exposure limit on top of a CDO's Diversity Score requirement suggests that others share our view.

That raises the question of whether one *needs* default correlation modeling to compare diversified credit portfolios. Take two portfolios, each comprised of 50 or more credits pretty evenly dispersed among 15 industries. Given this diversity, how likely is it that one portfolio is much more default correlated than another? Or, if a correlation methodology suggests that one portfolio *is* more default correlated than another, how likely is it that that analysis is correct? In comparing two diversified portfolios, we do not think it can be persuasively argued that one portfolio is better than the other because of its default correlation.

Yet, to analyze the risk of a credit portfolio, we see value in simulating default rates via the CSFB approach and creating a default probability distribution. To be most appropriate to the CSFB approach, credits in a portfolio must be homogenous, we must not be interested in tracking which individual credits default, and we must have a long default history of similar credits. Many credit portfolios generally fulfill these requirements.

On a theoretical level, we have a great deal of sympathy for the view that historical default rate fluctuations are caused by fluctuations in default probabilities rather than the workings of default correlation. Our view of rating agency ratings, internal bank rating systems, and statistical models of default probability is that they are all better indicators of *relative* default probability than *absolute* default probability. The theory behind CSFB's approach is in harmony with our experience.

We also think the CSFB approach is easy to grasp. For example, the difference between assuming that annual default probability is always 10% versus assuming that annual default probability varies according to a distribution like the one in Exhibit 31 is intuitive. One can look at various versions of Exhibit 31, with narrower or wider distributions, and easily understand what is being assumed about the volatility of default probability.

Finally, in preferring the CSFB approach over the KMV approach, we note that the efficacy of the CSFB approach could be improved by certain empirical studies. We now have long time series of annual default rates by rating thanks to the rating agencies' default studies. We also have models that predict the *absolute* level of future default rates for speculative grade bonds and loans.⁹ But we also know that credits can be grouped into more harmonious credit risk buckets by looking not only at their current rating, but also at their past ratings history, their rating outlook status, and their yield relative to that of other similarly-rated credits. Why couldn't empirical studies be conducted to determine historical default by *all* these default-predicting variables? Another avenue of study, helpful to this analysis, would be how default rates vary from one year to another and how one year's default rate is affected by the previous year's default rate. In other words, the time series auto-regressive correlation of default rates.

⁹ For example, Moody's prediction of speculative grade bond defaults and S&P LCD's prediction of speculative grade loan defaults.

If an analyst is faced with a heterogeneous portfolio or needs to keep track of which credits default or when they default, he has to consider the KMV approach. The approach is appropriate when looking at n^{th} to default swaps and other basket swaps with a relatively small number of underlyings.

However, in recommending the KMV approach for this application, we are skeptical about using historic market values or historic equity prices to make fine distinctions in the relationships of particular credits. The volatility of these variables is itself volatile from measurement period to measurement period. Using historic asset or equity data at the firm or even the industry level may create distinctions that are not stable or predictive. And the number of correlation estimates that must be made, one for every pair of credits in the portfolio, increases the chances for errors to affect the predictive ability of the model.

People who see the Merton model of default as immutably true as a physics formula tend to go along with its extension into default correlation. People who see the KMV implementation of the Merton model as a statistically fitted model that provides insight into relative default probability are more skeptical. We are in the later camp.

Conclusion

We wanted to provide not-overly-mathematical guide to default correlation. We also wanted to highlight some of the problems in applying default correlation to credit portfolios and discuss the virtues of proposed default correlation solutions.

We *defined* default correlation and discussed its causes in the context of systematic and unsystematic drivers of default. We used Venn diagrams to *picture* default probability and default correlation; and *derived mathematical formulas* for default correlation, joint probability of default, and the calculation of empirical default correlation.

We emphasized “*higher orders of default correlation*” and the insufficiency of pairwise default correlation to define default probabilities in a portfolio comprised of more than two credits. This is a point that we have never seen mentioned outside of our 1995 journal article.

We showed the calculation and results of *historic default correlation*. We showed that default correlations among well-diversified portfolios vary by the ratings of the credits and also by the time period over which defaults are examined. We described some *major problems* in measuring default correlation and therefore implementing a default correlation solution. First, there is no way to distinguish changing default probability from default correlation. Second, the way default correlation is commonly looked at ignores time series correlation of default probability. We feel that these problems in applying default correlation to actual portfolios have not been adequately explored.

We discussed the various ways analysts have attempted to incorporate default correlation into their analysis of credit risky portfolios:

- the ancient method of industry and single name exposure limits;
- Moody’s ad hoc method of assessing the trade-off between industry and single name diversity in their Diversity Score;
- the changing-default probability approach of CSFB;
- the historical market value approach of KMV.

In comparing well-diversified portfolios, we wondered whether *any* default correlation modeling is necessary. Given a certain level of single name and industry diversity, we doubt that typical portfolios have very different default correlations and we are skeptical of any measurement showing that they do.

However, we saw value in creating default probability distributions. We appreciated the CSFB method that focuses on observable default rates and were skeptical of making credit-by-credit distinctions in default correlation based on estimates of historical firm market value.

Finally, we wish that some of the work being directed at default correlation were instead focused on default probability in credit portfolios. We claim that the difference in performance between well-performing credit portfolios and poor-performing credit portfolios is not caused by default *correlation* but by default *probability*. High yielding, high-default probability assets, from telecom companies to dot coms are to blame for credit portfolio disasters, not default correlation. It seems to us that the focus on default correlation is sometimes besides the point. ♦

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