Asymptotic Validity of the Bayes-Inspired Indifference Zone Procedure: the Non-Normal Known Variance Case

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Indifference-Zone Ranking and Selection

- Ranking and Selection is a problem where we have to select the best alternative among $\{\mu_1, \mu_2, \ldots, \mu_k\}$ based on iid samples.

- If a procedure chooses $\hat{x}$, the probability of correct selection is

$$\mathrm{PCS}\,(\mu) = \mathbb{P}_{\mu}\left(\hat{x} \in \arg\max_x \mu_x\right).$$

- The preference zone is $\mathrm{PZ}\,(\delta) = \left\{\mu \in \mathbb{R}^k : \mu_{[k]} - \mu_{[k-1]} \geq \delta\right\}$.

- A procedure meets the IZ guarantee at $P^* \in (1/k, 1)$ and $\delta > 0$ if

$$\mathrm{PCS}\,(\mu) \geq P^* \text{ for all } \mu \in \mathrm{PZ}\,(\delta).$$

Conservatism Causes the Number of Samples Taken to Be Larger than Necessary

- A user wants to meet the IZ guarantee and have $\mathbb{E}\left[\# \text{ of samples}\right]$ as small as possible.
- The conservatism can be defined as $\text{PCS}\left(\mu\right) - P^*$. If this number is large, then $\mathbb{E}\left[\text{sample size}\right]$ may be large.

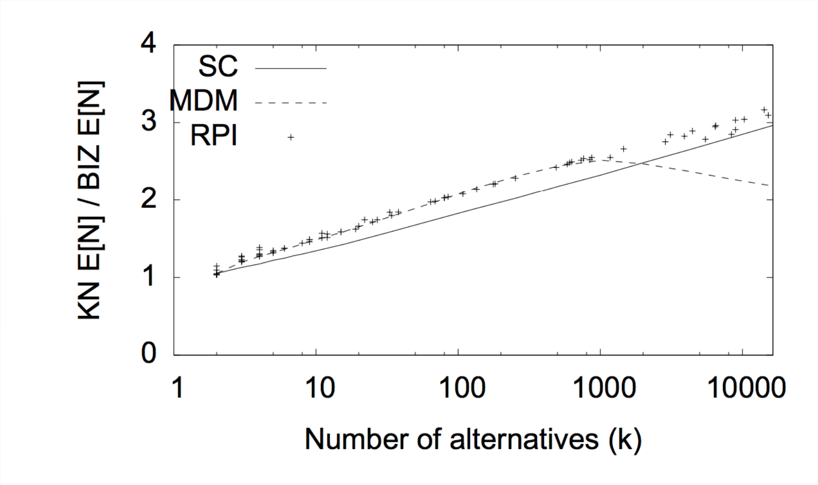
Sources of Conservatism

- The change from discrete time to continuous time often used to show IZ guarantees.
- The configuration under consideration is not a worst-case configuration.
- Bonferroni's inequality.

The Bayes-Inspired IZ (BIZ) Procedure

- BIZ is an elimination procedure that eliminates the use of Bonferroni's inequality, reducing conservatism.
- Its lower bound on worst-case probability of correct selection in the preference zone is tight in continuous time, and almost tight in the discrete time.
- The number of samples required by BIZ is significantly smaller than the KN procedure and the $\mathscr{P}_B^*$ procedure.

BIZ Requires Fewer Samples



The Bayes-Inspired IZ (BIZ) Procedure

- It satisfies the IZ guarantee under the following assumptions:
  - Normal samples.
  - Known variances.
  - Variances are either common across alternatives, or have an integer multiple structure.
- Based on empirical evidence, the BIZ procedure satisfies the IZ guarantee even when the last two assumptions are not true.
- Our contribution is to theoretically explain why it works. Specifically, we show asymptotic validity of the BIZ procedure.

Asymptotic Validity

Theorem

If samples are iid and the variances are finite and do not depend on δ, then

$$\inf_{a \in \mathrm{PZ}(1)} \lim_{\delta \to 0^+} \mathrm{PCS}(\delta) = P^*$$

where $\mu_k = a_k\delta, \ldots, \mu_1 = a_1\delta$.

