Pricing multiasset equity options with copulas: an empirical test

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Abstract

The market for equity exotic options on a basket of underlying assets (whether single stocks or stock indexes) has grown significantly through time thanks to the development of structured equity-linked bonds. When evaluating those options, the way in which dependence among underlying assets' returns is modeled is critical, and so is the definition of input parameters. Nevertheless, there is little empirical literature that tries to quantify the dispersion of theoretical fair price a trader may face when pricing an option the first time or when revaluing the option in his or her portfolio. This paper analyzes the effects of uncertain correlation inputs and of the choice between traditional standard methods assuming joint normality of asset returns and copula-based methods, through a Monte Carlo simulation applied to many different exotic contracts on a basket of five UK stocks. Implications for traders and risk managers and auditors, and the potential for further use of copulas in exotic options pricing are then discussed.

1. Introduction

The market for structured equity-linked bonds, that started with a guaranteed equity linked note in the United States in 1987, has since then developed both in terms of volumes and of sophistication. Derivatives desks inside major investment banks, in a quest for financial innovation and for the higher margins that innovators may sometimes attain, have invented each year new exotic options that could produce various types of payoffs for the final investors. Internal market reports from investment banks' research departments now consider a huge number of different alternative exotic equity-linked products: some are able to gain worldwide success, while others may become common and successful in certain countries and almost unknown in others.

In the great variety of the exotic options that an observer may find in the market one feature that is very common is the presence of more than one underlying asset. The kinds of multiasset exotic structures may vary from simple basket options whose payoff is linked to the overall performance of a basket of stock indices or single stocks, to cases such as the so-called conditional coupon structure where the investor receives a fixed coupon each year provided that none of a basket of stocks trespassed a certain barrier (e.g. none of the stocks went below 70% of the initial price). Whether or not closed-end pricing formulas are available, a potentially crucial issue in pricing these options is correlation among the different underlying assets. For instance, it is intuitive that the value of a basket call option would increase if correlation among underlying stocks or indices increase, since this would increase the volatility of the basket: therefore, the basket call would react as a simple call option whose value grows if implied volatility increases. The value of a conditional coupon structure would grow too for a very different reason. In fact, if underlying assets have lower correlation it would be more likely that at least one could touch the barrier and make the coupons disappear; if instead their correlation is higher it is more likely that all of them may grow together remaining distant from the barrier. The critical role of correlation in pricing these options is why they are often labeled as "correlation products", and raise two main problems.

The first problem is that the choice of correlation inputs becomes important for many different players inside the bank. Traders, risk managers and internal auditors are for different reasons interested in using the right correlation estimates, so to guarantee that the option is priced correctly. The fact that risk managers should control traders' work, and internal auditors should control risk managers is relevant since it means that traders' choice of correlation inputs must not only be right, but also must be clearly explainable to other parties inside the bank. Risk managers and auditors know in fact that a wrong set of correlation inputs would alter the value of the portfolio: at least potentially, the trader might hide losses by modifying correlation inputs so to increase the value of its position while using perfectly fair and certified implied volatility inputs.

Of course, if it were possible to extract implied correlations from traded multiasset options as implied volatilities are extracted from traded plain vanilla options, then risks would be much smaller. Unfortunately, while implied volatility can at least up to a certain extent be extracted from traded options' prices[1], it is almost impossible in practice to extract implied correlations. In fact, while when extracting implied volatility from a plain vanilla option's price the trader has one equation to be satisfied (the pricing formula) and one unknown term, in the case of a basket option with five underlying stocks the trader has only one equation (the pricing equation of the basket option), five unknown implied volatilities and ten unknown correlation coefficients. Moreover, the pricing algorithm may not be a closed-end formula, and the trader may also be uncertain about whether all market participants are using the same pricing technique. Despite the fact that the trader cannot infer them from the market, correlation values are critical for him since they influence the price of the option. Therefore, wrong correlation inputs would produce (a) a wrong price when the option is issued and offered inside a structured bond to the institutional client of the investment bank, (b) a wrong mark-to-market (or more precisely, mark-to-model) evaluation of the exotic at the end of each day when it has already been issued and (c) a wrong assessment of its risk profile and its Greeks (Delta, Gamma, Vega, Theta and Rho), since their values derive from the pricing formula and are equally sensible to input data. All the problems are relevant for the trader; the risk manager is concerned especially with the second and the third one. For the internal auditor the second is always crucial and the third is important as well if,

[1] *The problem of implied volatility remains an issue for long term exotic options where it is impossible to extract data from prices of traded options, whose maturity is typically much shorter. In this case, however, some information may be obtained from the OTC market. Some information providers have also tried through time to produce "average" implied volatilities by receiving data from individual investment banks and giving back an aggregate average volatility. This solution may be used for instance by the risk manager if he or she wants to control whether the implied volatility estimate used by the trader is correct.*

as it should be, he is actively controlling the efficiency of the risk management systems in place inside the bank.

The second problem is that if the dependence among underlying assets' returns plays a key role, one should question whether the classic assumption of multivariate normal distribution is suitable to price this products. The recent stream of contributions concerning copulas and their application to risk management can clearly be applied also to the pricing of multiasset exotic options. Yet, despite the growing literature on copulas on one hand and the relevance of the equity-linked exotic options market on the other hand, there have been very few contributions aimed at testing empirically the role of correlation and dependence when pricing these products. In our view, it is important both to test whether and how much using copulas may significantly change the fair price estimates for exotic products, and to discuss the implementation problems that their use may raise. It is in fact surprising that while they are widely recognized as a theoretically superior means to model dependence among returns, they do not appear to be really used in practice to price multiasset equity derivatives. One possible explanation could be that they might have only a modest impact on fair prices estimates (so that a simpler even if approximate method could be preferred). If this were not the case, other reasons should be found in order to explain why despite the growing interest also in practice as far as both risk measurement issues and credit derivatives evaluation are concerned, the diffusion of copulas in multiasset equity derivatives pricing decisions is still very limited. The aims of the paper are therefore the following:

(1) to evaluate the impact that using copulas may have on fair price estimates of different exotic options on a given basket of stocks;

(2) to discuss the problem of modeling dependence and defining proper input parameters in the context of equity exotic pricing;

(3) to analyze which are the risks deriving from uncertain dependence structure among assets and how the different players inside an investment bank may try to handle the problem.

The structure of the paper is the following. Section 2 defines the key elements about copulas and the main kinds of copulas that may be applied in a generic multivariate (and not only bivariate) setting. Section 3 describes the empirical test analyzing its aims, the choice of options' payoff that have been tested, the underlying asset and the data and procedures on which parameters' calibration has been based. Section 4 presents the results of the test, while Section 5 discusses its implications for the different players inside the bank. Section 6 concludes.

2. Copula functions

2.1. Definition of copula functions

An n-dimensional copula[2] is a multivariate distribution function (d.f.), C, with uniform distributed margins in [0,1] (U(0,1)) and the following properties:

1. C: $[0,1]^n \rightarrow [0,1]$
2. $\forall u_i \in [0,1] \quad C(u_1, \dots, u_n) = 0$ *if at least one of the* u_i *equals zero*
3. C is n-increasing
4. C has margins C_i which satisfy $C_i(u) = C(1, \dots, 1, u, 1, \dots 1) = u$ for all $u \in [0,1]$.

It is clear from the definition above that if $F_1, \dots, F_n$ are univariate distribution functions, $C(F_1(x_1), \dots, F_n(x_n))$ is a multivariate d.f. with margins $F_1, \dots, F_n$ because $u_i = F_i(x_i)$ is a uniform random variable, so the copulas are a useful tool to construct and simulate multivariate distributions.

The following theorem is known as Sklar's Theorem and it is the most important one about copulas because many practical applications are based on it.

Let F be an n-dimensional d.f. with continuous margins $F_1, \dots, F_n$, *then it has the following unique copula representation:*

$$F(x_1, \dots, x_n) = C(F_1(x_1), \dots, F_n(x_n)).$$

The following corollary can be obtained from the expression above.

Let F be an n-dimensional d.f. with continuous margins $F_1, \dots, F_n$, *and copula , then, for any* $(u_1, \dots, u_n)$ *in* $[0,1]^n$:

$$C(u_1, \dots, u_n) = F\left(F_1^{-1}(u_1), \dots, F_n^{-1}(u_n)\right).$$

where F_i^{-1} *is the generalized inverse of* F_i.

Therefore the use of copula function allows to overcome the issue of multivariate d.f. estimate, dividing it into two steps:

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[2] *See Nelsen (1998)*

- determine the margins $F_1, \ldots, F_n$ which represent the distribution of each marginal distribution (in our case, of each risk factor) and estimate their parameters;
- determine the copula function which completely describes the dependence structure of random variables.

