Copula's Conditional Dependence Measures for Portfolio Management and Value at Risk

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Abstract

Traditional portfolio theory based on multivariate normal distribution assumes that investors can benefit from diversification by investing in assets with lower correlations. However, this is not what happens in reality, since it is quite easy to see financial markets with different correlations but almost the same numbers of market crashes (if we define market crash as when returns are in their lowest qth percentile). In a similar fashion, recent empirical studies show that in volatile periods financial markets tend to be characterized by different level of dependence than occurs in quiet periods. In order to take into account this reality, we resort to copula theory and its conditional dependence measures, like Kendall's Tau and Tail dependence. The former satisfies most of the desired properties that a dependence measure must have and it can detect non-linear association that correlation cannot see. Tail dependence refers to the dependence that arises between random variables from extreme observations. We consider a portfolio made up of the five most important future contracts actually traded in American markets and we take into consideration the most volatile period of the last decade, that is between March 13th 2000 until June 9th 2000. We show how these conditional dependent measure can be easily implemented both in the traditional mean-variance framework and in multivariate VaR estimation, with a significant improvement over traditional multivariate correlation analysis.

1 Introduction

Traditional portfolio theory based on multivariate normal distribution assumes that investors can benefit from diversification by investing in assets with lower correlations. However this is not what happens in reality, since it is quite easy to see financial markets with different correlations but almost the same numbers of market crashes (if we define market crash as an event when returns are in their lowest qth percentile).

Correlation is a good measure of dependence in multivariate normal distributions but it has several shortcomings: a) The variances of the random variables must be finite for the correlation to exist, and for fat-tailed distributions this cannot be the case (a bivariate t-distribution with 2 degrees of freedom, for example); b) Independence between two random variables implies that linear correlation is zero, but the converse is true only for a multivariate normal distribution. This does not hold when only the marginals are Gaussian while the joint distribution is not normal, because correlation reflects linear association and not non-linear dependency ; c) Correlation is not invariant to strictly monotone transformations. This is because it depends not only on the joint distribution but also on the marginal distributions of the considered variables, so that changes of scales or other transformations in the marginals have an effect on correlation.

In order to overcome these problems we can resort to copula theory, since copulae capture those properties of the joint distribution which are invariant under strictly increasing transformation. A common dependence measure that can be expressed as a function of copula parameters and is scale invariant is Kendall's tau. It satisfies most of the desired properties that a dependence measure must have (see Nelsen 1999) and it measures concordance between two random variables: concordance arises if large values of one variable are associated with large values of the other, and small ones occur with small values of the other; if this is not true the two variables are said to be discordant. It is for this reason that concordance can detect nonlinear association that correlation cannot see.

As asset log return distributions are not normally distributed, the minimization of the portfolio's variance do not minimize portfolio risk and produce the wrong capital allocation. New risk measures have been proposed to obtain better capital allocations, but at the cost of simplicity and computational tractability: this is why most applied professionals skip them and prefer to rely on previous methods, similar to other financial fields (just think back to the Black & Sholes pricing formula and Garch(1,1), which are still by far the most used models for option pricing and volatility forecasting).

In order to satisfy this demand for understandable models, we propose here to use Kendall's tau dependence measure within the traditional mean-variance framework, in the place of the correlation coefficients: this solution has the advantage of keeping the model tractable but at the same time considering the non-linear dependency among the considered variables.

In a similar fashion, recent empirical studies show that in volatile periods financial markets tends to be characterized by different level of dependence than occurs in quiet periods. In order to take into account this reality, we propose to use the concept of Tail dependence, which refers to the dependence that arises between random variables from extreme observations. An important feature of copulae is that they allow for different degrees of tail dependence: Upper tail dependence exists when there is a positive probability of positive outliers occurring jointly, while lower tail dependence is symmetrically defined as the probability of negative outliers occurring jointly.

What we propose is a direct consideration of this concept in VaR models by means of copula theory, as tail dependence coefficients can be calculated as simple functions of copulae parameters: if we follow the well-known RiskMetrics multiple positions VaR model, tail dependence coefficients can be used in the place of linear correlation coefficients. The same approach can be followed with Kendall's Tau, too.

What we do in this work is to analyze the five most important future contracts actually traded in American markets (SP500, Dow Jones, Nasdaq100, Euro Dollar, T Bond Note) with high frequency data sampled at 5 – minutes frequency, taking into consideration the most volatile period of the last decade, that is between March 13 th 2000 till June 9th 2000. This period of time witnessed the falling of world financial markets following the burst of the high-tech bubble, with big intraday draw down returns. Thus, this sample is perfectly suited to highlight the importance of the copula-based dependence approach compared to the traditional correlation analysis.

We build up a portfolio and a multiple VaR position following both two approaches, using the initial part of the sample to estimate the assets' weights and the 95% (99 %) VaR, and the remaining part to compare the out-of-sample performances of the two approaches, both in terms of risk-returns measures and number of VaR exceedances of the effective portfolio losses. We show that our approach outperform the correlation based one both in terms of portfolio results and VaR back-testing.

The rest of the paper is organized as follows. In Section 2 we provide an outline of copula theory while in Section 3 we present its conditional dependence measures, that is Kendall's Tau and Tail Dependence; in Section 4 we introduce the main methods for copula estimation while in section 5 we show how to use copula dependence measure within portfolio management and multivariate VaR estimation. Section 6 presents the empirical results on the asset allocation problem and VaR estimation for a portfolio of five future contracts actually traded in American markets. We conclude in Section 7.

2 Copula Theory

2.1 Unconditional Copulas

An n-dimensional copula is basically a multivariate cumulative distribution function with uniform distributed margins in [0,1]. Let consider $X_1, \dots X_n$ to be random variables, W the conditioning variable and H the joint distribution function, and we further assume that the distribution function H has all the required derivatives, we have:

Definition 1 (Unconditional copula): The unconditional copula of $(X_1, \dots X_n)$, where $X_1 \sim F_1, \dots X_n \sim F_n$, and $F_1 \dots F_n$ are continuous, is the joint distribution function of $U_1 \equiv F_1(X_1) \dots U_n \equiv F_n(X_n)$.

The variables $U_1 \dots U_n$are the '*probability integral transforms*' of $X_1 \dots X_n$ which follow a Uniform(0,1) distribution, regardless of the original distribution, F. Thus a copula is a joint distribution of *Unif* (0,1) random variables.

Proposition 1 (Properties of a unconditional copula): A copula is a function $C$of n variables on the unit n-cube $[0,1]^n$ with the following properties:

1. The range of $C\ (u_1, u_2, ..., u_n)$ is the unit interval [0,1];
2. $C\ (u_1, u_2, ..., u_n) = 0$ if any $u_i = 0$, for i = *1, 2, ..., n*.
3. $C\ (1, ..., 1, u_i, 1, ..., 1) = u_i$, for all $u_i \in [0, 1]$
4. $C\ (u_1, u_2, ..., u_n)$ is n-increasing in the sense that for every $\mathbf{a} \leq \mathbf{b}$ in $[0, 1]^n$ the measure ΔC_a^b assigned by C to the n-box $[\mathbf{a}, \mathbf{b}] = [\mathrm{a}_1, \mathrm{b}_1] \cdot \ldots \cdot [\mathrm{a}_n, \mathrm{b}_n]$ is nonnegative, that is

$$\Delta C_a^b = \sum_{(\varepsilon_i, \dots, \varepsilon_n) \in \{0,1\}^n} (-1)^{\sum_{i=1}^{n} \varepsilon_i} C(\varepsilon_1 a_1 + (1-\varepsilon_1) b_1, ..., \varepsilon_n a_n + (1-\varepsilon_n) b_n) \geq 0$$

This definition shows that C is a multivariate distribution function with uniformly distributed margins. Copulae have many useful properties, such as uniform continuity and (almost everywhere) existence of all partial derivatives, just to mention a few (see Nelsen (1999), Theorem 2.2.4 and Theorem 2.2.7). Moreover, it can be shown that every copula is bounded by the so-called Fréchet-Hoeffding bounds,

$$\max(u_1 + \ldots + u_n - n + 1, 0) \leq \quad C\ (u_1, u_2, ..., u_n) \quad \leq \min\ (u_1, \ldots, u_n)$$

which are commonly denoted by $W$and M. In two dimensions, both of the Fréchet-Hoeffding bounds are copulas themselves, but as soon as the dimension increases, the Fréchet-Hoeffding lower bound W is no longer n-increasing. However, the inequality on the left-hand side cannot be improved, since for any $\boldsymbol{u}$ from the unit $\boldsymbol{n}$-cube, there exists a copula C_u such that $W(\boldsymbol{u}) = C_u(\boldsymbol{u})$ (see Nelsen (1999), Theorem 2.10.12).

We now present the Sklar's theorem, which justifies the role of copulas as dependence functions:

Theorem 1: **(Sklar's theorem):** Let H denote a n-dimensional distribution function with margins $F_1 \dots F_n$. Then there exists a n-copula C such that for all real $(x_1, \dots, x_n)$

$$H(x_1, \dots, x_n) = C(F(x_1), \dots, F(x_n))$$

If all the margins are continuous, then the copula is unique. Moreover, the converse of the above statement is also true and it is the most interesting for multivariate density modeling, since it implies that we may link together any $n \geq 2$ univariate distributions, of any type (not necessarily from the same family), with any copula in order to get a valid bivariate or multivariate distribution.

Corollary 1: Let $F_1^{(-1)}, \ldots, F_n^{(-1)}$ denote the generalized inverses of the marginal distribution functions, then for every $(u_1, \ldots, u_n)$ in the unit n-cube, exists a unique copula C : [0,1] $\times \ldots \times$ [0,1] $\rightarrow$ [0,1] such that

$$C\ (u_1, ..., u_n) = H\ (F_1^{(-1)}(u_1),\ \ldots,\ F_n^{(-1)}(u_n))$$

See Nelsen (1999) for a proof, Theorem 2.10.9 and the references given therein. From this corollary we know that given any two marginal distributions and any copula we have a joint distribution. A copula is thus a function that, when applied to univariate marginals, results in a proper multivariate pdf: since this pdf embodies all the information about the random vector, it contains all the information about the dependence structure of its components. Using copulas in this way splits the distribution of a random vector into individual components (marginals) with a dependence structure (the copula) among them without losing any information. It is important to highlight that this theorem does not require$F_1$and F_n to be identical or even to belong to the same distribution family.

By applying Sklar's theorem and using the relation between the distribution and the density function, we can derive the multivariate copula density $c(F_1(x_1),, \ldots, F_n(x_n))$, associated to a copula function $C(F_1(x_1),, \ldots, F_n(x_n))$:

$$f(x_1, ..., x_n) = \frac{\partial^n \left[C(F_1(x_1), \ldots, F_n(x_n))\right]}{\partial F_1(x_1), \ldots, \partial F_n(x_n)} \cdot \prod_{i=1}^{n} f_i(x_i) = c(F_1(x_1), \ldots, F_n(x_n)) \cdot \prod_{i=1}^{n} f_i(x_i)$$

where

$$c(F_1(x_1),\ ...,\ F_n(x_n)) = \frac{f(x_1, ..., x_n)}{\prod_{i=1}^{n} f_i(x_i)} \cdot\ , \tag{1}$$

The copula density will be later used to estimate its parameters to real market data.

2.2 Conditional Copulas

Time series analysis is often interested in random variables conditioned on some variables, this is why we now discuss how the existing results in copula theory may be extended to allow for conditioning variables. We consider here the bivariate case since it will be later used for empirical analysis. We will furthermore assume that the dimension of the conditioning variable, W, is 1.

Definition 2 (Conditional copula): The conditional copula of $(X_1, X_2)\,|W$, where $X_1|W \sim F_1$ and $X_2\ |W \sim F_2$, is the conditional joint distribution function of $U_1 \equiv F_1\ (X|\,W)$ and $U_2 \equiv F_2\ (Y\,|\,W)$ given W.

A two-dimensional conditional copula is derived from any distribution function such that the conditional joint distribution of the first two variables given the remaining variables is a copula for all values of the conditioning variables. It is simple to show that it has the following properties:

Proposition 1 (Properties of a conditional copula): A two-dimensional conditional copula has the following properties:

1. It is a function C : [0,1] $\times$ [0,1] $\times\ W \rightarrow$ [0,1]
2. $C(u_1,\, 0|w) = C(0,\, u_2|w) = 0$, for every u_1, u_2 in [0,1] and each $w \in W$
3. $C(u_1,\, 1|w) = u_1$ and $C(1,\, u_2|w) = u_2$, for every $u_1,\, u_2$ in [0,1] and each $w \in W$
4. $C(u_1,\, u_2|w)$ is grounded and n-increasing.

For the proof see Patton (2003). It is easy to observe that in this case a two-dimensional conditional copula is the conditional joint distribution of two *conditionally* Uniform(0,1) random variables.

Theorem 2 (Sklar's theorem for continuous conditional distributions): Let F_1 be the conditional distribution of $X_1| W$, F_1 be the conditional distribution of $F_1 \ | W$, and H be the joint conditional distribution of $(X_1, X_2 \ ) \,| W$. Assume that F_1 and F_2 are continuous in x_1 and x_2. Then there exists a unique conditional copula C such that

$$H(x_1, x_2|w) = C(F_1 \ (x_1|w),\ F_2 \ (x_2|w) \ |w),$$

Conversely, if we let F_1 be the conditional distribution of $X_1| W$, F_2 be the conditional distribution of $X_2 \ | W$, and C be a conditional copula, then the function H defined by equation (1.3) is a conditional bivariate distribution function with conditional marginal distributions F and G(Patton, 2003).

