Asymptotic Validity of a Fully Sequential Elimination Procedure for Indifference-Zone Ranking and Selection with Tight Bounds on Probability of Correct Selection

Saul Toscano

School of Operations Research and Information Engineering, Cornell University

Peter I. Frazier

School of Operations Research and Information Engineering, Cornell University

We consider the indifference-zone (IZ) formulation of the ranking and selection problem. Conservatism leads classical IZ procedures to take too many samples in problems with many alternatives. The Bayes-inspired Indifference Zone (BIZ) procedure, proposed in Frazier (2014), is less conservative than previous procedures, but its proof of validity requires strong assumptions. In this paper, we present a new proof of asymptotic validity that relaxes these assumptions.

1 Introduction

One common problem in simulation is that of choosing the best among several simulated systems. The problem of deciding how many samples to use from each alternative to support our selection as the best is the ranking and selection problem. An efficient solution to this problem has to balance between the time spent simulating and the quality of the selection.

This paper will consider the indifference-zone (IZ) formulation of the ranking and selection problem, in which we choose the best system with probability larger than a pre-specified threshold, whenever the distance between the best system and the others is sufficiently large. We say that a sampling procedure having this property satisfies the IZ guarantee and the set of system configurations under which the best alternative is better than the second best by at least some given $\delta > 0$ is called the preference zone (PZ). The seminal work dates back to Bechhofer (1954), with early work compiled in the monograph Bechhofer et al. (1968). The progress in the area has been summarized in Bechhofer et al. (1995), Swisher et al. (2003), Kim and Nelson (2006, 2007) and Frazier (2014).

The goal of an IZ algorithm is to take as few samples as possible while the IZ guarantee is satisfied. The first IZ procedures presented in Bechhofer (1954), Paulson (1964), Fabian (1974), Rinott (1978), Hartmann (1988, 1991), Paulson (1994) satisfy the IZ guarantee, but they usually take too many samples when there are many alternatives, in part because their probability of correct selection (PCS) is much larger than the probability specified by the user. A reason of this is the use of the Bonferroni's inequality which leads to sample more than necessary. More recent algorithms in Kim and Nelson (2001), Goldsman et al. (2002), Hong (2006) improve the performance but they still use the Bonferroni's inequality, and so the methods are inefficient when there are several systems. Procedures in Kim and Dieker (2011), Dieker and Kim (2012) do not use the Bonferroni's inequality only when compare groups of three alternatives.

Since the classic IZ procedures take too many samples with many alternatives, these methods are unpopular when there are more than a few hundred alternatives. However, Frazier (2014) presented a new sequential elimination IZ procedure, called BIZ (Bayes-inspired Indifference Zone), whose lower bound on worst-case probability of correct selection in the preference zone is tight in continuous time, and almost tight in the discrete time. In numerical experiments, the number of samples required by BIZ is significantly smaller than that of the most popular IZ procedures, especially on problems with many alternatives. Unfortunately, the proof that the BIZ procedure satisfies the IZ guarantee for the discrete-time case assumes that variances are known and have an integer multiple structure which is not very realistic. In practice, variances are unknown. However, asymptotically we can use a central limit theorem that allows us to prove the asymptotic validity of the BIZ procedure for the discrete-time case. Moreover, we only need to assume that the systems are independent, identically distributed and have finite variance.

Kim et al. (2006) also proves the asymptotical validity of a IZ procedure. Our proof is larger because the BIZ procedure is more sophisticated and the bound for the PCS is tighter. To prove the case when the variances are known, we use a theorem for Ergodic processes that shows how to standardize the output data to make them behave like Brownian motion processes in the limit. We also use an extension of the Continuous Mapping Theorem (Theorem 5.5 of Billingsley 1968) to see that the algorithm behaves like a sequential elimination IZ procedure with a Brownian motion process instead of the standardized sum of the output data in the limit, and then we use the results of the paper of Frazier [3] to prove the validity of this algorithm in the limit. Finally, we use a random change of time argument to prove the case when the variances are unknown.

This paper is organized as follows: In §2, we recall the indifference-zone ranking and selection problem. In §3, we recall the Bayes-inspired IZ (BIZ) procedure. In §4, we present the proof of the validity of the algorithm when the variances are known. In §5, we prove the case when the variances are unknown. In §6, we present some simulations showing the effectiveness of the algorithm. In §7, we conclude.

2 Indifference-Zone Ranking and Selection

Ranking and Selection is a procedure for selecting the best system among a finite set of alternatives, i.e. the system with the largest mean. The method selects a system as the best based on the samples that are observed sequentially over the time. If the best system is selected, we say that the procedure has made the *correct selection* (CS). We define the *probability of correct selection* as

$$\mathrm{PCS}\left(\mu,\lambda\right)=\mathbb{P}_{\mu,\lambda}\left(\hat{x}\in\arg\max_x\mu_x\right)$$

where $\hat{x}$ is the alternative chosen by the procedure and $\mathbb{P}_{\mu,\lambda}$ is the probability measure under which samples from system x have mean μ_x and finite variance λ_x^2.

In the Indifference-Zone Ranking and Selection, the procedure is indifferent in the selection of a system whenever the means of the populations are nearly the same. Formally, let $\mu=[\mu_k,\ldots,\mu_1]$ be the vector of the true means, the *indifference zone* is defined as the set $\left\{\mu\in\mathbb{R}^k:\mu_{[k]}-\mu_{[k-1]}<\delta\right\}$. The complement of the indifference zone is called the *preference zone* (PZ) and $\delta>0$ is called the indifference zone parameter. We say that a procedure meets the *indifference-zone (IZ) guarantee* at $P^*\in(1/k,1)$ and $\delta>0$ if

$$\mathrm{PCS}\left(\mu,\lambda\right)\geq P^*\text{ for all }\mu\in\mathrm{PZ}\left(\delta\right).$$

We assume $P^*>1/k$ because IZ guarantees can be meet by choosing $\hat{x}$ uniformly at random from among

$\{1, \ldots, k\}$.

3 The Bayes-inspired IZ (BIZ) Procedure

BIZ is an elimination procedure. This procedure maintains a set of alternatives that are in contention, and it takes samples from each alternative in this set at each point in time. At beginning, all alternatives are in contention, and over the time alternatives are eliminated. The procedure ends when there is only one alternative in the contention set and this remain alternative is chosen as the best.

Frazier (2014) showed that the BIZ procedure with known common variance satisfies the IZ guarantee when the systems follow the normal distribution , with tight bounds on worst-case preference-zone in continuous time. He also proved that this procedure retains the IZ guarantee when the systems follow the normal distribution, and the variances are known and are integer multiples of a common value. The continuous time version of this procedure also satisfies the IZ guarantee, with a tight worst-case preference-zone PCS bound.

The discrete-time BIZ procedure for unknown and/or heterogeneous sampling variances is given in Alg. 2. It takes a variable number of samples from each alternative, and n_{tx} is this number. This algorithm depends on a collection of integers $B_1, \ldots, B_k$, P^*, c, δ and n_0. n_0 is the number of samples to use in the first stage of samples, and 100 is the recommended value for n_0. B_x controls the number of samples taken from system x in each stage.

For each t, $x \in \{1, \ldots, k\}$, and subset $A \subset \{1, \ldots, k\}$, we define a function

$$q_{tx}(A) = \exp\left(\delta\beta_t \frac{Z_{tx}}{n_{tx}}\right) \Bigg/ \sum_{x' \in A} \exp\left(\delta\beta_t \frac{Z_{tx'}}{n_{tx'}}\right), \quad \beta_t = \frac{\sum_{x' \in A} n_{tx'}}{\sum_{x' \in A} \hat{\lambda}^2_{tx'}}$$

where $\hat{\lambda}^2_{tx'}$ is the sample variance of all samples from alternative x thus far and $Z_{tx} = Y_{n_{tx},x}$.

Algorithm 2: Discrete-time implementation of BIZ, for unknown and/or heterogeneous variances.

Require: $c \in [0, 1 - (P^*)^{\frac{1}{k-1}}]$, $\delta > 0$, $P^* \in (1/k, 1)$, $n_0 \geq 0$ an integer, $B_1, \ldots, B_k$ strictly positive integers. Recommended choices are $c = 1 - (P^*)^{\frac{1}{k-1}}$, $B_1 = \cdots = B_k = 1$ and n_0 between 10 and 30. If the sampling variances λ^2_x are known, replace the estimators $\widehat{\lambda}^2_{tx}$ with the true values λ^2_x, and set $n_0 = 0$.

For each x, sample alternative x n_0 times and set $n_{0x} \leftarrow n_0$. Let W_{0x} and $\widehat{\lambda}^2_{0x}$ be the sample mean and sample variance respectively of these samples. Let $t \leftarrow 0$.

Let $A \leftarrow \{1, \ldots, k\}$, $P \leftarrow P^*$.

while $x \in \max_{x \in A} q_{tx}(A) < P$ **do**

  while $\min_{x \in A} q_{tx}(A) \leq c$ **do**

    Let $x \in \arg\min_{x \in A} q_{tx}(A)$.

    Let $P \leftarrow P/(1 - q_{tx}(A))$.

    Remove x from A.

  end while

  Let $z \in \arg\min_{x \in A} n_{tx}/\widehat{\lambda}^2_{tx}$.

  For each $x \in A$, let $n_{t+1,x} = \text{ceil}\left(\widehat{\lambda}^2_{tx}(n_{tz} + B_z)/\widehat{\lambda}^2_{tz}\right)$.

  For each $x \in A$, if $n_{t+1,x} > n_{tx}$, take $n_{t+1,x} - n_{tx}$ additional samples from alternative x. Let $W_{t+1,x}$ and $\widehat{\lambda}^2_{t+1,x}$ be the sample mean and sample variance respectively of all samples from alternative x thus far.

  Increment t.

end while

Select $\hat{x} \in \arg\max_{x \in A} Z_{tx}/n_{tx}$ as our estimate of the best.

