On the Asymptotic Validity of a Fully Sequential Elimination Procedure for Indifference-Zone Ranking and Selection with Tight Bounds on Probability of Correct Selection

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We prove the validity of the sequential elimination IZ procedure proposed by Frazier [] when δ goes to 0.

Asymptotic Validity when the Variances are Known

Without loss of generality, suppose that the true means of the systems are indexed so that $\mu_k \geq \mu_{k-1} \geq \cdots \geq \mu_1$. We suppose that samples from system $x \in \{1 \ldots, k\}$ are normally distributed and independent, over time and across alternatives. We also suppose that the algorithm ends in at most $R(\delta) \in \mathbb{N}$ iterations, and $R(\delta) \to \infty$ as $\delta \to 0$ with probability 1. Furthermore, $R^{1/2}\delta \to \Delta$ with probability 1 where $\infty > \Delta > 0$ with probability 1. We also define $\lambda_z^2 := max_{i \in \{1 \ldots, k\}} \{\lambda_i^2\}$.

Lemma 1. If $x \in \{1 \ldots, k\}$ and $t \in [0,1]$, then

$$C_x(\delta, t) := \frac{Y_{ceil\left(\frac{\lambda_x^2}{\lambda_z^2}(n_0+tR)\right),x} - \frac{\lambda_x^2}{\lambda_z^2}(n_0+tR)\mu_x}{\frac{\lambda_x^2}{\lambda_z}\sqrt{R}} \Rightarrow W_x(t)$$

where $Y_{n,x} = \sum_{i=1}^{n} X_{xi}$ and W_x is a standard Brownian motion.

Proof. By the Lindeberg-Lévy central limit theorem and the fact that $\frac{ceil(nt)}{n} \to t$,

$$\frac{Y_{ceil(nt),x} - ceil\,(nt)\,(\mu_x)}{\lambda_x \sqrt{n}} \Rightarrow W_x\,(t)\,.$$

Consequently,

$$\frac{Y_{ceil(nt),x} - nt\,(\mu_x)}{\lambda_x \sqrt{n}} \Rightarrow W_x\,(t)$$

because $\frac{nt - ceil(nt)}{\lambda_x \sqrt{n}} \to 0$. Thus

$$\frac{Y_{ceil\left(\frac{\lambda_x^2}{\lambda_z^2} tR\right),x} - \frac{\lambda_x^2}{\lambda_z^2} tR\,(\mu_x)}{\frac{\lambda_x^2}{\lambda_z}\sqrt{R}} \Rightarrow W_x\,(t)$$

as $\delta \to 0$, and then

$$\frac{Y_{ceil\left(\frac{\lambda_x^2}{\lambda_z^2}(n_0 + tR)\right),x} - \frac{\lambda_x^2}{\lambda_z^2}\,(n_0 + tR)\,\mu_x}{\frac{\lambda_x^2}{\lambda_z}\sqrt{R}} \Rightarrow W_x\,(t)\,.$$

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Proposition 1. We have that for t sufficiently large

$$\begin{aligned}
\frac{W_{tx}}{n_{tx}} &= \frac{\lambda_x^2}{\lambda_z}\sqrt{R}\frac{\left(C_x\left(\delta, \frac{t}{R}\right) + \frac{\lambda_x^2}{\lambda_z^2}\,(n_0 + tR)\,\mu_x\right)}{\frac{\lambda_x^2}{\lambda_z^2}\,(n_0 + t)} \\
&= \lambda_z \sqrt{R}\frac{\left(C_x\left(\delta, \frac{t}{R}\right) + \frac{\lambda_x^2}{\lambda_z^2}\,(n_0 + t)\,\mu_x\right)}{(n_0 + t)}.
\end{aligned}$$

Now, we consider the set $D\,[0,1]^k$ which is the set of functions from $[0,1]$ to $\mathbb{R}^k$ that are right-continuous and have left-hand limits. We'll use the Skorokhod metric d on $D\,[0,1]^k$:

$$d\,(X, Y) = \inf_{\lambda \in \Lambda}\left\{\|\lambda - I\| \vee \|X - Y \circ \lambda\|\right\}$$

where Λ is the set of strictly increasing, continuous mappings of $[0,1]$ onto itself, and $\|\cdot\|$ is the uniform norm.