2.2. Elliptical copulas: the Gaussian copula

Elliptical distributions class provides a great range of multivariate distribution functions that share many of the tractable properties of the multivariate normal distribution and allow to design different dependence structures. Elliptical copulas are the copulas of elliptical distributions. Simulation from elliptical distributions is easy to perform, therefore, as a consequence of the Sklar's theorem, the simulation of elliptical copulas is also easy.

The most frequently used elliptical copulas are the Gaussian copula and the t-Student copula.

The Gaussian or normal copula is simply the copula derived from the multivariate normal distribution. Let f the standard univariate Gaussian c.d.f. and $f_{r,n}$ the standard multivariate normal c.d.f. with linear correlation matrix r, then the n-dimensional copula with correlation matrix r is the following:

$$C_r(u_1, \ldots, u_n) = f_{r,n}\left(f^{-1}(u_1), \ldots, f^{-1}(u_n)\right)$$

The Gaussian copula does not have upper tail dependence and, since elliptical copulas are symmetric, does not even have lower tail dependence.

The Gaussian copula is completely determined by the knowledge of the correlation matrix r and the parameters involved are simple to estimate.

To simulate random variables from Gaussian copula it is enough to simulate a vector from the standard multivariate normal distribution with correlation matrix Σ and then to transform this vector through a univariate c.d.f. so that you can obtain a vector from the chosen copula.

The matrix Σ, positive definite, can be easily determined with the Cholesky decomposition in order to calculate a matrix A such as $AA^{\mathrm{T}} = \Sigma$. Let $Z_1, \ldots, Z_n$ be independent standard normal variable and the vector $m \in \Re^n$, then the vector $m + AZ$

follows a multivariate normal distribution with mean vector m and covariance matrix Σ.

It is then possible to generate random variates from the n-dimensional Gaussian copula running the following algorithm:

- calculate the Cholesky decomposition A of the matrix Σ;
- simulate n indipendent standard normal random variates $z_1, \ldots, z_n$;
- set $x = Az$;
- determine the components $u_i = f(x_i), \qquad i = 1, \ldots, n$;
- the vector $(u_1, \ldots, u_n)^{\mathrm{T}}$ is a random variate from the n-dimensional Gaussian copula

2.3. Elliptical copulas: the t-Student Copula

The copula of the multivariate t-Student distribution is the t-Student copula. Defining by $T_{r,n}$ a multivariate t-Student distribution with n degrees of freedom and correlation matrix r, the corresponding copula is the following:

$$C_{r,n}(u_1, \ldots, u_n) = T_{r,n}\left(t_n^{-1}(u_1), \ldots, t_n^{-1}(u_n)\right)$$

where t_n is the univariate t-Student distribution with n degrees of freedom.

Because the t-Student distribution tends to the normal distribution when n goes to infinity, so the t-Student copula tends to the normal copula when $n \to +\infty$.

In contrast to the Gaussian copula, the t-Student copula has upper tail dependence increasing in r and decreasing in n. Therefore, the t-Student copula is more suitable to simulate events like stock market crashes or the joint default. Besides, for quite large values for n, the tail dependence is significantly different from 0 only when the correlation coefficient is close to 1. This suggests that, for moderate values of the correlation coefficient, a Student copula with a large number of degrees of freedom may be difficult to separate from the Gaussian copula.

The description of a Student copula is defined by two parameters: the correlation matrix r and the number of degrees of freedom n. The estimation of the parameter n is rather difficult and has an important role in the estimation of the correlation matrix, as we will see in the following section. Therefore the t-Student copula is more difficult to calibrate and use than the Gaussian copula.

Random variates from the n-dimensional t-Student copula can be generated through the following steps:

- calculate the Cholesky decomposition A of the matrix Σ;
- simulate n indipendent standard normal random variates $z_1, \ldots, z_n$;
- simulate a random variate, s, from c_n^2 distribution, indipendent of z;
- set $y = Az$
- set $x = \frac{\sqrt{n}}{\sqrt{s}} y$;
- determine the components $u_i = t_n(x_i), \qquad i = 1, \ldots, n$;
- the vector $(u_1, \ldots, u_n)^T$ is a random variate from the n-dimensional t-Student copula with n degrees of freedom

In this paper we will use the so-called Meta-t copula which is slightly different than the t- Student copula. Here we allow the marginals to have different degrees of freedom. (see for instance Demarta S. and McNeil A. 2005)

2.4. Archimedean copulas

Elliptical copulas are not the only possible type of copulas. Another important family is represented by Archimedean copulas, that include for instance the Gumbel and Clayton copula. This class of copulas, in contrast to elliptical copulas, have closed form expressions, because these copulas are not derived from a multivariate distribution function using the Sklar's theorem. As a consequence the Archimedean copulas are originally defined on two dimensions and their multivariate extension need some technical conditions to assert the n-copulas are proper. Therefore, even if they allow to model the dependence structure between variables in different and even more flexible ways than elliptical copulas, their application is currently confined in practice to bivariate problems (an example being the valuation of a credit derivative whose price depends also on the risk of joint default of the underlying bond and of the protection seller, that can be modelled through copulas). Unfortunately, multiasset equity options typically imply much more than two underlying assets, and therefore our test will be restricted to elliptical copulas.

3. The design of the empirical test

3.1. General aims of the test

The purpose of the empirical test is to check the differences among fair prices for a set of multiasset exotic options that can be obtained according to copulas as opposed to the standard assumption of joint normality of asset returns. Therefore, we will check how close or distant copula-based prices will be relative to the prices obtained through the simpler method, and how close or distant they are among themselves, if slightly different assumptions about the trader's behaviour in fitting the copulas are made. In particular, we have restricted the choice of copula functions to the Gaussian copula and t-Student copulas. The choice to avoid Archimedean copulas derives from the fact that fitting those copulas on joint distributions of more than two underlying assets is extremely complex, and at least at present it is more than unlikely that they might be applied in practice to price multiasset options where the number of underlying assets may range from three to even twenty or more assets. In particular, we have compared three copulas obtained through a two-step optimization procedure but differing among themselves for the assumptions on marginal return distributions and for the set of historical data adopted to fit copula parameters.

At the same time, as far as the traditional pricing method is concerned, we will check the impact of different estimates of the linear correlation matrix, depending on the size of the sample and on the frequency of return data that are used. As a whole, the test will give a picture of how stable or unstable fair prices may be depending on the assumptions the trader (or risk manager) is making, and of the degree of uncertainty that the different players inside the bank who are concerned with dependence and correlation on equity exotic products may typically face.

3.2. Underlying assets, data sample, and the set of exotic options

The underlying assets we have considered for our test are five UK stocks, and precisely B SkyB, BG Group, British Airways, Vodafone and Rolls Royce. In order to estimate parameters we used a six-year historical time series of daily closing prices and returns from January 1st, 1999 to December 31st, 2004. All option valuations were conducted with market data on December 31st, 2004; zero coupon risk-free rates were derived from the UK swap curve on that date, while dividend yields were estimated based on historical average dividend yields for the stocks in the sample.

As far as the sample of exotic options is concerned, we tried to build a sample of different payoff structures, a large part of which is actually commonly used in practice. All options had a remaining maturity of 5 years (i.e., all options were assumed to expire on December 31st, 2009). Yet, for most options we distinguished between a brand new option evaluated at the date of issue, and an already existing option with 8 years of initial maturity, issued on December 1st 2001 and evaluated after 3 years. Those options will be labelled as having a 3+5 maturity. The reason why we differentiated between new and existing options is that we wanted to test whether they may show a different sensitivity to the correlation among underlying assets. For instance, an option whose payoff is based on the worst performing asset within a basket could be expected to have a different sensitivity to changes in correlation among assets when it is issued (and the worst performing asset is still unknown) or after three years (when there will be a *pro tempore* ranking of assets' performance so that the option's payoff is likely to be linked especially to the correlation of the two or three worst performing assets, while being relatively less sensitive to correlation between the best performing ones). The underlying research question is therefore whether – at least for some kinds of payoffs – sensitivity to correlation or to dependence structure among asset returns is only a temporary effect or whether instead it is a permanent one.

The specific types of options we considered are described in detail in Table 1. In some cases the name of the option is consistent with a very well-defined industry standard, while in others a single widely accepted name for the particular payoff did not exist. In any case, Table 1 should provide a clear enough description of the payoff to avoid misunderstandings.