The Bayes-Inspired IZ (BIZ) Procedure

The BIZ procedure can be viewed as the composition of three maps:

- The mapping from $(Y_{tx} : t \in \mathbb{N}, x \in \{1, \ldots, k\})$, where Y_{tx} is the sum of the first t samples from alternative x, onto $(Z_{tx} : t \in \mathbb{N}, x \in \{1, \ldots, k\})$ where $Z_{tx} = Y_{n_{tx},x}$ is the sum of samples from alternative x observed by stage t.

- The second maps the previous time-changed random walk through a non-linear mapping for each t, x and subset $A \subset \{1, \ldots, k\}$, to $\left(q'_{tx}(A) : t \in \mathbb{N}, A \subset \{1, \ldots, k\}, x \in A\right)$, where

$$q'_{tx}(A) = \exp\left(\delta\beta_t \frac{Z_{tx}}{n_{tx}}\right) \Big/ \sum_{x' \in A} \exp\left(\delta\beta_t \frac{Z_{tx'}}{n_{tx'}}\right)$$

The Bayes-Inspired IZ (BIZ) Procedure

- The third map, h, maps the paths of $\left(q'_{tx}(A) : t \in \mathbb{N}, A \subset \{1, \ldots, k\}, x \in A\right)$ onto selection decisions.
  - Finds first time τ_1 that $q'_{tx}(A_0) \geq P_0$ or $q'_{tx}(A_0) \leq c$. In the first case, $x_0 \in \arg\max_x q'_{tx}(A_0)$ is selected as the best. In the second case, $x_0 \in \arg\min_x q'_{tx}(A_0)$ is eliminated from A_0, resulting in new parameters A_1 and P_1.
  - This process is repeated until an alternative is selected as the best.

Proof Outline

- The same selection decision is obtained if we apply the map h to $\left(q_{tx}(A) : t \in \delta^2\mathbb{N}, A \subset \{1, \ldots, k\}, x \in A\right)$ where $q_{tx}(A) := q'\left(\left(Z_{\frac{t}{\delta^2}x} : x \in A\right), \delta, t\right)$.

- The discrete-time process is interpolated by the continuous-time process $(q_{tx}(A) : t \geq 0, A \subset \{1, \ldots, k\}, x \in A)$.

- The selection decision between the continuous-time process and the discrete-time process goes to zero as $\delta \to 0$.

- Applying the BIZ selection map h to the previous process, produces a selection decision that satisfies the indifference-zone guarantee as $\delta \to 0$.

Proof Outline

- The following centralized version of $Z_{\frac{t}{\delta^2}x}$

$$\mathscr{C}_x(\delta, t) := \frac{Y_{n_x(t),x} - t\lambda_x^2\mu_x}{\frac{\lambda_x^2}{\lambda_z}\delta}$$

  converges to a Brownian motion as $\delta \to 0$.

- We construct $f(\cdot, \delta)$ that takes as input the process $(\mathscr{C}_x(\delta, t) : x \in \{1, \ldots, k\}, t \in \mathbb{R})$, and returns 1 if the correction selection was made, and 0 otherwise.

Proof Outline

- f has a continuity property that causes $f(\mathscr{C}(\delta,\cdot),\delta) \Rightarrow g(W)$ where g is the selection decision from applying the BIZ procedure in continuous time.

- The BIZ procedure satisfies the IZ guarantee when applied in continuous time, and so $\mathbb{E}[g(W)] \geq P^*$ with equality for the worst configuration in the preference zone.

