Without loss of generality, suppose that the true means of the systems are indexed so that $\mu_k \geq \mu_{k-1} \geq \cdots \geq \mu_1$. Assume $c > 0$.

Assumption 1 There exist finite constants μ_i and v_i^2 such that the probability of distribution of $C_i\left(t, r\right) \equiv \frac{\sum_{j=1}^{floor(rt)} X_{ij} - rt\mu_i}{v_i \sqrt{r}}$ over $D\left[0, 1\right]$ converges to that of a standard Brownian motion process, $W(t)$, for t on the unit interval, as r increases; i.e.

$$C_i\left(, r\right) \Longrightarrow W\left(\right)$$

as $r \to \infty$.

Note: $D\left[0, 1\right]$ is the Skorohod space, i.e. it's the space of real-valued functions on $\left[0, 1\right]$ that are right-continuous and have left-hand limits.

Under Assumption 1, μ_i is the mean, and $v_i^2 = \lim_{r\to\infty} r\mathrm{Var}\left(\bar{X}_i\left(r\right)\right)$ where $\bar{X}_i\left(r\right)$ is the sample mean of the first r observations from system i.

Lemma 1. For $i \neq l$, if $\mathbf{X}_i$ and $\mathbf{X}_l$ satisfy Assumption 1 and are independent, then there exists a constant v_{il}^2 such that

$$\frac{t_l\left(Y_{\text{floor}(rt_i),i} - rt_i\mu_i\right) - t_i\left(Y_{\text{floor}(rt_l),l} - rt_l\mu_l\right)}{v_{il}\sqrt{r}} \Longrightarrow t_l W_1\left(t_i\right) - t_i W_2\left(t_l\right)$$

as $r \to \infty$.

Proof. First, note that

$$v_{il}^2 := \lim_{r,s\to\infty} \mathrm{Var}\left(\sqrt{r}\bar{X}_i\left(r\right) + \sqrt{s}\bar{X}_l\left(s\right)\right) = v_i^2 + v_l^2$$

because of the independence of $\mathbf{X}_i$ and $\mathbf{X}_l$. Now,

$$\begin{aligned}
\frac{t_l\left(Y_{\text{floor}(rt_i),i}-rt_i\mu_i\right)-t_i\left(Y_{\text{floor}(rt_l),l}-rt_l\mu_l\right)}{v_{il}\sqrt{r}} &= \\
t_l\frac{\sum_{j=1}^{\text{floor}(rt_i)}X_{ij}-rt_i\mu_i}{v_{il}\sqrt{r}}-t_i\frac{\sum_{j=1}^{\text{floor}(rt)}X_{lj}-rt_l\mu_l}{v_{il}\sqrt{r}} &= \\
t_l\left(\frac{v_i}{v_{il}}\right)C_i\left(t,r\right)-t_i\left(\frac{v_l}{v_{il}}\right)C_l\left(t,r\right).
\end{aligned}$$

Because we assume that $\mathbf{X}_i$ and $\mathbf{X}_l$ are independent, so are $C_i\left(t,r\right)$ and $C_l\left(t,r\right)$. Assumption 1 implies $C_i\left(,r\right)\Rightarrow W_i\left(\right)$ and $C_l\left(,r\right)\Rightarrow W_l\left(\right)$ where W_i and W_l are independent standard Brownian motion processes. By Theorem 3.2 of Billingsley, $\left(C_i\left(,r\right),C_l\left(,r\right)\right)\Rightarrow\left(W_i\left(\right),W_l\left(\right)\right)$. By the Continuous Mapping Theorem,

$$t_l\left(\frac{v_i}{v_{il}}\right)C_i\left(t,r\right)-t_i\left(\frac{v_l}{v_{il}}\right)C_l\left(t,s\right)\Rightarrow t_l\left(\frac{v_i}{v_{il}}\right)W_i\left(t\right)-t_i\left(\frac{v_l}{v_{il}}\right)W_l\left(t\right).$$

Lemma 2 (Fabian 1974). Let $W(t,\Delta)$ be a Brownian motion process on $[0,\infty)$, with $E\left[W(t,\Delta)\right]=\Delta t$ and $Var\left[W\left(t,\Delta\right)\right]=t$, where $\Delta>0$. Let

$$\begin{aligned}
L &= -B \\
U &= B
\end{aligned}$$

for some $B > 0$. Let $R = (L, U)$ and let T^* be the first time that $W(t, \Delta) \notin R$. Finally, let A be the event that $W(T^*, \Delta) \leq -B$. Then,

$$\mathbb{P}\{\mathsf{A}\} = \frac{e^{-2B\Delta}}{1 + e^{-2B\Delta}}.$$

Theorem. If samples from system $x \in \{1 \ldots, k\}$ are normally distributed and independent, over time and across alternatives, then $\lim_{\delta \to 0} Pr\{\text{BIZ selects } k\} \geq P*$ provided $\mu_k \geq \mu_{k-1} + \delta$. We also suppose that the algorithm ends in at most $R(\delta) \in \mathbb{N}$ iterations, and $R(\delta) \to \infty$ as $\delta \to 0$ with probability 1. Furthermore, $\sqrt{R}\delta \to \Delta$ with probability 1 where $\infty > \Delta > 0$ with probability 1. Suppose $B_1 = \cdots = B_k = 1$.

Proof. Suppose the variances are known. We begin by considering the case of only two systems, denoted k and i, with $\mu_k \geq \mu_i + \delta*$. Let $A = \{k, i\}$. Let

$$T(\delta) = \min\left\{t \leq R, t \in \mathbb{N} : \min_{x \in A} \hat{q}_{tx}(A) \leq c \text{ or } \max_{x \in A} \hat{q}_{tx}(A) \geq P*\right\}.$$

Thus $T(\delta)$ is the stage at which the procedure terminates.

Now, let $a_t, b_t \in A$ such that $\exp\left(\delta\beta_t \frac{W_{ta_t}}{n_{ta_t}}\right) \geq \exp\left(\delta\beta_t \frac{W_{tb_t}}{n_{tb_t}}\right)$, so

$$\begin{aligned}
\min_{x\in A}\hat{q}_{tx}(A) &\leq c \\
\Leftrightarrow \min_{x\in A}\exp\left(\delta\beta_t\frac{W_{tx}}{n_{tx}}\right) &\leq c\sum_{x'\in A}\exp\left(\delta\beta_t\frac{W_{tx'}}{n_{tx'}}\right) \\
\Leftrightarrow \exp\left(\delta\beta_t\frac{W_{tb_t}}{n_{tb_t}}\right)(1-c) &\leq c\exp\left(\delta\beta_t\frac{W_{ta_t}}{n_{ta_t}}\right) \\
\Leftrightarrow \exp\left(\delta\beta_t\left(\frac{W_{tb_t}}{n_{tb_t}}-\frac{W_{ta_t}}{n_{ta_t}}\right)\right) &\leq \frac{c}{1-c}
\end{aligned}$$

an

$$\begin{aligned}
\max_{x\in A}\hat{q}_{tx}(A) &\geq P* \\
\Leftrightarrow \max_{x\in A}\exp\left(\delta\beta_t\frac{W_{tx}}{n_{tx}}\right) &\geq P*\sum_{x'\in A}\exp\left(\delta\beta_t\frac{W_{tx'}}{n_{tx'}}\right) \\
\Leftrightarrow \exp\left(\delta\beta_t\frac{W_{ta_t}}{n_{ta_t}}\right)(1-P^*) &\geq P^*\exp\left(\delta\beta_t\frac{W_{tb_t}}{n_{tb_t}}\right) \\
\Leftrightarrow \exp\left(\delta\beta_t\left(\frac{W_{ta_t}}{n_{ta_t}}-\frac{W_{tb_t}}{n_{tb_t}}\right)\right) &\geq \frac{P^*}{1-P^*},
\end{aligned}$$

thus

$$\begin{aligned}
T(\delta) &= \min\left\{t\leq R, t\in\mathbb{N}:\exp\left(\delta\beta_t\left|\frac{W_{tk}}{n_{tk}}-\frac{W_{ti}}{n_{ti}}\right|\right)\geq\min\left\{\frac{P^*}{1-P^*},\frac{1-c}{c}\right\}\right\} \\
&= \min\left\{t\leq R, t\in\mathbb{N}:\left|\frac{W_{tk}}{n_{tk}}-\frac{W_{ti}}{n_{ti}}\right|\geq\frac{\log\left[\min\left\{\frac{P^*}{1-P^*},\frac{1-c}{c}\right\}\right]}{\delta\beta_t}\right\}
\end{aligned}$$

Now, let's prove that $n_{ti}\to\infty, n_{tk}\to\infty$ as $t\to\infty$. For each t, $n_{t+1,x} =$

$n_{tx} + B_x$ for some $x \in A$. If $n_{t+1,i} = n_{ti} + B_i$ for a finite number of $t's$, then $n_{tk} \to \infty$ as $t \to \infty$. So, there exists l_0 such that $n_{li}/\hat{\lambda}_{li}^2 > n_{lk}/\hat{\lambda}_{lk}^2$ if $l > l_0$. Since $\hat{\lambda}_{ti}^2 \to \sigma_i^2$ and $\hat{\lambda}_{tk}^2 \to \sigma_k^2$ as $t \to \infty$, then $n_{ti} \to \infty$ as $t \to \infty$. Similarly, we get the same result in the other cases. Thus, $n_{ti} \to \infty, n_{tk} \to \infty$ as $t \to \infty$.

