Exponential Lower Bounds for Polytopes in Combinatorial Optimization

Vasilis Livanos and Manuel Torres

University of Illinois at Urbana-Champaign

May 2, 2018

Main Question

Main Question

- Let's prove P $=$ NP!

Main Question

- Let's prove P = NP!
- Idea: Can we write a polynomial-size LP (Linear Program) for an NP-Complete problem (e.g. TSP)?

Main Question

- Let's prove $\mathsf{P} = \mathsf{NP}$!
- Idea: Can we write a polynomial-size LP (Linear Program) for an NP-Complete problem (e.g. TSP)?
- Another idea: Encode a problem as a polytope that is the convex hull of the vertices corresponding to the solutions of the problem.

Necessary Background

Necessary Background

Definition (Extended Formulation)

Let $P = \{x : Ax \leq b\}$ and $Q = \{(x, y) : Bx + Cy \leq d\}$. Then Q is an *extended formulation (EF)* of P if and only if

$$P = \{x : \exists y, (x, y) \in Q\}$$

The size of an extended formulation is the number of facets (faces of maximal dimension) of Q.

Necessary Background

Definition (Extended Formulation)

Let $P = \{x : Ax \leq b\}$ and $Q = \{(x, y) : Bx + Cy \leq d\}$. Then Q is an *extended formulation (EF)* of P if and only if

$$P = \{x : \exists y, (x, y) \in Q\}$$

The size of an extended formulation is the number of facets (faces of maximal dimension) of Q.

Definition (Extension Complexity)

The *extension complexity* of P, denoted $\mathrm{xc}(P)$, is the minimum size EF of P.

Extended Formulations Can Help

Extended Formulations Can Help

- There are polytopes that have exponential size (i.e. number of facets), but have an EF with polynomial size.

Extended Formulations Can Help

- There are polytopes that have exponential size (i.e. number of facets), but have an EF with polynomial size.
    1. The permutahedron polytope, which is the convex hull of the permutation group on $\{1, 2, \ldots, n\}$.

Extended Formulations Can Help

- There are polytopes that have exponential size (i.e. number of facets), but have an EF with polynomial size.
  1. The permutahedron polytope, which is the convex hull of the permutation group on $\{1, 2, \ldots, n\}$.
  2. The spanning tree polytope.

Extended Formulations Can Help

- There are polytopes that have exponential size (i.e. number of facets), but have an EF with polynomial size.
  1. The permutahedron polytope, which is the convex hull of the permutation group on $\{1, 2, \ldots, n\}$.
  2. The spanning tree polytope.
  3. The matching polytope for planar graphs.

Extended Formulations Can Help

- There are polytopes that have exponential size (i.e. number of facets), but have an EF with polynomial size.
  1. The permutahedron polytope, which is the convex hull of the permutation group on $\{1, 2, \ldots, n\}$.
  2. The spanning tree polytope.
  3. The matching polytope for planar graphs.
- Thus, even though these polytopes have exponential size, we can write a polynomial-size LP for them, through their EF.

Prior Work

Prior Work

Theorem (Yannakakis, '91)

Every symmetric LP for TSP has extension complexity $2^{\Omega(n)}$*. Thus, it also has an exponential number of inequalities.*

Prior Work

Theorem (Yannakakis, '91)

Every symmetric LP for TSP has extension complexity $2^{\Omega(n)}$. Thus, it also has an exponential number of inequalities.

Corollary

No symmetric LP can be used to solve TSP in polynomial time.

Prior Work

Theorem (Yannakakis, '91)

Every symmetric LP for TSP has extension complexity $2^{\Omega(n)}$. Thus, it also has an exponential number of inequalities.

Corollary

No symmetric LP can be used to solve TSP in polynomial time.

- Can we use an asymmetric LP to solve TSP in polynomial time?

