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ASYMPTOTIC VALIDITY OF THE BAYES-INSPIRED INDIFFERENCE ZONE PROCEDURE: THE GENERAL DISTRIBUTION KNOWN VARIANCE CASE

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ABSTRACT

We consider the indifference-zone (IZ) formulation of the ranking and selection problem in which the goal is to choose an alternative with the largest mean with guaranteed probability, as long as the difference between this mean and the second largest exceeds a threshold. Conservatism leads classical IZ procedures to take too many samples in problems with many alternatives. The Bayes-inspired Indifference Zone (BIZ) procedure, proposed in Frazier (2014), is less conservative than previous procedures, but its proof of validity requires strong assumptions, specifically that samples are normal, and variances are known with an integer multiple structure. In this paper, we show asymptotic validity of a slight modification of the original BIZ procedure as the difference between the best alternative and the second best goes to zero, when the variances are known and finite, and samples are independent and identically distributed, but not necessarily normal.

1 INTRODUCTION

There are many applications where we have to choose the best alternative among a finite number of simulated alternatives. For example, in inventory problems, we may want to choose the best inventory policy (s,S) for a finite number of values of s and S. This is called the ranking and selection problem. A good procedure for addressing this problem should be both efficient and accurate, i.e. it should balance the number of samples it takes with the quality of its selection.

This paper considers the indifference-zone (IZ) formulation of the ranking and selection problem, in which we require that a procedure satisfy the IZ guarantee, i.e., that the best system be chosen with probability larger than some threshold P^* given by the user, when the distance between the best system and the others is larger than some other user-specified threshold $\delta > 0$. The set of problem configurations satisfying this constraint on the difference in means is called the preference zone. The paper **?** is considered the seminal work, and early work is presented in the monograph **?**. Some compilations of the theory developed in the area can be found in **?**, **?**, **?** and **?**. Other approaches, beyond the indifference-zone approach, include the Bayesian approach (**?**), the optimal computing budget allocation approach (**?**), and the large deviations approach (**?**).

A good IZ procedure satisfies the IZ guarantee and requires as few samples as possible. The first IZ procedures presented in **?**, **?**, **?**, **?**, **?**, **?**, **?** satisfy the IZ guarantee, but they usually take too many samples when there are many alternatives, in part because they are conservative: their probability of correct selection (PCS) is much larger than the probability specified by the user (**?**). One reason for this is that these procedures use Bonferroni's inequality, which leads then to sample more than necessary. The Bonferonni-based bounds underlying these procedures become looser, and the tendency to take more samples than necessary increases, as the number of alternatives grow. More recently, new algorithms were developed in **?**, **?**, **?**, and they improve performance but they still use Bonferroni's inequality, and so the methods are inefficient when there are many alternatives. Procedures in **?**, **?** do not use Bonferroni's inequality when there are only three alternatives, but again use Bonferroni's inequality when comparing more than three alternatives.

In addition to Bonferroni's inequality, two other common sources of conservatism in indifference-zone ranking and selection procedures are the change from discrete time to continuous time often used to show IZ guarantees, and the fact that typically, the configuration under consideration is not a worst-case configuration (**?**). The difference between worst and typical cases tends to contribute the most to conservatism, with Bonferonni's inequality contributing second-most, and the continuous/discrete time difference contributing the least (**?**). Although the difference between the worst and typical cases is the largest contributor to conservatism, all indifference zone procedures must meet the PCS guarantee for all configurations in the preference zone, including worst-case configurations, and so this source of conservatism is fundamental to the indifference-zone formulation. Thus, eliminating the use of Bonferroni's inequality remains an important route for reducing conservatism while still retaining the indifference-zone guarantee.

? presents a new sequential elimination IZ procedure, called BIZ (Bayes-inspired Indifference Zone), that eliminates the use of Bonferroni's inequality, reducing conservatism. This procedure's lower bound on worst-case probability of correct selection in the preference zone is tight in continuous time, and almost tight in the discrete time. In numerical experiments, the number of samples required by BIZ is significantly smaller than that of procedures like the KN procedure of **?** and the $\mathscr{P}_B^*$ procedure of **?**, especially on problems with many alternatives. Unfortunately, the proof from (**?**) that the BIZ procedure satisfies the IZ guarantee for the discrete-time case assumes that (1) samples are normally distributed; (2) variances are known; and (3) the variances are either common across alternatives, or have an unrealistic integer multiple structure.

The contribution of this work is to prove the asymptotic validity of the BIZ procedure as δ goes to zero, retaining the assumption of known variances, but replacing assumptions (1) and (3) by the much weaker assumption of independent and identically distributed finite variance samples. Thus, our proof allows a much broader set of sampling distributions than that allowed by (**?**), including non-normal samples and general heterogeneous variances. We also show that this bound on worst-case PCS is asymptotically tight as δ goes to zero, showing that the BIZ procedure successfully eliminates conservatism due to Bonferonni's inequality in this more general setting, just as was demonstrated by **?** for more restricted settings.

To simplify our analysis, we analyze a slight modification of the version of the BIZ procedure presented in (**?**), which keeps a certain parameter λ_z^2 fixed rather than letting it vary as did (**?**). Numerical experiments on typical cases show little difference in performance between the version of BIZ we analyze and the version in (**?**). We conjecture that a proof technique similar to the one presented here can be used to show asymptotic validity of the BIZ procedure when the variances are unknown, and we present numerical experiments that support this belief.