The Sklar's theorem for conditional distributions implies that the conditioning variable(s), W, must be the same for both marginal distributions and the copula: if we do not use the same conditioning variable for F, G and C, the function H will not be, in general, a joint conditional distribution function. The only case when $\hat{H}$will be the joint distribution of $(X_1, X_2 \ ) \mid (W_1{,}W_2)$ is when $F \ (x_1|W_1) = F \ (x_1|W_1{,}W_2)$ and $F(x_2, \ |W_2) = F(x_2, \ |W_1{,}W_2)$, that is when some variables affect the conditional distribution of one variable but not the other.

It is straightforward to see that we can derive an equivalent to Corollary 1 for the conditional case, so that it is possible to extract the implied conditional copula from any bivariate conditional distribution.

2.3 Elliptical Copulas

The class of elliptical distributions provides useful examples of multivariate distributions because they share many of the tractable properties of the multivariate normal distribution. Furthermore, they allow to model multivariate extreme events and forms of non-normal dependencies. Elliptical copulas are simply the copulas of elliptical distributions. If we follow Fang, Kotz and Ng (1987), we have the following definition:

Definition 3 (Elliptical distribution): let $\mathbf{X}$ be a n-dimensional vector of random variables and, for some $\mu \in \Re^n$and some n x n nonnegative definite symmetric matrix Σ , the characteristic function $\varphi_{\ X-\mu}$ is a function of the of the quadratic form $\mathbf{t}^T\Sigma\ \mathbf{t}$, then $\mathbf{X}$ has an elliptical distribution with parameters $(\mu,\ \Sigma,\ \varphi\ )$and we write $\mathbf{X} \sim \mathbf{E}_n\ (\mu,\ \Sigma,\ \varphi)$.

We present now two copulae belonging to the elliptical family and that will be later used in empirical applications, the Gaussian and T-copula.

2.3.1 Gaussian copula

The Gaussian copula is the copula of the multivariate normal distribution. In fact, the random vector X=(X_1,. . . ,X_n) is multivariate normal if the univariate margins $F_1, \ldots, F_n$ are Gaussians, and the dependence structure among the margins is described by the following copula function

$$C(u_1, ..., u_n; \Sigma) = \Phi^K(\Phi^{-1}(u_1), ..., \Phi^{-1}(u_n); \Sigma) \tag{2}$$

where Φ^k is the standard multivariate normal distribution function with linear correlation matrix Σ and Φ^{-1} is the inverse of the standard univariate Gaussian. When $n = 2$, expression (**??**) can be written as:

$$C(u_1, u_2; \rho) \ = \int_{-\infty}^{\Phi^{-1}(u)} \int_{-\infty}^{\Phi^{-1}(u)} \frac{1}{2\pi\sqrt{(1-\rho^2)}} \exp\left\{\frac{-(r^2 - 2\rho rs + s^2)}{2(1-\rho^2)}\right\} dr\, ds,$$

where ρ is the linear correlation coefficient between the two random variables.

The copula density is derived by applying equation (**??**):

$$c(\Phi(x_1),\ ...,\ \Phi(x_n)) = \frac{f^{Gaussian}(x_1,...,x_n)}{\prod\limits_{i=1}^{n} f_i^{Gaussian}(x_i)} = \frac{\frac{1}{(2\pi)^{n/2}|\Sigma|^{1/2}}\exp\left(-\frac{1}{2}x'\Sigma^{-1}x\right)}{\prod\limits_{i=1}^{n}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}x_i^2\right)} \ = \frac{1}{|\Sigma|^{1/2}}\exp\left(-\frac{1}{2}\zeta'(\Sigma^{-1}-I)\zeta\right)$$

where $\zeta = (\Phi^{-1}(u_1), ..., \Phi^{-1}(u_K))'$ is the vector of the Gaussian univariate inverse distribution functions, and $u_i = \Phi\ (x_i)$.

2.3.2 T - copula

The copula of the multivariate standardized t-Student distribution is the t-Student copula and is defined as follows:

$$C(u_1, \ldots, u_n; \Sigma, \nu) = T_{R,\nu}(t_\nu^{-1}(\hat{u}_1), \ldots, t_\nu^{-1}(\hat{u}_n))' \tag{3}$$

where $T_{\Sigma,\nu}$ is the standardized multivariate Student's t distribution function, Σ is the correlation matrix, ν are the degrees of freedom, $t_\nu^{-1}\ (u)$ denotes the inverse of the Student's t cumulative distribution function. When $n = 2$, expression (**??**) can be written as:

$$C^{Student'st}(u,\ v,\rho) = \int\limits_{-\infty}^{t_\nu^{-1}(u)}\int\limits_{-\infty}^{t_\nu^{-1}(v)} \frac{1}{2\pi\sqrt{(1-\rho^2)}}\exp\left\{1 + \frac{r^2 - 2\rho rs + s^2)}{2(1-\rho^2)}\right\} dr\, ds,$$

where ρ is the linear correlation coefficient between the two random variables. The copula density is again derived by applying equation (**??**):

$$c(t_\upsilon^(x_1),\ ...,\ t_\upsilon^(x_n)) = \frac{f^{Student}(x_1,...,x_n)}{\prod\limits_{i=1}^{n} f_i^{Student}(x_i)} \ = \ |\Sigma|^{-1/2}\frac{\Gamma\left(\frac{\upsilon+n}{2}\right)}{\Gamma\left(\frac{\upsilon}{2}\right)}\left[\frac{\Gamma\left(\frac{\upsilon}{2}\right)}{\Gamma\left(\frac{\upsilon+1}{2}\right)}\right]^n \frac{\left(1+\frac{\zeta'\Sigma^{-1}\zeta}{\upsilon}\right)^{-\frac{\upsilon+n}{2}}}{\prod\limits_{i=1}^{n}\left(1+\frac{\zeta_i^2}{2}\right)^{-\frac{\upsilon+1}{2}}}, \tag{4}$$

where $\zeta = (t_\upsilon^{-1}(u_1), ..., t_\upsilon^{-1}(u_K))'$ is the vector of the T-student univariate inverse distribution functions and .

2.4 Archimedean Copulas

Archimedean copulae provide analytical tractability and a large spectrum of different dependence measure. These copulae can be used in a wide range of applications for the following reasons: a) The ease with which they can be constructed; b) The many parametric families of copulas belonging to this class; c) The great variety of different dependence structures; d) The nice properties possessed by the members of this class. An Archimedean copula can be defined as follows:

Definition 4 (Archimedean copula): let consider a function φ : [0; 1] $\rightarrow$ [0;1] which is continuous, strictly decreasing φ'$(u) < 0$, convexφ"$(u) > 0$, and for which $\varphi\ (0) = \infty$ and $\varphi\ (1) = 0$. We then define the pseudo inverse of $\varphi^{[-1]}$: [0; ∞] $\rightarrow$ [0;1] such that :

$$\varphi^{[-1]}(t) = \left\{ \begin{array}{ll} \varphi^{-1}(t) & for\ 0 \le t \le \varphi(0) \\ 0 & for\ \varphi(0) \le t \le \infty \end{array} \right\}$$

As φ is convex , the function C: [0; 1]^2 $\rightarrow$ [0;1] defined as

$$C(u_1, u_2) = \varphi^{-1}[\varphi(u_1) + \varphi(u_2)] \tag{5}$$

is an Archimedean copula and φ is called the "generator" of the copula. Moreover, if $\varphi\ (0) = \infty$, the pseudo inverse describes an ordinary inverse function (that is $\varphi^{[-1]} = \varphi^{(-1)}$) and we call φ and

C, a strict generator and a strict Archimedean copula, respectively.

The multivariate extension can be found in Embrechtrs, Lindskog and McNeil (2001) as well as in Joe (1997): for all $n \geq 2$, the function C: $[0; 1]^n \rightarrow [0;1]$ defined as $C(u_1, \ldots, u_n) = \varphi^{-1}$ $[\varphi(u_1)$ $+ \ldots + \varphi(u_n)]$, is an n-dimensional Archimedean copula if and only if $\varphi^{-1}$is completely monotone on $[0, \infty)$.

We present now some of the most important multivariate Archimedean copulas.

1) Clayton (or Cook Johnson) copula: it corresponds to family B4 of Joe(1997). Let consider the generator φ (t) = ($\mathrm{t}^{-\alpha}$ - 1) / α , with $\alpha \in$ [-1, ∞) $\setminus \{0\}$ and inverse φ $^{-1}$(t) = (1 + t) $^{-1/\alpha}$.By using (**??**) we get,

$$C(u_1, \ldots, u_n) = \max \left[\left(\sum_{j=1}^{N} u_j^{-\alpha} - n + 1 \right)^{-1/\alpha} , \quad 0 \right]$$

However, if α ¿ 0 then we have φ (0) = ∞ , and the above expression become

$$C(u_1, \ldots, u_n) = \left(\sum_{j=1}^{N} u_j^{-\alpha} - n + 1 \right)^{-1/\alpha}$$

2) Gumbel copula: it corresponds to family B6 of Joe(1997). The generator is φ (t) = (- ln t)$^\alpha$, with $\alpha \geq 1$ and the inverse is φ $^{-1}$ (t) = exp{- t $^{1/\alpha}$} . We have,

$$C(u_1, \ldots, u_n) = \exp\Big\{ - [(-\ln u_1)^\alpha + ... + (-\ln u_N)^\alpha]^{1/\alpha} \Big\}$$

3) Survival (or rotated) Gumbel copula: The rotation allows us to take a copula exhibiting greater dependence in the negative (positive) quadrant and create one with greater dependence in the positive (negative) quadrant. The distribution function in this case is

$$C(u_1, \ldots, u_n) = \left(\sum_{j=1}^{N} u_j - n + 1 \right) + \exp\Big\{ - [(-\ln(1 - u_1)^\alpha + ... + (-\ln(1 - u_N)^\alpha]^{1/\alpha} \Big\}$$

3 Dependence Measures

Traditional portfolio theory based on multivariate normal distribution assumes that investors can benefit from diversification by investing in assets with lower correlations; however, correlation is a good measure of dependence in multivariate normal distributions but it presents several shortcomings: a) The variances of X_1 and X_2 must be finite for the correlation to exist, and for fat-tailed distributions this cannot be the case (a bivariate t-distribution with 2 degrees of freedom, for example); b) Independence between two random variables implies that linear correlation is zero, but the converse is true only for a multivariate normal distribution. This does not hold when only the marginals are Gaussian while the joint distribution is not normal, because correlation reflects linear association and not non-linear dependency ; c) Correlation is *not* invariant to strictly monotone transformations. This is because it depends not only on the joint distribution but also on the marginal distributions of the considered variables, so that changes of scales or other transformations in the marginals have an effect on correlation. d) Given the marginal distributions F_1 and F_2 for two random variables X and Y , all linear correlations between –1 and +1 cannot be attained through suitable specification of the joint distribution F (see Höffding, 1940).

Moreover, there are also statistical problems with correlation, as a single observation can have an arbitrarily high influence on the linear correlation. Thus linear correlation is not a robust measure.

3.1 Kendall's Tau

In order to avoid the above problems, we have to turn to rank correlation; however, we first need to define the concepts of concordance and discordance:

Definition 5 (Concordance): Observations $(x_i; y_i)$ and $(x_j\ ; y_j)$ are concordant, if $x_i < y_i$ and $y_i < y_i$, or if $x_i > x_j$ and $y_i > y_j$. Analogously, $(x_i; y_i)$ and $(x_j\ ; y_j)$ are discordant if $x_i < y_i$ and $y_i > y_j$ or if $x_i > x_j$ and $y_i < y_j$. Alternatively, $(x_i; y_i)$ and $(x_j\ ; y_j)$ are concordant, if $(x_i$ - $x_j)(y_i - y_j) > 0$ and discordant if $(x_i$ - $x_j)(y_i$ - $y_j) < 0$.

Concordance arises if large values of one variable are associated with large values of the other, and small ones occur with small values of the other; if this is not true the two variables are said to be discordant. It is for this reason that concordance can detect nonlinear association that correlation cannot see.

If we have $(X_1; Y_1)$ and $(X_2; Y_2)$ independent and identically distributed random vectors, the population version of Kendall's tau τ(X; Y) rank correlation is defined by

$$\tau(X, Y) = P\left[(X_1 - X_2)(Y_1 - Y_2) > 0\right] - P\left[(X_1 - X_2)(Y_1 - Y_2) < 0\right]$$

that is, $\tau\ (X, Y\ )$ is an estimate of the probability of concordance minus the probability of discordance. The sample version of the measure of dependence known as Kendall's tau is defined in terms of concordance as follows:

Definition 6 (Kendall's Tau - sample version): Let us denote a random sample of n observations $\{(\mathrm{X}_1, \mathrm{Y}_1); (\mathrm{X}_2, \mathrm{Y}_2), \ldots, (\mathrm{X}_n; \mathrm{Y}_n)\}$ from a vector $(\mathbf{X}, \ \mathbf{Y}\ )$ of continuous random variables. There are distinct pairs $(\mathrm{X}_i; \mathrm{Y}_i)$ and $(\mathrm{X}_j; \mathrm{Y}_j)$ of observations in the sample, and each pair is either concordant or discordant; if we call c the number of concordant pairs, and d the number of discordant pairs, then an estimate of Kendall's rank correlation for the sample is given by

$$\tau = \tfrac{c-d}{c+d} = (c-d)/\begin{pmatrix} n \\ 2 \end{pmatrix}$$,which can be alternatively expressed as follows

$$\hat{\tau} = \begin{pmatrix} n \\ 2 \end{pmatrix}^{-1} \sum_{i<j} sign\ \left[(X_i - X_j)(Y_i - Y_j)\right]$$

$\tau\ (X, Y\ )$ can be considered a measure of the degree of monotonic dependence between X and Y , whereas linear correlation measures the degree of linear dependence only.