Let $\lambda_r^2 = \min\{\lambda_i^2, \lambda_k^2\}$ and $\lambda_s^2 = \max\{\lambda_i^2, \lambda_k^2\}$. Then $n_{1,s} = n_0 + 1$, $n_{1r} = \max\left\{n_0, ceil\left(\frac{\lambda_r^2}{\lambda_s^2}(n_0+1)\right)\right\}$. Note that $\frac{n_{ts}}{\lambda_s^2} \le \frac{n_{tr}}{\lambda_r^2}$ for all t, because by induction

$$\frac{n_{t+1,r}}{\lambda_r^2} \ge \frac{\frac{\lambda_r^2}{\lambda_s^2}(n_{ts}+1)}{\lambda_r^2} = \frac{n_{ts}+1}{\lambda_s^2}.$$

So, $n_{tr} = \max\left\{n_{t-1,r}, ceil\left(\frac{\lambda_r^2}{\lambda_s^2}(n_{t-1,s}+1)\right)\right\} = \max\left\{n_{t-1,r}, ceil\left(\frac{\lambda_r^2}{\lambda_s^2}(n_0+t)\right)\right\}$. Thus there exists t_0 such that $n_{t_0,r} = ceil\left(\frac{\lambda_r^2}{\lambda_s^2}(n_0+t_0)\right)$ and $n_{tr} = n_0$ if $t < t_0$. Observe that if $t > t_0$, $n_{tr} = ceil\left(\frac{\lambda_r^2}{\lambda_s^2}(n_0+t)\right)$. Furthermore, $n_{ts} = n_0 + t$.

Suppose $i = s$ or $i = r$ and $t_0 = 1$. If $0 \le t \le 1$, let

$$\begin{aligned}
D_i(t,R) &= \frac{\sum_{j=1}^{n_{floor(tR),i}} X_{ij} - (n_0 + (tR))\mu_i}{v_i\sqrt{R}} \\
&= \frac{\sum_{j=1}^{n_0+floor(tR)} X_{ij} - (n_0 + (tR))\mu_i}{v_i\sqrt{R}} \\
&= \frac{\sum_{j=1}^{n_0} X_{ij} - n_0\mu_i}{v_i\sqrt{R}} + \frac{\sum_{j=1}^{floor(tR)} X_{i,n_0+j} - (tR)\mu_i}{v_i\sqrt{R}} \\
&\Rightarrow W_i().
\end{aligned}$$

Suppose $\lambda_i^2 \leq \lambda_k^2$. Assume $t_0 = 1$. Let

$$\begin{aligned}
D_{ik}(t,R) &= \frac{n_{tR,i}\left(\sum_{j=1}^{n_{floor(tR),k}} X_{kj} - (n_0 + (tR))\mu_k\right)}{v_{ik}R\sqrt{R}} \\
&\quad - \frac{n_{tR,k}\left(\sum_{j=1}^{n_{floor(tR),i}} X_{kj} - ceil\left(\frac{\lambda_i^2}{\lambda_k^2}(n_0 + (tR))\right)\mu_i\right)}{v_{ik}R\sqrt{R}} \\
&= \frac{ceil\left(\frac{\lambda_i^2}{\lambda_k^2}(n_0 + tR)\right)\left(\sum_{j=1}^{(n_0+floor(tR))} X_{kj} - (n_0 + (tR))\mu_k\right)}{v_{ik}R\sqrt{R}} \\
&\quad - \frac{(n_0 + tR)\left(\sum_{j=1}^{ceil\left(\frac{\lambda_i^2}{\lambda_k^2}(n_0+floor(tR))\right)} X_{kj} - ceil\left(\frac{\lambda_i^2}{\lambda_k^2}(n_0 + (tR))\mu_i\right)\right)}{v_{ik}R\sqrt{R}} \\
&\Rightarrow \frac{v_k}{v_{ik}}\frac{\lambda_i^2}{\lambda_k^2} tW_k(t) - \frac{v_i}{v_{ik}} tW_i(t) = tW().
\end{aligned}$$

Now, let $N = \max\{n_{Rk}, n_{Ri}\}$, then

$$\begin{aligned}
\frac{W_{T(\delta)k}}{n_{T(\delta)k}} &< \frac{W_{T(\delta)i}}{n_{T(\delta)i}} \\
\Leftrightarrow \frac{\mu_k - \mu_i}{R^{3/2}v_{ik}}\left(n_{T(\delta)k}n_{T(\delta)i}\right) &+ \frac{n_{T(\delta)i}\left(\sum_{j=1}^{n_{T(\delta)k}} X_{kj} - n_{T(\delta)k}\mu_k\right) - n_{T(\delta)k}\left(\sum_{j=1}^{n_{T(\delta)i}} X_{ij} - n_{T(\delta)i}\mu_i\right)}{R^{3/2}v_{ik}} < 0.
\end{aligned}$$

Then

$$\begin{aligned}
\mathbb{P}\left\{\frac{W_{T(\delta)k}}{n_{T(\delta)k}} < \frac{W_{T(\delta)i}}{n_{T(\delta)i}}\right\} &\leq \mathbb{P}\left\{\frac{n_{T(\delta)i}\left(\sum_{j=1}^{n_{T(\delta)k}} X_{kj} - n_{T(\delta)k}\mu_k\right) - n_{T(\delta)k}\left(\sum_{j=1}^{n_{T(\delta)i}} X_{ij} - n_{T(\delta)i}\mu_i\right)}{R^{3/2}v_{ik}}\right. \\
&\quad \left. + \frac{\delta}{R^{3/2}v_{ik}}\left(n_{T(\delta)k}n_{T(\delta)i}\right) < 0\right\} \\
&= E\left[\mathbb{P}\left\{\frac{n_{T(\delta)i}\left(\sum_{j=1}^{n_{T(\delta)k}} X_{kj} - n_{T(\delta)k}\mu_k\right) - n_{T(\delta)k}\left(\sum_{j=1}^{n_{T(\delta)i}} X_{ij} - n_{T(\delta)i}\mu_i\right)}{R^{3/2}v_{ik}}\right.\right. \\
&\quad \left.\left. + \frac{\delta}{R^{3/2}v_{ik}}\left(n_{T(\delta)k}n_{T(\delta)i}\right) < 0\right\} \mid \Delta\right]
\end{aligned}$$

Define

$$\begin{aligned}
\hat{T}(\delta) &= \min\left\{t \in \left\{\frac{1}{R}, \ldots, 1\right\} : \left|D_{ik}(t,\delta) + \frac{ceil\left(\frac{\lambda_i^2}{\lambda_k^2}(n_0 + tR)\right)(n_0 + tR)\,\delta}{R^{3/2}v_{ik}}\right|\right. \\
&\quad \left. \geq \frac{ceil\left(\frac{\lambda_i^2}{\lambda_k^2}(n_0 + tR)\right)(n_0 + tR)\log\left[\min\left\{\frac{P^*}{1-P^*}, \frac{1-c}{c}\right\}\right]}{R^{3/2}v_{ik}\beta_{floor(tR)}\delta}\right\}
\end{aligned}$$

where $\beta_{tR} = \frac{n_0 + tR + ceil\left(\frac{\lambda_i^2}{\lambda_k^2}(n_0+tR)\right)}{\lambda_i^2 + \lambda_k^2}$.

Clearly, $\hat{T}(\delta) = T(\delta)/R$. Also, define the stopping time of the corresponding continuous-time process as

$$\begin{aligned}
\tilde{T}(\delta) &= \min\left\{1 \geq t \geq \frac{1}{R} : \left|D_{ik}(t,\delta) + \frac{ceil\left(\frac{\lambda_i^2}{\lambda_k^2}(n_0 + tR)\right)(n_0 + tR)\,\delta}{R^{3/2}v_{ik}}\right|\right. \\
&\quad \left. \geq \frac{ceil\left(\frac{\lambda_i^2}{\lambda_k^2}(n_0 + tR)\right)(n_0 + tR)\log\left[\min\left\{\frac{P^*}{1-P^*}, \frac{1-c}{c}\right\}\right]}{R^{3/2}v_{ik}\beta_{floor(tR)}\delta}\right\}
\end{aligned}$$

Note that for fixed δ, $D_{ik}\left(\hat{T}(\delta),\delta\right)$ corresponds to the right-hand limit of a point of discontinuity of $D_{ik}(,\delta)$. We can show that $\hat{T}(\delta)\to\tilde{T}(\delta)$ with probability 1 as $\delta\to 0$, making use of the fact that $1/R\to 0$ with probability 1. Thus, in the limit, we can focus on $D_{ik}\left(\tilde{T}(\delta),\delta\right)$.

Now, condition on Δ. By Assumption 1, Lemma 1, and the CMT we have that

$$\begin{aligned} D_{ik}(t,\delta)+\frac{ceil\left(\frac{\lambda_i^2}{\lambda_k^2}(n_0+tR)\right)(n_0+tR)\,\delta}{R^{3/2}v_{ik}} &\Rightarrow tW(t)+t^2\frac{\lambda_i^2}{\lambda_k^2}\Delta. \\ &= t\left(W(t)+t\frac{\lambda_i^2}{\lambda_k^2}\Delta\right) \end{aligned}$$