This algorithm generalizes the BIZ procedure with known common variance. In that case, we have that$B_1 = \cdots = B_k = 1$ and $n_{tx} = t$. The algorithm 2 can be generalized to the continuous case (See appendix B and Frazier (2014)).

4 Asymptotic Validity when the Variances are Known

In this section we prove that the BIZ procedure satisfies the IZ guarantee when the variances are known and $\delta \to 0$, where the vector of the true means $\mu = [\mu_k, \ldots, \mu_1]$ is equal to δa for some $a \in \mathbb{R}^k$ and $\delta > 0$. Without loss of generality suppose that the true means satisfy that $\mu_k > \mu_{k-1} \geq \cdots \geq \mu_1$. We suppose that samples from system $x \in \{1 \ldots, k\}$ are identically distributed and independent, over time and across alternatives. We also define $\lambda_z^2 := \max_{i \in \{1 \ldots, k\}} \lambda_i^2$ and we suppose that $\min_{i \in \{1 \ldots, k\}} \lambda_i^2 > 0$ and $\max_{i \in \{1 \ldots, k\}} \lambda_i^2 < \infty$.

The original BIZ procedure maps $\left(Y_{n_x(t),x} : t \in \mathbb{N}, x \in \{1, \ldots, k\}\right)$ to a selection decision, where $n_x(t) = \text{ceil}\left(\frac{\lambda_x^2}{\lambda_z^2}(n_0 + t)\right)$ and $Y_{n_x(t),t}$ is the sum of samples observed by time t. In more detail, it works by transforming $\left(Y_{n_x(t),x} : A \subset \{1, \ldots, k\}, x \in A\right)$ to $(q'_{tx}(A) : t \in \mathbb{N}, A \subset \{1, \ldots, k\}, x \in A)$. This q' process is given by time process with jumps at $t \in \mathbb{N}$,

$$q'_{tx}(A) = q'\left((Z_{tx}, t : x \in A), \delta, A \subset \{1, \ldots, k\}\right) = \exp\left(\delta \beta_t \frac{Z_{tx}}{n_{tx}}\right) \Bigg/ \sum_{x' \in A} \exp\left(\delta \beta_t \frac{Z_{tx}}{n_{tx}}\right)$$

where the variances are fixed, and $\beta_t = 1/\lambda_z^2$, $n_{tx} = \text{ceil}\left(\lambda_x^2 (n_0 + t)/\lambda_z^2\right)$. The same distribtuion over selection decision will result if we apply the same same selection process to the following time process with jumps at $t \in \left\{0, \delta^2, 2\delta^2, \ldots\right\} = \delta^2 \mathbb{N}$,

$$q'_{tx}(A) = q'\left(\left(Z_{\frac{t}{\delta^2} x}, \frac{t}{\delta^2} : x \in A\right), \delta, A \subset \{1, \ldots, k\}\right).$$

Later, we are going to see that we can approximate this by the continuos time process

$$q_{tx}(A) = q'\left(\left(Z_{\frac{t}{\delta^2} x}, \frac{t}{\delta^2} : x \in A\right), \delta, A \subset \{1, \ldots, k\}\right) \tag{1}$$

where $t \geq 0$, and we will show the difference in selection decisions vanishes as $\delta \to 0$.

After that, we would like to use a kind of central limit theorem for $Z_{\frac{t}{\delta^2} x} = Y_{n_x(t/\delta^2),x}$ when δ goes to zero to work with a Brownian motion instead. Specifically, we are going to see that the the standardized sum of the output data converges to a Brownian motion in the sense of $D_\infty := D[0, \infty)$, which is the set of functions from $[0, \infty)$ to $\mathbb{R}$ that are right-continuous and have left-hand limits, with the Skorohod topology. The definition and the properties of this topology may be found in Chapter 3 of Billingsley 1999 and the appendix. We will first see this result because it will give us a clue for proving the asymptotic validity of the Algorithm 2.

We briefly recall the definition of convergence of random paths in the sense of D_∞. Suppose that we have a sequence of random paths $(\mathcal{X}_n)_{n \geq 0}^\infty$ such that $\mathcal{X}_n : \Omega \to D_\infty$ where $(\Omega, \mathcal{F}, \mathbb{P})$ is our probability space. We say that $\mathcal{X}_n \Rightarrow \mathcal{X}_0$ in the sense of D_∞ if $P_n \Rightarrow P_0$ where $P_n : \mathcal{D}_\infty \to [0, 1]$ are defined as $P_n[A] = \mathbb{P}\left[\mathcal{X}_n^{-1}(A)\right]$ for all $n \geq 0$ and $\mathcal{D}_\infty$ are the Borel subsets for the Skorohod topology.

The following lemma shows that the standardized sum of the output data with t changed by t/δ^2 converges to a Brownian motion in the sense of D_∞.

Lemma 1. For each $x \in \{1\ldots,k\}$, we define

$$\mathcal{C}_x(\delta,t) = \frac{Y_{\text{ceil}\left(\frac{\lambda_x^2}{\lambda_z^2}\left(n_0+t\frac{1}{\delta^2}\right)\right),x} - \frac{\lambda_x^2}{\lambda_z^2}\left(n_0+t\frac{1}{\delta^2}\right)\mu_x}{\frac{\lambda_x^2}{\lambda_z\delta}}$$

where $Y_{n,x}$ is the sum of the first n samples from alternative x, and $\lambda_z^2 := \max_{i\in\{1\ldots,k\}}\lambda_i^2$. Let $x \in \{1\ldots,k\}$, then

$$\mathcal{C}_x(\delta,\cdot) \Rightarrow W_x(\cdot)$$

as $\delta \to 0$ in the sense of $D[0,\infty)$, where W_x is a standard Brownian motion.

Proof. By Theorem 19.1 of Billingsley 1999,

$$\frac{Y_{\text{floor}\left(\frac{\lambda_x^2}{\lambda_z^2}\left(\cdot\frac{1}{\delta^2}\right)\right),x} - \text{floor}\left(\frac{\lambda_x^2}{\lambda_z^2}\left(\cdot\frac{1}{\delta^2}\right)\right)\mu_x}{\frac{\lambda_x^2}{\lambda_z}\sqrt{\frac{1}{\delta^2}}} \Rightarrow W_x(\cdot)$$

in the sense of $D[0,\infty)$.

Fix $w \in \Omega$. Observe that

$$\frac{Y_{\text{floor}\left(\frac{\lambda_x^2}{\lambda_z^2}\left(\cdot\frac{1}{\delta^2}\right)\right),x} - \text{floor}\left(\frac{\lambda_x^2}{\lambda_z^2}\left(\cdot\frac{1}{\delta^2}\right)\right)\mu_x}{\frac{\lambda_x^2}{\lambda_z}\sqrt{\frac{1}{\delta^2}}} - \frac{Y_{\text{ceil}\left(\frac{\lambda_x^2}{\lambda_z^2}\left(\cdot\frac{1}{\delta^2}\right)\right),x} - \text{ceil}\left(\frac{\lambda_x^2}{\lambda_z^2}\left(\cdot\frac{1}{\delta^2}\right)\right)\mu_x}{\frac{\lambda_x^2}{\lambda_z}\sqrt{\frac{1}{\delta^2}}} \to 0$$

uniformly in $[0,s]$ for all $s \geq 0$ and then by Theorem A.2

$$\frac{Y_{\text{ceil}\left(\frac{\lambda_x^2}{\lambda_z^2}\left(\cdot\frac{1}{\delta^2}\right)\right),x} - \text{ceil}\left(\frac{\lambda_x^2}{\lambda_z^2}\left(\cdot\frac{1}{\delta^2}\right)\right)\mu_x}{\frac{\lambda_x^2}{\lambda_z}\sqrt{\frac{1}{\delta^2}}} \Rightarrow W_x(\cdot)$$

in the sense of $D[0,\infty)$.

Since $\frac{\frac{\lambda_x^2}{\lambda_z^2}t\frac{1}{\delta^2} - ceil\left(\frac{\lambda_x^2}{\lambda_z^2}t\frac{1}{\delta^2}\right)}{\frac{\lambda_x^2}{\lambda_z}\sqrt{\frac{1}{\delta^2}}} \to 0$ uniformly on $[0,s]$ for every $s \geq 0$, then by Theorem A.2

$$\frac{Y_{\text{ceil}\left(\frac{\lambda_x^2}{\lambda_z^2}\left(\cdot\frac{1}{\delta^2}\right)\right),x} - \left(\frac{\lambda_x^2}{\lambda_z^2}\left(\cdot\frac{1}{\delta^2}\right)\right)\mu_x}{\frac{\lambda_x^2}{\lambda_z}\sqrt{\frac{1}{\delta^2}}} \Rightarrow W_x(\cdot).$$

Finally, observe that for fixed $\omega \in \Omega$,

$$
\begin{aligned}
& \frac{Y_{\text{ceil}\left(\frac{\lambda_x^2}{\lambda_z^2}\left(\cdot\frac{1}{\delta^2}\right)\right),x} - \left(\frac{\lambda_x^2}{\lambda_z^2}\left(\cdot\frac{1}{\delta^2}\right)\right)\mu_x}{\frac{\lambda_x^2}{\lambda_z}\sqrt{\frac{1}{\delta^2}}} - \frac{Y_{\text{ceil}\left(\frac{\lambda_x^2}{\lambda_z^2}\left(\cdot\frac{1}{\delta^2}\right)+n_0\frac{\lambda_x^2}{\lambda_z^2}\right),x} - \left(n_0\frac{\lambda_x^2}{\lambda_z^2} + \frac{\lambda_x^2}{\lambda_z^2}\left(\cdot\frac{1}{\delta^2}\right)\right)\mu_x}{\frac{\lambda_x^2}{\lambda_z}\sqrt{\frac{1}{\delta^2}}} \\
= \; & \frac{Y_{\text{ceil}\left(\frac{\lambda_x^2}{\lambda_z^2}\left(\cdot\frac{1}{\delta^2}\right)\right),x} - Y_{\text{ceil}\left(\frac{\lambda_x^2}{\lambda_z^2}\left(\cdot\frac{1}{\delta^2}\right)+n_0\frac{\lambda_x^2}{\lambda_z^2}\right),x} + \left(n_0\frac{\lambda_x^2}{\lambda_z^2}\right)\mu_x}{\frac{\lambda_x^2}{\lambda_z}\sqrt{\frac{1}{\delta^2}}} \\
\to \; & 0
\end{aligned}
$$

uniformly in $[0,t]$ for all $t \geq 0$, and so by Theorem A.2 the result follows.