The generalisation of Kendall's Tau to n ¿ 2 dimensions is analogous to the procedure for linear correlation, where we have a n x $n$matrix of pairwise correlation. The main properties of this dependence measure are reported here:

Theorem 3 (Kendall's Tau properties): Let X and Y be random variables with continuous distributions $\mathrm{F}_1$and F_2, joint distribution H and copula C. The following are true:

(1) $\tau\ (X, Y\ ) = \tau\ (Y, X\ )$.

(2) If X and $Y$are independent then $\tau\ (X, Y\ ) = 0$.

(3) $-1 \le \tau\ (X, Y\ ) \le +1$

(4) For T : $\mathbf{R} \rightarrow \mathbf{R}$ strictly monotonic on the range of X, $\tau\ (X, Y\ )$ satisfies

$$\tau(T(X), Y) = \begin{cases} \tau(X, Y) & if\ \ T \text{ increasing} \\ -\tau(X, Y) & if\ \ T \text{ decreasing} \end{cases}$$

Proof: (1) to (3) follows from the fact that $\tau\ (X, \ \ Y\ )$ is a measure of concordance, while for (4) see Embrechts, McNeil and Straumann (1999) and references therein.

Kendall's Tau can be expressed in terms of copula functions, thus simplifying calculus.

Proposition 2 (Kendall's Tau for copulae): $\tau(X,Y) = 4\int_0^1 \int_0^1 C(u,v)dC(u,v) - 1$

Proof: See Nelsen (1999).

Evaluating Kendall's Tau requires the evaluation of a double integral and for elliptical distributions this is not an easy task: this problem was solved by Lindskog, McNeil, and Schmock (2002) who proved this theorem:

Proposition 3 (Kendall's Tau for elliptical distributions): Let $\mathbf{X} \sim \mathbf{E}_n$ (μ, Σ, φ). If i,j $\in$ {1, . . . , p} satisfy $\boldsymbol{P}\{X_i = \mu_i\}$ ¡ 1 and $\boldsymbol{P}\{X_j = \mu_j\}$ ¡ 1, then

$$\tau(X_i, X_j) = \left(1 - \sum_{x \in R} (P\{X_i = x\})^2\right) \frac{2}{\pi} \arcsin \rho_{ij} \tag{6}$$

where the sum extends over all atoms of the distribution of X_i and ρ_{ij} is the correlation coefficient.

If in addition rank(Σ) $\geq$ 2, then (**??**) simplifies to

$$\tau(X_i, X_j) = \left(1 - (P\{X_i = \mu_i\})^2\right) \frac{2}{\pi} \arcsin \rho_{ij}$$

which further simplifies if $\boldsymbol{P}\{X_i = \mu_i\}$=0,

$$\tau(X, Y) = \frac{2}{\pi} \arcsin \rho_{xy} \tag{7}$$

Proof: See Lindskog, McNeil, and Schmock (2002)

For an Archimedean copula the situation is simpler, in that $\tau(X,Y)$ can be evaluated directly from the generator of the copula. Archimedean copulas are easy to work with because expressions with a function of only one argument (the generator) can often be employed rather than expressions with a function of two arguments (the copula).

Proposition 4 (Kendall's Tau for Archimedean Copulae): Let X and Y be random variables with an Archimedean copula C having generator φ. The population version of τ (X, Y) for X and Y is given by

$$\tau(X, Y) = 1 + 4 \int_0^1 \frac{\varphi(t)}{\varphi'(t)} dt$$

Proof: See Genest and MacKay (1986)

Straightforward calculations show that for Clayton and Gumbel copulae we have the following results:

Clayton: $\tau(X,Y) = \alpha$ / (α + 2)

Gumbel = Rotated gumbel = $\tau(X,Y) = 1$ - α^{-1}

3.2 Tail Dependence

As we previously saw, linear correlation present many problems, mainly when we work with heavy-tailed distributions. Copulas functions present the nice property to be able to model Tail dependence: this measure refers to the dependence that arises between random variables from extreme observations; it is easy to see that in volatile periods financial markets tends to be characterized by different level of dependence than occurs in quite periods. An important feature of copulae is that they allow for different degrees of tail dependence: *Upper* tail dependence exists when there is a positive probability of positive outliers occurring jointly, while *lower* tail dependence is symmetrically defined as the probability of negative outliers occurring jointly.

Definition 7 (Upper tail dependence): Let $(X_1,\ X_2)$ be a bivariate vector of continuous random variables with marginal distribution functions F_1 and F_2, the coefficient of *upper tail dependence* of X and Y is:

$$\lambda^U = \lim_{u\to 1} \Pr[X_2 > F_2^{-1}(u)|X_1 > F_1^{-1}(u)] = \lim_{u\to 1} \Pr[X_1 > F_1^{-1}(u)|X_2 > F_2^{-1}(u)]$$

$$= \lim_{u\to 1} \frac{\Pr[X_2>F_2^{-1}(u),X_1>F_1^{-1}(u)]}{\Pr[X_1>F_1^{-1}(u)]} \quad = \lim_{u\to 1} \frac{1-\Pr[X_1\le F_1^{-1}(u)]-\Pr[X_2\le F_2^{-1}(u)]+\Pr[X_2\le F_2^{-1}(u),X_1\le F_1^{-1}(u)]}{1-\Pr[X_1\le F_1^{-1}(u)]}$$

$$= \lim_{u\to 1} \frac{(1-2u+C(u,u))}{(1-u)} = \lambda^U$$

provided a limit $\lambda_U \in [0,1]$ exists. Here, $F^{-1}(u) = \inf\{x|F(x) \geq u\}$, $u \in (0,1)$.

X_1 and X_2 are said to be *asymptotically dependent* in the upper tail if $\lambda^U \in (0,\ 1]$ and *asymptotically independent* if $\lambda^U = 0$. Upper tail dependence exists when there is a positive probability of positive outliers occurring jointly.

Definition 8 (Lower tail dependence): Let $(X_1,\ X_2)$ be a bivariate vector of continuous random variables with marginal distribution functions F_1 and F_2, the coefficient of *lower tail dependence* of X_1 and X_2 is:

$$\lambda^L = \lim_{u\to 0} \Pr[X_2 \le F_2^{-1}(u)|X_1 \le F_1^{-1}(u)] = \lim_{u\to 0} \Pr[X_1 \le F_1^{-1}(u)|X_2 \le F_2^{-1}(u)] = \lim_{u\to 0} \frac{C(u,u)}{u} = \lambda^L$$

provided a limit $\lambda^L \in [0,1]$ exist. X_1 and $X_2$are said to be asymptotically dependent in the lower tail if $\lambda^L \in (0,\ 1]$ and asymptotically independent if $\lambda^L = 0$. Lower tail dependence exists when there is a positive probability of negative outliers occurring jointly.

3.2.1 Tail dependence for Elliptical copulae.

When dealing with copulae without closed form expressions, such as the Gaussian and the Student's t copula, an alternative version is provided by Embrechts, Lindskog and McNeil (2003). First by applying De l'Hopital theorem, we obtain:

$$\lambda^U = \lim_{\mathrm{u}\to 1}\ (1 - 2\mathrm{u}\ +\ \mathrm{C}\ (\mathrm{u,u}))\ /\ (1 - \mathrm{u})\ = -\lim_{u\to 1}\left[-2 + \frac{\partial}{\partial x_1}C(x_1,x_2)|_{x_1=x_2=u} + \frac{\partial}{\partial x_2}C(x_1,x_2)|_{x_1=x_2=v}\right]$$

Then, by the definition of copula function and by the expression of the copula density, we have:

$$\mathrm{P}(V \le v|U = u) = \tfrac{\partial}{\partial u}C(u,v), \quad \mathrm{P}(\mathrm{U} \le u|V = v) = \tfrac{\partial}{\partial v}C(u,v)$$
$$\mathrm{P}(V > v|U = u) = 1 - \tfrac{\partial}{\partial u}C(u,v), \quad \mathrm{P}(\mathrm{U} > u|V = v) = 1 - \tfrac{\partial}{\partial v}C(u,v)$$

Rearranging the expression above we obtain:

$$\lambda^U = \lim_{\mathrm{u}\to 1}\ [\ \mathrm{P}(V > u|U = u)] + \lim_{\mathrm{u}\to 1}\ [\ \mathrm{P}(U > u|V = u)]$$

In the case of exchangeable copulas (i.e. $C(u,v) = C(v,u)$) we can simplify as follows:

$$\lambda^U = 2\lim_{\mathrm{u}\to 1}\ [\ \mathrm{P}(V > u|U = u)]$$

Finally, by using the probability integral problem, in the case of a distribution function F with infinite right endpoint (such as the Gaussian of the Student's t distribution) we get:

$$\lambda^U = 2 \lim_{\mathrm{u}\to\infty} [\ \mathrm{P}(\mathrm{F}^{-1}(V) > u | F^{-1}(U) = u)] = 2 \lim_{\mathrm{u}\to\infty} [\ \mathrm{P}(Y > u | X = u)]$$

where we used Sklar's Theorem $(X_1,\ X_2) \sim C(F(x_1),\ F(x_2))$.

Let apply these results to Gaussian and T-Copula:

1) Gaussian Copula. Using standard result in the statistic theory , we have [1]

$$\lambda^U = 2 \lim_{\mathrm{x}\to\infty} [\ \mathrm{P}(X_2 > x | X_1 = x)] =$$

$$= 2 \lim_{\mathrm{x}\to\infty} \left[1 - \Phi\left(\frac{x-\rho x}{\sqrt{1-\rho^2}}\right)\right] = 2 \lim_{\mathrm{x}\to\infty} \left[1 - \Phi\left(x\frac{\sqrt{1-\rho}}{\sqrt{1+\rho}}\right)\right] = 0$$

Given the radial symmetry property of the Gaussian distribution, we can conclude that also the lower tail dependence coefficient is null, so confirming the asymptotic independence in tail of the Gaussian copula.

2) Student's t Copula. It can be shown that in this case we have (see Embrechts, McNeil, Straumann, 1999):

$$\lambda^U = 2 \lim_{\mathrm{x}\to\infty} [\ \mathrm{P}(X_2 > x | X_1 = x)] =$$

$$= 2 - 2\mathrm{t}_{\nu+1} \left[\sqrt{\nu+1} \cdot \frac{\sqrt{1-\rho}}{\sqrt{1+\rho}}\right] = \lambda^L$$

which is increasing in ρ and decreasing in ν. As the number of degrees of freedom goes to infinity, λ^U tends to 0 for $\rho < 1$. This can be explained by considering the asymptotic behavior of a Student's t distributed random variable which converges in distribution to a standard normal variate.

3.2.2 Tail dependence for Archimedean copulae.

For copulas of the Archimedean class, the formulas for the coefficient of tail dependence are the following:

$$\lambda^U = 2 - \lim_{u\to 1} \frac{\partial C(u,u)}{\partial u}, \qquad \lambda^L = \lim_{u\to 0} \frac{\partial C(u,u)}{\partial u}$$

We report here the tail dependence coefficients for the Gumbel and rotated Gumbel copula that we will use later for empirical applications:

1) Gumbel Copula: This copula has only positive upper tail dependence, which means the probability that both variables are in their right tails is positive. Remembering that $C^{Gumbel}(u_1,\ u_2) = \exp\{-[(-\ln u_1)^\alpha + (-\ln u_2)^\alpha]^{1/\alpha}\}$, the tail dependence coefficients are:

$$\lambda_U = 2 - 2^{\ 1/\alpha}\ \lambda_L = 0\ , \text{ where } \alpha \text{ is the copula parameter}$$

2) Survival (or rotated Gumbel): A *Gumbel Survival* copula is its mirror image and has positive lower tail dependence: the probability that both variables are in their left tails is positive. Needless to say, the *Gumbel Survival* copula is far more interesting for financial modeling. In this case we have,

$$\lambda_U = 0\ \lambda_L = 2 - 2^{\ 1/\beta}\ , \text{ where } \beta \text{ is the copula parameter}$$

[1]For a formal proof see Casella and Berger (2002)

4 Estimation Procedures

We will consider a random sample represented by the time series $X = (x_{1t}, x_{2t,} \dots x_{Nt}), t$ =1... T, where N stands for the number of underlying assets included and T represents the number of observations available.

4.1 Exact maximum likelihood method (or one stage method)

Let f be the density of the joint distribution F:

$$f(x; \alpha_1, ..., \cdot\alpha_n; \theta) = c(F_1(x_{1;}\alpha_1), ..., F_1(x_{n;}\alpha_n)) \cdot \prod_{i=1}^{n} f_i(x_i; \alpha_i) \tag{8}$$

where f_i is the univariate density of the marginal distribution F_i and c is the density of the copula given by the following expression:

$$c(u_1, ..., u_2; \theta) = \frac{\partial^n C(u_1, u_2, ..., u_n; \theta)}{\partial u_1 \partial u_2 ... \partial u_n}$$

We suppose to have a set of T empirical data of n financial asset log-returns, and let $\Theta = (\alpha_1, \dots, \alpha_{n;}\theta)$ be the parameter vector to estimate, where α_i, i = 1, ...,n is the vector of parameters of the marginal distribution $F_i$and θ is the vector of the copula parameters. The log-likelihood function is the following:

$$l(\Theta) = \sum_{t=1}^{T} \log(c(F_1(x_{1,t}; \alpha_1), \dots, F_n(x_{n,t}; \alpha_n); \theta)) + \sum_{t=1}^{T} \sum_{i=1}^{n} \log f_i(x_{i,t}; \alpha_{i,t}) \tag{9}$$

The ML estimator $\hat{\Theta}$of the parameter vector is the one which maximize (**??**): $\hat{\Theta}$ =arg max $l(\hat{\Theta})$

Let $\hat{\theta}_{ML}$be the maximum likelihood estimator. Then it verifies the property of asymptotic normality and we have (Durrelman et al. 2000):

$$\sqrt{T}(\hat{\theta}_{ML} - \theta_0) \to N(0, \mathrm{I}^{-1}(\theta_0))$$

with $\mathrm{I}(\theta_0)$ the information matrix of Fischer.