Let

$$\begin{aligned} A(\delta) &= \frac{n_0 ceil\left(\frac{\lambda_i^2}{\lambda_k^2}(n_0+tR)\right)\log\left[\min\left\{\frac{P^*}{1-P^*},\frac{1-c}{c}\right\}\right]}{R^{3/2}v_{ik}\beta_{tR}\delta} \\ &= \frac{(\lambda_i^2+\lambda_k^2)\,n_0 ceil\left(\frac{\lambda_i^2}{\lambda_k^2}(n_0+tR)\right)\log\left[\min\left\{\frac{P^*}{1-P^*},\frac{1-c}{c}\right\}\right]}{\left(n_0+tR+ceil\left(\frac{\lambda_i^2}{\lambda_k^2}(n_0+tR)\right)\right)R^{3/2}v_{ik}\delta}\xrightarrow{\delta\to 0}0 \end{aligned}$$

and

$$
\begin{aligned}
tB(\delta) &= \frac{\left(\lambda_i^2+\lambda_k^2\right) ceil\left(\frac{\lambda_i^2}{\lambda_k^2}\left(n_0+tR\right)\right)(tR)\log\left[\min\left\{\frac{P^*}{1-P^*},\frac{1-c}{c}\right\}\right]}{\left(n_0+tR+ceil\left(\frac{\lambda_i^2}{\lambda_k^2}\left(n_0+tR\right)\right)\right)R^{3/2}v_{ik}\delta} \\
&= \frac{\left(\lambda_i^2+\lambda_k^2\right) ceil\left(\frac{\lambda_i^2}{\lambda_k^2}\left(n_0+tR\right)\right)(t)\log\left[\min\left\{\frac{P^*}{1-P^*},\frac{1-c}{c}\right\}\right]}{\left(n_0+tR+ceil\left(\frac{\lambda_i^2}{\lambda_k^2}\left(n_0+tR\right)\right)\right)R^{1/2}v_{ik}\delta} \\
&\xrightarrow[\delta\to 0]{} \frac{\left(\lambda_i^2+\lambda_k^2\right)\frac{\lambda_i^2}{\lambda_k^2}(t)\log\left[\min\left\{\frac{P^*}{1-P^*},\frac{1-c}{c}\right\}\right]}{\left(1+\frac{\lambda_i^2}{\lambda_k^2}\right)v_{ik}\Delta} = Bt.
\end{aligned}
$$

Note that the stopping time $\tilde{T}(\delta)$ is the first time t at which the event

$$
\left\{\left|D_{ik}(t,\delta)+\frac{ceil\left(\frac{\lambda_i^2}{\lambda_k^2}\left(n_0+tR\right)\right)\left(n_0+tR\right)\delta}{R^{3/2}v_{ik}}\right|-A(\delta)-tB(\delta)\geq 0\right\}
$$

occurs. Define the mapping $s_\delta : D[0,1]\to\mathbb{R}$ such that $s_\delta(Y)=Y(T_{Y,\delta})$, where

$$
T_{Y,\delta}=\inf\left\{t:|Y(t)|-A(\delta)-B(\delta)t\geq 0\right\}
$$

for every $Y\in D[0,1]$ and $\delta>0$. Similarly, define $s(Y)=Y(T_Y)$, where

$$
T_Y=\inf\left\{t>0:|Y(t)|-Bt\geq 0\right\}
$$

for every $Y \in D[0,1]$. Note that

$$
\begin{aligned}
s_\delta\left(D_{ik}(t,\delta)+\frac{ceil\left(\frac{\lambda_i^2}{\lambda_k^2}(n_0+tR)\right)(n_0+tR)\delta}{R^{3/2}v_{ik}}\right) &= D_{ik}\left(\tilde{T}(\delta),\delta\right) \\
+\frac{ceil\left(\frac{\lambda_i^2}{\lambda_k^2}\left(n_0+\tilde{T}(\delta)R\right)\right)\left(n_0+\tilde{T}(\delta)R\right)\delta}{R^{3/2}v_{ik}},
\end{aligned}
$$

$s(tW(t,\Delta)) = T_{\mathfrak{W}()}W\left(T_{\mathfrak{W}()},\Delta\right)$ where $\mathfrak{W}(t) = tW(t,\Delta)$.

We need to show that

$$s_\delta(G_{ik}(t,\delta)) \Rightarrow s(tW(t,\Delta))$$

as $\delta \to 0$, where

$$G_{ik}(t,\delta) \equiv D_{ik}(t,\delta)+\frac{ceil\left(\frac{\lambda_i^2}{\lambda_k^2}(n_0+tR)\right)(n_0+tR)\delta}{R^{3/2}v_{ik}}$$

for $t \in [0,1]$ and $\delta > 0$. If $\mathbb{P}(tW(t,\Delta)\in\mathrm{D}[0,1]\text{-}D_s)) = 1$ where $D_s \equiv \{x \in D[0,1]$: for some sequence $\{x_n\} \subset D[0,1]$ with $lim_n d(x_n,x) = 0$, the sequence $\{s_{\delta_n}(x_n)\}$ does not converge to $s(x)\}$, and $d(X,Y)$ is the infimum of those positive w for which there exists $\lambda \in \Lambda$ such that $sup_{t\in[0,1]}|X(t) - Y(\lambda(t))| \leq w$ and $sup_{t\in[0,1]}|\lambda(t) - t| \leq w$ (Λ is the class of strictly increasing, continuous mappings of $[0,1]$ onto itself such that for every $\lambda \in \Lambda$, we have $\lambda(0) = 0$ and $\lambda(1) = 1$), then by Theorem 5.5 of Billingsley 1968 we conclude that this is true. By Kim et al. (2005), we know that $\mathbb{P}(W(t,\Delta)\in\mathrm{D}[0,1]\text{-}D_s)) = 1$, thus it follows this is true (prove).

Now, unconditioning on Δ gives

$$\begin{aligned}
\lim sup_{\delta\to 0}\mathbb{P}\left(ICS\right) &\leq E\left[\mathbb{P}\left[\tilde{T}\left(\delta\right)W\left(\tilde{T}\left(\delta\right),\frac{\lambda_i^2}{\lambda_k^2}\Delta\right)<0\mid\Delta\right]\right] \\
&= E\left[\frac{e^{-2B\Delta\frac{\lambda_i^2}{\lambda_k^2}}}{1+e^{-2B\Delta\frac{\lambda_i^2}{\lambda_k^2}}}\right] \\
&= E\left[\frac{e^{-\frac{2\frac{\lambda_i^4}{\lambda_k^2}\log\left[\min\left\{\frac{P^*}{1-P^*},\frac{1-c}{c}\right\}\right]}{v_{ik}}}}{1+e^{-\frac{2\frac{\lambda_i^4}{\lambda_k^2}\log\left[\min\left\{\frac{P^*}{1-P^*},\frac{1-c}{c}\right\}\right]}{v_{ik}}}}\right] \\
&= E\left[\frac{1}{1+e^{\frac{2\frac{\lambda_i^4}{\lambda_k^2}\log\left[\min\left\{\frac{P^*}{1-P^*},\frac{1-c}{c}\right\}\right]}{v_{ik}}}}\right] \\
&= E\left[\frac{1}{1+\left(\min\left\{\frac{P^*}{1-P^*},\frac{1-c}{c}\right\}\right)^{2\frac{\lambda_i^4}{\lambda_k^2 v_{ik}}}}\right] \\
&\leq \frac{1}{1+\left(\frac{P^*}{1-P^*}\right)^{2\frac{\lambda_i^4}{\lambda_k^2 v_{ik}}}}\leq\left(1-P^*\right)\leq\left(1-P^*\right)^{1/(k-1)}
\end{aligned}$$

the last equality follows because, if $a = 2\frac{\lambda_i^4}{\lambda_k^2 v_{ik}}$,

$$\begin{aligned}
\frac{1}{1+\left(\frac{P^*}{1-P^*}\right)^a} &\leq 1-P^* \\
\Leftrightarrow \frac{\left(1-P^*\right)^a}{\left(1-P^*\right)^a+P^{*a}} &\leq 1-P^* \\
\Leftrightarrow \left(1-P^*\right)^a &\leq \left(1-P^*\right)^a+P^{*a} \\
\Leftrightarrow \left(1-P^*\right)^a\left(1-1+P^*\right) &\leq P^{*a} \\
\Leftrightarrow \left(1-P^*\right)^{a-1} &\leq P^{*(a-1)} \\
\Leftrightarrow 1 &\leq 2P^*.
\end{aligned}$$

PROOF OF THE GENERAL CASE. First, suppose $c = 0$, and let $\lambda_z^2 = max_i\left\{\lambda_i^2\right\}$.

$\sum_{j=1}^{n_{floor(tR),i}} X_{ij} - \left(n_0 + (tR)\right)\mu_i$

Let

$$\begin{aligned}
C_x\left(\delta,t\right) &= \frac{Y_{ceil\left(\frac{\lambda_x^2}{\lambda_z^2}(n_0+tR)\right),x} - \frac{\lambda_x^2}{\lambda_z^2}\left(n_0+tR\right)\mu_x}{\frac{\lambda_x^2}{\lambda_z}\sqrt{R}} \\
&\Rightarrow W_x\left(t\right)
\end{aligned}$$

and

$$\mathfrak{C}\left(\delta\right) = \left(C_x\left(\delta,t\right) : t\in[0,1]; x\in A\right),$$

then

$$\mathfrak{C}\left(\delta\right) \Rightarrow \left(W_x(t) : t\in[0,1]; x\in A\right)$$

Let

$$
\begin{aligned}
T(\delta) &= \min\left\{t\in\left\{\frac{1}{R},\ldots,1\right\}: max_x q_{tR,x}(A)\geq P\right\}.\\
&= \min\left\{t\in\left\{\frac{1}{R},\ldots,1\right\}: max_x exp\left(\delta\beta_{tR}\frac{W_{tR,x}}{n_{tR,x}}\right)\geq P\sum_{x'\in A}exp\left(\delta\beta_{tR}\frac{W_{tR,x'}}{n_{tR,x'}}\right)\right\}\\
&= \min\left\{t\in\left\{\frac{1}{R},\ldots,1\right\}: \exists x \text{ s.t. } \left(\delta\beta_{tR}\frac{Y_{n_{tR,x}}}{n_{tR,x}}\right)\geq log\left(P\sum_{x'\in A}exp\left(\delta\beta_{tR}\frac{W_{n_{tR,x'}}}{n_{tR,x'}}\right)\right)\right\}\\
&= \min\left\{t\in\left\{\frac{1}{R},\ldots,1\right\}: \exists x \text{ s.t. } exp\left(\frac{\lambda_x^2}{\lambda_z n_{tR,x}}\sqrt{R}\delta\beta_{tR}C_x(\delta,t)+\mu_x\delta\beta_{tR}\right)\geq\right.\\
&\quad \left. P\sum_{x'\in A}exp\left(\left(\frac{\lambda_{x'}^2}{\lambda_z n_{tR,x'}}\sqrt{R}\delta\beta_{tR}C_{x'}(\delta,t)+\delta\beta_{tR}\mu_{x'}\right)\right)\right\}
\end{aligned}
$$

