Exponential Lower Bounds for Polytopes in Combinatorial Optimization

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Main Question

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- Let's prove $\mathsf{P} = \mathsf{NP}$!
- Idea: Can we write a polynomial-size LP (Linear Program) for an NP-Complete problem (e.g. TSP)?
- Another idea: Encode a problem as a polytope that is the convex hull of the vertices corresponding to the solutions of the problem.

Necessary Background

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Definition (Extended Formulation)

Let $P = \{x : Ax \leq b\}$ and $Q = \{(x, y) : Bx + Cy \leq d\}$. Then Q is an *extended formulation (EF)* of P if and only if

$$P = \{x : \exists y, (x, y) \in Q\}$$

The size of an extended formulation is the number of facets (faces of maximal dimension) of Q.

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Definition (Extension Complexity)

The *extension complexity* of P, denoted $\mathrm{xc}(P)$, is the minimum size EF of P.

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  3. The matching polytope for planar graphs.

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  1. The permutahedron polytope, which is the convex hull of the permutation group on $\{1, 2, \cdots, n\}$.
  2. The spanning tree polytope.
  3. The matching polytope for planar graphs.
- Thus, even though these polytopes have exponential size, we can write a polynomial-size LP for them, through their EF.

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Corollary

No symmetric LP can be used to solve TSP in polynomial time.

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Corollary

No symmetric LP can be used to solve TSP in polynomial time.

- Can we use an asymmetric LP to solve TSP in polynomial time?

More Technical Background

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Definition (Nonnegative Rank)

Let $M \in \mathbb{R}_+^{m \times n}$. The *nonnegative rank* of M is defined as $\text{rank}_+(M) = \min\{r : M = TU, T \in \mathbb{R}_+^{m \times r}, U \in \mathbb{R}_+^{r \times n}\}$.

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Definition (Slack Matrix)

Let $A \in \mathbb{R}^{m \times d}$, $b \in \mathbb{R}^m$, and $V = \{v_1, \ldots, v_n\} \subseteq \mathbb{R}^d$. Let $P = \{x \in \mathbb{R}^d : Ax \leq b\} = \text{conv}(V)$. Then $S \in \mathbb{R}_+^{m \times n}$, where each entry is defined as $S_{ij} = b_i - A_i v_j$ with $i \in [m]$, $j \in [n]$, is the *slack matrix* of P with respect to $Ax \leq b$ and V.

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$$\mathrm{xc}(P) = \mathrm{rank}_+(S)$$

- If we want to lower bound $\mathrm{xc}(P)$, it suffices to lower bound $\mathrm{rank}_+(S)$.

Lower Bound on Nonnegative Rank

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Definition (Support Matrix)

The *support matrix* of a matrix M with nonnegative entries is

$$\mathrm{suppmat}(M)_{ij} = \begin{cases} 0 & M_{ij} = 0 \\ 1 & M_{ij} \neq 0 \end{cases}$$

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Let M be a matrix with nonnegative entries, and $\mathrm{suppmat}(M)$ its support matrix. Then, $\mathrm{rank}_+(M)$ is lower bounded by the rectangle covering bound for $\mathrm{suppmat}(M)$.

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- This provides a connection between $\operatorname{xc}(P)$ and communication complexity.

Reminder: Rectangle Covers

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Definition (Rectangle Cover)

Let $M \in \{0,1\}^{2^n \times 2^n}$ with indices corresponding to bit strings. A *rectangle* is a subset of $\{0,1\}^n \times \{0,1\}^n$. We say that $R \subseteq \{0,1\}^n \times \{0,1\}^n$ is a *b-monochromatic rectangle* for f if $M_{xy} = b$ for all $(x,y) \in R$. We say that a set $\mathcal{R}$ of b-monochromatic rectangles is a *b-rectangle cover* if $\{(x,y) \in \{0,1\}^n \times \{0,1\}^n : M_{xy} = b\} \subseteq \bigcup_{R \in \mathcal{R}} R$.

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Definition

Let M be a $2^n \times 2^n$ matrix, where each row (resp. column) is indexed by a n-bit string a (resp. b), with

$$M_{ab} := \left(1 - a^T b\right)^2$$

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Theorem (De Wolf, '03)

Every 1-monochromatic rectangle cover of $\text{suppmat}(M)$ *has size* $2^{\Omega(n)}$.

TSP(n) Polytope

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Definition (TSP Polytope)

Let $K_n = (V_n, E_n)$ be the complete graph with n vertices. The *Traveling Salesman Problem polytope* TSP(n) is defined as the convex hull of the characteristic vectors of all $F \subseteq E_n$ such that F is a Hamiltonian cycle of K_n. In other words,

$$\mathrm{TSP}(n) := \mathrm{conv}\left(\left\{\chi^F \in \mathbb{R}^{E_n} \mid F \subseteq E_n \text{ is a Hamiltonian cycle of } K_n\right\}\right).$$

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- Intuitively, the TSP(n) polytope is the polytope that has every Hamiltonian cycle of K_n as a vertex.

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Lemma

For every n, there exists a positive integer $q = O(n^2)$ such that $\mathrm{COR}(n)$ *is contained in a face of* $\mathrm{TSP}(q)$.

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Theorem

The extension complexity of TSP(n) *is* $\mathrm{xc}\,(\mathrm{TSP}(n)) = 2^{\Omega(\sqrt{n})}$.

QUESTIONS ?