On the Asymptotic Validity of a Fully Sequential Elimination Procedure for Indifference-Zone Ranking and Selection with Tight Bounds on Probability of Correct Selection

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We prove the validity of the sequential elimination IZ procedure proposed by Frazier [] when δ goes to 0. Specifically, we analyze Algorithm 2, when $B_1 = \cdots = B_k = 1$:

Algorithm 2: Discrete-time implementation of BIZ, for unknown and/or heterogeneous variances.

Require: $c \in [0, 1 - (P^*)^{\frac{1}{k-1}}]$, $\delta > 0$, $P^* \in (1/k, 1)$, $n_0 \geq 0$ an integer, $B_1, \ldots, B_k$ strictly positive integers. Recommended choices are $c = 1 - (P^*)^{\frac{1}{k-1}}$, $B_1 = \cdots = B_k = 1$ and n_0 between 10 and 30. If the sampling variances λ_x^2 are known, replace the estimators $\widehat{\lambda}_{tx}^2$ with the true values λ_x^2, and set $n_0 = 0$. To compute $\widehat{q}_{tx}(A)$, use

$$q'_{t,x}(A) = \exp\left(\gamma \delta Y'_{tx}\right) \Big/ \sum_{x' \in A} \exp\left(\gamma \delta Y'_{tx'}\right) = \exp\left(\frac{\delta}{\lambda_x^2} Y_{n_x(t),x}\right) \Big/ \sum_{x' \in A} \exp\left(\frac{\delta}{\lambda_{x'}^2} Y_{n'_x(t),x'}\right), \qquad (1)$$

where $Y_{n_x(t),x}$ is the sum of the first $n_x(t)$ samples.

1: For each x, sample alternative x n_0 times and set $n_{0x} \leftarrow n_0$. Let W_{0x} and $\widehat{\lambda}_{0x}^2$ be the sample mean and sample variance respectively of these samples. Let $t \leftarrow 0$.

2: Let $A \leftarrow \{1, \ldots, k\}$, $P \leftarrow P^*$, $t \leftarrow 1$.

3: **while** $x \in \max_{x \in A} \widehat{q}_{tx}(A) < P$ **do**

4:     **while** $\min_{x \in A} \widehat{q}_{tx}(A) \leq c$ **do**

5:         Let $x \in \arg\min_{x \in A} \widehat{q}_{tx}(A)$.

6:         Let $P \leftarrow P/(1 - \widehat{q}_{tx}(A))$.

7:         Remove x from A.

8:     **end while**

9:     Let $z \in \arg\min_{x \in A} n_{tx}/\widehat{\lambda}_{tx}^2$.

10:     For each $x \in A$, let $n_{t+1,x} = \text{ceil}\left(\widehat{\lambda}_{tx}^2(n_{tz} + B_z)/\widehat{\lambda}_{tz}^2\right)$.

11:     For each $x \in A$, if $n_{t+1,x} > n_{tx}$, take $n_{t+1,x} - n_{tx}$ additional samples from alternative x. Let $W_{t+1,x}$ and $\widehat{\lambda}_{t+1,x}^2$ be the sample mean and sample variance respectively of all samples from alternative x thus far.

12:     Increment t.

13: **end while**

14: Select $\hat{x} \in \arg\max_{x \in A} W_{tx}/n_{tx}$ as our estimate of the best.

1   Introduction

This paper is organized as follows: In §2, we present the proof of the validity of the algorithm when the variances are known. In §3, we prove the case when the variances are unknown.

To prove the case when the variances are known, we use a Functional Central Limit Thoerem that shows how to standardize the output data to make them behave like Brownian motion processes in the limit. We also use an extension of the Continuous Mapping Theorem (Theorem 5.5 of Billingsley 1968) to see that the algorithm behaves like a sequential elimination IZ procedure with a Brownian motion process instead of the standardize of the sum of the output data in the limit. Finally, we use the results of the paper of Frazier [] to prove the validity of this algorithm in the limit.

2   Asymptotic Validity when the Variances are Known

Without loss of generality, suppose that the true means of the systems are indexed so that $\mu_k \geq \mu_{k-1} \geq \cdots \geq \mu_1$. We suppose that samples from system $x \in \{1 \ldots, k\}$ are identically distributed and independent, over time and across alternatives. We define $\lambda_z^2 := \max_{i \in \{1 \ldots, k\}} \lambda_i^2$. We suppose that $\min_{i \in \{1 \ldots, k\}} \lambda_i^2 > 0$.

Lemma 1. If $x \in \{1 \ldots, k\}$, then

$$C_x(\delta, \cdot) := \frac{Y_{\mathrm{ceil}\left(\frac{\lambda_x^2}{\lambda_z^2}(n_0 + tR(\delta))\right),x} - \frac{\lambda_x^2}{\lambda_z^2}(n_0 + tR(\delta))\mu_x}{\frac{\lambda_x^2}{\lambda_z}\sqrt{R(\delta)}} \Rightarrow W_x(\cdot)$$

where $Y_{n,x}$ is the sum of the first n samples and W_x is a standard Brownian motion.

Proof. We define

$$C_{x\delta}(t) := \frac{Y_{\mathrm{ceil}\left(\frac{\lambda_x^2}{\lambda_z^2}tR\right),x} - \mathrm{ceil}\left(\frac{\lambda_x^2}{\lambda_z^2}tR\right)\mu_x}{\frac{\lambda_x^2}{\lambda_z}\sqrt{R}}$$

where $t \in [0,1]$. Since $\frac{\lambda_x^2}{\lambda_z^2}R(\delta)\delta^2 \xrightarrow{P} \frac{\lambda_x^2}{\lambda_z^2}\Delta^2$ in probability, then by the Theorem 17.2 of Billingsley 1968,

$$C_{x\delta}(\cdot) \Rightarrow W(\cdot)$$

as $\delta \to 0$.

