Asymptotic Validity of a Fully Sequential Elimination Procedure for Indifference-Zone Ranking and Selection with Tight Bounds on Probability of Correct Selection

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We prove the validity of the sequential elimination IZ procedure proposed by Frazier [3] when δ goes to 0. Specifically, if k is the number of systems, we analyze Algorithm 2, when $B_1 = \cdots = B_k = 1$:

Algorithm 2: Discrete-time implementation of BIZ, for unknown and/or heterogeneous variances.

Require: $c \in [0, 1-(P^*)^{\frac{1}{k-1}}]$, $\delta > 0$, $P^* \in (1/k, 1)$, $n_0 \geq 0$ an integer, $B_1, \ldots, B_k$ strictly positive integers. Recommended choices are $c = 1-(P^*)^{\frac{1}{k-1}}$, $B_1 = \cdots = B_k = 1$ and n_0 between 10 and 30. If the sampling variances λ_x^2 are known, replace the estimators $\widehat{\lambda}_{tx}^2$ with the true values λ_x^2, and set $n_0 = 0$. We define $q_{tx}(A)$ where $A = \{1, \ldots, k\}$ as

$$q_{tx}(A) = \exp\left(\delta \frac{\sum_{x' \in A} n_{tx'}}{\sum_{x' \in A} \hat{\lambda}^2{}_{tx'}} \frac{Y_{n_{tx},x}}{n_{tx}}\right) \Big/ \sum_{x' \in A} \exp\left(\delta \frac{\sum_{x'' \in A} n_{tx''}}{\sum_{x'' \in A} \hat{\lambda}^2{}_{tx''}} \frac{Y_{n_{tx'},x'}}{n_{tx'}}\right)\right), \tag{1}$$

where $Y_{n_x(t),x}$ is the sum of the first $n_x(t)$ samples.

For each x, sample alternative x n_0 times and set $n_{0x} \leftarrow n_0$. Let W_{0x} and $\widehat{\lambda}_{0x}^2$ be the sample mean and sample variance respectively of these samples. Let $t \leftarrow 0$.
Let $A \leftarrow \{1, \ldots, k\}$, $P \leftarrow P^*$.
while $x \in \max_{x \in A} q_{tx}(A) < P$ **do**
  while $\min_{x \in A} q_{tx}(A) \leq c$ **do**
    Let $x \in \arg\min_{x \in A} q_{tx}(A)$.
    Let $P \leftarrow P/(1 - q_{tx}(A))$.
    Remove x from A.
  end while
  Let $z \in \arg\min_{x \in A} n_{tx}/\widehat{\lambda}_{tx}^2$.
  For each $x \in A$, let $n_{t+1,x} = \text{ceil}\left(\widehat{\lambda}_{tx}^2 (n_{tz} + B_z)/\widehat{\lambda}_{tz}^2\right)$.
  For each $x \in A$, if $n_{t+1,x} > n_{tx}$, take $n_{t+1,x} - n_{tx}$ additional samples from alternative x. Let $W_{t+1,x}$ and $\widehat{\lambda}_{t+1,x}^2$ be the sample mean and sample variance respectively of all samples from alternative x thus far.
  Increment t.
end while
Select $\hat{x} \in \arg\max_{x \in A} W_{tx}/n_{tx}$ as our estimate of the best.

1 Introduction

One common problem in simulation is that of choosing the best among several simulated systems. The problem of deciding how many samples to use from each alternative to support our selection as the best is the ranking and selection problem. An efficient solution to this problem has to balance between the time spent simulating and the quality of the selection.

This paper will consider the indifference-zone (IZ) formulation of the ranking and selection problem, in which we choose the best system with high probability, whenever the distance between the best system and the others is sufficiently large. The seminal work dates back to Bechhofer (1954), with early work compiled in the monograph Bechhofer et al. (1968). Since that time, papers in the are have been published (see, e.g., Paulson (1964), Fabian (1974), Rinott (1978), Hartmann (1988), Paulson (1994), Netson et al. (2001), Goldsman et al. (2002), Hong (2006), Andrad'ottir and Kim (2010)). This progress in the area has been summarized in Bechhofer et al. (1995), and more recent work has been done in Swisher et al. (2003), Kim and Nelson (2006, 2007), Frazier (2014).

The goal of an IZ algorithm is to take as few samples as possible while the IZ guarantee is satisfied. The first IZ procedures presented in Bechhofer (1954), Paulson (1964), Fabian (1974), Rinott (1978), Hartmann (1988, 1991), Paulson (1994) satisfy the IZ guarantee, but they usually take too many samples when there are many alternatives in part because their probability of correct selection (PCS) is much larger than the probability specified by the user. A reason of this is the use of the Bonferroni's inequality which leads to sample more than necessary. More recent algorithms in Kim and Nelson (2001), Goldsman et al. (2002), Hong (2006) improve the performance but they still use the Bonferroni's inequality, and so the methods are unefficient when there are several systems. Procedures in Kim and Dieker (2011), Dieker and Kim (2012) don't use the Bonferroni's inequality only when compare groups of three alternatives.

Since the classic IZ procedures require to take too many samples with many alternatives, these methods are unpopular when there are more than a few hundred alternatives. However, Frazier (2014) presented a new sequential elimination IZ procedure, called BIZ (Bayes-inspired Indifference Zone), whose lower bound on worst-case probability of correct selection in the preference zone is tight in continuous time, and almost tight in the discrete time. In numerical experiments, the number of samples required by BIZ is significantly smaller than that of the most popular IZ procedures, especially on problems with many alternatives. Unfortunately, the proofs of the theorical results for the discrete-time case assume that variances are known and have an integer multiple structure which is not very realistic. However, asymptotically we can use a central limit theorem that allows us to prove the asymptotic validity of the BIZ procedure for the discrete-time case. Kim et al. (2006) also proves the asymptotiv validity of a IZ procedure. They first prove the case when there are only two systems and then they need to use the Bonferroni's inequality and so their proof is shorter than our proof since we have to analyze all the times when an elimination occurs. However, their proof is still large since they have to use a very specific proposition of Kim et al. (2005).

This paper is organized as follows: In §2, we present the proof of the validity of the algorithm when the variances are known. In §3, we prove the case when the variacnes are unknown. In §4, we present some simulations showing the effectiveness of the algorithm.

To prove the case when the variances are known, we use a theorem for Ergodic processes that shows how to standardize the output data to make them behave like Brownian motion processes in the limit. We also use an extension of the Continuous Mapping Theorem (Theorem 5.5 of Billingsley 1968) to see that the algorithm behaves like a sequential elimination IZ procedure with a Brownian motion process instead of the standardized sum of the output data in the limit, and then we use the results of the paper of Frazier [3] to prove the validity of this algorithm in the limit. Finally, we use a random change of time argument to prove the case when the variances are unknown.

2 Asymptotic Validity when the Variances are Known

Without loss of generality suppose that the true means satisfy that $\mu_k > \mu_{k-1} \geq \cdots \geq \mu_1$. We suppose that samples from system $x \in \{1 \ldots, k\}$ are identically distributed and independent, over time and across alternatives. We also define $\lambda_z^2 := \max_{i \in \{1 \ldots, k\}} \lambda_i^2$. We suppose that $\min_{i \in \{1 \ldots, k\}} \lambda_i^2 > 0$. We are going to suppose that $\delta > 0$ and $\mu = \delta a$ for some $a \in \mathbb{R}^k$.

Now we are going to see that the the standardized sum of the output data converges to a Brownian motion in the sense of $D_\infty := D[0, \infty)$, which is the set of functions from $[0, \infty)$ to $\mathbb{R}$ that are right-continuous and have left-hand limits, with the Skorohod topology. The definition and the properties of this topology may be found in Chapter 3 of Billingsley 1999 and the appendix.

We are now going to explain the meaning of convergence of random paths in the sense of D_∞. Suppose that we have a sequence of random paths $(\mathcal{X}_n)_{n \geq 0}^{\infty}$ such that $\mathcal{X}_n : \mathcal{F} \to D_\infty$ where $(\Omega, \mathcal{F}, \mathbb{P})$ is our space of probability. We say that $\mathcal{X}_n \Rightarrow \mathcal{X}_0$ in the sense of D_∞ if $P_n \Rightarrow P_0$ where $P_n : \mathcal{D}_\infty \to [0, 1]$ are defined as $P_n [A] = \mathbb{P}\left[\mathcal{X}_n^{-1}(A)\right]$ for all $n \geq 0$ and $\mathcal{D}_\infty$ are the Borel subsets for the Skorohod topology.

The following lemma shows that the the standardized sum of the output data converges to a Brownian motion in the sense of D_∞.

Lemma 1. If $x \in \{1 \ldots, k\}$, then

$$\mathcal{C}_x(\delta, \cdot) := \frac{Y_{\mathrm{ceil}\left(\frac{\lambda_x^2}{\lambda_z^2}\left(n_0 + \cdot \frac{1}{\delta^2}\right)\right), x} - \frac{\lambda_x^2}{\lambda_z^2}\left(n_0 + \cdot \frac{1}{\delta^2}\right)\mu_x}{\frac{\lambda_x^2}{\lambda_z \delta}} \Rightarrow W_x(\cdot)$$

as $\delta \to 0$ in the sense of $D[0, \infty)$, where $Y_{n,x}$ is the sum of the first n samples and W_x is a standard Brownian motion.

