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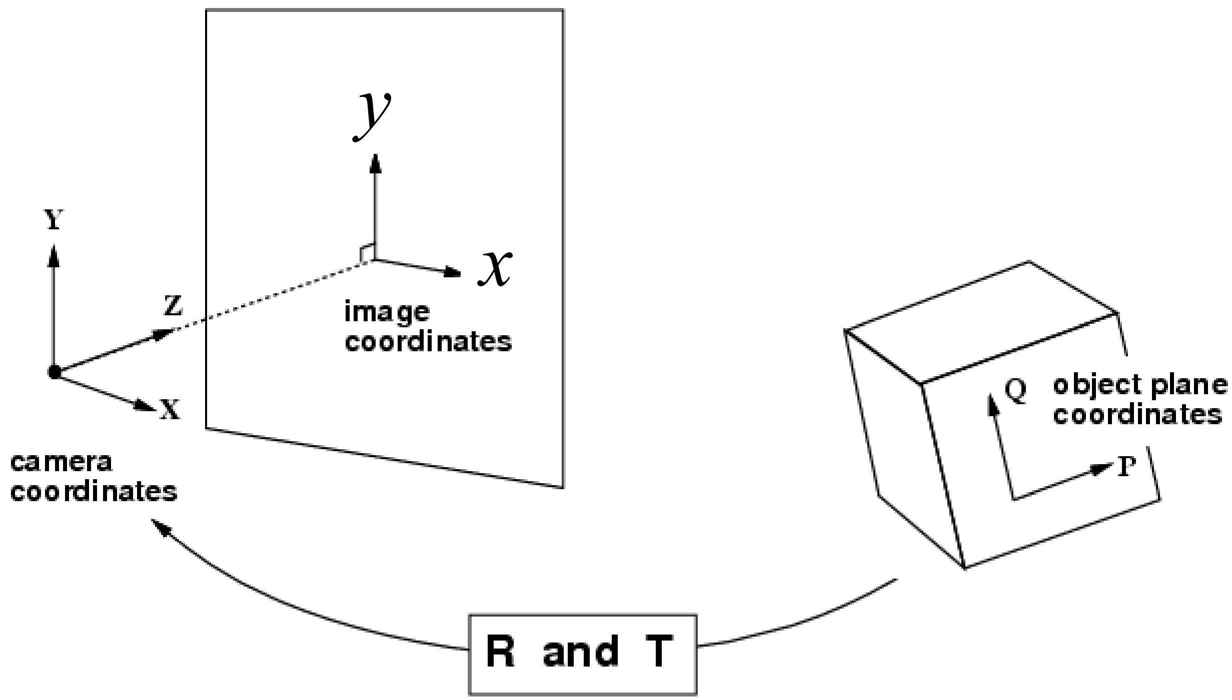
Lecture 16:

Planar Homographies

Motivation:
Points on Planar Surface



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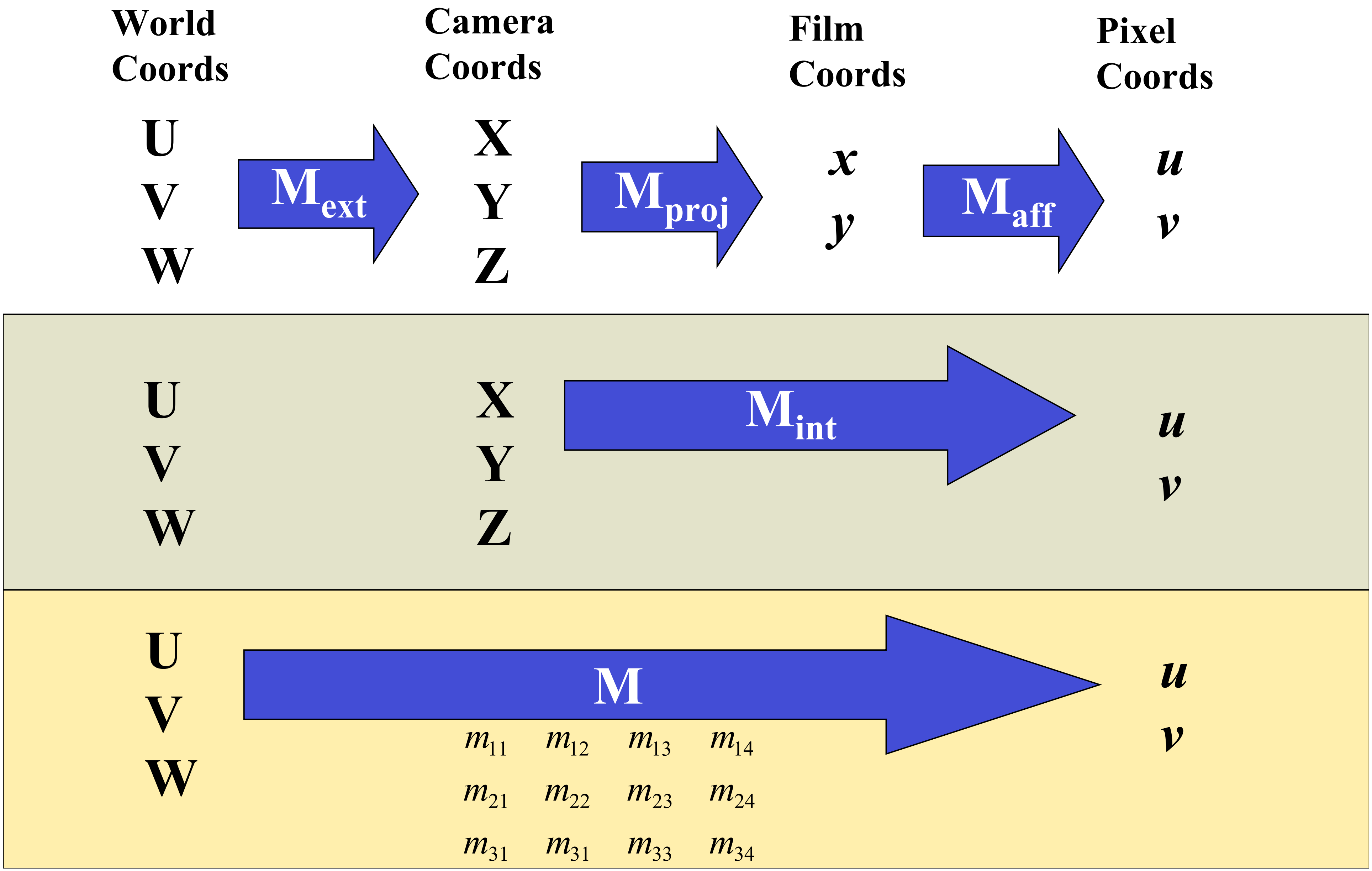
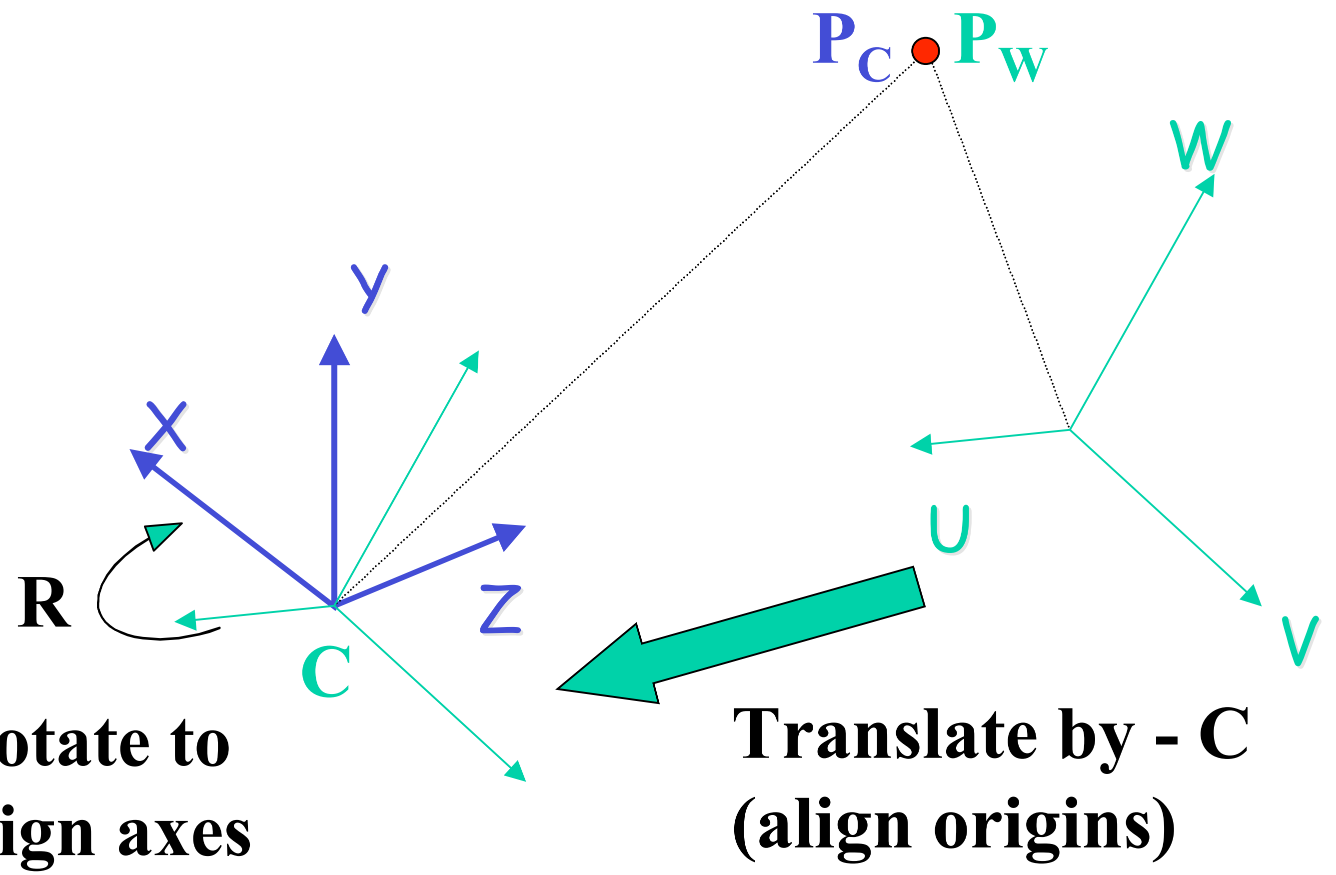
Review : Forward Projection



