On the Asymptotic Validity of a Fully Sequential Elimination Procedure for Indifference-Zone Ranking and Selection with Tight Bounds on Probability of Correct Selection

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We prove the validity of the sequential elimination IZ procedure proposed by Frazier [3] when δ goes to 0. Specifically, if k is the number of systems, we analyze Algorithm 2, when $B_1 = \cdots = B_k = 1$:

Algorithm 2: Discrete-time implementation of BIZ, for unknown and/or heterogeneous variances.

Require: $c \in [0, 1-(P^*)^{\frac{1}{k-1}}]$, $\delta > 0$, $P^* \in (1/k, 1)$, $n_0 \geq 0$ an integer, $B_1, \ldots, B_k$ strictly positive integers. Recommended choices are $c = 1-(P^*)^{\frac{1}{k-1}}$, $B_1 = \cdots = B_k = 1$ and n_0 between 10 and 30. If the sampling variances λ_x^2 are known, replace the estimators $\widehat{\lambda}_{tx}^2$ with the true values λ_x^2, and set $n_0 = 0$. We define $q_{tx}(A)$ where $A = \{1, \ldots, k\}$ as

$$q_{tx}(A) = \exp\left(\frac{\delta}{\lambda_x^2} Y_{n_x(t),x}\right) \Big/ \sum_{x' \in A} \exp\left(\frac{\delta}{\lambda_{x'}^2} Y_{n'_x(t),x'}\right), \tag{1}$$

where $Y_{n_x(t),x}$ is the sum of the first $n_x(t)$ samples.

For each x, sample alternative x n_0 times and set $n_{0x} \leftarrow n_0$. Let W_{0x} and $\widehat{\lambda}_{0x}^2$ be the sample mean and sample variance respectively of these samples. Let $t \leftarrow 0$.

Let $A \leftarrow \{1, \ldots, k\}$, $P \leftarrow P^*$, $t \leftarrow 1$.

while $x \in \max_{x \in A} q_{tx}(A) < P$ **do**

  while $\min_{x \in A} q_{tx}(A) \leq c$ **do**

    Let $x \in \arg\min_{x \in A} q_{tx}(A)$.

    Let $P \leftarrow P/(1 - q_{tx}(A))$.

    Remove x from A.

  end while

  Let $z \in \arg\min_{x \in A} n_{tx}/\widehat{\lambda}_{tx}^2$.

  For each $x \in A$, let $n_{t+1,x} = \text{ceil}\left(\widehat{\lambda}_{tx}^2 (n_{tz} + B_z)/\widehat{\lambda}_{tz}^2\right)$.

  For each $x \in A$, if $n_{t+1,x} > n_{tx}$, take $n_{t+1,x} - n_{tx}$ additional samples from alternative x. Let $W_{t+1,x}$ and $\widehat{\lambda}_{t+1,x}^2$ be the sample mean and sample variance respectively of all samples from alternative x thus far.

  Increment t.

end while

Select $\hat{x} \in \arg\max_{x \in A} W_{tx}/n_{tx}$ as our estimate of the best.

1 Introduction

This paper is organized as follows: In §2, we present the proof of the validity of the algorithm when the variances are known. In §3, we prove the case when the variances are unknown.

To prove the case when the variances are known, we use a theorem for Ergodic processes that shows how to standardize the output data to make them behave like Brownian motion processes in the limit. We also use an extension of the Continuous Mapping Theorem (Theorem 5.5 of Billingsley 1968) to see that the algorithm behaves like a sequential elimination IZ procedure with a Brownian motion process instead of the standardized sum of the output data in the limit. Finally, we use the results of the paper of Frazier [3] to prove the validity of this algorithm in the limit.

2 Asymptotic Validity when the Variances are Known

Without loss of generality, suppose that the true means of the systems are indexed so that $\mu_k > \mu_{k-1} > \cdots > \mu_1$. We suppose that samples from system $x \in \{1 \ldots, k\}$ are identically distributed and independent, over time and across alternatives. We also define $\lambda_z^2 := \max_{i\in\{1\ldots,k\}} \lambda_i^2$. We suppose that $\min_{i\in\{1\ldots,k\}} \lambda_i^2 > 0$ and $c \neq 1/k$. We are going to suppose that $\delta > 0$ and $\mu = \delta a$ for some a such that $a_k - a_i \geq 1$ for all i.

Now we are going to see that the standardized sum of the output data converges to a Brownian motion in $D[0,\infty)$, which is the set of functions from $[0,\infty)$ to $\mathbb{R}$ that are right-continuous and have left-hand limits, with the Skorohod topology. We are going to define the Skorohod topology on $D[0,\infty)$ by defining a metric on the space. The Skorohod metricric d_t on $D\left[0,t\right]$ is:

$$d_t\left(X,Y\right) = \inf_{\lambda\in\Lambda_t}\left\{\|\lambda - I\| \vee \|X - Y\circ\lambda\|\right\}$$

where Λ_t is the set of strictly increasing, continuous mappings of $[0,t]$ onto itself, and $\|\cdot\|$ is the uniform norm, and I is the identity map. Note that uniform convergence on $[0,t]$ implies Skorohod convergence.

We define the Skorohod topology on $D\left[0,\infty\right)$. For $X \in D\left[0,\infty\right)$, let X^m be the element of $D_\infty := D[0,\infty)$ defined by

$$X^m\left(t\right) = g_m\left(t\right)X\left(t\right)$$

where

$$g_m\left(t\right) = \begin{cases} 1 & \text{if } t \leq m-1, \\ m-t & \text{if } m-1 \leq t \leq m, \\ 0 & \text{if } t \geq m. \end{cases}$$

And now take

$$d_\infty\left(X,Y\right) = \sum_{m=1}^{\infty} 2^{-m}\left(1 \wedge d_m\left(X^m, Y^m\right)\right)$$

which is the Skorohod metric on $D\left[0,\infty\right)$. By Theorem 16.2 of Billingsley1999, there is convergence $d_\infty\left(x_n, x\right) \to 0$ in D_∞ if and only if $d_t\left(x_n,x\right) \to 0$ for each continuity point t of x.