More Technical Background

More Technical Background

Definition (Nonnegative Rank)

Let $M \in \mathbb{R}_+^{m \times n}$. The *nonnegative rank* of M is defined as $\text{rank}_+(M) = \min\{r : M = TU, T \in \mathbb{R}_+^{m \times r}, U \in \mathbb{R}_+^{r \times n}\}$.

More Technical Background

Definition (Nonnegative Rank)

Let $M \in \mathbb{R}_+^{m \times n}$. The *nonnegative rank* of M is defined as $\text{rank}_+(M) = \min\{r : M = TU, T \in \mathbb{R}_+^{m \times r}, U \in \mathbb{R}_+^{r \times n}\}$.

Definition (Slack Matrix)

Let $A \in \mathbb{R}^{m \times d}$, $b \in \mathbb{R}^m$, and $V = \{v_1, \ldots, v_n\} \subseteq \mathbb{R}^d$. Let $P = \{x \in \mathbb{R}^d : Ax \leq b\} = \text{conv}(V)$. Then $S \in \mathbb{R}_+^{m \times n}$, where each entry is defined as $S_{ij} = b_i - A_i v_j$ with $i \in [m]$, $j \in [n]$, is the *slack matrix* of P with respect to $Ax \leq b$ and V.

Yannakakis's Factorization Theorem

Yannakakis's Factorization Theorem

Theorem (Factorization Theorem, Yannakakis '91)

Let $P = \{x \in \mathbb{R}^d \mid Ax \leq b\} = \mathrm{conv}(V)$ *be a polytope, and* S *be its slack matrix with respect to* $Ax \leq b$ *and* V. *Then*

$$\mathrm{xc}(P) = \mathrm{rank}_+(S)$$

Yannakakis's Factorization Theorem

Theorem (Factorization Theorem, Yannakakis '91)

Let $P = \{x \in \mathbb{R}^d \mid Ax \leq b\} = \mathrm{conv}(V)$ *be a polytope, and* S *be its slack matrix with respect to* $Ax \leq b$ *and* V*. Then*

$$\mathrm{xc}(P) = \mathrm{rank}_+(S)$$

- If we want to lower bound $\mathrm{xc}(P)$, it suffices to lower bound $\mathrm{rank}_+(S)$.

Lower Bound on Nonnegative Rank

Lower Bound on Nonnegative Rank

Definition (Support Matrix)

The *support matrix* of a matrix M with nonnegative entries is

$$\mathrm{suppmat}(M)_{ij} = \begin{cases} 0 & M_{ij} = 0 \\ 1 & M_{ij} \neq 0 \end{cases}$$

Lower Bound on Nonnegative Rank

Definition (Support Matrix)

The *support matrix* of a matrix M with nonnegative entries is

$$\mathrm{suppmat}(M)_{ij} = \begin{cases} 0 & M_{ij} = 0 \\ 1 & M_{ij} \neq 0 \end{cases}$$

Theorem (Yannakakis, '91)

Let M be a matrix with nonnegative entries, and $\mathrm{suppmat}(M)$ *its support matrix. Then,* $\mathrm{rank}_+(M)$ *is lower bounded by the rectangle covering bound for* $\mathrm{suppmat}(M)$.

Lower Bound on Nonnegative Rank

Definition (Support Matrix)

The *support matrix* of a matrix M with nonnegative entries is

$$\mathrm{suppmat}(M)_{ij} = \begin{cases} 0 & M_{ij} = 0 \\ 1 & M_{ij} \neq 0 \end{cases}$$

Theorem (Yannakakis, '91)

Let M be a matrix with nonnegative entries, and $\mathrm{suppmat}(M)$ its support matrix. Then, $\mathrm{rank}_+(M)$ is lower bounded by the rectangle covering bound for $\mathrm{suppmat}(M)$.

- This provides a connection between $\mathrm{xc}(P)$ and communication complexity.