We now define

$$\tilde{C}_x(\delta, t) := \frac{Y_{\mathrm{ceil}\left(\frac{\lambda_x^2}{\lambda_z^2}tR\right),x} - \frac{\lambda_x^2}{\lambda_z^2}tR\mu_x}{\frac{\lambda_x^2}{\lambda_z}\sqrt{R}}.$$

Consequently, as $\delta \to 0$,

$$\tilde{C}_x(\delta, \cdot) \Rightarrow W(\cdot)$$

because $\dfrac{\frac{\lambda_x^2}{\lambda_z^2}tR - ceil\left(\frac{\lambda_x^2}{\lambda_z^2}tR\right)}{\frac{\lambda_x^2}{\lambda_z}\sqrt{R}} \to 0$ (see proof of Theorem 9.1 of Billingsley 1968).

Observe that for $\epsilon > 0$ and δ sufficiently small

$$\left| \frac{-Y_{\text{ceil}\left(\frac{\lambda_x^2}{\lambda_z^2} tR\right),x} + Y_{\text{ceil}\left(n_0 \frac{\lambda_x^2}{\lambda_z^2} + \frac{\lambda_x^2}{\lambda_z^2} tR\right),x}}{\frac{\lambda_x^2}{\lambda_z^2}\sqrt{R}} \right| < \epsilon \left(n_0 \frac{\lambda_x^2}{\lambda_z^2} + 2 \right)$$

and then

$$C_x\left(\delta, \cdot\right) \Rightarrow W\left(\cdot\right).$$

∎

Now we are going to define new algorithms that are almost the same than the one proposed by Frazier, but instead of $q_{tx}^{\delta}(A)$, these algorithms use new functions $q_{tx}^{Y,\delta}(A)$ which depend on a function Y that is in $D[0,1]^k$ and $D[0,1]^k$ is the set of functions from $[0,1]^k$ to $\mathbb{R}$ that are right-continuous and have left-hand limits. We'll use the Skorohod metric d on $D[0,1]^k$:

$$d\left(X,Y\right) = \inf_{\lambda \in \Lambda}\left\{\|\lambda - I\| \vee \|X - Y \circ \lambda\|\right\}$$

where Λ is the set of strictly increasing, continuous mappings of $[0,1]$ onto itself, and $\|\cdot\|$ is the uniform norm, and I is the identity map.

First we are going to suppose that $\delta > 0$ and $\mu_k = \delta, \mu_{k-1} = \cdots = \mu_1 = 0$.

Suppose Δ is given. Let $Y \in D[0,1]^k$ and $t \in [0,1]$, we define

$$q_{tx}^{Y,\delta}(A) : \quad = \quad \exp\left(\delta\sqrt{R}\frac{Y_x(t)}{\lambda_z} + \delta\beta_{tR}\mu_x\right) / \sum_{x' \in A} \exp\left(\delta\sqrt{R}\frac{Y_{x'}(t)}{\lambda_z} + \delta\beta_{tR}\mu_{x'}\right)$$

where $\beta_{tR} = \frac{(n_0 + tR)}{\lambda_z^2}$.

We then define for $0 < a < P^* - \frac{1}{k}$

$$
\begin{aligned}
T_{Y,\delta}^0 &= 0 \\
A_0^{Y,\delta} &= \{1, \ldots, k\} \\
P_0^{Y,\delta} &= P^* - a \\
T_{Y,\delta}^{n+1}\left(P_n^{Y,\delta}\right) &= \inf\left\{t \in [T_{Y,\delta}^n, 1] : \min_{x \in A_n^{Y,\delta}} q_{tx}^{Y,\delta}\left(A_n^{Y,\delta}\right) \leq c \text{ or } \max_{x \in A_n^{Y,\delta}} q_{tx}^{Y,\delta}\left(A_n^{Y,\delta}\right) \geq P_n^{Y,\delta}\right\} \\
Z_{n+1}^{Y,\delta} &\in \arg\min_{x \in A_n^{Y,\delta}} q_{T_{Y,\delta}^{n+1},x}^{Y,\delta}\left(A_n^{Y,\delta}\right) \\
A_{n+1}^{Y,\delta} &= A_n - \left\{Z_{n+1}^{Y,\delta}\right\} \\
P_{n+1}^{Y,\delta} &= P_n^{Y,\delta} / \left(1 - \min_{x \in A_n^{Y,\delta}} q_{T_{Y,\delta}^{n+1} x}^{Y,\delta}\left(A_n^{Y,\delta}\right)\right).
\end{aligned}
$$

Now, let

$$M_{Y,\delta} = \inf\left\{n = 1, \ldots, k-1 : \max_{x \in A_{n-1}^{Y,\delta}} q_{T_{Y,\delta}^n, x}\left(A_{n-1}^{Y,\delta}\right) \geq P_{n-1}^{Y,\delta}\right\}$$

and

$$f\left(Y,\delta\right)=\begin{cases}1 & \text{if } \mathrm{k}\in A_{M-1}^{Y,\delta} \text{ and } \lambda_k^2\left(Y_k\left(T_{Y,\delta}^{M_{Y,\delta}}\right)\right)+\delta\frac{\lambda_k^2}{\lambda_z\sqrt{R}}\left(n_0+T_{Y,\delta}^{M_{Y,\delta}}R\right)\geq\lambda_x^2\left(Y_x\left(T_{Y,\delta}^{M_{Y,\delta}}\right)\right)\ \forall x\in A_{M-1}^{Y,\delta}\\ 0 & \text{otherwise}\end{cases}.$$

