Traffic State Estimation Method with Efficient Data Fusion Based on the Aw–Rascle–Zhang Model

Toru Seo and Alexandre M. Bayen

***Abstract*—Higher-order traffic flow models describe the dynamics of non-equilibrium traffic (e.g., capacity drop, traffic oscillation). Therefore, a traffic state estimation (TSE) method based on higher-order models could have notable advantages over conventional method based on an equilibrium (first-order) model. First, they can encompass non-equilibrium traffic phenomena inherent to the nonlinearity of traffic flow. Second, they can directly fuse heterogeneous traffic data, such as flow measured by detectors and speed measured by probe vehicles. These features would be useful for large volumes of data. This article proposes a data-assimilation-based TSE method for capturing non-equilibrium traffic dynamics based on the Aw–Rascle–Zhang model, which is known as a physically-consistent higher-order model but not received as much attention as other models. The proposed method can efficiently and simply fuse heterogeneous data. Additionally, it is computationally efficient because it uses the extended Kalman filter with an analytical derivative. The features of the proposed method were investigated empirically and quantitatively using experimental dataset collected from actual traffic. We investigate the empirical relationship between the accuracy and amount of available data, and compare the proposed method with a conventional TSE method.**

I. Introduction

Traffic state estimation (TSE) refers to inference of the state of unobserved (or observed with noises) traffic using limited data; for a comprehensive survey on TSE, see [1]. In a TSE method, a traffic flow model is often used as the core of the inference process. Among other models, *the Lighthill–Whitham–Richards (LWR) model* [2], [3] (also known as the first-order model) is often used for TSE. The LWR model assumes that traffic is at *equilibrium*, that is, the traffic state (i.e., speed and density) follows the *fundamental diagram (FD)* (i.e., speed–density relation). Thanks to this assumption, the LWR model can be easily solved and utilized for TSE [4]–[9].

Unfortunately, the equilibrium assumption in the LWR model naturally leads to a problem: non-equilibrium traffic is ignored in conventional LWR-based TSE methods. Because non-equilibrium traffic may cause disturbances and degrade efficiency (e.g., capacity drop, traffic oscillation), it is valuable to take it into consideration explicitly in the model [10], to improve the estimation process and result. To account for this, higher-order traffic flow models, which explicitly describe the dynamics of non-equilibrium traffic, have been employed, such as Payne–Whitham model [11], [12], pseudo-higher-order-models [13], [14], phase transition model [15]. Among other higher-order models, *the Aw–Rascle–Zhang (ARZ) model* [16], [17] is one of the canonical and physically reasonable model [16]–[18]. However, the ARZ model has not extensively been considered in the TSE context. In particular, to our knowledge, incorporation of the data assimilation approaches such as [4] with the ARZ model has not been investigated well. Some studies used the ARZ model for estimation purposes as an initial/boundary value problem [19], [20], which is in fact simulation rather than TSE.

Higher-order traffic flow models would also be beneficial for TSE in for other reasons. First, the models can benefit from high *data density* and *data heterogeneity* in these days. They make it possible to properly consider spatiotemporal correlations among observations. Non-equilibrium traffic has strong spatiotemporal correlations (i.e., dynamics) [21], so measurements from densely distributed sensors (e.g., densely installed detectors or probe vehicles with a high penetration rate) must be affected by these correlations. Higher-order models can explicitly capture such dynamics. On the other hand, it would be difficult for LWR-based TSE methods to capture these correlations by modifying their error terms. Second, the models make it possible to properly fuse heterogeneous data. Recently, global positioning system (GPS)-equipped probe vehicles have become widely available as mobile sensors. They measure speed instead of flow, which is measured using conventional detectors. Higher-order models can directly fuse such heterogeneous measurements from probe vehicles and detectors. On the other hand, fusing such heterogeneous measurements could be problematic for LWR-based TSE methods, because the actual traffic state and sensor measurements are typically in non-equilibrium states. Moreover, information regarding non-equilibrium traffic is discarded by LWR-besed methods.