Then for δ sufficiently small we have that

$$
\begin{aligned}
T(\delta) &= \min\left\{t\in\left\{\frac{1}{R},\ldots,1\right\}: \exists x \text{ s.t. } exp\left(\frac{\lambda_x^2}{\lambda_z n_{tR,x}}\sqrt{R}\delta\beta_{tR}C_x(\delta,t)\right)\left(\frac{1}{P}-1\right)\geq\right.\\
&\quad \sum_{x'<x}exp\left(\left(\frac{\lambda_{x'}^2}{\lambda_z n_{tR,x'}}\sqrt{R}\delta\beta_{tR}C_{x'}(\delta,t)-\delta^2\beta_{tR}\right)\right)+\\
&\quad \left.\sum_{x>x'}exp\left(\left(\frac{\lambda_{x'}^2}{\lambda_z n_{tR,x'}}\sqrt{R}\delta\beta_{tR}C_{x'}(\delta,t)+\delta^2\beta_{tR}\right)\right)\right\}
\end{aligned}
$$

Now,

$$
\begin{aligned}
\mathbb{P}(CS) &= \mathbb{P}\left(\frac{Y_{n_{TR,k}}}{n_{TRk}}\geq\frac{Y_{n_{TR,x}}}{n_{TRx}}\ \forall x\in A\right)\\
&= \mathbb{P}\left(\frac{\lambda_k^2}{\lambda_z n_{tR,k}}\sqrt{R}\delta\beta_{tR}C_k(\delta,t)+\delta\beta_{tR}(\mu_k-\mu_x)\geq\frac{\lambda_x^2}{\lambda_z n_{tR,x}}\sqrt{R}\delta\beta_{tR}C_x(\delta,t)\ \ \forall x\in A\right)\\
&\geq \mathbb{P}\left(\frac{\lambda_k^2}{\lambda_z n_{tR,k}}\sqrt{R}\delta\beta_{tR}C_k(\delta,t)+\delta^2\beta_{tR}\geq\frac{\lambda_x^2}{\lambda_z n_{tR,x}}\sqrt{R}\delta\beta_{tR}C_x(\delta,t)\ \ \forall x\in A\right)
\end{aligned}
$$

Let

$$f\left(Y,\delta\right) = \begin{cases} 1 & \text{if } \frac{\lambda_k^2}{\lambda_z n_{tR,k}}\sqrt{R}Y_k\left(t\right)+\delta \geq \frac{\lambda_x^2}{\lambda_z n_{tR,x}}\sqrt{R}Y_x\left(t\right)\ \forall x\in A \\ 0 & \text{otherwise} \end{cases}$$

where

$$\begin{aligned} T_{Y,\delta} = \ & \inf\Bigg\{t\in[0,1]:\exists x \text{ s.t. } \ exp\left(\frac{\lambda_x^2}{\lambda_z n_{tR,x}}\sqrt{R}\delta\beta_{tR}Y_x\left(t\right)\right)\left(\frac{1}{P}-1\right)\geq \\ & \sum_{x'<x} exp\left(\left(\frac{\lambda_{x'}^2}{\lambda_z n_{tR,x'}}\sqrt{R}Y_{x'}\left(t\right)\delta\beta_{tR}-\delta^2\beta_{tR}\right)\right)+ \\ & \sum_{x>x'} exp\left(\left(\frac{\lambda_{x'}^2}{\lambda_z n_{tR,x'}}\sqrt{R}Y_{x'}\left(t\right)\delta\beta_{tR}+\delta^2\beta_{tR}\right)\right)\Bigg\} \end{aligned}$$

and $Y\in D\left[0,1\right]^k$ where $D\left[0,1\right]^k$ is the set of functions from $[0,1]$ to $\mathbb{R}^k$ that are right-continuous and have left-hand limits. We'll use the Skorokhod metric d on $D\left[0,1\right]^k$:

$$d\left(X,Y\right)=\inf_{\lambda\in\Lambda}\left\{\|\lambda-I\|\vee\|X-Y\circ\lambda\|\right\}$$

where Λ is the set of strictly increasing, continuous mappings of $[0,1]$ onto itself, and $\|\cdot\|$ is the uniform norm.

Then

$$f\left(\mathfrak{C}\left(\delta\right),\delta\right) = \begin{cases} 1 & \text{if } \frac{\lambda_k^2}{\lambda_z n_{tR,k}}\sqrt{R}C_k\left(\delta,t\right)+\delta \geq \frac{\lambda_x^2}{\lambda_z n_{tR,x}}\sqrt{R}C_x\left(\delta,t\right)\ \ \forall x\in A \\ 0 & \text{otherwise} \end{cases}$$

Observe that

$$\begin{aligned} \frac{\lambda_x^2}{\lambda_z n_{tR,x}}\sqrt{R}\delta\beta_{tR} &= \frac{\lambda_x^2}{\lambda_z}\sqrt{R}\delta\frac{\sum_{x'}\frac{n_{tR,x'}}{n_{tR,x}}}{\sum_{x'}\lambda_{x'}^2} \to \frac{\lambda_x^2}{\lambda_z}\Delta\frac{\sum_{x'}\frac{\lambda_{x'}^2}{\lambda_x^2}}{\sum_{x'}\lambda_{x'}^2} \\ &= \frac{\Delta}{\lambda_z} \end{aligned}$$

and

$$\delta^2\beta_{tR} = \delta^2\frac{\sum_{x'}n_{tRx'}}{\sum_{x'}\lambda_{x'}^2} \to \frac{\sum_{x'}\lambda_{x'}^2\left(t\Delta^2\right)}{\lambda_z^2\sum_{x'}\lambda_{x'}^2} = \frac{t\Delta^2}{\lambda_z^2} =: A\left(t\right)$$

where $\lambda_z^2 = max\left\{\lambda_i^2\right\}$.

Then

$$exp\left(\left(\frac{\lambda_x^2}{\lambda_z n_{tR,x}}\sqrt{R}Y_x\left(t\right)\delta\beta_{tR}-\delta^2\beta_{tR}\right)\right) \to exp\left(Y_x\left(t\right)\frac{\Delta}{\lambda_z}-A\left(t\right)\right).$$

Let

$$
\begin{aligned}
T_Y \;=\; & \inf\{t \in [0,1] : \exists x \text{ s.t. } \; exp\left(Y_x(t)\frac{\Delta}{\lambda_z}\right)\left(\frac{1}{P}-1\right) \geq \\
& \sum_{x'<x} exp\left(Y_{x'}(t)\frac{\Delta}{\lambda_z} - A(t)\right) + \\
& \sum_{x>x'} exp\left(Y_{x'}(t)\frac{\Delta}{\lambda_z} + A(t)\right)\Bigg\}
\end{aligned}
$$

and

$$
g(Y) \;=\; \begin{cases} 1 & \text{if } Y_k(T_Y) + \frac{1}{\lambda_z} T_Y \Delta \geq Y_x(T_Y) \;\; \forall x \in A \\ 0 & \text{otherwise} \end{cases}
$$

where $\lambda_z^2 = max_i\{\lambda_i^2\}$.

We need to show that given Δ

$$
f(\mathfrak{C}(\delta), \delta) \Rightarrow g(W).
$$

Let $\{\delta_n\} \to 0$ and $f_n := f(\cdot, \delta_n)$.

Proposition. Let $D_s \equiv \{x \in D[0,1]^k :$ for some sequence $\{x_n\} \subset D[0,1]^k$ with $lim_n d(x_n, x) = 0$, the sequence $\{f_n(x_n)\}$ does not converge to $\{g(x)\}\}$, then $\mathbb{P}\left(W(t) \in \mathrm{D}[0,1]^k\text{-}D_s\right) = 1$.

Proof. Let $\{x_n\} \subset D[0,1]^k$ such that $x_n \to W$. Then there exists $\lambda_n \in \Lambda$ such that $\sup_{t\in[0,1]} \|\lambda_n(t) - t\| \leq d(x_n, W) + \frac{1}{n}$ and $\sup_{t\in[0,1]} \|x_n(t) - W(\lambda_n(t))\| \leq d(x_n, W) + \frac{1}{n}$. Taking $g_n \equiv \sup_{t\in[0,1]} \|W(t) - W(\lambda_n(t))\|$, we see from the uniform continuity of W on $[0,1]^k$ and the definition of g_n that $\lim_{n\to\infty} g_n = 0$. Moreover, if we take $\epsilon_n = 3n^{-1} + 3\sup\{d(x_l, W) + g_l : l = n, n+1, \ldots\}$, then $\{\epsilon_n\}$ is a monotonically decreasing sequence of positive numbers with limit zero.