The following lemma shows that the standardized sum of the output data converges to a Brownian motion in D_∞.

Lemma 1. If $x \in \{1 \ldots, k\}$, then

$$C_x(\delta, \cdot) := \frac{Y_{\text{ceil}\left(\frac{\lambda_x^2}{\lambda_z^2}\left(n_0 + \cdot \frac{1}{\delta^2}\right)\right), x} - \frac{\lambda_x^2}{\lambda_z^2}\left(n_0 + \cdot \frac{1}{\delta^2}\right)\mu_x}{\frac{\lambda_x^2}{\lambda_z \delta}} \Rightarrow W_x(\cdot)$$

in the sense of $D[0, \infty)$, where $Y_{n,x}$ is the sum of the first n samples and W_x is a standard Brownian motion.

Proof. By the Theorem 19.1 of Billingsley 1999 and the sandwich theorem,

$$\frac{Y_{\text{ceil}\left(\frac{\lambda_x^2}{\lambda_z^2}\left(\cdot \frac{1}{\delta^2}\right)\right), x} - \text{ceil}\left(\frac{\lambda_x^2}{\lambda_z^2}\left(\cdot \frac{1}{\delta^2}\right)\right)\mu_x}{\frac{\lambda_x^2}{\lambda_z}\sqrt{\frac{1}{\delta^2}}} \Rightarrow W_x(\cdot).$$

Since $\frac{\frac{\lambda_x^2}{\lambda_z^2} t \frac{1}{\delta^2} - ceil\left(\frac{\lambda_x^2}{\lambda_z^2} t \frac{1}{\delta^2}\right)}{\frac{\lambda_x^2}{\lambda_z}\sqrt{\frac{1}{\delta^2}}} \to 0$ uniformly on $[0, r]$ for every r, then it also converges to 0 on $D[0, \infty)$ and so

$$\frac{Y_{\text{ceil}\left(\frac{\lambda_x^2}{\lambda_z^2}\left(\cdot \frac{1}{\delta^2}\right)\right), x} - \left(\frac{\lambda_x^2}{\lambda_z^2}\left(\cdot \frac{1}{\delta^2}\right)\right)\mu_x}{\frac{\lambda_x^2}{\lambda_z}\sqrt{\frac{1}{\delta^2}}} \Rightarrow W_x(\cdot).$$

Observe that for $\epsilon > 0$ and δ sufficiently small

$$\left| \frac{-Y_{\text{ceil}\left(\frac{\lambda_x^2}{\lambda_z^2} t \frac{1}{\delta^2}\right), x} + Y_{\text{ceil}\left(n_0 \frac{\lambda_x^2}{\lambda_z^2} + \frac{\lambda_x^2}{\lambda_z^2} t \frac{1}{\delta^2}\right), x}}{\frac{\lambda_x^2}{\lambda_z}\sqrt{\frac{1}{\delta^2}}} \right| < \epsilon\left(n_0 \frac{\lambda_x^2}{\lambda_z^2} + 2\right)$$

and then

$$C_x(\delta, \cdot) \Rightarrow W_x(\cdot).$$

■

Now we are going to define new algorithms that are almost the same as the continuous-time procedure proposed by Frazier, but these algorithms use new functions $q_{tx}^{Y,\delta}(A)$ which depend on δ and a function C that is in $D[0, \infty)^k$. More explicity, if we define

$$q'_{tx}(A) = \exp\left(\frac{\delta}{\lambda_x^2} Y'_{n_x(t), x}\right) \Bigg/ \sum_{x' \in A} \exp\left(\frac{\delta}{\lambda_{x'}^2} Y'_{n'_x(t), x'}\right)$$

where Y'_{tx} is a Brownian motion under $\mathbb{P}_{\mu,\lambda}$ starting from 0, with drift μ_x, volatility λ_x, and independence across x, the procedure proposed by Frazier is defined by first setting

$$\tau_0 = 0, P_0 = P^*, A_0 = \{1, \ldots, k\}$$

then defining recursively, for $n = 0, 1, \ldots, k-2$,

$$\begin{aligned}
\tau_{n+1} &= \inf\left\{t \geq \tau_n : \ \min_{x\in A_n} q_{tx}^{'}\left(A_n\right) \leq c \text{ or } \max_{x\in A_n} q_{tx}^{'}\left(A_n\right) \geq P_n\right\} \\
Z_{n+1} &\in \arg\min_{x\in A_n} q_{\tau_{n+1},x}^{'}\left(A_n\right) \\
A_{n+1} &= A_n \backslash \{Z_{n+1}\} \\
P_{n+1} &= P_n \Big/ \left(1 - \min_{x\in A_n} q_{\tau_{n+1},x}^{'}\left(A_n\right)\right)
\end{aligned}$$

and finally letting the selected alternative $\hat{x}$ be the single entry in A_{k-1}. We also define

$$M = \inf\left\{n = 1, \ldots, k-1 : \max_{x\in A_n} q_{\tau_n,x}^{'}\left(A_{n-1}\right) \geq P_{n-1}\right\}.$$

