

# On the Asymptotic Validity of a Fully Sequential Elimination Procedure for Indifference-Zone Ranking and Selection with Tight Bounds on Probability of Correct Selection

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We prove the validity of the sequential elimination IZ procedure proposed by Frazier [3] when  $\delta$  goes to 0. Specifically, if  $k$  is the number of systems, we analyze Algorithm 2, when  $B_1 = \dots = B_k = 1$ :

**Algorithm 2: Discrete-time implementation of BIZ, for unknown and/or heterogeneous variances.**

**Require:**  $c \in [0, 1 - (P^*)^{\frac{1}{k-1}}]$ ,  $\delta > 0$ ,  $P^* \in (1/k, 1)$ ,  $n_0 \geq 0$  an integer,  $B_1, \dots, B_k$  strictly positive integers.

Recommended choices are  $c = 1 - (P^*)^{\frac{1}{k-1}}$ ,  $B_1 = \dots = B_k = 1$  and  $n_0$  between 10 and 30. If the sampling variances  $\lambda_x^2$  are known, replace the estimators  $\hat{\lambda}_{tx}^2$  with the true values  $\lambda_x^2$ , and set  $n_0 = 0$ . We define  $q_{tx}(A)$  where  $A = \{1, \dots, k\}$  as

$$q_{tx}(A) = \exp\left(\frac{\delta}{\lambda_x^2} Y_{n_x(t),x}\right) \bigg/ \sum_{x' \in A} \exp\left(\frac{\delta}{\lambda_{x'}^2} Y_{n_{x'}(t),x'}\right), \quad (1)$$

where  $Y_{n_x(t),x}$  is the sum of the first  $n_x(t)$  samples.

- 1: For each  $x$ , sample alternative  $x$   $n_0$  times and set  $n_{0x} \leftarrow n_0$ . Let  $W_{0x}$  and  $\hat{\lambda}_{0x}^2$  be the sample mean and sample variance respectively of these samples. Let  $t \leftarrow 0$ .
- 2: Let  $A \leftarrow \{1, \dots, k\}$ ,  $P \leftarrow P^*$ ,  $t \leftarrow 1$ .
- 3: **while**  $x \in \max_{x \in A} q_{tx}(A) < P$  **do**
- 4:   **while**  $\min_{x \in A} q_{tx}(A) \leq c$  **do**
- 5:     Let  $x \in \arg \min_{x \in A} q_{tx}(A)$ .
- 6:     Let  $P \leftarrow P/(1 - q_{tx}(A))$ .
- 7:     Remove  $x$  from  $A$ .
- 8:   **end while**
- 9:   Let  $z \in \arg \min_{x \in A} n_{tx}/\hat{\lambda}_{tx}^2$ .
- 10:   For each  $x \in A$ , let  $n_{t+1,x} = \text{ceil}\left(\hat{\lambda}_{tx}^2(n_{tz} + B_z)/\hat{\lambda}_{tz}^2\right)$ .
- 11:   For each  $x \in A$ , if  $n_{t+1,x} > n_{tx}$ , take  $n_{t+1,x} - n_{tx}$  additional samples from alternative  $x$ . Let  $W_{t+1,x}$  and  $\hat{\lambda}_{t+1,x}^2$  be the sample mean and sample variance respectively of all samples from alternative  $x$  thus far.
- 12:   Increment  $t$ .
- 13: **end while**
- 14: Select  $\hat{x} \in \arg \max_{x \in A} W_{tx}/n_{tx}$  as our estimate of the best.

# 1 Introduction

This paper is organized as follows: In §2, we present the proof of the validity of the algorithm when the variances are known. In §3, we prove the case when the variances are unknown.

To prove the case when the variances are known, we use a theorem for Ergodic processes that shows how to standardize the output data to make them behave like Brownian motion processes in the limit. We also use an extension of the Continuous Mapping Theorem (Theorem 5.5 of Billingsley 1968) to see that the algorithm behaves like a sequential elimination IZ procedure with a Brownian motion process instead of the standardized sum of the output data in the limit. Finally, we use the results of the paper of Frazier [3] to prove the validity of this algorithm in the limit.