Table 1. A description of the exotic options analyzed in the empirical test

Option	Option description
Asian basket option (5 yrs, strikes equal to initial prices)	New Asian basket options with 5 years of maturity. The option's payoff is the maximum between the mean percentage return of each stock (if the mean is positive) and zero (basket feature). The performance of each stock is calculated as the difference between the average price of the stock at the end of each month (Asian feature) and the stock's initial price.
Asian basket option (3+5 yrs, strikes equal to initial prices)	Already issued Asian basket options; the option has been issued 3 years before valuation date and has 5 years of remaining maturity. The option's payoff is calculated as in the previous case, withthe only difference that relevant initial prices are not prices on October 1st, 2004 but prices on October 1st, 2001, since the option has been issued on that day.

Table 1 (continued). A description of the exotic options analyzed in the empirical test

Option	Option description
Asian basket option (3+5 yrs, strikes equal to average prices during the first 3 years)	Already issued Asian basket option. The option is identical to the preceding one, apart from the fact that the strike price is set equal for each stock to the average price at the end of the month during the first free years. Therefore, the option is at the money at the beginning of the simulation as the first 5-year Asian option.
Asian best option (5 yrs, 40% participation rate)	New option with 5 years of maturity where the final payoff is 40% of the return of the highest performing stock (if positive). The performance of each stock is calculated as the difference between the average price of the stock at the end of each month (Asian feature) and the stock's initial price.
Asian best option (3+5 yrs, 40% participation rate)	Already issued Asian best option identical to the preceding one but with 3 years of past history and 5 years of remaining maturity
Napoleon option (based on monthly basket returns, annual coupon 12%)	5-year option where the investor receives each year a coupon equal to 12% minus the worst monthly performance of the stock basket during the year. The coupon is floored at zero if the basket has a minimum monthly performance lower than - 12%.
Conditional coupon structure (5 yrs, 8% annual coupon, barrier=60%)	New 5-year option that pays each year a fixed 8% coupon if none of the underlying assets at the end of each month has ever touched a barrier equal to 60% of the initial price on October 1st, 2004.
Conditional coupon structure (3+5 yrs, 8% annual coupon, barrier=70%)	Already issued option with 5 years of remaining maturity that pays each year a fixed 8% coupon if none of the underlying assets at the end of each month has ever touched a barrier equal to 70% of the initial price on October 1st, 2001.
Fixed 80% coupon minus worst performance (5 yrs)	New 5-year option that pays at maturity a coupon equal to 80% plus the negative performance of the worst performing stock within the basket. The coupon is capped at 80% (if all stocks had positive returns) and floored at 0 (if one stock decreased by more than 80%)[3]
Fixed 80% coupon minus worst performance (3+5 yrs)	Already issued option with 3 years of past history and 5 years of remaining maturity that pays at maturity a coupon equal to 80% plus the negative performance of the worst performing stock within the basket. The coupon is capped at 80% (if all stocks had positive returns) and floored at 0 (if one stock decreased by more than 80%)
Fixed 25% coupon minus put on basket performance (5 yrs)	New 5-year option that pays at maturity a coupon equal to 25% plus the performance (only if it is negative) of the basket of stocks. The coupon is floored at 0 (if the basket value decreased by more than 25%)[4]
Fixed 25% coupon minus put on basket performance (3+5 yrs)	Already issued option with 3 years of past history and 5 years of remaining maturity that pays at maturity a coupon equal to 25% plus the performance (only if it is negative) of the basket of stocks. The coupon is floored at 0 (if the basket value decreased by more than 25%)

[3] *The zero floor can be interpreted as the result of a spread position between two different worst put options: the investor would be selling a put on the worst of all assets and then buying another put on the worst of all assets whose strike price is fixed 80% belowcurrent prices.*
[4] *Again, the position can be conceived as the result between a short ATM basket put and a long OTM basket put whose strike price is fixed at a 25% decrease from current basket prices.*

3.3. Pricing methodology

All the options described in the previous paragraph have been priced through a Monte Carlo simulation, calculating therefore the fair value of the option as the discounted value of the expected payoff in a risk-neutral world. In each of the 10,000 simulation runs we have simulated daily correlated returns for each of the stocks and reproduced the daily price of each asset so to precisely calculate the payoff of each contract. In each simulation run we have compared seven different pricing alternatives: four of them were based on a simple multivariate Monte Carlo simulation under a joint normality assumption, and three were based on different copulas. The risk-neutral expected return for each asset was obviously the same under each pricing method. In all the cases, since the simulation requires to extract first a vector of uncorrelated standard normal random variables and then to transform them into correlated returns, we have used the same uncorrelated vectors so to guarantee that differences in prices may not derive from uncorrelated random variables sampling errors. In practice, for each extraction of a daily random uncorrelated vector we have transformed the same vector into different correlated vectors according to each of the seven method tested, and reproduced jointly seven alternative return and price paths according to each methodology. Uncorrelated random variables were extracted using a Latin Hypercube algorithm, that enabled us to reproduce in the best possible way the whole joint multivariate distribution in our simulation.

The four alternative payoffs under the standard assumption of multivariate normality (MVN) were obtained by using four different input historical correlation matrixes: two 6-year linear correlation matrixes based respectively on daily and weekly stock returns, and two equivalent linear correlation matrixes based on 3 years of data only. The purpose was to check the effects of the uncertainty that even if the simpler model is adopted a trader or a risk manager may face in feeding the simple model with the "right" inputs. Volatility was set equal to historical volatility in all four cases. In fact, even if one could argue that consistent volatilities should have been used (e.g. weekly 3-year sample volatilities should be combined with weekly 3-year sample correlation coefficients), our assumption about the practical trader's behaviour is that the trader would know an implied volatility value for all the underlying assets, and would then have to decide which correlation inputs he should use. Since we wanted to investigate the correlation problem only, we decided to test it by maintaining the same level of implied volatility throughout the four correlation scenarios. We simply used 6-year, weekly data historical volatility as a proxy for implied volatility, but this should not alter results in any way. Annualized historical 6-year volatilities based on weekly returns for B SkyB, BG Group, British Airways, Vodafone and Rolls Royce were equal respectively to 47.24%, 34.36%, 50.42%, 45.81%, 42.95%.

For any of the four MVN cases, historical correlation matrixes have been reproduced by simulating random uncorrelated standard normal variables and then transforming them into correlated random variables through the classic Cholesky decomposition. Therefore we had four different Cholesky matrixes (one for each historical correlation matrix) that were applied at the same time to produce different multivariate return paths. Correlation matrixes in the four different cases are reported in Tables 2 through 5.

Table 2. Correlation matrix based on weekly returns (6-year sample)

	B SkyB	BG Group	British Airways	Vodafone	Rolls Royce
B SkyB	1,000	0,099	0,281	0,380	0,180
BG Group	0,099	1,000	0,194	0,056	0,208
British Airways	0,281	0,194	1,000	0,248	0,548
Vodafone	0,380	0,056	0,248	1,000	0,166
Rolls Royce	0,180	0,208	0,548	0,166	1,000

Table 3. Correlation matrix based on daily returns (6-year sample)

	B SkyB	BG Group	British Airways	Vodafone	Rolls Royce
B SkyB	1,000	0,234	0,338	0,416	0,234
BG Group	0,234	1,000	0,204	0,221	0,210
British Airways	0,338	0,204	1,000	0,344	0,391
Vodafone	0,416	0,221	0,344	1,000	0,263
Rolls Royce	0,234	0,210	0,391	0,263	1,000

Table 4. Correlation matrix based on weekly returns (3-year sample)

	B SkyB	BG Group	British Airways	Vodafone	Rolls Royce
B SkyB	1,000	0,237	0,519	0,402	0,400
BG Group	0,237	1,000	0,235	0,186	0,301
British Airways	0,519	0,235	1,000	0,432	0,645
Vodafone	0,402	0,186	0,432	1,000	0,318
Rolls Royce	0,400	0,301	0,645	0,318	1,000

Table 5. Correlation matrix based on daily returns (3-year sample)

	B SkyB	BG Group	British Airways	Vodafone	Rolls Royce
B SkyB	1,000	0,417	0,536	0,537	0,401
BG Group	0,417	1,000	0,371	0,453	0,351
British Airways	0,536	0,371	1,000	0,506	0,567
Vodafone	0,537	0,453	0,506	1,000	0,448
Rolls Royce	0,401	0,351	0,567	0,448	1,000

As far as copulas are concerned, we tested three different solutions. While we always used the six-year return sample, the first one is the copula based on daily data and considering t-Student marginal distributions. The second one is calculated again on daily data but assumes that marginal distributions are normal (as the trader or risk manager may be likely to assume in order to speed up the calibration process). The third one is instead calculated allowing marginal data to be t-Student distributed but uses weekly instead of daily data. The choice of the frequency of data may be relevant since on one hand the calibration of copula parameters through a log-likelihood maximization would support the choice of a sample with more data (i.e., daily instead than weekly, given a certain size of the time horizon), but serial cross-autocorrelation problems tat may emerge with non-syncronously traded stocks or stock indexes[5] would probably force the trader or risk manager to use weekly data frequently.