4.2 The Inference Functions for Margins method (IFM – or two stage method)

According to the IFM method, the parameters of the marginal distributions are estimated separately from the parameters of the copula. In other words, the estimation process is divided into the following two steps:
estimating the parameters α_i, i = 1, ...,n of the marginal distributions F_i using the ML method:

$$\hat{\alpha}_i = \arg\max l^i(\alpha_i) = \arg\max \sum_{t=1}^{T} \log f_i(x_{i,t}; \alpha_i)$$

where l^i is the log-likelihood function of the marginal distribution F_i;
estimating the copula parameters θ , given the estimations performed in step 1):

$$\hat{\theta} = \arg\max l^c(\theta) = \arg\max \sum_{t=1}^{T} \log(c(F_1(x_{1,t}; \hat{\alpha}_1), \dots, F_n(x_{n,t}; \hat{\alpha}_n); \theta)) \tag{10}$$

where l^c is the log-likelihood function of the copula. Like the ML estimator it verifies the properties of asymptotic normality (Joe and Xu, 1996):

$$\sqrt{T}(\hat{\theta}_{IFM} - \theta_0) \to N(0, V^{-1}(\theta_0))$$

where $V(\theta_0)$ id the Information matrix of Godambe. If we define the score function in the following way $g(\theta) = (\partial_{\alpha_1} l^1, ..., \partial_{\alpha_N} l^N, \partial_\theta l^c)$, the Godambe information matrix takes the form (Joe, 1997):

$$V(\theta_0) = D^{-1} M (D^{-1})^T$$

where $\mathbf{D} = \mathbf{E}[\partial g(\theta)^T \ / \ \partial \ \theta]$ and $\mathbf{M} = \mathbf{E} \ [g(\theta)^T \ g(\theta)]$. Note that the IFM method could be viewed as a special case of the GMM with an identity weight matrix.

4.3 The Canonical Maximum Likelihood (CML) method

The CML method differs from the IFL method because no assumptions are made about the parametric form of the marginal distributions. The estimation process is performed in two steps:

1. Transforming the dataset $(x_{1t}, x_{2t} \ldots x_{Nt}), t = 1 \ldots T$ into uniform variates $(\hat{u}_{1t}, \hat{u}_{2t}, \ldots, \hat{u}_{Tt})$ using the empirical distributions $\hat{F}_n(\cdot)$defined as follows:

$$\hat{F}_n(\cdot) = \frac{1}{T} \sum_{t=1}^{T} 1_{\{X_{nt} \leq \ \bullet)} \tag{11}$$

   where $1_{\ \{X \leq \bullet\}}$represent the indicator function.

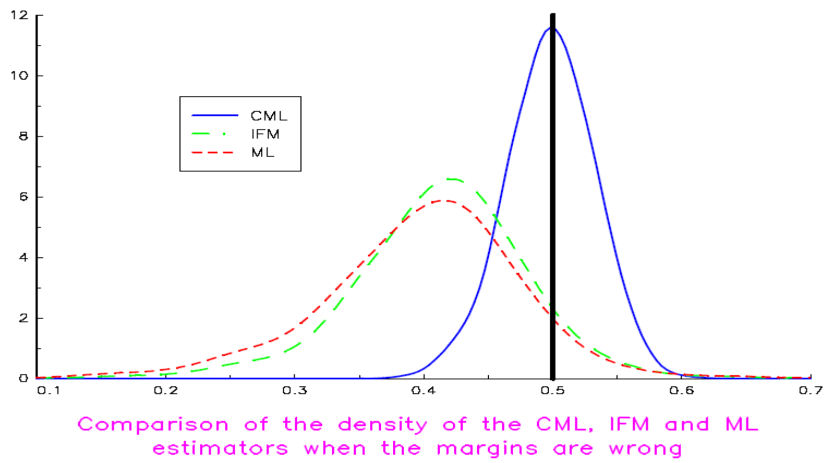
2. Estimating the copula parameters as follows:

$$\hat{\theta} = \arg\max \sum_{t=1}^{T} \log(c(\hat{u}_{1,\mathrm{t}}, \ ..., \hat{u}_{n,t}); \theta) \tag{12}$$

The previous Godambe information matrix or the semi-parametric covariance matrix by Genest, Ghoudi and Rivest (1995) can be used for inference purposes: if the marginal distributions are correctly specified, the two approaches give the same results.

Remark 1: CML is the best estimator, because there are no assumptions on the margins. If we use wrong margins, MLE and IFM will 'modify' the dependence function. Look at this nice example taken by Roncalli (2000): Bivariate distribution F with Normal copula (ρ = 0,5) and two t-student margins ($F_1 = t_1$ and $F_2 = t_2$). If we fit the distribution with a Normal copula and two Gaussian margins, we get:

Figure 1: CML vs IFM and ML



5 Portfolio management and VaR applications

5.1 Portfolio management

5.1.1 Introduction

The mean-variance approach for portfolio selection first proposed by Markowitz (1952) is very intuitive and, due to its simplicity, is by far the most used in the financial sector. The advantage of using the variance for describing portfolio risk is principally due to the simplicity of the computation, but from the point of view of risk measurement the variance is not a satisfactory measure. First, the variance is a symmetric measure and consider gains and losses in the same way. Second, the variance is inappropriate to describe the risk of low probability events, as for example the default risk. Finally, mean-variance decisions are usually not consistent with the expected utility approach, unless returns are normally distributed or a quadratic utility index is chosen.

As already suggested by Markowitz (1959), other risk measures can be used in the mean-risk approach, such as value-at-risk (VaR) and expected-shortfall (ES). However, it is possible to show that under the assumption that returns are normally distributed, the efficient frontiers resulting from the mean-VaR and from the mean-ES optimization are subsets of the mean-variance efficient frontier. The equivalence of these optimization problems under multivariate normal distribution was first stated by Rockafellar and Uryasev (1999, Proposition 4.1). Leippold (2001) and, for a more general framework, Leippold, Vanini, and Troiani (2002) considered the impact of value-at-risk and expected-shortfall limits on the mean-variance portfolio allocation and has shown for multivariate Gaussian returns that VaR and ES constraints reduces the mean-variance set of efficient portfolio allocations. However, when a risk-free asset is available, the set of efficient portfolios resulting from mean-VaR or from mean-ES portfolio selection are identical to the mean-variance efficient frontier. Finally, Hürlimann (2002) proved the equivalence of mean-ES and mean-variance analysis for a more general class of distribution function for the returns, that is the class of elliptical distributions.

For these reasons, we decide to rely on the traditional mean-variance approach. However, we present here a possible variation by using Kendall's Tau instead of linear correlation, in order to improve its performance when dealing with assets *not* elliptically distributed, that is the usual case when we are work with high frequency datasets.

Moreover, as we've seen in section 3, linear correlation is a good measure of dependence in multivariate normal distributions but it presents four important shortcomings: a) The variances of the random variables must be finite for the correlation to exist; b) Independence between two random variables implies that linear correlation is zero, but the converse is true only for a multivariate normal distribution; c) Correlation is not invariant to strictly monotone transformations. d) Given the marginal distributions F_1 and F_2 for two random variables X and Y , all linear correlations between -1 and +1 cannot be attained through suitable specification of the joint distribution F.

Kendall's Tau prove to be a good alternative, where the main advantages of rank correlation over ordinary correlation are the invariance under monotonic transformations and the fact that perfect dependence corresponds to correlations of + 1 and –1. Another advantage is that rank correlation is quite robust against outliers. The main disadvantage is that rank correlations do not lend themselves to the same elegant variance-covariance manipulations as do linear correlations, since they are not moment-based measures. However, as we've seen in paragraph 3.1, Kendall's Tau can be easily obtained as a function of copula parameters, and in some cases like multivariate distribution which possesses a simple closed-form copula, like the Gumbel copula, moments may be difficult to determine while calculation of rank correlation is a much more easier work.

5.1.2 Copula's conditional Kendall's Tau for portfolio management

The classical optimization problem of a portfolio manager is the following one:

$$
\begin{array}{lll}
\sigma^2_{p,t+1} & = \min\limits_{w_{t+1}} \; w'_{t+1}\Sigma_{t+1}\, w_{t+1} & (a) \\
s.t.: & w'_{t+1}\,1 = 1 & (b) \\
 & w'_{t+1}\,\mu_{t+1} = g & (c)
\end{array}
$$

where g = portfolio target return;

w_{t+1} = optimal vector of weights at time t+1;

Σ_{t+1} = forecasted variance-covariance matrix of asset returns at time t+1;

μ_{t+1} = vector of expected returns at time t+1.

and the usual variance covariance matrix $\Sigma_{t+1}$is made up as follows ($2 \cdot 2$ example):

$$
\begin{array}{ll}
\sigma^2_{i,t+1} & \rho_{ij,t+1}\sigma_{j,t+1}\cdot\sigma_{i,t+1} \\
\rho_{ij,t+1}\sigma_{i,t+1}\cdot\sigma_{j,t+1} & \sigma^2_{j,t+1}
\end{array}
$$

where $\rho_{\;ij,t+1} \cdot \sigma_{i,t+1} \cdot \sigma_{j,t+1}$ is the covariance between asset i and asset j at time t+1.

As a direct consequence of what we said in paragraphs 3.1 and 5.1.1, we want to propose here to modify the covariance matrix of asset returns Σ_{t+1} as follows:

$$
\begin{array}{ll}
\sigma^2_{i,t+1} & \tau_{ij,t+1}\sigma_{j,t+1}\cdot\sigma_{i,t+1} \\
\tau_{ij,t+1}\sigma_{i,t+1}\cdot\sigma_{j,t+1} & \sigma^2_{j,t+1}
\end{array}
$$

where we substituted the linear correlation $\rho_{\;ij,t+1}$ with Kendall's Tau $\tau_{\;ij,t+1.}$

The general procedure we followed to get the forecasts of both variances and Kendall's Tau coefficients (or linear correlation's) is a three-step approach similar to Engle and Sheppard (2001) and Tse and Tsui (2002) for dynamic conditional correlation models, and Embrechts et al. (2003a,b) and Ling Hu (2003) for copula modeling. It's basically the semi-parametric CML method described in paragraph 4.3:

Definition 9 (Global minimum variance portfolio): The steps to estimate the optimal vector $\mathbf{w}_{t+1}$ are:

1. *Estimate an ARMA – GARCH* model for the conditional means and variances, by using t-student or normal distributed errors, where in the latter case we have QML properties; →<u>Get the forecasted variances</u> $\sigma_{i,t+1}$, $\sigma_{j,t+1}$;

2. *Estimate the standardized residuals and compute the empirical CDF* , assuming independence:

$$\hat{F}_n(\cdot) = \frac{1}{T}\sum_{t=1}^{T} 1_{\{X_{nt}\leq\ \bullet)}$$

   As financial returns are usually not i.i.d., we have to use the standardized residuals to estimate copulas and use the asymptotic results in Genest et al. (1995) for i.i.d. observations.

3. *Compute the log-likelihood function of the copula* (**??**), with u_1 and u_2 replaced by the empirical CDFs $\hat{F}_1(\cdot)$ and $\hat{F}_2(\cdot)$ respectively, where we assume that the copulas' parameters θ follow a dynamic similar to that used for ARMA-GARCH models. → <u>Get the forecasted linear correlations</u> $\rho_{ij,t+1}$ <u>or Kendall's Tau</u> $\tau_{ij,t+1}$, build Σ_{t+1} and estimate the global minimum variance portfolio minimizing (a) s.t.(b)

It is important to remember that Sklar's theorem for conditional distributions (**Theorem 2**) implies that the conditioning variable(s) must be the same for both marginal distributions and the copula: if we do not use the same conditioning variable for F_1, F_2 and C, the function H will not be, in general, a joint conditional distribution function (Patton, 2003).

5.2 Value at risk

5.2.1 Introduction: the univariate case.

Value at Risk or VaR is a concept developed in the field of risk management that is defined as the minimum amount of money that one could expect to lose with a given probability over a specific period of time. While VaR is widely used, it is, nonetheless, a controversial concept, primarily due to the diverse methods used in obtaining VaR, the widely divergent values so obtained and the fear that management will rely too heavily on VaR with little regard for other kinds of risks. The VaR concept embodies three factors:

1. A given *time horizon*. A risk manager might be concerned about possible losses over one day, one week, etc.
2. VaR is associated with a probability. The stated VaR represents the minimum possible loss over a given period of time with a given probability.
3. The actual amount of money invested.

If we call $\Delta V(l)$ the change in the value of the assets in the financial position from t to t + l and $F_l(x)$ the cumulative distribution function of $\Delta V(l)$, VaR is formally defined as follows:

Definition 10 (VaR): We define the VaR of a long position over time horizon l with probability p as

$$p = Pr\ [\Delta V(l) \leq \mathrm{VaR}_t(p,\, l)] = F_l(\mathrm{VaR}_t(p,\, l)).$$

where VaR is defined as a negative value (loss). Many finance textbooks define the VaR as a positive value.