World to Camera Transformation



$$P_C = R\,(\,P_W - C\,)$$
$$= R\,P_W + T$$

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Perspective Matrix Equation

(Camera Coordinates)

$$x = f\frac{X}{Z}$$

$$y = f\frac{Y}{Z}$$

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

$$p = M_{\text{int}} \cdot P_{\text{C}}$$

Film to Pixel Coords

2D affine transformation from film coords (x,y) to pixel coordinates (u,v):

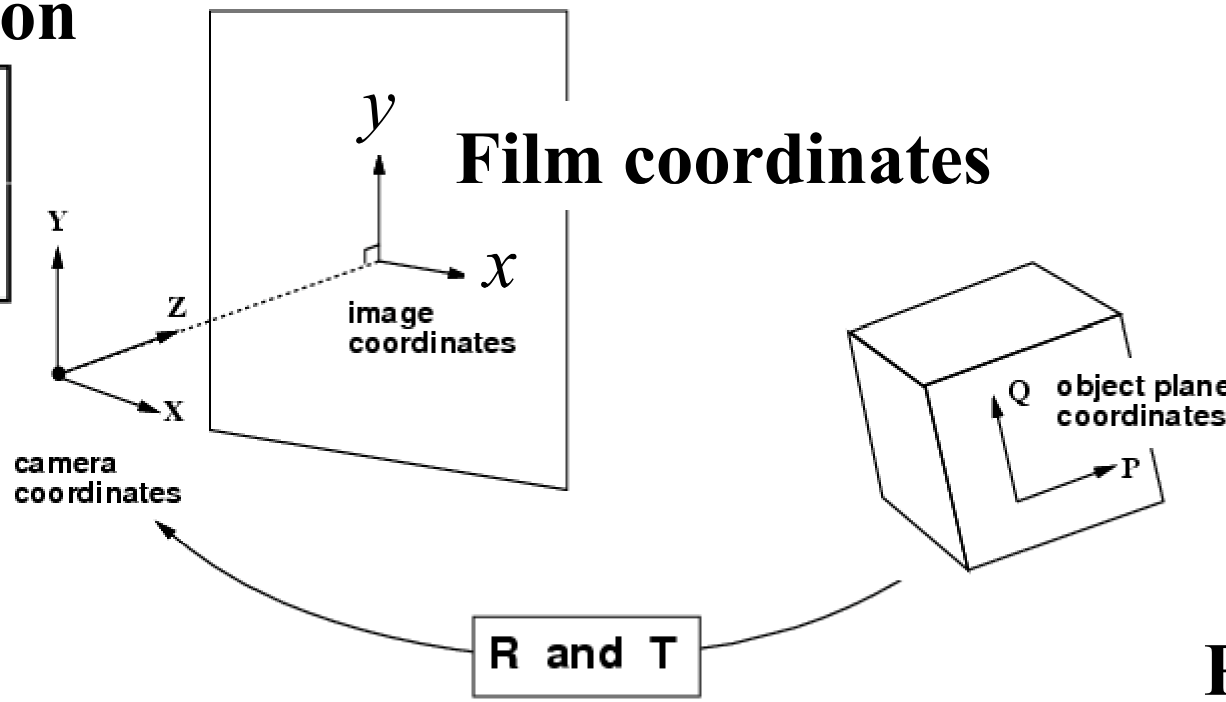
$$\begin{pmatrix} u' \\ v' \\ w' \end{pmatrix} = \underbrace{\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & 1 \end{pmatrix}}_{\mathbf{M_{aff}}} \underbrace{\begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{\mathbf{M_{proj}}} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

$$\mathbf{u} = \mathbf{M_{int}}\, \mathbf{P_C} = \mathbf{M_{aff}}\, \mathbf{M_{proj}}\, \mathbf{P_C}$$

Projection of Points on Planar Surface

Perspective projection

$$\begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$



$$\begin{bmatrix} p \\ q \\ 0 \\ 1 \end{bmatrix}$$

Point on plane

Rotation + Translation

$$\begin{bmatrix} r_{11} & r_{12} & r_{13} & t_x \\ r_{21} & r_{22} & r_{23} & t_y \\ r_{31} & r_{32} & r_{33} & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Projection of Planar Points

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \sim \begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} & t_x \\ r_{21} & r_{22} & r_{23} & t_y \\ r_{31} & r_{32} & r_{33} & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p \\ q \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \sim \begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & t_x \\ r_{21} & r_{22} & t_y \\ r_{31} & r_{32} & t_z \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p \\ q \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \sim \begin{bmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & t_x \\ r_{21} & r_{22} & t_y \\ r_{31} & r_{32} & t_z \end{bmatrix} \begin{bmatrix} p \\ q \\ 1 \end{bmatrix}$$

Projection of Planar Points (cont)

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \sim \begin{bmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & t_x \\ r_{21} & r_{22} & t_y \\ r_{31} & r_{32} & t_z \end{bmatrix} \begin{bmatrix} p \\ q \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \sim \begin{bmatrix} fr_{11} & fr_{12} & ft_x \\ fr_{21} & fr_{22} & ft_y \\ r_{31} & r_{32} & t_z \end{bmatrix} \begin{bmatrix} p \\ q \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \sim \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{bmatrix} p \\ q \\ 1 \end{bmatrix}$$

Homography H (planar projective transformation)

Projection of Planar Points (cont)

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \sim \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{bmatrix} p \\ q \\ 1 \end{bmatrix}$$

Homography H (planar projective transformation)

Punchline: For planar surfaces, 3D to 2D perspective projection reduces to a 2D to 2D transformation.

Punchline2: This transformation is INVERTIBLE!

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Special Case : Frontal Plane

What if the planar surface is perpendicular to the optic axis (Z axis of camera coord system)?

Then world rotation matrix simplies:

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \sim \begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} & t_x \\ r_{21} & r_{22} & r_{23} & t_y \\ r_{31} & r_{32} & r_{33} & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p \\ q \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Frontal Plane

So the homography for a frontal plane simplies:

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \sim \begin{bmatrix} fr_{11} & fr_{12} & ft_x \\ fr_{21} & fr_{22} & ft_y \\ r_{31} & r_{32} & t_z \end{bmatrix} \begin{bmatrix} p \\ q \\ 1 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \sim \begin{bmatrix} f\cos\theta & -f\sin\theta & ft_x \\ f\sin\theta & f\cos\theta & ft_y \\ 0 & 0 & t_z \end{bmatrix} \begin{bmatrix} p \\ q \\ 1 \end{bmatrix}$$

Similarity Transformation!