This paper is organized as follows: In 2, we recall the indifference-zone ranking and selection problem. In 3, we recall the Bayes-inspired IZ (BIZ) procedure from **?**. In 4, we present the proof of the validity of the algorithm when the variances are known. In 5, we present some numerical experiments. In 6, we conclude.

2 INDIFFERENCE-ZONE RANKING AND SELECTION

Ranking and Selection is a problem where we have to select the best system among a finite set of alternatives, i.e. the system with the largest mean. The method selects a system as the best based on the samples that are observed sequentially over time. We suppose that samples are identically distributed and independent, over time and across alternatives, and each alernative x has mean μ_x. We define $\mu = (\mu_1, \ldots, \mu_k)$.

If the best system is selected, we say that the procedure has made the *correct selection* (CS). We define the *probability of correct selection* as

$$\mathrm{PCS}(\mu) = \mathbb{P}_\mu\left(\hat{x} \in \arg\max_x \mu_x\right)$$

where $\hat{x}$ is the alternative chosen by the procedure and $\mathbb{P}_\mu$ is the probability measure under which samples from system x have mean μ_x and finite variance λ_x^2.

In the Indifference-Zone Ranking and Selection, the procedure is indifferent in the selection of a system whenever the means of the populations are nearly the same. Formally, let $\mu = [\mu_k, \ldots, \mu_1]$ be the vector of the true means, the *indifference zone* is defined as the set $\left\{\mu \in \mathbb{R}^k : \mu_{[k]} - \mu_{[k-1]} < \delta\right\}$. The complement of the indifference zone is called the *preference zone* (PZ) and $\delta > 0$ is called the indifference zone parameter. We say that a procedure meets the *indifference-zone (IZ) guarantee* at $P^* \in (1/k, 1)$ and $\delta > 0$ if

$$\mathrm{PCS}(\mu) \geq P^* \text{ for all } \mu \in \mathrm{PZ}(\delta).$$

We assume $P^* > 1/k$ because IZ guarantees can be meet by choosing $\hat{x}$ uniformly at random from among $\{1, \ldots, k\}$.

3 THE BAYES-INSPIRED IZ (BIZ) PROCEDURE

BIZ is an elimination procedure. This procedure maintains a set of alternatives that are candidates for the best system, and it takes samples from each alternative in this set at each point in time. At beginning, all alternatives are possible candidates for the best system, and over the time alternatives are eliminated. The procedure ends when there is only one alternative in the contention set and this remain alternative is chosen as the best.

Frazier (**?**) showed that the BIZ procedure with known common variance satisfies the IZ guarantee when the systems follow the normal distribution , with tight bounds on worst-case preference-zone in continuous time. He also proved that this procedure retains the IZ guarantee when the systems follow the normal distribution, and the variances are known and are integer multiples of a common value. The continuous time version of this procedure also satisfies the IZ guarantee, with a tight worst-case preference-zone PCS bound.

The discrete-time BIZ procedure for unknown and/or heterogeneous sampling variances is given below. It takes a variable number of samples from each alternative, and n_{tx} is this number. This algorithm depends on a collection of integers $B_1, \ldots, B_k$, P^*, c, δ and n_0. n_0 is the number of samples to use in the first stage of samples, and 100 is the recommended value for n_0. B_x controls the number of samples taken from system x in each stage. The procedure presented is a slightly modification of the original BIZ procedure where $z \in \arg\max_{x\in A} \widehat{\lambda}_{tx}^2$, instead of $z \in \arg\min_{x\in A} n_{tx}/\widehat{\lambda}_{tx}^2$.

For each t, $x \in \{1, \ldots, k\}$, and subset $A \subset \{1, \ldots, k\}$, we define a function

$$q'_{tx}(A) = \exp\left(\delta \beta_t \frac{Z_{tx}}{n_{tx}}\right) \Big/ \sum_{x' \in A} \exp\left(\delta \beta_t \frac{Z_{tx'}}{n_{tx'}}\right), \quad \beta_t = \frac{\sum_{x'\in A} n_{tx'}}{\sum_{x'\in A} \hat{\lambda}_{tx'}^2}$$

where $\hat{\lambda}_{tx'}^2$ is the sample variance of all samples from alternative x thus far and $Z_{tx} = Y_{n_{tx},x}$.

Algorithm: Discrete-time implementation of BIZ, for unknown and/or heterogeneous variances.

Require: $c \in [0, 1 - (P^*)^{\frac{1}{k-1}}]$, $\delta > 0$, $P^* \in (1/k, 1)$, $n_0 \geq 0$ an integer, $B_1, \ldots, B_k$ strictly positive integers. Recommended choices are $c = 1 - (P^*)^{\frac{1}{k-1}}$, $B_1 = \cdots = B_k = 1$ and n_0 between 10 and 30. If the sampling variances λ_x^2 are known, replace the estimators $\widehat{\lambda}_{tx}^2$ with the true values λ_x^2, and set $n_0 = 0$.

1: For each x, sample alternative x n_0 times and set $n_{0x} \leftarrow n_0$. Let W_{0x} and $\widehat{\lambda}_{0x}^2$ be the sample mean and sample variance respectively of these samples. Let $t \leftarrow 0$.Let $z \in \arg\max_{x\in A} \widehat{\lambda}_{tx}^2$.
2: Let $A \leftarrow \{1, \ldots, k\}$, $P \leftarrow P^*$.
3: **while** $x \in \max_{x\in A} q'_{tx}(A) < P$ **do**
4:     **while** $\min_{x\in A} q'_{tx}(A) \leq c$ **do**
5:         Let $x \in \arg\min_{x\in A} q_{tx}(A)$.