The aim of this article is thus to provide a TSE method based on a higher-order model, highlighting the model's advantage of exploiting rich and heterogeneous data in modern contexts. First, we propose a data-assimilation-based TSE method for capturing the dynamics of non-equilibrium traffic based on a physical traffic flow model, namely, the ARZ model. The proposed method is build to fuse heterogeneous data in an efficient and straightforward manner. Second, we empirically investigate the performance of these features using a real-world dataset. Specifically, we show the empirical and quantitative relationships between estimation accuracy and the amount of available data. Moreover, we compare the proposed method with an LWR-based TSE method to confirm the proposed method's advantages.

The rest of this article is organized as follows. Section II describes the formulation of the proposed TSE method. Section III describes the empirical validation of the method. Section IV concludes this article.

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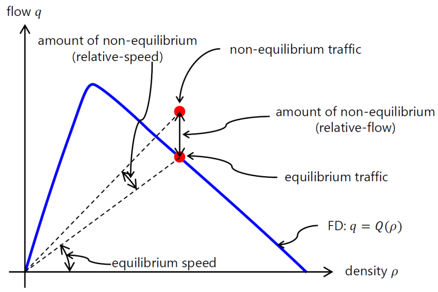
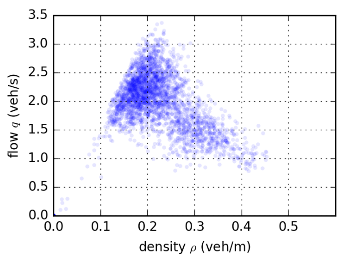


(a) Conceptual FD and traffic state.



(b) Traffic state in NGSIM dataset. Each dot represents a traffic state in a 2sec×50m-sized domain in the dataset.

Fig. 1: Equilibrium and non-equilibrium traffic in flow–density plane.

II. Estimation method

A. The ARZ model

1) Continuum partial differential equation model: The ARZ model is commonly written as follows:

$$\partial_t \rho + \partial_x(\rho v) = 0, \tag{1a}$$

$$\partial_t(\rho(v - V(\rho))) + \partial_x(\rho(v - V(\rho))v) = -\frac{\rho(v - V(\rho))}{\tau}, \tag{1b}$$

where ρ is the density, v is the speed, $V(\cdot)$ is the density–speed FD, and τ is the relaxation time [16], [17]. Eq. (1a) represents the conservation law of traffic. Eq. (1b) is the momentum equation, which describes the spatiotemporal dynamics of non-equilibrium traffic. The amount of non-equilibrium can be measured as *relative-flow*, defined as $\rho(v - V(\rho))$, which is the difference between the actual flow and equilibrium flow at a set density (Fig. 1a). Note that the LWR model can be obtained by assuming $v = V(\rho)$, or equivalently, by taking the limit $\tau \to 0$.

In the ARZ model, the momentum equation allows traffic states to shift from an equilibrium point. If non-equilibrium traffic state is given as a boundary condition, it decays as traffic advances. The decaying speed is characterized by τ. Therefore, the ARZ model describes the decaying propagation of non-equilibrium traffic by considering driver behavior. At the same time, the model maintains some of the advantages of the LWR model. That is, it has physically preferable features such as non-negative speed and anisotropic nature of traffic [16], [17].