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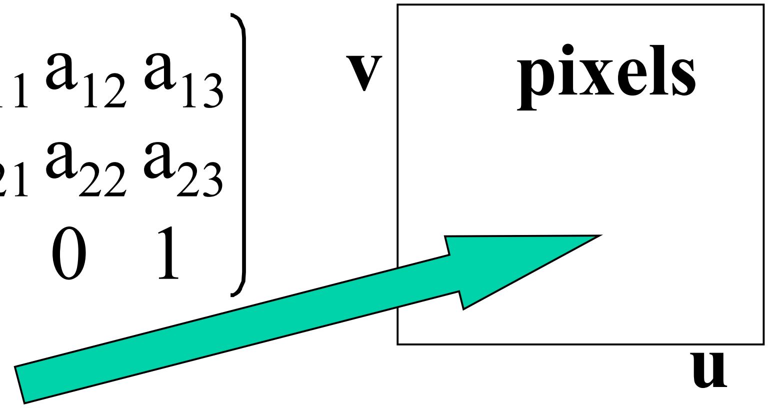
Convert to Pixel Coords

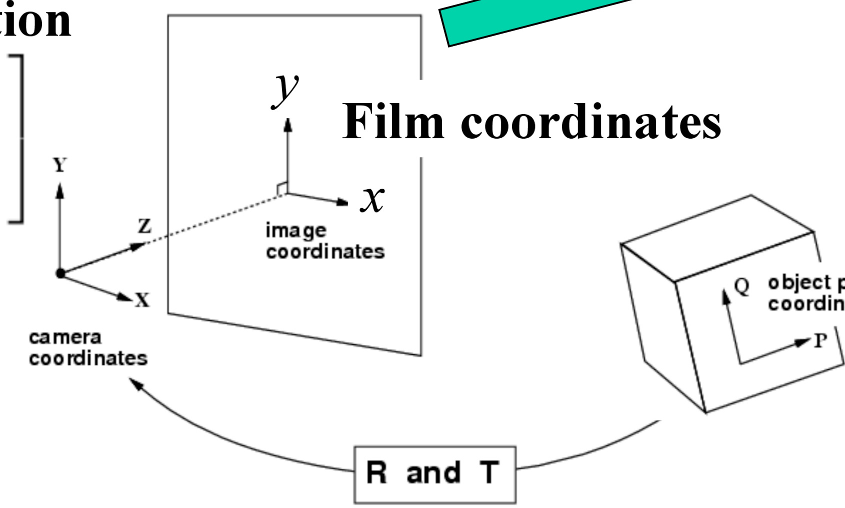
Internal camera parameters

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & 1 \end{pmatrix}$$



Perspective projection

$$\begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$



$$\begin{bmatrix} p \\ q \\ 0 \\ 1 \end{bmatrix}$$

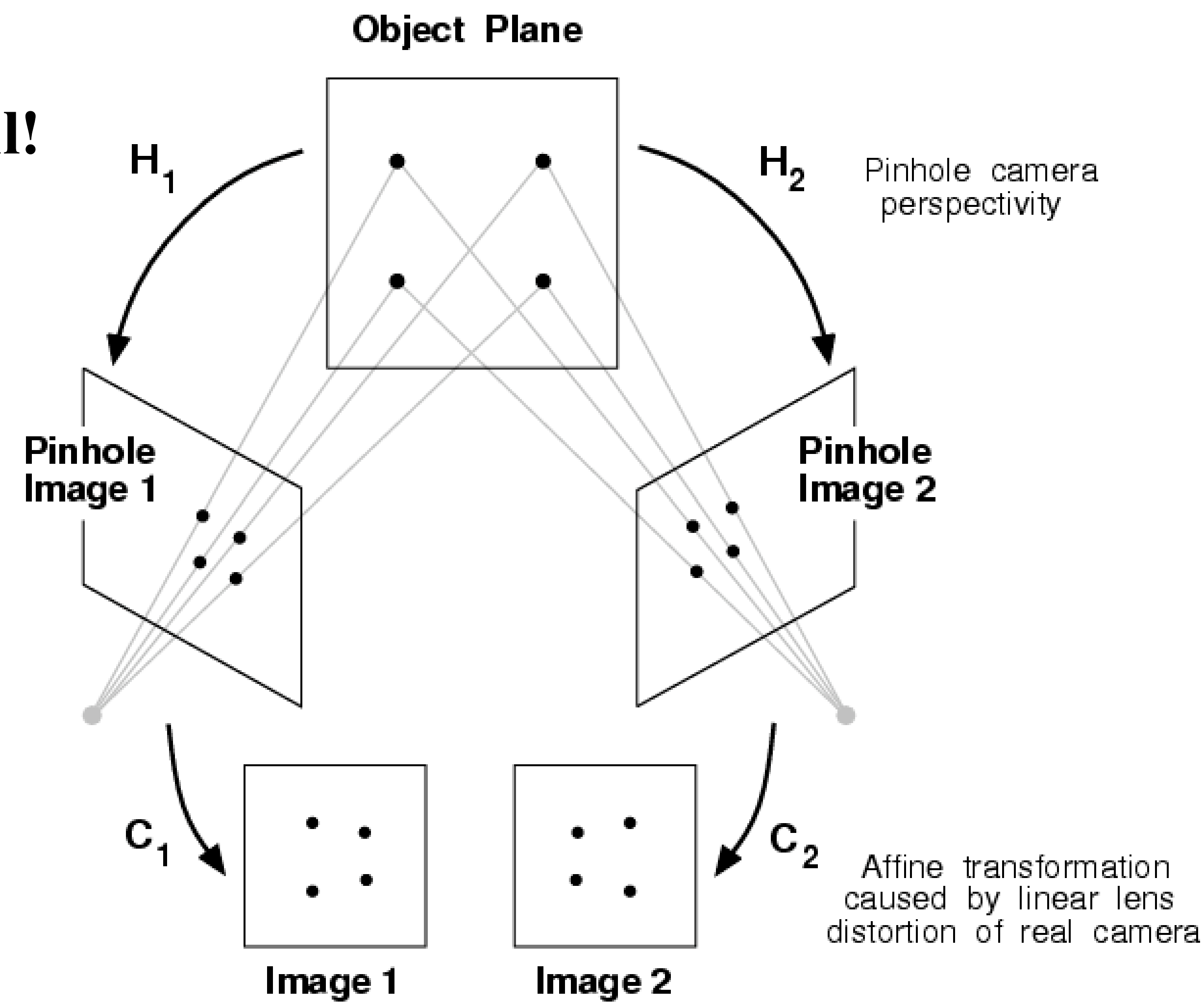
Point on plane

Rotation + Translation

$$\begin{bmatrix} r_{11} & r_{12} & r_{13} & t_x \\ r_{21} & r_{22} & r_{23} & t_y \\ r_{31} & r_{32} & r_{33} & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Planar Projection Diagram

Here's where transformation groups get useful!