Proof. By the Theorem 19.1 of Billingsley 1999,

$$\frac{Y_{\mathrm{floor}\left(\frac{\lambda_x^2}{\lambda_z^2}\left(\cdot \frac{1}{\delta^2}\right)\right), x} - \mathrm{floor}\left(\frac{\lambda_x^2}{\lambda_z^2}\left(\cdot \frac{1}{\delta^2}\right)\right)\mu_x}{\frac{\lambda_x^2}{\lambda_z}\sqrt{\frac{1}{\delta^2}}} \Rightarrow W_x(\cdot)$$

in the sense of $D[0, \infty)$.

Fix $w \in \Omega$. Observe that

$$\frac{Y_{\mathrm{floor}\left(\frac{\lambda_x^2}{\lambda_z^2}\left(\cdot \frac{1}{\delta^2}\right)\right), x} - \mathrm{floor}\left(\frac{\lambda_x^2}{\lambda_z^2}\left(\cdot \frac{1}{\delta^2}\right)\right)\mu_x}{\frac{\lambda_x^2}{\lambda_z}\sqrt{\frac{1}{\delta^2}}} - \frac{Y_{\mathrm{ceil}\left(\frac{\lambda_x^2}{\lambda_z^2}\left(\cdot \frac{1}{\delta^2}\right)\right), x} - \mathrm{ceil}\left(\frac{\lambda_x^2}{\lambda_z^2}\left(\cdot \frac{1}{\delta^2}\right)\right)\mu_x}{\frac{\lambda_x^2}{\lambda_z}\sqrt{\frac{1}{\delta^2}}} \to 0$$

uniformly in $[0,s]$ for all $s \geq 0$ and then by Theorem A.2

$$\frac{Y_{\text{ceil}\left(\frac{\lambda_x^2}{\lambda_z^2}\left(\cdot\frac{1}{\delta^2}\right)\right),x} - \text{ceil}\left(\frac{\lambda_x^2}{\lambda_z^2}\left(\cdot\frac{1}{\delta^2}\right)\right)\mu_x}{\frac{\lambda_x^2}{\lambda_z}\sqrt{\frac{1}{\delta^2}}} \Rightarrow W_x(\cdot)$$

in the sense of $D[0,\infty)$.

Since $\frac{\frac{\lambda_x^2}{\lambda_z^2}t\frac{1}{\delta^2} - ceil\left(\frac{\lambda_x^2}{\lambda_z^2}t\frac{1}{\delta^2}\right)}{\frac{\lambda_x^2}{\lambda_z}\sqrt{\frac{1}{\delta^2}}} \to 0$ uniformly on $[0,s]$ for every $s \geq 0$, then by Theorem A.2

$$\frac{Y_{\text{ceil}\left(\frac{\lambda_x^2}{\lambda_z^2}\left(\cdot\frac{1}{\delta^2}\right)\right),x} - \left(\frac{\lambda_x^2}{\lambda_z^2}\left(\cdot\frac{1}{\delta^2}\right)\right)\mu_x}{\frac{\lambda_x^2}{\lambda_z}\sqrt{\frac{1}{\delta^2}}} \Rightarrow W_x(\cdot).$$

Finally, observe that for fixed $\omega \in \Omega$,

$$\begin{aligned}
&\frac{Y_{\text{ceil}\left(\frac{\lambda_x^2}{\lambda_z^2}\left(\cdot\frac{1}{\delta^2}\right)\right),x} - \left(\frac{\lambda_x^2}{\lambda_z^2}\left(\cdot\frac{1}{\delta^2}\right)\right)\mu_x}{\frac{\lambda_x^2}{\lambda_z}\sqrt{\frac{1}{\delta^2}}} - \frac{Y_{\text{ceil}\left(\frac{\lambda_x^2}{\lambda_z^2}\left(\cdot\frac{1}{\delta^2}\right)+n_0\frac{\lambda_x^2}{\lambda_z^2}\right),x} - \left(n_0\frac{\lambda_x^2}{\lambda_z^2} + \frac{\lambda_x^2}{\lambda_z^2}\left(\cdot\frac{1}{\delta^2}\right)\right)\mu_x}{\frac{\lambda_x^2}{\lambda_z}\sqrt{\frac{1}{\delta^2}}} \\
= &\frac{Y_{\text{ceil}\left(\frac{\lambda_x^2}{\lambda_z^2}\left(\cdot\frac{1}{\delta^2}\right)\right),x} - Y_{\text{ceil}\left(\frac{\lambda_x^2}{\lambda_z^2}\left(\cdot\frac{1}{\delta^2}\right)+n_0\frac{\lambda_x^2}{\lambda_z^2}\right),x} + \left(n_0\frac{\lambda_x^2}{\lambda_z^2}\right)\mu_x}{\frac{\lambda_x^2}{\lambda_z}\sqrt{\frac{1}{\delta^2}}} \\
\to\ &0
\end{aligned}$$

uniformly in $[0,t]$ for all $t \geq 0$, and so by Theorem A.2 the result follows.

∎

Now, we use the topology product in $D^k[0,\infty)$ for $k \in \mathbb{N}$. This topology may be described as the one under which $\left(Z_n^1,\ldots,Z_n^k\right) \to \left(Z_0^1,\ldots,Z_0^k\right)$ if and only if $Z_n^i \to Z_0^i$ for all $i \in \{1,\ldots,k\}$. See the Miscellany of Billingsley 1968. The following corollary follows from the previous result and independence.

Corollary 1. We have that

$$\mathcal{C}(\delta,\cdot) := \left(\mathcal{C}_x(\delta,\cdot)\right)_{x\in A} \Rightarrow W(\cdot) := \left(W_x(\cdot)\right)_{x\in A}$$

as $\delta \to 0$ in the sense of D_∞^k.

Now we are going to define new algorithms that are almost the same as the continuous-time procedure proposed by Frazier, but these algorithms use new functions $q_{tx}^{Y,\delta}(A)$ which depend on δ and a function C that is in $D[0,\infty)^k$. More explicity, if we define $\mathbb{P}_{\mu,\lambda}$ as the probability measure under which samples from system x have mean μ_x and variance λ_x^2, and

$$q_{tx}^{'}(A) = \exp\left(\frac{\delta}{\lambda_x^2}Y_{n_x(t),x}^{'}\right) \Big/ \sum_{x'\in A}\exp\left(\frac{\delta}{\lambda_{x'}^2}Y_{n_x'(t),x'}^{'}\right)$$

where $Y_{tx}^{'}$ is a Brownian motion under $\mathbb{P}_{\mu,\lambda}$, with drift μ_x, volatility λ_x, and independence across x, the procedure proposed by Frazier is defined by first setting

$$\tau_0 = 0, P_0 = P^*, A_0 = \{1, \ldots, k\}$$

then defining recursively, for $n = 0, 1, \ldots, k-2$,

$$\begin{aligned}
\tau_{n+1} &= \inf\left\{t \geq \tau_n : \ \min_{x\in A_n} q^{'}_{tx}(A_n) \leq c \text{ or } \max_{x\in A_n} q^{'}_{tx}(A_n) \geq P_n\right\} \\
Z_{n+1} &\in \arg\min_{x\in A_n} q^{'}_{\tau_{n+1},x}(A_n) \\
A_{n+1} &= A_n \setminus \{Z_{n+1}\} \\
P_{n+1} &= P_n \Big/ \left(1 - \min_{x\in A_n} q^{'}_{\tau_{n+1},x}(A_n)\right)
\end{aligned}$$

and finally letting the selected alternative $\hat{x}$ be the single entry in A_{k-1}. We also define

$$M = \inf\left\{n = 1, \ldots, k-1 : \max_{x\in A_n} q^{'}_{\tau_n,x}(A_{n-1}) \geq P_{n-1}\right\}. \tag{2}$$

If $C \in D[0,\infty)^k$, we define new functions $q^{C,\delta}_{tx}(A)$ by

$$\begin{aligned}
q^{C,\delta}_{tx}(A) &= \exp\left(\frac{\lambda_x^2 C_x(t)\beta_\delta(t)}{\lambda_z \mathrm{ceil}\left(\frac{\lambda_x^2}{\lambda_z^2}\left(n_0 + \frac{t}{\delta^2}\right)\right)} + \delta^2\lambda_x^2 \frac{\left(n_0 + t\frac{1}{\delta^2}\right)\beta_\delta(t)}{\lambda_z^2 \mathrm{ceil}\left(\frac{\lambda_x^2}{\lambda_z^2}\left(n_0 + \frac{t}{\delta^2}\right)\right)} a_x\right) \\
&\quad \times \left[\sum_{x'\in A} \exp\left(\frac{\lambda_{x'}^2 C_{x'}(t)\beta_\delta(t)}{\lambda_z \mathrm{ceil}\left(\frac{\lambda_x^2}{\lambda_z^2}\left(n_0 + \frac{t}{\delta^2}\right)\right)} + \delta^2\lambda_{x'}^2 \frac{\left(n_0 + t\frac{1}{\delta^2}\right)\beta_\delta(t)}{\lambda_z^2 \mathrm{ceil}\left(\frac{\lambda_{x'}^2}{\lambda_z^2}\left(n_0 + \frac{t}{\delta^2}\right)\right)} a_x\right)\right]^{-1}
\end{aligned} \tag{3}$$

where $\beta_\delta(t) = \sum_{x'\in A} \mathrm{ceil}\left(\frac{\lambda_{x'}^2}{\lambda_z^2}\left(n_0 + \frac{t}{\delta^2}\right)\right) / \sum_{x'\in A} \lambda_{x'}^2$.