Let $P \leftarrow P/(1-q_{tx}(A))$.
Remove x from A.
end while
For each $x \in A$, let $n_{t+1,x} = \text{ceil}\left(\widehat{\lambda}^2_{tx}(n_{tz}+B_z)/\widehat{\lambda}^2_{tz}\right)$.
For each $x \in A$, if $n_{t+1,x} > n_{tx}$, take $n_{t+1,x} - n_{tx}$ additional samples from alternative x. Let $W_{t+1,x}$ and $\widehat{\lambda}^2_{t+1,x}$ be the sample mean and sample variance respectively of all samples from alternative x thus far.
Increment t.
end while
Select $\hat{x} \in \arg\max_{x\in A} Z_{tx}/n_{tx}$ as our estimate of the best.

This algorithm generalizes the BIZ procedure with known common variance. In that case, we have that$B_1 = \cdots = B_k = 1$ and $n_{tx} = t$. The algorithm 2 can be generalized to the continuous case (See **?**).

4 ASYMPTOTIC VALIDITY WHEN THE VARIANCES ARE KNOWN

In this section we prove that the BIZ procedure satisfies asymptotically the IZ guarantee when the variances are known. This means that we consider a collection of ranking and selection problems parametrized by $\delta > 0$. For the problem given δ, we suppose that the vector of the true means $\mu = [\mu_k, \ldots, \mu_1]$ is equal to δa for some fixed $a \in \mathbb{R}^k$ that does not depend on δ and $a_k > a_{k-1} \geq \cdots \geq a_1$, $a_k - a_{k-1} > 1$. Moreover, the variances of the alternatives are finite, strictly greater than zero and do not depend on δ. We also suppose that samples from system $x \in \{1\ldots,k\}$ are identically distributed and independent, over time and across alternatives. We also define $\lambda_z^2 := \max_{i\in\{1\ldots,k\}} \lambda_i^2$.

Any ranking and selection algorithm can be viewed as mapping from paths of the k-dimensional discrete-time random walk $(Y_{tx} : t \in \mathbb{N}, x \in \{1,\ldots,k\})$ onto selection decisions. Our proof uses this viewpoint, noting that the BIZ procedure's mapping from paths onto selections decisions is the composition of three simpler maps.

The first is the mapping from the raw discrete-time random walk $(Y_{tx} : t \in \mathbb{N}, x \in \{1,\ldots,k\})$ onto a time changed version of this random walk, written as $(Z_{tx} : t \in \mathbb{N}, x \in \{1,\ldots,k\})$, where we recall $Z_{tx} = Y_{n_x(t),t}$ is the sum of the samples from alternative x observed by stage t.

The second maps this time-changed random walk through a non-linear mapping for each t,x and subset $A \subset \{1,\ldots,k\}$, to obtain $\left(q'_{tx}(A) : t \in \mathbb{N}, A \subset \{1,\ldots,k\}, x \in A\right)$, where

$$q'_{tx}(A) = \exp\left(\delta\beta_t \frac{Z_{tx}}{n_{tx}}\right) \Bigg/ \sum_{x'\in A} \exp\left(\delta\beta_t \frac{Z_{tx}}{n_{tx}}\right) := q'\left((Z_{tx} : x \in A), \delta, t\right)$$

where we note that $n_x(t)$ and β_t are deterministic in the version of the known-variance BIZ procedure that we consider here.

The third maps the paths of $\left(q'_{tx}(A) : t \in \mathbb{N}, A \subset \{1,\ldots,k\}, x \in A\right)$ onto selection decisions. Specifically, this mapping begins with $A_0 = \{1,\ldots,k\}$, $P_0 = P^*$, and finds the first time τ_1 that $q'_{tx}(A_0)$ falls above the threshold P_0, or below the threshold c. If the first case occurs, the alternative with the largest $q'_{\tau_1,x}(A_0)$ is selected as the best. If the second case occurs, the alternative with the smallest $q'_{\tau_1,x}(A_0)$ is eliminated, resulting in a new set A_1, a new selection threshold P_1 is calculated from P_0 and the eliminated alternative's value of $q'_{\tau_1,x}(A_0)$, and the process continues. This process is repeated until an alternative is selected as the best. Call this mapping h, so that the BIZ selection decision is $h\left(\left(q'_{tx}(A) : t \in \mathbb{N}, A \subset \{1,\ldots,k\}, x \in A\right)\right)$.

4.1 Proof Outline

Based on this view of the BIZ procedure as a composition of three maps, we outline the main ideas of our proof here.

Our proof first notes that the same selection decision is obtained if we apply the BIZ selection map h to a time-changed version of $\left(q'_{tx}(A) : t \in \mathbb{N}, A \subset \{1,\ldots,k\}, x \in A\right)$, specifically to

$$\left(q_{tx}(A) : t \in \delta^2\mathbb{N}, A \subset \{1,\ldots,k\}, x \in A\right),$$

where $q_{tx}(A) := q'\left(\left(Z_{\frac{t}{\delta^2}x} : x \in A\right), \delta, t\right)$.