2) Discretization: Eq. (1) can be written in the following conservative form [16]:

$$\partial_t U + \partial_x F(U) = R(U) \tag{2}$$

with

$$U = \begin{pmatrix} \rho \\ y \end{pmatrix}, F(U) = \begin{pmatrix} y + \rho V(\rho) \\ \frac{y^2}{\rho} + yV(\rho) \end{pmatrix}, R(U) = \begin{pmatrix} 0 \\ -\frac{y}{\tau} \end{pmatrix}. \tag{3}$$

where y is the relative-flow. Then, Eq. (2) is discretized to solve it numerically. Applying the Lax–Friedrich (LxF) scheme [22], one obtains

$$\begin{aligned} U_j^{n+1} =& \frac{1}{2}(U_{j+1}^n + U_{j-1}^n) - \frac{\Delta t}{2\Delta x}(F(U_{j+1}^n) - F(U_{j-1}^n)) \\ &+ \frac{\Delta t}{2}(R(U_{j+1}^n) + R(U_{j-1}^n)), \end{aligned} \tag{4}$$

where Δt is the time discretization step, Δx is the space discretization step, subscripts denote spatial indices, and superscripts denote temporal indices.

Note that the LxF scheme might be outperformed by other schemes in terms of numerical accuracy. However, it possesses a key feature (i.e., it is basically differentiable) used in the filtering process described in the next section.

B. EKF for TSE

The EKF is a classical extension of Kalman filtering for differentiable non-linear systems [23], and a well-known approach for data assimilation because of its computational efficiency. It is based on a state–space model with a nonlinear system model, a nonlinear observation model, and additive zero-mean Gaussian noise; that is,

$$\boldsymbol{x}^n = \boldsymbol{f}(\boldsymbol{x}^{n-1}) + \boldsymbol{\nu}^n, \tag{5a}$$

$$\boldsymbol{z}^n = \boldsymbol{h}^n(\boldsymbol{x}^n) + \boldsymbol{\omega}^n, \tag{5b}$$

where $\boldsymbol{x}^n$ is the state vector, $\boldsymbol{f}$ is the system model, $\boldsymbol{\nu}^n$ is system noise such that $\boldsymbol{\nu}^n \sim \mathcal{N}(0, Q^n)$ with variance-covariance matrix Q^n, $\boldsymbol{z}^n$ is the observation vector, $\boldsymbol{h}^n$ is the observation model, and $\boldsymbol{\omega}^n$ is observation noise such that $\boldsymbol{\omega}^n \sim \mathcal{N}(0, R^n)$ with variance-covariance matrix R^n. Eq. (5a) is the system equation, representing the dynamics of the state variables $\boldsymbol{x}$ of the system modeled as $\boldsymbol{f}$ with modeling error $\boldsymbol{\nu}$. In the TSE context, $\boldsymbol{x}$ can represent traffic states; and $\boldsymbol{f}$ can represent a traffic flow model. Eq. (5b) is the observation equation, representing observed variables $\boldsymbol{z}$ obtained from the system with variables $\boldsymbol{x}$, using the observations modeled as $\boldsymbol{h}$ with measurement error $\boldsymbol{\omega}$. In the TSE context, $\boldsymbol{z}$ can be data obtained by detectors and probe vehicles.

The EKF-based estimation method can be described as follows:

Initialization step: Give initial conditions $\boldsymbol{x}^{0|0}$ and $W^{0|0}$, where W is variance-covariance matrix of the state distribution (i.e., uncertainty of the state).

Estimation step: For n in $1, \dots, n^{\max}$, do:

Prediction step: Calculate a prior state and its uncertainty for the current time step, based on the posterior state and the uncertainty of the previous time step:

$$\boldsymbol{x}^{n|n-1} = \boldsymbol{f}(\boldsymbol{x}^{n-1|n-1}), \tag{6}$$

$$W^{n|n-1} = \hat{F}^n W^{n-1|n-1} \hat{F}^{n\top} + Q^n, \tag{7}$$

where $\hat{F}^n$ is Jacobian matrix of $\boldsymbol{f}$ with the posterior state at time step $n-1$, namely,

$$\hat{F}^n = \left.\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\right|_{\boldsymbol{x}^{n-1|n-1}}. \tag{8}$$