3.4. Estimating parameters for copulas

In order to estimate the optimal copula parameters for each of the three copula cases we considered, the first step has been represented by estimating marginal distributions' parameters. While this was elementary when assuming normal marginal distributions, in the other two cases we used t-Student distributions to describe the historical return

[5] *The problem of serial cross-autocorrelation is described for instance in Pope and Yadav (1994), with reference to the estimation of ex post tracking error.*

distribution of each of the five stocks. In these cases it was then necessary to estimate the marginal distribution parameters. The density function of a t-Student distribution with n degrees of freedom is given by

$$h_n(x) = \frac{\Gamma[(n+1)/2]}{\Gamma(n/2)\Gamma(1/2)\sqrt{n}}\left(1+\frac{x^2}{n}\right)^{-\frac{n+1}{2}}$$

and in the general case $x \rightarrow \frac{x-m}{s}$ the distribution becomes

$$f_n(x) = \frac{\Gamma[(n+1)/2]}{\Gamma(n/2)\Gamma(1/2)}\left(\sqrt{n}\,s\right)^n \left(ns^2 + (x-m)^2\right)^{-\frac{n+1}{2}}$$

In $f_n(x)$, m is the mean of the distribution but s is not the standard deviation because the variance of this distribution is infinite; s is referred to as the scale parameter. For every series of asset returns it is necessary to find the mean, m, the scale parameter, s, and the degrees of freedom, n. Using the above density distribution function $f_n(x)$ we can calculate the log likelihood function as a function of the parameters, m, s and n. The parameters are then taken in such way that they maximize the likelihood function. This result was achieved by standard numerical optimization.

The final results that were obtained as far as marginal distribution parameters are concerned are reported in Table 6 (separating parameters on daily returns used for the first copula and those on weekly returns used for the third one).

Table 6. Parameters of marginal distribution functions for the t-Student copula

		B SkyB	BG Group	British Airways	Vodafone	Rolls Royce
Daily data	Mean	-0.0005	0.0001	-0.0008	-0.0007	-0.0003
	Scale parameter	0.0195	0.0177	0.0248	0.0238	0.0188
	Degrees of freedom	3	6	5	6	4
Weekly data	Mean	-0.0002	0.0018	-0.0037	-0.0012	-0.0002
	Scale parameter	0.0393	0.0281	0.0579	0.0464	0.0419
	Degrees of freedom	3	4	4	7	4

After estimating parameters of the marginal distribution functions, the second logical step is represented by estimating copula parameters. The classical estimation method is again the maximum likelihood method. The density of the joint distribution F is given by

$$f_q(x_1,\dots,x_n) = c(F_1(x_1),\dots,F_n(x_n))\prod_{i=1}^{n} f_i(x_i)$$

where f_i is the density of the marginal distribution F_i and c is the copula density.

Set $X = \{(x_1^t,\dots,x_n^t)\}_{t=1}^{T}$, the likelihood function will be

$$L(q) = \prod_{t=1}^{T} f_q(x_1^t,\dots,x_n^t)$$

from which the estimator

$$\hat{q} = \{q_{max} \mid ln\, L(q_{max}) \geq ln\, L(q) \quad \forall q \in \Theta\}.$$

The function that should be maximized is represented by the logarithm of the likelihood function $l(q)$. So we obtain

$$l(q) = \sum_{t=1}^{T} \ln c(F_1(x_1^t),\dots,F_n(x_n^t)) + \sum_{t=1}^{T}\sum_{i=1}^{n} \ln f_i(x_i^t)$$

where q is the vector including the parameters of the n marginal distributions and the parameters of the copula.

For a t-Student copula the function is

$$l(q) = -\frac{T}{2}\ln|r| - \left(\frac{n+n}{2}\right)\sum_{t=1}^{T}\ln\left(1+\frac{1}{n}V_t^{\mathrm{T}} r^{-1} V_t\right) + \left(\frac{n+1}{2}\right)\sum_{t=1}^{T}\sum_{i=1}^{n}\ln\left(1+\frac{V_i^2}{n}\right)$$

where $V_t = \left(t_{n_1}^{-1}(u_1^t),\dots,t_{n_n}^{-1}(u_n^t)\right)^T$.

This method can be computational intensive in the case of high dimensional distributions, because it requires to estimate jointly the parameters of the marginal distributions and the parameters of the dependence structure. Since copulas allow to split the parameters in *specific* parameters for the margins and in *common* parameters for the joint structure the log of the maximum likelihood function can be written in the following way:

$$l(q) = \sum_{t=1}^{T} \ln c(F_1(x_1^t;q_1),\dots,F_n(x_n^t;q_n);a) + \sum_{t=1}^{T}\sum_{i=1}^{n} \ln f_i(x_i^t;q_i)$$

where $q = (q_1, \ldots, q_n, a)$. q_i and a are respectively the parameters of the margins and the copula.

So we can perform the estimation of the univariate marginal distributions as a first step and then determine a given the previous estimates through

$$\hat{a} := \arg\max \sum_{t=1}^{T} \ln c\left(F_1\left(x_1^t; \hat{q}_1\right), \ldots, F_n\left(x_n^t; \hat{q}_n\right); a\right)$$

This two-steps method is called the method of *inference functions for margins* or IMF method.

However in the case of a t-Student copula the estimation of the parameters could require numerical optimisation of the likelihood function because it does not exist a closed form expression as in the case of a Gaussian copula.

The best ideal choice would have been to estimate jointly the number of degrees of freedom and the correlation matrix using a simulation. The procedure would have implied to simulate some matrixes through extractions of random numbers between 0 and 1 and a number between 4 and 21 (for n), to calculate the likelihood function for each matrix and n, and to select the combination with the maximum output. Unfortunately the positive constraint on the logarithm and the need to obtain a positive definite matrix would have forced to an excessively high number of simulations in order to peg the estimate. For this reason, and considering that we wanted to replicate a method that could be applied in practice, we decided to resort to a simpler method, even if it does not allow to estimate the degrees of freedom and correlation parameters jointly. More precisely, we decided to apply the IFM method and, after the estimation of the margins parameters, to calculate the correlation matrix using an iterative algorithm, that does not require optimisation:

1. let $\hat{r}_0$ the IFM estimate of the correlation matrix for the Gaussian copula[6]
2. $\hat{r}_{m+1}$ is obtained using the following equation

$$\hat{r}_{m+1} = \frac{1}{T}\left(\frac{n+n}{n}\right)\sum_{t=1}^{T} \frac{V_t V_t^T}{1 + \frac{1}{n} V_t^{\mathrm{T}} \hat{r}_m^{-1} V_t}$$

3. repeat the second step until convergence $\hat{r}_{m+1} = \hat{r}_m \left(:= \hat{r}_\infty\right)$

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6 $\hat{r}_{IFM} = \frac{1}{T}\sum_{t=1}^{T} V_t V_t^T$ where $V_t = \left(f^{-1}\left(u_1^t\right), \ldots, f^{-1}\left(u_n^t\right)\right)^T$ and $u_1^t = F_1\left(x_1^t\right), \ldots, u_n^t = F_n\left(x_n^t\right)$.

4. the IFM estimate of the correlation matrix for the Student copula is $\hat{r}_{IFM} = \hat{r}_{\infty}$

This procedure does not allow to estimate at the same time the number of degrees of freedom for copula, which are considered as given. So we repeated the analysis for any possible number of degrees of freedom between 4 and 20 (so to define a "champion" correlation matrix for any reasonable number of degrees of freedom) and then compared the log-likelihood ratio of each set of parameters so to find the best one among "champion" sets.

As the calibration of copula parameters is subordinated to the definition of the parameters of the marginal distributions, we obtained at the end with this procedure three different optimal copula parameters, depending on the different assumptions on marginal distributions. Results are reported in Tables 7 to 9. Please note that the optimal number of degrees of freedom varies in the three cases.