If $C \in D\left[0,\infty\right)^k$, we define new functions $q_{tx}^{C,\delta}\left(A\right)$ by

$$q_{tx}^{C,\delta}\left(A\right) = \exp\left(\frac{C_x\left(t\right)}{\lambda_z} + \delta^2 \frac{\left(n_0 + t\frac{1}{\delta^2}\right)}{\lambda_z^2} a_x\right) \Big/ \sum_{x'\in A} \exp\left(\frac{C_{x'}\left(t\right)}{\lambda_z} + \delta^2 \frac{\left(n_0 + t\frac{1}{\delta^2}\right)}{\lambda_z^2} a_{x'}\right).$$

The reason of this definition is because $C\left(\delta,\cdot\right) = \left(C_x\left(\delta,\cdot\right)\right)_{x\in A} \in D\left[0,\infty\right)^k$ and

$$\frac{\delta}{\lambda_x^2} Y_{n_x(\cdot),x} = \frac{C_x\left(\delta,\cdot\right)}{\lambda_z} + \delta^2 \frac{\left(n_0 + \cdot\frac{1}{\delta^2}\right)}{\lambda_z^2} a_x \Rightarrow W_x^{'}\left(\cdot\right) := \frac{W_x\left(\cdot\right)}{\lambda_z} + \frac{\cdot}{\lambda_z^2} a_x$$

where $W_x^{'}$ is a Brownian motion starting from 0, with drift $\frac{\mu_x}{\lambda_z^2}$ and volatility $\frac{1}{\lambda_z}$.

Note that the functions $q_{tx}^{C,\delta}\left(A\right)$ define a new algorithm for each $C \in D\left[0,\infty\right)^k$ and $\delta > 0$.

We also define a new function

$$f\left(C,\delta\right) = \begin{cases} 1 & \text{if } k \text{ is chosen} \\ 0 & \text{otherwise} \end{cases}$$

using the algorithm that $\left(C,\delta\right)$ induces.

Now, we also define new functions $q_{tx}^{C}\left(A\right)$ to analyze the limit of $f\left(C,\delta\right)$ when δ goes to zero:

$$q_{tx}^{C}\left(A\right) := \exp\left(\frac{C_x\left(t\right)}{\lambda_z} + \frac{1}{\lambda_z^2} t a_x\right) \Big/ \sum_{x'\in A} \exp\left(\frac{C_{x'}\left(t\right)}{\lambda_z} + \frac{1}{\lambda_z^2} t a_{x'}\right)$$

and we define the functions

$$g\left(Y\right) = \begin{cases} 1 & \text{if } k \text{ is chosen} \\ 0 & \text{otherwise} \end{cases}.$$

We want to prove that

$$f\left(C\left(\delta,\cdot\right),\delta\right) \Rightarrow g\left(W\right).$$

In order to prove this, we will prove the Lemma 2 which will allow us to use the Theorem 5.5 of Billingsley 1968 that implies the desired result.

Lemma 2. Let $\{\delta_n\} \subset (0,\infty)$ such that $\delta_n \to 0$. If $D_s \equiv \{x \in D[0,\infty)^k :$ for all sequences $\{x_n\} \subset D[0,\infty)^k$ such that $\lim_n d(x_n, x) = 0$ the sequence $\{f(x_n, \delta_n)\}$ converges to $\{g(x)\}\}$, then $\mathbb{P}(W \in D_s) = 1$.

First, we are going to prove the following three propositions.

Proposition 1. Suppose $\{f_n\}$ and $\{g_n\}$ are two sequences of functions on D_∞ such that $f_n \to f$ and $g_n \to g$ in the sense of D_∞. If f and g are continuous, then

$$\min(f_n, g_n) \to \min(f, g)$$

in the sense of D_∞.

Proof. Let $t^* > 0$. We will prove that $\min(f_n, g_n) \to \min(f, g)$ in the sense of D_{t^*} and the theorem will follow.

Since f and g are uniformly continuous in $[0, t^*]$ and $f_n \to f$ and $g_n \to g$ in $[0, t^*]$, then f_n and g_n converge uniformly to f and g, respectively. Consequently, $(f_n, g_n) \to (f, g)$ uniformly in $[0, t^*]$. Let $a_f^+ = \max_{t\in[0,t^*]} f(t)$, $a_f^- = \min_{t\in[0,t^*]} f(t)$, $a_g^+ = \max_{t\in[0,t^*]} g(t)$ and $a_g^- = \min_{t\in[0,t^*]} g(t)$. Let N such that if $n \geq N$, then $|f_n(t) - f(t)| < 1$ and $|g_n(t) - g(t)| < 1$ for all $t \in [0, t^*]$. Consequently, if $n \geq N$, $f_n(t) \in \left[a_f^- - 1, a_f^+ + 1\right]$ and $g_n(t) \in \left[a_g^- - 1, a_g^+ + 1\right]$ for all t in $[0, t^*]$. Since $\min(x, y)$ is continuous, then it is uniformly continuous in $A = \left[a_f^- - 1, a_f^+ + 1\right] \times \left[a_g^- - 1, a_g^+ + 1\right]$.

Let $\epsilon > 0$. Let $\delta > 0$ such that if $\|(u, v)\|_2 < \delta$ and $(u := (u_1, u_2), v := (v_1, v_2)) \in A$, then

$$|\min(u_1, u_2) - \min(v_1, v_2)| < \epsilon.$$

Let $M > N$ such that if $n > M$, then

$$\|(f_n(t), g_n(t)) - (f(t), g(t))\|_2 < \delta$$

for all $t \in [0, t^*]$. Consequently, if $n > M$,

$$|\min(f_n(t), g_n(t)) - \min(f(t), g(t))| < 1$$

for all $t \in [0, t^*]$. Since uniform convergence implies Skorohod convergente, then $\min(f_n, g_n) \to \min(f, g)$ in the sense of D_{t^*}.