Lemma 2. Suppose Δ is given. Let $Y \in D\left[0,1\right]^k$ and $t \in [0,1]$

$$q_{tx}^{Y,\delta}\left(A\right): \;=\; \exp\left(\delta\beta_{tR}\lambda_z\sqrt{R}\frac{Y_x\left(t\right)}{\left(n_0+tR\right)}\right)/\sum_{x'\in A}\exp\left(\delta\beta_{tR}\lambda_z\sqrt{R}\frac{Y_{x'}\left(t\right)}{\left(n_0+tR\right)}+\beta_{tR}\delta^2 I_{x',x}\right)$$

where $\beta_{tR}=\frac{(n_0+tR)}{\lambda_z^2}$ and

$$I_{x',x}=\begin{cases}0 & \text{if } x'=x\\ 1 & \text{if } x'>x\\ -1 & \text{if } x'<x\end{cases}\cdot$$

We then define

$$\begin{aligned}
T_{Y,\delta}^{0} &= 0\\
A_0^{Y,\delta} &= \{1,\ldots,k\}\\
P_0^{Y,\delta} &= P^*\\
T_{Y,\delta}^{n+1} &= \inf\left\{t\in\left[T_{Y,\delta}^{n},1\right]:\ \min\nolimits_{x\in A_n^{Y,\delta}} q_{tx}^{Y,\delta}\left(A_n^{Y,\delta}\right)\le c \text{ or } \max\nolimits_{x\in A_n^{Y,\delta}} q_{tx}^{Y,\delta}\left(A_n^{Y,\delta}\right)\ge P_n^{Y,\delta}\right\}\\
Z_{n+1}^{Y,\delta} &\in \arg\min\nolimits_{x\in A_n^{Y,\delta}} q_{T_{Y,\delta}^{n+1},x}^{Y,\delta}\left(A_n^{Y,\delta}\right)\\
A_{n+1}^{Y,\delta} &= A_n-\left\{Z_{n+1}^{Y,\delta}\right\}\\
P_{n+1}^{Y,\delta} &= P_n^{Y,\delta}/\left(1-\min\nolimits_{x\in A_n^{Y,\delta}} q_{T_{Y,\delta}^{n+1}x}^{Y,\delta}\left(A_n^{Y,\delta}\right)\right).
\end{aligned}$$

Now, let

$$M_{Y,\delta}=\inf\left\{n=1,\ldots,k-1:\max\nolimits_{x\in A_{n-1}^{Y,\delta}} q_{T_{Y,\delta}^{n},x}\left(A_n^{Y,\delta}\right)\ge P_{n-1}^{Y,\delta}\right\}$$

and

$$f\left(Y,\delta\right) \;=\; \begin{cases}1 & \text{if k} \in \arg\max\nolimits_{x\in A_{M-1}^{Y,\delta}} \frac{\frac{\lambda_k^2}{\lambda_z}\left(Y_k\left(T_{Y,\delta}^{M_{Y,\delta}}\right)\right)}{}+\delta\frac{\lambda_k^2}{\lambda_z^2\sqrt{R}}\left(n_0+T_{Y,\delta}^{M_{Y,\delta}}R\right)\ge\frac{\frac{\lambda_x^2}{\lambda_z}\left(Y_x\left(T_{Y,\delta}^{M_{Y,\delta}}\right)\right)}{}\\ 0 & \text{otherwise}\end{cases}.$$

Now, we also define

$$q_{tx}^{Y}\left(A\right) := \exp\left(\Delta\frac{Y_{x}\left(t\right)}{\lambda_{z}}\right) / \sum_{x'\in A}\exp\left(\Delta\frac{Y_{x'}\left(t\right)}{\lambda_{z}} + t\frac{\Delta^{2}}{\lambda_{z}^{2}}I_{x',x}\right)$$

$$\begin{aligned}
T_{Y}^{0} &= 0 \\
A_{0}^{Y} &= \{1,\ldots,k\} \\
P_{0}^{Y} &= P^{*} \\
T_{Y}^{n+1} &= \inf\left\{t\in[T_{Y}^{n},1]:\ \min_{x\in A_{n}^{Y}} q_{tx}^{Y}\left(A_{n}^{Y}\right)\leq c \text{ or } \max_{x\in A_{n}^{Y}} q_{tx}^{Y}\left(A_{n}^{Y}\right)\geq P_{n}^{Y}\right\} \\
Z_{n+1}^{Y} &\in \arg\min_{x\in A_{n}^{Y}} q_{T_{Y}^{n+1},x}^{Y}\left(A_{n}^{Y}\right) \\
A_{n+1}^{Y} &= A_{n}-\left\{Z_{n+1}^{Y}\right\} \\
P_{n+1}^{Y} &= P_{n}^{Y,\delta}/\left(1-\min_{x\in A_{n}^{Y,}} q_{T_{Y}^{n+1}x}^{Y}\left(A_{n}^{Y}\right)\right).
\end{aligned}$$