Table 7. Parameters for copula #1 (optimized on daily data with t-Student marginals)

Copula type	t-Student with 6 degrees of freedom				
Correlation matrix	B SkyB	BG Group	British Airways	Vodafone	Rolls Royce
B SkyB	1,000	0,214	0,313	0,348	0,247
BG Group	0,214	1,000	0,181	0,201	0,198
British Airways	0,313	0,181	1,000	0,290	0,343
Vodafone	0,348	0,201	0,290	1,000	0,262
Rolls Royce	0,247	0,198	0,343	0,262	1,000

Table 8. Parameters for copula #2 (optimized on daily data with normal marginals)

Copula type	t-Student with 7 degrees of freedom				
Correlation matrix	B SkyB	BG Group	British Airways	Vodafone	Rolls Royce
B SkyB	1,000	0,164	0,235	0,276	0,169
BG Group	0,164	1,000	0,145	0,166	0,152
British Airways	0,235	0,145	1,000	0,237	0,267
Vodafone	0,276	0,166	0,237	1,000	0,197
Rolls Royce	0,169	0,152	0,267	0,197	1,000

Table 9. Parameters for copula #3 (optimized on weekly data with t-Student marginals)

Copula type	t-Student with 4 degrees of freedom				
Correlation matrix	B SkyB	BG Group	British Airways	Vodafone	Rolls Royce
B SkyB	1,000	0,157	0,308	0,324	0,230
BG Group	0,157	1,000	0,164	0,062	0,188
British Airways	0,308	0,164	1,000	0,239	0,465
Vodafone	0,324	0,062	0,239	1,000	0,202
Rolls Royce	0,230	0,188	0,465	0,202	1,000

4. Empirical results: the impact of the dependence structure

The empirical test we conducted can be divided into three parts. The first one is related to the impact of the choice of different historical inputs for the linear correlation coefficient matrix on the price of the exotic options described in Section 3. The second one refines the analysis by applying a homogeneous shock to linear correlation coefficients so to derive the "correlation Greek" estimating how sensitive different payoff structures may be to changes in (linear) correlation among asset returns. The third section investigates the impact on estimated fair prices that may derive from assuming that a different method such as copulas should be used to model dependence. In a sense, the three steps try to answer to practical problems the risk manager may face when evaluating the value and the risk profile of a trading desk portfolio. First of all, how relevant may be the impact of a few "simple" choices about historical correlation coefficients' estimation (e.g. the length of the sample or the frequency of data)? Secondly, which is the exposure of the option book either to changes or to poor estimates of correlation coefficients? And thirdly, how substantially would the price of each option change if copulas were adopted to model dependence? How stable prices may be depending on the choices that may be done in calibrating the copula parameters? Furthermore, may those differences in fair prices deriving from adopting copulas instead than assuming multivariate normal returns offset each other when the bank trades different kinds of exotic options? These questions will be analyzed in different steps in the next few pages by studying the set of options described in Section 3.

4.1. The impact of linear correlation inputs

The first step of the analysis is therefore to assess the impact of a few simple choices concerning the frequency of the data or the data sample size that the trader and/or the risk manager may have to take if they wanted to price correlation-sensitive exotic options by using the same set of implied volatilities (that are assumed to be equal to historical volatilities in this case, and kept equal in all correlation scenarios) and different sets of historical correlation coefficients. Results are reported in Tables 10 and 11.

Table 10. The effect of different estimates of linear correlations under multivariate normality assumption

Option type	6 years, weekly returns	6 years, daily returns	3 years, weekly returns	3 years, daily returns	$(P_{max}-P_{min}) / P_{avg}$
Asian basket option (5 yrs, strikes equal to initial prices)	17,935	18,559	19,599	20,658	14,19%
Asian basket option (3+5 yrs, strikes equal to initial prices)	8,759	9,045	9,826	10,401	17,27%
Asian basket option (3+5 yrs, strikes equal to average prices during the first 3 years)	15,741	16,065	16,735	17,362	9,84%
Asian best option (5 yrs, 40% participation rate)	32,649	31,825	30,269	28,530	13,37%
Asian best option (3+5 yrs, 40% participation rate)	19,896	19,842	18,664	18,175	8,99%
Napoleon option (based on monthly basket returns, annual coupon 12%)	6,124	5,333	4,165	3,359	58,27%
Conditional coupon structure (5 yrs, 8% annual coupon, barrier=60%)	5,540	5,904	6,966	7,808	34,61%
Conditional coupon structure (3+5 yrs, 8% annual coupon, barrier=70%)	3,559	4,021	4,625	5,517	44,18%
Fixed 80% coupon minus worst performance (5 yrs)	14,618	15,383	17,816	19,520	29,12%
Fixed 80% coupon minus worst performance (3+5 yrs)	13,156	14,178	16,195	18,131	32,28%
Fixed 25% coupon minus put on basket performance (5 yrs)	12,598	12,347	11,961	11,617	8,09%
Fixed 25% coupon minus put on basket performance (3+5 yrs)	12,336	12,123	11,674	11,357	8,25%
Average					23,20%

Table 10 immediately points out that even if the trader simply adopted a measure of historical linear correlation coefficients, the choice of the data that should be used to estimate those coefficients may impact the fair price estimate substantially. The range of fluctuation of estimated fair prices is in some cases striking.

Table 11 enables then to assess whether in this case the choice of the length of the historical sample or the choice between a weekly or a daily return data is more critical: the choice of sample length appears to be the crucial one, since changes between the 3-year and the 6-year sample are substantial. Anyway, the choice between monthly and weekly returns to estimate correlation may maintain a significant impact on prices.

Table 11. The effect of different sample size and return frequency.

	Daily vs weekly returns (1)		3-yr vs 6-yr sample	
	6-yr sample	3-yr sample	Weekly returns	Daily returns
Option type	(a)	(b)	(c)	(d)
Asian basket option (5 yrs, strikes equal to initial prices)	3,42%	5,26%	8,87%	10,71%
Asian basket option (3+5 yrs, strikes equal to initial prices)	3,22%	5,69%	11,48%	13,94%
Asian basket option (3+5 yrs, strikes equal to average prices during the first 3 years)	2,04%	3,68%	6,12%	7,76%
Asian best option (5 yrs, 40% participation rate)	-2,56%	-5,92%	-7,57%	-10,92%
Asian best option (3+5 yrs, 40% participation rate)	-0,27%	-2,65%	-6,39%	-8,77%
Napoleon option (based on monthly basket returns, annual coupon 12%)	-13,79%	-21,44%	-38,07%	-45,44%
Conditional coupon structure (5 yrs, 8% annual coupon, barrier=60%)	6,36%	11,40%	22,81%	27,78%
Conditional coupon structure (3+5 yrs, 8% annual coupon, barrier=70%)	12,18%	17,58%	26,04%	31,36%
Fixed 80% coupon minus worst performance (5 yrs)	5,10%	9,13%	19,72%	23,71%
Fixed 80% coupon minus worst performance (3+5 yrs)	7,48%	11,29%	20,71%	24,47%
Fixed 25% coupon minus put on basket performance (5 yrs)	-2,01%	-2,91%	-5,19%	-6,09%
Fixed 25% coupon minus put on basket performance (3+5 yrs)	-1,75%	-2,75%	-5,52%	-6,52%
Average	1,62%	2,36%	4,42%	5,17%
Average of differences in absolute value	*5,01%*	*8,31%*	*14,87%*	*18,12%*
Maximum	12,18%	17,58%	26,04%	31,36%
Minimum	-13,79%	-21,44%	-38,07%	-45,44%

(1) The column reports the difference between the price based on daily correlations and the price based on weekly correlations divided by the average of the two prices

(2) The column reports the difference between the price based on the 3-year sample correlations and the price based on the 6-year sample correlations divided by the average of the two prices

These differences in prices can be quite easily explained observing that on average daily-data-based correlation coefficients appeared to be higher than their weekly equivalents, and at the same time correlation coefficients based on the 3-year sample resulted to be higher than those based on the 6-year sample. As a consequence, some options which intuitively would react positively to an increase in correlation (e.g the Asian basket, or the conditional coupon structure) show a positive sign in Table 11, while others (such as the best option, which is worth more if correlation among asset returns is lower) show the opposite sign. As a consequence, the risk deriving from the choice of correlation inputs may be reduced at portfolio level if the bank is issuing options with opposite sensitivity to correlation.

Thirdly, at least in the specific case and analyzing the impact in terms of percentage change, there is no clear systematic difference between the impact of correlation inputs changes over new (5 years) as opposed to already issued ("3+5") options. This result should in any case be taken with remarkable caution considering that (i) pre-issued options may be particularly sensitive to one or two coefficients alone, and (ii) correlation coefficients for each pair of assets do not always change in the same direction when moving from an historical sample to another. An exception in this case is represented by Asian-type options: Asian options in fact, by linking the final performance also to the previous fixings of underlying assets, tend in general to show a smoother behaviour for all greeks and even in this case appear to be less sensitive to the choice of correlation inputs as time passes and more underlying assets' price fixings are registered.