## 2 Asymptotic Validity when the Variances are Known

Without loss of generality, suppose that the true means of the systems are indexed so that  $\mu_k \geq \mu_{k-1} \geq \dots \geq \mu_1$ . We suppose that samples from system  $x \in \{1 \dots, k\}$  are identically distributed and independent, over time and across alternatives. We also define  $\lambda_z^2 := \max_{i \in \{1, \dots, k\}} \lambda_i^2$ . We suppose that  $\min_{i \in \{1, \dots, k\}} \lambda_i^2 > 0$  and  $c \neq 1/k$ . First we are going to suppose that  $\delta > 0$  and  $\mu = \delta a$  for some  $a$  such that  $a_k - a_i \leq 1$  for all  $i$ .

Now we are going to see that the standardized sum of the output data converges to a Brownian motion in  $D[0, \infty)$ , which is the set of functions from  $[0, \infty)$  to  $\mathbb{R}$  that are right-continuous and have left-hand limits, with the Skorohod topology. We are going to define the Skorohod topology on  $D[0, \infty)$  by defining a metric on the space. The Skorohod metric  $d_t$  on  $D[0, t]$  is:

$$d_t(X, Y) = \inf_{\lambda \in \Lambda_t} \{ \|\lambda - I\| \vee \|X - Y \circ \lambda\| \}$$

where  $\Lambda_t$  is the set of strictly increasing, continuous mappings of  $[0, t]$  onto itself, and  $\|\cdot\|$  is the uniform norm, and  $I$  is the identity map. Note that uniform convergence on  $[0, t]$  implies Skorohod convergence.

We define the Skorohod topology on  $D[0, \infty)$ . For  $X \in D[0, \infty)$ , let  $X^m$  be the element of  $D_\infty := D[0, \infty)$  defined by

$$X^m(t) = g_m(t) X(t)$$

where

$$g_m(t) = \begin{cases} 1 & \text{if } t \leq m-1, \\ m-t & \text{if } m-1 \leq t \leq m, \\ 0 & \text{if } t \geq m. \end{cases}$$

And now take

$$d_\infty(X, Y) = \sum_{m=1}^{\infty} 2^{-m} (1 \wedge d_m(X^m, Y^m))$$

which is the Skorohod metric on  $D[0, \infty)$ . By Theorem 16.2 of Billingsley 1999, there is convergence  $d_\infty(x_n, x) \rightarrow 0$  in  $D_\infty$  if and only if  $d_t(x_n, x) \rightarrow 0$  for each continuity point  $t$  of  $x$ .

The following lemma shows that the standardized sum of the output data converges to a Brownian motion in  $D_\infty$ .

**Lemma 1.** If  $x \in \{1 \dots, k\}$ , then

$$C_x(\delta, \cdot) := \frac{Y_{\text{ceil}\left(\frac{\lambda_x^2}{\lambda_z^2}\left(n_0 + \cdot \frac{1}{\delta^2}\right)\right), x} - \frac{\lambda_x^2}{\lambda_z^2}\left(n_0 + \cdot \frac{1}{\delta^2}\right) \mu_x}{\frac{\lambda_x^2}{\lambda_z} \delta} \Rightarrow W_x(\cdot)$$

in the sense of  $D[0, \infty)$ , where  $Y_{n,x}$  is the sum of the first  $n$  samples and  $W_x$  is a standard Brownian motion.

**Proof.** By the Theorem 19.1 of Billingsley 1999 and the sandwich theorem,

$$\frac{Y_{\text{ceil}\left(\frac{\lambda_x^2}{\lambda_z^2}\left(\cdot \frac{1}{\delta^2}\right)\right), x} - \text{ceil}\left(\frac{\lambda_x^2}{\lambda_z^2}\left(\cdot \frac{1}{\delta^2}\right)\right) \mu_x}{\frac{\lambda_x^2}{\lambda_z} \sqrt{\frac{1}{\delta^2}}} \Rightarrow W_x(\cdot).$$

Since  $\frac{\frac{\lambda_x^2}{\lambda_z^2} t \frac{1}{\delta^2} - \text{ceil}\left(\frac{\lambda_x^2}{\lambda_z^2} t \frac{1}{\delta^2}\right)}{\frac{\lambda_x^2}{\lambda_z} \sqrt{\frac{1}{\delta^2}}} \rightarrow 0$  uniformly on  $[0, r]$  for every  $r$ , then it also converges to 0 on  $D[0, \infty)$  and so

$$\frac{Y_{\text{ceil}\left(\frac{\lambda_x^2}{\lambda_z^2}\left(\cdot \frac{1}{\delta^2}\right)\right), x} - \left(\frac{\lambda_x^2}{\lambda_z^2}\left(\cdot \frac{1}{\delta^2}\right)\right) \mu_x}{\frac{\lambda_x^2}{\lambda_z} \sqrt{\frac{1}{\delta^2}}} \Rightarrow W_x(\cdot).$$