Reminder: Rectangle Covers

Reminder: Rectangle Covers

Definition (Rectangle Cover)

Let $M \in \{0,1\}^{2^n \times 2^n}$ with indices corresponding to bit strings. A *rectangle* is a subset of $\{0,1\}^n \times \{0,1\}^n$. We say that $R \subseteq \{0,1\}^n \times \{0,1\}^n$ is a *b-monochromatic rectangle* for f if $M_{xy} = b$ for all $(x,y) \in R$. We say that a set $\mathcal{R}$ of b-monochromatic rectangles is a *b-rectangle cover* if $\{(x,y) \in \{0,1\}^n \times \{0,1\}^n : M_{xy} = b\} \subseteq \bigcup_{R \in \mathcal{R}} R$.

A Matrix of Exponential Nonnegative Rank

A Matrix of Exponential Nonnegative Rank

Definition

Let M be a $2^n \times 2^n$ matrix, where each row (resp. column) is indexed by a n-bit string a (resp. b), with

$$M_{ab} := \left(1 - a^T b\right)^2$$

A Matrix of Exponential Nonnegative Rank

Definition

Let M be a $2^n \times 2^n$ matrix, where each row (resp. column) is indexed by a n-bit string a (resp. b), with

$$M_{ab} := \left(1 - a^T b\right)^2$$

Theorem (De Wolf, '03)

Every 1-monochromatic rectangle cover of $\operatorname{suppmat}(M)$ *has size* $2^{\Omega(n)}$.

CUT(n) and COR(n) Polytopes

Definition (cut polytope)

Let $K_n = (V, E)$ be the complete graph on n vertices. Let $\delta(S)$ denote the cut of $S \subseteq V$. Then let $\chi^{\delta(S)} \in \mathbb{R}^{|E|}$ such that

$$\chi_e^{\delta(S)} = \begin{cases} 1 & e \in \delta(S) \\ 0 & e \notin \delta(S) \end{cases}.$$

Then $\mathrm{CUT}(n) := \mathrm{conv}\left(\left\{\chi^{\delta(S)} \in \mathbb{R}^{|E|} \mid S \subseteq V\right\}\right)$

CUT(n) and COR(n) Polytopes

Definition (cut polytope)

Let $K_n = (V, E)$ be the complete graph on n vertices. Let $\delta(S)$ denote the cut of $S \subseteq V$. Then let $\chi^{\delta(S)} \in \mathbb{R}^{|E|}$ such that

$$\chi_e^{\delta(S)} = \begin{cases} 1 & e \in \delta(S) \\ 0 & e \notin \delta(S) \end{cases}.$$

Then $\mathrm{CUT}(n) := \mathrm{conv}\left(\left\{\chi^{\delta(S)} \in \mathbb{R}^{|E|} \mid S \subseteq V\right\}\right)$

Definition (correlation polytope)

We have $\mathrm{COR}(n) := \mathrm{conv}\left(\left\{bb^T \in \mathbb{R}^{n \times n} \mid b \in \{0,1\}^n\right\}\right)$

Connection Between CUT(n) and COR(n)

Definition (linearly isomorphic polytopes)

Two polytopes $P \subseteq \mathbb{R}^n$ and $Q \subseteq \mathbb{R}^m$ are called *linearly isomorphic* if there exists a linear invertible function $f : \mathbb{R}^n \to \mathbb{R}^m$ such that $f(P) = Q$.

Connection Between CUT(n) and COR(n)

Definition (linearly isomorphic polytopes)

Two polytopes $P \subseteq \mathbb{R}^n$ and $Q \subseteq \mathbb{R}^m$ are called *linearly isomorphic* if there exists a linear invertible function $f : \mathbb{R}^n \to \mathbb{R}^m$ such that $f(P) = Q$.

Lemma (De Simone, '90)

For all n, COR(n) *is linearly isomorphic to* CUT($n+1$)*.*

Connection Between CUT(n) and COR(n)

Definition (linearly isomorphic polytopes)

Two polytopes $P \subseteq \mathbb{R}^n$ and $Q \subseteq \mathbb{R}^m$ are called *linearly isomorphic* if there exists a linear invertible function $f : \mathbb{R}^n \to \mathbb{R}^m$ such that $f(P) = Q$.