The reason of this definition is because $\mathcal{C}(\delta,\cdot) = (\mathcal{C}_x(\delta,\cdot))_{x\in A} \in D[0,\infty)^k$ and

$$\frac{\delta}{n_{\cdot x}}\beta_\delta(\cdot) Y_{n_{\cdot x},x} = \frac{\lambda_x^2 \beta_\delta(\cdot)\mathcal{C}_x(\delta,\cdot)}{\lambda_z \mathrm{ceil}\left(\frac{\lambda_x^2}{\lambda_z^2}\left(n_0 + \frac{t}{\delta^2}\right)\right)} + \delta^2\lambda_x^2\beta_\delta(\cdot)\frac{\left(n_0 + \cdot\frac{1}{\delta^2}\right)}{\lambda_z^2 \mathrm{ceil}\left(\frac{\lambda_x^2}{\lambda_z^2}\left(n_0 + \frac{t}{\delta^2}\right)\right)} a_x \Rightarrow W^{'}_x(\cdot) := \frac{W_x(\cdot)}{\lambda_z} + \frac{\cdot}{\lambda_z^2} a_x$$

in the sense of D_∞ where $W^{'}_x$ is a Brownian motion starting from 0, with drift $\frac{a_x}{\lambda_z^2}$ and volatility $\frac{1}{\lambda_z^2}$ for all $x \in A$.

Note that the functions $q^{C,\delta}_{tx}(A)$ define a new algorithm for each $C \in D[0,\infty)^k$ and $\delta > 0$.

We also define a new function

$$f(C,\delta) = \begin{cases} 1 & \text{if } k \text{ is chosen} \\ 0 & \text{otherwise} \end{cases}$$

using the algorithm that (C,δ) induces.

Now, we also define new functions $q^{C}_{tx}(A)$ to analyze the limit of $f(C,\delta)$ when δ goes to zero:

$$q^{C}_{tx}(A) := \exp\left(\frac{C_x(t)}{\lambda_z} + \frac{1}{\lambda_z^2} t a_x\right) / \sum_{x'\in A} \exp\left(\frac{C_{x'}(t)}{\lambda_z} + \frac{1}{\lambda_z^2} t a_{x'}\right)$$

and we define the functions

$$g(Y) = \begin{cases} 1 & \text{if } k \text{ is chosen} \\ 0 & \text{otherwise} \end{cases} .$$

We want to prove that

$$f(\mathcal{C}(\delta,\cdot),\delta) \Rightarrow g(W)$$

as $\delta \to 0$ in distribution.

In order to prove this, we will prove the Lemma 2 which will allow us to use the Theorem 5.5 of Billingsley 1968 that implies the desired result.

Lemma 2. Let $\{\delta_n\} \subset (0,\infty)$ such that $\delta_n \to 0$. If $D_s \equiv \{Z \in D[0,\infty)^k$: if $\{Z_n\} \subset D[0,\infty)^k$ and $\lim_n d_\infty(Z_n, Z) = 0$, then the sequence $\{f(Z_n,\delta_n)\}$ converges to $\{g(Z)\}\}$, then $\mathbb{P}(W \in D_s) = 1$.

First, we are going to prove the following five results.

Proposition 1. Suppose $\{f_n\}$ and $\{g_n\}$ are two sequences of functions on D_∞ such that $f_n \to f$ and $g_n \to g$ in D_∞. If f and g are continuous, then

$$\min(f_n, g_n) \to \min(f,g)$$

in D_∞.

Proof. Let $t^* > 0$. We will prove that $\min(f_n,g_n) \to \min(f,g)$ in D_{t^*} and the theorem will follow from Theorem 16.2 of Billingsley 1999.

Since f and g are uniformly continuous in $[0,t^*]$ and $f_n \to f$ and $g_n \to g$ in $[0,t^*]$, then f_n and g_n converge uniformly to f and g, respectively. Consequently, $(f_n,g_n) \to (f,g)$ uniformly in $[0,t^*]$. Let $a_f^+ = \max_{t\in[0,t^*]}|f(t)|$, $a_f^- = -a_f^+$, $a_g^+ = \max_{t\in[0,t^*]}|g(t)|$ and $a_g^- = -a_g^+$. Let N such that if $n \geq N$, implies $|f_n(t) - f(t)| < 1$ and $|g_n(t) - g(t)| < 1$ for all $t \in [0,t^*]$. Consequently, if $n \geq N$, $f_n(t) \in \left[a_f^- - 1, a_f^+ + 1\right]$ and $g_n(t) \in \left[a_g^- - 1, a_g^+ + 1\right]$ for all t in $[0,t^*]$. Since $\min(x,y)$ is continuous, then it is uniformly continuous in $A = \left[a_f^- - 1, a_f^+ + 1\right] \times \left[a_g^- - 1, a_g^+ + 1\right]$.

Let $\epsilon > 0$, then there exists $\delta > 0$ such that if $\|(u,v)\|_2 < \delta$ and $(u := (u_1,u_2), v := (v_1,v_2)) \in A$, then

$$|\min(u_1,u_2) - \min(v_1,v_2)| < \epsilon. \tag{4}$$

Let $M > N$ such that if $n > M$, then

$$\|(f_n(t), g_n(t)) - (f(t), g(t))\|_2 < \delta$$

for all $t \in [0,t^*]$. Consequently, if $n > M$, by (4),

$$|\min(f_n(t), g_n(t)) - \min(f(t), g(t))| < \epsilon$$

for all $t \in [0, t^*]$. Since uniform convergence implies Skorohod convergente, then $\min(f_n, g_n) \to \min(f, g)$ in D_{t^*}, and the result follows.

■

Proposition 2. Suppose $\{f_n\}$ is a sequence of functions in D_∞ such that $f_n \to f$ in D_∞ , f is continuous, and $\{T_n\} \subset [0, \infty)$ is a sequence such that $T_n \to T$. We define $T(a) := \inf\{t \geq T : f(t) \geq a\}$ for each $a \in \mathbb{R}$. Suppose $T(0) \in \mathbb{R}$. Furthermore, we suppose that there exists $\{\epsilon_n\} \subset (0, \infty)$ such that $\epsilon_n \to 0$, $\epsilon_n \geq \epsilon_{n+1}$, and

$$\|f_n(t) - f(t)\|_2 < \epsilon_n \tag{5}$$

for all $t \in [0, T(0)]$. We also suppose that $\lim \sup_n T(\epsilon_n) \leq T(0)$. Thus we have that

$$\inf\{t \geq T_n : f_n(t) \geq 0\} \to \inf\{t \geq T : f(t) \geq 0\}$$

if $T(0) > T$ or $T_n = T$.

Proof. We are going to suppose $T(0) > T$, and the case $T_n = T$ can be proved using almost the same ideas. We introduce the notation $T_n(a) : \inf\{t \geq T_n : f_n(t) \geq a\}$ for $a \in \mathbb{R}$. Since $T(0) > T$, we can take N such that if $n > N$, then $T_n < T + T(0) - T = T(0)$ and $\epsilon > \epsilon_n$ where $\epsilon := sup_n \epsilon_n$. Let $n > N$. Note that $T_n(0) \leq T_n(\epsilon_n)$.

We also have that

$$\begin{aligned} T(0) &\leq \lim \inf_n T(\epsilon_n) \\ &\leq \lim \sup_n T(\epsilon_n) \end{aligned} \tag{6}$$

Now, since $\lim \sup_n T(\epsilon_n) \leq T(0)$,

$$\begin{aligned} T(0) &\geq \lim \sup_n T(\epsilon_n) \\ &\geq \lim \inf_n T(\epsilon_n) \\ &\geq T(0) \end{aligned}$$

and so

$$\lim \inf_n T_n(0) \leq \lim \sup_n T_n(0) \leq \lim \sup_n T_n(\epsilon_n) = \lim_n T(\epsilon_n) = T(0)$$

Now, let's prove that $\lim \inf_n T_n(0) \geq T(0)$. Since $T(0) > T$, there exists M such that $T(0) - \frac{1}{m} \geq T$ if $m > M$, and let $t_m = T(0) - \frac{1}{m}$ and $\alpha_m = \max\{f(t) : t \in [T, t_m]\}$. Note that $\alpha_m < 0$ because $t_m < T(0)$. Since $\epsilon_n \to 0$, there exists N such that if $n > N$, then

$$\epsilon_n \leq -\alpha_m$$

Thus, since $\|f_n(t) - f(t)\|_2 < \epsilon_n$ if $t \in [T, t_m]$, then

$$f_n(t) < f(t) + \epsilon_n \leq f(t) - \alpha_m \leq 0 \tag{7}$$

If $T \leq T_n(0)$ for all $n > N$, then $T_n(0) \geq t_m$ by (7) and so $\lim \inf_n T_n(0) \geq t_m$. Taking the limit $m \to \infty$, we conclude that $\lim \inf_n T_n(0) \geq T(0)$.