Observe that for  $\epsilon > 0$  and  $\delta$  sufficiently small

$$\left| \frac{-Y_{\text{ceil}\left(\frac{\lambda_x^2}{\lambda_z^2} t \frac{1}{\delta^2}\right), x} + Y_{\text{ceil}\left(n_0 \frac{\lambda_x^2}{\lambda_z^2} + \frac{\lambda_x^2}{\lambda_z^2} t \frac{1}{\delta^2}\right), x}}{\frac{\lambda_x^2}{\lambda_z} \sqrt{\frac{1}{\delta^2}}} \right| < \epsilon \left( n_0 \frac{\lambda_x^2}{\lambda_z^2} + 2 \right)$$

and then

$$C_x(\delta, \cdot) \Rightarrow W_x(\cdot).$$

■

Now we are going to define new algorithms that are almost the same as the continuous-time procedure proposed by Frazier, but these algorithms use new functions  $q_{tx}^{Y, \delta}(A)$  which depend on  $\delta$  and a function  $C$  that is in  $D[0, \infty)^k$ . More explicity, if we define

$$q'_{tx}(A) = \exp\left(\frac{\delta}{\lambda_x^2} Y'_{n_x(t), x}\right) \bigg/ \sum_{x' \in A} \exp\left(\frac{\delta}{\lambda_{x'}^2} Y'_{n_{x'}(t), x'}\right)$$

where  $Y'_{tx}$  is a Brownian motion under  $\mathbb{P}_{\mu, \lambda}$  starting from 0, with drift  $\mu_x$ , volatility  $\lambda_x$ , and independence across  $x$ , the procedure proposed by Frazier is defined by first setting

$$\tau_0 = 0, P_0 = P^*, A_0 = \{1, \dots, k\}$$

then defining recursively, for  $n = 0, 1, \dots, k-2$ ,

$$\begin{aligned}\tau_{n+1} &= \inf \left\{ t \geq \tau_n : \min_{x \in A_n} q'_{tx}(A_n) \leq c \text{ or } \max_{x \in A_n} q'_{tx}(A_n) \geq P_n \right\} \\ Z_{n+1} &\in \arg \min_{x \in A_n} q'_{\tau_{n+1},x}(A_n) \\ A_{n+1} &= A_n \setminus \{Z_{n+1}\} \\ P_{n+1} &= P_n / \left( 1 - \min_{x \in A_n} q'_{\tau_{n+1},x}(A_n) \right)\end{aligned}$$

and finally letting the selected alternative  $\hat{x}$  be the single entry in  $A_{k-1}$ . We also define

$$M = \inf \left\{ n = 1, \dots, k-1 : \max_{x \in A_n} q'_{\tau_n,x}(A_{n-1}) \geq P_{n-1} \right\}.$$

If  $C \in D[0, \infty)^k$ , we define new functions  $q^{C,\delta}_{tx}(A)$  by

$$q^{C,\delta}_{tx}(A) = \exp \left( \frac{C_x(t)}{\lambda_z} + \delta^2 \frac{(n_0 + t \frac{1}{\delta^2})}{\lambda_z^2} a_x \right) / \sum_{x' \in A} \exp \left( \frac{C_{x'}(t)}{\lambda_z} + \delta^2 \frac{(n_0 + t \frac{1}{\delta^2})}{\lambda_z^2} a_{x'} \right).$$

The reason of this definition is because  $C(\delta, \cdot) = (C_x(\delta, \cdot))_{x \in A} \in D[0, \infty)^k$  and

$$\frac{\delta}{\lambda_x^2} Y_{n_x(\cdot),x} = \frac{C_x(\delta, \cdot)}{\lambda_z} + \delta^2 \frac{(n_0 + \cdot \frac{1}{\delta^2})}{\lambda_z^2} a_x \Rightarrow W'_x(\cdot) := \frac{W_x(\cdot)}{\lambda_z} + \frac{\cdot}{\lambda_z^2} a_x$$

where  $W'_x$  is a Brownian motion starting from 0, with drift  $\frac{\mu_x}{\lambda_z^2}$  and volatility  $\frac{1}{\lambda_z}$ .

Note that the functions  $q^{C,\delta}_{tx}(A)$  define a new algorithm for each  $C \in D[0, \infty)^k$  and  $\delta > 0$ .