The holder of short position suffers a loss, when the value of the asset increases [i.e. $\Delta V(l)$ ¿ 0]. Hence the VaR for a short position is defined as,

$$p = Pr\ [\Delta V(l) \geq \mathrm{VaR}_t(p,\, l)] = 1 \text{ - } Pr[\Delta V(l) \leq \mathrm{VaR}_t(p,\, l)] = 1\text{-}F_l(\mathrm{VaR}_t(p,\, l)).$$

The previous definition show that VaR is concerned with tail behaviour of the cdf F_l(x). For any univariate cdf F_l(x) and probability p, such that 0 ¡ p ¡ 1, the quantity $x_p = \inf\{x|F_l(\mathrm{x}) \geq p\}$ is called the p-th quantile of F_l(x), where inf denotes the smallest real number satisfying $F_l(\mathrm{x}) \geq \quad p$. If the cdf is known, then VaR is simply its p-th quantile times the value of the financial position: however this is not known in practice and must be estimated.

If we consider an ARMA-GARCH model to model the mean r_t and volatility σ_t and we assume that the residuals are Gaussian, then the conditional distribution of r_{t+1} given the information available at time t is N[$\hat{r}_{t+1}$,$\hat{\sigma}_{t+1}$] where $\hat{r}_{t+1}$ and $\hat{\sigma}_{t+1}$ are 1-step ahead forecast. It directly follows that the 5 % quantile is then $\hat{r}_{t+1}-$ 1.65$\hat{\sigma}_{t+1}$, and in general

$$VaR_{t+1} = \hat{r}_{i,t+1} - z_\alpha \hat{\sigma}_{i,t+1} \tag{13}$$

If we instead assume that the residuals follow a standardized Student-t distribution with υ degrees of freedom, then the quantile is

$$VaR_{t+1} = \hat{r}_{t+1} - t^*_{\upsilon,a} \hat{\sigma}_{t+1} \tag{14}$$

where $t^*_{\upsilon,a}$ is the α-th quantile of a standardized Student-t distribution with υ degrees of freedom.

5.2.2 The multivariate case.

The concept of VaR is a very appealing one because it can be developed for any kind of portfolio and can be aggregated across portfolios of different kinds of instruments. This does not imply that estimating VaR for a portfolio is a simple process, because the dependences across asset classes must be accounted for: so far, correlation represented the most used tool for evaluating dependence, but, as we have seen, it cannot be the best way to do it due to nonlinear dependence among financial assets.

When we deal with a portfolio of assets, VaR estimation can become very difficult due to the complexity of joint multivariate modeling. Some approaches have been proposed, see Giot and Laurent (2003) for a review, but most of them are rather complicated to implement and can give similar results to simpler methods. Moreover, recent empirical studies show that in volatile periods, financial markets tend to be characterized by different levels of dependence than occur in queit periods. In order to take into account this reality, we propose to use the concept of Tail dependence, which refers to the dependence that arises between random variables from extreme observations. An important feature of copulas is that they allow for different degrees of tail dependence: *Upper* tail dependence exists when there is a positive probability of positive outliers occurring jointly, while *lower* tail dependence is symmetrically defined as the probability of negative outliers occurring jointly.

What we propose is a direct consideration of this concept in VaR models by means of copula theory, as tail dependence coefficients can be calculated as simple functions of copulae parameters: if we follow the well-known RiskMetrics multiple positions VaR model, lower tail dependence coefficients can be used at the place of linear correlation coefficients.For these reasons, we want to present here the following two methodologies:

Definition 11 (A modified RiskMetrics multiple positions model): The RiskMetrics model is by far the most used by financial professionals and we consider it here for this reason, however with a slight modification to take into account copula's conditional dependence measures, such as Kendall's Tau and Tail dependence.

Let consider the global minimum variance portfolio made up of m assets and estimated as described in paragraph 5.1. The euro amount invested in asset i is $W_{i,t+1} = w_{i,t+1}W$, where W is the total euro amount, and $w_{i,t+1}$ is the optimal share of asset i in the portfolio at time t+1: in this case the $VaR_{i,t+1}$ for the single asset i is

$$VaR_{i,t+1} = W_{i,t+1}\cdot\ (\hat{r}_{i,t+1} - \mathrm{z}_\alpha \hat{\sigma}_{i,t+1}) \text{ or } VaR_{i,t+1} = W_{i,t+1}\cdot\ (\hat{r}_{i,t+1} - t^*_{\upsilon,a}\quad \hat{\sigma}_{i,t+1})$$

as in paragraph 5.2.1. Then, the traditional RiskMetrics generalization of VaR when dealing with a portfolio of m assets is (see Tsay 2002, and Longerstaey and More 1995),

$$VaR_{p,t+1} = \sqrt{\sum_{i=1}^{m} VaR^2_{i,t+1} + 2\sum_{i<j}^{m} \rho_{ij}\, VaR_{i,t+1}\, VaR_{j,t+1}} \tag{15}$$

where ρ_{ij} is the correlation coefficient between returns of the i^{th} and j^{th} instruments and VaR_i is the VaR of the i^{th} instrument.

In order to preserve this simple framework, we propose to modify (10) by substituting the correlation ρ_{ij} with the following conditional dependence measures, estimated via a dynamic copula specification:

a) *the forecasted conditional correlation* $\hat{\rho}_{ij,t+1}$:

$$VaR_{p,t+1} = \sqrt{\sum_{i=1}^{m} VaR^2_{i,t+1} + 2\sum_{i<j}^{m} \hat{\rho}_{ij,t+1}\, VaR_{i,t+1}\, VaR_{j,t+1}} \tag{16}$$

b) *the forecasted conditional Kendall's Tau* $\hat{\tau}_{ijt+1}$:

$$VaR_{p,t+1} = \sqrt{\sum_{i=1}^{m} VaR^2_{i,t+1} + 2\sum_{i<j}^{m} \hat{\tau}_{ij,t+1}\, VaR_{i,t+1}\, VaR_{j,t+1}} \tag{17}$$

c) *the forecasted conditional lower Tail dependence*$\hat{\lambda}_{ij,t+1}$:

$$VaR_{p,t+1} = \sqrt{\sum_{i=1}^{m} VaR^2_{i,t+1} + 2\sum_{i<j}^{m} \hat{\lambda}_{ij,t+1}\, VaR_{i,t+1}\, VaR_{j,t+1}} \tag{18}$$

Definition 12 (Multivariate copula modeling): Consider again the global minimum variance portfolio made up of m assets estimated as described in paragraph 5.1. The euro amount invested in asset i is $W_{i,t+1} = w_{i,t+1}W$, where W is the total euro amount, and $w_{i,t+1}$ is the optimal share of asset i in the portfolio at time t+1: Let $\boldsymbol{w}_{t+1}$ be the optimal vector of shares for time t+1, $\boldsymbol{R}_{t+1}$ the vector of real returns, μ_{t+1} the vector of expected returns computed by an ARMA model, and Σ_{t+1} the forecasted variance-covariance matrix estimated via a dynamic parametric copula and GARCH models, as previously explained in paragraph 5.1. Then,

$$\boldsymbol{R}_{p,t+1} = \boldsymbol{w}_{t+1}\text{'}\boldsymbol{R}_{t+1}\ ,\ E[\boldsymbol{R}_p] = \boldsymbol{w}_{t+1}\text{'}\ \mu_{t+1}\ ,\ V[\boldsymbol{R}_p] = \sigma^2_p = \boldsymbol{w}_{t+1}\text{'}\ \Sigma_{t+1}\ \boldsymbol{w}_{t+1}$$

The VaR of the portfolio at level α is the minimal amount that can be lost with probability less than α, $VaR_{p,t+1}(\alpha) = W \cdot \ q_{p,t+1}(\alpha)$, where $q_{p,t+1}(\alpha)$ is determined by $Pr[\boldsymbol{R}_{p,t+1} < q_{p,t+1}(\alpha)] = \alpha$. Under the joint normal assumption, the one-step-ahead VaR$_{t+1}$ computed in t is given by,

$$VaR_{p,t+1} = [\boldsymbol{w}'_{t+1}\mu_{t+1} + z_\alpha(\boldsymbol{w}'_{t+1}\Sigma_{t+1}\boldsymbol{w}_{t+1})^{1/2}] \cdot \quad W \tag{19}$$

with z$_\alpha$ the left α %-quantile of the N(0,1) distribution. Under the assumption of multivariate Student-t innovations, the one-step-ahead VaR is obtained by replacing z$_\alpha$ by $t^*_{\upsilon,a}$, instead, where $t^*_{\upsilon,a}$ is the p-th quantile of a standardized Student-t distribution with υ degrees of freedom,

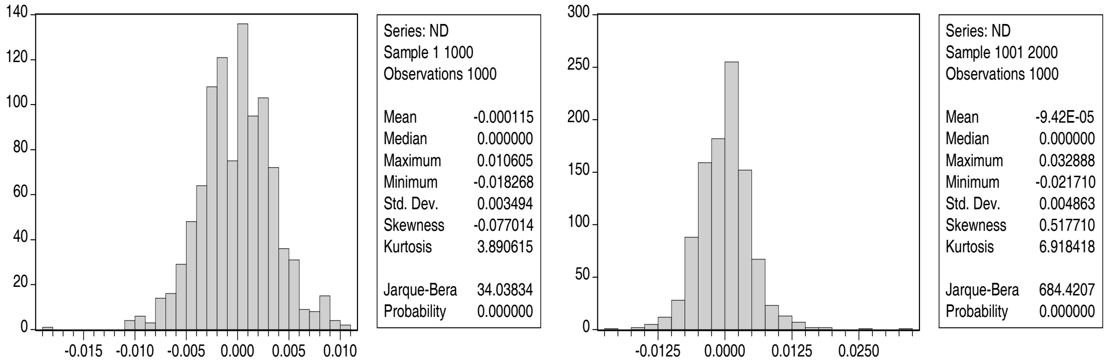
$$VaR_{p,t+1} = [\boldsymbol{w}'_{t+1}\mu_{t+1} + t^*_{\upsilon,a}(\boldsymbol{w}'_{t+1}\Sigma_{t+1}\boldsymbol{w}_{t+1})^{1/2}] \cdot W \tag{20}$$

It is again clear that this approach emphasizes the semi-parametric approach we've followed so far, where we consider a parametric copula, a parametric joint distribution (normal or T-Student), but we do not model the marginals, even though we emphasis that they are connected by equation (**??**). This choice is mostly justified by the fact that as far as we are concerned with portfolio allocation and VaR, dependences across assets play the main role in today's financial industry: moreover, misspecification of marginals can lead to dangerous biases in dependence measures estimation, as we've seen in paragraph 4.3 . This is why the semi-parametric approach is quickly becoming the major standard in joint multivariate modeling.

We would also like to highlight a few comments regarding skewed models, both univariate and multivariate, that we have excluded so far:

Remark 2: some portfolio managers pointed out that skewed models present some hidden dangers due to the fact that a positive or negative skewness is highly dependent on the particular sample of time considered in the analysis. For example, assets usually presented positive skewness till the year 2000, while later we observed a shift to negative skewness due to falling markets: if you had estimated the VaR quantiles using a skewed model and market data till the beginning of 2000, you would have found conservative estimates for the left tail and aggressive ones for the right tail, when in reality financial professionals needed just the opposite! Moreover, as all practitioners know, this shift in skewness is present not only in macrodata but also at the high frequency level, where asset prices can follow micro-trends which are just the opposite the general macro trend. A simple

Figure 2: Nasdaq100 distributions



example is given by the first 2000 observation of our Nasdaq100 future dataset (Figure 2): the first 1000 presented negative skewness (due to market falling), while observations spanning from 1001 to 2000 present positive skewness (due to the market rebound)!

For these reasons financial professional prefer to use symmetric distributions which are more stable against these changes in skewness, rather than skewed models, which can provide better data fitting but completely wrong estimates if markets conditions are changing.

6 Empirical analysis

We consider the five most important future contracts actually traded in American markets (Sp500, Dow Jones, Nasdaq100, Euro Dollar, T Bond Note) with high frequency data sampled at $5 -$ minutes frequency, taking into consideration the most volatile period of the last decade, that is between March the 13 th 2000 till June the 09th 2000. This period of time saw the falling of world financial markets following the bursting of the high-tech bubble, with big intraday draw down returns. Thus, this sample is perfectly suited to highlight the importance of the copula-based dependence approach compared to the traditional correlation analysis.

We build up a portfolio and a multiple VaR position following both the two approaches, using the initial part of the sample to estimate the assets' weights and the 95% (99 %) VaR, and the remaining part to compare the out-of-sample performances of the two approaches, both in terms of risk measures and number of VaR exceedances of the effective portfolio losses. We show that our approach outperform the correlation based one both in terms of portfolio results and VaR back-testing.

We analyse these futures by considering the intraday data included between 9.30 and 15.00 (New York time): we chose this time sample because the Euro$ future contract is traded until 15.00 in the afternoon. For the same reason, some days that were not present in all five futures where cancelled out in order to have a common sample among all futures. Moreover, the first trade which took place every day after 9.30 in the morning has been deleted in order to analyse only the intraday behaviour of the considered series. The descriptive statistics of the (raw) log-returns are presented in Table 1.

6.1 Univariate modeling

As pointed out in previous paragraphs, you need i.i.d observations in order to estimate copulae. To reach this goal, an AR(6) - GARCH(1,1) filter with t-distributed errors was applied to the raw 5-minutes futures returns. The six lags were found to be significant on almost all cases[2] and they

[2]For sake of brevity, we do not report here the estimated coefficients. However, these results are available from the authors.