Lemma (De Simone, '90)

For all n, COR(n) *is linearly isomorphic to* CUT($n+1$).

Corollary

$\mathrm{xc}(\mathrm{COR}(n)) = \mathrm{xc}(\mathrm{CUT}(n+1))$.

CUT(n) has Exponential Extension Complexity

Theorem

The extension complexity of CUT(n) *is* $2^{\Omega(n)}$.

CUT(n) has Exponential Extension Complexity

Theorem

The extension complexity of CUT(n) *is* $2^{\Omega(n)}$.

Proof Sketch.

1. $\mathrm{xc}(\mathrm{COR}(n)) = \mathrm{xc}(\mathrm{CUT}(n+1))$

$\square$

CUT(n) has Exponential Extension Complexity

Theorem

The extension complexity of CUT(n) *is* $2^{\Omega(n)}$.

Proof Sketch.

1. $\mathrm{xc}(\mathrm{COR}(n)) = \mathrm{xc}(\mathrm{CUT}(n+1))$
2. $\mathrm{xc}(\mathrm{COR}(n)) = \mathrm{rank}_+(S)$ where S is the slack matrix of COR(n)

$\square$

CUT(n) has Exponential Extension Complexity

Theorem

The extension complexity of CUT(n) *is* $2^{\Omega(n)}$.

Proof Sketch.

1. $\mathrm{xc}(\mathrm{COR}(n)) = \mathrm{xc}(\mathrm{CUT}(n+1))$
2. $\mathrm{xc}(\mathrm{COR}(n)) = \mathrm{rank}_+(S)$ where S is the slack matrix of COR(n)
3. $\mathrm{rank}_+(S) \geq \mathrm{rank}_+(M)$

$\square$

$\mathrm{CUT}(n)$ has Exponential Extension Complexity

Theorem

The extension complexity of $\mathrm{CUT}(n)$ *is* $2^{\Omega(n)}$.

Proof Sketch.

1. $\mathrm{xc}(\mathrm{COR}(n)) = \mathrm{xc}(\mathrm{CUT}(n+1))$
2. $\mathrm{xc}(\mathrm{COR}(n)) = \mathrm{rank}_+(S)$ where S is the slack matrix of $\mathrm{COR}(n)$
3. $\mathrm{rank}_+(S) \geq \mathrm{rank}_+(M)$
4. $\mathrm{rank}_+(M) = 2^{\Omega(n)}$.

$\square$

Reductions

Lemma

Let P and F be polytopes. If F is a face of P, then $\mathrm{xc}(P) \geq \mathrm{xc}(F)$.

STAB(G) Reduces to COR(n)

Definition (Stable Set Polytope)

The *stable set polytope* STAB(G) is defined as the convex hull of the characteristic vectors of all stable sets in $G = (V, E)$. That is,

$$\mathrm{STAB}(G) := \mathrm{conv}\left(\left\{\chi^S \in \mathbb{R}^V \mid S \text{ is a stable set of } G\right\}\right).$$

Reduction from STAB(G) to COR(n)

- We will construct a graph H_n

Reduction from STAB(G) to COR(n)

- We will construct a graph H_n
- Let K_n be the complete graph on the vertices $[n]$

Reduction from STAB(G) to COR(n)

- We will construct a graph H_n
- Let K_n be the complete graph on the vertices $[n]$
- For an edge $\{i, j\}$ in K_n where $i < j$, label the edge ij

Reduction from STAB(G) to COR(n)

- We will construct a graph H_n
- Let K_n be the complete graph on the vertices $[n]$
- For an edge $\{i,j\}$ in K_n where $i < j$, label the edge ij
- The vertex set of H_n is

$$V = \{ii, \overline{ii} : i \in [n]\} \cup \{ij, \overline{ij}, \underline{ij}, \underline{\overline{ij}}, : i,j \in [n], i < j\}.$$

Reduction from STAB(G) to COR(n) (cont.)