We also define a new function

$$f(C, \delta) = \begin{cases} 1 & \text{if } k \text{ is chosen} \\ 0 & \text{otherwise} \end{cases}$$

using the algorithm that  $(C, \delta)$  induces.

Now, we also define new functions  $q^C_{tx}(A)$  to analyze the limit of  $f(C, \delta)$  when  $\delta$  goes to zero:

$$q^C_{tx}(A) := \exp \left( \frac{C_x(t)}{\lambda_z} + \frac{1}{\lambda_z^2} t a_x \right) / \sum_{x' \in A} \exp \left( \frac{C_{x'}(t)}{\lambda_z} + \frac{1}{\lambda_z^2} t a_{x'} \right)$$

and we define the functions

$$g(Y) = \begin{cases} 1 & \text{if } k \text{ is chosen} \\ 0 & \text{otherwise} \end{cases}.$$

We want to prove that

$$f(C(\delta, \cdot), \delta) \Rightarrow g(W).$$

In order to prove this, we will prove the Lemma 2 which will allow us to use the Theorem 5.5 of Billingsley 1968 that implies the desired result.

**Lemma 2.** Let  $\{\delta_n\} \subset (0, \infty)$  such that  $\delta_n \rightarrow 0$ . If  $D_s \equiv \{x \in D[0, \infty)^k : \text{for all sequences } \{x_n\} \subset D[0, \infty)^k \text{ such that } \lim_n d(x_n, x) = 0 \text{ the sequence } \{f(x_n, \delta_n)\} \text{ converges to } \{g(x)\}\}$ , then  $\mathbb{P}(W \in D_s) = 1$ .

First, we are going to prove the following two propositions.

**Proposition 1.** Suppose  $\{f_n\}$  and  $\{g_n\}$  are two sequences of functions on  $D_\infty$  such that  $f_n \rightarrow f$  and  $g_n \rightarrow g$  in the sense of  $D_\infty$ , then

$$\min(f_n, g_n) \rightarrow \min(f, g)$$

in the sense of  $D_\infty$ .

**Proof.** Since  $f_n \rightarrow f$  and  $g_n \rightarrow g$ , then there exist elements  $\lambda_n, \beta_n$  of  $\Lambda_\infty$  such that

$$\begin{aligned} \sup_{t < \infty} |\lambda_n t - t| &\rightarrow 0 \\ \sup_{t < \infty} |\beta_n t - t| &\rightarrow 0 \end{aligned}$$

and, for each  $m$ ,

$$\begin{aligned} \sup_{t \leq m} |f_n(\lambda_n t) - f(t)| &\rightarrow 0 \\ \sup_{t \leq m} |g_n(\lambda_n t) - g(t)| &\rightarrow 0. \end{aligned}$$

Define

**Proposition 2.** Suppose  $\{f_n^{\delta_n}\}$  is a sequence of functions on  $D_\infty$  such that  $f_n^{\delta_n} \rightarrow f$  in the sense of  $D_\infty$ ,  $\delta_n \rightarrow 0$ ,  $\{\delta_n\} \subset (0, \infty)$  and  $\{T_n\} \subset [0, \infty)$  is a sequence such that  $T_n \rightarrow T$ . We define  $T(a) := \inf\{t \geq T : f(t) \geq a\}$  for each  $a \in \mathbb{R}$ . Moreover,  $f$  is continuous and  $T(\delta) \in \mathbb{R}$  where  $\delta = \sup_n \delta_n$ . Suppose there exists  $\{\epsilon_n\} \subset (0, \infty)$  such that  $\epsilon_n \rightarrow 0$  and

$$\left\| f_n^{\delta_n}(t) - f(t) \right\|_2 < \epsilon_n$$

for all  $t \in [0, T(\delta)]$ . We also suppose that  $\limsup_n T(\epsilon_n + \delta_n) \leq T(0)$ . Thus we have that

$$\inf\{t \geq T_n : f_n^{\delta_n}(t) \geq 0\} \rightarrow \inf\{t \geq T : f(t) \geq 0\}$$

if  $T(0) > T$  or  $T_n = T$ .

**Proof.** We are going to suppose  $T(0) > T$ , and the case when  $T_n = T$  can be proved using almost the same ideas. We introduce the notation  $T_n(a) : \inf\{t \geq T_n : f_n(t) \geq a\}$   $a \in \mathbb{R}$ . Let  $N$  such that if  $n > N$ ,  $T_n < T + T(0) - T = T(0)$  and  $\delta > \epsilon_n + \delta_n$ . Let  $n > N$ . Note that  $T_n(0) \leq T_n(\delta_n)$ .