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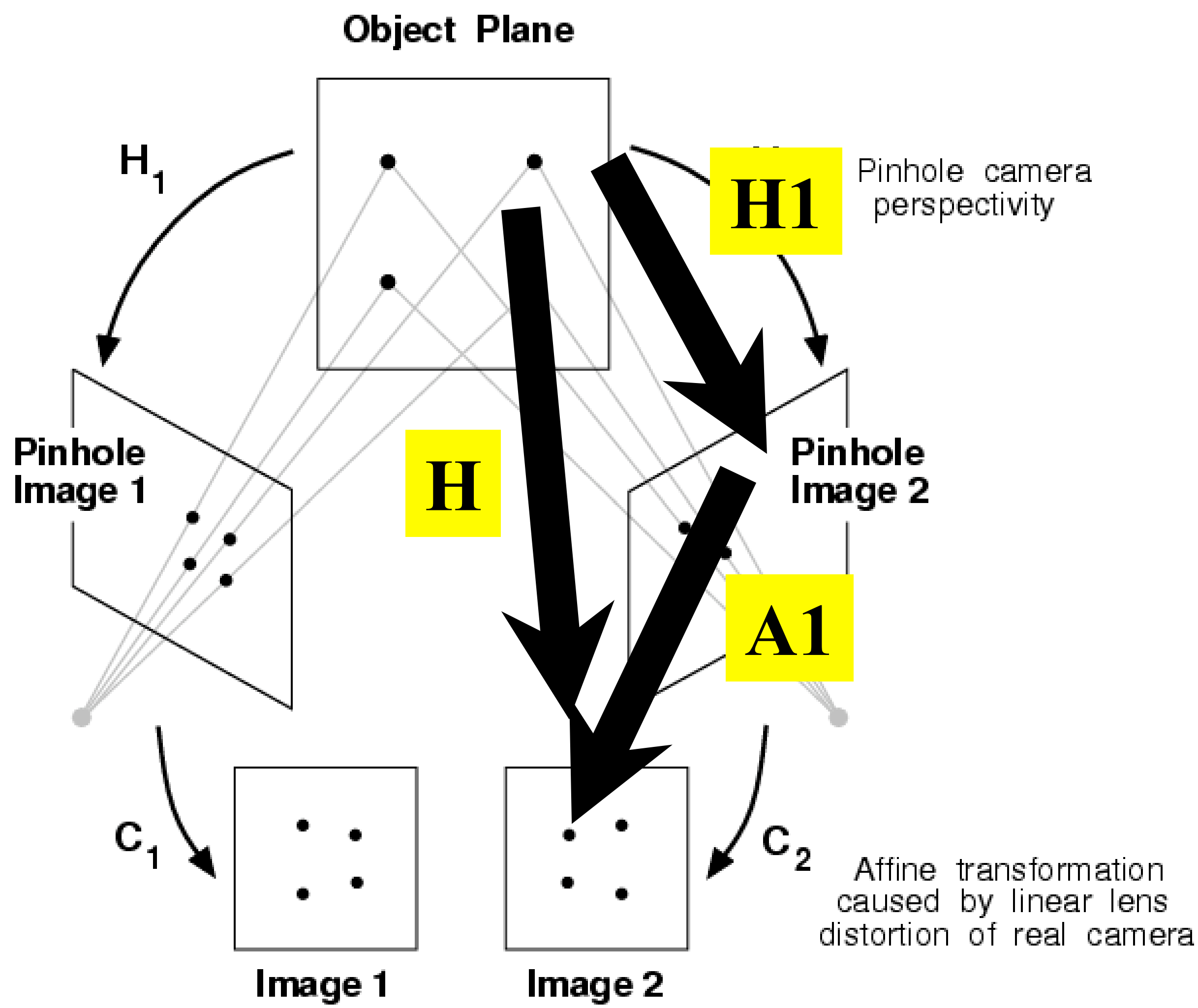
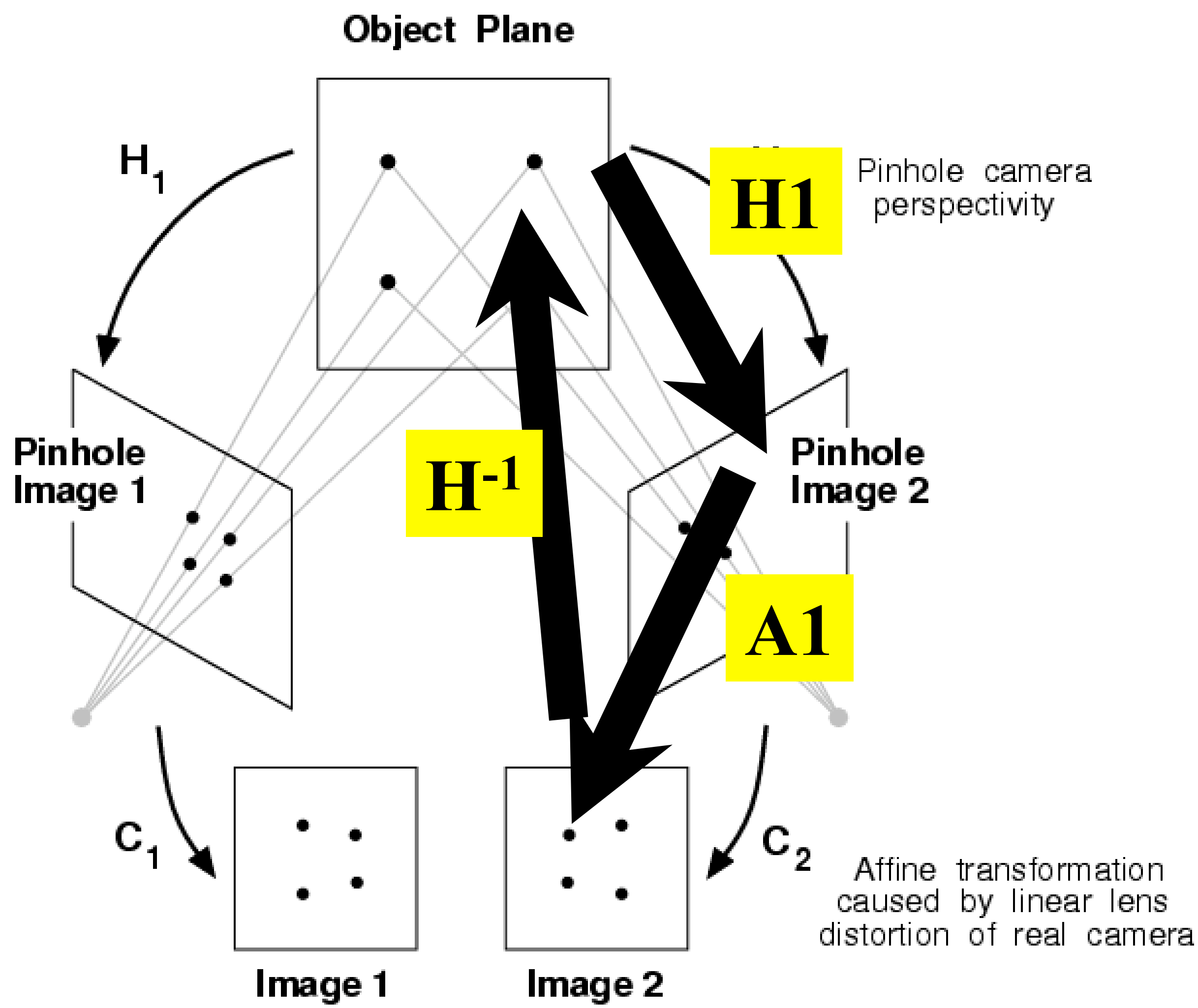
General Planar Projection