■

Now, we use the product topology in $D^k[0,\infty)$ for $k \in \mathbb{N}$. This topology may be described as the one under which $\left(Z_n^1,\ldots,Z_n^k\right) \to \left(Z_0^1,\ldots,Z_0^k\right)$ if and only if $Z_n^i \to Z_0^i$ for all $i \in \{1,\ldots,k\}$. See the Miscellany of Billingsley 1968. The following corollary follows from the previous result and independence.

Corollary 1. We have that

$$
\mathcal{C}(\delta,\cdot) := \left(\mathcal{C}_x(\delta,\cdot)\right)_{x\in A} \Rightarrow W(\cdot) := \left(W_x(\cdot)\right)_{x\in A}
$$

as $\delta \to 0$ in the sense of D_∞^k.

We now gain the insight that we were looking for. We will use $\mathcal{C}_x(\delta,\cdot)$ instead of $Z_{\frac{t}{\delta^2}x}$ in (1), which is the definition of $q_{tx}(A)$,

$$
q_{tx}(A) = q'\left(\left(C_x(\delta,t)\frac{\lambda_x^2}{\delta\lambda_z^2} + \frac{\lambda_x^2}{\lambda_z^2}\left(n_0+\frac{t}{\delta^2}\right)\delta a_x, \frac{t}{\delta^2} : x\in A\right), \delta, A\subset\{1,\ldots,k\}\right),
$$

the previous equation is true beucase

$$
Z_{\frac{t}{\delta^2}x} = C_x(\delta,t)\frac{\lambda_x^2}{\delta\lambda_z^2} + \frac{\lambda_x^2}{\lambda_z^2}\left(n_0+\frac{t}{\delta^2}\right).
$$

Observe that $\mathcal{C}(\delta,\cdot) \in D[0,\infty)^k$ and the convergence of Lemma 1 is in $D[0,\infty)^k$, consequently the following natural step is to generalize the continuous-time procedure for the heterogeneous variance setting, see Appendix B, to the space $D[0,\infty)^k$. Specifically, for each $F \in D[0,\infty)^k$, we define $q_{tx}^{F,\delta}(A)$ as

$$
q_{tx}^{F,\delta}(A) = q'\left(\left(F_x(t)\frac{\lambda_x^2}{\delta\lambda_z^2} + \frac{\lambda_x^2}{\lambda_z^2}\left(n_0+\frac{t}{\delta^2}\right)\delta a_x, \frac{t}{\delta^2} : x\in A\right), \delta, A\subset\{1,\ldots,k\}\right).
$$

Note that if we replace F by $\mathcal{C}(\delta,t)$, we get $q_{tx}(A)$ in the previous equation. Using the continuous-time procedure for the heterogeneous variance setting with the paths $\left(q_{tx}^{F,\delta}(A) : t \geq 0\right)$, we see that there exists a selection decision for each $F \in D[0,\infty)^k$. Since we are interested in the right selection, (i.e. in choosing k) for each $F \in D\left[0,\infty\right)^k$, we define the following function

$$f\left(F,\delta\right) \quad = \quad \begin{cases} 1 & \text{if } k \text{ is chosen} \\ 0 & \text{otherwise} \end{cases}$$

Observe that f is a function that depends in the path $\left\{q_{tx}^{F,\delta}\left(A\right) : t \geq 0\right\}$. We are doing all this because we want to see that $f\left(\mathcal{C}\left(\delta,\cdot\right),\delta\right)$ looks like the right or wrong selection decision of an algorithm that depends on a standard Brownian motion instead of $\mathcal{C}\left(\delta,\cdot\right)$, and this will give us the proof because we will be able to use the theory developed by Frazier (2014) to prove the theorem. Thus, the following step is to find the limit of $q_{tx}^{F,\delta}\left(A\right)$ when δ goes to zero, which is called $q_{tx}^{F}\left(A\right)$ and equal to

$$q_{tx}^{F}\left(A\right) := \exp\left(\frac{F_x\left(t\right)}{\lambda_z} + \frac{1}{\lambda_z^2}ta_x\right) / \sum_{x' \in A} \exp\left(\frac{F_{x'}\left(t\right)}{\lambda_z} + \frac{1}{\lambda_z^2}ta_{x'}\right).$$

Using the same previous idea, for each $F \in D\left[0,\infty\right)^k$, we define the new functions

$$g\left(F\right) \quad = \quad \begin{cases} 1 & \text{if } k \text{ is chosen} \\ 0 & \text{otherwise} \end{cases}.$$

So, we want to prove that

$$f\left(\mathcal{C}\left(\delta,\cdot\right),\delta\right) \Rightarrow g\left(W\right)$$

as $\delta \to 0$ in distribution.

In order to prove this, we first prove Lemma 2, which will allow us to use Theorem 5.5 of Billingsley 1968, which implies the desired result.

Lemma 2. Let $\{\delta_n\} \subset (0,\infty)$ such that $\delta_n \to 0$. If $D_s \equiv \{Z \in D\left[0,\infty\right)^k :$ if $\{Z_n\} \subset D\left[0,\infty\right)^k$ and $\lim_n d_\infty\left(Z_n, Z\right) = 0$, then the sequence $\{f\left(Z_n,\delta_n\right)\}$ converges to $\{g\left(Z\right)\}\}$, then $\mathbb{P}\left(W \in D_s\right) = 1$.

First, we are going to prove the following five results.

Proposition 1. Suppose $\{f_n\}$ and $\{g_n\}$ are two sequences of functions on D_∞ such that $f_n \to f$ and $g_n \to g$ in D_∞. If f and g are continuous, then

$$\min\left(f_n, g_n\right) \to \min\left(f,g\right)$$

in D_∞.

Proof. Let $t^* > 0$. We will prove that $\min\left(f_n, g_n\right) \to \min\left(f,g\right)$ in D_{t^*} and the theorem will follow from Theorem 16.2 of Billingsley 1999.

Since f and g are uniformly continuous in $[0,t^*]$ and $f_n \to f$ and $g_n \to g$ in $[0,t^*]$, then f_n and g_n converge uniformly to f and g, respectively. Consequently, $\left(f_n,g_n\right) \to \left(f,g\right)$ uniformly in $[0,t^*]$. Let $a_f^+ =$

$\max_{t\in[0,t^*]}|f(t)|$, $a_f^- = -a_f^+$, $a_g^+ = \max_{t\in[0,t^*]}|g(t)|$ and $a_g^- = -a_g^+$. Let N such that if $n \geq N$, implies $|f_n(t) - f(t)| < 1$ and $|g_n(t) - g(t)| < 1$ for all $t \in [0, t^*]$. Consequently, if $n \geq N$, $f_n(t) \in \left[a_f^- - 1, a_f^+ + 1\right]$ and $g_n(t) \in \left[a_g^- - 1, a_g^+ + 1\right]$ for all t in $[0, t^*]$. Since $\min(x, y)$ is continuous, then it is uniformly continuous in $A = \left[a_f^- - 1, a_f^+ + 1\right] \times \left[a_g^- - 1, a_g^+ + 1\right]$.

Let $\epsilon > 0$, then there exists $\delta > 0$ such that if $\|(u, v)\|_2 < \delta$ and $(u := (u_1, u_2), v := (v_1, v_2)) \in A$, then

$$|\min(u_1, u_2) - \min(v_1, v_2)| < \epsilon. \tag{2}$$

Let $M > N$ such that if $n > M$, then

$$\|(f_n(t), g_n(t)) - (f(t), g(t))\|_2 < \delta$$

for all $t \in [0, t^*]$. Consequently, if $n > M$, by (2),

$$|\min(f_n(t), g_n(t)) - \min(f(t), g(t))| < \epsilon$$

for all $t \in [0, t^*]$. Since uniform convergence implies Skorohod convergente, then $\min(f_n, g_n) \to \min(f, g)$ in D_{t^*}, and the result follows.

■

Proposition 2. Suppose $\{f_n\}$ is a sequence of functions in D_∞ such that $f_n \to f$ in D_∞ , f is continuous, and $\{T_n\} \subset [0, \infty)$ is a sequence such that $T_n \to T$. We define $T(a) := \inf\{t \geq T : f(t) \geq a\}$ for each $a \in \mathbb{R}$. Suppose $T(0) \in \mathbb{R}$. Furthermore, we suppose that there exists $\{\epsilon_n\} \subset (0, \infty)$ such that $\epsilon_n \to 0$, $\epsilon_n \geq \epsilon_{n+1}$, and

$$\|f_n(t) - f(t)\|_2 < \epsilon_n \tag{3}$$

for all $t \in [0, T(0)]$. We also suppose that $\lim\sup_n T(\epsilon_n) \leq T(0)$. Thus we have that

$$\inf\{t \geq T_n : f_n(t) \geq 0\} \to \inf\{t \geq T : f(t) \geq 0\}$$

if $T(0) > T$ or $T_n = T$.

Proof. We are going to suppose $T(0) > T$, and the case $T_n = T$ can be proved using almost the same ideas. We introduce the notation $T_n(a) : \inf\{t \geq T_n : f_n(t) \geq a\}$ for $a \in \mathbb{R}$. Since $T(0) > T$, we can take N such that if $n > N$, then $T_n < T + T(0) - T = T(0)$ and $\epsilon > \epsilon_n$ where $\epsilon := sup_n \epsilon_n$. Let $n > N$. Note that $T_n(0) \leq T_n(\epsilon_n)$.