Table 1: **Descriptive statistics of futures log-returns**

	Dow Jones (DJ)	**S&P 500 (SP)**	**Nasdaq100 (ND)**	**Euro-Dollar (EC)**	**T-Bond Note (US)**
Mean	2.98e^{-05}	1.89e^{-05}	-6.74e^{-05}	2.44e^{-06}	-3.13e^{-06}
Maximum	0.0071	0.0072	0.0329	0.0048	0.0039
Minimum	-0.0096	-0.0102	-0.0217	-0.0054	-0.0036
Std. Dev.	0.0014	0.0015	0.0040	0.0007	0.0006
Skewness	0.0282	-0.0074	0.2612	-0.1794	0.1062
Kurtosis	5.4337	5.7315	6.7976	7.0795	5.6609
Jarque-Bera	839.6	1057.0	2081.7	2375.9	1009.4
P-value JB	0.0000	0.0000	0.0000	0.0000	0.0000
Observations	3400	3400	3400	3400	3400

can be explained by means of two well known reasons:

1. *Bid-ask bounce*;
2. *Delays in news delivery to the public.* As it is known, quote data are in some cases delayed by 10/15 minutes, according to the used trading platform and similarly some financial news is made public with a delay ranging from a few minutes to half an hour. It is a well known fact that a single Bloomberg or Reuters platform, with the most important news features, costs about \$30.000 - \$100.000 per year and not all traders can afford that.

The Ljung-Box Q-statistics relative to the futures log-returns, standardized residuals and standardized squared residuals, plus the BDS test for independence applied to the standardized residuals, are reported in Table 2. We remind that the BDS test is a portmanteau test for time based dependence in a series and can be used to check whether the residuals are independent and identically distributed. Under the null hypothesis of independence, we expect this statistic to be close to zero (see Brock, Dechert, Scheinkman and LeBaron - 1996).

Table 2: **Tests statistics: futures log-returns and standardized residuals**

	LB(50) - Raw returns	LB(50) - Raw returns squared	LB(50) - Std residuals	LB(50) - Std residuals squared	BDS(50) (P-Value)
Dow Jones (DJ)	**90.41** (*)	**540.1** (*)	32.64	5.33	0.32
S&P 500 (SP)	**85.57** (*)	**1174.5** (*)	38.54	11.10	0.31
Nasdaq100(ND)	**73.78** (*)	**1391.4** (*)	27.71	19.45	0.28
Euro-Dollar (EC)	**65.96** (**)	**147.2** (*)	45.22	26.49	0.39
T-Bond N. (US)	40.54	**250.7** (*)	36.94	27.29	0.34

(*) H_0 rejected at 1% level ; (**) H_0 rejected at 10% level

All standardized residuals display the desired i.i.d. properties necessary for copula estimation, so we can now estimate the empirical CDF (*Step 2* in Proposition 5),

$$\hat{F}_n(\cdot) = \frac{1}{T}\sum_{t=1}^{T} 1_{\{X_{nt} \leq \ \bullet)}$$

and proceed to compute the log-likelihood functions of the bivariate dynamic copulae (*Step 3* in Proposition 5).

6.2 Multivariate modeling

6.2.1 Copula modeling.

The elliptical copulae we use here are the Normal copula and the T-copula, whose density in the bivariate case are reported here:

1) Normal copula:

$$c(\Phi(x_1),\ \Phi(x_2);\rho) = \frac{1}{\sqrt{(1-\rho^2)}} \cdot \exp\left(-\frac{1}{2}\cdot\frac{[(\Phi^{-1}(u_1))^2 + (\Phi^{-1}(u_2))^2 - 2\rho\cdot\Phi^{-1}(u_1)\cdot\Phi^{-1}(u_2)]}{(1-\rho^2)}\right)\cdot\exp\left(\frac{1}{2}[(\Phi^{-1}(u_1))^2 + (\Phi^{-1}(u_2))^2]\right)$$

where ρ is the correlation coefficient and Φ^{-1} is the inverse of the standard univariate Gaussian distribution function.

2) T-copula:

$$c(t_\upsilon(x_1),\ t_\upsilon(x_2);\rho,\ \upsilon) = \frac{\Gamma\left(\frac{\upsilon+2}{2}\right)\Gamma\left(\frac{\upsilon}{2}\right)}{\Gamma\left(\frac{\upsilon+1}{2}\right)^2\sqrt{1-\rho^2}}\left(1+\left[\frac{(t_\upsilon^{-1}(u_1))^2 + (t_\upsilon^{-1}(u_2))^2 - 2\rho\cdot t_\upsilon^{-1}(u_1)\cdot t_\upsilon^{-1}(u_2)}{(1-\rho^2)\cdot\upsilon}\right]\right)^{-\frac{\upsilon+n}{2}}$$

$$\cdot\left[\left(1+\frac{t_\upsilon^{-1}(u_1))^2}{\upsilon}\right)\left(1+\frac{(t_\upsilon^{-1}(u_2))^2}{\upsilon}\right)\right]^{\frac{\upsilon+1}{2}}$$

where ρ is the correlation coefficient, $t_\upsilon^{-1}$is the inverse of the standard univariate T-student distribution function, while υ are the degrees of freedom.

The Archimedean copula we use is the *Mixed-Gumbel*: mixed copulae are a common way to model both upper and tail dependence with copulae that would otherwise model only one of the two dependence measure (see Embrechts and Dias 2003, Ling Hui 2003, Patton 2003). The simple Gumbel copula model only upper tail dependence, while the Rotated Copula only the lower tail dependence: if we mix them we can model both tail dependences, nesting symmetry as a special case and not by construction as for elliptical copulae. The mixed-gumbel CDF and Pdf are reported here:

3) Mixed-Gumbel copula (we express copula parameters as a function of tail dependence coefficients λ^U, λ^L)**:**

CDF: $C(u_1;\ u_2;\ \theta) = 0.5 * C^{Gumbel}(u_1,\ u_2;\ \lambda^U) + 0.5 * C^{RotatedGumbel}(u_1,\ u_2;\ \lambda^L)$

where $C^{Gumbel}(u_1,\ u_2;\ \lambda^U) = \exp\{ - ((- \log \mathrm{u})^{\theta_1} + (- \log \mathrm{u})^{\theta_1})^{1/\theta_1}\}$,

$C^{RotatedGumbel}\ (u_1, u_2;\ \lambda^L) = u_1 + u_2 - 1 + C^{Gumbel}(1 - u_1,\ 1 - u_2;\ \lambda^L)$

and $\theta_1 = f(\lambda^U)$, $\theta_2 = f(\lambda^L)$

PDF: $\mathrm{c}(u_1;\ u_2;\ \theta) = 0.5 * c^{Gumbel}(u_1, u_2;\ \lambda^U) + 0.5 * \mathrm{c}^{RotatedGumbel}(u_1,\ u_2;\ \lambda^L)$

where

$$c^{Gumbel}(u_1,u_2;\lambda^U) = \frac{C^{Gumbel}(u_1,u_2)(logu_1\cdot logu_2)^{\theta_1-1}}{uv((-logu_1)_1^\theta + (-\log u_2)_1^\theta)^{2-1/\theta_1}}\cdot((-logu_1)_1^\theta + (-logu_2)_1^\theta)^{1/\theta_1} + \theta_1 - 1)$$

$c^{RotatedGumbel}\ (u_1, u_2;\ \lambda^L) = c^{Gumbel}(1 - u_1, 1- u_2;\ \lambda^L)$

and $\theta_1 = f(\lambda^U)$, $\theta_2 = f(\lambda^L)$

As we'll see in the next paragraph, the mixed-Gumbel prove to be the best copula in some cases, confirming previous results in Embrechts and Dias(2003) and Ling Hui (2003): however, it presents the serious limitation to model only *positive* dependence by construction. To overcome this problem, we consider also the mixed-Normal copula, which is a mixture of the rotated-Gumbel and the Normal copula. This choice is justified by the fact that modeling lower tail dependence is by far the most interesting for financial purposes. We use this copula as a substitute for the mixed-Gumbel when two assets present negative dependence or changes of dependence within the sample (like the Euro$ and T-bond note futures, for example). Its density function is the following:

4) Mixed-Normal copula:

PDF: $c(u_1; u_2; \theta) = 0.5 * c^{Normal}(u_1, u_2; \rho) + 0.5 * c^{RotatedGumbel}(u_1, u_2; \lambda^L)$

where the functional form of $c^{Normal}(u_1, u_2; \rho)$ and $c^{RotatedGumbel}(u_1, u_2; \lambda^L)$ were stated above.

We explicitly consider time dynamics by allowing the parameters of the copula to evolve through time. Following Patton (2003) and Dias and Embrechts (2003), we suggest these general evolution equations as starting points for the afore-mentioned copulae:

Table 3: **Copulas parameters dynamic specifications**

Elliptical copulae		*Archimedean copulae*	
Copula	*General specification*	*Copula*	*General specification*
Normal copula	$\rho_t = \Lambda\left(\omega + \alpha \cdot \rho_{t-1} + \sum_{i=1}^{6} \beta_i \cdot \lvert u_{t-i} - v_{t-i}\rvert\right)$	*Mixed - Gumbel*	$\lambda_t^U = \tilde{\Lambda}\left(\omega + \beta \cdot \rho_{t-1} + \sum_{i=1}^{6} \alpha_i \cdot \lvert u_{t-i} - v_{t-i}\rvert\right)$ $\lambda_t^L = \tilde{\Lambda}\left(\omega + \beta \cdot \rho_{t-1} + \sum_{i=1}^{6} \alpha_i \cdot \lvert u_{t-i} - v_{t-i}\rvert\right)$
T-copula	$\rho_t = \Lambda\left(\omega + \alpha \cdot \rho_{t-1} + \sum_{i=1}^{6} \beta_i \cdot \lvert u_{t-i} - v_{t-i}\rvert\right)$	*Mixed - Normal*	$\rho_t = \Lambda\left(\omega + \beta \cdot \rho_{t-1} + \sum_{i=1}^{6} \alpha_i \cdot \lvert u_{t-i} - v_{t-i}\rvert\right)$ $\lambda_t^L = \tilde{\Lambda}\left(\omega + \beta \cdot \rho_{t-1} + \sum_{i=1}^{6} \alpha_i \cdot \lvert u_{t-i} - v_{t-i}\rvert\right)$

where $\Lambda(x) \equiv (1\text{-}e^{-x})/(1+e^{-x})$ is the modified logistic transformation, designed to keep ρ_t in (-1, 1) at all times, while $\tilde{\Lambda}_t \equiv 1/(1+e^{-x})$ is the simple logistic transformation, used to keep λ^U and λ^L in (0,1) at all times. Table 3 shows that we assume copula parameters follow an ARMA(1,6)-type process: we include ρ_{t-1} as a regressor to capture any persistence in the dependence parameter, and the differences between u_{t-j} and v_{t-j} over the previous six observations to capture any variation in dependence. We tried some variants for the last regressor, but we didn't find any significant improvement, so we stick to this simple specification.

6.2.2 Copula estimation results

The estimation results for each pair of assets and the best copula specification are reported in Table $4-7$ (the best model among all copulae for a specified pair of assets, is highlighted in bold fonts).

Looking at tables $4-7$, we see that the best results are achieved by the T-copula in most of the cases and the mixed-Gumbel copula in one case. However, the normal copula does not perform much worse and presents the important feature to be very easy to compute! The mixed-normal presents better results than the normal copula in some cases, but it is always worse than the T-copula; moreover its computation can be quite intensive.

We now look how these copulae perform both in terms of out-of-sample portfolio results and VaR back-testing.

Table 4: **Normal Copula estimation results**

NORMAL copula (3400 observation – Pentium IV 2400 Mhz)						
Pair of assets	*Best dynamic specification*	*N.param.*	*Log-likelihood*	*AIC*	*BIC*	*Time to compute*
SP / DJ	$\rho_t = \Lambda\ (\omega + \beta \cdot \rho_{t-1} + \alpha_1 \cdot \lvert u_{t-1} - v_{t-1}\rvert)$	3	1854.63	-1.089	-1.084	6 sec
ND / DJ	$\rho_t = \Lambda\ (\omega + \beta \cdot \rho_{t-1} + \alpha_1 \cdot \lvert u_{t-1} - v_{t-1}\rvert)$	3	951.53	-0.558	-0.553	4 sec
SP / ND	$\rho_t = \Lambda\ (\omega + \beta \cdot \rho_{t-1} + \alpha_1 \cdot \lvert u_{t-1} - v_{t-1}\rvert)$	3	1561.76	-0.917	-0.912	5 sec
DJ / EC	$\rho_t = \Lambda\ (\omega + \beta \cdot \rho_{t-1} + \alpha_1 \cdot \lvert u_{t-1} - v_{t-1}\rvert)$	3	123.49	-0.071	-0.065	7 sec
SP / EC	$\rho_t = \Lambda\ (\omega + \beta \cdot \rho_{t-1} + \alpha_1 \cdot \lvert u_{t-1} - v_{t-1}\rvert)$	3	134.96	-0.078	-0.072	7 sec
ND / EC	$\rho_t = \Lambda\ (\omega + \beta \cdot \rho_{t-1} + \alpha_1 \cdot \lvert u_{t-1} - v_{t-1}\rvert)$	3	107.65	-0.062	-0.056	7 sec
DJ / US	$\rho_t = \Lambda\ (\omega + \beta \cdot \rho_{t-1} + \alpha_1 \cdot \lvert u_{t-1} - v_{t-1}\rvert)$	3	183.96	-0.106	-0.101	7 sec
SP / US	$\rho_t = \Lambda\ (\omega + \beta \cdot \rho_{t-1} + \alpha_1 \cdot \lvert u_{t-1} - v_{t-1}\rvert)$	3	210.57	-0.122	-0.117	8 sec
ND / US	$\rho_t = \Lambda\ (\omega + \beta \cdot \rho_{t-1} + \alpha_1 \cdot \lvert u_{t-1} - v_{t-1}\rvert)$	3	204.58	-0.119	-0.113	7 sec
EC / US	$\rho_t = \Lambda\ (\omega + \beta \cdot \rho_{t-1} + \alpha_1 \cdot \lvert u_{t-1} - v_{t-1}\rvert)$	3	59.41	-0.033	-0.028	6 sec