This discrete-time process is interpolated by the continuous-time process

$$\left(q_{tx}(A) : t \in \mathbb{R}, A \subset \{1,\ldots,k\}, x \in A\right). \tag{1}$$

If we apply the BIZ selection map h to this continuous-time process, the selection decision will differ from BIZ's selection decision for $\delta > 0$, but we show that this difference vanishes as $\delta \to 0$. Thus, our proof focuses on showing that, as $\delta \to 0$, applying the BIZ selection map h to (**??**) produces a selection decision that satisfies the indifference-zone guarantee.

To accomplish this, we use a functional central limit theorem for $Z_{\frac{t}{\delta^2}x}$, which shows that a centralized version of $Z_{\frac{t}{\delta^2}x}$ converges to a Brownian motion as δ goes to 0. This centralized version of $Z_{\frac{t}{\delta^2}x}$ is

$$\mathscr{C}_x(\delta,t) := \frac{Y_{n_x(t),x} - t\lambda_x^2\mu_x}{\frac{\lambda_x^2}{\lambda_z\delta}}.$$

Rewriting $Z_{\frac{t}{\delta^2}x}$ in terms of $\mathscr{C}_x(\delta,t)$ and substituting into the definition of $q_{tx}(A)$ provides the expression

$$q_{tx}(A) = q\left(\left(\mathscr{C}_x(\delta,t)\frac{\lambda_x^2}{\delta\lambda_z^2} + \frac{\lambda_x^2}{\lambda_z^2}\left(n_0 + \frac{t}{\delta^2}\right)\delta a_x : x \in A\right), \delta, t\right). \tag{2}$$

We will construct a mapping $f(\cdot,\delta)$ that takes as input the process $(\mathscr{C}_x(\delta,t) : x \in \{1,\ldots,k\}, t \in \mathbb{R})$, calculates (**??**) from it, applies the BIZ selection map h to (**??**), and then returns 1 if the correct selection was made, and 0 otherwise. Thus, the correct selection event that results from applying the BIZ selection map h to (**??**) is the result of applying the mapping $f(\cdot,\delta)$ to the paths $t \mapsto \mathscr{C}_x(\delta,t)$.

With these pieces in place, the last part of our proof is to observe that (1) $\mathscr{C}(\delta,\cdot)$ converges to a multivariate Brownian motion W as δ goes to 0; (2) the function f has a continuity property that causes

$$f(\mathscr{C}(\delta,\cdot),\delta) \Rightarrow g(W)$$

where g is the selection decision from applying the BIZ procedure in continuous time; and (3) the BIZ procedure satisfies the IZ guarantee when applied in continuous time (Theorem 1 in (**?**)), and so $E[g(W)] \geq P^*$ with equality for the worst configurations in the preference zone.

4.2 Preliminaries for the Proof of the Main Theorem

In this section, we present preliminary results and definitions used in the proof of the main theorem: first, a central limit theorem Corollary **??**; second, definitions of the functions $f(\cdot,\delta)$ and $g(\cdot)$; third, a continuity result Lemma **??**; and fourth, a result Lemma **??** that allows us to change from discrete-time processes to continuous-time processes.

First, we are going to see that the centralized sum of the output data $\mathscr{C}_x(\delta,t)$ converges to a Brownian motion in the sense of $D_\infty := D[0,\infty)$, which is the set of functions from $[0,\infty)$ to $\mathbb{R}$ that are right-continuous and have left-hand limits, with the Skorohod topology. The definition and the properties of this topology may be found in Chapter 3 of **?**.

We briefly recall the definition of convergence of random paths in the sense of D_∞. Suppose that we have a sequence of random paths $(\mathscr{X}_n)_{n\geq 0}^{\infty}$ such that $\mathscr{X}_n : \Omega \to D_\infty$ where $(\Omega, \mathscr{F}, \mathbb{P})$ is our probability space. We say that $\mathscr{X}_n \Rightarrow \mathscr{X}_0$ in the sense of D_∞ if $P_n \Rightarrow P_0$ where $P_n : \mathscr{D}_\infty \to [0,1]$ are defined as $P_n[A] = \mathbb{P}\left[\mathscr{X}_n^{-1}(A)\right]$ for all $n \geq 0$ and $\mathscr{D}_\infty$ are the Borel subsets for the Skorohod topology.

The following lemma shows that the centralized sum of the output data with t changed by t/δ^2 converges to a Brownian motion in the sense of D_∞.

Lemma 1 *x*. Let $x \in \{1\ldots,k\}$, then

$$\mathscr{C}_x(\delta,\cdot) \Rightarrow W_x(\cdot)$$

as $\delta \to 0$ in the sense of $D[0,\infty)$, where W_x is a standard Brownian motion.

Proof. By Theorem 19.1 of **?**,

$$\frac{Y_{n_x(t),x} - \mathrm{floor}\left(\frac{\lambda_x^2}{\lambda_z^2}\left(\cdot\frac{1}{\delta^2}\right)\right)\mu_x}{\frac{\lambda_x^2}{\lambda_z}\sqrt{\frac{1}{\delta^2}}} \Rightarrow W_x(\cdot)$$

in the sense of $D[0,\infty)$.