So, we only need to prove that $T \leq T_n(0)$ for n large. We proceed by contradiction. Let $s > 0$ be any number. Take N such that if $n > N$, then $\epsilon_n < s$. Since we are supposing that for all N' there exists $n > N'$ such that $T > T_n(0)$, and $T_n(0) \geq T_n$, $T_n \to T$, then we have that for every $\epsilon > 0$ there exists n large such that $T - \epsilon < T_n(0) < T$. By (5),

$$\begin{aligned} f(T_n(0)) &\geq f_n(T_n(0)) - \epsilon_n \\ &\geq -\epsilon_n > -s \end{aligned}$$

Let $a > 0$, since f is continuous there exists $\epsilon > 0$ such that $f(x) < f(T) + a$ if $|x - T| < \epsilon$. Thus, there exists $n > N$ such that $T - \epsilon < T_n(0) < T$ and so

$$-s < f(T_n(0)) < f(T) + a$$

Since s and a are arbitrary,

$$f(T) \geq 0$$

which is a contradiction because $T(0) > T$. Consequently, $T \leq T_n(0)$ and so

$$\lim_n T_n(0) = T(0).$$

$\blacksquare$

Proposition 3. Let $\mathcal{S}$ be the event such that W is continuous and $\mathbb{P}[\mathcal{S}] = 1$. Fix $\omega \in \mathcal{S}$ and let $\{Z_n\} \subset D[0,\infty)^k$ be a sequence of functions such that $Z_n \to W$ in $D^k[0,\infty)$, then

$$\begin{aligned} f_{Z_n}(\cdot) : &= \max\left\{c - \min_{x \in A} q_{\cdot x}^{Z_n,\delta_n}(A), \max_{x \in A} q_{\cdot x}^{Z_n,\delta_n}(A) - P\right\} \\ &\to f_W(\cdot) := \max\left\{c - \min_{x \in A} q_{\cdot x}^{W}(A), \max_{x \in A} q_{\cdot x}^{W}(A) - P\right\} \end{aligned}$$

in $D[0,\infty)$ for any $P \in (0,1)$ and $A \subset \{1,\ldots,k\}$. Furthermore, if $m \in \mathbb{N}$, there exists a sequence $\{\epsilon_k\}$ such that $\epsilon_k \downarrow 0$ and

$$|f_{Z_k}(t) - f_W(t)| < \epsilon_k$$

for all $t \in [0,m]$.

Proof.

Since $Z_n \to W$ in $D[0,\infty)$, by Theorem 16.2 of Billingsley 1999, for each $s \geq 0$ there exist functions λ_s^n in Λ_s such that

$$\lim_n Z_n(\lambda_s^n t) = W(t)$$

uniformly in t and

$$\lim_n \lambda_s^n t = t$$

uniformly in t. Then

$$\lim_n \frac{\lambda_x^2 \beta_\delta\left(\lambda_s^n t\right)}{\lambda_z \text{ceil}\left(\frac{\lambda_x^2}{\lambda_z^2}\left(n_0+\frac{\lambda_s^n(t)}{\delta^2}\right)\right)} Z_n^x\left(\lambda_s^n t\right)+\delta_n^2 \lambda_x^2 \frac{\left(n_0+\lambda_s^n(t) \frac{1}{\delta_n^2}\right) \beta_\delta\left(\lambda_s^n(t)\right)}{\lambda_z^2 \text{ceil}\left(\frac{\lambda_x^2}{\lambda_z^2}\left(n_0+\frac{\lambda_s^n(t)}{\delta^2}\right)\right)} a_x=W_x(t) \frac{1}{\lambda_z}+\frac{t}{\lambda_z^2} a_x$$

uniformly in t, and so

$$\lim_n \exp \left(\frac{\lambda_x^2 \beta_\delta\left(\lambda_s^n t\right)}{\lambda_z \text{ceil}\left(\frac{\lambda_x^2}{\lambda_z^2}\left(n_0+\frac{\lambda_s^n(t)}{\delta^2}\right)\right)} Z_n^x\left(\lambda_s^n t\right)+\delta_n^2 \lambda_x^2 \frac{\left(n_0+\lambda_s^n(t) \frac{1}{\delta_n^2}\right) \beta_\delta\left(\lambda_s^n(t)\right)}{\lambda_z^2 \text{ceil}\left(\frac{\lambda_x^2}{\lambda_z^2}\left(n_0+\frac{\lambda_s^n(t)}{\delta^2}\right)\right)}\right)=\exp \left(W_x(t) \frac{1}{\lambda_z}+\frac{t}{\lambda_z^2} a_x\right)$$

uniformly in t since exp is uniformly continuous in $[0, s]$. Consequently,

$$q_{\lambda_s^n(t) x}^{Z_n, \delta_n}(A) \rightarrow q_{t x}^W(A)$$

uniformly in $t \in[0, s]$. Thus, $q_{\cdot x}^{Z_n, \delta_n}(A) \rightarrow q_{\cdot x}^W(A)$ in $D[0, s]$ for any set $A \subset\{1, \ldots, k\}$ and $s \geq 0$. Consequently, $q_{\cdot x}^{Z_n, \delta_n}(A) \rightarrow q_{\cdot x}^W(A)$ in $D[0, \infty)$ by Theorem 16.2 of Billingsley 1999. By Proposition 1, $\min_{x \in A} q_{\cdot x}^{Z_n, \delta_n}(A) \rightarrow \min_{x \in A} q_{\cdot x}^W(A)$ and $\max_{x \in A} q_{\cdot x}^{Z_n, \delta_n}(A) \rightarrow \max_{x \in A} q_{\cdot x}^W(A)$ in $D[0, \infty)$, and so by Proposition 1,

$$f_{Z_n}:=\max \left\{c-\min_{x \in A} q_{\cdot x}^{Z_n, \delta_n}(A), \max_{x \in A} q_{\cdot x}^{Z_n, \delta_n}(A)-P\right\} \rightarrow f_W:=\max \left\{c-\min_{x \in A} q_{\cdot x}^W(A), \max_{x \in A} q_{\cdot x}^W(A)-P\right\}$$

in $D[0, \infty)$ for any $P \in(0,1)$. Now, let's prove the second part. Fix $m \in \mathbb{N}$. By the definition of $d_m\left(f_{Z_n}, f_W\right)$, we have that there exists $\lambda_n \in \Lambda_{\infty}$ such that

$$\begin{aligned}
\sup_{t \leq m}\left\|\lambda_n(t)-t\right\|_2 & \leq d_m\left(f_{Z_n}, f_W\right)+\frac{1}{n} \\
\sup_{t \leq m}\left\|f_{Z_n}(t)-f_W\left(\lambda_n t\right)\right\|_2 & \leq d_m\left(f_{Z_n}, f_W\right)+\frac{1}{n}
\end{aligned}$$

for all n. Taking $g_n \equiv \sup_{t \leq m}\left\|f_W(t)-f_W\left(\lambda_n t\right)\right\|_2$, we see from the uniform continuity of f_W on $[0, m]$ (f_W is uniformly continuous because it's continuos in a compact set) and the definition of g_n that $\lim_{n \rightarrow \infty} g_n=0$. Moreover, if we take $\epsilon_n=2 n^{-1}+2 \sup \left\{d_m\left(f_{Z_l}, f_W\right)+g_l: l=n, n+1, \ldots\right\}$, then $\left\{\epsilon_n\right\}$ is a monotonically decreasing sequence of positive numbers with limit zero by the previous result.

From the definition of ϵ_n we have that $d_m\left(f_{Z_n}, f_W\right)<\epsilon_n / 2-\frac{1}{n}$ and $g_n<\epsilon_n / 2$ for $n=1,2, \ldots$ Consequently, we have

$$\begin{aligned}
\left\|f_{Z_n}(t)-f_W(t)\right\| & \leq\left\|f_{Z_n}(t)-f_W\left(\lambda_n t\right)\right\|+\left\|f_W\left(\lambda_n t\right)-f_W(t)\right\| \\
& <d_m\left(f_{Z_n}, f_W\right)+\frac{1}{n}+g_n \\
& <\epsilon_n
\end{aligned}$$

for all $t \in[0, m]$.