Now, we also define

$$q_{tx}^Y\left(A\right):=\exp\left(\Delta\frac{Y_x\left(t\right)}{\lambda_z}+\frac{1}{\lambda_z^2}\Delta^2 t I_{\{x=k\}}\right)\Big/\sum_{x'\in A}\exp\left(\Delta\frac{Y_{x'}\left(t\right)}{\lambda_z}+\frac{1}{\lambda_z^2}\Delta^2 t I_{\{x'=k\}}\right)$$

$$\begin{aligned}
T_Y^0 &= 0\\
A_0^Y &= \{1,\ldots,k\}\\
P_0^Y &= P^*\\
T_Y^{n+1}\left(P_n^Y\right) &= \inf\left\{t\in[T_Y^n,1]:\ \min_{x\in A_n^Y}q_{tx}^Y\left(A_n^Y\right)\leq c \text{ or } \max_{x\in A_n^Y}q_{tx}^Y\left(A_n^Y\right)\geq P_n^Y\right\}\\
Z_{n+1}^Y &\in \arg\min_{x\in A_n^Y}q_{T_Y^{n+1},x}^Y\left(A_n^Y\right)\\
A_{n+1}^Y &= A_n^Y-\left\{Z_{n+1}^Y\right\}\\
P_{n+1}^Y &= P_n^{Y,\delta}\Big/\left(1-\min_{x\in A_n^Y,}q_{T_Y^{n+1}x}^Y\left(A_n^Y\right)\right).
\end{aligned}$$

Now, let

$$M_Y=\inf\left\{n=1,\ldots,k-1:\max_{x\in A_{n-1}^Y}q_{T_Y^n,x}\left(A_{n-1}^Y\right)\geq P_{n-1}^Y\right\}$$

and

$$g\left(Y\right)=\begin{cases}1 & \text{if } \mathrm{k}\in A_{M-1}^Y \text{ and } Y_k\left(T_Y^{M_Y}\right)+\Delta\frac{1}{\lambda_z}T_Y^{M_Y}\geq\frac{\lambda_x^2}{\lambda_k^2}\left(Y_x\left(T_Y^{M_Y}\right)\right)\ \forall x\in A_{M-1}^Y\\ 0 & \text{otherwise}\end{cases}.$$

We define for $x\in A_{M-1}^{Y,\delta}$

$$H_x\left(C\left(\delta,\cdot\right)\right)=\lambda_k^2\left(C_k\left(\delta,T_{C(\delta,\cdot),\delta}^{M_{C(\delta,\cdot),\delta}}\right)\right)+\delta\frac{\lambda_k^2}{\lambda_z\sqrt{R}}\left(n_0+T_{C(\delta,\cdot),\delta}^{M_{C(\delta,\cdot),\delta}}R\right)-\lambda_x^2\left(Y_x\left(T_{C(\delta,\cdot),\delta}^{M_{C(\delta,\cdot),\delta}}\right)\right)$$

and for $x\in A_{M-1}^Y$

$$S_x\left(W\left(\cdot\right),\Delta\right)=W_k\left(T_W^{M_W}\right)+\Delta\frac{1}{\lambda_z}T_W^{M_W}-\frac{\lambda_x^2}{\lambda_k^2}\left(W_x\left(T_W^{M_W}\right)\right)$$

Now, we want to prove that for δ sufficiently small $A_{M-1}^{Y,\delta}\subset A_{M-1}^Y$ and if $x\in A_{M-1}^{Y,\delta}$ for δ sufficiently small

$$H_x\left(C\left(\delta,\cdot\right)\right)\Rightarrow S_x\left(W\left(\cdot\right),\Delta\right)$$

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as $\delta \to 0$.

We want to prove that

$$f\left(C\left(\delta,\cdot\right)\right) \Rightarrow g\left(W\right).$$

In order to prove this, we will prove the following lemma which will allow us to use the Theorem 5.5 of Billingsley 1968 that implies the desired result.

Lemma 2. Let $D_s \equiv \{x \in D\left[0,1\right]^k : \text{ for some sequence } \{x_n\} \subset D\left[0,1\right]^k \text{ with } lim_n d\left(x_n, x\right) = 0$, the sequence $\{f_n\left(x_n\right)\}$ converges to $\{g\left(x\right)\}\}$, then $\mathbb{P}\left(W \in D_s\right) = 1$.

Proof. We will use the following property: if $0 < T < 1$ where T is a stopping time, then by the local version of the law of the iterated logarithm for Brownian motion

$$\lim \sup_{u \to 0+} \frac{\left[W_x\left(T+u\right) - W_x\left(T\right)\right]}{\sqrt{2u \ln\left[\ln\left(1/u\right)\right]}} = 1 \tag{2}$$

almost surely for each system x. Furthermore, note that W is continuous on $\left[0,1\right]^k$ and so W is also uniformly continuous on $\left[0,1\right]^k$.

Let $\{Z_n\} \subset D\left[0,1\right]^k$ such that $Z_n \to W$. Furthermore, $\delta_n^2 \frac{\left(n_0 + tR(\delta_n)\right)}{\lambda_z^2} \to \frac{\Delta^2 t}{\lambda_z^2}$ in $D\left[0,1\right]$ and $\frac{\delta_n \sqrt{R(\delta_n)}}{\lambda_z} \to \frac{\Delta}{\lambda_z}$ in $D\left[0,1\right]$ because uniformly convergence implies convergence in the Skorohod topology. Consequently, there exist functions λ_n in Λ such that

$$\lim_n Z_n\left(\lambda_n t\right) = W\left(t\right)$$

uniformly in t and

$$\lim_n \lambda_n t = t$$

uniformly in t. Then

$$\lim_n \frac{\delta_n \sqrt{R\left(\delta_n\right)}}{\lambda_z} Z_n\left(\lambda_n t\right) + \delta_n^2 \frac{\left(n_0 + \lambda_n\left(t\right) R\left(\delta_n\right)\right)}{\lambda_z^2} = W\left(t\right) \frac{\Delta}{\lambda_z} + \frac{\Delta^2 t}{\lambda_z^2}$$

uniformly in t, and so

$$\lim_n \exp\left(\frac{\delta_n \sqrt{R\left(\delta_n\right)}}{\lambda_z} Z_n\left(\lambda_n t\right) + \delta_n^2 \frac{\left(n_0 + \lambda_n\left(t\right) R\left(\delta_n\right)\right)}{\lambda_z^2}\right) = \exp\left(W\left(t\right) \frac{\Delta}{\lambda_z} + \frac{\Delta^2 t}{\lambda_z^2}\right)$$

uniformly in t since exp is uniformly continuous in $\left[0,1\right]$. Consequently,