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Proposition 2. Suppose $\{f_n\}$ is a sequence of functions on D_∞ such that $f_n \to f$ in the sense of D_∞, f is continuous, and $\{T_n\} \subset [0,\infty)$ is a sequence such that $T_n \to T$. We define $T(a) := \inf\{t \geq T : f(t) \geq a\}$ for each $a \in \mathbb{R}$. Suppose $T(0) \in \mathbb{R}$. Suppose there exists $\{\epsilon_n\} \subset (0,\infty)$ such that $\epsilon_n \to 0$, $\epsilon_n \geq \epsilon_{n+1}$, and

$$\|f_n(t) - f(t)\|_2 < \epsilon_n$$

for all $t \in [0, T(0)]$. We also suppose that $\lim\sup_n T(\epsilon_n) \leq T(0)$. Thus we have that

$$\inf\{t \geq T_n : f_n(t) \geq 0\} \to \inf\{t \geq T : f(t) \geq 0\}$$

if $T(0) > T$ or $T_n = T$.

Proof. We are going to suppose $T(0) > T$, and the case when $T_n = T$ can be proved using almost the same ideas. We introduce the notation $T_n(a) : \inf\{t \geq T_n : f_n(t) \geq a\}$ $a \in \mathbb{R}$. Let N such that if $n > N$, $T_n < T + T(0) - T = T(0)$ and $\epsilon > \epsilon_n$ where $\epsilon := sup_n \epsilon_n$. Let $n > N$. Note that $T_n(0) \leq T_n(\epsilon_n)$.

We also have that

$$\begin{aligned} T(0) &\leq \lim\inf_n T(\epsilon_n) \\ &\leq \lim\sup_n T(\epsilon_n) \end{aligned} \tag{2}$$

Now, since $\lim\sup_n T(\epsilon_n) \leq T(0)$,

$$\begin{aligned} T(0) &\geq \lim\sup_n T(\epsilon_n) \\ &\geq \lim\inf_n T(\epsilon_n) \\ &\geq T(0) \end{aligned}$$

and so

$$\lim\inf_n T_n(0) \leq \lim\sup_n T_n(0) \leq \lim_n T(\epsilon_n) = T(0)$$

Now, let's prove that $\lim\inf_n T_n(0) \geq T(0)$. Let M such that $T(0) - \frac{1}{m} \geq T$ if $m > M$, and let $t_m = T(0) - \frac{1}{m}$ and $\alpha_m = \max\{f(t) : t \in [T, t_m]\}$. Note that $\alpha_m < 0$ because $t_m < T(0)$. Let N such that if $n > N$, then

$$\epsilon_n \leq -\alpha_m$$

Thus, if $t \in [T, t_m]$

$$f_n(t) < f(t) + \epsilon_n \leq 0$$

If $T \leq T_n(0)$, then $T_n(0) \geq t_m$ and so $\lim\inf_n T_n(0) \geq t_m$. Consequently, $\lim\inf_n T_n(0) \geq T(0)$.

Let's prove that $T \leq T_n(0)$ for n large. Let $s > 0$ be any number. Let N such that if $n > N$, then $\epsilon_n < s$. Suppose that for every $\epsilon > 0$ there exists $n > N$ such that $T - \epsilon < T_n(0) < T$. Note that

$$\begin{aligned} f(T_n(0)) &\geq f_n(T_n(0)) - \epsilon_n \\ &\geq -\epsilon_n > -s \end{aligned}$$

Let $a > 0$, since f is continuous there exists $\epsilon > 0$ such that $f(x) < f(T) + a$ if $|x - T| < \epsilon$. Thus, there exists $n > N$ such that $T - \epsilon < T_n(0) < T$ and so

$$-s < f(T_n(0)) < f(T) + a$$

Since s and a are arbitrary,

$$f(T) \geq 0$$

which is a contradiction. Consequently, $T \leq T_n(0)$ and so

$$\lim_n T_n(0) = T(0).$$

$\blacksquare$

Corollary 1. Let $\{Z_n\} \subset D[0,\infty)^k$ be a sequence of functions such that $Z_n \to W$ a.s. in $D[0,\infty)$, then

$$f_{Z_n}(\cdot) := \max\left\{c - \min_{x\in A} q_{\cdot x}^{Z_n,\delta_n}(A), \max_{x\in A} q_{\cdot x}^{Z_n,\delta_n}(A) - P\right\} \overset{a.s.}{\to} f_W := \max\left\{c - \min_{x\in A} q_{\cdot x}^{W}(A), \max_{x\in A} q_{\cdot x}^{W}(A) - P\right\}$$

in $D[0,\infty)$ for any $P \in (0,1)$ and $A \subset \{1,\dots,k\}$.

Furthermore, if $m \in \mathbb{N}$, there exists almost surely a sequence $\{\epsilon_k\}$ such that $\epsilon_k \downarrow 0$ and

$$|f_{Z_k}(t) - f_W(t)| < \epsilon_k$$

for all $t \in [0,m]$.

Proof. We will only consider the event $\mathcal{S}$ such that W is continuous with probability 1.