From the definition of ϵ_n we have $d(x_n, W) < \epsilon_n/2$ and $g_n < \epsilon_n/2$ for $n = 1, 2, \ldots$ Consequently, we have

$$\begin{aligned}\|x_n(t) - W(t)\| &\leq \|x_n(t) - W(\lambda_n(t))\| + \|W(\lambda_n(t)) - W(t)\| \\ &< \epsilon_n\end{aligned}$$

Consequently,

$$\|W(T_{x_n,\delta_n}) - x_n(T_{x_n,\delta_n})\| < \epsilon_n.$$

Now,

$$\frac{\lambda_x^2}{\lambda_z n_{tR,x}} \sqrt{R}\delta\beta_{tR} \to \frac{\Delta}{\lambda_z}$$

and

$$\delta_n^2 \beta_{tR_n} \to A(t)$$

then for all $\epsilon > 0$ and n sufficiently large

$$\begin{aligned}
\left|\frac{\lambda_x^2}{\lambda_z n_{tR,x}}\sqrt{R}\delta\beta_{tR}x_{n_x}(t) - \frac{\Delta}{\lambda_z}W_x(t)\right| &\leq \left|\frac{\lambda_x^2}{\lambda_z n_{tR,x}}\sqrt{R}\delta\beta_{tR}x_{n_x}(t) - \frac{\Delta}{\lambda_z}x_{n_x}(t)\right| \\
&\quad + \left|\frac{\Delta}{\lambda_z}W_x(t) - \frac{\Delta}{\lambda_z}x_{n_x}(t)\right| \\
&\leq \frac{\Delta}{\lambda_z}\epsilon_n + \epsilon\left|x_{n_x}(t)\right|,
\end{aligned}$$

furthermore

$$\begin{aligned}
\left|\frac{\lambda_x^2}{\lambda_z n_{tR,x}}\sqrt{R}\delta\beta_{tR}x_{n_x}(t) - \delta_n^2\beta_{tR_n} - \frac{\Delta}{\lambda_z}W_x(t) + A(t)\right| &\leq \left|\frac{\lambda_x^2}{\lambda_z n_{tR,x}}\sqrt{R}\delta\beta_{tR}x_{n_x}(t) - \frac{\Delta}{\lambda_z}x_{n_x}(t)\right| \\
&\quad + \left|\frac{\Delta}{\lambda_z}W_x(t) - \frac{\Delta}{\lambda_z}x_{n_x}(t)\right| \\
&\quad + \left|A(t) - \delta_n^2\beta_{tR_n}\right| \\
&\leq \frac{\Delta}{\lambda_z}\epsilon_n + \epsilon\left|x_{n_x}(t)\right| + \epsilon,
\end{aligned}$$

Now consider T_{x_n,δ_n}. Since $x_{n_i}(t) - W_i(t) < \epsilon_n$ and $W_i(t) - x_{n_i}(t) < \epsilon_n$, consequently

$$T_{W-\epsilon_n e,\delta_n} \leq T_{x_n,\delta_n} \leq T_{W+\epsilon_n e,\delta_n}$$

Observe that

$$t^* = \lim\ \inf_n T_{W-\epsilon_n e,\delta_n} \geq T_W$$

and

$$t_* = \lim\ \sup_n T_{W+\epsilon_n e,\delta_n} \leq T_W.$$

Then

$$t_* \leq T_W \leq t^* \leq \lim\, \inf_n T_{W+\epsilon_n e,\delta_n} \leq t_*$$

thus

$$t_* = t^* = T_W = \lim_n T_{W-\epsilon_n e,\delta_n} = \lim_n T_{W+\epsilon_n e,\delta_n}.$$

Then

$$\lim_n T_{x_n,\delta_n} = T_W$$

and so

$$\lim_n x_{n_i}\left(T_{x_n,\delta_n}\right) = \lim_n W_i\left(T_{x_n,\delta_n}\right) = W_i\left(T_W\right)$$

by the continuity of W_i.

Therefore

$$\lim_n f_n\left(x_n\right) = \lim_n f\left(x_n,\delta_n\right) = g\left(W\right).$$

$\blacksquare$

Now,

$$
\begin{aligned}
\underline{lim}_{\delta\to 0}\mathbb{P}\left(CS\right) &\geq \underline{lim}_{\delta\to 0}\mathbb{P}\left(f\left(\mathfrak{C}\left(\delta\right),\delta\right)=1\right)\\
&= \underline{lim}_{\delta\to 0}E\left(\mathbb{P}\left(f\left(\mathfrak{C}\left(\delta\right),\delta\right)=1\mid\Delta\right)\right)\\
&= E\left(\underline{lim}_{\delta\to 0}\mathbb{P}\left(f\left(\mathfrak{C}\left(\delta\right),\delta\right)=1\mid\Delta\right)\right)\\
&= E\left(\mathbb{P}\left(g\left(W\right)=1\mid\Delta\right)\right)\\
&= E\left(\mathbb{P}\left(W_k\left(T_W\right)+\frac{1}{\lambda_z}T_W\Delta\geq W_x\left(T_W\right)\ \forall x\in A\mid\Delta\right)\right)\\
&=
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{P}\left(g\left(W\right)=1\right) &= \mathbb{P}\left(\lambda_k W_k\left(T_W\right)+\frac{\lambda_k^2}{\lambda_z^2}T_W\Delta\geq\frac{\lambda_x\lambda_k^2 W_x\left(T_W\right)}{\lambda_x^2}\ \forall x\in A\right)\\
&= E\left(\lambda_k\lambda_x^2 W_k\left(T_W\right)+\frac{\lambda_k^2}{\lambda_z^2}\lambda_x^2 T_W\Delta\geq\lambda_x\lambda_k^2 W_x\left(T_W\right)\ \forall x\in A\mid\Delta\right)
\end{aligned}
$$

$$
\begin{aligned}
\hat{T}(\delta) &= \min\left\{t \in \left\{\frac{1}{R},\ldots,1\right\} : max_x q_{tR,x}(A) \geq P\right\}. \\
&= \min\left\{t \in \left\{\frac{1}{R},\ldots,1\right\} : max_x exp\left(\delta\beta_{tR}\frac{W_{tR,x}}{n_{tR,x}}\right) \geq P\sum_{x'\in A} exp\left(\delta\beta_{tR}\frac{W_{tR,x'}}{n_{tR,x'}}\right)\right\} \\
&= \min\left\{t \in \left\{\frac{1}{R},\ldots,1\right\} : \exists x \text{ s.t. } \left(\delta\beta_{tR}\frac{Y_{n_{tR,x}}}{n_{tR,x}}\right) \geq log\left(P\sum_{x'\in A} exp\left(\delta\beta_{tR}\frac{W_{n_{tR,x'}}}{n_{tR,x'}}\right)\right)\right\} \\
&= \min\left\{t \in \left\{\frac{1}{R},\ldots,1\right\} : \exists x \text{ s.t. } \frac{\lambda_x\sqrt{R}}{n_{tR,x}}\left(\frac{Y_{n_{tR,x}-\left(n_{tR,x}\right)\mu_X}}{\lambda_x\sqrt{R}}\right)(\delta\beta_{tR}) + \mu_x\delta\beta_{tR} \geq \right. \\
&\qquad \left. log\left(P\sum_{x'\in A} exp\left(\left(\frac{\lambda_{x'}\sqrt{R}}{n_{tR,x'}}\left(\delta\beta_{tR}\frac{Y_{n_{tR,x'}-\left(n_{tR,x'}\right)\mu_{X'}}}{\lambda_{x'}\sqrt{R}}\right) + \delta\beta_{tR}\mu_{x'}\right)\right)\right)\right\}
\end{aligned}
$$

and

$$T(\delta) = \hat{T}(\delta) R$$

Now, we know that

$$\frac{Y_{n_{tR,x}-\left(n_{tR,x}\right)\mu_X}}{\lambda_x\sqrt{R}} \Rightarrow W_x(t)$$

as $\delta \rightarrow 0$.

Now, if $\lambda_r^2 = \max_{i\in A-\{k\}}\left\{\lambda_i^2\right\}$, then \lim inf_{\delta\rightarrow0}

$$
\begin{aligned}
\mathbb{P}\left(f\left(\mathfrak{C}\left(\delta\right)\right)=1\right) &= \mathbb{P}\left(\frac{W_{Tk}}{n_{Tk}} \geq \frac{W_{Tx}}{n_{Tx}}\ \forall x \in A\right) \\
&= E\left[\prod_{i\in A-\{k\}} \mathbb{P}\left\{\frac{\left(\sum_{j=1}^{n_{T(\delta)k}} X_{kj} - n_{T(\delta)k}\mu_k\right)}{n_{T(\delta)k}} - \frac{\lambda_i^2\left(\sum_{j=1}^{n_{T(\delta)i}} X_{ij} - n_{T(\delta)i}\mu_i\right)}{\lambda_i^2 n_{T(\delta)i}}\right.\right. \\
&\quad \left.\left. +\delta > 0\right\} \mid \Delta, X_{k1}, \ldots, X_{kn_{Tk}}, T\}\right] \\
&\geq E\left[\prod_{i\in A-\{k\}} \mathbb{P}\left\{\frac{\left(\sum_{j=1}^{n_{T(\delta)k}} dX_{kj} - n_{T(\delta)k}\mu_k\right)}{n_{T(\delta)k}} - \frac{\lambda_r^2\left(\sum_{j=1}^{n_{T(\delta)i}} X_{ij} - n_{T(\delta)i}\mu_i\right)}{\lambda_i^2 n_{T(\delta)i}}\right.\right. \\
&= \left.\left. +\delta > 0\right\} \mid \Delta, X_{k1}, \ldots, X_{kn_{Tk}}, T\}\right] \\
&= E\left[\left(\mathbb{P}\left\{\frac{\left(\sum_{j=1}^{n_{T(\delta)k}} X_{kj} - n_{T(\delta)k}\mu_k\right)}{n_{T(\delta)k}} - \frac{\left(\sum_{j=1}^{n_{T(\delta)r}} X_{rj} - n_{T(\delta)r}\mu_r\right)}{n_{T(\delta)r}}\right.\right.\right. \\
&= \left.\left.\left. +\delta > 0\right\} \mid \Delta, X_{k1}, \ldots, X_{kn_{Tk}}\ , T\}\right)^{k-1}\right] \\
&= E\left[\left(\mathbb{P}\left\{\frac{n_{T(\delta)r}\left(\sum_{j=1}^{n_{T(\delta)k}} X_{kj} - n_{T(\delta)k}\mu_k\right) - n_{T(\delta)k}\left(\sum_{j=1}^{n_{T(\delta)r}} X_{rj} - n_{T(\delta)r}\mu_r\right)}{R^{3/2}v_{rk}}\right.\right.\right. \\
&\quad \left.\left.\left. +\frac{\delta}{R^{3/2}v_{rk}}\left(n_{T(\delta)k}n_{T(\delta)r}\right) > 0\right\} \mid \Delta, X_{k1}, \ldots, X_{kn_{Tk}}, T\right)^{k-1}\right] \\
&\geq E\left[\left(\mathbb{P}\left\{\frac{n_{T(\delta)r}\left(\sum_{j=1}^{n_{T(\delta)k}} X_{kj} - n_{T(\delta)k}\mu_k\right) - n_{T(\delta)k}\left(\sum_{j=1}^{n_{T(\delta)r}} X_{rj} - n_{T(\delta)r}\mu_r\right)}{R^{3/2}v_{rk}}\right.\right.\right. \\
&\quad \left.\left.\left. +\frac{\delta}{R^{3/2}v_{rk}}\left(n_{T(\delta)k}n_{T(\delta)r}\right) > 0\right\} \mid \Delta, X_{k1}, \ldots, X_{kn_{Tk}}, T\right)\right]^{k-1} \\
&= E\left[\left(\mathbb{P}\left\{\frac{n_{T(\delta)r}\left(\sum_{j=1}^{n_{T(\delta)k}} X_{kj} - n_{T(\delta)k}\mu_k\right) - n_{T(\delta)k}\left(\sum_{j=1}^{n_{T(\delta)r}} X_{rj} - n_{T(\delta)r}\mu_r\right)}{R^{3/2}v_{rk}}\right.\right.\right. \\
&\quad \left.\left.\left. +\frac{\delta}{R^{3/2}v_{rk}}\left(n_{T(\delta)k}n_{T(\delta)r}\right) > 0\right\} \mid \Delta\right]^{k-1}\right. \\
&\geq
\end{aligned}
$$