■

Proposition 4. Let $W=\left(W_1, \ldots, W_k\right)$ be a $k-$dimensional independent standard Brownian motion. Let $q_{t x}^W(A):=\exp \left(\frac{1}{\lambda_z^2}\left(\lambda_z W_x(t)+t a_x\right)\right) / \sum_{x^{\prime} \in A} \exp \left(\frac{1}{\lambda_z^2}\left(\lambda_z W_{x^{\prime}}(t)+t a_{x^{\prime}}\right)\right)$ for all $x \in A$ and $A \subset\{1, \ldots, k\}$. Fix

$m \in \{1,\ldots,k-1\}$. We have that for all $N \in \mathbb{N}$, there exists t such that $\tau_m + \frac{1}{N} \geq t > \tau_m$ and $q_{tx}^{W}(A_{m-1}) > P_{m-1}$ for some $x \in A_{m-1}$ almost surely given that $M = m$. (See the definitions in p. 4).

Proof. The proof has the same spirit than the proof of Lemma 4 of Frazier [3], there Frazier proved that $\tau_m < \infty$ almost surely. Furthermore, $M \leq k-1$ almost surely. We should note that the Lemma 4 supposes that all systems have the same variance which is the case here because $\mathrm{Var}\left(\lambda_z W_x(t) + t a_x\right) = \lambda_z^2 t$. We condition on the event $\mathcal{S}$ such that the previous properties hold and has probability 1.

Define $a = \lambda_z \log\left(\frac{P_{m-1}(k-1)}{1-P_{m-1}}\right)$. If $m=1$, then $P_{m-1} = P^* \in (0,1)$, and so a is finite. If $m>1$, then

$$P^* \leq P_{m-1} \leq P^*/(1-c)^{m-1} \leq P^*/(1-c)^{k-2} \leq P^*/(P^*)^{k-2/k-1} < 1$$

and so $P_{m-1} \in (0,1)$ and a is finite.

We define $T^* = \inf\left\{t \geq \tau_m : q_{t,x}^{W}(A_{m-1}) > P_{m-1} \text{ for some } x \in A_{m-1}\right\}$. Fix $n \in \mathbb{N}$, $n > 2$, and define $\tau = \tau_m + \frac{t^*}{nN}$ for a derministic $t^* \in \{1,\ldots,n-1\}$. Let x be any $\mathcal{F}_{\tau_m}$-measurable random variable that is almost surely in A_{m-1} and we define $\Gamma_{t,x} = W_x(t) + t\frac{a_x}{\lambda_z}$. Consider the event $a < \Gamma_{\tau+\frac{1}{nN},x} - \Gamma_{\tau+\frac{1}{nN},y}$ for each $y \in A_{m-1} - \{x\}$. On this event, $q^{W}_{\tau+\frac{1}{mN},y}(A_{m-1})/q^{W}_{\tau+\frac{1}{mN},x}(A_{m-1}) = \exp\left(\frac{1}{\lambda_z}\left(\Gamma_{\tau+\frac{1}{nN},y} - \Gamma_{\tau+\frac{1}{nN},x}\right)\right) < \exp(-a/\lambda_z)$ for $y \in A_{m-1} - \{x\}$ and

$$q_{\tau+\frac{1}{nN},x}(A_{m-1}) = \left[1 + \sum_{y \in A_{m-1}-\{x\}} q_{\tau+\frac{1}{nN},y}(A_{m-1})/q_{\tau+\frac{1}{nN},x}(A_{m-1})\right]^{-1} > \left[1 + (k-1)\exp(-a/\lambda_z)\right]^{-1} = P_{m-1}.$$

Thus, on the event considered, $T^* \leq \tau + \frac{1}{nN}$.

We now define $\tilde{x} \in \arg\max_{x \in A_{m-1}} \Gamma_{\tau,x}$, which is $\mathcal{F}_\tau$-measurable and is almost surely in A_{m-1}. Then we have that

$$\begin{aligned}
\mathbb{P}\left\{T^* \leq \tau + \frac{1}{nN} \mid \mathcal{F}_\tau, T^* > \tau\right\} &\geq \mathbb{P}\left\{a < \Gamma_{\tau+\frac{1}{nN},\tilde{x}} - \Gamma_{\tau+\frac{1}{nN},x}\ \forall x \in A_{m-1} - \{\tilde{x}\} \mid \mathcal{F}_\tau, T^* > \tau\right\} \\
&\geq \mathbb{P}\left\{\Gamma_{\tau+\frac{1}{nN},\tilde{x}} \geq \Gamma_{\tau,\tilde{x}}, \Gamma_{\tau,\tilde{x}} - \Gamma_{\tau+\frac{1}{nN},x} > a\ \forall x \in A_{m-1} - \{\tilde{x}\} \mid \mathcal{F}_\tau, T^* > \tau\right\} \\
&\geq \mathbb{P}\left\{\Gamma_{\tau+\frac{1}{nN},\tilde{x}} \geq \Gamma_{\tau,\tilde{x}}, \Gamma_{\tau,x} - \Gamma_{\tau+\frac{1}{nN},x} > a\ \forall x \in A_{m-1} - \{\tilde{x}\} \mid \mathcal{F}_\tau, T^* > \tau\right\} \\
&= \mathbb{P}\left\{\Gamma_{\tau+\frac{1}{nN},\tilde{x}} \geq \Gamma_{\tau_m,\tilde{x}} \mid \mathcal{F}_\tau\right\} \prod_{x \in A_{m-1}\setminus\{\tilde{x}\}} \mathbb{P}\left\{\Gamma_{\tau,x} - \Gamma_{\tau+\frac{1}{nN},x} > a \mid \mathcal{F}_\tau\right\}.
\end{aligned}$$

Note that $\Gamma_{t,x}$ is a Brownian motion, and so $\mathbb{P}\left\{\Gamma_{\tau+\frac{1}{nN},\tilde{x}} \geq \Gamma_{\tau_m,\tilde{x}} \mid \mathcal{F}_{\tau_m}\right\}$ is the probability of a conditionally $N\left(\frac{a_{\tilde{x}}}{nN}\frac{1}{\lambda_z}, 1\right)$ random variable $\Gamma_{\tau+\frac{1}{nN},\tilde{x}} - \Gamma_{\tau,\tilde{x}}$, exceeding 0. This probability is $\Phi\left(\frac{a_{\tilde{x}}}{nN}\frac{1}{\lambda_z}\right)$, which is bounded below by $\Phi\left(\frac{\min_x a_x}{N}\frac{x_0}{\lambda_z}\right)$ where $x_0 = 0$ if $\min_x a_x \geq 0$, and $x_0 = \frac{1}{3}$ otherwise. Here, Φ is the normal cumulative distribution function. Similarly, the probability $\mathbb{P}\left\{\Gamma_{\tau,x} - \Gamma_{\tau+\frac{1}{nN},x} > a \mid \mathcal{F}_{\tau_m}\right\}$ is the probability of a conditionally $N\left(-a - \frac{a_x}{nN}\frac{1}{\lambda_z}, 1\right)$ random variable, $\Gamma_{\tau,x} - \Gamma_{\tau+\frac{1}{nN},x} - a$, exceeding 0. This probability is $\Phi\left(-a - \frac{a_x}{nN}\frac{1}{\lambda_z}\right)$, and is bounded below by $\Phi\left(-a - \frac{\max_x a_x}{N}\frac{x_0^+}{\lambda_z}\right)$ where $x_0^+ = 0$ if $\max_x a_x \leq 0$ and $x_0^+ = \frac{1}{3}$ otherwise.

Thus, replacing τ with $\tau_m + \frac{t^*}{nN}$,

$$\mathbb{P}\left\{T^* \leq \tau_m + \frac{t^*}{nN} + \frac{1}{nN} \mid \mathcal{F}_{\tau_m+\frac{t^*}{nN}}, T^* > \tau_m + \frac{t^*}{nN}\right\} \geq \Phi\left(\frac{\min_x a_x}{N}\frac{x_0}{\lambda_z}\right)\Phi\left(-a - \frac{\max_x a_x}{N}\frac{x_0^+}{\lambda_z}\right)^{k-1}$$

Let ϵ be the quantity on the righ-hand side of this inequality, then $\epsilon < 1$ and it does not depend on t^* and n.