Now, let's prove that  $T_n(\delta_n) \leq T(\epsilon_n + \delta_n)$ . Observe that  $T_n < T(0) \leq T(\epsilon_n + \delta_n)$  and  $\delta_n \leq f(T(\epsilon_n + \delta_n)) - \epsilon_n < f_n^{\delta_n}(T(\epsilon_n + \delta_n))$  because  $T(\epsilon_n + \delta_n) \leq T(\delta)$ , and so  $T_n(\delta_n) \leq T(\epsilon_n + \delta_n)$ .

We also have that

$$\begin{aligned} T(0) &\leq \liminf_n T(\delta_n + \epsilon_n) \\ &\leq \limsup_n T(\delta_n + \epsilon_n) \end{aligned} \tag{2}$$

Now, since  $\limsup_n T(\delta_n + \epsilon_n) \leq T(0)$ ,

$$\begin{aligned} T(0) &\geq \limsup_n T(\delta_n + \epsilon_n) \\ &\geq \liminf_n T(\delta_n + \epsilon_n) \\ &\geq T(0) \end{aligned}$$

and so

$$\liminf_n T_n(0) \leq \limsup_n T_n(0) \leq \lim_n T(\delta_n + \epsilon_n) = T(0)$$

Now, let's prove that  $\liminf_n T_n(0) \geq T(0)$ . Let  $M$  such that  $T(0) - \frac{1}{m} \geq T$  if  $m > M$ , and let  $t_m = T(0) - \frac{1}{m}$  and  $\alpha_m = \max \{f(t) : t \in [T, t_m]\}$ . Note that  $\alpha_m < 0$  because  $t_m < T(0)$ . Let  $N$  such that if  $n > N$ , then

$$\epsilon_n \leq -\alpha_m$$

Thus, if  $t \in [T, t_m]$

$$f_n^{\delta_n}(t) < f(t) + \epsilon_n \leq 0$$

If  $T \leq T_n(0)$ , then  $T_n(0) \geq t_m$  and so  $\liminf_n T_n(0) \geq t_m$ . Consequently,  $\liminf_n T_n(0) \geq T(0)$ .

Let's prove that  $T \leq T_n(0)$  for  $n$  large. Let  $s > 0$  be any number. Let  $N$  such that if  $n > N$ , then  $\epsilon_n < s$ . Suppose that for every  $\epsilon > 0$  there exists  $n > N$  such that  $T - \epsilon < T_n(0) < T$ . Note that

$$\begin{aligned} f(T_n(0)) &\geq f_n(T_n(0)) - \epsilon_n \\ &\geq -\epsilon_n > -s \end{aligned}$$

Let  $a > 0$ , since  $f$  is continuous there exists  $\epsilon > 0$  such that  $f(x) < f(T) + a$  if  $|x - T| < \epsilon$ . Thus, there exists  $n > N$  such that  $T - \epsilon < T_n(0) < T$  and so

$$-s < f(T_n(0)) < f(T) + a$$

Since  $s$  and  $a$  are arbitrary,

$$f(T) \geq 0$$

which is a contradiction. Consequently,  $T \leq T_n(0)$ .

Consequently,

$$\lim_n T_n(0) = T(0).$$

■

**Proof of Lemma 2.** We will use the following property: if  $T$  is a stopping time, then by the local version of the law of the iterated logarithm for Brownian motion

$$\limsup_{u \rightarrow 0^+} \frac{[W_x(T+u) - W_x(T)]}{\sqrt{2u \ln[\ln(1/u)]}} = 1 \quad (3)$$

almost surely for each system  $x$ . Furthermore, note that  $W$  is almost surely continuous on  $[0, t]^k$  if  $t > 0$  and so  $W$  is also almost surely uniformly continuous on  $[0, t]^k$  because  $[0, t]^k$  is compact. We will only consider the event  $\mathcal{S}$  such that the previous properties hold which has probability one.