Filtering step: Calculate a posterior state and the uncertainty for the current time step, based on the prior state and observation of the current time step:

$$\boldsymbol{x}^{n|n} = \boldsymbol{x}^{n|n-1} + K^n(\boldsymbol{z}^n - \boldsymbol{h}^n(\boldsymbol{x}^{n|n-1})), \tag{9}$$
$$W^{n|n} = W^{n|n-1} - K^n \hat{H}^n W^{n|n-1}, \tag{10}$$

where K^n is the Kalman gain represented as $W^{n|n-1}\hat{H}^{n\top}(\hat{H}^n W^{n|n-1}\hat{H}^{n\top} + R^n)^{-1}$, and $\hat{H}^n$ is the Jacobian matrix of $\boldsymbol{h}$ with the prior state at time n, namely,

$$\hat{H}^n = \left.\frac{\partial \boldsymbol{h}^n}{\partial \boldsymbol{x}}\right|_{\boldsymbol{x}^{n|n-1}}. \tag{11}$$

In what follows, a TSE method based on the ARZ model is constructed by specifying the aforementioned terms.

1) System model: The ARZ model discretized by the LxF scheme is used as the system model. The state vector is defined as

$$\boldsymbol{x}^n = \left(\rho_0^n, y_0^n, \rho_1^n, y_1^n, \dots, \rho_j^n, y_j^n, \dots, \rho_{J-1}^n, y_{J-1}^n\right)^\top \tag{12}$$

which is a vector representing the spatial distribution of a traffic state at time n. The J denotes the spatial discretization count (i.e., number of cells). Then, the system model $\boldsymbol{f}$ can be constructed based on the ARZ model discretized by the LxF scheme. By combining (4) with (12), one obtains

$$f_{2j}(\boldsymbol{x}) = \frac{1}{2}(x_{2j-2} + x_{2j+2}) - \frac{\Delta t}{2\Delta x}(x_{2j+3} + x_{2j+2}V(x_{2j+2}) - x_{2j-1} - x_{2j-2}V(x_{2j-2})), \tag{13a}$$
$$f_{2j+1}(\boldsymbol{x}) = \frac{1}{2}(x_{2j-1} + x_{2j+3}) - \frac{\Delta t}{2\Delta x}\left(\frac{(x_{2j+3})^2}{x_{2j+2}} + x_{2j+3}V(x_{2j+2}) - \frac{(x_{2j-1})^2}{x_{2j-2}} - x_{2j-1}V(x_{2j-2})\right) - \frac{\Delta t}{2\tau}(x_{2j+3} + x_{2j-1}), \tag{13b}$$

where f_j and x_j indicate the j-th element of vectors $\boldsymbol{f}$ and $\boldsymbol{x}$, respectively. The first element of a vector is indexed as 0; thus x_{2j} corresponds to ρ_j. Assuming that the FD is differentiable, the Jacobian $\hat{F}$ of the system model can be obtained analytically.[1]

2) Observation model: Detectors and probe vehicles are considered to be available sensors. We assume that detectors observe flow and speed (and converted to $\tilde{\rho}$ and $\tilde{y}$ by the definition), and probe vehicles observe speed ($\tilde{v}$). Thus, observation vector can be defined as

$$\boldsymbol{z}^n = \left(\tilde{\rho}_0^n, \tilde{y}_0^n, \tilde{v}_0^n, \dots, \tilde{\rho}_j^n, \tilde{y}_j^n, \tilde{v}_j^n, \dots, \tilde{\rho}_{J-1}^n, \tilde{y}_{J-1}^n, \tilde{v}_{J-1}^n\right)^\top. \tag{14}$$

The observation model $\boldsymbol{h}^t$ is a mapping from $\boldsymbol{x}$ to $\boldsymbol{z}$ as shown in (5b). Therefore, by comparing (12) and (14), the

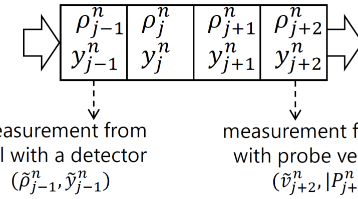


Fig. 2: Diagram of the state and observation variables.