$$q_{\lambda_n(t)x}^{Z_n,\delta_n}\left(A\right) \to q_{tx}^W\left(A\right)$$

uniformly in t. Thus, $q_{\cdot x}^{Z_n,\delta_n}\left(A\right) \to q_{\cdot x}^W\left(A\right)$ in $D\left[0,1\right]$ for any set $A \subset \{1,\ldots,k\}$. Then there exists $\lambda_{n,A} \in \Lambda$ such that $\sup_{t\in[0,1]} \|\lambda_{n,A}\left(t\right) - t\| \le d\left(q_{\cdot x}^{Z_n,\delta_n}\left(A\right), q_{\cdot x}^W\left(A\right)\right) + \frac{1}{n}$ and $\sup_{t\in[0,1]} \left\|q_{tx}^{Z_n,\delta_n}\left(A\right) - q_{\lambda_{n,A}(t)x}^W\left(A\right)\right\| \le d\left(q_{\cdot x}^{Z_n,\delta_n}\left(A\right), q_{\cdot x}^W\left(A\right)\right) + \frac{1}{n}$. Taking $g_n^A \equiv \sup_{t\in[0,1]} \left\|q_{tx}^W\left(A\right) - q_{\lambda_n(t)x}^W\left(A\right)\right\|$, we see from the uniform continuity of W on $\left[0,1\right]^k$ and the definition of g_n^A that $\lim_{n\to\infty} g_n^A = 0$. Moreover, if we take $\epsilon_n^A = 3n^{-1} + 3\sup\left\{d\left(q_{\cdot x}^{Z_l,\delta_l}\left(A\right), q_{\cdot x}^W\left(A\right)\right) + g_l^A : l = n, n+1, \ldots\right\}$, then $\{\epsilon_n^A\}$ is a monotonically decreasing sequence of positive numbers with limit zero.

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From the definition of ϵ_n we have $d\left(q_{\cdot x}^{Z_n,\delta_n}(A), q_{\cdot x}^W(A)\right) < \epsilon_n/2$ and $g_n^A < \epsilon_n/2$ for $n = 1, 2, \ldots$ Consequently, we have

$$\left\|q_{tx}^{Z_n,\delta_n}(A) - q_{tx}^W(A)\right\| \leq \left\|q_{tx}^{Z_n,\delta_n}(A) - q_{\lambda_n(t)x}^W(A)\right\| + \left\|q_{\lambda_n(t)x}^W(A) - q_{tx}^W(A)\right\|$$
$$< \epsilon_n^A$$

for all $t \in [0,1]$ and $x \in A$.

We will show that $\mathbb{P}\left(W \in D[0,1]^k\text{-}D_s \mid M_W = i\right) = 1$ for $i \in \{1, \ldots, k-1\}$ so that the desired conclusion follows.

Suppose first that $M_W = 1$. Let's prove that $T_{Z_n,\delta_n}^1(P) \to T_W^1(P)$ as $n \to \infty$. Since $M_W = 1$, then $\max_{x \in A} q_{T_W^1 x}^W(A) = P$ almost surely because W is continuous almost surely. Let $q_x^* = \min_{t \in [0,T_W^1]} q_{tx}^W$ (the minimum exists because q_{tx}^W is continuous) and let $q^* = \min_x q_x^*$. Let N such that if $n > N$ then $\epsilon_n^A < q^* - c$ (note that $q^* - c > 0$ because $M_W = 1$ and N is a random variable). If $n > N$ and $t \leq T_W^1(P)$ then

$$q_{tx}^W < q_{tx}^{Z_n,\delta_n} + q^* - c$$
$$\Rightarrow c < q_{tx}^{Z_n,\delta_n} + q^* - q_{tx}^W \leq q_{tx}^{Z_n,\delta_n}.$$

Suppose $T_W^1(P) > \epsilon > 0$, and let $q_x^+ = \max_{t \in [0,T_W^1-\epsilon]} q_{tx}^W$ and $q^+ = \max_x q_x^*$. Let N_2^* such that if $n > N_2^*$, then $\delta_n < r < P - q^+$ where $r \in \mathbb{Q}$. Let N_2^{**} such that if $n > N_2^{**}$, then $\epsilon_n^A < P - q^+ - r$. Thus, if $n > N_2 := \max\{N_2^*, N_2^{**}\}$ and $t \leq T_W^1(P) - \epsilon$, then

$$q_{tx}^{Z_n,\delta_n}(A) < q_{tx}^W(A) + \epsilon_n^A$$
$$< q_{tx}^W(A) + P - q^+ - r$$
$$\leq P - r \leq P - \delta_n$$

and so $T_{Z_n,\delta_n}^1(P) > T_W^1(P) - \epsilon$ if $n > N_3 := \max\{N_1, N_2\}$.

Case 1. $T_W^1(P) = 1$, so $\left|T_W^1(P) - T_{Z_n,\delta_n}^1(P)\right| < \epsilon$ if $n > N_3$ by the previous equation.

If $\epsilon_2 \geq T_W^1(P) = 1$, then $\left|T_W^1(P) - T_{Z_n,\delta_n}^1(P)\right| < \epsilon_2$ if $n > N_3$.

Case 2. $T_W^1(P) = 0$. Note that $q_{0x}^W(A) = \frac{1}{k} < P$ and so this case is not possible.

Case 3. $0 < T_W^1(P) < 1$.