Now,

Suppose $c > 0$.

(I haven't erased the following in case I need it later)

Observe that,

$$\mu_k - \mu_i + \frac{v_k}{\sqrt{n_{T(\delta)k}}} C_k\left(1, n_{T(\delta)k}\right) - \frac{v_i}{\sqrt{n_{T(\delta)i}}} C_i\left(1, n_{T(\delta)i}\right) \Rightarrow \mu_k - \mu_i$$

when $\delta \to 0$ if $n_{T(\delta)j}$ is independent of $C_j(1,n)$ for n sufficiently large and $j = k, i$.

Suppose $i = r$ and $t_0 > 1$. Let $s_0 \leq t \leq 1$ where $floor(s_0 R) \geq t_0$ and $floor(sR) < t_0$ if $s < s_0$. Let

$$\begin{aligned} D_i(t,R) &= \frac{\sum_{j=1}^{n_{floor(tR),i}} X_{ij} - \frac{\lambda_i^2}{\lambda_k^2}\left(n_0 + (tR)\right)\mu_i}{v_i\sqrt{R}} \\ &= \frac{\sum_{j=1}^{ceil\left(\frac{\lambda_i^2}{\lambda_k^2}(n_0 + floor(tR))\right)} X_{ij} - \left(n_0 + (tR)\right)\mu_i}{v_i\sqrt{R}} \end{aligned}$$

and if $t < s_0$

$$\begin{aligned} D_i(t,R) &= \frac{\sum_{j=1}^{n_{floor(tR),i}} X_{ij} - n_0\mu_i}{v_i\sqrt{R}} \\ &= \frac{\sum_{j=1}^{n_0} X_{ij} - n_0\mu_i}{v_i\sqrt{R}}, \end{aligned}$$

Suppose $D_i(t,R) \Rightarrow W()$ if $t \geq s_0$.

$$D_{ik}\left(t,R\right)=\frac{\sum_{j=1}^{n}}{v_{ik}\sqrt{R}}$$

By Assumption 1, Lemma 1, and the CMT we have that

$$C_{ik}\left(\frac{n_{t,i}}{N},\frac{n_{t,k}}{N},\delta\right)+\frac{n_{t,k}n_{t,i}\delta}{N^{1/2}v_{ik}}\Rightarrow n_{ti}W_{1}\left(n_{ti}\right)-n_{tk}W_{2}\left(n_{tk}\right)+\frac{n_{t,k}n_{t,i}\Delta}{v_{ik}}.$$

Let

$$A\left(\delta\right)=\frac{\log\left[\min\left\{\frac{P^{*}}{1-P^{*}},\frac{1-c}{c}\right\}\right]}{N^{1/2}v_{ik}\delta}\rightarrow\frac{\log\left[\min\left\{\frac{P^{*}}{1-P^{*}},\frac{1-c}{c}\right\}\right]}{N^{1/2}v_{ik}\delta}$$

$$\beta_{t}=\sum$$

Observe that,

$$\mu_{k}-\mu_{i}+\frac{v_{k}}{\sqrt{n_{T(\delta)k}}}C_{k}\left(1,n_{T(\delta)k}\right)-\frac{v_{i}}{\sqrt{n_{T(\delta)i}}}C_{i}\left(1,n_{T(\delta)i}\right)\Rightarrow\mu_{k}-\mu_{i}$$

when $\delta\rightarrow0$ if $n_{T(\delta)j}$ is independent of $C_{j}\left(1,n\right)$ for n sufficiently large and $j=k,i$.

Let ICS denote the even that an incorrect selection is made. Then,

$$\begin{aligned}
\lim\ \inf_{\delta\to 0}\mathbb{P}\{ICS\} &= \lim\ \inf_{\delta\to 0}\mathbb{P}\left\{\frac{W_{T(\delta)k}}{n_{T(\delta)k}} < \frac{W_{T(\delta)i}}{n_{T(\delta)i}}\right\} \\
&= \mathbb{P}\{\mu_k < \mu_i\} = 0.
\end{aligned}$$

However, it's very likely $n_{T(\delta)k}$ is not independent of $C_k(1,n)$. Thus, let's give other proof. First, we'll show that n_{ti} is independent of $C_i(1,n)$ if $n \geq 1$ and $t \geq 0$. Note that

$$\begin{aligned}
C_i(1,n) &= \frac{\sqrt{n}\sum_{j=1}^{n} X_{ij}}{v_i n} - \frac{\sqrt{n}}{v_i}\mu_i \\
&= \frac{\sqrt{n}\bar{X}_i(n)}{v_i} - \frac{\sqrt{n}}{v_i}\mu_i.
\end{aligned}$$

Since $n_{0i} = n_0$, then $C_i(1,n)$ is independent of n_{0i}. Note that

$$n_{1,i} = \text{ceil}\left(\hat{\lambda_{0i}^2}\left(n_0 + B_z\right)/\hat{\lambda_{0z}^2}\right),$$

if $z = i$, then $n_{1,i}$ is independent of $C_i(1,n)$. Suppose $z = k$. Note that

$$\begin{aligned}
\hat{\lambda_{1i}^2} &= \frac{1}{n-1}\sum_{j=1}^{n_{1i}}\left(X_{ij}-\bar{X}_i\left(n_{1i}\right)\right)^2 \\
&= \frac{1}{n-1}\left(\left(X_{i1}-\bar{X}_i\left(n_{1i}\right)\right)^2+\sum_{j=2}^{n_{ti}}\left(X_{ij}-\bar{X}_i\left(n_{1i}\right)\right)^2\right) \\
&= \frac{1}{n-1}\left(\left[\sum_{j=2}^{n_{1i}}\left(X_{ij}-\bar{X}_i\left(n_{1i}\right)\right)^2\right]+\sum_{j=2}^{n_{1i}}\left(X_{ij}-\bar{X}_i\left(n_{1i}\right)\right)^2\right).
\end{aligned}$$

Then $\hat{\lambda}_{1i}^2$ can be written as a function only of $\left(X_{i2}-\bar{X}_i\left(n_{0i}\right),\ldots,X_{in_{ti}}-\bar{X}_i\left(n_{ti}\right),n_{ti}\right)$.

Suppose $C_i\left(1,n\right)$ is independent of n_{ti}. Observe that

$$n_{t+1,i}=\operatorname{ceil}\left(\hat{\lambda_{ti}^2}\left(n_{tz}+B_z\right)/\hat{\lambda_{tz}^2}\right),$$

if $z=i$, then $n_{t+1,i}=\operatorname{ceil}\left(\left(n_{tz}+B_z\right)\right)$ and $C_i\left(1,n\right)$ are independent random variables. Now suppose $z=k$. Note that

$$\begin{aligned}
\hat{\lambda_{ti}^2} &= \frac{1}{n-1}\sum_{j=1}^{n_{ti}}\left(X_{ij}-\bar{X}_i\left(n_{ti}\right)\right)^2 \\
&= \frac{1}{n-1}\left(\left(X_{i1}-\bar{X}_i\left(n_{ti}\right)\right)^2+\sum_{j=2}^{n_{ti}}\left(X_{ij}-\bar{X}_i\left(n_{ti}\right)\right)^2\right) \\
&= \frac{1}{n-1}\left(\left[\sum_{j=2}^{n_{ti}}\left(X_{ij}-\bar{X}_i\left(n_{ti}\right)\right)\right]^2+\sum_{j=2}^{n_{ti}}\left(X_{ij}-\bar{X}_i\left(n_{ti}\right)\right)^2\right).
\end{aligned}$$