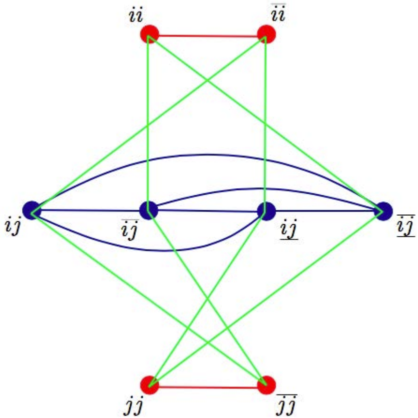

Figure: the set of edges and vertices added to H_n for some $i, j \in [n]$ with $i < j$.

Reduction from STAB(G) to COR(n) (cont.)

Lemma

STAB(H_n) *contains a face that is an extension of* COR(n).

Extension Complexity of STAB(G) is $2^{\Omega(\sqrt{n})}$

Theorem

For all n, there exists a graph G_n with n vertices such that $\mathrm{xc}(\mathrm{STAB}(G_n)) = 2^{\Omega(\sqrt{n})}$.

Extension Complexity of STAB(G) is $2^{\Omega(\sqrt{n})}$

Theorem

For all n, there exists a graph G_n with n vertices such that $\mathrm{xc}(\mathrm{STAB}(G_n)) = 2^{\Omega(\sqrt{n})}$.

Proof Sketch.

1. Let G_n be H_ℓ with $n - O(\ell^2)$ isolated vertices

□

Extension Complexity of STAB(G) is $2^{\Omega(\sqrt{n})}$

Theorem

For all n, there exists a graph G_n with n vertices such that $\mathrm{xc}(\mathrm{STAB}(G_n)) = 2^{\Omega(\sqrt{n})}$.

Proof Sketch.

1. Let G_n be H_ℓ with $n - O(\ell^2)$ isolated vertices $\Rightarrow \ell = \Omega(\sqrt{n})$

$\square$

Extension Complexity of STAB(G) is $2^{\Omega(\sqrt{n})}$

Theorem

For all n, there exists a graph G_n with n vertices such that $\mathrm{xc}(\mathrm{STAB}(G_n)) = 2^{\Omega(\sqrt{n})}$.

Proof Sketch.

1. Let G_n be H_ℓ with $n - O(\ell^2)$ isolated vertices $\Rightarrow \ell = \Omega(\sqrt{n})$
2. $\mathrm{xc}(\mathrm{STAB}(G_n)) \geq \mathrm{xc}(\mathrm{STAB}(H_\ell))$

$\square$

Extension Complexity of STAB(G) is $2^{\Omega(\sqrt{n})}$

Theorem

For all n, there exists a graph G_n with n vertices such that $\mathrm{xc}(\mathrm{STAB}(G_n)) = 2^{\Omega(\sqrt{n})}$.

Proof Sketch.

1. Let G_n be H_ℓ with $n - O(\ell^2)$ isolated vertices $\Rightarrow \ell = \Omega(\sqrt{n})$
2. $\mathrm{xc}(\mathrm{STAB}(G_n)) \geq \mathrm{xc}(\mathrm{STAB}(H_\ell))$
3. $\mathrm{xc}(\mathrm{STAB}(H_\ell)) \geq \mathrm{xc}(\mathrm{COR}(\ell))$

□

Extension Complexity of STAB(G) is $2^{\Omega(\sqrt{n})}$

Theorem

For all n, there exists a graph G_n with n vertices such that $\mathrm{xc}(\mathrm{STAB}(G_n)) = 2^{\Omega(\sqrt{n})}$.

Proof Sketch.