We also have that

$$\begin{aligned} T(0) &\leq \lim\inf_n T(\epsilon_n) \\ &\leq \lim\sup_n T(\epsilon_n) \end{aligned} \tag{4}$$

Now, since $\lim\,\sup_n T(\epsilon_n) \leq T(0)$,

$$\begin{aligned} T(0) &\geq \lim\,\sup_n T(\epsilon_n) \\ &\geq \lim\,\inf_n T(\epsilon_n) \\ &\geq T(0) \end{aligned}$$

and so

$$\lim\,\inf_n T_n(0) \leq \lim\,\sup_n T_n(0) \leq \lim\,\sup_n T_n(\epsilon_n) = \lim_n T(\epsilon_n) = T(0)$$

Now, let's prove that $\lim\,\inf_n T_n(0) \geq T(0)$. Since $T(0) > T$, there exists M such that $T(0) - \frac{1}{m} \geq T$ if $m > M$, and let $t_m = T(0) - \frac{1}{m}$ and $\alpha_m = \max\{f(t) : t \in [T, t_m]\}$. Note that $\alpha_m < 0$ because $t_m < T(0)$. Since $\epsilon_n \to 0$, there exists N such that if $n > N$, then

$$\epsilon_n \leq -\alpha_m$$

Thus, since $\|f_n(t) - f(t)\|_2 < \epsilon_n$ if $t \in [T, t_m]$, then

$$f_n(t) < f(t) + \epsilon_n \leq f(t) - \alpha_m \leq 0 \tag{5}$$

If $T \leq T_n(0)$ for all $n > N$, then $T_n(0) \geq t_m$ by (5) and so $\lim\,\inf_n T_n(0) \geq t_m$. Taking the limit $m \to \infty$, we conclude that $\lim\,\inf_n T_n(0) \geq T(0)$.

So, we only need to prove that $T \leq T_n(0)$ for n large. We proceed by contradiction. Let $s > 0$ be any number. Take N such that if $n > N$, then $\epsilon_n < s$. Since we are supposing that for all N' there exists $n > N'$ such that $T > T_n(0)$, and $T_n(0) \geq T_n$, $T_n \to T$, then we have that for every $\epsilon > 0$ there exists n large such that $T - \epsilon < T_n(0) < T$. By (3),

$$\begin{aligned} f(T_n(0)) &\geq f_n(T_n(0)) - \epsilon_n \\ &\geq -\epsilon_n > -s \end{aligned}$$

Let $a > 0$, since f is continuous there exists $\epsilon > 0$ such that $f(x) < f(T) + a$ if $|x - T| < \epsilon$. Thus, there exists $n > N$ such that $T - \epsilon < T_n(0) < T$ and so

$$-s < f(T_n(0)) < f(T) + a$$

Since s and a are arbitrary,

$$f(T) \geq 0$$

which is a contradiction because $T(0) > T$. Consequently, $T \leq T_n(0)$ and so

$$\lim_n T_n(0) = T(0).$$

$\blacksquare$

Proposition 3. Let $\mathcal{S}$ be the event such that W is continuous and $\mathbb{P}[\mathcal{S}] = 1$. Fix $\omega \in \mathcal{S}$ and let $\{Z_n\} \subset D[0,\infty)^k$ be a sequence of functions such that $Z_n \to W$ in $D^k[0,\infty)$, then

$$\begin{aligned} f_{Z_n}(\cdot): &= \max\left\{c - \min_{x\in A} q_{\cdot x}^{Z_n,\delta_n}(A), \max_{x\in A} q_{\cdot x}^{Z_n,\delta_n}(A) - P\right\} \\ &\to f_W(\cdot) := \max\left\{c - \min_{x\in A} q_{\cdot x}^{W}(A), \max_{x\in A} q_{\cdot x}^{W}(A) - P\right\} \end{aligned}$$

in $D[0,\infty)$ for any $P \in (0,1)$ and $A \subset \{1,\ldots,k\}$. Furthermore, if $m \in \mathbb{N}$, there exists a sequence $\{\epsilon_k\}$ such that $\epsilon_k \downarrow 0$ and

$$|f_{Z_k}(t) - f_W(t)| < \epsilon_k$$

for all $t \in [0,m]$.

Proof.

Since $Z_n \to W$ in $D[0,\infty)$, by Theorem 16.2 of Billingsley 1999, for each $s \geq 0$ there exist functions λ_s^n in Λ_s such that

$$\lim_n Z_n(\lambda_s^n t) = W(t)$$

uniformly in t and

$$\lim_n \lambda_s^n t = t$$

uniformly in t. Then

$$\lim_n \frac{\lambda_x^2 \beta_\delta(\lambda_s^n t)}{\lambda_z \mathrm{ceil}\left(\frac{\lambda_x^2}{\lambda_z^2}\left(n_0 + \frac{\lambda_s^n(t)}{\delta^2}\right)\right)} Z_n^x(\lambda_s^n t) + \delta_n^2 \lambda_x^2 \frac{\left(n_0 + \lambda_s^n(t)\frac{1}{\delta_n^2}\right)\beta_\delta(\lambda_s^n(t))}{\lambda_z^2 \mathrm{ceil}\left(\frac{\lambda_x^2}{\lambda_z^2}\left(n_0 + \frac{\lambda_s^n(t)}{\delta^2}\right)\right)} a_x = W_x(t)\frac{1}{\lambda_z} + \frac{t}{\lambda_z^2} a_x$$

uniformly in t, and so

$$\lim_n \exp\left(\frac{\lambda_x^2 \beta_\delta(\lambda_s^n t)}{\lambda_z \mathrm{ceil}\left(\frac{\lambda_x^2}{\lambda_z^2}\left(n_0 + \frac{\lambda_s^n(t)}{\delta^2}\right)\right)} Z_n^x(\lambda_s^n t) + \delta_n^2 \lambda_x^2 \frac{\left(n_0 + \lambda_s^n(t)\frac{1}{\delta_n^2}\right)\beta_\delta(\lambda_s^n(t))}{\lambda_z^2 \mathrm{ceil}\left(\frac{\lambda_x^2}{\lambda_z^2}\left(n_0 + \frac{\lambda_s^n(t)}{\delta^2}\right)\right)}\right) = \exp\left(W_x(t)\frac{1}{\lambda_z} + \frac{t}{\lambda_z^2} a_x\right)$$

uniformly in t since exp is uniformly continuous in $[0,s]$. Consequently,

$$q_{\lambda_s^n(t)x}^{Z_n,\delta_n}(A) \to q_{tx}^{W}(A)$$

uniformly in $t \in [0,s]$. Thus, $q_{\cdot x}^{Z_n,\delta_n}(A) \to q_{\cdot x}^{W}(A)$ in $D[0,s]$ for any set $A \subset \{1,\ldots,k\}$ and $s \geq 0$. Consequently, $q_{\cdot x}^{Z_n,\delta_n}(A) \to q_{\cdot x}^{W}(A)$ in $D[0,\infty)$ by Theorem 16.2 of Billingsley 1999. By Proposition 1, $\min_{x\in A} q_{\cdot x}^{Z_n,\delta_n}(A) \to \min_{x\in A} q_{\cdot x}^{W}(A)$ and $\max_{x\in A} q_{\cdot x}^{Z_n,\delta_n}(A) \to \max_{x\in A} q_{\cdot x}^{W}(A)$ in $D[0,\infty)$, and so by Proposition 1,

$$f_{Z_n} := \max\left\{c - \min_{x\in A} q_{\cdot x}^{Z_n,\delta_n}(A), \max_{x\in A} q_{\cdot x}^{Z_n,\delta_n}(A) - P\right\} \to f_W := \max\left\{c - \min_{x\in A} q_{\cdot x}^{W}(A), \max_{x\in A} q_{\cdot x}^{W}(A) - P\right\}$$

in $D[0,\infty)$ for any $P\in(0,1)$. Now, let's prove the second part. Fix $m\in\mathbb{N}$. By the definition of $d_m(f_{Z_n},f_W)$, we have that there exists $\lambda_n\in\Lambda_\infty$ such that

$$\begin{aligned}\sup_{t\le m}\|\lambda_n(t)-t\|_2 &\le d_m(f_{Z_n},f_W)+\frac{1}{n}\\ \sup_{t\le m}\|f_{Z_n}(t)-f_W(\lambda_n t)\|_2 &\le d_m(f_{Z_n},f_W)+\frac{1}{n}\end{aligned}$$

for all n. Taking $g_n\equiv\sup_{t\le m}\|f_W(t)-f_W(\lambda_n t)\|_2$, we see from the uniform continuity of f_W on $[0,m]$ (f_W is uniformly continuous because it's continuos in a compact set) and the definition of g_n that $\lim_{n\to\infty}g_n=0$. Moreover, if we take $\epsilon_n=2n^{-1}+2\sup\{d_m(f_{Z_l},f_W)+g_l:l=n,n+1,\ldots\}$, then $\{\epsilon_n\}$ is a monotonically decreasing sequence of positive numbers with limit zero by the previous result.

From the definition of ϵ_n we have that $d_m(f_{Z_n},f_W)<\epsilon_n/2-\frac{1}{n}$ and $g_n<\epsilon_n/2$ for $n=1,2,\ldots$ Consequently, we have

$$\begin{aligned}\|f_{Z_n}(t)-f_W(t)\| &\le \|f_{Z_n}(t)-f_W(\lambda_n t)\|+\|f_W(\lambda_n t)-f_W(t)\|\\ &< d_m(f_{Z_n},f_W)+\frac{1}{n}+g_n\\ &< \epsilon_n\end{aligned}$$

for all $t\in[0,m]$.