General Planar Projection



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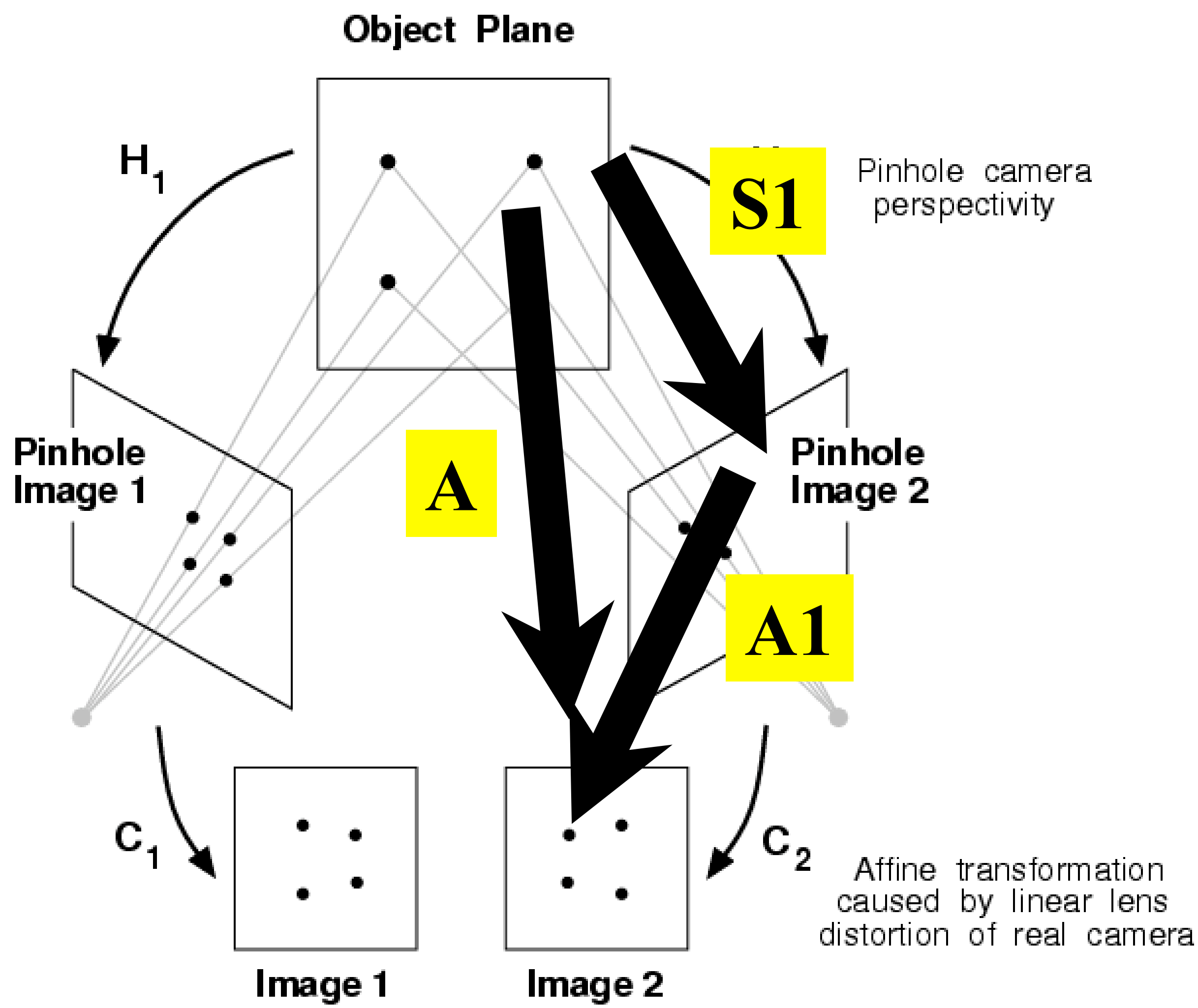
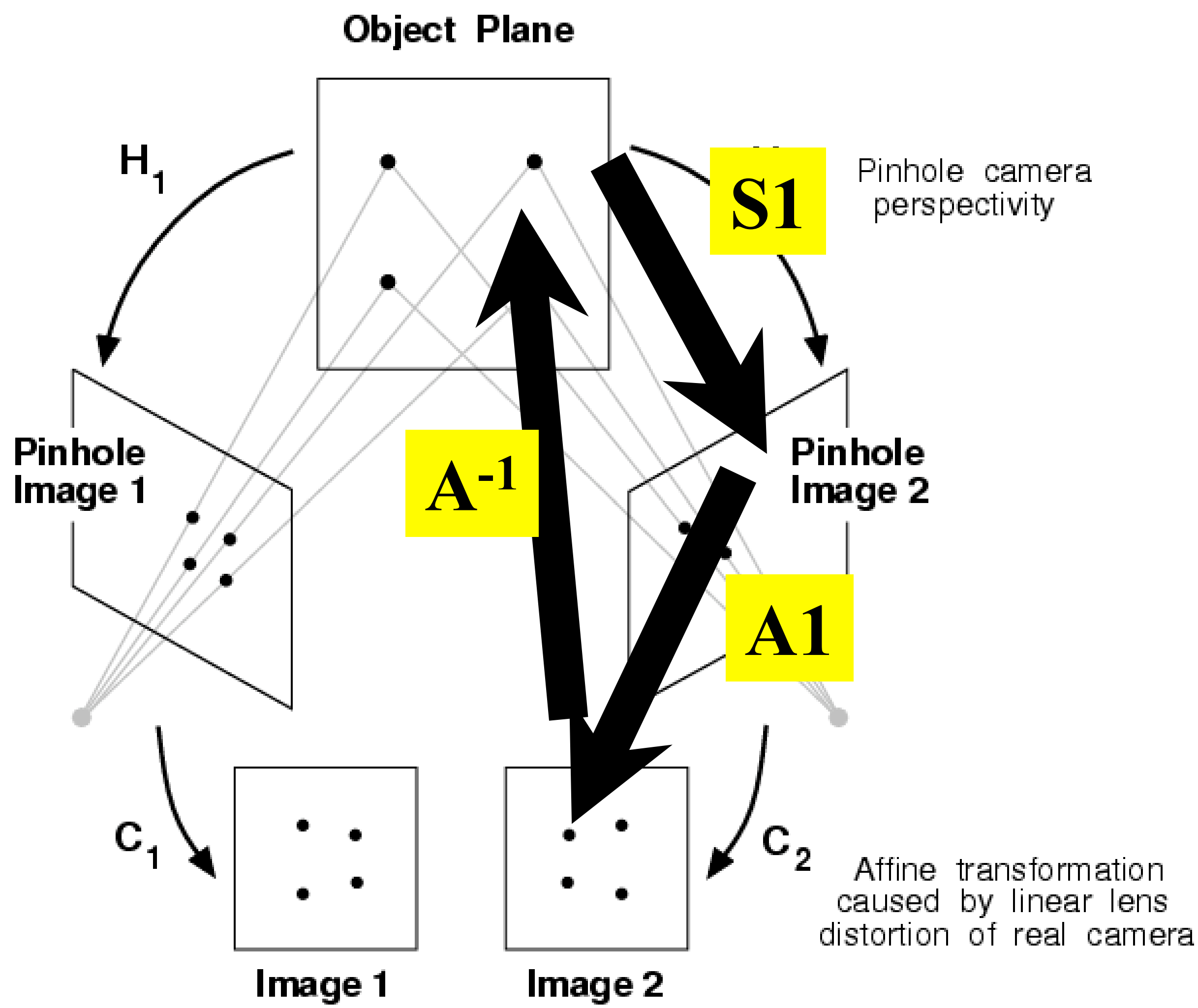
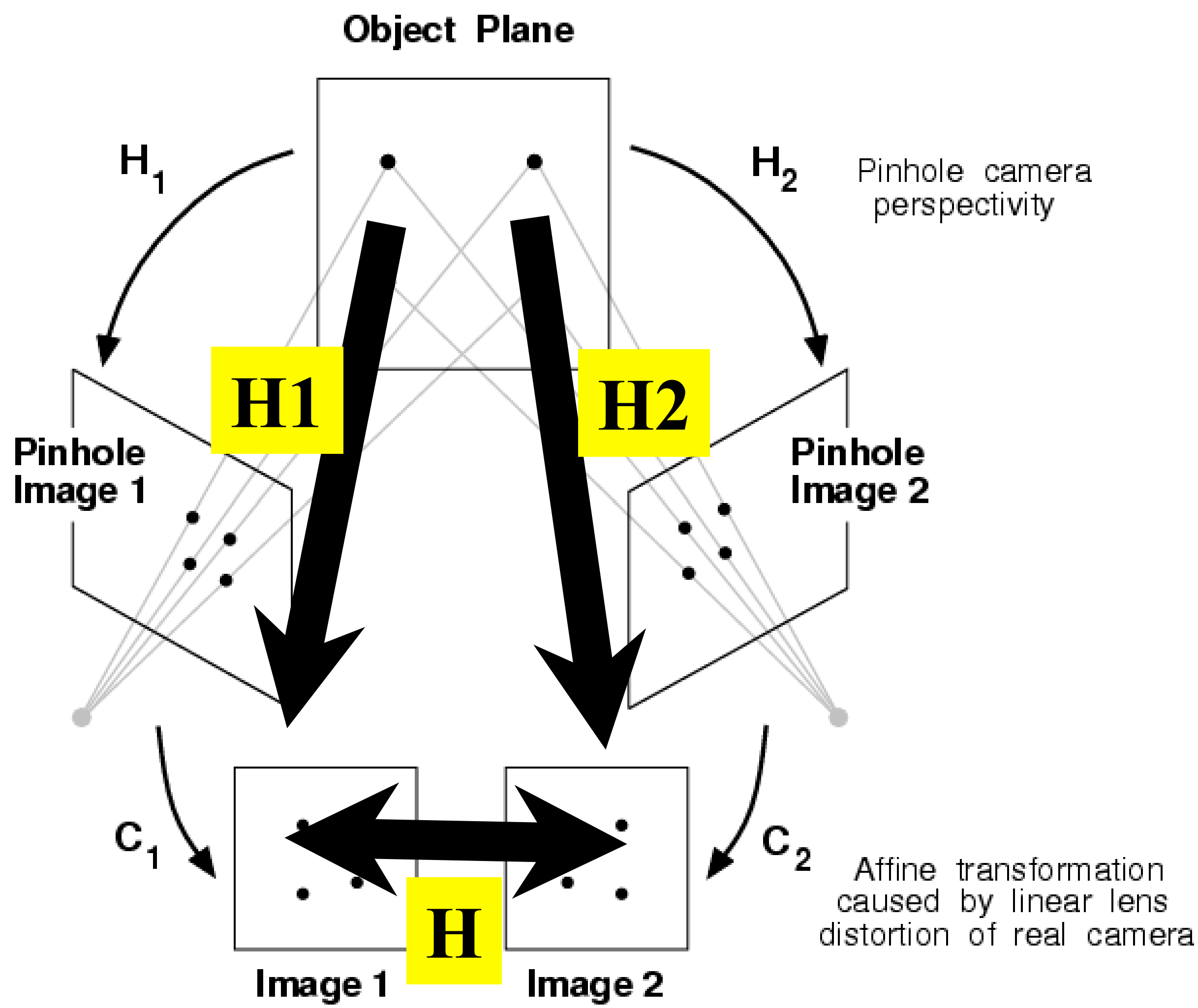
Frontal Plane Projection



