For each of the following problems, provide your answer and show the steps taken to solve the problem.

Problem 1. Maximum Likelihood Estimation (50 points) Given a dataset {x1, x2, ..., xN } of size N, derive the maximum likelihood estimate (as a function of x1, ..., xN) for: (a) The lower and upper limits, a and b, of a uniform distribution,

f(x; a, b) = {

$$\begin{matrix} \frac{1}{b-a}, & \text{if } a \le x \le b \\ 0, & \text{otherwise} \end{matrix}$$

1.

(assuming each xi ∈ R). Show all of your work. (25 points)

to find the upper limit of a uniform distribution we have:

$$b_{MLE} = argmax P_b(x_1, x_2, \ldots, x_N)$$

$$= argmax \prod_{i=1}^{N} \frac{1}{b-a}$$

$$= argmax (\frac{1}{b-a})^N$$

From this expression we can see that the larger b, the larger the maximum likelihood estimation. Therefore, we can conclude that $b = max(x_{i}) $

Similarly for the lower limit,

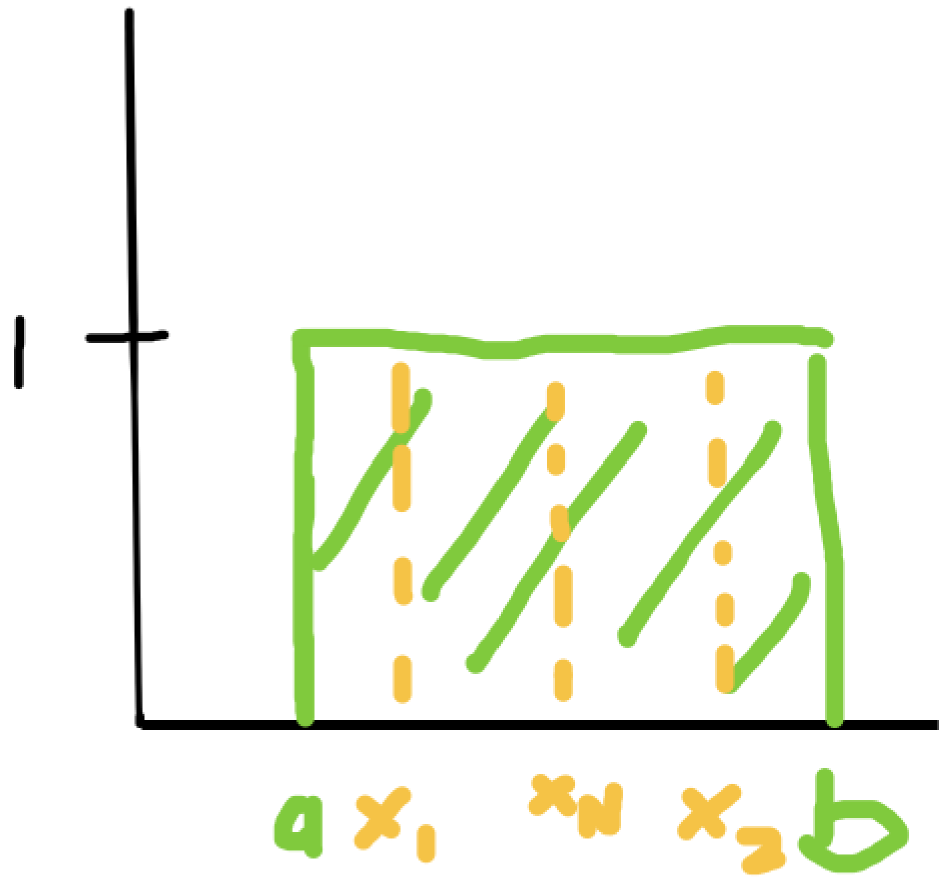
$$a_{MLE} = argmax P_a(x_1, x_2, \ldots, x_N)$$

$$= argmax \prod_{i=1}^{N} \frac{1}{b-a}$$

$$= argmax (\frac{1}{b-a})^N$$

The smaller a the larger the MLE, Therefore, we can conclude that $a = min(x_i)$

To test this, we can draw the uniform distribution and some x values:



uniform distribution

If some value of x was greater than b, the probability would be 0 and if some value of x was less than a, the probability would also be 0. Therefore to maximize the likelihood, we want all values of x to fall within the range a and b.

I didn’t have time to finish :(

b. The λ parameter of a Poisson distribution, f(x; λ) = (e −λ λ x x! , x ≥ 0 0 x < 0

2.

(assuming each xi ≥ 0). Show all of your work. (25 points) Hints: (i) plotting some sample data may be helpful and calculus should not be required (a); (ii) maximizing the log likelihood provides the same parameter values and often provides a simpler path to a solution (b); (iii) log(ab) = log a + log b; (iv) log e a = a.

Problem 2. Bayesian Parameter Estimation (50 points) The density function of an exponential distribution is given by fλ(x) = λe−λx. The MLE for the parameter λ can be calculated as λ = n Σixi . We will now consider Bayesian parameter estimation for this distribution.

a. Using a prior distribution from the Gamma distribution, fα,β(λ) = β αλ α−1 e −λβ Γ(α) , with parameters α and β, show that the posterior distribution for λ, after updating using three datapoints x1, x2, x3, is also a Gamma distribution and show its new parameter values, α 0 and β 0 , in terms of α, β, x1, x2, and x3. (25

points)

b. If our prior parameters are $\alpha = 2$ and $\beta = 1$, and our data sample consists of $x1 = 3.7$, $x2 = 4.5$, $x3 = 4.8$: Compute the posterior probability of a new datapoint $x4 = 3.8$ under the fully Bayesian estimation of λ. You can either leave your answer in terms of the Gamma function, or provide the exact answer. (13 points) Hints: (i) You shouldn't have to solve the complicated integral; (ii) Since the Gamma distribution normalizes to 1, R λ λ α–1 e –$\lambda\beta$dλ = $\Gamma(\alpha)$ $\beta\alpha$; (iii) The Gamma function is related to the factorial function as

$\Gamma(x) = (x - 1)!$ for positive integers x.

c. If we have the same prior and datapoints as in (b), what is the probability of a new datapoint $x4 = 3.8$ using maximum a posteriori estimation of λ? (12 points) Hint: (i) The mode of the Gamma distribution (i.e., the λ that attains its maximal probability) is $\alpha-1$