■

Proposition 4. Let $W=(W_1,\ldots,W_k)$ be a $k-$dimensional independent standard Brownian motion. Let $q^W_{tx}(A):=\exp\left(\frac{1}{\lambda_z^2}(\lambda_z W_x(t)+ta_x)\right)/\sum_{x'\in A}\exp\left(\frac{1}{\lambda_z^2}(\lambda_z W_{x'}(t)+ta_{x'})\right)$ for all $x\in A$ and $A\subset\{1,\ldots,k\}$. Fix $m\in\{1,\ldots,k-1\}$. We have that for all $N\in\mathbb{N}$, there exists t such that $\tau_m+\frac{1}{N}\ge t>\tau_m$ and $q^W_{tx}(A_{m-1})>P_{m-1}$ for some $x\in A_{m-1}$ almost surely given that $M=m$. (See the definitions in p. 4).

Proof. The proof has the same spirit than the proof of Lemma 4 of Frazier [3], there Frazier proved that $\tau_m<\infty$ almost surely. Furthermore, $M\le k-1$ almost surely. We should note that the Lemma 4 supposes that all systems have the same variance which is the case here because $\mathrm{Var}(\lambda_z W_x(t)+ta_x)=\lambda_z^2 t$. We condition on the event $\mathcal{S}$ such that the previous properties hold and has probability 1.

Define $a=\lambda_z\log\left(\frac{P_{m-1}(k-1)}{1-P_{m-1}}\right)$. If $m=1$, then $P_{m-1}=P^*\in(0,1)$, and so a is finite. If $m>1$, then

$$P^*\le P_{m-1}\le P^*/(1-c)^{m-1}\le P^*/(1-c)^{k-2}\le P^*/(P^*)^{k-2/k-1}<1$$

and so $P_{m-1}\in(0,1)$ and a is finite.

We define $T^*=\inf\left\{t\ge\tau_m:q^W_{t,x}(A_{m-1})>P_{m-1}\text{ for some }x\in A_{m-1}\right\}$. Fix $n\in\mathbb{N}$, $n>2$, and define $\tau=\tau_m+\frac{t^*}{nN}$ for a deterministic $t^*\in\{1,\ldots,n-1\}$. Let x be any $\mathcal{F}_{\tau_m}$-measurable random variable that is almost surely in A_{m-1} and we define $\Gamma_{t,x}=W_x(t)+t\frac{a_x}{\lambda_z}$. Consider the event $a<\Gamma_{\tau+\frac{1}{nN},x}-\Gamma_{\tau+\frac{1}{nN},y}$ for each $y\in A_{m-1}-\{x\}$. On this event, $q^W_{\tau+\frac{1}{mN},y}(A_{m-1})/q^W_{\tau+\frac{1}{mN},x}(A_{m-1})=\exp\left(\frac{1}{\lambda_z}\left(\Gamma_{\tau+\frac{1}{nN},y}-\Gamma_{\tau+\frac{1}{nN},x}\right)\right)<\exp(-a/\lambda_z)$ for $y\in A_{m-1}-\{x\}$ and

$$q_{\tau+\frac{1}{nN},x}(A_{m-1})=\left[1+\sum_{y\in A_{m-1}-\{x\}}q_{\tau+\frac{1}{nN},y}(A_{m-1})/q_{\tau+\frac{1}{nN},x}(A_{m-1})\right]^{-1}>[1+(k-1)\exp(-a/\lambda_z)]^{-1}=P_{m-1}.$$

Thus, on the event considered, $T^* \le \tau + \frac{1}{nN}$.

We now define $\tilde{x} \in \arg\max_{x \in A_{m-1}} \Gamma_{\tau,x}$, which is $\mathcal{F}_\tau$-measurable and is almost surely in A_{m-1}. Then we have that

$$\begin{aligned}
\mathbb{P}\left\{T^* \le \tau + \frac{1}{nN} \mid \mathcal{F}_\tau, T^* > \tau\right\} &\ge \mathbb{P}\left\{a < \Gamma_{\tau+\frac{1}{nN},\tilde{x}} - \Gamma_{\tau+\frac{1}{nN},x} \ \forall x \in A_{m-1} - \{\tilde{x}\} \mid \mathcal{F}_\tau, T^* > \tau\right\} \\
&\ge \mathbb{P}\left\{\Gamma_{\tau+\frac{1}{nN},\tilde{x}} \ge \Gamma_{\tau,\tilde{x}}, \Gamma_{\tau,\tilde{x}} - \Gamma_{\tau+\frac{1}{nN},x} > a \ \forall x \in A_{m-1} - \{\tilde{x}\} \mid \mathcal{F}_\tau, T^* > \tau\right\} \\
&\ge \mathbb{P}\left\{\Gamma_{\tau+\frac{1}{nN},\tilde{x}} \ge \Gamma_{\tau,\tilde{x}}, \Gamma_{\tau,x} - \Gamma_{\tau+\frac{1}{nN},x} > a \ \forall x \in A_{m-1} - \{\tilde{x}\} \mid \mathcal{F}_\tau, T^* > \tau\right\} \\
&= \mathbb{P}\left\{\Gamma_{\tau+\frac{1}{nN},\tilde{x}} \ge \Gamma_{\tau_m,\tilde{x}} \mid \mathcal{F}_\tau\right\} \prod_{x \in A_{m-1} \setminus \{\tilde{x}\}} \mathbb{P}\left\{\Gamma_{\tau,x} - \Gamma_{\tau+\frac{1}{nN},x} > a \mid \mathcal{F}_\tau\right\}.
\end{aligned}$$

Note that $\Gamma_{t,x}$ is a Brownian motion, and so $\mathbb{P}\left\{\Gamma_{\tau+\frac{1}{nN},\tilde{x}} \ge \Gamma_{\tau_m,\tilde{x}} \mid \mathcal{F}_{\tau_m}\right\}$ is the probability of a conditionally $N\left(\frac{a_{\tilde{x}}}{nN}\frac{1}{\lambda_z}, 1\right)$ random variable $\Gamma_{\tau+\frac{1}{nN},\tilde{x}} - \Gamma_{\tau,\tilde{x}}$, exceeding 0. This probability is $\Phi\left(\frac{a_{\tilde{x}}}{nN}\frac{1}{\lambda_z}\right)$, which is bounded below by $\Phi\left(\frac{\min_x a_x}{N}\frac{x_0}{\lambda_z}\right)$ where $x_0 = 0$ if $\min_x a_x \ge 0$, and $x_0 = \frac{1}{3}$ otherwise. Here, Φ is the normal cumulative distribution function. Similarly, the probability $\mathbb{P}\left\{\Gamma_{\tau,x} - \Gamma_{\tau+\frac{1}{nN},x} > a \mid \mathcal{F}_{\tau_m}\right\}$ is the probability of a conditionally $N\left(-a - \frac{a_x}{nN}\frac{1}{\lambda_z}, 1\right)$ random variable, $\Gamma_{\tau,x} - \Gamma_{\tau+\frac{1}{nN},x} - a$, exceeding 0. This probability is $\Phi\left(-a - \frac{a_x}{nN}\frac{1}{\lambda_z}\right)$, and is bounded below by $\Phi\left(-a - \frac{\max_x a_x}{N}\frac{x_0^+}{\lambda_z}\right)$ where $x_0^+ = 0$ if $\max_x a_x \le 0$ and $x_0^+ = \frac{1}{3}$ otherwise.

Thus, replacing τ with $\tau_m + \frac{t^*}{nN}$,

$$\mathbb{P}\left\{T^* \le \tau_m + \frac{t^*}{nN} + \frac{1}{nN} \mid \mathcal{F}_{\tau_m + \frac{t^*}{nN}}, T^* > \tau_m + \frac{t^*}{nN}\right\} \ge \Phi\left(\frac{\min_x a_x}{N}\frac{x_0}{\lambda_z}\right)\Phi\left(-a - \frac{\max_x a_x}{N}\frac{x_0^+}{\lambda_z}\right)^{k-1}$$

Let ϵ be the quantity on the righ-hand side of this inequality, then $\epsilon < 1$ and it does not depend on t^* and n.

By repeated application of this inequality, we have that $\mathbb{P}\left\{T^* > T_W^1(P) + \frac{1}{N} \mid \mathcal{F}_{\tau_m}\right\} \le (1-\epsilon)^n$ for all $n > 2$. This is true because

$$\begin{aligned}
\mathbb{P}\left\{T^* > T_W^1(P) + \frac{1}{N} \mid \mathcal{F}_{\tau_m + \frac{n-1}{nN}}, T^* > \tau_m + \frac{n-1}{nN}\right\} &< (1-\epsilon) \\
\Rightarrow \mathbb{P}\left\{T^* > \tau_m + \frac{1}{N} \mid \mathcal{F}_{T_W^1(P) + \frac{n-2}{nN}}, T^* > \tau_m + \frac{n-2}{nN}\right\} &< (1-\epsilon)\,\mathbb{P}\left\{T^* > \tau_m + \frac{n-1}{nN} \mid \mathcal{F}_{\tau_m + + \frac{n-2}{nN}}, T^* > \tau_m + \frac{n-2}{nN}\right\} \\
&\le (1-\epsilon)^2 \\
&\vdots \\
\Rightarrow \mathbb{P}\left\{T^* > \tau_m + \frac{1}{N} \mid \mathcal{F}_{T_W^1(P)}\right\} &\le (1-\epsilon)^n
\end{aligned}$$

and $(1-\epsilon)^n$ vanishes in the limit as $n \to \infty$. Then, $\mathbb{P}\left\{T^* > \tau_m + \frac{1}{N} \mid \mathcal{F}_{T_W^1(P)}\right\} = 0$ and then

$$\mathbb{P}\left\{\tau_m \le T^* \le \tau_m + \frac{1}{N} \mid \mathcal{F}_{T_W^1(P)}\right\} = 1$$

for all N.

■

The following proposition can be proved using a similar argument.

