Chapter 1

Conclusion and Further Research

1.1 Quasi-linear Differential Problems

Throughout this theis, it has been possible to either use techniques from Applied Mathematics to construct solution strategies on a case by case basis, or use Statistical methods to find a semi-parametric fit. As differential equations become more complex, both approaches begin to rapidly become non-viable. In this section, quasi-linear differential equation models will be touched upon.

The difference between a quasi-linear and a linear differential equation is that the coefficients in a quasi-linear equation are allowed to depend on the unknown function. Instead of an ODE such as $y' = \beta(t)y$, one would have an ODE such as $y' = \beta(y,t)y$. Though quasi-linear problems tend to be reminiscent of linear ones, they are nonetheless substanially more complicated, and require more technical knowledge, and even ingenuity to tackle.

For a quasi-linear variation of a linear ODE, consider the Van Der Pol Equation:

$$y''(t) + \beta(1 - y(t))^2 y'(t) + y(t)$$

This ODE has no obvious solution.

1.1.1 Inviscid Burger's Equation

Even if a solution exists, an estimation strategy might be difficult to derive. Consider the inviscid Burger's Equation:

$$\frac{\partial u(x,t)}{\partial t} + \beta u(x,t)\frac{\partial u(x,t)}{\partial x} = 0$$

This equation is identical to the Transport Equation except that the rate term is equal to $\beta u(x,t)$. The solution is given by:

$$u(x,t) = f(s)$$

Here $f(\cdot)$ is some arbitary function as before, and s is implicitly defined as the solution of the equation $x = \beta f(s)t + s$. Since $s = x - \beta ut$, this can be written as:

$$u(x,t) = f(x - \beta ut)$$

Fitting this model is substantially trickier than the Transport Equation. There is no clean seperation between the problem of estimating $f(\cdot)$ and β since $u(x,t)$ appears on the righthand side and scales β.

A further complication is that $u(x,t)$ might only define a *relation*, instead of a function. There might be multiple values of u associated with a given (x,t) that satisfy the solution equation. Physically speaking, multiple values correspond to shock waves.

1.1.2 Discussion

We see that the level of knowledge required to devise fitting strategies can increase substantially even with seemingly modest increases in the complexity of the associated differential equation.

Consider the following quasi-linear model of genetic drift in a population proposed[1] by R.A. Fisher:[8]

$$\frac{\partial u(x,t)}{\partial t} + \beta_1 \frac{\partial u(x,t)}{\partial x} = \beta_2 u(x,t)(1 - u(x,t)) \tag{1.1}$$

This problem is similar to the previous PDEs we discussed, it even admits travelling wave solutions of the form $f(x + ct)$ as Fisher himself noted. Nonetheless, it is a much more difficult problem despite the apparently modest increase in complexity. One would likely have to consult a textbook that covers non-linear PDEs that can generate waves in fair degree of detail to be able to devise a fitting strategy. This is quite a specialised subject!

The overall result is that as the complexity of the differential equation increases, more and more time will be needed to model it correctly.

[1] In the *Annals of Eugenics...*

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