Then $\hat{\lambda_{ti}^2}$ can be written as a function only of $\left(X_{i2}-\bar{X}_i\left(n_{ti}\right),\ldots,X_{in_{ti}}-\bar{X}_i\left(n_{ti}\right),n_{ti}\right)$. Suppose n_{ti} is given. The joint pdf of the sample $X_{i1},\ldots,X_{in_{ti}}$ is given by

$$f\left(x_1,\ldots,x_{n_{ti}}\right)=\frac{1}{\left(2\pi\right)^{n_{ti}/2}}e^{-(1/2)\sum_{i=1}^{n_{ti}}x_i^2}.$$

Make the transformation $y_1=\bar{x}\left(n_{ti}\right),y_2=x_2-\bar{x}\left(n_{ti}\right),\ldots,y_{n_{ti}}=x_{n_{ti}}-\bar{x}\left(n_{ti}\right)$. This is a linear transformation with a Jacobian equal to $1/n_{ti}$. We have

$$\begin{aligned}f\left(y_1,\ldots,y_{n_{ti}}\right) &= \frac{n_{ti}}{\left(2\pi\right)^{n_{ti}/2}}e^{-(1/2)\left(y_1-\sum_{i=2}^{n_{ti}}y_i\right)^2}e^{-(1/2)\sum_{i=2}^{n_{ti}}(y_i+y_1)^2}\\ &= \left[\left(\frac{n_{ti}}{2\pi}\right)^{1/2}e^{\left(-n_{ti}y_1^2\right)/2}\right]\left[\frac{n_{ti}^{1/2}}{\left(2\pi\right)^{(n_{ti}-1)/2}}e^{-(1/2)\left[\sum_{i=2}^{n_{ti}}y_i^2+\left(\sum_{i=2}^{n_{ti}}y_i\right)^2\right]}\right].\end{aligned}$$

Then y_1 is independent of $\hat{\lambda_{ti}^2}$ given n_{ti}. If $n_{ti}\geq n$, then $\bar{X}_i\left(n\right)$ is independent of $\hat{\lambda_{ti}^2}$ because X_{ij} is independent of $X_{ij'}$ for $j'<j$. If $n_{ti}<n$,

$$\begin{aligned}\bar{X}_i\left(n\right) &= \frac{\sum_{j=1}^{n_{ti}}X_{ij}}{n_{ti}}\frac{n_{ti}}{n}+\frac{\sum_{j=n_{ti+1}}^{n}X_{ij}}{n}\\ &= Y_1\frac{n_{ti}}{n}+\frac{\sum_{j=n_{ti+1}}^{n}X_{ij}}{n}\end{aligned}$$

and then $\bar{X}_i\left(n\right)$ is independent of $\hat{\lambda_{ti}^2}$. Thus, $C_i\left(1,n\right)$ is independent of $\hat{\lambda_{ti}^2}$ given n_{ti}. So, by induction,

$$\begin{aligned}f\left(C_i\left(1,n\right),\hat{\lambda_{ti}^2}\right) &= \int f\left(C_i\left(1,n\right),\hat{\lambda_{ti}^2}\mid n_{ti}\right)f\left(n_{ti}\right)dn_{ti}\\ &= f\left(C_i\left(1,n\right)\right)f\left(\hat{\lambda_{ti}^2}\right),\end{aligned}$$

then $C_i(1,n)$ is independent of $\hat{\lambda}_{ti}^2$. Consequently, $C_i(1,n)$ is independent of $n_{t,i+1}$. Thus

$$\mu_k - \mu_i + \frac{v_k}{\sqrt{n_{tk}}} C_k(1, n_{tk}) - \frac{v_i}{\sqrt{n_{ti}}} C_i(1, n_{ti}) \Rightarrow \mu_k - \mu_i$$

as $t \to \infty$. Let $Y_t = C_k\left(1, n_{T(1/t)k}\right)$ and $X_t = C_k(1, n_{tk})$, if $d(Y_t, X_t) \to 0$ in probability, then $Y_t \Rightarrow W()$. Thus the result would be proved. Let's show that $d(Y_t, X_t) \to 0$. ???Let $\epsilon > 0$.

$$\mu_k - \mu_i + \frac{v_k}{\sqrt{n_{T(\delta)k}}} C_k(1, n_{tk}) - \frac{v_i}{\sqrt{n_{T(\delta)i}}} C_i\left(1, n_{T(\delta)i}\right) \Rightarrow \mu_k - \mu_i$$

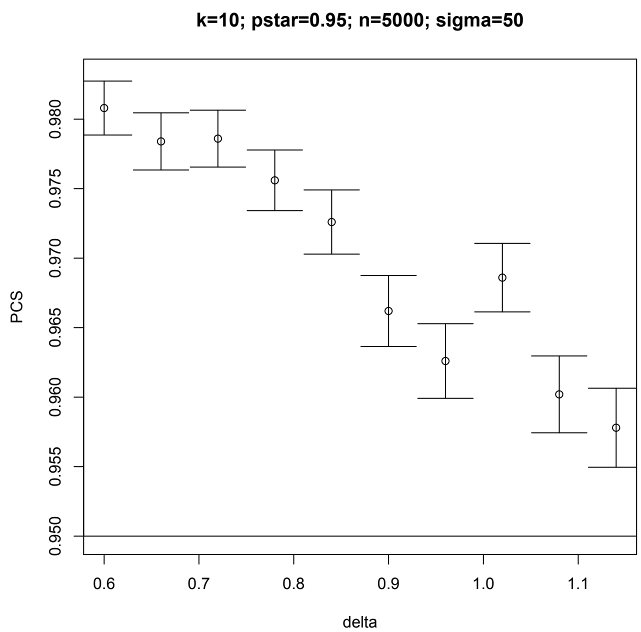
$$\mathbb{P}\left\{\frac{W_{T(\delta)k}}{n_{T(\delta)k}} < \frac{W_{T(\delta)i}}{n_{T(\delta)i}}\right\} = \mathbb{P}\left\{\frac{W_{T(\delta)k}}{n_{T(\delta)k}} < \frac{W_{T(\delta)i}}{n_{T(\delta)i}}\right\}$$

Now consider $k \geq 2$ systems and let CS be the event that k is selected and let ICS_i be the event that an incorrect selection is made when systems

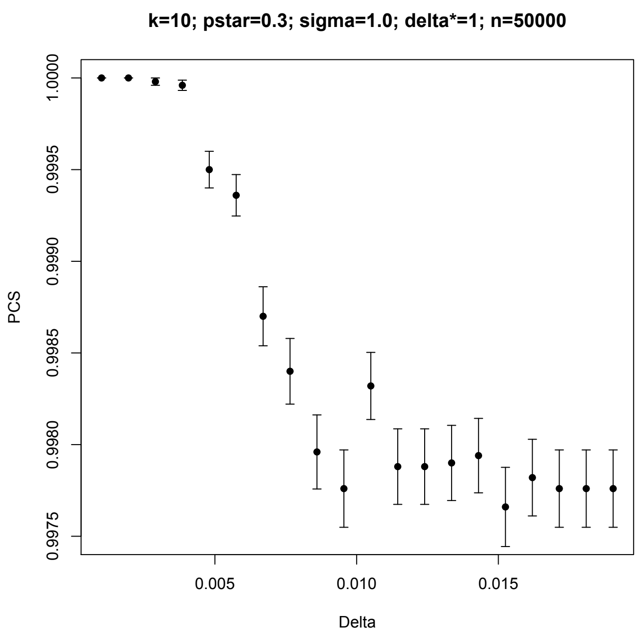
k and i are considered in isolation.

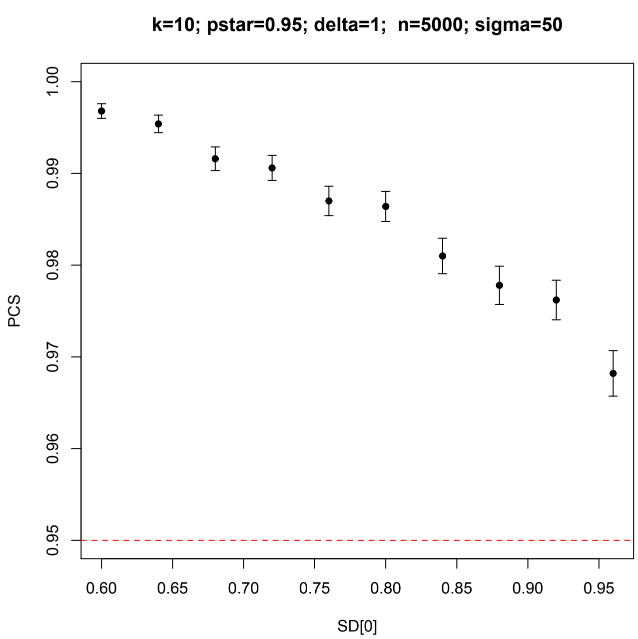
Theorem. If samples from system $x \in \{1 \ldots, k\}$ are normally distributed and independent, over time and across alternatives, then $\lim_{\min_x \sigma_x^2 \to \infty} Pr\{\text{BIZ selects } k\} \geq P*$ provided $\mu_k \geq \mu_{k-1} + \delta*$ for some $\delta* > 0$.