observation model can be specified as

$$h_{3j}^n(\boldsymbol{x}) = x_{2j}, \tag{15a}$$
$$h_{3j+1}^n(\boldsymbol{x}) = x_{2j+1}, \tag{15b}$$
$$h_{3j+2}^n(\boldsymbol{x}) = \frac{x_{2j+1}}{x_{2j}} + V(x_{2j}), \tag{15c}$$

if cell j at time n is observed by a detector or probe vehicles (otherwise 0). Therefore, the observation model is dynamic depending on location of probe vehicles. Assuming that the FD is differentiable again, the Jacobian $\hat{H}^n$ of the observation model can be obtained analytically.

The measurement variables and noise matrix R^n are specified as follows. For detectors, we assume that they can obtain density and relative-flow with Gaussian noises with constant variances which are given by their technical specifications. For GPS probe vehicles, we assume that probe vehicles are randomly sampled with an unknown rate; and each of them measures its trajectory. The trajectories are converted into mean speeds of the probe vehicles in each cell, denoted as $\tilde{v}_j^n$, using Edie's generalized definition of the traffic state [24]. Note that, except for in very limited and unrealistic cases, $\tilde{v}_j^n$ is not identical to v_j^n even if the trajectory measurement is exact—this is an inevitable measurement error of the mean speeds by the probe vehicles. Following the results of [25], [26], the measurement error was approximated as $\phi/\sqrt{|P_j^n|}$, where ϕ is a pre-determined constant that is approximately the variance of mean speed of each vehicles, and $|P_j^n|$ is the number of probe vehicles in cell j at time n.

C. Discussion

Using the proposed method, the density ρ and relative-flow y are directly estimated from detector measurements $(\tilde{\rho}, \tilde{y})$ and probe vehicle measurements $\tilde{v}$ (Fig. 2). Then, we can obtain all of the traffic variables (ρ, q, v) and the amount of non-equilibrium y by transforming the estimate using the definition.

In the data assimilation framework, a posterior state is estimated based on the model predictions, observations, and their errors. This means that neither the model or observations are considered to be perfect. This is a beneficial feature that compensates for the limitations of the ARZ model and the LxF scheme. For example, the backward moving wave and endogenous occurrence of non-equilibrium traffic, which are not considered in the ARZ model, can be realized by the observations. Additionally, the large diffusion in the LxF scheme can be corrected if spatiotemporally dense observations are available (e.g., high penetration rate of probe vehicles).

[1]Note that some commonly utilized FDs such as the triangular FD are not differentiable

III. Empirical verification

A. Data and calibration

We used part of the NGSIM dataset [27] in our empirical verifications. It contains the trajectories of all vehicles on a freeway segment in an actual environment. Specifically, the data were recorded on a segment of US-101 from 8:07 am to 8:19 am. The road segment is 550 m long and consists of five lanes, with one lane for on/off-ramps of which in/out-flows were negligible. The Eulerian traffic state (i.e., cell) with a time–space resolution (2 s and 50 m) was derived from the raw trajectory data using Edie's generalized definition, and was considered as the ground truth state (Fig. 3). Several forward and backward moving waves of non-equilibrium traffic, as well as free-flowing regions and jam waves can be found.

The FD for the proposed method was calibrated as follows. For this verification, we used the Greenshields FD [28] as the FD's functional form because it is a good fit to the dataset [15]. It can be represented as $V_{\mathrm{G}}(\rho) = (1 - \rho/\rho_{\max})\, v_{\max}$, where $\rho_{\max}$ is the jam density and $v_{\max}$ is the free-flow speed. The FD parameters were estimated using regression of the function to nearly-stationary traffic data (which can be presumed to be nearly non-equilibrium traffic) extracted from the dataset. The nearly-stationary traffic state is defined as the mean speed and density in each cell where the coefficient of variation of the speed of each vehicle in the cell is less than 25%. The estimated parameter values were $\rho_{\max} = 0.45$ (veh/m) and $v_{\max} = 20.60$ (m/s).