Fix $w \in \Omega$. Observe that

$$\frac{Y_{\mathrm{floor}\left(\frac{\lambda_x^2}{\lambda_z^2}\left(\cdot\frac{1}{\delta^2}\right)\right),x} - \mathrm{floor}\left(\frac{\lambda_x^2}{\lambda_z^2}\left(\cdot\frac{1}{\delta^2}\right)\right)\mu_x}{\frac{\lambda_x^2}{\lambda_z}\sqrt{\frac{1}{\delta^2}}} - \frac{Y_{\mathrm{ceil}\left(\frac{\lambda_x^2}{\lambda_z^2}\left(\cdot\frac{1}{\delta^2}\right)\right),x} - \mathrm{ceil}\left(\frac{\lambda_x^2}{\lambda_z^2}\left(\cdot\frac{1}{\delta^2}\right)\right)\mu_x}{\frac{\lambda_x^2}{\lambda_z}\sqrt{\frac{1}{\delta^2}}} \to 0$$

uniformly in $[0,s]$ for all $s \geq 0$ and then by Theorem A.2

$$\frac{Y_{\mathrm{ceil}\left(\frac{\lambda_x^2}{\lambda_z^2}\left(\cdot\frac{1}{\delta^2}\right)\right),x} - \mathrm{ceil}\left(\frac{\lambda_x^2}{\lambda_z^2}\left(\cdot\frac{1}{\delta^2}\right)\right)\mu_x}{\frac{\lambda_x^2}{\lambda_z}\sqrt{\frac{1}{\delta^2}}} \Rightarrow W_x(\cdot)$$

in the sense of $D[0,\infty)$.

Since $\frac{\frac{\lambda_x^2}{\lambda_z^2}t\frac{1}{\delta^2} - ceil\left(\frac{\lambda_x^2}{\lambda_z^2}t\frac{1}{\delta^2}\right)}{\frac{\lambda_x^2}{\lambda_z}\sqrt{\frac{1}{\delta^2}}} \to 0$ uniformly on $[0,s]$ for every $s \geq 0$, then by Theorem A.2

$$\frac{Y_{\mathrm{ceil}\left(\frac{\lambda_x^2}{\lambda_z^2}\left(\cdot\frac{1}{\delta^2}\right)\right),x} - \left(\frac{\lambda_x^2}{\lambda_z^2}\left(\cdot\frac{1}{\delta^2}\right)\right)\mu_x}{\frac{\lambda_x^2}{\lambda_z}\sqrt{\frac{1}{\delta^2}}} \Rightarrow W_x(\cdot).$$

Finally, observe that for fixed $\omega \in \Omega$,

$$\begin{aligned}
&\frac{Y_{\mathrm{ceil}\left(\frac{\lambda_x^2}{\lambda_z^2}\left(\cdot\frac{1}{\delta^2}\right)\right),x} - \left(\frac{\lambda_x^2}{\lambda_z^2}\left(\cdot\frac{1}{\delta^2}\right)\right)\mu_x}{\frac{\lambda_x^2}{\lambda_z}\sqrt{\frac{1}{\delta^2}}} - \frac{Y_{\mathrm{ceil}\left(\frac{\lambda_x^2}{\lambda_z^2}\left(\cdot\frac{1}{\delta^2}\right)+n_0\frac{\lambda_x^2}{\lambda_z^2}\right),x} - \left(n_0\frac{\lambda_x^2}{\lambda_z^2} + \frac{\lambda_x^2}{\lambda_z^2}\left(\cdot\frac{1}{\delta^2}\right)\right)\mu_x}{\frac{\lambda_x^2}{\lambda_z}\sqrt{\frac{1}{\delta^2}}} \\
= \quad &\frac{Y_{\mathrm{ceil}\left(\frac{\lambda_x^2}{\lambda_z^2}\left(\cdot\frac{1}{\delta^2}\right)\right),x} - Y_{\mathrm{ceil}\left(\frac{\lambda_x^2}{\lambda_z^2}\left(\cdot\frac{1}{\delta^2}\right)+n_0\frac{\lambda_x^2}{\lambda_z^2}\right),x} + \left(n_0\frac{\lambda_x^2}{\lambda_z^2}\right)\mu_x}{\frac{\lambda_x^2}{\lambda_z}\sqrt{\frac{1}{\delta^2}}} \\
\to \quad & 0
\end{aligned}$$

uniformly in $[0,t]$ for all $t \geq 0$, and so by Theorem A.2 the result follows.

□

Now, we use the product topology in $D^k[0,\infty)$ for $k \in \mathbb{N}$. This topology may be described as the one under which $\left(Z_n^1,\ldots,Z_n^k\right) \to \left(Z_0^1,\ldots,Z_0^k\right)$ if and only if $Z_n^i \to Z_0^i$ for all $i \in \{1,\ldots,k\}$. See the Miscellany of **?**. The following corollary follows from the previous result and independence.

Corollary 1 We have that

$$\mathscr{C}(\delta,\cdot) := \left(\mathscr{C}_x(\delta,\cdot)\right)_{x\in A} \Rightarrow W(\cdot) := \left(W_x(\cdot)\right)_{x\in A}$$

as $\delta \to 0$ in the sense of D_∞^k.

Now that we have obtained this functional central limit theorem for $\mathscr{C}(\delta,\cdot)$, we now continue along the proof outline and define the function $f(\cdot,\delta)$ that was sketched there. This function has three parts: first, computing a "non-centralized" path from an arbitrary input "centralized" path in $D[0,\infty)^k$; second, applying the BIZ selection map h to this non-centralized path; and third, reporting whether selection was correct or not.

To accomplish the first part, for each $F \in D[0,\infty)^k$, we define $q_{tx}^{F,\delta}(A)$ as

$$q_{tx}^{F,\delta}(A) = q'\left(\left(F_x(t)\frac{\lambda_x^2}{\delta\lambda_z^2} + \frac{\lambda_x^2}{\lambda_z^2}\left(n_0 + \frac{t}{\delta^2}\right)\delta a_x : x \in A\right), \delta, A \subset \{1,\ldots,k\}\right).$$

Note that if we replace F by $\mathscr{C}(\delta,t)$, we get $q_{tx}(A)$ in (**??**).