Numerical Experiments

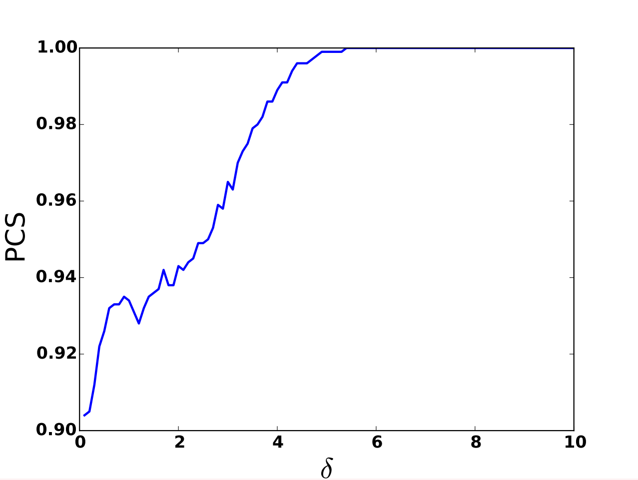


Figure: Known variance case. 100 alternatives and $P^* = 0.9$. PCS converges to P^* as δ goes to 0. Typical behavior, where the PCS is above P^* for all values of δ.

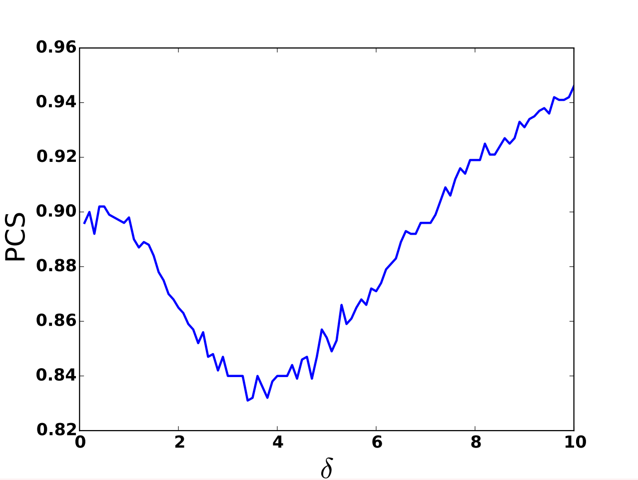


Figure: Unknown variance case. 100 alternatives and $P^* = 0.9$. PCS converges to P^* as δ goes to 0. Atypical behavior, n_0 is small and the variance of the best alternative is much larger than the other variances.

Numerical Experiments

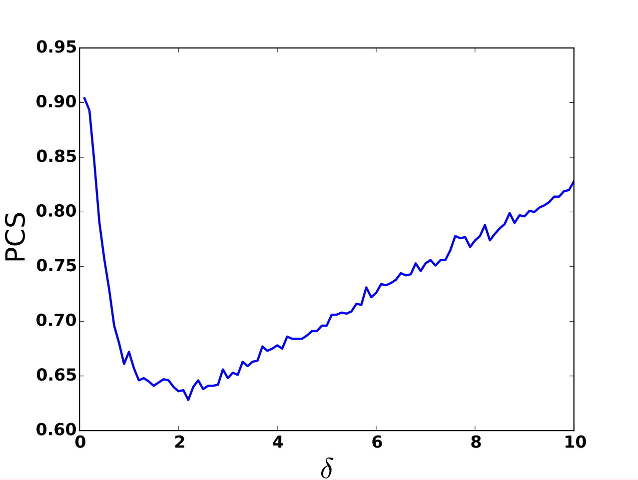


Figure: Known variance case. 100 alternatives and $P^* = 0.9$. PCS converges to P^* as δ goes to 0. Atypical behavior, n_0 is small and the variance of the best alternative is much larger than the other variances.

Conclusion

- BIZ is a fully sequential IZ procedure with elimination that eliminates one common source of conservatism: Bonferroni's inequality.
- We have proved the asymptotic validity of the Bayes-inspired Zone procedure when the variances are known. This implies that BIZ procedure satisfies the IZ guarantee when the means of the systems are very similar.
- Theoretical results require unrealistic assumptions on the sampling variances, but empirical results suggest that behavior is robust to violations of these assumptions in the problem regimes tested.

Future Work

- Asymptotic validity when the variances are unknown.
- Probability of good selection guarantee:

$$\forall \mu, \ \mathrm{PGS}(\mu) := \mathbb{P}(\mu_k - \mu_{\hat{x}} \leq \delta) \quad \geq \quad P^*$$

Thank you!!