Proposition 5. Let $W = (W_1, \ldots, W_k)$ be a $k-$dimensional independent standard Brownian motion. Let $q_{tx}^W(A) := \exp\left(\frac{1}{\lambda_z^2}(\lambda_z W_x(t) + ta_x)\right) / \sum_{x' \in A} \exp\left(\frac{1}{\lambda_z^2}\left(\lambda_z W_{x'}(t) + ta_{x'}\right)\right)$ for all $x \in A$ and $A \subset \{1, \ldots, k\}$. Fix $m \in \{1, \ldots, k-1\}$. We have that for all $N \in \mathbb{N}$, there exists t such that $\tau_m - \frac{1}{N} \leq t < \tau_m$ and $q_{tx}^W(A_{m-1}) < c$ for some $x \in A_{m-1}$ almost surely given that $M = m$.

We are now ready to prove the lemma 2.

Lemma 2. Let $\{\delta_n\} \subset (0, \infty)$ such that $\delta_n \to 0$. If $D_s \equiv \{Z \in D[0,\infty)^k$: if $\{Z_n\} \subset D[0,\infty)^k$ and $\lim_n d_\infty(Z_n, Z) = 0$, then the sequence $\{f(Z_n, \delta_n)\}$ converges to $\{g(Z)\}\}$, then $\mathbb{P}(W \in D_s) = 1$.

Proof of Lemma 2. By Lemma 4 and 6 of Frazier, $\tau_n < \infty$ for $n = 0, 1, \ldots, k-1$, $M \leq k-1$,$\tau_n = \tau_M$ for all $n \geq M$, $\hat{x} \in \arg\max_{x \in A_{m-1}} (\lambda_z W_x(\tau_M) + a_x \tau_M)$ and $\tau_{M-1} < \tau_M$ with probability 1. Then there exists a measurable set $\mathcal{L}$ such that $\mathbb{P}[\mathcal{L}] = 1$ and the previous properties are true, W is continuous and Proposition 4 and 5 hold. We will show that $\mathbb{P}(W \in D_s \mid M_W = i, \mathcal{L}) = 1$ for $i \in \{1, \ldots, k-1\}$ so that the desired conclusion follows. It will be useful to use the same notation than in Proposition 2:

$$T_W^m(a) \quad = \quad \begin{cases} \inf\left\{t \geq \tau_{m-1} : \max_{x \in A_{m-1}} q_{tx}^W(A_{m-1}) \geq a\right\} & \text{if } m \geq M_W \\ \inf\left\{t \geq \tau_{m-1} : \min_{x \in A_{m-1}} q_{tx}^W(A_{m-1}) \leq a\right\} & \text{if } m < M_W \end{cases}$$

and

$$T_n^m(a) := \inf\left\{t \geq \tau_{m-1} : \max_{x \in A_{m-1}} q_{tx}^{Z_n,\delta_n}(A_{m-1}) \geq a \text{ or } \min_{x \in A_{m-1}} q_{tx}^{Z_n,\delta_n}(A_{m-1}) \leq c\right\}$$

for any $a \in \mathbb{R}$ and $1 \leq m \leq k-1$.

Case I. First, we fix $\omega \in \mathcal{L} \bigcap \{M_W = 1\}$. Let $\{Z_n\} \subset D[0,\infty)^k$ such that $Z_n \to W$. We want to prove that $f(Z_n, \delta_n) \to g(W)$. Let's prove that $\tau_1^{Z_n,\delta_n} \to \tau_1^W$ as $n \to \infty$.

First, we are going to prove that $T_W^1(P_0) = \lim_n T_W^1(P_0 + \epsilon_n)$. Note that $\lim_n T_W^1(P_0 + \epsilon_n) = \inf_n T_W^1(P_0 + \epsilon_n)$ and so we have to prove that for all $M > 0$ there exists n such that $T_W^1(P_0 + \epsilon_n) < T_W^1(P_0) + M$. Equivalently, we should prove that for all $N \in \mathbb{N}$ there exists $t \in \left(T_W^1(P_0), T_W^1(P_0) + \frac{1}{N}\right]$ such that for some $x \in A_0$, $q_{t,x}^W(A_0) > P_0$. However, this is true by proposition 4.

By Proposition 2 and 3,

$$T_W^1(P_0) = \lim_n T_n^1(P_0). \tag{6}$$

By Proposition 3, we have that $f_{Z_n}(\cdot) \to f_W(\cdot)$ uniformly in $\left[0, T_W^1(P_0) + 1\right]$ and so it should be the case that $\max_{x \in A_0} q_{T_n^1(P_0)x}^{Z_n,\delta_n}(A_0) \geq P_0$ for n large.

Recall that $\hat{x} \in \gamma := \arg\max_{x \in A_0}\left(\lambda_z W_x\left(T_W^1(P_0)\right) + a_x T_W^1(P_0)\right)$.

Let $\epsilon = \left(q_{T_W^1(P_0),\hat{x}}^W(A_0) - \arg\max_{x \in A_0 - \gamma} q_{T_W^1(P_0),x}^W(A_0)\right)/4$. Since the function $q_{\cdot,x}^W(A_0)$ is uniformly continuous in $\left[0, T_W^1(P_0) + 1\right]$ for all $x \in A_0$, then there exists $\delta > 0$ such that if $|t - s| < \delta$ and $t, s \in \left[0, T_W^1(P_0) + 1\right]$, then

$$\left|q_{t,x}^W(A_0) - q_{s,x}^W(A_0)\right| < \frac{\epsilon}{2}$$

for all $x \in A_0$. By the proof of Proposition 3, $q_{\cdot,x}^{Z_n,\delta_n}(A_0) \to q_{\cdot,x}^W(A_0)$ in $\left[0, T_W^1(P_0) + 1\right]$ with the Skorohod topology, and then $q_{\cdot,x}^{Z_n,\delta_n}(A_0) \to q_{\cdot,x}^W(A_0)$ uniformly in $\left[0, T_W^1(P_0) + 1\right]$ because $q_{\cdot,x}^W(A_0)$ is uniformly continuous in $\left[0, T_W^1(P_0) + 1\right]$ for all $x \in A_0$. Consequently, there exists N_1 such that if $n > N_1$, then

$\left|q_{t,x}^{W}\left(A_0\right)-q_{t,x}^{Z_n,\delta_n}\left(A_0\right)\right|<\frac{\epsilon}{2}$ for all $t\in\left[0,T_W^1\left(P_0\right)+1\right]$ and all $x\in A_0$. By (6), there exists N_2 such that if $n>N_2$, then $\left|T_W^1\left(P_0\right)-T_n^1\left(P_0\right)\right|<\delta$ and $T_n^1\left(P_0\right)\in\left[0,T_W^1\left(P_0\right)+1\right]$. So if $n>\max\left\{N_1,N_2\right\}$, we have that

$$\begin{aligned}\left|q_{T_W^1(P_0),x}^{W}\left(A_0\right)-q_{T_n^1(P_0),x}^{Z_n,\delta_n}\left(A_0\right)\right| &= \left|q_{T_W^1(P_0),x}^{W}\left(A_0\right)-q_{T_n^1(P_0),x}^{W}\left(A_0\right)\right|+\left|q_{T_n^1(P_0),x}^{W}\left(A_0\right)-q_{T_n^1(P_0),x}^{Z_n,\delta_n}\left(A_0\right)\right|\\ &< \frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon\end{aligned}$$

for all $x\in A_0$.

By Proposition 3, there exists a sequence $\left\{\epsilon_n\right\}_{n>N^*}$ decreasing to zero for some N^* such that if $n>N^*$.

$$\left|q_{T_W^1(P_0),\hat{x}}^{W}\left(A_0\right)-\max_{x\in A_0}q_{T_n^1(P_0),x}^{Z_n,\delta_n}\left(A_0\right)\right|<\epsilon_n.$$

Let $y_n\in\arg\max q_{T_n^1(P_0),y}^{Z_n,\delta_n}\left(A_0\right)$. Thus if $n>\max\left\{N^*,N_1,N_2\right\}$, then

$$\begin{aligned}\left|q_{T_W^1(P_0),\hat{x}}^{W}\left(A_0\right)-q_{T_W^1(P_0),y_n}^{W}\left(A_0\right)\right| &\leq \left|q_{T_W^1(P_0),\hat{x}}^{W}\left(A_0\right)-q_{T_n^1(P_0),y_n}^{Z_n,\delta_n}\left(A_0\right)\right|+\left|-q_{T_W^1(P_0),y_n}^{W}\left(A_0\right)+q_{T_n^1(P_0),y_n}^{Z_n,\delta_n}\left(A_0\right)\right|\\ &< \epsilon_n+\epsilon\end{aligned}$$

Take N_3 such that if $n>N_3$, then $\epsilon_n<\epsilon$. Consequently, if $n>\max\left\{N_3,N_1,N_2,N^*\right\}$, we have that

$$\left|q_{T_W^1(P_0),\hat{x}}^{W}\left(A_0\right)-q_{T_W^1(P_0),y_n}^{W}\left(A_0\right)\right| < \frac{q_{T_W^1(P_0),\hat{x}}^{W}\left(A_0\right)-\arg\max_{x\in A_0-\gamma}q_{T_W^1(P_0),x}^{W}\left(A_0\right)}{2}$$

if $y_n\neq\hat{x}$, then

$$\left|q_{T_W^1(P_0),\hat{x}}^{W}\left(A_0\right)-q_{T_W^1(P_0),y_n}^{W}\left(A_0\right)\right|<\frac{q_{T_W^1(P_0),\hat{x}}^{W}\left(A_0\right)-q_{T_W^1(P_0),y_n}^{W}\left(A_0\right)}{2}$$

which is a contradiction. Consequently, $y_n=\hat{x}$ for n large. Consequently, $f\left(Z_n,\delta_n\right)$ converges to $g\left(W\right)$.