Graphs Delta goes to 0

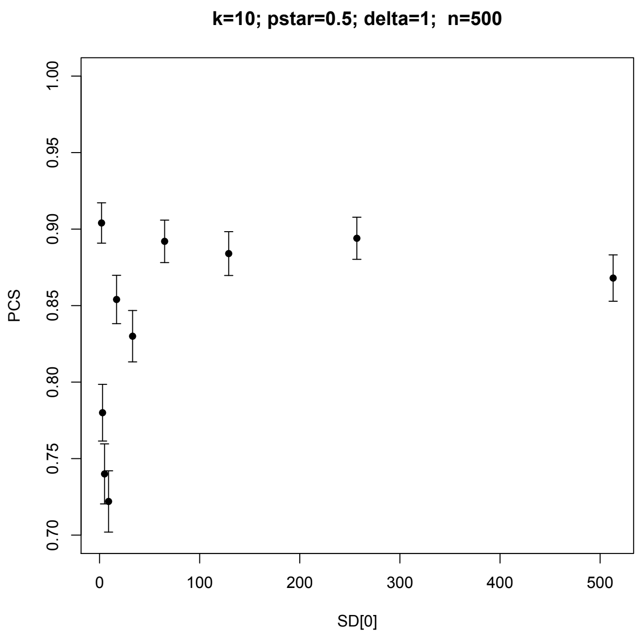


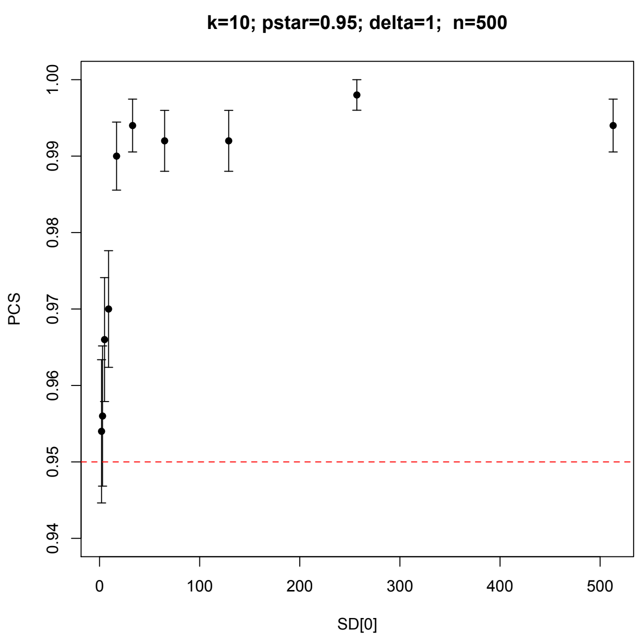
Delta goes to 0.





Variances go to infinity.


k=10; pstar=0.5; delta=1; n=500
PCS
SD[0]
0.70
0.75
0.80
0.85
0.90
0.95
1.00
0
100
200
300
400
500


k=10; pstar=0.95; delta=1; n=500
PCS
SD[0]
0.94
0.95
0.96
0.97
0.98
0.99
1.00
0
100
200
300
400
500

Theorem. If samples from system $x \in \{1 \ldots, k\}$ are normally distributed and independent, over time and across alternatives, then $\lim_{\delta \to 0} Pr \{\text{BIZ selects } k\} \geq P*$ (or $=1$??) provided $\mu_k \geq \mu_{k-1} + \delta^*$.

Prooof. We begin by considering the case of only two systems, denoted k and i, with $\mu_k \geq \mu_i + \delta*$. Let $A = \{k, i\}$. For $\delta \leq \delta*$, let

$$T(\delta) = \min \{t \in \mathbb{N} : \min_{x \in A} \hat{q}_{tx}(A) \leq c \text{ or } \max_{x \in A} \hat{q}_{tx}(A) \geq P*\}.$$

Thus $T(\delta)$ is the stage at which the procedure terminates.

Now,

$$\begin{aligned}
\min_{x \in A} \hat{q}_{tx}(A) &\leq c \\
\Leftrightarrow \min_{x \in A} \exp\left(\delta\beta_t \frac{W_{tx}}{n_{tx}}\right) &\leq c \sum_{x' \in A} \exp\left(\delta\beta_t \frac{W_{tx'}}{n_{tx'}}\right) \\
\Leftrightarrow \min_{x \in A} \frac{W_{tx}}{n_{tx}} &\leq \log\left[c \sum_{x' \in A} \exp\left(\delta\beta_t \frac{W_{tx'}}{n_{tx'}}\right)\right] / (\delta\beta_t) \\
\Leftrightarrow \left(\frac{W_{tk}}{n_{tk}} + \frac{W_{ti}}{n_{ti}} - \left|\frac{W_{tk}}{n_{tk}} - \frac{W_{ti}}{n_{ti}}\right|\right) &\leq 2\log\left[c \sum_{x' \in A} \exp\left(\delta\beta_t \frac{W_{tx'}}{n_{tx'}}\right)\right] / (\delta\beta_t)
\end{aligned}$$

and

$$
\begin{aligned}
\max_{x\in A}\hat{q}_{tx}(A) &\geq P* \\
\Leftrightarrow \max_{x\in A}\exp\left(\delta\beta_t\frac{W_{tx}}{n_{tx}}\right) &\geq P*\sum_{x'\in A}\exp\left(\delta\beta_t\frac{W_{tx'}}{n_{tx'}}\right) \\
\Leftrightarrow \max_{x\in A}\frac{W_{tx}}{n_{tx}} &\geq \log\left[P*\sum_{x'\in A}\exp\left(\delta\beta_t\frac{W_{tx'}}{n_{tx'}}\right)\right]/(\delta\beta_t) \\
\Leftrightarrow \left(\frac{W_{tk}}{n_{tk}}+\frac{W_{ti}}{n_{ti}}+\left|\frac{W_{tk}}{n_{tk}}-\frac{W_{ti}}{n_{ti}}\right|\right) &\geq 2\log\left[P*\sum_{x'\in A}\exp\left(\delta\beta_t\frac{W_{tx'}}{n_{tx'}}\right)\right]/(\delta\beta_t).
\end{aligned}
$$

Let $d(\delta):=2\log\left(\sum_{x'\in A}\exp\left(\delta\beta_t\frac{W_{tx'}}{n_{tx'}}\right)\right)/(\delta\beta_t)$, then

$$
\begin{aligned}
T(\delta) = \min\Bigg\{t\in\mathbb{N}: &\left(\frac{W_{tk}}{n_{tk}}+\frac{W_{ti}}{n_{ti}}+\left|\frac{W_{tk}}{n_{tk}}-\frac{W_{ti}}{n_{ti}}\right|\right)\geq 2\frac{\log(P*)}{\delta\beta_t}+d(\delta) \text{ or} \\
&\left(\frac{W_{tk}}{n_{tk}}+\frac{W_{ti}}{n_{ti}}-\left|\frac{W_{tk}}{n_{tk}}-\frac{W_{ti}}{n_{ti}}\right|\right)\leq 2\frac{\log c}{\delta\beta_t}+d(\delta)\Bigg\}.
\end{aligned}
$$

Now, let's prove that $n_{ti}\to\infty, n_{tk}\to\infty$ as $t\to\infty$. For each t, $n_{t+1,x}=n_{tx}+B_x$ for some $x\in A$. If $n_{t+1,i}=n_{ti}+B_i$ for a finite number of $t's$, then $n_{tk}\to\infty$ as $t\to\infty$. So, there exists l_0 such that $n_{li}/\hat{\lambda}_{li}^2>n_{lk}/\hat{\lambda}_{lk}^2$ if $l>l_0$. Since $\hat{\lambda}_{ti}^2\to\sigma_i^2$ and $\hat{\lambda}_{tk}^2\to\sigma_k^2$ as $t\to\infty$, then $n_{ti}\to\infty$ as $t\to\infty$. Similarly, we get the same result in the other cases. Thus, $n_{ti}\to\infty, n_{tk}\to\infty$ as $t\to\infty$. Furthermore, it's easy to see that $T(\delta)\to\infty$ as $\delta\to 0$.

Now, we'll show that $T(\delta)$ is finite. Observe that there exists t_0 such that if $t>t_0$

$$W_{ti}/n_{ti}<W_{tk}/n_{tk},$$

otherwise there would exists t such that

$$\mu_k - \delta/2 < W_{tk}/n_{tk} \leq W_{ti}/n_{ti} < \delta/2 + \mu_i$$

which is a contradiction. Moreover, there exists $t_1 > t_0$ such that if $t > t_1$

$$\beta_t \left(W_{tk}/n_{tk} - W_{ti}/n_{ti} \right) > \log \left(P* / \left(1 - P* \right) \right) / \delta$$

since $\beta_t \to \infty$ as $t \to \infty$ and $P* > \frac{1}{2}$. Consequently, if $t > t_1$

$$\begin{aligned} \exp\left(\delta\beta_t \frac{W_{tk}}{n_{tk}}\right) / \exp\left(\delta\beta_t \frac{W_{ti}}{n_{ti}}\right) &> \frac{P*}{1-P*} \\ \Rightarrow \exp\left(\delta\beta_t \frac{W_{tk}}{n_{tk}}\right) &> P*\exp\left(\delta\beta_t \frac{W_{ti}}{n_{ti}}\right) \\ &\quad + P*\exp\left(\delta\beta_t \frac{W_{tk}}{n_{tk}}\right) \end{aligned}$$

and so $T(\delta) \leq t_1(\delta) < \infty$.

By the strong law of large numbers, almost surely

$$\frac{W_{T(\delta)k}}{n_{T(\delta)k}} - \frac{W_{T(\delta)i}}{n_{T(\delta)i}} \to \mu_k - \mu_i$$

as $\delta \to 0$. Thus,

$$\frac{W_{T(\delta)k}}{n_{T(\delta)k}} - \frac{W_{T(\delta)i}}{n_{T(\delta)i}} \Rightarrow \mu_k - \mu_i$$

as $\delta \to 0$.

Let ICS denote the even that an incorrect selection is made. Then,

$$\begin{aligned}
\liminf_{\delta \to 0} \mathbb{P}\{ICS\} &= \liminf_{\delta \to 0} \mathbb{P}\left\{ \frac{W_{T(\delta)k}}{n_{T(\delta)k}} < \frac{W_{T(\delta)i}}{n_{T(\delta)i}} \right\} \\
&= \mathbb{P}\{\mu_k < \mu_i\} = 0.
\end{aligned}$$