1. Let G_n be H_ℓ with $n - O(\ell^2)$ isolated vertices $\Rightarrow \ell = \Omega(\sqrt{n})$
2. $\mathrm{xc}(\mathrm{STAB}(G_n)) \geq \mathrm{xc}(\mathrm{STAB}(H_\ell))$
3. $\mathrm{xc}(\mathrm{STAB}(H_\ell)) \geq \mathrm{xc}(\mathrm{COR}(\ell))$
4. $\mathrm{xc}(\mathrm{COR}(\ell)) = 2^{\Omega(\ell)}$

$\square$

Extension Complexity of STAB(G) is $2^{\Omega(\sqrt{n})}$

Theorem

For all n, there exists a graph G_n with n vertices such that $\mathrm{xc}(\mathrm{STAB}(G_n)) = 2^{\Omega(\sqrt{n})}$.

Proof Sketch.

1. Let G_n be H_ℓ with $n - O(\ell^2)$ isolated vertices $\Rightarrow \ell = \Omega(\sqrt{n})$
2. $\mathrm{xc}(\mathrm{STAB}(G_n)) \geq \mathrm{xc}(\mathrm{STAB}(H_\ell))$
3. $\mathrm{xc}(\mathrm{STAB}(H_\ell)) \geq \mathrm{xc}(\mathrm{COR}(\ell))$
4. $\mathrm{xc}(\mathrm{COR}(\ell)) = 2^{\Omega(\ell)}$
5. $2^{\Omega(\ell)} = 2^{\Omega(\sqrt{n})}$.

$\square$

TSP(n) Polytope

TSP(n) Polytope

Definition (TSP Polytope)

Let $K_n = (V_n, E_n)$ be the complete graph with n vertices. The *Traveling Salesman Problem polytope* TSP(n) is defined as the convex hull of the characteristic vectors of all $F \subseteq E_n$ such that F is a Hamiltonian cycle of K_n. In other words,

$$\mathrm{TSP}(n) := \mathrm{conv}\left(\left\{\chi^F \in \mathbb{R}^{E_n} \mid F \subseteq E_n \text{ is a Hamiltonian cycle of } K_n\right\}\right).$$

TSP(n) Polytope

Definition (TSP Polytope)

Let $K_n = (V_n, E_n)$ be the complete graph with n vertices. The *Traveling Salesman Problem polytope* TSP(n) is defined as the convex hull of the characteristic vectors of all $F \subseteq E_n$ such that F is a Hamiltonian cycle of K_n. In other words,

$$\mathrm{TSP}(n) := \mathrm{conv}\left(\left\{\chi^F \in \mathbb{R}^{E_n} \mid F \subseteq E_n \text{ is a Hamiltonian cycle of } K_n\right\}\right).$$

- Intuitively, the TSP(n) polytope is the polytope that has every Hamiltonian cycle of K_n as a vertex.

TSP(n) Polytope

TSP(n) Polytope

Lemma

For every n, there exists a positive integer $q = O(n^2)$ such that $\mathrm{COR}(n)$ *is contained in a face of* $\mathrm{TSP}(q)$.

TSP(n) Polytope

Lemma

For every n, there exists a positive integer $q = O(n^2)$ such that $\mathrm{COR}(n)$ is contained in a face of $\mathrm{TSP}(q)$.

Theorem

The extension complexity of $\mathrm{TSP}(n)$ is $\mathrm{xc}\left(\mathrm{TSP}(n)\right) = 2^{\Omega\left(\sqrt{n}\right)}$.

Subsequent Work

- Rhothvoß showed that the matching polytope has extension complexity $2^{\Omega(n)}$

Subsequent Work

- Rhothvoß showed that the matching polytope has extension complexity $2^{\Omega(n)}$
- In this work, he also showed that the extension complexity of TSP(n) is $2^{\Omega(n)}$.

Subsequent Work

- Rhothvoß showed that the matching polytope has extension complexity $2^{\Omega(n)}$
- In this work, he also showed that the extension complexity of $\mathrm{TSP}(n)$ is $2^{\Omega(n)}$.
- Braverman and Moitra showed that any linear program that attains an $O(n^{1-\epsilon})$ approximation for Max-Clique has size $2^{\Omega(n^{\epsilon})}$.

QUESTIONS ?