Let $\{Z_n\} \subset D[0,\infty)^k$ such that $Z_n \to W$ almost surely. We Note that $\delta_n^2 \frac{\left(n_0 + t\frac{1}{\delta_n^2}\right)}{\lambda_z^2} a_x \to \frac{t}{\lambda_z^2} a_x$ in $D[0,\infty)$ and $\frac{\delta_n \sqrt{\frac{1}{\delta_n^2}}}{\lambda_z} \to \frac{1}{\lambda_z}$ in $D[0,\infty)$ because uniformly convergence implies convergence in the Skorohod topology. Consequently, for each $s > 0$ there exist functions λ_s^n in Λ_s such that

$$\lim_n Z_n(\lambda_s^n t) = W(t)$$

uniformly in t and

$$\lim_n \lambda_s^n t = t$$

uniformly in t. Then

$$\lim_n \frac{\delta_n \sqrt{\frac{1}{\delta_n^2}}}{\lambda_z} Z_n(\lambda_s^n t) + \delta_n^2 \frac{\left(n_0 + \lambda_s^n(t)\frac{1}{\delta_n^2}\right)}{\lambda_z^2} a_x = W(t)\frac{1}{\lambda_z} + \frac{t}{\lambda_z^2} a_x$$

uniformly in t, and so

$$\lim_n \exp\left(\frac{\delta_n \sqrt{\frac{1}{\delta_n^2}}}{\lambda_z} Z_n(\lambda_n t) + \delta_n^2 \frac{\left(n_0 + \lambda_s^n(t)\frac{1}{\delta_n^2}\right)}{\lambda_z^2} a_x\right) = \exp\left(W(t)\frac{1}{\lambda_z} + \frac{t}{\lambda_z^2} a_x\right)$$

uniformly in t since exp is uniformly continuous in $[0,s]$. Consequently,

$$q_{\lambda_s^n(t)x}^{Z_n,\delta_n}(A) \to q_{tx}^{W}(A)$$

uniformly in $t \in [0,s]$. Thus, $q_{\cdot x}^{Z_n,\delta_n}(A) \to q_{\cdot x}^{W}(A)$ in $D[0,s]$ for any set $A \subset \{1,\dots,k\}$ and $s \geq 0$.

Consequently, $q_{\cdot x}^{Z_n,\delta_n}(A) \to q_{\cdot x}^{W}(A)$ in $D[0,\infty)$. By Proposition 1, $\min_{x\in A} q_{\cdot x}^{Z_n,\delta_n}(A) \to \min_{x\in A} q_{\cdot x}^{W}(A)$ and $\max_{x\in A} q_{\cdot x}^{Z_n,\delta_n}(A) \to \max_{x\in A} q_{\cdot x}^{W}(A)$, and so

$$f_{Z_n} := \max\left\{c - \min_{x\in A} q_{\cdot x}^{Z_n,\delta_n}(A), \max_{x\in A} q_{\cdot x}^{Z_n,\delta_n}(A) - P_s\right\} \to f_W := \max\left\{c - \min_{x\in A} q_{\cdot x}^{W}(A), \max_{x\in A} q_{\cdot x}^{W}(A) - P_s\right\}$$

in $D[0,\infty)$ for any P_s of the algorithm.

Then for each $m \geq 0$ there exists $\lambda_n \in \Lambda_\infty$ such that

$$\begin{aligned}\sup_{t\leq m}\|\lambda_n(t) - t\|_2 &\leq d(f_{Z_n}, f_W) + \frac{1}{n}\\ \sup_{t\leq m}\|f_{Z_n}(t) - f_W(\lambda_n t)\|_2 &\leq d(f_{Z_n}, f_W) + \frac{1}{n}.\end{aligned}$$

Taking $g_n \equiv \sup_{t\leq m}\|f_W(t) - f_W(\lambda_n t)\|_2$, we see from the uniform continuity of f_W on $[0,m]^k$ (f_W is uniformly continuous because it's continuos in a compact set) and the definition of g_n that $\lim_{n\to\infty} g_n = 0$. Moreover, if we take $\epsilon_n = 2n^{-1} + 2\sup\{d(f_{Z_l}, f_W) + g_l : l = n, n+1, \ldots\}$, then $\{\epsilon_n\}$ is a monotonically decreasing sequence of positive numbers with limit zero.

From the definition of ϵ_n we have $d(f_{Z_n}, f_W) < \epsilon_n/2$ and $g_n < \epsilon_n/2$ for $n = 1, 2, \ldots$ Consequently, we have

$$\begin{aligned}\|f_{Z_n}(t) - f_W(t)\| &\leq \|f_{Z_n}(t) - f_W(\lambda_n t)\| + \|f_W(\lambda_n t) - f_W(t)\|\\ &< \epsilon_n\end{aligned}$$

for all $t \in [0, m]$.

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Proposition 3. Let $W = (W_1, \ldots, W_k)$ be a $k-$dimensional Brownian motion where each W_i is a standard Brownian motion and $W_i \amalg W_j$ for all $i \neq j$. Let $q_{tx}^{W}(A) := \exp\left(\frac{1}{\lambda}\left(W_x(t) + t\frac{a_x}{\lambda_z}\right)\right) / \sum_{x'\in A}\exp\left(\frac{1}{\lambda}\left(W_{x'}(t) + t\frac{a_{x'}}{\lambda_z}\right)\right)$ for all $x \in A$ and $A \subset \{1, \ldots, k\}$. Fix $m \in \{1, \ldots, k-1\}$. We have that for all $N \in \mathbb{N}$, there exists t such that $\tau_m + \frac{1}{N} \geq t > \tau_m$ and $q_{tx} > P_{m-1}$ for some $x \in A_{m-1}$ almost surely given that $M = m$.

Proof. We know that $\tau_m < \infty$ almost surely by Lemma 4 of the paper of Frazier [3]. We will only consider the event $\mathcal{S}$ such that the previous property holds with probability 1.

Define $a = \lambda_z \log\left(\frac{P_{m-1}(k-1)}{1-P_{m-1}}\right)$. If $m = 1$, then $P_{m-1} = P^* \in (0,1)$, and so a is finite. If $m > 1$, then

$$P^* \leq P_{m-1} \leq P^*/(1-c)^{m-1} \leq P^*/(1-c)^{k-2} \leq P^*/(P^*)^{k-2/k-1} < 1$$

and so $P_{m-1} \in (0,1)$ and a is finite.