Frontal Plane Projection



General Planar Projection



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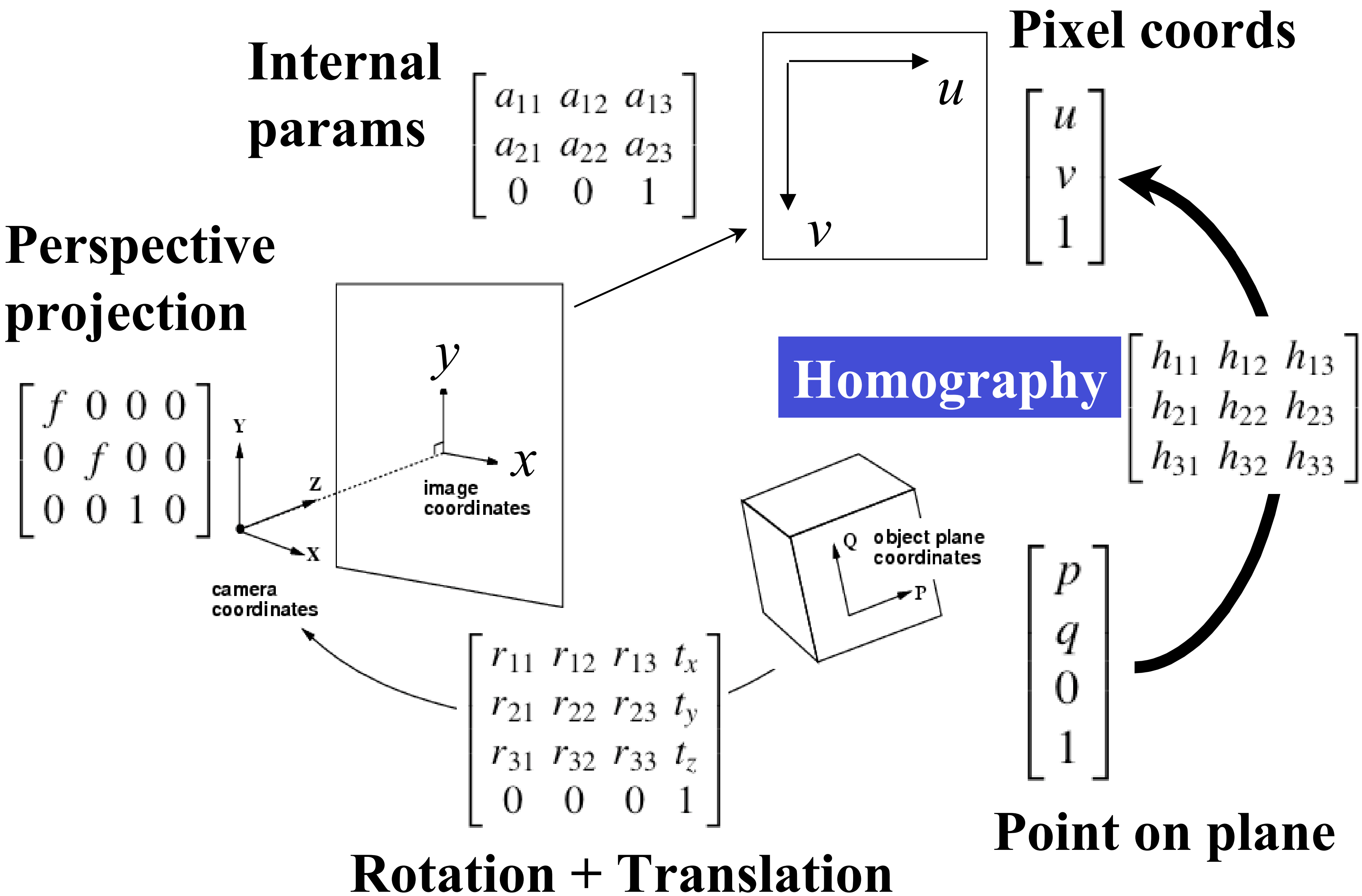
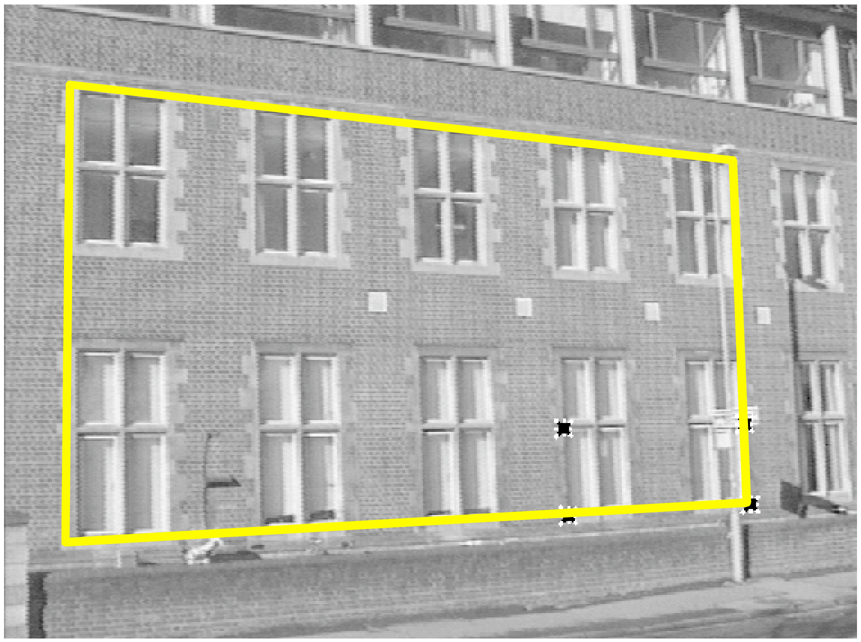
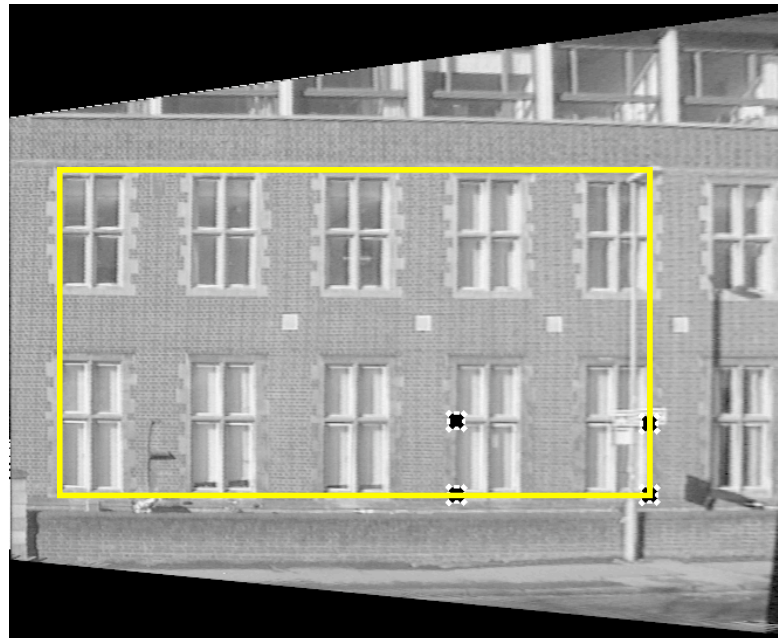
Summary: Planar Projection