By repeated application of this inequality, we have that $\mathbb{P}\left\{T^* > T_W^1(P) + \frac{1}{N} \mid \mathcal{F}_{\tau_m}\right\} \leq (1-\epsilon)^n$ for all $n > 2$. This is true because

$$\begin{aligned}
\mathbb{P}\left\{T^* > T_W^1(P) + \frac{1}{N} \mid \mathcal{F}_{\tau_m + \frac{n-1}{nN}}, T^* > \tau_m + \frac{n-1}{nN}\right\} &< (1-\epsilon) \\
\Rightarrow \mathbb{P}\left\{T^* > \tau_m + \frac{1}{N} \mid \mathcal{F}_{T_W^1(P) + \frac{n-2}{nN}}, T^* > \tau_m + \frac{n-2}{nN}\right\} &< (1-\epsilon)\,\mathbb{P}\left\{T^* > \tau_m + \frac{n-1}{nN} \mid \mathcal{F}_{\tau_m ++ \frac{n-2}{nN}}, T^* > \tau_m + \frac{n-2}{nN}\right\} \\
&\leq (1-\epsilon)^2 \\
&\vdots \\
\Rightarrow \mathbb{P}\left\{T^* > \tau_m + \frac{1}{N} \mid \mathcal{F}_{T_W^1(P)}\right\} &\leq (1-\epsilon)^n
\end{aligned}$$

and $(1-\epsilon)^n$ vanishes in the limit as $n \to \infty$. Then, $\mathbb{P}\left\{T^* > \tau_m + \frac{1}{N} \mid \mathcal{F}_{T_W^1(P)}\right\} = 0$ and then

$$\mathbb{P}\left\{\tau_m \leq T^* \leq \tau_m + \frac{1}{N} \mid \mathcal{F}_{T_W^1(P)}\right\} = 1$$

for all N.

■

The following proposition can be proved using a similar argument.

Proposition 5. Let $W = (W_1, \ldots, W_k)$ be a $k-$dimensional independent standard Brownian motion. Let $q_{tx}^W(A) := \exp\left(\frac{1}{\lambda_z^2}(\lambda_z W_x(t) + t a_x)\right) / \sum_{x' \in A} \exp\left(\frac{1}{\lambda_z^2}(\lambda_z W_{x'}(t) + t a_{x'})\right)$ for all $x \in A$ and $A \subset \{1, \ldots, k\}$. Fix $m \in \{1, \ldots, k-1\}$. We have that for all $N \in \mathbb{N}$, there exists t such that $\tau_m - \frac{1}{N} \leq t < \tau_m$ and $q_{tx}^W(A_{m-1}) < c$ for some $x \in A_{m-1}$ almost surely given that $M = m$.

We are now ready to prove the lemma 2.

Lemma 2. Let $\{\delta_n\} \subset (0, \infty)$ such that $\delta_n \to 0$. If $D_s \equiv \{Z \in D[0,\infty)^k$: if $\{Z_n\} \subset D[0,\infty)^k$ and $\lim_n d_\infty(Z_n, Z) = 0$, then the sequence $\{f(Z_n, \delta_n)\}$ converges to $\{g(Z)\}\}$, then $\mathbb{P}(W \in D_s) = 1$.

Proof of Lemma 2. By Lemma 4 and 6 of Frazier, $\tau_n < \infty$ for $n = 0, 1, \ldots, k-1$, $M \leq k-1$,$\tau_n = \tau_M$ for all $n \geq M$, $\hat{x} \in \arg\max_{x \in A_{m-1}}(\lambda_z W_x(\tau_M) + a_x \tau_M)$ and $\tau_{M-1} < \tau_M$ with probability 1. Then there exists a measurable set $\mathcal{L}$ such that $\mathbb{P}[\mathcal{L}] = 1$ and the previos properties are true, W is continuous and Proposition 4 and 5 hold. We will show that $\mathbb{P}(W \in D_s \mid M_W = i, \mathcal{L}) = 1$ for $i \in \{1, \ldots, k-1\}$ so that the desired conclusion follows. It will be useful to use the same notation than in Proposition 2:

$$T_W^m(a) = \begin{cases} \inf\left\{t \geq \tau_{m-1} : \max_{x \in A_{m-1}} q_{tx}^W(A_{m-1}) \geq a\right\} & \text{if } m \geq M_W \\ \inf\left\{t \geq \tau_{m-1} : \min_{x \in A_{m-1}} q_{tx}^W(A_{m-1}) \leq a\right\} & \text{if } m < M_W \end{cases}$$

and

$$T_n^m(a) := \inf\left\{t \geq \tau_{m-1} : \max_{x \in A_{m-1}} q_{tx}^{Z_n,\delta_n}(A_{m-1}) \geq a \text{ or } \min_{x \in A_{m-1}} q_{tx}^{Z_n,\delta_n}(A_{m-1}) \leq c\right\}$$

for any $a \in \mathbb{R}$ and $1 \leq m \leq k-1$.

Case I. First, we fix $\omega \in \mathcal{L} \bigcap \{M_W = 1\}$. Let $\{Z_n\} \subset D[0,\infty)^k$ such that $Z_n \to W$. We want to prove that $f(Z_n, \delta_n) \to g(W)$. Let's prove that $\tau_1^{Z_n,\delta_n} \to \tau_1^W$ as $n \to \infty$.

First, we are going to prove that $T_W^1(P_0) = \lim_n T_W^1(P_0 + \epsilon_n)$. Note that $\lim_n T_W^1(P_0 + \epsilon_n) = \inf_n T_W^1(P_0 + \epsilon_n)$ and so we have to prove that for all $M > 0$ there exists n such that $T_W^1(P_0 + \epsilon_n) < T_W^1(P_0) + M$. Equivalently, we should prove that for all $N \in \mathbb{N}$ there exists $t \in \left(T_W^1(P_0), T_W^1(P_0) + \frac{1}{N}\right]$ such that for some $x \in A_0$, $q_{t,x}^W(A_0) > P_0$. However, this is true by proposition 4.

By Proposition 2 and 3,

$$T_W^1(P_0) = \lim_n T_n^1(P_0). \tag{8}$$

By Proposition 3, we have that $f_{Z_n}(\cdot) \to f_W(\cdot)$ uniformly in $\left[0, T_W^1(P_0) + 1\right]$ and so it should be the case that $\max_{x\in A_0} q_{T_n^1(P_0)x}^{Z_n,\delta_n}(A_0) \geq P_0$ for n large.

Recall that $\hat{x} \in \gamma := \arg\max_{x\in A_0}\left(\lambda_z W_x\left(T_W^1(P_0)\right) + a_x T_W^1(P_0)\right)$.

Let $\epsilon = \left(q_{T_W^1(P_0),\hat{x}}^W(A_0) - \arg\max_{x\in A_0-\gamma} q_{T_W^1(P_0),x}^W(A_0)\right)/4$. Since the function $q_{\cdot,x}^W(A_0)$ is uniformly continuous in $\left[0, T_W^1(P_0) + 1\right]$ for all $x \in A_0$, then there exists $\delta > 0$ such that if $|t - s| < \delta$ and $t, s \in \left[0, T_W^1(P_0) + 1\right]$, then

$$\left|q_{t,x}^W(A_0) - q_{s,x}^W(A_0)\right| < \frac{\epsilon}{2}$$

for all $x \in A_0$. By the proof of Proposition 3, $q_{\cdot,x}^{Z_n,\delta_n}(A_0) \to q_{\cdot,x}^W(A_0)$ in $\left[0, T_W^1(P_0) + 1\right]$ with the Skorohod topology, and then $q_{\cdot,x}^{Z_n,\delta_n}(A_0) \to q_{\cdot,x}^W(A_0)$ uniformly in $\left[0, T_W^1(P_0) + 1\right]$ because $q_{\cdot,x}^W(A_0)$ is uniformly continuous in $\left[0, T_W^1(P_0) + 1\right]$ for all $x \in A_0$. Consequently, there exists N_1 such that if $n > N_1$, then $\left|q_{t,x}^W(A_0) - q_{t,x}^{Z_n,\delta_n}(A_0)\right| < \frac{\epsilon}{2}$ for all $t \in \left[0, T_W^1(P_0) + 1\right]$ and all $x \in A_0$. By (8), there exists N_2 such that if $n > N_2$, then $\left|T_W^1(P_0) - T_n^1(P_0)\right| < \delta$ and $T_n^1(P_0) \in \left[0, T_W^1(P_0) + 1\right]$. So if $n > \max\{N_1, N_2\}$, we have that

$$\begin{aligned}
\left|q_{T_W^1(P_0),x}^W(A_0) - q_{T_n^1(P_0),x}^{Z_n,\delta_n}(A_0)\right| &= \left|q_{T_W^1(P_0),x}^W(A_0) - q_{T_n^1(P_0),x}^W(A_0)\right| + \left|q_{T_n^1(P_0),x}^W(A_0) - q_{T_n^1(P_0),x}^{Z_n,\delta_n}(A_0)\right| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon
\end{aligned}$$

for all $x \in A_0$.