Note that $T_{Z_n,\delta_n}^1(P) \leq T_{Z_n,\delta_n}^1(P + \delta_n) \leq T_W^1(P + \delta_n + \epsilon_n)$ because if $t < T_{Z_n,\delta_n}^1(P + \delta_n)$

$$q_{tx}^W(A) - \epsilon_n < q_{tx}^{Z_n,\delta_n}(A) < P + \delta_n.$$

Furthermore,

$$T_W^1(P) \leq \liminf_n T_W^1(P + \delta_n + \epsilon_n)$$
$$\leq \limsup_n T_W^1(P + \delta_n + \epsilon_n).$$

Now, let $x = \arg\max q_{T_W^1 x}^W$. By (2), for any $\varepsilon \in (0,1)$, there exists a monotonically decreasing sequence $\{s_k : k = 1, 2, \ldots\} \subset \left(T_W^1(P), 1\right)$ such that $\lim_n s_n = T_W^1(P)$; and for $k = 1, 2, \ldots$ we have

$$W_x\left(s_k\right) - W_x\left(T_W^1\left(P\right)\right) \; > v_k \equiv \;\; (\varepsilon)\sqrt{2\left(s_k - T_W^1\left(P\right)\right)\ln\ln\left(1/\left(s_k - T_W^1\left(P\right)\right)\right)}.$$

Notice that $v_k > 0$ and $\lim_k v_k = 0$. Let $\alpha > 0$. Let δ such that if $\delta_3 < \delta$, then $\alpha > -\log\left(1-\delta_3\right)$. Pick $\delta_3 < \delta$. Let K such that if $k > K$, $\dfrac{\sum_{x'\in A}\exp\left(\Delta\frac{W_{x'}\left(T_W^1\right)}{\lambda_z}+\frac{1}{\lambda_z^2}\Delta^2 T_W^1 I_{\left\{x'=k\right\}}\right)}{\sum_{x'\in A}\exp\left(\Delta\frac{W_{x'}\left(s_k\right)}{\lambda_z}+\frac{1}{\lambda_z^2}\Delta^2 s_k I_{\left\{x'=k\right\}}\right)} > 1-\delta_3$. Pick $k > K$. Let N_4 such that if $n > N_4$, then $\frac{\epsilon_n}{P} < v_k\frac{\Delta}{\lambda_z}+\alpha$ and $\frac{\delta_n}{P} < \left(v_k\frac{\Delta}{\lambda_z}+\alpha\right)^2/2$. Observe that

$$
\begin{aligned}
q_{s_k x}^{W} &= \exp\left(\Delta\frac{W_x\left(s_k\right)}{\lambda_z}+\frac{1}{\lambda_z^2}\Delta^2 s_k I_{\left\{x=k\right\}}\right) \Big/ \sum_{x'\in A}\exp\left(\Delta\frac{W_{x'}\left(s_k\right)}{\lambda_z}+\frac{1}{\lambda_z^2}\Delta^2 s_k I_{\left\{x'=k\right\}}\right) \\[2mm]
&> \frac{\exp\left(\Delta\frac{W_x\left(T_W^1\right)+v_k}{\lambda_z}+\frac{1}{\lambda_z^2}\Delta^2 T_W^1 I_{\left\{x=k\right\}}\right)}{\sum_{x'\in A}\exp\left(\Delta\frac{W_{x'}\left(T_W^1\right)}{\lambda_z}+\frac{1}{\lambda_z^2}\Delta^2 T_W^1 I_{\left\{x'=k\right\}}\right)} \times \frac{\sum_{x'\in A}\exp\left(\Delta\frac{W_{x'}\left(T_W^1\right)}{\lambda_z}+\frac{1}{\lambda_z^2}\Delta^2 T_W^1 I_{\left\{x'=k\right\}}\right)}{\sum_{x'\in A}\exp\left(\Delta\frac{W_{x'}\left(s_k\right)}{\lambda_z}+\frac{1}{\lambda_z^2}\Delta^2 s_k I_{\left\{x'=k\right\}}\right)} \\[2mm]
&\geq q_{T_W^1 x}^{W}\exp\left(v_k\frac{\Delta}{\lambda_z}\right)\left(1-\delta_3\right) \geq q_{T_W^1 x}^{W}\exp\left(v_k\frac{\Delta}{\lambda_z}\right)\exp\left(\alpha\right) \\[2mm]
&\geq P\left(1+v_k\frac{\Delta}{\lambda_z}+\alpha+\frac{\left(v_k\frac{\Delta}{\lambda_z}+\alpha\right)^2}{2}\right) > \left(P+\epsilon_n+\delta_n\right).
\end{aligned}
$$

Consequently,
$$s_k > T_W^1\left(P+\epsilon_n+\delta_n\right)$$

if $n > N_4$, and so
$$s_k \geq \limsup_n T_W^1\left(P+\delta_n+\epsilon_n\right).$$

Since $s_k \to T_W^1$, we must have that

$$
\begin{aligned}
T_W^1\left(P\right) &\geq \limsup_n T_W^1\left(P+\delta_n+\epsilon_n\right) \\
&\geq \liminf_n T_W^1\left(P+\delta_n+\epsilon_n\right) \\
&\geq T_W^1\left(P\right)
\end{aligned}
$$

Consequently,
$$T_W^1\left(P\right) = \lim_n T_W^1\left(P+\delta_n+\epsilon_n\right).$$

Now,
$$T_{Z_n,\delta_n}^1\left(P\right) \geq T_{Z_n,\delta_n}^1\left(P-\delta_n\right) \geq T_W^1\left(P-\delta_n-\epsilon_n\right)$$

because if $t < T_W^1\left(P-\delta_n-\epsilon_n\right)$

$$q_{tx}^{Z_n,\delta_n}\left(A\right) - \epsilon_n < q_{tx}^{W}\left(A\right) < P-\delta_n-\epsilon_n.$$

Similarly we can prove that
$$\lim_n T_W^1\left(P-\delta_n-\epsilon_n\right) = T_W^1\left(P\right).$$

Consequently,
$$T_W^1(P) = \lim_n T_{Z_n,\delta_n}^1(P).$$

Let $x = \arg\max_x q_{T_W^1,x}^W$. Let $\epsilon = > 0$. Let N such that if $n > N$, then $2\epsilon_n + 2\epsilon < -\max_{y \in A - \{x\}} q_{T_W^1(P),y}^W(A) + q_{T_W^1(P),x}^W(A)$

$$\left| q_{T_W^1(P),x}^W(A) - q_{T_{Z_n,\delta_n}^1(P),x}^{Z_n,\delta_n}(A) \right| < \epsilon + \epsilon_n$$

for all $x \in A$, and so if $z \in A - \{x\}$,

$$
\begin{aligned}
q_{T_{Z_n,\delta_n}^1(P),z}^{Z_n,\delta_n}(A) &< \epsilon + \epsilon_n + q_{T_W^1(P),z}^W(A) \\
&< -\epsilon - \epsilon_n + q_{T_W^1(P),x}^W(A) \\
&< q_{T_{Z_n,\delta_n}^1(P),x}^{Z_n,\delta_n}(A)
\end{aligned}
$$

and so $x = \arg\max_x q_{T_{Z_n,\delta_n}^1(P),x}^{Z_n,\delta_n}(A)$.