Applying Homographies to Remove Perspective Distortion

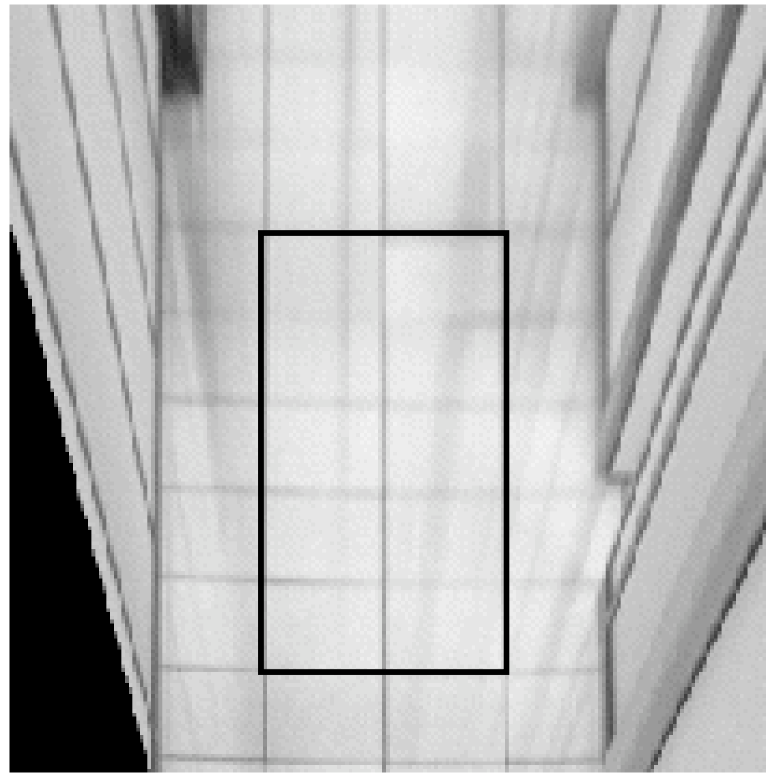




from Hartley & Zisserman

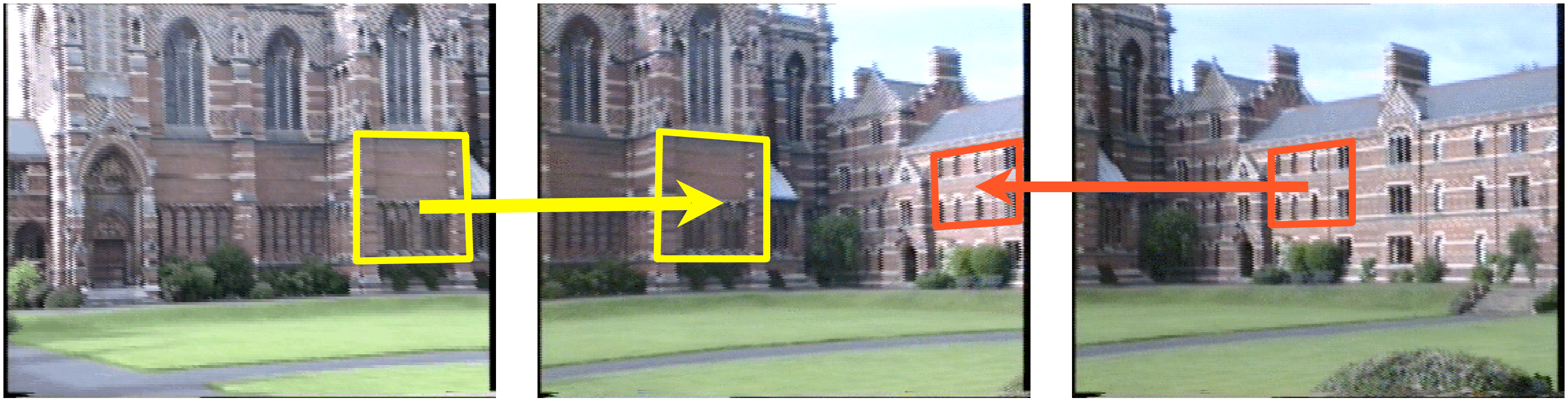
4 point correspondences suffice for the planar building facade

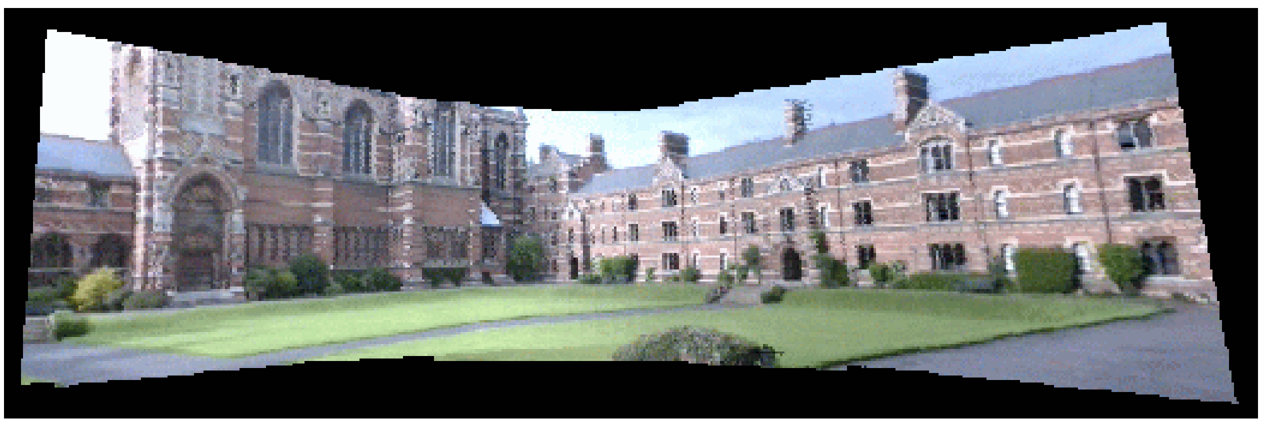
Homographies for Bird's-eye Views



from Hartley & Zisserman

Homographies for Mosaicing





from Hartley & Zisserman

Two Practical Issues

How to estimate the homography given four or more point correspondences (will derive L.S. solution now)

How to (un)warp image pixel values to produce a new picture (last class)

Estimating a Homography

Matrix Form:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} \sim \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Equations:

$$x' = \frac{h_{11}x + h_{12}y + h_{13}}{h_{31}x + h_{32}y + h_{33}}$$

$$y' = \frac{h_{21}x + h_{22}y + h_{23}}{h_{31}x + h_{32}y + h_{33}}$$

Degrees of Freedom?

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} \sim \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

There are 9 numbers $h_{11},\ldots,h_{33}$, so are there 9 DOF?

No. Note that we can multiply all h_{ij} by nonzero k without changing the equations:

$$x' = \frac{kh_{11}x + kh_{12}y + kh_{13}}{kh_{31}x + kh_{32}y + kh_{33}} \qquad \Rightarrow \qquad x' = \frac{h_{11}x + h_{12}y + h_{13}}{h_{31}x + h_{32}y + h_{33}}$$

$$y' = \frac{kh_{21}x + kh_{22}y + kh_{23}}{kh_{31}x + kh_{32}y + kh_{33}} \qquad \qquad y' = \frac{h_{21}x + h_{22}y + h_{23}}{h_{31}x + h_{32}y + h_{33}}$$

Enforcing 8 DOF

One approach: Set $h_{33} = 1$.

$$x' = \frac{h_{11}x + h_{12}y + h_{13}}{h_{31}x + h_{32}y + 1}$$

$$y' = \frac{h_{21}x + h_{22}y + h_{23}}{h_{31}x + h_{32}y + 1}$$

Second approach: Impose unit vector constraint

$$x' = \frac{h_{11}x + h_{12}y + h_{13}}{h_{31}x + h_{32}y + h_{33}}$$

$$y' = \frac{h_{21}x + h_{22}y + h_{23}}{h_{31}x + h_{32}y + h_{33}}$$

Subject to the constraint: $h_{11}^2 + h_{12}^2 + h_{13}^2 + h_{21}^2 + h_{22}^2 + h_{23}^2 + h_{31}^2 + h_{32}^2 + h_{33}^2 = 1$

L.S. using Algebraic Distance

Setting $h_{33} = 1$

$$x' = \frac{h_{11}x + h_{12}y + h_{13}}{h_{31}x + h_{32}y + 1}$$

$$y' = \frac{h_{21}x + h_{22}y + h_{23}}{h_{31}x + h_{32}y + 1}$$

Multiplying through by denominator

$$(h_{31}x + h_{32}y + 1)x' = h_{11}x + h_{12}y + h_{13}$$

$$(h_{31}x + h_{32}y + 1)y' = h_{21}x + h_{22}y + h_{23}$$

Rearrange

$$h_{11}x + h_{12}y + h_{13} - h_{31}xx' - h_{32}yx' = x'$$

$$h_{21}x + h_{22}y + h_{23} - h_{31}xy' - h_{32}yy' = y'$$

Algebraic Distance, h_{33}=1 (cont)

$$
\begin{array}{rccc}
 & \mathbf{2N \times 8} & \mathbf{8 \times 1} & & \mathbf{2N \times 1} \\
\begin{array}{r}\textbf{Point 1}\\[3em]\textbf{Point 2}\\[3em]\textbf{Point 3}\\[3em]\textbf{Point 4}\end{array} &
\begin{bmatrix}
x_1 & y_1 & 1 & 0 & 0 & 0 & -x_1x'_1 & -y_1x'_1 \\
0 & 0 & 0 & x_1 & y_1 & 1 & -x_1y'_1 & -y_1y'_1 \\
x_2 & y_2 & 1 & 0 & 0 & 0 & -x_2x'_2 & -y_2x'_2 \\
0 & 0 & 0 & x_2 & y_2 & 1 & -x_2y'_2 & -y_2y'_2 \\
x_3 & y_3 & 1 & 0 & 0 & 0 & -x_3x'_3 & -y_3x'_3 \\
0 & 0 & 0 & x_3 & y_3 & 1 & -x_3y'_3 & -y_3y'_3 \\
x_4 & y_4 & 1 & 0 & 0 & 0 & -x_4x'_4 & -y_4x'_4 \\
0 & 0 & 0 & x_4 & y_4 & 1 & -x_4y'_4 & -y_4y'_4
\end{bmatrix} &
\begin{bmatrix} h_{11} \\ h_{12} \\ h_{13} \\ h_{21} \\ h_{22} \\ h_{23} \\ h_{31} \\ h_{32} \end{bmatrix} & = &
\begin{bmatrix} x'_1 \\ y'_1 \\ x'_2 \\ y'_2 \\ x'_3 \\ y'_3 \\ x'_4 \\ y'_4 \end{bmatrix} \\
\textbf{additional points} & \vdots & & & \vdots
\end{array}
$$