It must be noted that the differences in Table 11 linked to data frequency and sample size may have in practice a different relevance on traders' or structuring desks behaviours. In fact, while the choice between weekly against monthly data might represent a pure discretionary one, most traders would likely prefer to use a longer sample (e.g. the 6-year sample) rather than a shorter one if they were to price a long-dated option. Using a shorter sample is instead more frequent when some of the underlying assets have only a short return series (e.g. because they are stocks that were listed through an IPO only a couple of years before the exotic option is issued). Results in Table 10 and 11 suggest therefore that sometimes by including an asset with a shorter return series the trader may impact (deliberately or inadvertently) on the choice of correlation coefficients used to price the option, and this could impact estimated fair prices in a relevant way. In particular, if the trader expects his or her counterparty to evaluate correlation through historical data, by including a short-history stock such as a stock deriving from a recent IPO he might increase, in some cases (e.g. the Asian basket option, or the fixed coupon minus worst performance option) the fair price estimated by the counterparty, and possibly increase his offer price.

In any case, even if we consider the most favorable case when any sample size is available, the differences between the "fair" prices that can be obtained remains remarkable, especially if we consider that (a) we are assuming no uncertainty about implied volatility inputs (i.e. volatility is the same across all simulations) and dividend yields and (b) we are adopting the same pricing method with the same, easy assumption on the shape of the joint distribution of assets' returns. Both conditions may not hold true in practice, where traders may use different implied volatilities and might be tempted to use alternative models to define the dependence structure among assets' returns.

4.2. Estimating the sensitivity to changes in linear correlation

While the analysis of the effects of different choices in estimating historical correlation data may give the idea of the uncertainty the risk manager may face, it does not provide a view on the sensitivity that different options may have to changes in the level of correlation. Therefore, we have estimated a "correlation Greek" by measuring the sensitivity of each option's value to a homogeneous change in the level of correlation coefficients. When an option is valued through Monte Carlo simulation, greeks are usually estimated numerically through a central difference[7], so that for instance the Delta (Δ) of a generic exotic option on one underlying asset is estimated as

$$\Delta = \lim_{h \to 0} \frac{V(S+h) - V(S-h)}{2h}$$

where V is the value of the option, S is the initial price of the underlying, and h is a small change in the price of the underlying asset. $V(S+h)$ and $V(S-h)$ are both estimated through a Monte Carlo simulation using the same vector of standard normal random returns (but producing different price paths since the initial price is different). We applied the same technique to correlation to derive a correlation Greek (that we will label as τ) by applying a homogeneous shock to all the linear correlation coefficients

[7] *See for instance Wilmott (1998), pp. 620-622; 681-682.*

among returns of underlying assets (except, obviously, the main diagonal coefficients). The correlation greek τ is therefore estimated as

$$t = \frac{V(\mathbf{?}+h)-V(\mathbf{?}-h)}{2h}$$

where *V(*r +h*)* is the value of the option dependent on the correlation coefficient matrix r +h obtained by increasing all coefficients lower than one by *h*, and *V(*r -h*)* is the value of the option dependent on the correlation coefficient matrix r -h obtained by decreasing all coefficients lower than one by *h*. In particular, we used a value of *h*=0,001, and applied it to the 6-year sample, weekly data historical correlation matrix. For the two perturbed correlation matrixes, using the same volatility vector, two different Cholesky decompositions were performed so to simulate jointly and using the same vectors of independent standard normal draws two different and parallel paths for the set of five underlying assets. We also kept track of actual ex post correlation coefficients in each simulation run, since one might argue that even with 10,000 simulations the actual ex post difference in correlation coefficients may be close, but not exactly equal, to the desired one. For each option we therefore calculated two different τ values, where the first one (t_{exp}) has as the denominator the expected value of *h* (0,001) while the second (t_{act}) has as the denominator the actual difference across average correlation coefficients in the two parallel simulation runs. We used the second as a control, even if in practice results are very similar. Results are reported in columns from two to four in Table 12.

It is possible to note that in all cases the sign of the coefficient is as expected, and enables to clearly identify the "correlation-bullish" and "correlation-bearish" positions the trader may be assuming. For an Asian basket option the sensitivity to increases in correlation coefficients is positive since this increases the volatility of the basket; for the same reason the sensitivity is negative in the Napoleon option and in the case of a fixed coupon minus a put on the basket's return. In the conditional coupon structure (which can be considered as a string of digital options linked to the worst performing asset), and in the short worst put option case the sensitivity is positive since an increase in correlation increases the chances that all assets may have a positive (or at least not so negative) return. For a similar reason the effect on the Asian best is the opposite, so that an increase in correlation reduces the value of the option.

Table 12. The estimated values of the "correlation greek" τ (10,000 simulation paths, shock h=0,001).

Option type	Estimate of τ			Impact of a 0.01 change in correlation coefficients	
	Expected sign	τ (expected h)	τ (actual h)	Absolute impact (τ/100)	Percentage impact on the price*
Asian basket option (5 yrs, strikes equal to initial prices)	+	12,670	12,679	0,127	0,72%
Asian basket option (3+5 yrs, strikes equal to initial prices)	+	7,993	7,998	0,080	0,94%
Asian basket option (3+5 yrs, strikes equal to average prices during the first 3 years)	+	7,730	7,735	0,077	0,50%
Asian best option (5 yrs, 40% participation rate)	-	-19,185	-19,198	-0,192	-0,59%
Asian best option (3+5 yrs, 40% participation rate)	-	-10,230	-10,237	-0,102	-0,52%
Napoleon option (based on monthly basket returns, annual coupon 12%)	-	-19,121	-19,134	-0,191	-3,13%
Conditional coupon structure (5 yrs, 8% annual coupon, barrier=60%)	+	8,843	8,849	0,088	1,59%
Conditional coupon structure (3+5 yrs, 8% annual coupon, barrier=70%)	+	5,879	5,883	0,059	1,65%
Fixed 80% coupon minus worst performance (5 yrs)	+	23,265	23,281	0,233	1,61%
Fixed 80% coupon minus worst performance (3+5 yrs)	+	21,275	21,289	0,213	1,63%
Fixed 25% coupon minus put on basket performance (5 yrs)	-	-5,275	-5,279	-0,053	-0,41%
Fixed 25% coupon minus put on basket performance (3+5 yrs)	-	-4,915	-4,918	-0,049	-0,40%

*The fair price based on 6-year sample, weekly data has been taken as a benchmark.

Interestingly, the comparison of τ values for the same kind of option with or without three years of past history shows how the sensitivity to correlation may change for different payoff structures as the option gets closer to maturity. Although the explanation may be intuitive, this clarifies how the choice of different option structures may make pricing (and hedging) more or less uncertain for the trader as time passes. For instance, the fact that already issued options have lower τ in the case of the Asian basket depends on the fact that since a certain number of critical prices have already been registered, the option is less sensitive to the volatility of the underlying basket of assets (and consequently to increases correlation that increase basket volatility). The different values for τ for "new" and already existing best/worst-type options can be explained considering that due to the different performance of the five underlying assets during the first three-year period (which is relevant for already existing options) the best or worst performing asset may already have emerged, and in this case correlation coefficients among all stocks may be less crucial than at the date of issue, while perhaps only the correlation between a couple of assets may still be relevant. Therefore, a best/worst-type option may become less correlation-sensitive through time.

Since the problem of correlation uncertainty is relevant both for pricing and for hedging, this simple analysis suggests that a careful choice of the payoff structure may help the trader to identify those contracts where hedging uncertainty problems due to correlation may diminish through time. Asian-type options make correlation less relevant through time since early fixing dates make the payoff more and more stable and less sensitive to volatility of the underlying through time; best/worst-type options may be multiasset options at the beginning but become through time linked to only one or a few underlying assets, therefore reducing the number of key pricing inputs, the number of Delta hedges that would be necessary at least in theory, and the number of key correlation coefficients that may significantly alter hedging ratios.

Finally, the last two columns of Table 12 present the same values in form of a "τ/100" coefficient, expressing the impact in basis points of a change of 0.01 in correlation coefficients, and reports the impact of such a change on each option's values in percentage terms.