Now, we suppose that $M_W = 2$. Let's prove that $T_W^1(A) = \lim_n T_{Z_n,\delta_n}^1$. We know that $T_W^1(A) < 1$.
Case 1. $0 < T_W^1(P) < 1$. Like in the previos proof, we conclude that

$$T_W^1(P) = \lim_n T_{Z_n,\delta_n}^1(P).$$

Furthermore, if $x = \arg\min_{x \in A_1^W} q_{T_W^1,x}^W(A)$. Let N such that if $n > N$, then $2\epsilon_n + 2\epsilon < \min_{y \in A_1^W - \{x\}} q_{T_W^1,y}^W(A) - q_{T_W^1,x}^W(A)$ for some $\epsilon > 0$ and

$$\left| q_{T_{Z_n,\delta_n}z}^{Z_n,\delta_n} - q_{T_W^1,z}^W \right| < \epsilon$$

for all $z \in A - \{x\}$.
Then if $z \in A - \{x\}$,

$$
\begin{aligned}
q_{T_{Z_n,\delta_n}z}^{Z_n,\delta_n}(A) &> -\epsilon_n - \epsilon + q_{T_W^1,z}^W(A) > q_{T_W^1,x}^W(A) + \epsilon_n + \epsilon \\
&> q_{T_{Z_n,\delta_n}x}^{Z_n,\delta_n}(A)
\end{aligned}
$$

for n sufficiently large and so $x = \arg\min_{x \in A_n^Y} q_{T_{Z_n,\delta_n},x}^{Z_n,\delta_n}(A)$. Consequently, $A_1^{Z_n,\delta_n} = A_1^W$ for n sufficiently large. Furthermore,

Case 2. $T_W^1(P) = 0$, then $\min_{x \in A_n^Y} q_{0x}^W(A_n^W) = 1/k \le c$. Suppose $c > \frac{1}{k}$. Let N such that if $n > N$, then $\epsilon_n < c - \frac{1}{k}$. Thus if $n > N$,

$$q_{0x}^{Z_n,\delta_n} < \epsilon_n + \frac{1}{k} < c$$

and so $T_{Z_n,\delta_n}^1 = 0$. If $c = \frac{1}{k}, \dots \dots$ (suppose $c \ne 1/k$). We also can see that $A_1^{Z_n,\delta_n} = A_1^W$ for n sufficiently large.
Case 3. $T_W^1(P) = 1$. It's almost the same than the previous cases.

Now, let's prove that $T_W^2\left(P_1^W\right) = \lim_n T_{Z_n,\delta_n}^2\left(P_1^{Z_n,\delta_n}\right)$. By the above argument, we kno that there exists N such that if $n > N$, then $A_1^{Z_n,\delta_n} = A_1^W$ and

$$P_1^{Z_n,\delta_n} < P_1^W.$$

Now, suppose $T_W^2\left(P_1^W\right) - T_W^1\left(P\right) > \epsilon > 0$, and let $q_x^+ = \max_{t\in\left[T_W^1(P),T_W^2\left(P_1^W\right)-\epsilon\right]} q_{tx}^W$ and $q^+ = \max_x q_x^*$. Let N_2^* such that if $n > N_2^*$, then $\delta_n < r < P_1^W - q^+$ where $r \in \mathbb{Q}$. Let N_2^{**} such that if $n > N_2^{**}$, then $\epsilon_n^{A_1^W} < P_1^W - q^+ - r$. Thus, if $n > N_2 := \max\left\{N_2^*, N_2^{**}\right\}$ and $T_W^1\left(P\right) \le t \le T_W^2\left(P_1^W\right) - \epsilon$, then

$$
\begin{aligned}
q_{tx}^{Z_n,\delta_n}\left(A_1^W\right) &< q_{tx}^W\left(A_1^W\right) + \epsilon_n^{A_1^W} \\
&< q_{tx}^W\left(A_1^W\right) + P_1^W - q^+ - r \\
&\le P_1^W - r \le P_1^W - \delta_n
\end{aligned}
$$

and so $T_{Z_n,\delta_n}^2\left(P\right) > T_W^2\left(P\right) - \epsilon$ if $n > N_3 := \max\left\{N_1, N_2\right\}$.

Case 1. $T_W^2\left(P_1^W\right) = 1$, so $\left|T_W^2\left(P\right) - T_{Z_n,\delta_n}^1\left(P\right)\right| < \epsilon$ if $n > N_3$.

If $\epsilon_2 \ge T_W^1\left(P\right) = 1$, then $\left|T_W^1\left(P\right) - T_{Z_n,\delta_n}^1\left(P\right)\right| < \epsilon_2$ if $n > N_3$.

Case 2. $T_W^2\left(P_1^W\right) = 0 = T_W^1\left(P\right)$. This is impossible because $P_1^W \le \frac{1}{k} \le c$ and $P_1^W \ge P^* > \frac{1}{k}$.

Case 3. $0 < T_W^2\left(P_1^W\right) < 1$.