We define $T^* = \inf\left\{t \geq \tau_m : q_{t,x}^{W}(A_{m-1}) > P_{m-1} \text{ for some } x \in A_{m-1}\right\}$. Fix $n \in \mathbb{N}$, $n > 2$, and define $\tau = \tau_m + \frac{t^*}{nN}$ for a derministic $t^* \in \{1, \ldots, n-1\}$. Let x be any $\mathcal{F}_{\tau_m}$-measurable random variable that is almost surely in A_{m-1} and we define $\Gamma_{t,x} = W_x(t) + t\frac{a_x}{\lambda_z}$. Consider the event $a < \Gamma_{\tau+\frac{1}{nN},x} - \Gamma_{\tau+\frac{1}{nN},y}$ for each $y \in A_{m-1} - \{x\}$. On this event, $q_{\tau+\frac{1}{mN},y}^{W}(A_{m-1}) / q_{\tau+\frac{1}{mN},x}^{W}(A_{m-1}) = \exp\left(\frac{1}{\lambda_z}\left(\Gamma_{\tau+\frac{1}{nN},y} - \Gamma_{\tau+\frac{1}{nN},x}\right)\right) < \exp(-a/\lambda_z)$

for $y \in A_{m-1} - \{x\}$ and

$$q_{\tau+\frac{1}{nN},x}\left(A_{m-1}\right) = \left[1 + \sum_{y \in A_{m-1}-\{x\}} q_{\tau+\frac{1}{nN},y}\left(A_{m-1}\right)/q_{\tau+\frac{1}{nN},x}\left(A_{m-1}\right)\right]^{-1} > \left[1+(k-1)\exp\left(-a/\lambda_z\right)\right]^{-1} = P_{m-1}.$$

Thus, on the event considered, $T^* \leq \tau + \frac{1}{nN}$.

We now define $\tilde{x} \in \arg\max_{x \in A_{m-1}} \Gamma_{\tau,x}$, which is $\mathcal{F}_\tau$-measurable and is almost surely in A_{m-1}. Then we have that

$$\begin{aligned}
\mathbb{P}\left\{T^* \leq \tau + \frac{1}{nN} \mid \mathcal{F}_\tau, T^* > \tau\right\} &\geq \mathbb{P}\left\{a < \Gamma_{\tau+\frac{1}{nN},\tilde{x}} - \Gamma_{\tau+\frac{1}{nN},x}\ \forall x \in A_{m-1} - \{\tilde{x}\} \mid \mathcal{F}_\tau, T^* > \tau\right\} \\
&\geq \mathbb{P}\left\{\Gamma_{\tau+\frac{1}{nN},\tilde{x}} \geq \Gamma_{\tau,\tilde{x}}, \Gamma_{\tau,\tilde{x}} - \Gamma_{\tau+\frac{1}{nN},x} > a\ \forall x \in A_{m-1} - \{\tilde{x}\} \mid \mathcal{F}_\tau, T^* > \tau\right\} \\
&\geq \mathbb{P}\left\{\Gamma_{\tau+\frac{1}{nN},\tilde{x}} \geq \Gamma_{\tau,\tilde{x}}, \Gamma_{\tau,x} - \Gamma_{\tau+\frac{1}{nN},x} > a\ \forall x \in A_{m-1} - \{\tilde{x}\} \mid \mathcal{F}_\tau, T^* > \tau\right\} \\
&= \mathbb{P}\left\{\Gamma_{\tau+\frac{1}{nN},\tilde{x}} \geq \Gamma_{\tau_m,\tilde{x}} \mid \mathcal{F}_\tau\right\} \prod_{x \in A_{m-1}\setminus\{\tilde{x}\}} \mathbb{P}\left\{\Gamma_{\tau,x} - \Gamma_{\tau+\frac{1}{nN},x} > a \mid \mathcal{F}_\tau\right\}.
\end{aligned}$$

Note that $\Gamma_{t,x}$ is a Brownian motion, and so $\mathbb{P}\left\{\Gamma_{\tau+\frac{1}{nN},\tilde{x}} \geq \Gamma_{\tau_m,\tilde{x}} \mid \mathcal{F}_{\tau_m}\right\}$ is the probability of a conditionally $N\left(\frac{a_{\tilde{x}}}{nN}\frac{1}{\lambda_z}, 1\right)$ random variable $\Gamma_{\tau+\frac{1}{nN},\tilde{x}} - \Gamma_{\tau,\tilde{x}}$, exceeding 0. This probability is $\Phi\left(\frac{a_{\tilde{x}}}{nN}\frac{1}{\lambda_z}\right)$, which is bounded below by $\Phi\left(\frac{\min_x a_x}{N}\frac{x_0}{\lambda_z}\right)$ where $x_0 = 0$ if $\min_x a_x \geq 0$, and $x_0 = \frac{1}{3}$ otherwise. Here, Φ is the normal cumulative distribution function. Similarly, the probability $\mathbb{P}\left\{\Gamma_{\tau,x} - \Gamma_{\tau+\frac{1}{nN},x} > a \mid \mathcal{F}_{\tau_m}\right\}$ is the probability of a conditionally $N\left(-a - \frac{a_x}{nN}\frac{1}{\lambda_z}, 1\right)$ random variable, $\Gamma_{\tau,x} - \Gamma_{\tau+\frac{1}{nN},x} - a$, exceeding 0. This probability is $\Phi\left(-a - \frac{a_x}{nN}\frac{1}{\lambda_z}\right)$, and is bounded below by $\Phi\left(-a - \frac{\max_x a_x}{N}\frac{x_0^+}{\lambda_z}\right)$ where $x_0^+ = 0$ if $\max_x a_x \leq 0$ and $x_0^+ = \frac{1}{3}$ otherwise.