A column of this kind could be easily used to determine which could be the reasonable size of the mark-up over the fair price that the seller (or buyer) of the exotic option may be willing to adopt if he is uncertain about the level of correlation coefficients. The risk manager too may be willing to use a similar approach when assessing whether the "mark-to-model" profits the trader may argue he has made through an exotic option portfolio could disappear in case of a slight change in correlation levels, or if – simply – they are resistant to the potential errors in correlation estimation. If a trader who just sold a 5-year Asian basket option such as the one evaluated in the test reported a (theoretical) profit equal to 40 basis points it might be useful to know that if "real" correlation coefficients were higher by 0.03 than the trader's estimates almost all such "mark-to-model" profit would disappear.

4.3. A comparison between copula-based and multivariate normal simulations

The third step of the analysis has been represented by introducing copulas so to check their impact on estimated fair prices, by simulating 10,000 paths with the same vector of raw independent random standard normal draws used for the multivariate normal case. Overall results can be summarized in Table 13, which reports the fair price for all the options depending on whether the dependence structure between assets' returns had been modeled either by assuming multivariate normally distributed returns or through copulas. In the case of the multivariate normality assumption (MVN) we maintain then distinction based on how the historical linear correlation matrix had been estimated, while copulas differ depending on the assumptions and estimation of marginal distribution parameters, as explained in Section 3.4.

Table 13. Fair prices for all option types depending on simulation method.

Simulation method	Multivariate normal				t-Student copula		
Option type	6 years, weekly returns	6 years, daily returns	3 years, weekly returns	3 years, daily returns	t-Student marginals, daily data	Normal marginals, daily data	t-Student marginals, weekly data
Asian basket option (5 yrs, strikes equal to initial prices)	17,935	18,559	19,599	20,658	20,540	17,316	20,037
Asian basket option (3+5 yrs, strikes equal to initial prices)	8,759	9,045	9,826	10,401	9,882	8,263	9,532
Asian basket option (3+5 yrs, strikes equal to average prices during the first 3 years)	15,741	16,065	16,735	17,362	17,223	15,332	16,819
Asian best option (5 yrs, 40% participation rate)	32,649	31,825	30,269	28,530	36,517	33,520	36,733
Asian best option (3+5 yrs, 40% participation rate)	19,896	19,842	18,664	18,175	21,570	20,617	21,113
Napoleon option (based on monthly basket returns, annual coupon 12%)	6,124	5,333	4,165	3,359	5,630	7,031	6,174
Conditional coupon structure (5 yrs, 8% annual coupon, barrier=60%)	5,540	5,904	6,966	7,808	5,351	5,013	5,220
Conditional coupon structure (3+5 yrs, 8% annual coupon, barrier=70%)	3,559	4,021	4,625	5,517	3,572	3,152	3,363
Fixed 80% coupon minus worst performance (5 yrs)	14,618	15,383	17,816	19,520	14,282	13,293	14,039
Fixed 80% coupon minus worst performance (3+5 yrs)	13,156	14,178	16,195	18,131	13,222	12,147	12,867
Fixed 25% coupon minus put on basket performance (5 yrs)	12,598	12,347	11,961	11,617	12,728	12,801	12,763
Fixed 25% coupon minus put on basket performance (3+5 yrs)	12,336	12,123	11,674	11,357	12,443	12,478	12,463

Table 13 clearly shows that on one hand using copulas instead than a MVN simulation has a remarkable effect on estimated fair price, but on the other hand that the effect of “model” choice (copulas against multivariate normality assumption) is almost as relevant as input choice in a simple MVN framework. As a consequence, input choice issues should be considered as crucial as model issues by a trader or by a risk manager.

Table 14. Fair values obtained through copulas versus standard multivariate normal method.

	Differences versus multivariate normal method (MVN)			$(P_{max}-P_{min}) / P_{avg}$			
	t-Student marginals, daily data vs 6-yrs sample, daily data MVN	Normal marginals, daily data, vs 6-yrs sample, daily data MVN	t-Student marginals, weekly data vs 6-yrs sample, weekly data MVN	All MVN simulations	MVN simulations based on 6-yrs sample	All copulas	Copulas with t-Student marginals
Asian basket option (5 yrs, strikes equal to initial prices)	10,7%	-6,7%	11,7%	14,19%	3,42%	16,71%	2,48%
Asian basket option (3+5 yrs, strikes equal to initial prices)	9,3%	-8,6%	8,8%	17,27%	3,22%	17,55%	3,61%
Asian basket option (3+5 yrs, strikes equal to average prices during the first 3 years)	7,2%	-4,6%	6,8%	9,84%	2,04%	11,49%	2,37%
Asian best option (5 yrs, 40% participation rate)	14,7%	5,3%	12,5%	13,37%	2,56%	9,03%	0,59%
Asian best option (3+5 yrs, 40% participation rate)	8,7%	3,9%	6,1%	8,99%	0,27%	4,52%	2,14%
Napoleon option (based on monthly basket returns, annual coupon 12%)	5,6%	31,8%	0,8%	58,27%	13,79%	22,30%	9,20%
Conditional coupon structure (5 yrs, 8% annual coupon, barrier=60%)	-9,4%	-15,1%	-5,8%	34,61%	6,36%	6,50%	2,48%
Conditional coupon structure (3+5 yrs, 8% annual coupon, barrier=70%)	-11,2%	-21,6%	-5,5%	44,18%	12,18%	12,50%	6,03%
Fixed 80% coupon minus worst performance (5 yrs)	-7,2%	-13,6%	-4,0%	29,12%	5,10%	7,13%	1,72%
Fixed 80% coupon minus worst performance (3+5 yrs)	-6,7%	-14,3%	-2,2%	32,28%	7,48%	8,43%	2,73%
Fixed 25% coupon minus put on basket performance (5 yrs)	3,1%	3,7%	1,3%	8,09%	2,01%	0,57%	0,27%
Fixed 25% coupon minus put on basket performance (3+5 yrs)	2,6%	2,9%	1,0%	8,25%	1,75%	0,29%	0,17%

To analyse more precisely the impact of copulas on estimated fair price, it may be useful to express it in percentage terms, and to discuss the variability that the choice of a simple parameter such as the number of degrees of freedom may have in determining the final estimated value. Data are reported in Table 14, where the simpler standard multivariate normal method is represented through the 6-year sample, weekly data case.

Results from Table 14 can be analyzed from different perspectives. First of all, using copulas seems to have a relevant effect on estimated prices, even if differences are in some cases lower than the ones that may be produced under the MVN assumption by using different historical samples to estimate linear correlation coefficients. Model risk, in a sense, seems as relevant as “input risk” in making the fair price estimate subjective.

Secondly, the assumptions on marginal distributions play a relevant role, since the copula with normal marginals produces fair values which are quite distant from the values produced with the same daily 6-year sample using copulas and t-Student marginals. Using the optimized copula under the assumption of normal marginals produces effects which are quite similar to those of a reduction in correlation coefficients: e.g., the values of long basket option positions and short worst put options decrease, the values of long best option and short basket option positions increase. Instead, using copulas with t-Student marginals produces completely different effects, since apart from the dependence structure one is allowing marginal return distributions to be fat-tailed (so that for instance the overall effect on a long basket option position such as the first three options is positive). In practice, using copulas is clearly different from a simple change in correlation parameters, since it implies a completely different “mechanics” of price co-movements, and it enables to more properly model even marginal distributions. As a consequence, it could be questioned to which extent it may be useful to resort to copulas in order to model the dependence structure among assets while maintaining much simpler assumptions (namely, normality) on marginal return distributions. The simulation with normal marginal returns seems in fact from a theoretical point of view to miss some of the opportunities for greater flexibility that the use of copulas gives to the trader or risk manager, and on the other side produces results which are quite distant from the fully developed copula-based simulations.

Thirdly, already issued options do not show a significantly lower variability of estimated fair price (in percentage). This is true both in the case of options on basket returns, and for best/worst type options, whose interpretation is more complex since after three years a clear best/worst stock may have emerged and in that case the way in which the marginal distribution of the asset is modeled may be as relevant as the choices concerning the dependence structure of all assets.

5. Implications for traders, risk managers and auditors

The empirical test whose results have been just displayed has many implications for at least three different players who may be concerned about correlation and correlation risk in a multiasset exotic options' desk.