Note that $T_{Z_n,\delta_n}^2\left(P_1^{Z_n}\right) \le T_{Z_n,\delta_n}^2\left(P_1^{Z_n} + \delta_n\right) \le T_W^2\left(P_1^{Z_n} + \delta_n + \epsilon_n\right)$ because if $T_{Z_n,\delta_n}^1\left(P\right) \le t < T_{Z_n,\delta_n}^2\left(P_1^{Z_n} + \delta_n\right)$

$$q_{tx}^W\left(A_1^W\right) - \epsilon_n < q_{tx}^{Z_n,\delta_n}\left(A_1^W\right) < P_1^{Z_n} + \delta_n$$

Furthermore, for n sufficiently large and $\delta_2 > 0$ sufficiently small (THIS PARAGRAPH CAN BE REMOVED...)

$$
\begin{aligned}
T_W^2\left(P_1^W\right) - \delta_2 \le T_W^2\left(P_1^{Z_n}\right) &\le \liminf_n T_W^2\left(P_1^{Z_n} + \delta_n + \epsilon_n\right) \\
&\le \limsup_n T_W^2\left(P_1^{Z_n} + \delta_n + \epsilon_n\right)
\end{aligned}
$$

because $P_1^{Z_n} \to P_1^W$ and so for n sufficiently large

$$P_1^W < P_1^W - \max_{t\in\left[T_W^1, T_W^2\left(P_1^W\right)-\delta_2\right]} q_{tx}^W\left(A_1^W\right) + P_1^{Z_n}.$$

Now, let $x = \arg\max q_{T_W^2\left(P_1^W\right)x}^W$. By (2), for any $\varepsilon \in (0,1)$, there exists a monotonically decreasing sequence $\left\{s_k : k = 1,2,\ldots\right\} \subset \left(T_W^2\left(P_1^W\right), 1\right)$ such that $\lim_n s_n = T_W^2\left(P_1^W\right)$; and for $k = 1,2,\ldots$ we have

$$W_x\left(s_k\right) - W_x\left(T_W^2\left(P_1^W\right)\right) > v_k \equiv (\varepsilon)\sqrt{2\left(s_k - T_W^2\left(P_1^W\right)\right)\ln\ln\left(1/\left(s_k - T_W^2\left(P_1^W\right)\right)\right)}.$$

Notice that $v_k > 0$ and $\lim_k v_k = 0$. Let $\alpha > 0$. Let δ such that if $\delta_3 < \delta$, then $\alpha > -\log\left(1 - \delta_3\right)$. Pick $\delta_3 < \delta$. Let K such that if $k > K$, $\dfrac{\sum_{x'\in A_1^W}\exp\left(\Delta\frac{W_{x'}\left(T_W^2\left(P_1^W\right)\right)}{\lambda_z} + \frac{1}{\lambda_z^2}\Delta^2 T_W^2\left(P_1^W\right)I_{\left\{x'=k\right\}}\right)}{\sum_{x'\in A_1^W}\exp\left(\Delta\frac{W_{x'}\left(s_k\right)}{\lambda_z} + \frac{1}{\lambda_z^2}\Delta^2 s_k I_{\left\{x'=k\right\}}\right)} > 1 - \delta_3$. Pick $k > K$. Let

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N_4 such that if $n > N_4$, then $\frac{\epsilon_n}{P_1^W} < v_k \frac{\Delta}{\lambda_z} + \alpha$ and $\frac{\delta_n}{P_1^W} < \left(v_k \frac{\Delta}{\lambda_z} + \alpha\right)^2 /2$. Observe that

$$
\begin{aligned}
q_{s_k x}^W \left(A_1^W\right) &= \exp\left(\Delta \frac{W_x(s_k)}{\lambda_z} + \frac{1}{\lambda_z^2} \Delta^2 s_k I_{\{x=k\}}\right) \Big/ \sum_{x' \in A_1^W} \exp\left(\Delta \frac{W_{x'}(s_k)}{\lambda_z} + \frac{1}{\lambda_z^2} \Delta^2 s_k I_{\{x'=k\}}\right) \\[2mm]
&> \frac{\exp\left(\Delta \frac{W_k(T_W^2(P_1^W))+v_k}{\lambda_z} + \frac{1}{\lambda_z^2} \Delta^2 T_W^2\left(P_1^W\right) I_{\{x=k\}}\right)}{\sum_{x' \in A_1^W} \exp\left(\Delta \frac{W_{x'}(T_W^2(P_1^W))}{\lambda_z} + \frac{1}{\lambda_z^2} \Delta^2 T_W^2\left(P_1^W\right) I_{\{x'=k\}}\right)} \\[2mm]
&\quad \times \frac{\sum_{x' \in A_1^W} \exp\left(\Delta \frac{W_{x'}(T_W^2(P_1^W))}{\lambda_z} + \frac{1}{\lambda_z^2} \Delta^2 T_W^2\left(P_1^W\right) I_{\{x'=k\}}\right)}{\sum_{x' \in A_1^W} \exp\left(\Delta \frac{W_{x'}(s_k)}{\lambda_z} + \frac{1}{\lambda_z^2} \Delta^2 s_k I_{\{x'=k\}}\right)} \\[2mm]
&\geq q_{T_W^2(P_1^W)x}^W \exp\left(v_k \frac{\Delta}{\lambda_z}\right)(1 - \delta_3) \geq q_{T_W^1 x}^W \exp\left(v_k \frac{\Delta}{\lambda_z}\right)\exp(\alpha) \\[2mm]
&\geq P_1^W \left(1 + v_k \frac{\Delta}{\lambda_z} + \alpha + \frac{\left(v_k \frac{\Delta}{\lambda_z} + \alpha\right)^2}{2}\right) > P_1^W + \epsilon_n + \delta_n.
\end{aligned}
$$

Consequently,

$$
s_k > T_W^2\left(P_1^W + \epsilon_n + \delta_n\right)
$$

if $n > N_4$, and so

$$
s_k \geq \limsup_n T_W^2\left(P_1^W + \delta_n + \epsilon_n\right).
$$

Since $s_k \to T_W^1$, we must have that

$$
\begin{aligned}
T_W^2\left(P_1^W\right) &\geq \limsup_n T_W^2\left(P_1^W + \delta_n + \epsilon_n\right) \\
&\geq \liminf_n T_W^2\left(P_1^W + \delta_n + \epsilon_n\right) \\
&\geq T_W^2\left(P_1^W\right)
\end{aligned}
$$