Let  $\{Z_n\} \subset D[0, \infty)^k$  such that  $Z_n \rightarrow W$ . Note that  $\delta_n^2 \frac{(n_0 + t \frac{1}{\delta_n^2})}{\lambda_z^2} a_x \rightarrow \frac{t}{\lambda_z^2} a_x$  in  $D[0, \infty)$  and  $\frac{\delta_n \sqrt{\frac{1}{\delta_n^2}}}{\lambda_z} \rightarrow \frac{1}{\lambda_z}$  in  $D[0, \infty)$  because uniform convergence implies convergence in the Skorohod topology. Consequently, for each  $s > 0$  there exist functions  $\lambda_s^n$  in  $\Lambda_s$  such that

$$\lim_n Z_n(\lambda_s^n t) = W(t)$$

uniformly in  $t$  and

$$\lim_n \lambda_s^n t = t$$

uniformly in  $t$ . Then

$$\lim_n \frac{\delta_n \sqrt{\frac{1}{\delta_n^2}}}{\lambda_z} Z_n(\lambda_s^n t) + \delta_n^2 \frac{(n_0 + \lambda_s^n(t) \frac{1}{\delta_n^2})}{\lambda_z^2} a_x = W(t) \frac{1}{\lambda_z} + \frac{t}{\lambda_z^2} a_x$$

uniformly in  $t$ , and so

$$\lim_n \exp \left( \frac{\delta_n \sqrt{\frac{1}{\delta_n^2}}}{\lambda_z} Z_n(\lambda_s^n t) + \delta_n^2 \frac{(n_0 + \lambda_s^n(t) \frac{1}{\delta_n^2})}{\lambda_z^2} a_x \right) = \exp \left( W(t) \frac{1}{\lambda_z} + \frac{t}{\lambda_z^2} a_x \right)$$

uniformly in  $t$  since  $\exp$  is uniformly continuous in  $[0, s]$ . Consequently,

$$q_{\lambda_s^n(t)x}^{Z_n, \delta_n}(A) \rightarrow q_{tx}^W(A)$$

uniformly in  $t \in [0, s]$ . Thus,  $q_x^{Z_n, \delta_n}(A) \rightarrow q_x^W(A)$  in  $D[0, s]$  for any set  $A \subset \{1, \dots, k\}$  and  $s \geq 0$ . Consequently,  $q_x^{Z_n, \delta_n}(A) \rightarrow q_x^W(A)$  in  $D[0, \infty)$ . By Proposition 1,  $\min_{x \in A} q_x^{Z_n, \delta_n}(A) \rightarrow \min_{x \in A} q_x^W(A)$  and  $\max_{x \in A} q_x^{Z_n, \delta_n}(A) \rightarrow \max_{x \in A} q_x^W(A)$ , and so

$$f_{Z_n} := \max \left\{ c - \min_{x \in A} q_x^{Z_n, \delta_n}(A), \max_{x \in A} q_x^{Z_n, \delta_n}(A) - P_s \right\} \rightarrow f_W := \max \left\{ c - \min_{x \in A} q_x^W(A), \max_{x \in A} q_x^W(A) - P_s \right\}$$

in  $D[0, \infty)$  for any  $P_s$  of the algorithm.

Then for each  $m \geq 0$  there exists  $\lambda_n \in \Lambda_\infty$  such that  $\sup_{t \leq m} \|\lambda_n(t) - t\|_2 \leq d(f_{Z_n}, f_W) + \frac{1}{n}$  and  $\sup_{t \leq m} \|f_{Z_n}(t) - f_W(t)\|_2 \leq d(f_{Z_n}, f_W) + \frac{1}{n}$ . Taking  $g_n \equiv \sup_{t \leq m} \|f_W(t) - f_W(\lambda_n t)\|_2$ , we see from the uniform continuity of  $f_W$  on  $[0, m]^k$  ( $f_W$  is uniformly continuous because it's continuous in a compact set) and the definition of  $g_n$  that  $\lim_{n \rightarrow \infty} g_n = 0$ . Moreover, if we take  $\epsilon_n = 2n^{-1} + 2\sup \{d(f_{Z_l}, f_W) + g_l : l = n, n+1, \dots\}$ , then  $\{\epsilon_n\}$  is a monotonically decreasing sequence of positive numbers with limit zero.

From the definition of  $\epsilon_n$  we have  $d(f_{Z_n}, f_W) < \epsilon_n/2$  and  $g_n < \epsilon_n/2$  for  $n = 1, 2, \dots$ . Consequently, we have

$$\begin{aligned} \|f_{Z_n}(t) - f_W(t)\| &\leq \|f_{Z_n}(t) - f_W(\lambda_n t)\| + \|f_W(\lambda_n t) - f_W(t)\| \\ &< \epsilon_n \end{aligned}$$

for all  $t \in [0, m]$ .

