For each of the following problems, provide your answer and show the steps taken to solve the problem.

Problem 1. Maximum Likelihood Estimation (50 points) Given a dataset {x1, x2, ..., xN } of size N, derive the maximum likelihood estimate (as a function of x1, ..., xN) for: (a) The lower and upper limits, a and b, of a uniform distribution,

f(x; a, b) = {

$$\begin{array}{ll} \frac{1}{b-a}, & \text{if } a \leq x \leq b \\ 0, & \text{otherwise} \end{array}$$

1.

(assuming each xi ∈ R). Show all of your work. (25 points)

to find the upper limit of a uniform distribution we have:

$$b_{MLE} = argmaxP_b(x_1, x_2, \ldots, x_N)$$

$$= argmax \prod_{i=1}^{N} \frac{1}{b-a}$$

$$= argmax(\frac{1}{b-a})^N$$

From this expression we can see that the larger b, the larger the maximum likelihood estimation. Therefore, we can conclude that $b = max(x_{i}) $

Similarly for the lower limit,

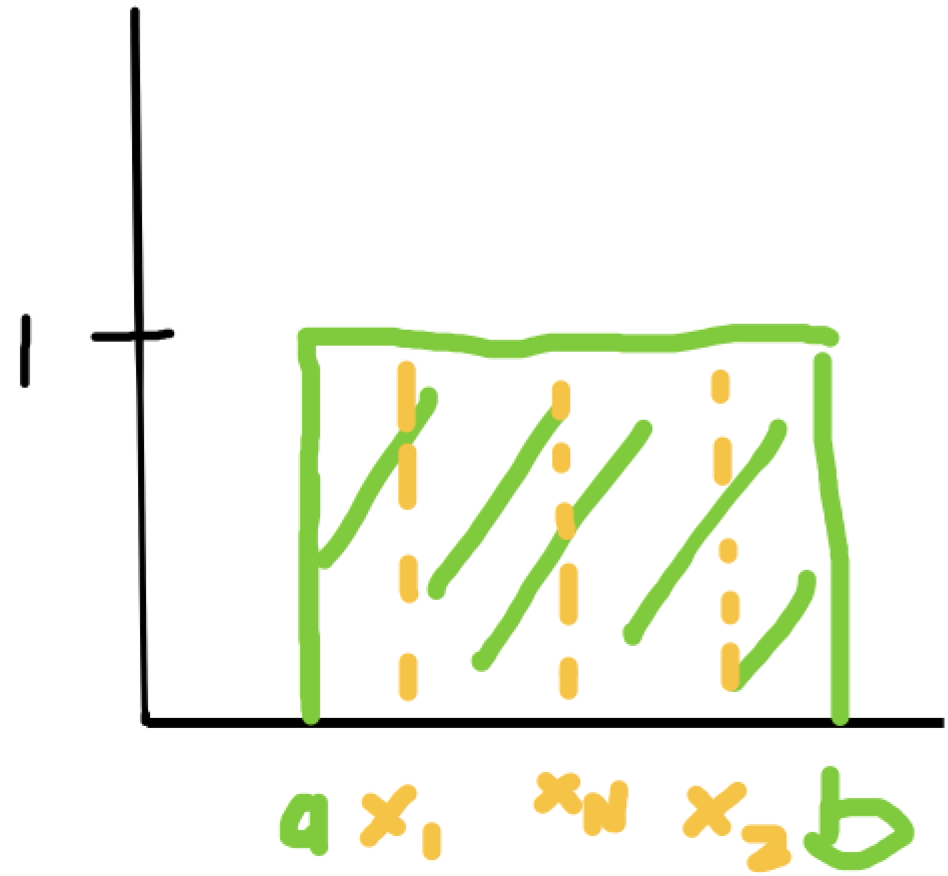
$$a_{MLE} = argmaxP_a(x_1, x_2, \ldots, x_N)$$

$$= argmax \prod_{i=1}^{N} \frac{1}{b-a}$$

$$= argmax(\frac{1}{b-a})^N$$

The smaller a the larger the MLE, Therefore, we can conclude that $a = min(x_i)$

To test this, we can draw the uniform distribution and some x values:



uniform distribution

If some value of x was greater than b, the probability would be 0 and if some value of x was less than a, the probability would also be 0. Therefore to maximize the likelihood, we want all values of x to fall within the range a and b.

b. The λ parameter of a Poisson distribution,

$$f(x;\lambda) = \begin{cases} \frac{e^{-\lambda}\lambda^x}{x!}, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0 \end{cases}$$