Table 5: **T - Copula estimation results**

T - copula (3400 observation – Pentium IV 2400 Mhz)						
Pair of assets	*Best dynamic specification*	*N.par.*	*Log-likelihood*	*AIC*	*BIC*	*Time to compute*
SP / DJ	$\rho_t = \Lambda\ (\omega + \beta \cdot \rho_{t-1} + \alpha_1 \cdot \lvert u_{t-1} - v_{t-1}\rvert + \alpha_2 \cdot \lvert u_{t-3} - v_{t-3}\rvert)$	4	**1875.21**	**-1.100**	**-1.091**	49 sec
ND / DJ	$\rho_t = \Lambda\ (\omega + \beta \cdot \rho_{t-1} + \alpha_1 \cdot \lvert u_{t-1} - v_{t-1}\rvert + \alpha_2 \cdot \lvert u_{t-3} - v_{t-3}\rvert)$	4	**975.44**	**-0.571**	**-0.562**	42 sec
SP / ND	$\rho_t = \Lambda\ (\omega + \alpha_1 \cdot \lvert u_{t-1} - v_{t-1}\rvert + \alpha_2 \cdot \lvert u_{t-3} - v_{t-3}\rvert)$	3	1588.75	-0.932	-0.923	27 sec
DJ / EC	$\rho_t = \Lambda\ (\omega + \beta \cdot \rho_{t-1} + \alpha_1 \cdot \lvert u_{t-1} - v_{t-1}\rvert)$	3	**125.55**	**-0.072**	**-0.064**	37 sec
SP / EC	$\rho_t = \Lambda\ (\omega + \beta \cdot \rho_{t-1} + \alpha_1 \cdot \lvert u_{t-1} - v_{t-1}\rvert)$	3	**140.15**	**-0.080**	**-0.073**	38 sec
ND / EC	$\rho_t = \Lambda\ (\omega + \beta \cdot \rho_{t-1} + \alpha_1 \cdot \lvert u_{t-1} - v_{t-1}\rvert)$	3	**111.66**	**-0.063**	**-0.056**	32 sec
DJ / US	$\rho_t = \Lambda\ (\omega + \beta \cdot \rho_{t-1} + \alpha_1 \cdot \lvert u_{t-1} - v_{t-1}\rvert)$	3	**203.20**	**-0.117**	**-0.110**	31 sec
SP / US	$\rho_t = \Lambda\ (\omega + \beta \cdot \rho_{t-1} + \alpha_1 \cdot \lvert u_{t-1} - v_{t-1}\rvert)$	3	**239.40**	**-0.138**	**-0.131**	40 sec
ND / US	$\rho_t = \Lambda\ (\omega + \beta \cdot \rho_{t-1} + \alpha_1 \cdot \lvert u_{t-1} - v_{t-1}\rvert)$	3	**226.17**	**-0.131**	**-0.123**	28 sec
EC / US	$\rho_t = \Lambda\ (\omega + \beta \cdot \rho_{t-1} + \alpha_1 \cdot \lvert u_{t-1} - v_{t-1}\rvert)$	3	**60.82**	**-0.033**	**-0.026**	37 sec

Table 6: **Mixed - Gumbel Copula estimation results (positive dependence only)**

Mixed - Gumbel copula (3400 observations – Pentium IV 2400 Mhz)						
Assets	*Best dynamic specification*	*N.par.*	*Log-likelihood*	*AIC*	*BIC*	*Time to comput*
SP / DJ	$\lambda_t^U = \tilde{\Lambda}\ (\omega + \beta \cdot \rho_{t-1} + \alpha_1 \cdot \lvert u_{t-1} - v_{t-1}\rvert)$ $\lambda_t^L = \tilde{\Lambda}\ (\omega + \beta \cdot \rho_{t-1} + \alpha_1 \cdot \lvert u_{t-1} - v_{t-1}\rvert + \alpha_2 \cdot \lvert u_{t-3} - v_{t-3}\rvert)$	7	1851.12	-1.085	-1.075	40 sec
ND / DJ	$\lambda_t^U = \tilde{\Lambda}\ (\omega + \beta \cdot \rho_{t-1} + \alpha_1 \cdot \lvert u_{t-1} - v_{t-1}\rvert)$ $\lambda_t^L = \tilde{\Lambda}\ (\omega + \beta \cdot \rho_{t-1} + \alpha_1 \cdot \lvert u_{t-1} - v_{t-1}\rvert)$	6	960.91	-0.562	-0.551	21 sec
SP / ND	$\lambda_t^U = \tilde{\Lambda}\ (\omega + \beta \cdot \rho_{t-1} + \alpha_1 \cdot \lvert u_{t-1} - v_{t-1}\rvert)$ $\lambda_t^L = \tilde{\Lambda}\ (\omega + \beta \cdot \rho_{t-1} + \alpha_1 \cdot \lvert u_{t-1} - v_{t-1}\rvert)$	6	**1589.23**	**-0.932**	**-0.923**	80 sec

Table 7: **Mixed - Normal Copula estimation results**

Mixed - Normal copula (3400 observations – Pentiumn IV 2400 Mhz)						
Assets	*Best dynamic specification*	*N.par*	*Log-likelihood*	*AIC*	*BIC*	*Time to comput*
DJ / EC	$\rho_t = \Lambda\ (\omega + \beta \cdot \rho_{t-1} + \alpha_1 \cdot \lvert u_{t-1} - v_{t-1} \rvert)$ $\lambda_t^L = \tilde{\Lambda}\ (\omega)$	4	114.56	-0.065	-0.058	12 sec
SP / EC	$\rho_t = \Lambda\ (\omega + \beta \cdot \rho_{t-1} + \alpha_1 \cdot \lvert u_{t-1} - v_{t-1} \rvert)$ $\lambda_t^L = \tilde{\Lambda}\ (\omega)$	4	133.39	-0.076	-0.069	40 sec
ND / EC	$\rho_t = \Lambda\ (\omega + \beta \cdot \rho_{t-1} + \alpha_1 \cdot \lvert u_{t-1} - v_{t-1} \rvert)$ $\lambda_t^L = \tilde{\Lambda}\ (\omega)$	4	104.00	-0.059	-0.052	10 sec
DJ / US	$\rho_t = \Lambda\ (\omega + \beta \cdot \rho_{t-1} + \alpha_1 \cdot \lvert u_{t-1} - v_{t-1} \rvert)$ $\lambda_t^L = \tilde{\Lambda}\ (\omega + \alpha_1 \cdot \lvert u_{t-1} - v_{t-1} \rvert)$	5	187.27	-0.107	-0.098	22 sec
SP / US	$\rho_t = \Lambda\ (\omega + \beta \cdot \rho_{t-1} + \alpha_1 \cdot \lvert u_{t-1} - v_{t-1} \rvert)$ $\lambda_t^L = \tilde{\Lambda}\ (\omega + \alpha_1 \cdot \lvert u_{t-1} - v_{t-1} \rvert)$	5	219.02	-0.126	-0.117	91 sec
ND / US	$\rho_t = \Lambda\ (\omega + \beta \cdot \rho_{t-1} + \alpha_1 \cdot \lvert u_{t-1} - v_{t-1} \rvert)$ $\lambda_t^L = \tilde{\Lambda}\ (\omega + \alpha_1 \cdot \lvert u_{t-1} - v_{t-1} \rvert + \alpha_2 \cdot \lvert u_{t-3} - v_{t-3} \rvert)$	6	206.38	-0.118	-0.107	92 sec
EC / US	$\rho_t = \Lambda\ (\omega + \beta \cdot \rho_{t-1} + \alpha_1 \cdot \lvert u_{t-1} - v_{t-1} \rvert)$ $\lambda_t^L = \tilde{\Lambda}\ (\omega + \alpha_1 \cdot \lvert u_{t-1} - v_{t-1} \rvert)$	5	58.82	-0.032	-0.023	14 sec

6.3 Portfolio management results

We consider a portfolio composed of the five futures, and we use an iterative procedure where AR(6)-GARCH(1,1) and Copulae models are estimated to predict one-day-ahead variances $\sigma_{i,t+1}$, correlations $\rho_{i,t+1}$ and Kendall's Tau $\tau_{i,t+1}$. We build the variance-covariance matrix Σ_{t+1} and we then estimate the global minimum variance portfolio minimizing (5) under (5a). The first estimation sample is given by the first 2000 observations . The predicted optimal vector of shares $\boldsymbol{w}_{t+1}$ is then used to estimate the real portfolio return and this result is recorded for later assessment using risk-returns measures. At the j-th iteration where j goes from 2000 to 3399 (for a total of 1400 observations), the estimation sample is augmented to include one more observation, and the forecasted optimal vector $\boldsymbol{w}_{t+j}$and observed portfolio return $\boldsymbol{R}_{p,t+j} = \boldsymbol{w}_{t+j}\text{'}\boldsymbol{R}_{t+j}$ are recorded. We repeat this procedure until all days have been included in the estimation sample.

6.3.1 Portfolio results: Assets with positive dependence only

We first consider a portfolio composed of futures which present positive dependence only, that is the DowJones, Nasdaq, and S&P500. By doing this, we are able to compare the performances of the Mixed-Gumbel copula against the Normal and T-Copula. Besides the usual statistics, we present here two other important measures of risk, the 1% Value at Risk (VaR) and the 1% Expected Shortfall (ES). The 1%VaR is defined as the negative of the first empirical percentile of the realised portfolio returns, $VaR\ (X;\ 0.01) = -\ F_n^{-1}\ (0.01)$, where F_n is the empirical distribution function of portfolio returns X, where we use the n out-of-sample observations. While VaR present some advantages over traditional risk measures, it has been showed that it is not a coherent measure of risk (Artzner et al.,1999); moreover VaR may underestimate the risk of securities with fat-tailed properties and a high potential for large losses, and it may disregard the tail dependence of asset returns (Yamai and Yoshiba, 2002). An alternative to VaR that has gained some attention recently is the Expected Shortfall of a portfolio (Rockafellar, and Uryasev 1999, Rockafellar, and Uryasev 2001, Acerbi, Nordio, Sirtori 2001, Acerbi and Tasche 2002). The 1% expected shortfall is defined as the negative of average return on a portfolio given that the return has exceeded its 1% VaR, that is ES $(X;\ 0.01) \equiv E_n\ [X|X \leq\ VaR\ (X;\ 0.01)$, where E_n is the sample average. Yamai and Yoshiba (2002) show that Expected Shortfall suffers less in disregarding assets' fat tails and tail dependence than VaR does. We present the results relative to an unconditional portfolio too, which is composed of equally weighted futures, that is [1/3, 1/3, 1/3].

Empirical results and risk-returns measures associated with observed portfolios are reported in Table 8.

Table 8: **Portfolio returns summary statistics (positive dependence only)**

	NORMAL copula		T − Copula		Mixed − Gumbel	Unconditional Portfolio - (1/3 each)
	Conditional RHO	*Conditional TAU*	*Conditional RHO*	*Conditional TAU*	*Conditional TAU*	
Mean return	2.99e-07	-1.96e-06	1.19e-06	-5.85e-08	**1.90e-06**	-1.09e-05
Cumulated ret.	0.000418	-0.002743	0.001670	-0.000082	**0.002667**	-0.015953
Std. deviation	0.001013	0.001092	**0.000978**	0.001028	0.001000	0.001855
Skewness	-0.480047	-0.446088	-0.502499	-0.478087	-0.491423	-0.165399
Kurtosis	8.357190	8.513731	7.295340	8.442339	8.374764	6.979274
Sharpe ratio	-0.000525	-0.002554	0.000371	-0.000864	**0.001075**	-0.006293
VaR(1%)	0.002440	0.002842	0.002630	0.002538	**0.002438**	0.004694
ES (1%)	0.003938	0.004238	**0.003789**	0.004001	0.003896	0.006604

Table 8 shows that the Kendall's Tau – mixed Gumbel portfolio outperforms all other portfolio with the highest cumulated return and Sharpe Ratio, and the lowest Value at Risk; however, the T-copula presents the lowest Expected Shortfall and its results are not far distant from the Mixed-Gumbel (like in paragraph 6.2.2 where the AIC and BIC of the two are very close).

6.3.2 Portfolio results: all assets considered

We now present the results relative to portfolios composed of all five futures, thus including the ones with negative dependence. Besides Normal and T-copula portfolios, as the mixed-Gumbel copula can model only positive dependence, we decide to build a portfolio by using the forecasted mixed-Gumbel Kendall's Tau $\tau_{i,t+1}$ for positive dependence assets and Normal or Mixed-Normal or T-copula $\tau_{i,t+1}$ for the remaining assets. Moreover we present the results relative to an unconditional portfolio too, which is composed of equally weighted futures, that is [1/5, 1/5, 1/5, 1/5, 1/5].