We will show that  $\mathbb{P}(W \in D_s \mid M_W = i) = 1$  for  $i \in \{1, \dots, k-1\}$  so that the desired conclusion follows.

Suppose first that  $M_W = 1$ . Let's prove that  $T_{Z_n, \delta_n}^1(P) \rightarrow T_W^1(P)$  as  $n \rightarrow \infty$ .

Let  $x = \arg \max_{T_W^1(P)} W_x(A)$ . By (3), for any  $\varepsilon \in (0, 1)$ , there exists a monotonically decreasing sequence  $\{s_k : k = 1, 2, \dots\} \subset (T_W^1(P), \infty)$  such that  $\lim_n s_n = T_W^1(P)$ ; and for  $k = 1, 2, \dots$  we have

$$\min_{y \in A - \{x\}} \{W_x(s_k) - W_x(T_W^1(P)) - W_y(s_k) + W_y(T_W^1(P))\} > v_k(-\epsilon) \equiv (-\epsilon) \sqrt{2(s_k - T_W^1(P)) \ln \ln(1/(s_k - T_W^1(P)))}$$

Notice that  $v_k > 0$  and  $\lim_k v_k = 0$ . Now,

$$\begin{aligned} \sum_{x' \in A - \{x\}} \frac{q_{s_k, x'}^W}{q_{s_k, x}^W} &= \left( \sum_{x' \in A - \{x\}} \exp \left( \frac{W_{x'}(s_k) - W_x(s_k)}{\lambda_z} + \frac{1}{\lambda_z^2} s_k (a_{x'} - a_x) \right) \right) \\ &\leq \left( \sum_{x' \in A - \{x\}} \exp \left( \frac{W_{x'}(T_W^1(P)) - W_x(T_W^1(P))}{\lambda_z} + \frac{1}{\lambda_z^2} s_k (a_{x'} - a_x) + \epsilon \frac{v_k}{\lambda_z} \right) \right) \\ (a_{x'} - a_x \leq 0) &\leq \left( \sum_{x' \in A - \{x\}} \exp \left( \frac{W_{x'}(T_W^1(P)) - W_x(T_W^1(P))}{\lambda_z} + \frac{1}{\lambda_z^2} T_W^1(P) (a_{x'} - a_x) + \epsilon \frac{v_k}{\lambda_z} \right) \right) \\ &\leq \exp \left( \epsilon \frac{v_k}{\lambda_z} \right) \left( \frac{1}{P} - 1 \right) \end{aligned}$$

Then

$$\begin{aligned} q_{s_k, x}^W &= \left[ 1 + \sum_{x' \in A - \{x\}} \frac{q_{s_k, x'}^W}{q_{s_k, x}^W} \right]^{-1} \\ &\geq \left[ 1 + \exp \left( \epsilon \frac{v_k}{\lambda_z} \right) \left( \frac{1}{P} - 1 \right) \right]^{-1} \\ &= \frac{P}{P + \exp \left( \epsilon \frac{v_k}{\lambda_z} \right) (1 - P)} \\ &\geq P + \epsilon_n + \delta_n - \alpha \end{aligned}$$

for all  $0 < \alpha < 1$  and  $k, n$  sufficiently large. Then  $s_k \geq T_W(P + \epsilon_n + \delta_n - \alpha)$  and so if  $\alpha_n = 1/n$

$$\begin{aligned} T_W(P) &\geq \limsup_m \limsup_n T_W(P + \epsilon_n + \delta_n - \alpha_n) \\ &= \limsup_n \limsup_m T_W(P + \epsilon_n + \delta_n - \alpha_n) \\ &\geq \limsup_n T_W(P + \epsilon_n + \delta_n) \\ &\geq \liminf_n T_W(P + \epsilon_n + \delta_n) \\ &\geq T_W(P) \end{aligned}$$