To accomplish the second and third parts, we define $f(F,\delta)$ to be obtained by applying the BIZ selection map h to the process $\left(q_{tx}^{F,\delta}(A) : t \in \mathbb{R}, A \subset \{1,\ldots,k\}, x \in A\right)$, and then reporting whether the selection was correct. More precisely, $f(F,\delta)$ is defined to be

$$f(F,\delta) \quad = \quad \begin{cases} 1 & \text{if } h\left(\left(q_{tx}^{F,\delta}(A) : t \in \mathbb{R}, A \subset \{1,\ldots,k\}, x \in A\right)\right) = k, \\ 0 & \text{otherwise.} \end{cases}$$

We now construct a function $g(\cdot)$ that, when applied to the path of a k-dimensional standard Brownian motion, will be equal in distribution to the indicator of the correct selection event from the continuous-time BIZ procedure from (**?**) to a transformed problem that does not depend on δ.

We construct g analogously to $f(\cdot,\delta)$, but we replace the path $q_{tx}^{F,\delta}$ used in the construction of $f(\cdot,\delta)$ by a new path q_{tx}^F that doesn't depend on δ, and is obtained by taking the limit as $\delta \to 0$. This path is

$$q_{tx}^F(A) := \exp\left(\frac{F_x(t)}{\lambda_z} + \frac{1}{\lambda_z^2}ta_x\right) \Big/ \sum_{x' \in A}\exp\left(\frac{F_{x'}(t)}{\lambda_z} + \frac{1}{\lambda_z^2}ta_{x'}\right).$$

Then, g is defined to be

$$g(F) \quad = \quad \begin{cases} 1 & \text{if } h\left(\left(q_{tx}^F(A) : t \in \mathbb{R}, A \subset \{1,\ldots,k\}, x \in A\right)\right) = k, \\ 0 & \text{otherwise.} \end{cases}$$

In the proof of the main theorem, we will show that

$$f(\mathscr{C}(\delta,\cdot),\delta) \Rightarrow g(W)$$

as $\delta \to 0$ in distribution. We will use the following lemma, which shows a continuity property. A proof of Lemma **??** may be found in a full version of this paper (**?**), which will be submitted soon to arXiv.

Lemma 2 Let $\{\delta_n\} \subset (0,\infty)$ such that $\delta_n \to 0$. If $D_s \equiv \{Z \in D[0,\infty)^k :$ if $\{Z_n\} \subset D[0,\infty)^k$ and $\lim_n d_\infty (Z_n, Z) = 0$, then the sequence $\{f(Z_n,\delta_n)\}$ converges to $\{g(Z)\}\}$, then $\mathbb{P}(W \in D_s) = 1$.

The following lemma shows that the difference in the correct selection events obtained from applying the BIZ selection map h to the discrete-time and continuous-time versions of $q_{tx}(A)$ vanish as δ goes to 0. A proof of Lemma **??** may be found in a full version of this paper (**?**).

Lemma 3 $\lim_{\delta\to 0}\mathbb{P}\left(h\left(\left(q'_{tx}(A) : t \in \mathbb{N}, A \subset \{1,\ldots,k\}, x \in A\right)\right) = k\right) = \lim_{\delta\to 0}\mathbb{P}\left(f\left(\mathscr{C}(\delta,t),\delta\right) = 1\right)$.

4.3 The Main Result

Theorem 1 If samples from system $x \in \{1\ldots,k\}$ are identically distributed and independent, over time and across alternatives, then $\lim_{\delta\to 0}\text{PCS}(\delta) \geq P*$ provided $\mu_k = a_k\delta, \mu_{k-1} = a_{k-1}\delta, \ldots, \mu_1 = a_1\delta$, $a_k > a_{k-1} \geq \cdots \geq a_1$, $a_k - a_{k-1} \geq 1$, and the variances are finite and do not depend on δ.

Furthermore,

$$\inf_{a\in PZ(1)} \lim_{\delta\to 0}\text{PCS}(\delta) = P^*$$

where $PZ(1) = \left\{a \in \mathbb{R}^k : a_k - a_{k-1} > 1, a_k > a_{k-1} \geq \cdots \geq a_1\right\}$.

Proof. Using the definitions given at the beginning of this section, the selection decision of the discrete-time BIZ procedure for a particular $\delta > 0$ when $\mu_k = a_k\delta, \mu_{k-1} = a_{k-1}\delta, \ldots, \mu_1 = a_1\delta$ is given by

$$h\left(\left(q'_{tx}(A) : t \in \mathbb{N}, A \subset \{1,\ldots,k\}, x \in A\right)\right)$$

and the probability of correct selection $\text{PCS}(\delta)$ is

$$\text{PCS}(\delta) = \mathbb{P}\left(h\left(\left(q'_{tx}(A) : t \in \mathbb{N}, A \subset \{1,\ldots,k\}, x \in A\right)\right) = k\right).$$

By Lemma **??**, we have that

$$\lim_{\delta\to 0}\text{PCS}(\delta) = \lim_{\delta\to 0}\mathbb{P}\left(f\left(\mathscr{C}(\delta,t),\delta\right) = 1\right). \tag{3}$$