Algebraic Distance, h_{33}=1 (cont)

Linear equations

$$\overset{2N\text{x}8}{A}\;\overset{8\text{x}1}{h} = \overset{2N\text{x}1}{b}$$

Solve:

$$\overset{8\text{x}2N}{A^T}\;\overset{2N\text{x}8}{A}\;\overset{8\text{x}1}{h} = \overset{8\text{x}2N}{A^T}\;\overset{2N\text{x}1}{b}$$

$$\overbrace{(A^T\;A)}^{8\text{x}8}\;\overset{8\text{x}1}{h} = \overbrace{(A^T\;b)}^{8\text{x}1}$$

$$h = (A^T\;A)^{-1}\,(A^T\;b)$$

Matlab: $h = A \setminus b$

Caution: Numeric Conditioning

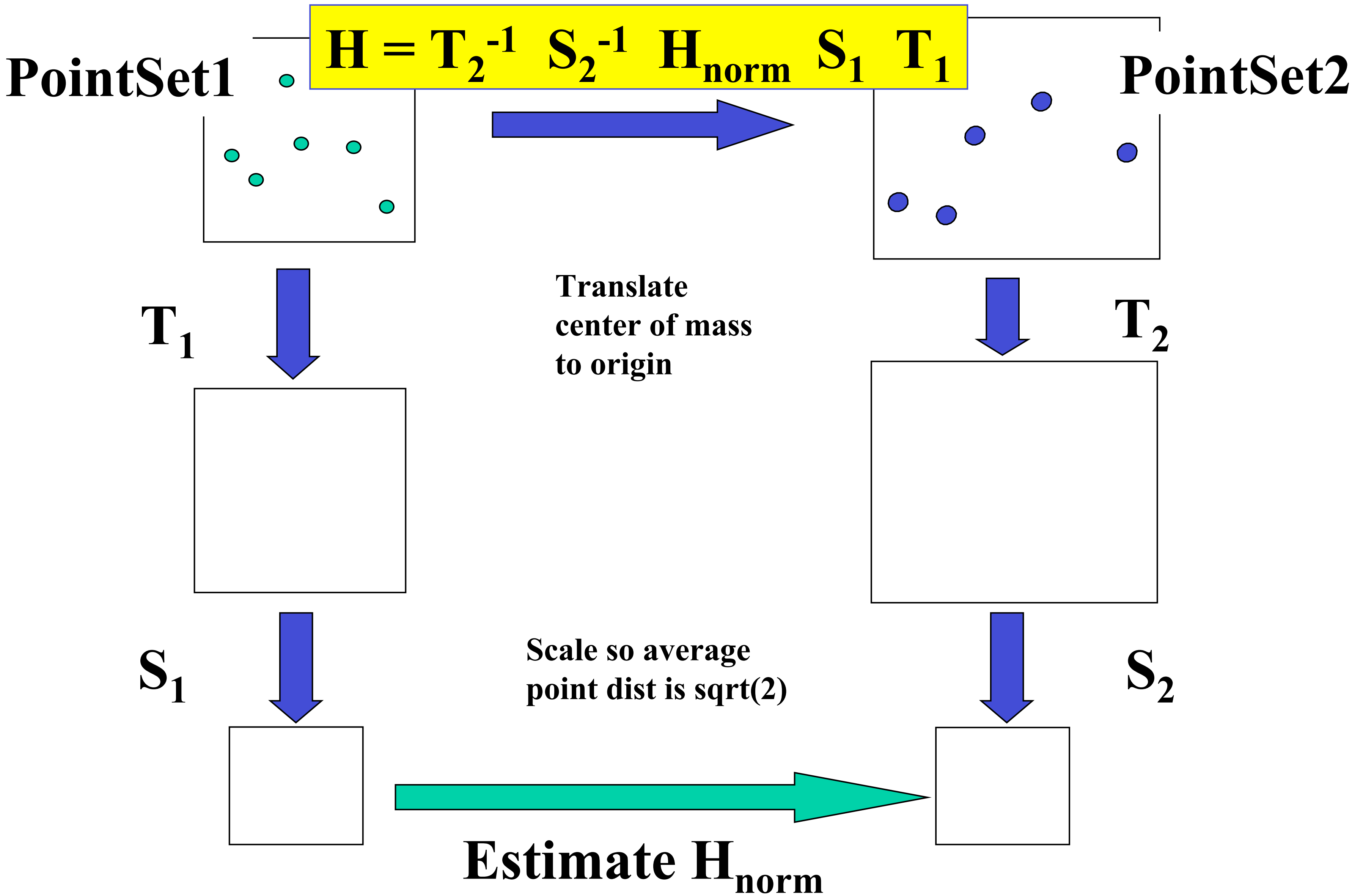
R.Hartley: "In Defense of the Eight Point Algorithm"

Observation: Linear estimation of projective transformation parameters from point correspondences often suffer from poor "conditioning" of the matrices involves. This means the solution is sensitive to noise in the points (<u>even if there are no outliers</u>).

To get better answers, precondition the matrices by performing a normalization of each point set by:

- translating center of mass to the origin
- scaling so that average distance of points from origin is sqrt(2).
- do this normalization to each point set independently

Hartley's PreConditioning



A More General Approach

What might be wrong with setting $h_{33} = 1$?

If h_{33} actually = 0, we can't get the right answer.

Algebraic Distance, ||h||=1

||h|| = 1

$$x' = \frac{h_{11}x + h_{12}y + h_{13}}{h_{31}x + h_{32}y + h_{33}}$$

$$y' = \frac{h_{21}x + h_{22}y + h_{23}}{h_{31}x + h_{32}y + h_{33}}$$

Multiplying through by denominator

$$(h_{31}x + h_{32}y + h_{33})x' = h_{11}x + h_{12}y + h_{13}$$

$$(h_{31}x + h_{32}y + h_{33})y' = h_{21}x + h_{22}y + h_{23}$$

Rearrange

$$h_{11}x + h_{12}y + h_{13} - h_{31}xx' - h_{32}yx' - h_{33}x' = 0$$

$$h_{21}x + h_{22}y + h_{23} - h_{31}xy' - h_{32}yy' - h_{33}y' = 0$$

Algebraic Distance, ||h||=1 (cont)

4 POINTS

$$\begin{array}{cccc} 2N \times 9 & 9 \times 1 & & 2N \times 1 \\ \begin{bmatrix} x_1 & y_1 & 1 & 0 & 0 & 0 & -x_1x'_1 & -y_1x'_1 & -x'_1 \\ 0 & 0 & 0 & x_1 & y_1 & 1 & -x_1y'_1 & -y_1y'_1 & -y'_1 \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \end{bmatrix} & \begin{bmatrix} h_{11} \\ h_{12} \\ h_{13} \\ h_{21} \\ h_{22} \\ h_{23} \\ h_{31} \\ h_{32} \\ h_{33} \end{bmatrix} & = & \begin{bmatrix} 0 \\ 0 \\ \\ \\ \\ \\ \end{bmatrix} \\ \vdots & & & \vdots \end{array}$$

additional points

Algebraic Distance, ||h||=1 (cont)

Homogeneous equations $\underset{2N\times 9}{A}\ \underset{9\times 1}{h} = \underset{2N\times 1}{0}$

Solve: $\underset{9\times 2N}{A^T}\ \underset{2N\times 9}{A}\ \underset{9\times 1}{h} = \underset{9\times 2N}{A^T}\ \underset{2N\times 1}{0}$

$$\overbrace{(A^T\ A)}^{9\times 9}\ \underset{9\times 1}{h} = \underset{9\times 1}{0}$$

SVD of $A^T A = U\ D\ U^T$

Let h be the column of U (unit eigenvector) associated with the smallest eigenvalue in D. (if only 4 points, that eigenvalue will be 0)