By Proposition 3, there exists a sequence $\{\epsilon_n\}_{n>N^*}$ decreasing to zero for some N^* such that if $n > N^*$.

$$\left|q_{T_W^1(P_0),\hat{x}}^W(A_0) - \max_{x\in A_0} q_{T_n^1(P_0),x}^{Z_n,\delta_n}(A_0)\right| < \epsilon_n.$$

Let $y_n \in \arg\max q_{T_n^1(P_0),y}^{Z_n,\delta_n}(A_0)$. Thus if $n > \max\{N^*, N_1, N_2\}$, then

$$\begin{aligned}
\left|q_{T_W^1(P_0),\hat{x}}^W(A_0) - q_{T_W^1(P_0),y_n}^W(A_0)\right| &\leq \left|q_{T_W^1(P_0),\hat{x}}^W(A_0) - q_{T_n^1(P_0),y_n}^{Z_n,\delta_n}(A_0)\right| + \left|-q_{T_W^1(P_0),y_n}^W(A_0) + q_{T_n^1(P_0),y_n}^{Z_n,\delta_n}(A_0)\right| \\
&< \epsilon_n + \epsilon
\end{aligned}$$

Take N_3 such that if $n > N_3$, then $\epsilon_n < \epsilon$. Consequently, if $n > \max\{N_3, N_1, N_2, N^*\}$, we have that

$$\left|q_{T_W^1(P_0),\hat{x}}^W(A_0) - q_{T_W^1(P_0),y_n}^W(A_0)\right| < \frac{q_{T_W^1(P_0),\hat{x}}^W(A_0) - \arg\max_{x\in A_0-\gamma} q_{T_W^1(P_0),x}^W(A_0)}{2}$$

if $y_n \neq \hat{x}$, then

$$\left|q_{T_W^1(P_0),\hat{x}}^W(A_0) - q_{T_W^1(P_0),y_n}^W(A_0)\right| < \frac{q_{T_W^1(P_0),\hat{x}}^W(A_0) - q_{T_W^1(P_0),y_n}^W(A_0)}{2}$$

which is a contradiction. Consequently, $y_n = \hat{x}$ for n large. Consequently, $f(Z_n, \delta_n)$ converges to $g(W)$.

Case II. Now, we fix we fix $\omega \in \mathcal{L} \bigcap \{M_W = 2\}$. Similarly, we take $\{Z_n\} \subset D[0,\infty)^k$ such that $Z_n \to W$. Let's prove that $\tau_1^{Z_n,\delta_n} \to \tau_1^W$. First, we are going to prove that $T_W^1(c) = \lim_n T_W^1(c - \epsilon_n)$. Note that $\lim_n T_W^1(c-\epsilon_n) = \inf_n T_W^1(c-\epsilon_n)$ and so we have to prove that for all $M > 0$ there exists n such that $T_W^1(c-\epsilon_n) < T_W^1(c) + M$. Equivalently, we should prove that for all $N \in \mathbb{N}$ there exists $t \in \left(T_W^1(c), T_W^1(c) + \frac{1}{N}\right]$ such that for some $x \in A_0$, $q_{t,x}^W(A_0) < c$. However, this is true by proposition 5. By Proposition 2 and 3, $T_W^1(c) = \lim_n T_n^1(P_0)$. Using similar arguments than the given in Case I, we can conclude that $f(Z_n, \delta_n)$ converges to $g(W)$.

The cases $M_W = i$ for $k-1 \geq i \geq 3$ can be proved in a similar way. Then we conclude that $\mathbb{P}(W \in D_s) = 1$.

■

By the extension of the CMT (Theorem 5.5 of Billingsley 1968), we have the following corollary.

Corollary 2. We have that

$$f(C(\delta,t),\delta) \Rightarrow g(W(t))$$

in distribution as $\delta \to 0$.

Theorem 1. If samples from system $x \in \{1\ldots,k\}$ are identically distributed and independent, over time and across alternatives, then $\lim_{\delta\to 0} Pr\{\text{BIZ selects } k\} \geq P*$ provided $\mu_k = a_k\delta, \mu_{k-1} = a_{k-1}\delta, \ldots, \mu_1 = a_1\delta$, and $\mu_k > \mu_{k-1} > \cdots > \mu_1$.

Furthermore,

$$\inf_{a\in PZ(1)} \lim_{\delta\to 0} \mathbb{P}(CS_\delta) = P^*$$

where $PZ(1) = \left\{a \in \mathbb{R}^k : a_k - a_{k-1} \geq 1\right\}$.

Proof. Define $\tau_0 = 0, A_0 = A, P_0 = P^*$ and

$$\begin{aligned}
\tilde{\tau}_{n+1} &= \inf\left\{t \in \{\tilde{\tau}_n, \tilde{\tau}_n + 1, \ldots\} : \ \min_{x\in A_n} q'_{tx}(A_n) \leq c \text{ or } \max_{x\in A_n} q'_{tx}(A_n) \geq P_n\right\} \\
Z_{n+1} &\in \arg\min_{x\in A_n} q'_{\tilde{\tau}_{n+1},x}(A_n) \\
A_{n+1} &= A_n \setminus \{Z_{n+1}\} \\
P_{n+1} &= P_n \Big/ \left(1 - \min_{x\in A_n} q'_{\tilde{\tau}_{n+1},x}(A_n)\right)
\end{aligned}$$

and

$$M = \inf\left\{n = 1, \ldots, k-1 : \max_{x\in A_n} q'_{\tilde{\tau}_n,x}(A_{n-1}) \geq P_{n-1}\right\}.$$

It can be proved that $\tilde{\tau}_M = \tilde{\tau}_n$ and $\max_{x\in A_{M-1}} q'_{\tilde{\tau}_M,x}(A_{M-1}) = \max_{x\in A_n} q'_{\tilde{\tau}_n,x}(A_n)$ for all $n \geq M$.

Similarly, we define

$$\begin{aligned}\hat{\tau}_{n+1}(\delta) &= \inf\left\{t\in\left\{\tau_n\delta^2,(\tau_n+1)\,\delta^2,\ldots\right\}:\ \min\nolimits_{x\in A_n} q'_{t/\delta^2,x}(A_n)\le c \text{ or } \max\nolimits_{x\in A_n} q'_{t/\delta^2,x}(A_n)\ge P_n\right\}\\ &= \inf\left\{t\in\left\{\tau_n\delta^2,(\tau_n+1)\,\delta^2,\ldots\right\}:\ \min\nolimits_{x\in A_n} q'_{t/\delta^2,x}(A_n)\le c \text{ or } \max\nolimits_{x\in A_n} q'_{t/\delta^2,x}(A_n)\ge P_n\right\}\end{aligned}$$

and then $\hat{\tau}_{n+1}(\delta)=\delta^2\tilde{\tau}_{n+1}$. The corresponding continuous hitting times are the times $(\tau_n(\delta))_{n=1}^{k-1}$ we defined in (3) with the functions $q_{t,x}^{\mathcal{C}(\delta,\cdot),\delta}$. Using that $\mathcal{C}(\delta,\cdot)$ is right-continuous, we can prove that $\hat{\tau}_n(\delta)-\tau_n(\delta)\to 0$ with probability 1, and so we can use $\mathcal{C}(\delta,\tau_n(\delta))$ instead of $\mathcal{C}(\delta,\hat{\tau}_n(\delta))$ in the limit.

Let CS_δ be the event of doing a correct selection given the configuration $\mu_k=a_k\delta,\mu_{k-1}=a_{k-1}\delta,\ldots,\mu_1=a_1\delta$. Then by the previous argument and the Corollary 2,

$$\begin{aligned}\lim\nolimits_{\delta\to 0}\mathbb{P}(CS_\delta) &= \lim\nolimits_{\delta\to 0}\mathbb{P}(f(\mathcal{C}(\delta,t),\delta)=1)\\ &= \mathbb{P}(g(W)=1)\\ &\ge P^*\end{aligned}$$

where the last inequality follows from the Theorem 2 of Frazier [3].

Furthermore, by the same theorem 2,

$$\inf\nolimits_{a\in PZ(1)}\mathbb{P}(g(W)=1)=P^*$$

where $PZ(1)=\left\{a\in\mathbb{R}^k: a_k-a_{k-1}\ge 1\right\}$.

■

3 Asymptotic Validity when the Variances are Unknown

We use a random change of time to prove that the new $\hat{\mathcal{C}}_x$ defined using the sample variances also converges to a Brownian motion in the sense of D_∞.

Lemma 3. We have that

$$\hat{\mathcal{C}}_x(\delta,\cdot):=\frac{Y_{\mathrm{ceil}\left(\frac{\hat{\lambda}^2_{t/\delta^2,x}}{\hat{\lambda}^2_{t/\delta^2,z}}\left(n_0+\cdot\frac{1}{\delta^2}\right)\right),x}-\frac{\hat{\lambda}^2_{t/\delta^2,x}}{\hat{\lambda}^2_{t/\delta^2,z}}\left(n_0+\cdot\frac{1}{\delta^2}\right)\mu_x}{\frac{\hat{\lambda}^2_{t/\delta^2,x}}{\hat{\lambda}_{t/\delta^2,z}}\frac{1}{\delta}}\Rightarrow W_x(\cdot)$$

in the sense of D_∞ for each $x\in A$, where $\hat{\lambda}^2_{t/\delta^2,x}=\frac{1}{n_{t/\delta^2,x}-1}\sum_{i=1}^{n_{t/\delta^2,x}}\left(X_{xi}-Y_{n_{t/\delta^2,x}}\right)^2$.