We also have, by Lemma **??** and an extension of the continuous mapping theorem (Theorem 5.5 of **?**),

$$f\left(\mathscr{C}(\delta,t),\delta\right) \Rightarrow g\left(W(t)\right)$$

in distribution as $\delta \to 0$. This implies that

$$\lim_{\delta\to 0}\mathbb{P}\left(f\left(\mathscr{C}(\delta,t),\delta\right) = 1\right) = \mathbb{P}\left(g(W) = 1\right). \tag{4}$$

The random variable $g(W)$ is equal in distribution to the indicator of the event of correct selection that results from applying the continuous-time BIZ procedure from (**?**) in a problem with indifference-zone parameter equal to 1, where each alternative's observation process has volatility λ_z and drift a_x. This can be seen by noting that the path $(q^W_{tx}(A) : t \geq 0)$ defined above is equal in distribution to the path $(q_{tx}(A) : t \geq 0)$ defined in equation (2) of **?**, and that the selection decision of the continuous-time algorithm in **?** is obtained by applying h to this path.

Theorem 1 in **?** states that

$$\mathbb{P}\left(g(W) = 1\right) \geq P^*. \tag{5}$$

Combining (**??**), (**??**), and (**??**), we have

$$\lim_{\delta \to 0} \mathrm{PCS}(\delta) \geq P^*.$$

Furthermore, Theorem 1 in **?** shows that

$$\inf_{a \in PZ(1)} \mathbb{P}(g(W) = 1) = P^* \tag{6}$$

where $PZ(1) = \left\{a \in \mathbb{R}^k : a_k - a_{k-1} \geq 1\right\}$.

Combining (**??**), (**??**), and (**??**), shows

$$\inf_{a \in PZ(1)} \lim_{\delta \to 0} \mathrm{PCS}(\delta) = P^*.$$

□

5 NUMERICAL EXPERIMENTS

We now use simulation experiments to illustrate and further investigate the phenomenon characterized by Theorem **??**. Using the version of BIZ described in Section **??** with maximum elimination ($c = 1 - (P^*)^{\frac{1}{k-1}}$), we estimate and then plot the PCS as a function of δ using 10,000 independent replications. We consider three different examples, setting $P^* = 0.9$ in all three.

Our first example, illustrated in Figure **??**, is a known variance slippage configuration. Specifically, we consider 100 systems with independent normally distributed samples, where $\mu_k = \delta, \mu_{k-1} = 0, \ldots, \mu_1 = 0$, δ is within the interval $[0.1, 10]$, and $\lambda_{100} = 1, \lambda_{99} = 1 + \frac{(0.5)(98)}{99}, \cdots, \lambda_1 = 0.5$. Here, $n_0 = 0$. Figure **??** shows that in this example the IZ guarantee is always satisfied, and the PCS approaches P^* as δ goes to zero. When δ is big enough, the PCS is almost one because the difference between the best system and the others is large enough to be easily identifiable by the BIZ procedure. Here, the points plotted have central confidence intervals of length at most 0.014.

Our second example, illustrated in Figure **??**, is an unknown variance slippage configuration. Although Theorem **??** applies only to the known-variance version of BIZ, we conjecture that the unknown-variance version of BIZ should exhibit similar behavior. In this example, we consider 100 systems with independent normally distributed samples, where $\mu_{100} = \delta, \mu_{99} = 0, \ldots, \mu_1 = 0$, δ is within the interval $[0.1, 10]$, and $\lambda_{100} = 10, \lambda_{99} = \cdots = \lambda_1 = 1$. We set $n_0 = 15$. In this example, we have intentionally chosen n_0 to be small, and have chosen a large variance for the best system, to cause BIZ to fail to meet the IZ guarantee for $\delta > 0$.

We should note that this is a difficult example, and still we have a good performance in most of the cases. However, if we want to always satisfy the IZ guarantee, we can just increment the parameter n_0. Here, the points plotted have central confidence intervals of length at most 0.014.

Now, we consider a pathological case when the variances are known. Specifically, we consider a slippage configuration with 100 systems normally distributed, where $\mu_{100} = \delta, \mu_{99} = 0, \ldots, \mu_1 = 0$, δ is within the interval $[0.1, 10]$, and $\lambda_{100} = 10, \lambda_{99} = \cdots = \lambda_1 = 1$. Here, $P^* = 0.9$ and $n_0 = 0$. This configurations was specially chosen to illustrate our theorem, and so it is harder than typical configurations. In fact, in most of the configurations the algorithm always satisfies the IZ guarantee. Furthermore, in practice, we would have run the algorithm with $n_0 > 0$, and the BIZ algorithm would have worked well even in this pathological example. Figure **??** shows that PCS converges to 0.9 as δ goes to zero. Specifically, PCS is equal to 0.904 when $\delta = 0.1$. This figure also shows that the bound is tight on probability of correct selection. We should also note that the number of samples required increases very fast. Here, the points plotted have central confidence intervals of length at most 0.014.

6 CONCLUSION

We have proved the asymptotic validity of the Bayes-inspired Zone procedure (**?**) when the variances are known, which is a new sequential elimination procedure. This algorithm is relevant because it takes fewer samples than other IZ procedures, especially for problems with large numbers of alternatives. Even though this proof does not guarantee that the algorithm will work for any sample, we know that it will work if the alternatives are not very different, which are the most difficult cases.