Case II. Now, we fix we fix $\omega\in\mathcal{L}\bigcap\left\{M_W=2\right\}$. Similarly, we take $\left\{Z_n\right\}\subset D\left[0,\infty\right)^k$ such that $Z_n\to W$. Let's prove that $\tau_1^{Z_n,\delta_n}\to\tau_1^{W}$. First, we are going to prove that $T_W^1\left(c\right)=\lim_n T_W^1\left(c-\epsilon_n\right)$. Note that $\lim_n T_W^1\left(c-\epsilon_n\right)=\inf_n T_W^1\left(c-\epsilon_n\right)$ and so we have to prove that for all $M>0$ there exists n such that $T_W^1\left(c-\epsilon_n\right)<T_W^1\left(c\right)+M$. Equivalently, we should prove that for all $N\in\mathbb{N}$ there exists $t\in\left(T_W^1\left(c\right),T_W^1\left(c\right)+\frac{1}{N}\right]$ such that for some $x\in A_0$, $q_{t,x}^{W}\left(A_0\right)<c$. However, this is true by proposition 5. By Proposition 2 and 3, $T_W^1\left(c\right)=\lim_n T_n^1\left(P_0\right)$. Using similar arguments than the given in Case I, we can conclude that $f\left(Z_n,\delta_n\right)$ converges to $g\left(W\right)$.

The cases $M_W=i$ for $k-1\geq i\geq 3$ can be proved in a similar way. Then we conclude that $\mathbb{P}\left(W\in D_s\right)=1$.

■

By the extension of the CMT (Theorem 5.5 of Billingsley 1968), we have the following corollary.

Corollary 2. We have that

$$f\left(C\left(\delta,t\right),\delta\right)\Rightarrow g\left(W\left(t\right)\right)$$

in distribution as $\delta \to 0$.

Theorem 1. If samples from system $x \in \{1 \ldots, k\}$ are identically distributed and independent, over time and across alternatives, then $\lim_{\delta \to 0} Pr\{\text{BIZ selects } k\} \geq P*$ provided $\mu_k = a_k\delta, \mu_{k-1} = a_{k-1}\delta, \ldots, \mu_1 = a_1\delta$, and $\mu_k > \mu_{k-1} > \cdots > \mu_1$.

Furthermore,

$$\inf_{a \in PZ(1)} \lim_{\delta \to 0} \mathbb{P}(CS_\delta) = P^*$$

where $PZ(1) = \left\{a \in \mathbb{R}^k : a_k - a_{k-1} \geq 1\right\}$.

Proof. By the comments given at the beginning of this section, we know that we can work with the algorithm defined by

$$q'_{tx}(A) = q'\left(\left(Z_{\frac{t}{\delta^2}x}, \frac{t}{\delta^2} : x \in A\right), \delta, A \subset \{1, \ldots, k\}\right)$$

where $t \in \delta^2\mathbb{N}$, instead of the algorithm defined by

$$q'_{tx}(A) = q'((Z_{tx}, t : x \in A), \delta, A \subset \{1, \ldots, k\})$$

where $t \in \mathbb{N}$.

Now, we define

$$\hat{\tau}_{n+1}(\delta) \quad = \quad \inf\left\{t \in \left\{\tau_n\delta^2, (\tau_n + 1)\delta^2, \ldots\right\} : \ \min_{x \in A_n} q'_{t/\delta^2,x}(A_n) \leq c \text{ or } \max_{x \in A_n} q'_{t/\delta^2,x}(A_n) \geq P_n\right\}$$

We denote the corresponding continuous hitting times by $(\tau_n(\delta))_n$, which are defined as $\tau_{n+1} = \inf\{t \geq \tilde{\tau}_n : \ \min_{x \in A_n} q_{tx}(A$

Using that $\mathcal{C}(\delta, \cdot)$ is right-continuous, we can prove that $\hat{\tau}_n(\delta) - \tau_n(\delta) \to 0$ with probability 1 as $\delta \to 0$, and so we can use $\mathcal{C}(\delta, \tau_n(\delta))$ instead of $\mathcal{C}(\delta, \hat{\tau}_n(\delta))$ in the limit.

Let CS_δ be the event of doing a correct selection given the configuration $\mu_k = a_k\delta, \mu_{k-1} = a_{k-1}\delta, \ldots, \mu_1 = a_1\delta$. Then by the previous argument and the Corollary 2,

$$\begin{aligned} \lim_{\delta \to 0} \mathbb{P}(CS_\delta) &= \lim_{\delta \to 0} \mathbb{P}(f(\mathcal{C}(\delta, t), \delta) = 1) \\ &= \mathbb{P}(g(W) = 1) \\ &\geq P^* \end{aligned}$$

where the last inequality follows from the Theorem 2 of Frazier [3].

Furthermore, by the same theorem 2,

$$\inf_{a \in PZ(1)} \mathbb{P}(g(W) = 1) = P^*$$

where $PZ(1) = \left\{a \in \mathbb{R}^k : a_k - a_{k-1} \geq 1\right\}$.

■

5 Asymptotic Validity when the Variances are Unknown

We use a random change of time to prove that the new $\hat{\mathcal{C}}_x$ defined using the sample variances also converges to a Brownian motion in the sense of D_∞.

Lemma 3. We have that

$$\hat{\mathcal{C}}_x(\delta, \cdot) := \frac{Y_{\mathrm{ceil}\left(\frac{\hat{\lambda}^2_{t/\delta^2,x}}{\hat{\lambda}^2_{t/\delta^2,z}}\left(n_0 + \cdot\frac{1}{\delta^2}\right)\right),x} - \frac{\hat{\lambda}^2_{t/\delta^2,x}}{\hat{\lambda}^2_{t/\delta^2,z}}\left(n_0 + \cdot\frac{1}{\delta^2}\right)\mu_x}{\frac{\hat{\lambda}^2_{t/\delta^2,x}}{\hat{\lambda}_{t/\delta^2,z}}\frac{1}{\delta}} \Rightarrow W_x(\cdot)$$

in the sense of D_∞ for each $x \in A$, where $\hat{\lambda}^2_{t/\delta^2,x} = \frac{1}{n_{t/\delta^2,x}-1}\sum_{i=1}^{n_{t/\delta^2,x}}\left(X_{xi} - Y_{n_{t/\delta^2,x}}\right)^2$.

Proof. Fix $x \in A$. Define $\Psi_\delta : [0,\infty) \to [0,\infty)$ by $\Psi_\delta(t) = -n_0\delta^2 + \frac{\lambda_z^2}{\lambda_x^2}\frac{\hat{\lambda}^2_{t/\delta^2,x}}{\hat{\lambda}^2_{t/\delta^2,z}}\left(t + \delta^2 n_0\right)$, then $\Psi_\delta \in D_\infty$. Now define $\varphi : D_\infty \times D_\infty \to D_\infty$ by

$$\varphi(\mathcal{X}, \mathcal{Y}) = \mathcal{X} \circ \mathcal{Y}$$

Using Lemma 1, we have that $(\mathcal{C}_x(\delta,\cdot), \Psi_\delta(\cdot)) \Rightarrow (W_x(\cdot), I(\cdot))$ as $\delta \to 0$ in the sense of D_∞. We can prove that φ is measurable, and since W_x is continuous almost surely, we can use a generalization of the Lemma from the chapter Random Change of Time of Billingsley (1999) to conclude that $\varphi(\mathcal{C}_x(\delta,\cdot), \Psi_\delta(\cdot)) \Rightarrow \varphi(W_x(\cdot), I(\cdot)) = W_x(\cdot)$ in the sense of D_∞ as we wanted to prove.

■

In the previous proof, we only have to replace λ_x^2 by its estimators and we get the same result. We should note that

$$\mathrm{ceil}\left(\frac{\hat{\lambda}^2_{t/\delta^2,x}}{\hat{\lambda}^2_{t/\delta^2,z}}\left(n_0 + \cdot\frac{1}{\delta^2}\right)\right) - n_{t/\delta^2,x} \to 0$$

for any x. Consequently, we will have that

$$\frac{Y_{n_{t/\delta^2,x},x} - n_{t/\delta^2,x}\mu_x}{\frac{\hat{\lambda}^2_{t/\delta^2,x}}{\hat{\lambda}_{t/\delta^2,z}}\frac{1}{\delta}} \Rightarrow W_x(\cdot)$$

So, we have the following theorem.

Theorem 2. If samples from system $x \in \{1\ldots,k\}$ are identically distributed and independent, over time and across alternatives, then $\lim_{\delta\to 0} Pr\{\text{BIZ selects } k\} \geq P*$ provided $\mu_k = a_k\delta, \mu_{k-1} = a_{k-1}\delta, \ldots, \mu_1 = a_1\delta$, and $\mu_k > \mu_{k-1} > \cdots > \mu_1$.

Furthermore,

$$\mathrm{inf}_{a\in PZ(1)}\mathrm{lim}_{\delta\to 0}\mathbb{P}(CS_\delta) = P^*$$

where $PZ(1) = \left\{a \in \mathbb{R}^k : a_k - a_{k-1} \geq 1\right\}$.

6 Probability of Good Selection

Ideally, we would like that the difference between the chosen system and the best system is practically insignificant. We thus define the probability of good selection as

$$\mathrm{PGS}\left(\mu\right)=\mathbb{P}\left(\mu_k-\mu_{\hat{x}}\leq\delta\right).$$

We would like to prove the PGS guarantee:

$$\forall\mu,\ \mathrm{PGS}\left(\mu\right)\geq P^*.$$

First, observe that if $\mu_k-\mu_{k-1}>\delta$, then $\mathrm{PGS}\left(\mu\right)=\mathrm{PCS}\left(\mu\right)$ and so $\mathrm{PGS}\left(\mu\right)\geq P^*$. Consequently, we only need to prove that if $\mu_k-\mu_{k-1}\leq\delta$, then we have that $\mathrm{PGS}\left(\mu\right)\geq P^*$.