Thus, replacing τ with $\tau_m + \frac{t^*}{nN}$,

$$\mathbb{P}\left\{T^* \leq \tau_m + \frac{t^*}{nN} + \frac{1}{nN} \mid \mathcal{F}_{\tau_m+\frac{t^*}{nN}}, T^* > \tau_m + \frac{t^*}{nN}\right\} \geq \Phi\left(\frac{\min_x a_x}{N}\frac{x_0}{\lambda_z}\right)\Phi\left(-a - \frac{\max_x a_x}{N}\frac{x_0^+}{\lambda_z}\right)^{k-1}$$

Let ϵ be the quantity on the righ-hand side of this inequality, then $\epsilon < 1$ and it does not depend on t^* and n.

By repeated application of this inequality, we have that $\mathbb{P}\left\{T^* > T_W^1(P) + \frac{1}{N} \mid \mathcal{F}_{\tau_m}\right\} \leq (1-\epsilon)^n$ for all $n > 2$. This is true because

$$\begin{aligned}
\mathbb{P}\left\{T^* > T_W^1(P) + \frac{1}{N} \mid \mathcal{F}_{\tau_m+\frac{n-1}{nN}}, T^* > \tau_m + \frac{n-1}{nN}\right\} &< (1-\epsilon) \\
\Rightarrow \mathbb{P}\left\{T^* > \tau_m + \frac{1}{N} \mid \mathcal{F}_{T_W^1(P)+\frac{n-2}{nN}}, T^* > \tau_m + \frac{n-2}{nN}\right\} &< (1-\epsilon)\,\mathbb{P}\left\{T^* > \tau_m + \frac{n-1}{nN} \mid \mathcal{F}_{\tau_m++\frac{n-2}{nN}}, T^* > \tau_m + \frac{n-2}{nN}\right\} \\
&\leq (1-\epsilon)^2 \\
&\vdots \\
\Rightarrow \mathbb{P}\left\{T^* > \tau_m + \frac{1}{N} \mid \mathcal{F}_{T_W^1(P)}\right\} &\leq (1-\epsilon)^n
\end{aligned}$$

and $(1-\epsilon)^n$ vanishes in the limit as $n \to \infty$. Then, $\mathbb{P}\left\{T^* > \tau_m + \frac{1}{N} \mid \mathcal{F}_{T_W^1(P)}\right\} = 0$ and then

$$\mathbb{P}\left\{\tau_m \leq T^* \leq \tau_m + \frac{1}{N} \mid \mathcal{F}_{T_W^1(P)}\right\} = 1$$

for all N.

■

Proposition 4. Let $W=(W_1,\ldots,W_k)$ be a $k-$dimensional Brownian motion where each W_i is a standard Brownian motion and $W_i \amalg W_j$ for all $i\neq j$. Let $q_{tx}^{W}(A) := \exp\left(\frac{1}{\lambda}\left(W_x(t)+t\frac{a_x}{\lambda_z}\right)\right)/\sum_{x'\in A}\exp\left(\frac{1}{\lambda}\left(W_{x'}(t)+t\frac{a_{x'}}{\lambda_z}\right)\right)$ for all $x\in A$ and $A\subset\{1,\ldots,k\}$. Fix $m\in\{1,\ldots,k-1\}$. We have that for all $N\in\mathbb{N}$, there exists t such that $\tau_m-\frac{1}{N}\leq t<\tau_m$ and $q_{tx}<c$ for some $x\in A_{m-1}$ almost surely given that $M>m$.

Proof of Lemma 2. Let $\{Z_n\}\subset D[0,\infty)^k$ such that $Z_n\to W$. We will show that $\mathbb{P}(W\in D_s \mid M_W=i)=1$ for $i\in\{1,\ldots,k-1\}$ so that the desired conclusion follows.

Suppose first that $M_W=1$. Let's prove that $T^1_{Z_n,\delta_n}\to T^1_W$ as $n\to\infty$.

First, we are going to prove that $T^1_W(P)=\lim_n T^1_W(P+\epsilon_n)$ with probability 1. Note that $\lim_n T^1_W(P+\epsilon_n)=\inf_n T^1_W(P+\epsilon_n)$ and so we have to prove that for all $M>0$ there exists n such that $T^1_W(P+\epsilon_n)<T^1_W(P)+M$. Equivalently, we should prove that for all N there exists $t\in\left(T^1_W(P),T^1_W(P)+\frac{1}{N}\right]$ such that for some $x\in A$, $q^W_{t,x}(A)>P$ where $\hat{x}=\arg\max q^W_{T^1_W x}(A)$. However, from proposition 2, there is an event $\mathcal{S}$ such that the previos property holds and has probability 1. We will only consider this event.

By Proposition 2,

$$T^1_W=\lim_n T^1_{Z_n,\delta_n}.$$

Let $x\in\gamma:=\arg\max_x q^W_{T^1_W,x}(A)$. Let $\epsilon>0$ such that $2\epsilon<-\max_{y\in A-\gamma}q^W_{T^1_W,y}(A)+q^W_{T^1_W,x}(A)$.

Let N such that if $n>N$

$$\left|q^W_{T^1_W,x}(A)-q^{Z_n,\delta_n}_{T^1_{Z_n,\delta_n},x}(A)\right|<\epsilon$$

for all $x\in A$.