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REFERENCES

Bechhofer, R. E. 1954. "A Single-Sample multiple-decision procedure for selecting the multinomial event which has the highest probability". *The Annals of Mathematical Statistics* 25 (1): 16–39.

Bechhofer, R. E., J. Kiefer, and M. Sobel. 1968. *Sequential Identification and Ranking Procedures*. Chicago: University of Chicago Press.

Billingsley, P. 1968. *Convergence of Probability Measures*. New York: John Wiley and Sons.

Billingsley, P. 1999. *Convergence of Probability Measures*. 2nd ed. New York: John Wiley and Sons.

Chen, C.-H., and L. H. Lee. 2011. *Stohcastic Simulation Optimization: An Optimal Computing Budget Allocation*, Volume 1. Singapore: World Scientific.

Dieker, A., and S.-H. Kim. 2012. "Selecting the best by comparing simulated systems in a group of three when variances are known and unequal". In *Proceedings of the 2012 Winter Simulation Conference.*, edited by IEEE.

Fabian, V. 1974. "Note on Anderson's sequential procedures with triangular boundary.". *The Annals of Mathematical Statistics* 2 (1): 170–176.

Frazier, P. 2012. "TUTORIAL: OPTIMIZATION VIA SIMULATION WITH BAYESIAN STATISTICS AND DYNAMIC PROGRAMMING". *Proceedings of the 2012 Winter Simulation Conference.*.

Frazier, P. I. 2014. "A Fully Sequential Elimination Procedure for Indifference-Zone Ranking and Selection with Tight Bounds on Probability of Correct Selection". *Operations Research*.

Glynn, P., and S. Juneja. 2004. "A large deviations perspective on ordinal optimization". In *Simulation Conference, 2004. Proceedings of the 2004 Winter*, Volume 1. IEEE.

Goldsman, D., S. Kim, W. Marshall, and B. Nelson. 2002. "Ranking and selection for steady-state simulation: Procedures and perspectives.". *INFORMS Journal on Computing* 14 (1): 2–19.

Hartmann, M. 1988. "An improvement on Paulson's sequential ranking procedure". *Sequential Analysis* 7 (4): 363–372.

Hartmann, M. 1991. "An improvement on Paulson's procedure for selecting the population with the largest mean from k normal populations with a common unknown variance". *Sequential Analysis* 10:1–16.

Hong, J. 2006. "Fully sequential indi?erence-zone selection procedures with variance-dependent sampling". *Naval Research Logistics* 53 (5): 464–476.

Kim, S., and B. Nelson. 2006a. "Selecting the best system". In *Handbook in Operations Research and Management Science: Simulation*, edited by e. S.G. Henderson, B.L. Nelson, 501–534. Elsevier, Amsterdam.

Kim, S., and B. Nelson. 2007. "Recent advances in ranking and selection". In *Proceedings of the 39th conference on Winter simulation: 40 years! The best is yet to come.*, edited by I. Press, 162–172. Piscataway, New Jersey.

Kim, S.-H., and A. Dieker. 2011. "Selecting the best by comparing simulated systems in a group of three.". *Proceedings of the 2011 Simulation Conference.*:3987–3997.

Kim, S.-H., and B. L. Nelson. 2001. "A fully sequential procedure for indiference-zone selection in simulation". *ACM Trans. Model. Comput. Simul.* 11 (3): 251–273.

Kim, S.-H., and B. L. Nelson. 2006b. "On the Asymptotic Validity of Fully Sequential Selection Procedures for Steady-State Simulation". *Operations Research* 54 (3): 475–488.

Paulson, E. 1964. "A sequential procedure for selecting the population with the largest mean from k normal populations". *The Annals of Mathematical Statistics* 35 (1): 174–180.

Paulson, E. 1994. "Sequential procedures for selecting the best one of k Koopman-Darmois populations". *Sequential Analysis* 13 (3).

R. E. Bechhofer, T.J. Santner, D. G. 1995. *Design and Analysis of Experiments for Statistical Selection, Screening and Multiple Comparisons.* New York: J.Wiley and Sons.

Rinott, Y. 1978. "On Two-Stage Selection Procedures and Related Probability-Inequalities". *Communications in Statistics-Theory and Methods* 7 (8): 799–811.

Swisher, J., S. Jacobson, and E. Yücesan. 2003. "Discrete-event simulation optimization using ranking, selection, and multiple comparison procedures: A survey.". *ACM Transactions on Modeling and Computer Simulation* 13 (2): 134–154.

Toscano-Palmerin, S., and P. I. Frazier. 2015. "Asymptotic Validity of the Bayes-Inspired Indifference Zone Procedure: The General Distribution Known Variance Case". Technical report. in preparation.

Wang, H., and S.-H. Kim. 2013. "Reducing the conservativeness of fully sequential indifference-zone procedures". *IEEE Transactions on automatic control* 58 (6): 1613–1619.

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(a)

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(b)

plot3.pdf

(c)

Figure 1: (a) Known variances and $P^* = 0.9$. In this example, our theorem is true: the IZ guarantee is always satisfied and the inequality is tight as δ goes to zero. (b) Unknown variances and $P^* = 0.9$. In this example, our theorem is true: the IZ guarantee is satisfied and the inequality is tight as delta goes to zero. If we want that the IZ guarantee be satisfied in all the cases, we can just increment n_0. (c) Known variances, $P^* = 0.9$ This was a hard example to find and it was specially chosen to illustrate our theorem. In practice, we would choose $n_0 > 0$ and the IZ guarantee will be satisfied.