(assuming each xi ≥ 0). Show all of your work. (25 points) Hints: (i) plotting some sample data may be helpful and calculus should not be required (a); (ii) maximizing the log likelihood provides the same parameter values and often provides a simpler path to a solution (b); (iii) log(ab) = log a + log b; (iv) log e a = a.

$$l(\lambda|x) = p(x|\lambda) = \prod_{i=1}^{N} p(x_i|\lambda) = \prod_{i=1}^{N} e^{-\lambda}\frac{\lambda}{x!}$$

Then, take the log likelihood

$$L(\lambda|x) = logl(\lambda|x) = \sum_{i=1}^{N} logp(x_i|\lambda)$$

$$= \sum_{i=1}^{N} loge^{-\lambda}\frac{\lambda^x}{x!}$$

$$= \sum_{i=1}^{N} loge^{-\lambda} + log\lambda^x - logx!$$

$$= -\lambda\sum_{i=1}^{N} +xlog\lambda - logx!$$

$$MLE = \lambda^* = argmax_\lambda L(\lambda|x)$$

$$= -\lambda\sum_{i=1}^{N} +xlog\lambda - logx!$$

Problem 2. Bayesian Parameter Estimation (50 points) The density function of an exponential distribution is given by $f_\lambda(x) = \lambda e^{-\lambda x}$. The MLE for the parameter λ can be calculated as $\lambda = \frac{n}{\sum_i x_i}$.
We will now consider Bayesian parameter estimation for this distribution.

a. Using a prior distribution from the Gamma distribution, $f_{\alpha,\beta}(\lambda) = \frac{\beta^\alpha \lambda^{\alpha-1} e^{-\lambda\beta}}{\Gamma(\alpha)}$, with parameters α and β, show that the posterior distribution for λ, after updating using three datapoints x1, x2, x3, is also a Gamma distribution and show its new parameter values, α0 and β0, in terms of α, β, x1, x2, and x3. (25 points)

$$p(x_1, x_2, x_3 \mid \lambda) = \lambda^3 e^{-\lambda(x_1+x_2+x_3)}$$

$$p(\lambda \mid x_1, x_2, x_3) \propto p(x_1, x_2, x_3 \mid \lambda) \cdot p(\lambda)$$

substituting the prior and likelihood functions we get:

$$p(\lambda \mid x_1, x_2, x_3) \propto \frac{\beta^\alpha}{\Gamma(\alpha)} \lambda^{\alpha-1} e^{-\lambda\beta} \cdot \lambda^3 e^{-\lambda(x_1+x_2+x_3)}$$

which simplifies to :

$$p(\lambda \mid x_1, x_2, x_3) \propto \frac{\beta^\alpha}{\Gamma(\alpha)} \lambda^{\alpha+2-1} e^{-\lambda(\beta+x_1+x_2+x_3)}$$

and we get the parameter values as:

$$\alpha_0 = \alpha + 2$$

$$\beta_0 = \beta + x_1 + x_2 + x_3$$

b. If our prior parameters are α = 2 and β = 1, and our data sample consists of x1 = 3.7, x2 = 4.5, x3 = 4.8: Compute the posterior probability of a new datapoint x4 = 3.8 under the fully Bayesian estimation of λ. You can either leave your answer in terms of the Gamma function, or provide the exact answer. (13 points) Hints: (i) You shouldn't have to solve the complicated integral; (ii) Since the Gamma distribution normalizes to 1, R λ λ α−1 e −λβdλ = Γ(α) βα ; (iii) The Gamma function is related to the factorial function as Γ(x) = (x − 1)! for positive integers x.

$$\alpha_0 = 2 + 2 = 4$$

$$\beta_0 = 1 + 3.7 + 4.5 + 4.8 = 14$$

To compute x_4, plug α_0 β_0 and x_4 into the density function

$$f(\lambda \mid \alpha_0, \beta_0) = \frac{\Gamma(\alpha_0)}{\beta_0^{\alpha_0}} \lambda^{\alpha_0 - 1} e^{-\beta_0 \lambda}$$

so we get:

$$f(\lambda \mid 4, 14) = \frac{\Gamma(4)}{14^4} \cdot (3.8)^{4-1} e^{-14 \cdot 3.8}$$

c. If we have the same prior and datapoints as in (b), what is the probability of a new datapoint x4 = 3.8 using maximum a posteriori estimation of λ? (12 points) Hint: (i) The mode of the Gamma distribution (i.e., the λ that attains its maximal probability) is α–1