Let us consider the trader first. The trader's typical activity is to price and then sell the exotic option (e.g. when the investment bank builds a structured bond for a defined counterparty) and then either hedge the position directly or hedge just part of the risk transferring the risk linked to individual equities to the desks that are in charge of it (through a sort of internal Delta hedge). Therefore, he may be concerned with (a) being wrong in pricing the exotic option when he sells it and (b) being wrong in hedging the position. Correlation is then important for two reasons. First, a poor correlation estimate may lead the trader to misprice the option and to hedge it poorly, since the hedging coefficients would be wrong as well. Second, correlations among underlying assets may change through time, and therefore a change in the level of correlation may produce a change in the option's price (and in hedging coefficients) that is very hard to hedge for the trader[8]. The huge bid-ask spread that is typical for the long-term exotic equity options which are embedded into structured equity-linked bonds can be at least partially explained with the need to compensate for those risks.

As far as the trader's viewpoint is concerned, our test suggests that if the trader believes in a multivariate normally distributed world but is concerned with correlation changes, a natural way to reduce the risk would be to try to balance inside the portfolio different kind of options with opposite exposure to correlation changes. For instance, while the value of an Asian best option would increase if correlation among underlying assets increases, a conditional coupon structure that pays a fixed coupon provided that the price of all the stocks included in the basket does not fall below a certain prespecified barrier has the opposite exposure. Anyway, this solution may not work in practice since it may be difficult to persuade many final customers to buy options on almost identical baskets so to hedge correlation risk almost entirely. Moreover, even if the trader sold an

[8] *A similar point has been made by Rebonato (1999, p. xiv) who stated (with reference to the different issue of OTC options' volatility smiles) that "a trader can hope to make money from a non-plain-vanilla options strategy if her view about the future evolution of some un-tradable key quantities, on which her hedge is based, turns out to be similar to what she assumed when pricing the option".*

Asian best option and a conditional coupon structure on the same set of underlying assets, the price of the former option would be driven after a while by the group of best performing assets, while the value of the latter would mostly depend on the price of the worst performing ones. The hedging problem could therefore be only partially tempered by proposing to final customers those options which either typically have a smoother behaviour in terms of Greeks (e.g. Asian-type options) or are likely to depend after a certain period mostly on a limited number of underlying assets (such as options with a best or worst feature, where even if the stock basket is large at the beginning after a while the performance is linked to a small subset only).

If instead the trader is not so convinced about the real joint return distribution, the impact on the fair price of different alternative assumptions may sometimes be reduced when the option's payoff is bounded in some manner, as it happens for instance for the last four options that can be conceived as spreads between a long position in an exotic option and a short position in a similar option with a different strike. Yet, if the trader uses more sophisticated methods – at least in order to control the prices he is quoting for new options – than the market, he could also plan to exploit the asymmetric information he has in his favour, so to suggest those options that may be overvalued under a simpler MVN pricing algorithm. The effect could be similar to what happens sometimes as far as volatility is concerned, so that some investment banks are said to suggest structured bonds where the investor is buying an option (e.g. an Asian basket call) when historical volatilities are higher than long-term implied volatilities, and to suggest structured bonds where the investor is selling an option (e.g. a put on the worst stock, or on the worst monthly return, as in the conditional coupon or in the Napoleon structure) when historical volatilities are lower than long-term implied volatilities. If this were the case, the uninformed customer who cannot observe implied volatilities but only historical ones would be lead to overestimate the bond's price as opposed to the "real" price the trader knows.

Obviously, a key issue is whether and to what extent the trader may be able to exploit this private superior information even internally, when providing the inputs to price the option. This is one of the key concerns for the risk manager. The results of our test unfortunately tell to the risk manager that the choice of both the set of inputs, even in the MVN, and especially of the assumptions on the joint distribution of asset returns may be critical. Therefore, the risk manager has in practice three alternative solutions.

The first one is to strive to detect both the best method and the best set of parameters in order to price any single option in his or her portfolio, continuously revising his or her optimal choice, and force the trader to use the “best” model. Unfortunately, this task may be too computationally intensive for a bank that has to revaluate a huge portfolio of options on different sets of underlying assets.

The second possible alternative solution is to define the best model once and then adopt it with only small “maintenance” costs based on a limited input revision. This would imply for instance to choose once the distribution function for marginal distributions and the copula to be adopted (e.g., a t-Student copula with 10 degrees of freedom) to simulate dependence among assets’ returns. After the initial choice, only a relatively minor estimation effort aimed at redefining marginal distribution and t-10 copula parameters would be needed. The same result could be achieved even if through time an industry standard emerged; in this case, many risk managers would probably assume the industry standard as a benchmark and refine only parameters within the “champion” model. In this case, of course, it remains questionable how frequently parameters could be revised. While in a very simplified setting a vector of implied volatilities and a correlation matrix would be sufficient to resume the trader’s estimates, in a more complicated setting the communication of all the underlying pricing assumption may become more complex, and inputs revision may inadvertently become slower. A second possible alternative might be to make some simplifying assumption on marginal distributions, such as assuming normal marginal distributions while concentrating calibration efforts on the copula structure. Yet, our test suggests that this choice may lead to completely different results from an “unconstrained” optimization of both marginal distribution and copula parameters.

The third possible solution would be to maintain the simple model for day close portfolio repricing and use the most sophisticated models as a pricing control tool when the option is issued on one side and as a measure to capture model risk on the other. Maximum differences or dispersion measures among theoretical prices (at single option level) or theoretical portfolio values (at desk level) may be considered as a rough proxy of the amount of model risk the trader is assuming, and is reflected on the chance of making both pricing mistakes and hedging mistakes. Since the trader responds to the existence of the greater pricing and hedging risk that a multiasset exotic option may generate by overpricing the option he sells relative to its fair value, a part of the overpricing should be considered as a provisioning against mispricing and mishedging. As a consequence, the markup against the theoretical fair price should be attributed to the derivatives desk only *pro rata* throughout the option's life span. To a certain extent, using the complex methodology just at the date of issue, using then a simpler (and easier to check) pricing methodology and forcing the trader to split the initial markup over time may be for the risk manager a more viable and less risky solution than attributing to the trader a result which is derived by the comparison of theoretical prices obtained through complex model whose inputs would be very hard to verify in an efficient manner for the risk manager.

The constraints deriving from the computational costs of (i) adapting to changing market conditions more sophisticated pricing models and (ii) having an independent check of the same inputs by an independent risk manager are even greater if we consider that the internal auditor too could be assigned the responsibility to check the consistency of pricing algorithms and risk management procedures. Again, the more complex are the methodologies and the tougher is the task for the auditor. Since complex methodologies typically require very skilled people whose cost may be higher than average, it is quite unlikely to assume that a certain set of competencies can be easily doubled or tripled inside the bank. Yet, a certain understanding of the most sophisticated method might be useful for a few auditors too, so to be able to check from time to time the methodology and the process that the front-office is adopting. Moreover, auditors too may sponsor the adoption of a provisioning system that may cover the bank through time from the risks deriving from inherently uncertain pricing and the consequent potentially poor hedging that is typical for most correlation products.

6. Concluding remarks

What are the implications of our study for the diffusion of copulas for pricing multiasset equity options? The differences emerging in section 4 point out that copulas may have a nontrivial impact on estimated fair prices, even if their effect is not always greater than the effect of alternative choice of input parameters even within a multivariate normal setting. Moreover, results are markedly different depending on how marginal distributions are modeled, so that the extent to which a trader or risk manager may want to exploit the greater flexibility that a copula-based simulation may grant in pricing an exotic option is critical. At the same time, this also points out that there may be still some arbitrary and critical choices the trader or the risk manager may make and that may have a relevant impact on the final prices. This is obviously bad news for those who may be in charge of controlling the reliability of the overall evaluations the bank is making, and who must face not only the problem of whether inputs have been carefully selected, but also of whether the model is appropriate. Of course, while it may be possible to check whether an implied volatility for an option with a given strike on a certain underlying asset is reasonable according to the market's average, it is very unlikely that a risk manager (or an auditor) may be able to check whether the number of degrees of freedom for a given copula is adequate or not and is agreed upon by other competitors.

As a consequence, further diffusion of copulas for pricing exotic options may become possible especially if clear best practices about parameters' estimation and "leading" copula functions will emerge. There is probably at present a lack in empirical literature on efficient methods for calibrating copulas in a high dimensional setting, as it is typically required in order to use them consistently and continuously to price multiasset derivatives.

At the same time, and even experimentally, the use of copulas may be important to point out to risk managers and auditors where the greatest mispricing risks may lie within the bank's derivatives portfolio. While a lot of improvements have been made in checking implied volatility inputs, the way in which dependence among returns is modeled, relevant parameters are calibrated, correlation risk is quantified and – if not reduced through a proper balanced portfolio with "correlation-bullish" and "correlation-bearish" options – at least covered by appropriate provisioning, represent a set of topics that need great attention for those banks that run huge portfolios of complex equity derivatives.

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