By Proposition 2,

$$T_W^1(P) = \lim_n T_{Z_n, \delta_n}^1(P).$$

Let  $x = \arg \max_x q_{T_W^1(P), x}^W(A)$ . Let  $\epsilon > 0$ . Let  $N$  such that if  $n > N$ , then  $2\epsilon_n + 2\epsilon < -\max_{y \in A - \{x\}} q_{T_W^1(P), y}^W(A) + q_{T_W^1(P), x}^W(A)$

$$\left| q_{T_W^1(P), x}^W(A) - q_{T_{Z_n, \delta_n}^1(P), x}^{Z_n, \delta_n}(A) \right| < \epsilon + \epsilon_n$$

for all  $x \in A$ , and so if  $z \in A - \{x\}$ ,

$$\begin{aligned} q_{T_{Z_n, \delta_n}^1(P), z}^{Z_n, \delta_n}(A) &< \epsilon + \epsilon_n + q_{T_W^1(P), z}^W(A) \\ &< -\epsilon - \epsilon_n + q_{T_W^1(P), x}^W(A) \\ &< q_{T_{Z_n, \delta_n}^1(P), x}^{Z_n, \delta_n}(A) \end{aligned}$$

and so  $x = \arg \max_x q_{T_{Z_n, \delta_n}^1(P), x}^{Z_n, \delta_n}(A)$ .

Now, we suppose that  $M_W = 2$ . By a similar argument than the previous one, we should have that  $T_W^1(A) = \lim_n T_{Z_n, \delta_n}^1$  and  $T_W^2(P_1^W) = \lim_n T_{Z_n, \delta_n}^2(P_1^{Z_n, \delta_n})$ . By a similar argument than before, we can see that  $\arg \max_x q_{T_{Z_n, \delta_n}^2(P_1^W), x}^{Z_n, \delta_n}(A_1^W) = \arg \max_x q_{T_W^2(P_1^W), x}^W(A_1^W)$  for  $n$  sufficiently large.

The cases  $M_W = i$  for  $k-1 \geq i \geq 3$  can be proved in a similar way.

Since almost surely  $M_Y \in \{1, \dots, k-1\}$  by Frazier [3], we conclude that

$$\mathbb{P}(W \in D[0, 1]^k - D_s) = 1.$$

■

By the extension of the CMT (Theorem 5.5 of Billingsley 1968), we have the following corollary.

**Corollary 1.** We have that

$$f(C(\delta, t), \delta) \Rightarrow g(W(t))$$

in distribution as  $\delta \rightarrow 0$ .

**Theorem 1.** If samples from system  $x \in \{1, \dots, k\}$  are identically distributed and independent, over time and across alternatives, then  $\lim_{\delta \rightarrow 0} Pr\{\text{BIZ selects } k\} \geq P^*$  provided  $\mu_k = a_k \delta, \mu_{k-1} = a_{k-1} \delta, \dots, \mu_1 = a_1 \delta$ . We also suppose  $B_1 = \dots = B_k = 1$ .

**Proof.** Let

$$\hat{T}_n(\delta) = \min \left\{ t \in \{0, \delta^2, 2\delta^2, \dots\} : \min_{x \in A_n^{Y, \delta}} q_{tx}^{C(\delta, \cdot), \delta} \left( A_n^{C(\delta, \cdot), \delta} \right) \leq c \text{ or } \max_{x \in A_n^{Y, \delta}} q_{tx}^{C(\delta, \cdot), \delta} \left( A_n^{C(\delta, \cdot), \delta} \right) \geq P_n^{C(\delta, \cdot), \delta} \right\}$$

and  $T_n(\delta)$  the usual stopping times of the algorithm. Then  $T_n(\delta) = \hat{T}_n(\delta)/\delta^2$ . Now, we can prove that  $\hat{T}_n(\delta) - T_{C(\delta, \cdot), \delta}^n(P_n^{C(\delta, \cdot), \delta}) \rightarrow 0$  with probability 1 as  $\delta \rightarrow 0$  using that  $C(\delta, \cdot)$  is right-continuous and  $\delta^2 \rightarrow 0$ . Consequently, we can use  $C(\delta, T_{C(\delta, \cdot), \delta}^n(P_n^{C(\delta, \cdot), \delta}))$  instead of  $C(\delta, \hat{T}_n(\delta))$ .

Let  $CS_\delta$  be the event of doing a correct selection given the configuration  $\mu_k = a_k\delta, \mu_{k-1}, \mu_{k-1} = a_{k-1}\delta, \dots, \mu_1 = a_1\delta$ . Then

$$\begin{aligned} \lim_{\delta \rightarrow 0} \mathbb{P}(CS_\delta) &= \lim_{\delta \rightarrow 0} \mathbb{P}(f(C(\delta, t), \delta) = 1) \\ &= \mathbb{P}(g(W) = 1) \\ &\geq P^* \end{aligned}$$

where the last inequality follows from the paper of Frazier [3]. ■

## References

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