The relaxation time τ was set to 40 s according to the calibration in [20] for the same dataset. The variances of the system noises were assumed to be 0.1 (veh/m)^2 for the density and 0.1 (veh/s)^2 for the relative-flow; the same values were assumed for the corresponding elements of $W^{0|0}$. The parameter of the measurement noise on $\tilde{v}_j^n$ by probe vehicles was assumed to be $\phi = 10$ (m/s).

Hereafter, p denotes the probe vehicle penetration rate, and d denotes the number of internal detectors.

B. Results on time–space diagram

The time–space diagrams of the TSE results using the proposed method are shown in Fig. 4. They show the TSE result with probe vehicle penetration rate $p = 5\%$ and number of internal detectors $d = 1$. According to Fig. 4, the proposed method with some sensors successfully estimated the traffic states. Especially, the dynamics of the non-equilibrium traffic (e.g., the forward and backward moving waves in Fig. 3c) were accurately reproduced.

C. Effect of the amount of available data

To quantify how the amount of data affects the results, we show the relationship between the proposed method's accuracy and the amount of data in Fig. 5. In each sub-figure, we visualize the mean absolute percentage error (MAPE) of the traffic state under certain probe vehicle penetration rates (p) and numbers of detectors (d) using contour plots (this visualization approach was proposed by [29]). The definition of MAPE for density is $\mathrm{MAPE}(\rho) = 1/S \sum_{\forall j,n} |\hat{\rho}_j^n - \rho_j^n| / \rho_j^n \times 100\%$, where S denotes the total number of combinations of (j, n) and $\hat{\rho}_j^n$ denotes an estimated density.

Fig. 5 shows that the accuracy of the density and speed estimates using the proposed method increased monotonically with the number of probe vehicles or detectors. The number of probe vehicles contributed more to the improvements in speed estimates than the density. This is a reasonable result, because the probe vehicle's measurements directly affect the speed state, but are only used as a constraint condition between the density and relative-flow states (c.f., Eq. (15c)).

D. Effect of the order of traffic flow models

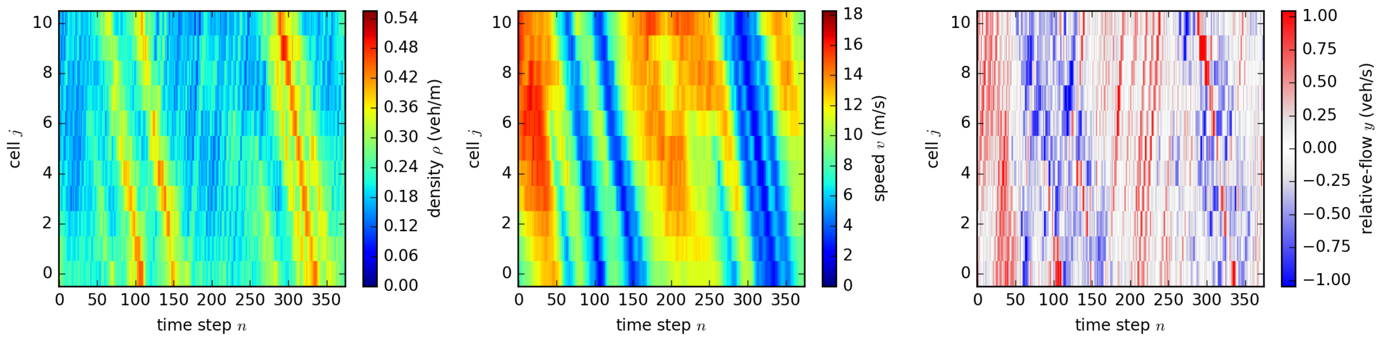
To assess the advantage of the ARZ model (i.e., the relaxation of the equilibrium assumption), we conducted a comparative analysis with an LWR-based TSE method. The LWR-based method used here is similar to the proposed ARZ-based method; that is, it uses the LxF scheme for the discretization and EKF for the data assimilation. The only essential difference is the choice of traffic flow model. Therefore, we can assess the effect of the traffic flow model based on this comparison.[2] The accuracy of the LWR-based method compared to the amount of available data is shown in Fig. 6, in the same manner as Fig. 5.[3]