Proof. Fix $x\in A$. Define $\Psi_\delta:[0,\infty)\to[0,\infty)$ by $\Psi_\delta(t)=-n_0\delta^2+\frac{\lambda_z^2}{\lambda_x^2}\frac{\hat{\lambda}^2_{t/\delta^2,x}}{\hat{\lambda}^2_{t/\delta^2,z}}\left(t+\delta^2 n_0\right)$, then $\Psi_\delta\in D_\infty$. Now define $\varphi: D_\infty\times D_\infty\to D_\infty$ by

$$\varphi(\mathcal{X},\mathcal{Y})=\mathcal{X}\circ\mathcal{Y}$$

Using Lemma 1, we have that $(\mathcal{C}_x(\delta,\cdot),\Psi_\delta(\cdot))\Rightarrow(W_x(\cdot),I(\cdot))$ as $\delta\to 0$ in the sense of D_∞. We can prove that φ is measurable, and since W_x is continuous almost surely, we can use a generalization of the Lemma from the chapter Random Change of Time of Billingsley (1999) to conclude that $\varphi\left(\mathcal{C}_x\left(\delta,\cdot\right),\Psi_\delta\left(\cdot\right)\right)\Rightarrow\varphi\left(W_x\left(\cdot\right),I\left(\cdot\right)\right)=W_x\left(\cdot\right)$ in the sense of D_∞ as we wanted to prove.

■

In the previous proof, we only have to replace λ_x^2 by its estimators and we get the same result. We should note that

$$\text{ceil}\left(\frac{\hat{\lambda}^2_{t/\delta^2,x}}{\hat{\lambda}^2_{t/\delta^2,z}}\left(n_0+\cdot\frac{1}{\delta^2}\right)\right)-n_{t/\delta^2,x}\to 0$$

for any x. Consequently, we will have that

$$\frac{Y_{n_{t/\delta^2,x},x}-n_{t/\delta^2,x}\mu_x}{\frac{\hat{\lambda}^2_{t/\delta^2,x}}{\hat{\lambda}_{t/\delta^2,z}}\frac{1}{\delta}}\Rightarrow W_x\left(\cdot\right)$$

So, we have the following theorem.

Theorem 2. If samples from system $x\in\{1\ldots,k\}$ are identically distributed and independent, over time and across alternatives, then $\lim_{\delta\to 0}Pr\left\{\text{BIZ selects } k\right\}\geq P*$ provided $\mu_k=a_k\delta,\mu_{k-1}=a_{k-1}\delta,\ldots,\mu_1=a_1\delta$, and $\mu_k>\mu_{k-1}>\cdots>\mu_1$.

Furthermore,

$$\inf_{a\in PZ(1)}\lim_{\delta\to 0}\mathbb{P}\left(CS_\delta\right)=P^*$$

where $PZ\left(1\right)=\left\{a\in\mathbb{R}^k:a_k-a_{k-1}\geq 1\right\}$.

4 Probability of Good Selection

Ideally, we would like that the difference between the choosen system and the best system is practically insignificant. We thus define the probability of good selection as

$$\text{PGS}\left(\mu\right)=\mathbb{P}\left(\mu_k-\mu_{\hat{x}}\leq\delta\right).$$

We would like to prove the PGS guarantee:

$$\forall\mu,\ \text{PGS}\left(\mu\right)\geq P^*.$$

First, observe that if $\mu_k-\mu_{k-1}>\delta$, then $\text{PGS}\left(\mu\right)=\text{PCS}\left(\mu\right)$ and so $\text{PGS}\left(\mu\right)\geq P^*$. Consequently, we only need to prove that if $\mu_k-\mu_{k-1}\leq\delta$, then we have that $\text{PGS}\left(\mu\right)\geq P^*$.

Inspired on Frazier (2014), we first construct a probability measure Q as follows. Let X^* be chosen uniformly at random from among $1,\ldots,k$ and let $\theta_{X^*}=\delta$ and $\theta_x=0$ if $x\neq X^*$. We then define a family of probability measures Q_u that i

5 Simulations

6 Conclusion

7 Appendix A: Skorohod topology

We are going to define the Skorohod topology on $D[0,\infty)$ by defining a metric on the space. The Skorohod metricric d_t on $D\,[0,t]$ for each $t \geq 0$ is:

$$d_t\left(\mathcal{X},\mathcal{Y}\right) = \inf_{\lambda\in\Lambda_t}\left\{\|\lambda - I\| \vee \|\mathcal{X} - \mathcal{Y}\circ\lambda\|\right\}$$

where Λ_t is the set of strictly increasing, continuous mappings of $[0,t]$ onto itself, and $\|\cdot\|$ is the uniform norm, and I is the identity map. Note that uniform convergence on $[0,t]$ implies Skorohod convergence.

We define the Skorohod topology on $D\,[0,\infty)$. For $\mathcal{X} \in D\,[0,\infty)$, let $\mathcal{X}^m$ be the element of $D_\infty := D[0,\infty)$ defined by

$$\mathcal{X}^m\left(t\right) = g_m\left(t\right)\mathcal{X}\left(t\right)$$

where

$$g_m\left(t\right) = \begin{cases} 1 & \text{if } t \leq m-1, \\ m-t & \text{if } m-1 \leq t \leq m, \\ 0 & \text{if } t \geq m. \end{cases}$$

And now take

$$d_\infty\left(\mathcal{X},\mathcal{Y}\right) = \sum_{m=1}^{\infty} 2^{-m}\left(1 \wedge d_m\left(\mathcal{X}^m,\mathcal{Y}^m\right)\right)$$

which is the Skorohod metric on $D\,[0,\infty)$. By Theorem 16.2 of Billingsley1999, there is convergence $d_\infty\left(x_n,x\right) \to 0$ in D_∞ if and only if $d_t\left(x_n,x\right) \to 0$ for each continuity point t of x.

We can also define

$$d^\circ_\infty\left(\mathcal{X},Y\right) = \sum_{m=1}^{\infty} 2^{-m}\left(1 \wedge d^\circ_m\left(\mathcal{X}^m,\mathcal{Y}^m\right)\right)$$

where $d^\circ_m\left(\mathcal{X}^m,\mathcal{Y}^m\right) = \inf_{\lambda\in\Lambda_m}\left\{\sup_{s<t}\left|\log\frac{\lambda t-\lambda s}{t-s}\right| \vee \|\mathcal{X} - \mathcal{Y}\circ\lambda\|\right\}$. This is also a metric and the proof is in Bilingsley 1999 and these two metrics are equivalent.

Theorem A.1 We have that $d_\infty\left(\mathcal{X}_n,\mathcal{Y}_n\right) \to 0$ if and only if $d^\circ_\infty\left(\mathcal{X}_n,\mathcal{Y}_n\right) \to 0$ and $d_t\left(\mathcal{X}_n,\mathcal{Y}_n\right) \to 0$ if and only if $d^\circ_t\left(\mathcal{X}_n,\mathcal{Y}_n\right) \to 0$ for all $t \geq 0$.

Theorem A.2 Suppose we have two sequences of random paths $\{\mathcal{X}_n = (X_n\left(t\right) : 0 \leq t < \infty)\}_{n\geq 0}^{\infty}$,$\{\mathcal{Y}_n = (Y_n\left(t\right) : 0 \leq$ such that $\mathcal{X}_n, \mathcal{Y}_n : \mathcal{F} \to D_\infty$ for all $n \geq 0$ where $(\Omega, \mathcal{F}, \mathbb{P})$ is our space of probability. Suppose that $\mathcal{X}_n \Rightarrow \mathcal{X}_0$ in the sense of $D\,[0,\infty)$ and $\mathcal{X}_n - \mathcal{Y}_n \to 0$ uniformly in $[0,m]$ for all $m \in \mathbb{N}$ with probability 1, then $\mathcal{Y}_n \Rightarrow \mathcal{X}_0$.

Proof. We only need to prove that $d_\infty\left(\mathcal{X}_n,\mathcal{Y}_n\right) \to 0$ almost surely and the result will follow by Theorem 3.1 of Billingsley 1999. Then we only need to prove that $d_m\left(\mathcal{Y}^m_n,\mathcal{X}^m_n\right) \to 0$ almost surely for all $m \in \mathbb{N}$. Fix

$\omega \in \Omega$ and $m \in \mathbb{N}$ such that $\mathcal{X}_n - \mathcal{Y}_n \to 0$ uniformly in $[0, m]$. Observe that

$$\begin{aligned}
d_m\left(\mathcal{Y}_n^m, \mathcal{X}_n^m\right) &= \inf_{\lambda \in \Lambda_m} \left\{\|\lambda - I\| \vee \|\mathcal{Y}_n^m - \mathcal{X}_n^m \circ \lambda\|\right\} \\
&\leq \|\mathcal{Y}_n^m - \mathcal{X}_n^m\| \\
&= \|g_m \mathcal{Y}_n - g_m \mathcal{X}_n\| \\
&= \sup_{t \leq m-1} \|Y_n(t) - X_n(t)\| \vee \sup_{t > m-1} \|Y_n(t) - X_n(t)\| (m - t) \\
&\leq \sup_{t \leq m} \|Y_n(t) - X_n(t)\| \\
&\to 0
\end{aligned}$$

as we wanted to prove.

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