By Corollary 1, there exist a sequence $\{\epsilon_n\}$ decreasing to zero and N_2 such that if $n>N_2$, then $2\epsilon_n+2\epsilon<-\max_{y\in A-\gamma}q^W_{T^1_W,y}(A)+q^W_{T^1_W,x}(A)$ and

$$\left|q^W_{T^1_W,x}(A)-q^{Z_n,\delta_n}_{T^1_{Z_n,\delta_n},x}(A)\right|<\epsilon_n$$

for all $x\in A$. Consequently if $n>N^*=\max\{N,N_2\}$, then

$$\begin{aligned}\left|q^W_{T^1_W,x}(A)-q^{Z_n,\delta_n}_{T^1_{Z_n,\delta_n},x}(A)\right| &\leq \left|q^W_{T^1_W,x}(A)-q^{Z_n,\delta_n}_{T^1_W,x}(A)\right|+\left|q^W_{T^1_W,x}(A)-q^{Z_n,\delta_n}_{T^1_{Z_n,\delta_n},x}(A)\right|\\ &< \epsilon+\epsilon_n\end{aligned}$$

for all $x\in A$.

If $z\in A-\gamma$,

$$\begin{aligned}q^{Z_n,\delta_n}_{T^1_{Z_n,\delta_n},z}(A) &< \epsilon+\epsilon_n+q^W_{T^1_W,z}(A)\\ &< -\epsilon-\epsilon_n+q^W_{T^1_W,x}(A)\\ &< q^{Z_n,\delta_n}_{T^1_{Z_n,\delta_n},x}(A)\end{aligned}$$

and so $x \in \arg\max_x q^{Z_n,\delta_n}_{T^1_{Z_n,\delta_n}(P),x}(A)$. By Lemma 6 of Frazier [3], if $n > N^*$, both procedures choose x as the best system. Consequently, $f(Z_n, \delta_n)$ converges to $g(W)$ almost surely given that $M_W = 1$.

Now, we suppose that $M_W = 2$. By a similar argument than the previous one, we should have that $T^1_W(A) = \lim_n T^1_{Z_n,\delta_n}$ and $T^2_W(P^W_1) = \lim_n T^2_{Z_n,\delta_n}\left(P^{Z_n,\delta_n}_1\right)$. By a similar argument than before, we can see that $\arg\max_x q^{Z_n,\delta_n}_{T^2_{Z_n,\delta_n}(P^W_1),x}\left(A^W_1\right) = \arg\max_x q^W_{T^2_W(P^W_1),x}\left(A^W_1\right)$ for n sufficiently large. Consequently, $f(Z_n, \delta_n)$ converges to $g(W)$ almost surely given that $M_W = 2$.

The cases $M_W = i$ for $k-1 \geq i \geq 3$ can be proved in a similar way.

Since almost surely $M_Y \in \{1, \ldots, k-1\}$ by Frazier [3], we conclude that

$$\mathbb{P}\left(W \in \mathrm{D}[0,1]^k\text{-}D_s\right) = 1.$$

$\blacksquare$

By the extension of the CMT (Theorem 5.5 of Billingsley 1968), we have the following corollary.

Corollary 2. We have that

$$f(C(\delta,t), \delta) \Rightarrow g(W(t))$$

in distribution as $\delta \to 0$.

Theorem 1. If samples from system $x \in \{1 \ldots, k\}$ are identically distributed and independent, over time and across alternatives, then $\lim_{\delta\to 0} Pr\{\text{BIZ selects } k\} \geq P*$ provided $\mu_k = a_k\delta, \mu_{k-1} = a_{k-1}\delta, \ldots, \mu_1 = a_1\delta$. We also suppose $B_1 = \cdots = B_k = 1$.

Proof. Let

$$\hat{T}_n(\delta) = \min\left\{t \in \left\{0, \delta^2, 2\delta^2, \ldots\right\}: \ \min_{x\in A^{Y,\delta}_n} q^{C(\delta,\cdot),\delta}_{tx}\left(A^{C(\delta,\cdot),\delta}_n\right) \leq c \text{ or } \max_{x\in A^{Y,\delta}_n} q^{C(\delta,\cdot),\delta}_{tx}\left(A^{C(\delta,\cdot),\delta}_n\right) \geq P^{C(\delta,\cdot),\delta}_n\right\}$$

and $T_n(\delta)$ the usual stopping times of the algorithm. Then $T_n(\delta) = \hat{T}_n(\delta)/\delta^2$. Now, we can prove that $\hat{T}_n(\delta) - T^n_{C(\delta,\cdot),\delta}\left(P^{C(\delta,\cdot),\delta}_n\right) \to 0$ with probability 1 as $\delta \to 0$ using that $C(\delta,\cdot)$ is right-continuous and $\delta^2 \to 0$. Consequently, we can use $C\left(\delta, T^n_{C(\delta,\cdot),\delta}\left(P^{C(\delta,\cdot),\delta}_n\right)\right)$ instead of $C\left(\delta, \hat{T}_n(\delta)\right)$.

Let CS_δ be the event of doing a correct selection given the configuration $\mu_k = a_k\delta, \mu_{k-1}, \mu_{k-1} = a_{k-1}\delta, \ldots, \mu_1 =$

$a_1\delta$. Then

$$\begin{aligned}\underline{lim}_{\delta\to 0}\mathbb{P}\left(CS_\delta\right) &= \underline{lim}_{\delta\to 0}\mathbb{P}\left(f\left(C\left(\delta,t\right),\delta\right)=1\right)\\ &= \mathbb{P}\left(g\left(W\right)=1\right)\\ &\geq P^*\end{aligned}$$

where the last inequality follows from the paper of Frazier [3].

■

3 Asymptotic Validity when the Variances are Unknown

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