With respect to the density estimates, both methods performed similarly when using a limited volume of data (i.e., $p \leq 10\%$ and $d \leq 1$). More precisely, the accuracy of the ARZ-based method was almost the same as for the LWR-based method when $p = 0$ and $d = 0$, and became slightly better as p or d increased. However, there were notable differences between the methods when using a large amount of data. The accuracy of the ARZ-based method always increased as p increased, whereas as the accuracy of the LWR-based method did not when p was large (e.g., $p > 15\%$). This is due to the FD-based mapping operation based on the equilibrium assumption. As discussed earlier, the accuracy of the speed estimates using probe vehicles improves as p increases. Meanwhile, the accuracy of the density estimates using the FD-based mapping (i.e., $\rho = V^{-1}(\tilde{v})$) was limited due to the "scatter" in the FD (Fig. 1b), which is regardless of the number of probe vehicles. Therefore, the accuracy of the density estimates will be bounded by the limitation of the FD-based mapping. This can be also considered as a consequence of ignoring the spatiotemporal correlations of non-equilibrium traffic in the LWR-based method.

With respect to the speed estimates, the difference between the methods is significant. Clearly, the ARZ-based method outperformed the LWR-based method. The LWR-based method could not properly utilize the data because of the equilibrium assumption. When there was limited data, the accuracy of the LWR-based method slightly increased as p increased, but did not necessarily increase as d increased. Furthermore, when there was a large amount of data, the accuracy did not necessarily increase even if p increased. This may be due to the error

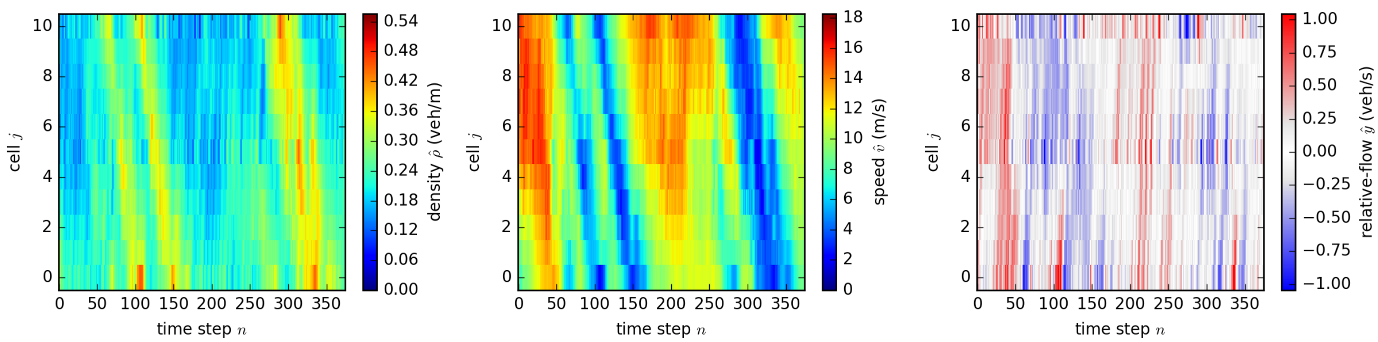
[2] Specifically, the state variable is density, ρ. It can be directly observed by detectors and indirectly observed by probe vehicles: $\tilde{v}$ observed by probe vehicles is converted to ρ using an FD-based mapping, namely, $V^{-1}(\tilde{v})$.