Now, let

$$M_{Y} = \inf\left\{n=1,\ldots,k-1:\max_{x\in A_{n-1}^{Y}} q_{T_{Y}^{n},x}\left(A_{n}^{Y}\right)\geq P_{n-1}^{Y}\right\}$$

and

$$g\left(Y\right) = \begin{cases} 1 & \text{if k} \in \arg\max_{x\in A_{M-1}^{Y}} \frac{Y_{k}\left(T_{Y}^{M_{Y}}\right)}{} + \Delta\frac{1}{\lambda_{z}}T_{Y}^{M_{Y}} \geq \frac{\frac{\lambda_{x}^{2}}{\lambda_{k}^{2}}\left(Y_{x}\left(T_{Y}^{M_{Y}}\right)\right)}{} \\ 0 & \text{otherwise}\end{cases}.$$

Proof. We have that if $\hat{\tau}_{M}$ is the continuous version of $\frac{\tau_{M}}{R}$, then

$$f\left(C\left(\delta,t\right),\delta\right) = \begin{cases} 1 & \text{if k} \in \arg\max_{x\in A_{M-1}^{Y,\delta}} \frac{\lambda_{x}^{2}}{\lambda_{z}}\sqrt{R}\left(C_{x}\left(\delta,\hat{\tau}_{M}\right)+\frac{\lambda_{x}^{2}}{\lambda_{z}^{2}}\left(n_{0}+\hat{\tau}_{M}R\right)\mu_{x}\right) \\ 0 & \text{otherwise}\end{cases}.$$

By lemma 1,

$$C\left(\delta,t\right)\Rightarrow W\left(t\right).$$

Theorem. If samples from system $x \in \{1 \dots, k\}$ are normally distributed and independent, over time and across alternatives, then $\lim_{\delta \to 0} Pr\{\text{BIZ selects } k\} \geq P*$ provided $\mu_k \geq \mu_{k-1} + \delta$. We also suppose that the algorithm ends in at most $R(\delta) \in \mathbb{N}$ iterations, and $R(\delta) \to \infty$ as $\delta \to 0$ with probability 1. Furthermore, $\sqrt{R}\delta \to \Delta$ with probability 1 where $\infty > \Delta > 0$ with probability 1. We also suppose $B_1 = \cdots = B_k = 1$.

Proof. Let

$$\begin{aligned} T_k(\delta) &= \min\left\{t \in \left\{\frac{\tau_{k-1}}{R}, \dots, 1\right\} : max_x q_{tR,x}(A) \geq P\right\}. \\ &= \min\left\{t \in \left\{\frac{\tau_{k-1}}{R}, \dots, 1\right\} : max_x exp\left(\delta\beta_{tR}\frac{W_{tR,x}}{n_{tR,x}}\right) \geq P \sum_{x' \in A} exp\left(\delta\beta_{tR}\frac{W_{tR,x'}}{n_{tR,x'}}\right)\right\}. \end{aligned}$$

Let

$$f(Y,\delta) = \begin{cases} 1 & \text{if } \frac{\lambda_k^2}{\lambda_z n_{tR,k}}\sqrt{R}Y_k(T_{Y,\delta}) + \delta \geq \frac{\lambda_x^2}{\lambda_z n_{tR,x}}\sqrt{R}Y_x(T_{Y,\delta}) \ \forall x \in A \\ 0 & \text{otherwise} \end{cases}$$

where

$$\begin{aligned} T_{Y,\delta} = \inf\Bigg\{t \in [0,1] : \exists x \text{ s.t. } & exp\left(\frac{\lambda_x^2}{\lambda_z n_{tR,x}}\sqrt{R}\delta\beta_{tR}Y_x(t)\right)\left(\frac{1}{P} - 1\right) \geq \\ & \sum_{x' < x} exp\left(\left(\frac{\lambda_{x'}^2}{\lambda_z n_{tR,x'}}\sqrt{R}Y_{x'}(t)\,\delta\beta_{tR} - \delta^2\beta_{tR}\right)\right) + \\ & \sum_{x > x'} exp\left(\left(\frac{\lambda_{x'}^2}{\lambda_z n_{tR,x'}}\sqrt{R}Y_{x'}(t)\,\delta\beta_{tR} + \delta^2\beta_{tR}\right)\right)\Bigg\} \end{aligned}$$