Exponential Lower Bounds for Polytopes in Combinatorial Optimization

Vasilis Livanos and Manuel Torres

University of Illinois at Urbana-Champaign

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Main Question

Question: can we write a polynomial size linear program for TSP?

Necessary Background

give definition of extended formulation, size of extended formulation, and extension complexity

Extended Formulations Can Help

Here we give the example of the permutahedron

Prior Work

- state the result of yannakakis regarding the exponential lower bounds on symmetric LPs for TSP
- maybe mention work of EFs in combinatorial optimization
- (really, most of the work we want to talk about came after Fiorini et al, so we don't really have much prior work to talk about, mainly work that came after)

More Technical Background

- give definition of nonnegative rank,
- give definition of slack matrix

Yannakakis's Factorization Theorem

give statement of theorem (maybe also have to give definition of an extension)

Lower Bound on Nonnegative Rank

state theorem regarding the lower bound of nonnegative rank by the rectangle cover bound of the support matrix

A Matrix of Exponential Nonnegative Rank

give the definition of M (and the equivalent characterization?)

Important Property of M

give statement of theorem showing that every 1-monochromatic rectangle cover of the support matrix of M has exponential size

CUT(n) and COR(n) Polytopes

Definition (cut polytope)

Let $G = (V, E)$ be a graph. Let $\delta(S)$ denote the cut of $S \subseteq V$. Then let $\chi^{\delta(X)} \in \mathbb{R}^{|E|}$ such that

$$\chi_e^{\delta(X)} = \begin{cases} 1 & e \in \delta(X) \\ 0 & e \notin \delta(X) \end{cases}.$$

Then $\mathrm{CUT}(n) := \mathrm{conv}\left(\left\{\chi^{\delta(X)} \in \mathbb{R}^{|E|} \mid X \subseteq V_n\right\}\right)$

Definition (correlation polytope)

We have $\mathrm{COR}(n) := \mathrm{conv}\left(\left\{bb^T \in \mathbb{R}^{n \times n} \mid b \in \{0,1\}^n\right\}\right)$

Connection Between CUT(n) and COR(n)

define linearly isomorphic, state theorem showing COR(n) is linearly isomorphic CUT($n+1$) (Question: why is it that two linearly isomorphic polytopes have some extension complexity?)

Definition (linearly isomorphic polytopes)

Two polytopes $P \subseteq \mathbb{R}^n$ and $Q \subseteq \mathbb{R}^m$ are called *linearly isomorphic* if there exists an invertible function $f : \mathbb{R}^n \to \mathbb{R}^m$ such that $f(P) = Q$.

Lemma

For all n, COR(n) *is linearly isomorphic to* CUT($n+1$).

CUT(n) has Exponential Extension Complexity

state theorem regarding exponential extension complexity of cut polytope

Theorem

There exists some constant $c > 0$ such that for all n,

$$\mathrm{xc}(\mathrm{CUT}(n+1)) \geq 2^{c(n)}.$$

CUT(n) has Exponential Extension Complexity (pf. sketch)

give a proof sketch of $\mathrm{xc}(\mathrm{CUT}(n)) = 2^{\Omega(n)}$

Reductions

state lemma regarding reductions

STAB(n) Reduces to COR(n)

give definition of STAB(n), state about reduction

Reduction from STAB(n) to COR(n)

show the picture of the reduction

Extension Complexity of STAB(n) is $2^{\Omega(\sqrt{n})}$

state theorem about this and give proof sketch

Subsequent Work

state the work of Rhothvoßon matching polytope and showing a better bound on the TSP polytope

Subsequent Work (cont.)

state the work on approximate EFs