[3] It should be noted that the LWR-based TSE method itself was fairly accurate, although it is based on simple schemes. For example, the MAPEs of the method were 13.8% for the density and 20.4% for the speed, when there were no additional sensors. This is a nearly comparable performance to existing LWR-based TSE methods with advanced techniques [4], [6], [7]. Therefore, the comparative analysis between the proposed ARZ-based method and the LWR-based method can be considered as valid.



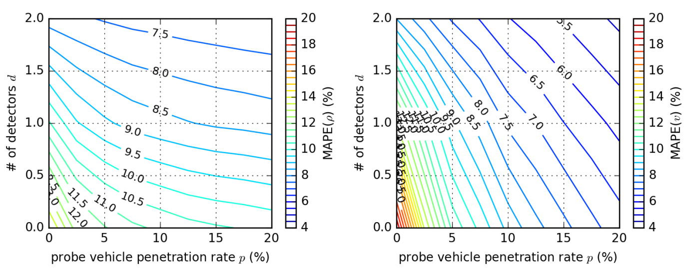
(a) density (b) speed (c) relative-flow

Fig. 3: Ground truth traffic state.



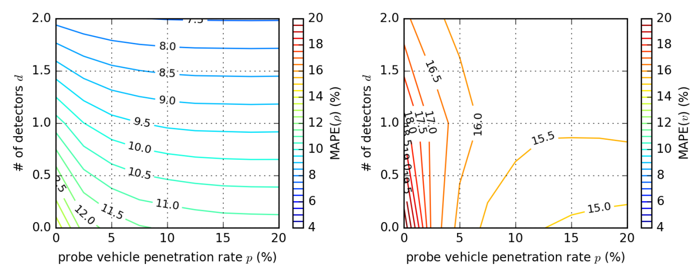
(a) density (b) speed (c) relative-flow

Fig. 4: TSE results by the proposed method with probe vehicles and detectors.



(a) density (b) speed

Fig. 5: Relation between estimation accuracy and available amount of data of the proposed ARZ-based TSE method.



(a) density (b) speed

Fig. 6: Relation between estimation accuracy and available amount of data of an LWR-based TSE method.

in the FD-based mapping, which is essentially the equilibrium assumption. Moreover, Fig. 6b shows that the accuracy of the LWR-based method *decreased* as d increased. This might be due to the locations of detectors, and can be also considered as a limitation of the LWR-based method. That is, if there is a significant amount of non-equilibrium traffic at the location of a detector, its measurements can negatively affect the estimate, which is assumed to be equilibrated. However, note that this "accuracy decreasing phenomena" may not always occur; it depends on the locations of detectors and the extent of non-equilibrium.

IV. Conclusion

This article proposed and verified a TSE method based on a higher-order traffic flow model and a data assimilation framework. The proposed method uses the ARZ model, a second order traffic flow model which has not been received much attention for TSE purposes. Leveraging the ARZ model's features combined with EKF, the proposed method can capture non-equilibrium traffic and fuse heterogeneous data (i.e., detector and probe vehicle data) in a straightforward manner.

The advantages of the proposed method were quantitatively evaluated through empirical analysis using actual traffic data.

We confirmed that the proposed method properly estimated the spatiotemporal dynamics of the traffic states and non-equilibrium traffic. Additionally, the data assimilation framework successfully compensated for the theoretical limitations of the ARZ model. Moreover, we confirmed the data fusion capabilities of the proposed method—its accuracy increased monotonically as the probe vehicle penetration rate or number of detectors increased. This can be considered as an advantage of the high order traffic flow model; because an LWR-based TSE method (for comparison purposes) could not fuse the heterogeneous data properly due to the equilibrium assumption, if there were a lot of data. These results suggest that the proposed method can effectively exploit rich and heterogeneous data in these days and the near future.

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