Consequently,

$$
T_W^2\left(P_1^W\right) = \lim_n T_W^2\left(P_1^W + \delta_n + \epsilon_n\right).
$$

Now,

$$
T_{Z_n,\delta_n}^2\left(P_1^{Z_n}\right) \geq T_{Z_n,\delta_n}^2\left(P_1^{Z_n} - \delta_n\right) \geq T_W^2\left(P_1^{Z_n} - \delta_n - \epsilon_n\right)
$$

because if $t < T_W^2\left(P_1^{Z_n} - \delta_n - \epsilon_n\right)$

$$
q_{tx}^{Z_n,\delta_n}\left(A_1^W\right) - \epsilon_n < q_{tx}^W\left(A_1^W\right) < P_1^{Z_n} - \delta_n - \epsilon_n.
$$

Similarly we can prove that

$$
\lim_n T_W^2\left(P_1^W - \delta_n - \epsilon_n\right) = T_W^2\left(P_1^W\right).
$$

Consequently,

$$
T_W^2\left(P_1^W\right) = \lim_n T_{Z_n,\delta_n}^2\left(P_1^W\right).
$$

By a similar argument than before, we can see that $\arg\max_x q_{T_{Z_n,\delta_n}^2(P_1^W),x}^{Z_n,\delta_n}\left(A_1^W\right) = \arg\max_x q_{T_W^2(P_1^W),x}^W\left(A_1^W\right)$

for n sufficiently large.

The cases $M_W = i$ for $k - 1 \geq i \geq 3$ can be proved in a similar way.

Now, suppose that $M_Y = \infty$, then the algorithm ends at step R. The algorithm will select $x = \arg\max q_{R,x}^W \left(A_m^W \right)$ for some m. Using a similar argument than above, we can see that

$$\arg\max_x q_{R,x}^W \left(A_m^W \right) = \arg\max_x q_{R,x}^{Z_n,\delta_n} \left(A_m^W \right)$$

for n sufficiently large.

Thus, we conclude that

$$\mathbb{P}\left(W \in D[0,1]^k \text{-} D_s \right) = 1.$$

∎

By the extension of the CMT (Theorem 5.5 of Billingsley 1968), we have the following corollary.

Corollary 1. We have that if $t \in [0,1]$,

$$f\left(C\left(\delta, t\right), \delta \right) \Rightarrow g\left(W\left(t \right) \right)$$

as $\delta \to 0$.

Theorem 1. If samples from system $x \in \{1 \ldots, k\}$ are normally distributed and independent, over time and across alternatives, then $\lim_{\delta \to 0} Pr\{\text{BIZ selects } k\} \geq P*$ provided $\mu_k = \delta, \mu_{k-1} = \cdots = \mu_1 = 0$. We also suppose that the algorithm ends in at most $R\left(\delta\right) \in \mathbb{N}$ iterations, and $R(\delta) \to \infty$ as $\delta \to 0$ with probability 1. Furthermore, $\sqrt{R}\delta \to \Delta$ with probability 1 where $\infty > \Delta > 0$ with probability 1. We also suppose $B_1 = \cdots = B_k = 1$.

Proof.

$$
\begin{aligned}
\underline{lim}_{\delta \to 0} \mathbb{P}\left(CS \right) &\geq \underline{lim}_{\delta \to 0} \mathbb{P}\left(f\left(C\left(\delta, t\right), \delta \right) = 1 \right) \\
&= \underline{lim}_{\delta \to 0} E\left(\mathbb{P}\left(f\left(C\left(\delta, t\right), \delta \right) = 1 \mid \Delta \right) \right) \\
&= E\left(\underline{lim}_{\delta \to 0} \mathbb{P}\left(f\left(C\left(\delta, t\right), \delta \right) = 1 \mid \Delta \right) \right) \\
&= E\left(\mathbb{P}\left(g\left(W \right) = 1 \mid \Delta \right) \right) \\
&\geq P^*
\end{aligned}
$$

where the last inequallity follows from the paper of Frazier [].

Theorem 2. If samples from system $x \in \{1 \ldots, k\}$ are normally distributed and independent, over time and across alternatives, then $\lim_{\delta \to 0} \mathbb{P}\left(CS_\delta \right) \geq P*$ provided provided $\mu_k = \delta, \mu_{k-1} = \cdots = \mu_1 = 0$. We suppose $B_1 = \cdots = B_k = 1$.

Proof. Let τ be the maximum number of iterations used of the algorithm. Note that

$$
\begin{aligned}
\underline{lim}_{\delta \to 0} \mathbb{P}\left(CS_\delta\right) &= \underline{lim}_{\delta \to 0} \mathbb{P}\left(CS_\delta, \tau = \frac{1}{\delta^2}\right) \\
&= \underline{lim}_{\delta \to 0} \mathbb{P}\left(CS_\delta, \tau < \frac{1}{\delta^2}\right)
\end{aligned}
$$

because

$$
\begin{aligned}
\mathbb{P}\left(CS_\delta, \tau < \infty\right) - \mathbb{P}\left(CS_\delta, \tau < \frac{1}{\delta^2}\right) &= \mathbb{P}\left(CS_\delta, \infty > \tau \geq \frac{1}{\delta^2}\right) \\
&\leq \mathbb{P}\left(\infty > \tau \geq \frac{1}{\delta^2}\right)
\end{aligned}
$$

and so

$$
\begin{aligned}
\underline{lim}_{\delta \to 0}\left\{\mathbb{P}\left(CS_\delta, \tau < \infty\right) - \mathbb{P}\left(CS_\delta, \tau < \frac{1}{\delta^2}\right)\right\} &\leq \underline{lim}_{\delta \to 0} \mathbb{P}\left(\infty > \tau \geq \frac{1}{\delta^2}\right) \\
&= \mathbb{P}\left(\infty > \tau \geq \bigcap_{n=1}^{\infty} n^2\right) \\
&= 0
\end{aligned}
$$