Exponential Lower Bounds for Polytopes in Combinatorial Optimization

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Main Question

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- Idea: Can we write a polynomial-size LP (Linear Program) for an NP-Complete problem (e.g. TSP)?

Necessary Background

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Definition (Extended Formulation)

Let $P = \{x : Ax \leq b\}$ and $Q = \{(x, y) : Bx + Cy \leq d\}$. Then Q is an *extended formulation (EF)* of P if and only if

$$P = \{x : \exists y, (x, y) \in Q\}$$

The size of an extended formulation is the number of facets (faces of maximal dimension) of Q.

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Definition (Extension Complexity)

The *extension complexity* of P, denoted $\mathrm{xc}(P)$, is the minimum size EF of P.

Extended Formulations Can Help

Here we give the example of the permutahedron maybe mention work of EFs in combinatorial optimization

Prior Work

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Theorem (Yannakakis, '91)

Every symmetric LP for TSP has extension complexity $2^{\Omega(n)}$. Thus, it also has an exponential number of inequalities.

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Corollary

No symmetric LP can be used to solve TSP in polynomial time.

More Technical Background

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Definition (Nonnegative Rank)

Let $M \in \mathbb{R}_+^{m \times n}$. The *nonnegative rank* of M is defined as $\text{rank}_+(M) = \min\{r : M = TU, T \in \mathbb{R}_+^{m \times r}, U \in \mathbb{R}_+^{r \times n}\}$.

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Definition (Slack Matrix)

Let $A \in \mathbb{R}^{m \times d}$, $b \in \mathbb{R}^m$, and $V = \{v_1, \ldots, v_n\} \subseteq \mathbb{R}^d$. Let $P = \{x \in \mathbb{R}^d : Ax \leq b\} = \text{conv}(V)$. Then $S \in \mathbb{R}_+^{m \times n}$, where each entry is defined as $S_{ij} = b_i - A_i v_j$ with $i \in [m]$, $j \in [n]$, is the *slack matrix* of P with respect to $Ax \leq b$ and V.

Yannakakis's Factorization Theorem

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Theorem (Factorization Theorem)

Let $P = \{x \in \mathbb{R}^d \mid Ax \leq b\} = \mathrm{conv}(V)$ *be a polytope, and* S *be its slack matrix with respect to* $Ax \leq b$ *and* V. *Then*

$$\mathrm{xc}(P) = \mathrm{rank}_+(S)$$

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$$\mathrm{xc}(P) = \mathrm{rank}_+(S)$$

- If we want to lower bound $\mathrm{xc}(P)$, it suffices to lower bound $\mathrm{rank}_+(S)$.

Lower Bound on Nonnegative Rank

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Definition (Support Matrix)

The *support matrix* of a matrix M with nonnegative entries is

$$\mathrm{suppmat}(M)_{ij} = \begin{cases} 0 & M_{ij} = 0 \\ 1 & M_{ij} \neq 0 \end{cases}$$

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Theorem

Let M be a matrix with nonnegative entries, and $\operatorname{suppmat}(M)$ *its support matrix. Then,* $\operatorname{rank}_+(M)$ *is lower bounded by the rectangle covering bound for* $\operatorname{suppmat}(M)$.

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- This provides a connection between $\mathrm{xc}(P)$ and communication complexity.

A Matrix of Exponential Nonnegative Rank

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Definition

Let M be a $2^n \times 2^n$ matrix, where each row (resp. column) is indexed by a n-bit string a (resp. b), with

$$M_{ab} := \left(1 - a^T b\right)^2$$

Important Property of M

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Theorem (De Wolf, '03)

Every 1*-monochromatic rectangle cover of* $\mathrm{suppmat}(M)$ *has size* $2^{\Omega(n)}$*.*

TSP(n) Polytope

Definition of TSP Polytope

TSP(n) Polytope

Face lemma and lower bound Theorem

QUESTIONS ?