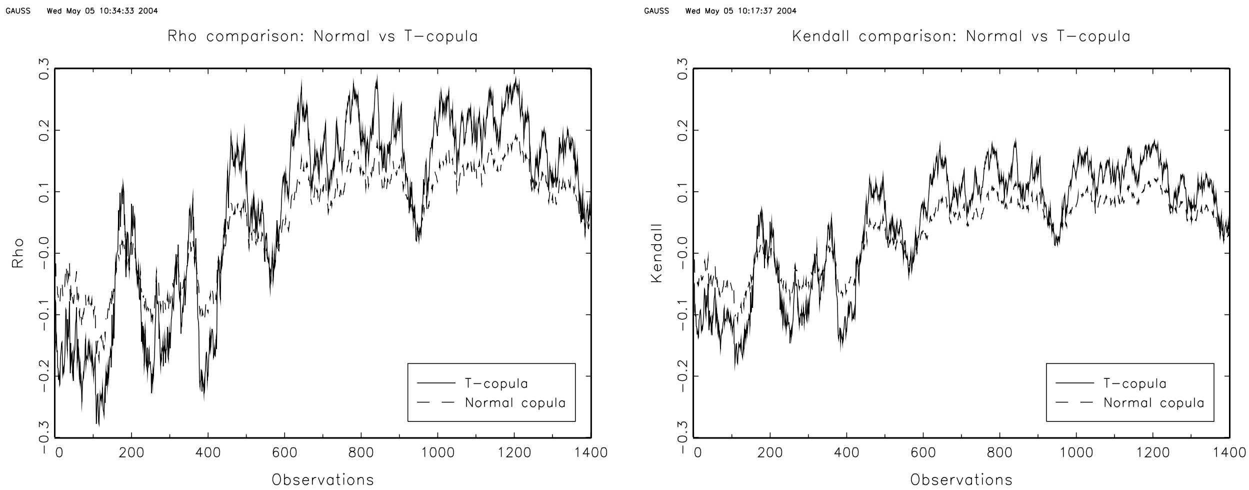
Empirical results and risk-returns measures associated with observed portfolios are reported in Table 9.

Table 9: **Portfolio returns summary statistics (all assets)**

	NORMAL copula		T − Copula		Mix−Gum + Normal	Mix−Gum +T−copula	Mix−Gum +Mix−Nor	Unc.Port (1/5 each)
	Conditional RHO	*Conditional TAU*	*Conditional RHO*	*Conditional TAU*	*Conditional TAU*	*Conditional TAU*	*Conditional TAU*	
Mean return	1.44E-05	1.39E-05	1.13E-05	1.44E-05	1.47E-05	1.45e-05	**1.48E-05**	1.68e-06
Cumulated ret.	0.020181	0.019496	0.015862	0.020111	0.020553	0.020262	**0.020689**	0.002349
Std.Deviation	0.000392	0.000393	0.000397	0.000390	0.000394	**0.000390**	0.000397	0.001067
Skewness	0.373465	0.336516	0.375887	0.379609	0.365647	**0.384651**	0.379876	-0.06015
Kurtosis	6.766149	6.741271	6.813594	6.883077	6.598502	6.924860	6.519878	6.497770
Sharpe ratio	0.034657	0.033350	0.026471	0.034682	**0.035183**	0.034990	0.035172	0.000795
VaR (1%)	0.000918	0.000921	0.000974	0.000906	0.000931	**0.000896**	0.000913	-0.002791
ES (1%)	0.001131	0.001145	0.001190	0.001125	0.001128	**0.001118**	0.001126	-0.003714

As easily expected, the three mixed-Gumbel based portfolios are the best and all three have analogous results. The biggest difference when considering all assets are the unsatisfactory results from correlation based portfolios with respect to Kendall's Tau: while the latter present similar results among all copulae, the former seem to be very sensitive to the particular copula used for estimation. Look, for example, at Figure 3 which reports the conditional linear correlation coefficient ρ_{t+i} and Kendall's Tau τ_{t+i} for Euro$ and T-bond Note, both estimated with the Normal and the T-copula: Kendall's Tau estimation appears to be more stable among different copulas.

Figure 3: Euro-dollar / T-bond Note conditional ρ and τ



6.3.3 Tests for superior portfolio performance

We now want to test whether the performances among portfolios built by using different copulae and dependence meausures in paragragh 6.3.2 are statistically significant. We present the results of the reality check of White (2000), as modified by Hansen (2001). This test compares all models jointly, and test whether a given benchmark portfolio performs as well as the best competing alternative model, chosen among many alternatives. We report the three estimates of the p-values discussed in Hansen (2001), and focus on the *consistent* p-value estimates. We reject the null hypothesis that the benchmark model (built with a specified copula) performs as well as the best competing alternative model, whenever the p-value is less than 0.10. The performance measure used is the sample mean of the realised return (a rejection of the null is presented in bold fonts):

Table 10: **Reality check P-values**

Benchmark portfolio	**Lower**	***Consistent***	**Upper**
Normal copula (RHO)	0.744	0.8575	0.8575
Normal copula (Kendall)	0.504	0.5455	0.5455
T-copula (RHO)	**0.0735**	**0.0735**	**0.0735**
T-copula (Kendall)	0.6925	0.757	0.8505
Mixed - Gumbel + T-copula (Kendall)	0.787	0.847	0.947
Mixed - Gumbel + Normal c. (Kendall)	0.8235	0.9235	0.9235
Mixed - Gumbel + Mixed - Normal c.(Kendall)	0.6295	0.7865	0.7865

These results support what we've have seen in the previous paragraph: different copulae give similar results and their performances are statistically just as good as their best alternative. What really matters is the used dependence measure, instead: for the portfolio built with linear correlations estimated by using the T-copula, we are able to reject the null that it performs as well as the best alternative.

6.4 Value at Risk results

We consider the portfolio composed of the five futures, and we use the previous iterative procedure to predict one-step ahead variances $\sigma_{i,t+1}$, correlations $\rho_{ij,t+1}$, Kendall's Tau $\tau_{ij,t+1}$and lower Tail dependences $\lambda^U_{ij,t+1}$. The latter are estimated by using the mixed-Gumbel copula for positive dependent assets and the T-copula for the remaining pairs of assets: in this last case they are zero in almost all cases except for the Euro$ -T bond Note. These forecasts are now used to estimate the portfolio Value at Risk, both with the *modified RiskMetrics multiple positions model* (11a) (11b) (11c) and with the *Multivariate copula* model (12a) and (12b). The predicted one-step-ahead VaR is then compared with the observed return and both results are recorded for later assessment using a back-testing procedure. The model is re-estimated for each observation ranging from 2000 to 3399, as explained in paragraph 6.3, and the procedure is iterated until all days have been included in the estimation sample.

We assess the performance of the models by computing Kupiec's (1995) LR tests on the empirical failure rates: this test is based on binomial theory and tests the difference between the observed and expected number of VaR exceedances of the effective portfolio losses. As VaR is based on a confidence level 1-p, when we observe N losses in excess of VaR out of T observations, hence we observe N/T proportion of excessive losses: the Kupiec's test answers the question whether $N/T$is statistically significantly different from p.

Following binomial theory, the probability of observing N failures out of T observations is (1-p)$^{T-N}p^N$, so that the test of the null hypothesis H_0: $p = p*$ is given by a likelihood ratio test statistic:

$$\text{LR} = 2\cdot\ln[(1 - p*)^{T-N} p*^N] + 2 \cdot \ln[(1 - T/N)^{T-N}\ (N/T)^N\ ]$$

which is distributed as χ^2 (1) under H_0 . It is well known that the power of this test, that is the ability to reject a bad model, rises with T: as we are working with 1400 observations, this test should work well.

The real VaR exceedances and the Kupiec test p-values are reported in Table 10 – 11 (when H_0 is not rejected at the 5 % level, the numbers are reported in bold font):

Table 11: **VaR back-testing analysis (modified RiskMetrics multiple positions model)**

	1%VaR		**5%VaR**	
	Real Exceed.(%)	*Kupiec Test: p-val.*	*Real Exceed. (%)*	*Kupiec Test: p-val.*
Normal copula (RHO)	**0.64**	**0.150**	**5.21**	**0.714**
Normal copula (Kendall)	0.50	0.037	**4.86**	**0.812**
T- copula (RHO)	2.64	0.000	7.79	0.000
T- copula (Kendall)	**0.64**	**0.150**	**5.36**	**0.535**
Mix.-Gum. + T-cop.(Kendall)	**0.71**	**0.258**	**5.43**	**0.468**
Mix.-Gum. + Normal (Kendall)	0.43	0.015	**5.07**	**0.902**
Mix-Gum + Mix-Norm.(Kendall)	0.29	0.002	**4.21**	**0.166**
Mix-Gum. + T-cop.(Tail Dep.)	0.29	0.002	**3.93**	**0.056**

Table 12: **VaR back-testing analysis (Multivariate copula modelling)**

	1%VaR				**5%VaR**			
	Normal quantile		*T-st. quan.(10 df)*		*Normal quantile*		*T-st. quan.(10 df)*	
	Real Exc.	*Kup. Test*	*Real Exc.*	*Kup. Test*	*Real Exc.*	*Kup. Test*	*Real Exc.*	*Kup. Test*
Normal copula (RHO)	**1.21**	**0.436**	**1.00**	**1.000**	**4.57**	**0.456**	**4.71**	**0.621**
Normal copula (Kendall)	**1.07**	**0.791**	**0.57**	**0.080**	**4.43**	**0.317**	**4.57**	**0.456**
T- copula (RHO)	3.79	0.000	3.07	0.000	7.00	0.001	7.29	0.000
T- copula (Kendall)	**1.36**	**0.203**	**1.14**	**0.599**	**4.79**	**0.711**	**5.00**	**1.000**
Mix.-Gum. + T-cop. (Kendall)	1.57	0.047	**1.14**	**0.599**	**4.71**	**0.621**	**4.79**	**0.711**
Mix.-Gum + Normal (Kendall)	**1.00**	**1.000**	**0.71**	**0.258**	**4.21**	**0.166**	**4.35**	**0.260**
Mix-Gum + Mix-Norm.(Kendall)	**0.71**	**0.258**	0.36	0.005	3.71	0.021	3.79	0.030

The VaR results confirm what we've already seen in paragraph 6.3 for portfolio management, that is the linear correlation measure is very sensitive to the copula used and can give poor VaR forecasts: as we can see from tables 10 – 11, the linear correlation based VaR is the only case where the Kupiec test reject the model because it is too aggressive, while all other rejections are due to too conservative estimates. As linear correlation estimation based on the T-copula is becoming quite popular, practitioners should be aware that this dependence measure is very sensitive to copula misspecification, while this is not the case for Kendall's Tau. Tail dependence, on the other hand, tend to be quite conservative instead.

When we compare the modified RiskMetrics model and the multivariate copula model we see the latter as being generally more precise and presenting fewer rejections: this is not surprising, as the RiskMetrics model assumes that log-returns of each asset follow a random-walk IGARCH(1,1) with no mean dynamics, which is a more restrictive framework than the multivariate copula model.

Finally, we note that within the multivariate copula model, the results among normal quantile and standardized T-student quantile are quite similar except for one case: this fact points out that the major benefit when estimating VaR comes from conditional dependence modeling. The choice of 10 degrees of freedom has been done by observing that portfolio returns estimated with the models presented in section 6.3, exhibit a T-student distribution with 10 degrees of freedom in almost all cases. Needless to say, lowering the d.o.f. would have determined more conservative estimates.

7 Conclusions

The aim of this paper has been to show the main shortcomings of traditional correlation analysis and presents possible solutions which has the advantage to keep risk-returns models tractable but at the same time consider the non-linear dependency among the considered variables.

Traditional portfolio theory based on multivariate normal distribution assumes that investors can benefit from diversification by investing in assets with lower correlations. However this is not what happens in reality, as one often finds financial markets with different correlations but almost the same number of market crashes (if we define market crash as when returns are in their lowest qth percentile).

In order to overcome these problems we can resort to copula theory, since copulae capture those properties of the joint distribution which are invariant under strictly increasing transformation. A common dependence measure that can be expressed as a function of copula parameters and is scale invariant is Kendall's Tau. It satisfies most of the desired properties that a dependence measure must have and it measures the concordance between two random variables. It is for this reason that it can detect non-linear association that correlation cannot see.

We showed how to use Kendall's Tau within the traditional mean-variance framework, in the place of the correlation coefficients: The empirical results confirm that linear correlation can give poor out-of-sample portfolio performances and it is very sensitive to the particular copula used. The best copula to model positive dependence was determined to be the Mixed-Gumbel, while for more general dependence the T-copula proved to be a good solution (when using Kendall's Tau). The normal copula presented the main advantage at being very easy to compute, with results very similar to the previous two.

In a similar fashion, recent empirical studies show that in volatile periods financial markets tend to be characterized by different levels of dependence than occurs in quiet periods. In order to take this reality into account , we proposed to use the concept of Tail dependence, which refers to the dependence that arises between random variables from extreme observations. We considered this measure within the well-known RiskMetrics VaR model at the place of linear correlation coefficients and we did the same with Kendall's Tau, too. Again, we found that linear correlation is very sensitive to the copula used and can give poor VaR forecasts: as linear correlation estimation based on the T-copula is becoming quite popular, practitioners should be aware that this dependence measure is very sensitive to copula misspecification, while this is not the case for Kendall's Tau. Tail dependence, on the other hand, gave a fairly conservative result, while Kendall's Tau exhibited the best outcomes.

We also proposed a more general framework than the RiskMetrics model, where we considered a parametric copula, a parametric joint distribution, but we exclude the marginals. This choice is mostly justified by the fact that as far as we are concerned with portfolio allocation and VaR, dependences across assets play the main role in today's financial industry: moreover, the misspecification of marginals can lead to dangerous biases in dependence measures estimation. This model was generally more precise and presented fewer rejections than the RiskMetrics model when using back-testing procedures: this is not a surprise, as the latter assumes that log-returns of each asset follow a random-walk IGARCH(1,1) with no mean dynamics, which is a more restrictive framework than the multivariate copula model.

Possible extensions that can be considered for future research include the passage from bivariate to multivariate copula, where many hints can be found in Joe (1997). Secondly, a solution to the trade-off between parsimonious modeling and unbiased estimates may be given by a copula *factor* model: for example, if we consider the dataset used in this work, it is quite clear that the Dow Jones, Nasdaq100 and S&P500 future on one hand, and Euro$ and T-Bond Note on the other, can be modelled by common factors. The direct consideration of transaction costs can surely help in detecting which factors are useful and which are not.

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