Inspired on Frazier (2014), we first construct a probability measure Q as follows. Let X^* be chosen uniformly at random from among $1,\ldots,k$ and let $\theta_{X^*}=\delta$ and $\theta_x=0$ if $x\neq X^*$. We then define a family of probability measures Q_u that i

7 Simulations

8 Conclusion

Appendix A: Skorohod topology

We are going to define the Skorohod topology on $D[0,\infty)$ by defining a metric on the space. The Skorohod metricric d_t on $D\left[0,t\right]$ for each $t\geq0$ is:

$$d_t\left(\mathcal{X},\mathcal{Y}\right)=\inf_{\lambda\in\Lambda_t}\left\{\|\lambda-I\|\vee\|\mathcal{X}-\mathcal{Y}\circ\lambda\|\right\}$$

where Λ_t is the set of strictly increasing, continuous mappings of $[0,t]$ onto itself, and $\|\cdot\|$ is the uniform norm, and I is the identity map. Note that uniform convergence on $[0,t]$ implies Skorohod convergence.

We define the Skorohod topology on $D\left[0,\infty\right)$. For $\mathcal{X}\in D\left[0,\infty\right)$, let $\mathcal{X}^m$ be the element of $D_\infty:=D[0,\infty)$ defined by

$$\mathcal{X}^m\left(t\right)=g_m\left(t\right)\mathcal{X}\left(t\right)$$

where

$$g_m\left(t\right)=\begin{cases}1 & \text{if } t\leq m-1,\\ m-t & \text{if } m-1\leq t\leq m,\\ 0 & \text{if } t\geq m.\end{cases}$$

And now take

$$d_\infty\left(\mathcal{X},\mathcal{Y}\right)=\sum_{m=1}^{\infty}2^{-m}\left(1\wedge d_m\left(\mathcal{X}^m,\mathcal{Y}^m\right)\right)$$

which is the Skorohod metric on $D\left[0,\infty\right)$. By Theorem 16.2 of Billingsley1999, there is convergence $d_\infty\left(x_n,x\right)\to0$ in D_∞ if and only if $d_t\left(x_n,x\right)\to0$ for each continuity point t of x.

We can also define

$$d_{\infty}^{\circ}\left(\mathcal{X},Y\right)=\sum_{m=1}^{\infty}2^{-m}\left(1\wedge d_{m}^{\circ}\left(\mathcal{X}^{m},\mathcal{Y}^{m}\right)\right)$$

where $d_{m}^{\circ}\left(\mathcal{X}^{m},\mathcal{Y}^{m}\right)=\inf_{\lambda\in\Lambda_{m}}\left\{\sup_{s<t}\left|\log\frac{\lambda t-\lambda s}{t-s}\right|\vee\|\mathcal{X}-\mathcal{Y}\circ\lambda\|\right\}$. This is also a metric and the proof is in Bilingsley 1999 and these two metrics are equivalent.

Theorem A.1 We have that $d_{\infty}\left(\mathcal{X}_{n},\mathcal{Y}_{n}\right)\to 0$ if and only if $d_{\infty}^{\circ}\left(\mathcal{X}_{n},\mathcal{Y}_{n}\right)\to 0$ and $d_{t}\left(\mathcal{X}_{n},\mathcal{Y}_{n}\right)\to 0$ if and only if $d_{t}^{\circ}\left(\mathcal{X}_{n},\mathcal{Y}_{n}\right)\to 0$ for all $t\geq 0$.

Theorem A.2 Suppose we have two sequences of random paths $\{\mathcal{X}_{n}=\left(X_{n}\left(t\right):0\leq t<\infty\right)\}_{n\geq 0}^{\infty}$,$\{\mathcal{Y}_{n}=\left(Y_{n}\left(t\right):0\leq$ such that $\mathcal{X}_{n},\mathcal{Y}_{n}:\mathcal{F}\to D_{\infty}$ for all $n\geq 0$ where $(\Omega,\mathcal{F},\mathbb{P})$ is our space of probability. Suppose that $\mathcal{X}_{n}\Rightarrow\mathcal{X}_{0}$ in the sense of $D\left[0,\infty\right)$ and $\mathcal{X}_{n}-\mathcal{Y}_{n}\to 0$ uniformly in $[0,m]$ for all $m\in\mathbb{N}$ with probability 1, then $\mathcal{Y}_{n}\Rightarrow\mathcal{X}_{0}$.

Proof. We only need to prove that $d_{\infty}\left(\mathcal{X}_{n},\mathcal{Y}_{n}\right)\to 0$ almost surely and the result will follow by Theorem 3.1 of Billingsley 1999. Then we only need to prove that $d_{m}\left(\mathcal{Y}_{n}^{m},\mathcal{X}_{n}^{m}\right)\to 0$ almost surely for all $m\in\mathbb{N}$. Fix $\omega\in\Omega$ and $m\in\mathbb{N}$ such that $\mathcal{X}_{n}-\mathcal{Y}_{n}\to 0$ uniformly in $[0,m]$. Observe that

$$\begin{aligned}
d_{m}\left(\mathcal{Y}_{n}^{m},\mathcal{X}_{n}^{m}\right) &= \inf_{\lambda\in\Lambda_{m}}\left\{\|\lambda-I\|\vee\|\mathcal{Y}_{n}^{m}-\mathcal{X}_{n}^{m}\circ\lambda\|\right\}\\
&\leq \|\mathcal{Y}_{n}^{m}-\mathcal{X}_{n}^{m}\|\\
&= \|g_{m}\mathcal{Y}_{n}-g_{m}\mathcal{X}_{n}\|\\
&= \sup_{t\leq m-1}\|Y_{n}\left(t\right)-X_{n}\left(t\right)\|\vee\sup_{t>m-1}\|Y_{n}\left(t\right)-X_{n}\left(t\right)\|\left(m-t\right)\\
&\leq \sup_{t\leq m}\|Y_{n}\left(t\right)-X_{n}\left(t\right)\|\\
&\to 0
\end{aligned}$$

as we wanted to prove.

■

Appendix B: The BIZ Procedure

B.1 The BIZ Procedure with Known Common Variance

We first define the Bayes-inspired indifference zone (BIZ) procedure for the case of known common variance, when $\lambda_{x}^{2}=\sigma^{2}<\infty$ for all x. Frazier (2014) showed that this procedure satisfies the IZ guarantee when the systems follow the normal distribution, with tight bounds on worst-case preference-zone in continuous time.

BIZ depends on $P^{*}\in(1/k,1)$ and $\delta>0$ for which we desire an IZ guarantee, and a parameter c such that

$$\begin{aligned}
c &\in \left[0,1-\left(P^{*}\right)^{\frac{1}{k-1}}\right] \text{ if } k>2\\
c &= 0 \text{ if } k=2
\end{aligned}$$

The parameter $c$controls how aggressively the alternatives are eliminated. It is recommended to choose c equal to $1-\left(P^{*}\right)^{\frac{1}{k-1}}$.

For each t, $x \in \{1, \ldots, k\}$, and subset $A \subset \{1, \ldots, k\}$, we define a function

$$\hat{q}_{tx}(A) = \exp\left(\frac{\delta}{\sigma^2} Y_{tx}\right) \Bigg/ \sum_{x' \in A} \exp\left(\frac{\delta}{\sigma^2} Y_{tx'}\right)$$

where Y_{tx} is the sum of the samples observed from alternative x by time t.

Then, the BIZ procedure for known common variance is defined by Alg. 1.

Algorithm 1: BIZ for known common sampling variance, in discrete time.

Require: $c \in [0, 1 - (P^*)^{\frac{1}{k-1}}]$, $\delta > 0$, $P^* \in (1/k, 1)$, common sampling variance $\sigma^2 > 0$.

Let $A \leftarrow \{1, \ldots, k\}$, $P \leftarrow P^*$, $Y_{0x} \leftarrow 0$ for each x.
while $\max_{x \in A} \hat{q}_{tx}(A) < P$ **do**
    while $\min_{x \in A} \hat{q}_{tx}(A) \leq c$ **do**
        Let $x \in \arg\min_{x \in A} \hat{q}_{tx}(A)$.
        Let $P \leftarrow P/(1 - \hat{q}_{tx}(A))$.
        Remove x from A.
    end while
    Sample from each $x \in A$ and add this sample to Y_{tx} to obtain $Y_{t+1,x}$. Then increment t.
end while
Select $\hat{x} \in \arg\max_{x \in A} Y_{tx}$ as our estimate of the best.

B.2 The BIZ Procedure with Heterogeneous and Unknown Sampling Variances

Frazier (2014) showed that this procedure retains the IZ guarantee when the systems follow the normal distribution, and the variances are known and are integer multiples of a common value. The continuous time version of this procedure also satisfies the IZ guarantee, with a tight worst-case preference-zone PCS bound.

The discrete-time BIZ procedure for unknown and/or heterogeneous sampling variances is given in Alg. 2. It takes a variable number of samples from each alternative, and n_{tx} is this number. This algorithm depends on a collection of integers $B_1, \ldots, B_k$, P^*, c, δ and n_0. n_0 is the number of samples to use in the first stage of samples, and 100 is the recommended value for n_0. B_x controls the number of samples taken from system x in each stage.

For each t, $x \in \{1, \ldots, k\}$, and subset $A \subset \{1, \ldots, k\}$, we define a function

$$q_{tx}(A) = \exp\left(\delta\beta_t \frac{Z_{tx}}{n_{tx}}\right) \Bigg/ \sum_{x' \in A} \exp\left(\delta\beta_t \frac{Z_{tx'}}{n_{tx'}}\right), \quad \beta_t = \frac{\sum_{x' \in A} n_{tx'}}{\sum_{x' \in A} \hat{\lambda}^2_{tx'}}$$

where $\hat{\lambda}^2_{tx'}$ is the sample variance of all samples from alternative x thus far and $Z_{tx} = Y_{n_{tx},x}$.

Algorithm 2: Discrete-time implementation of BIZ, for unknown and/or heterogeneous variances.

Require: $c \in [0, 1 - (P^*)^{\frac{1}{k-1}}]$, $\delta > 0$, $P^* \in (1/k, 1)$, $n_0 \geq 0$ an integer, $B_1, \ldots, B_k$ strictly positive integers. Recommended choices are $c = 1 - (P^*)^{\frac{1}{k-1}}$, $B_1 = \cdots = B_k = 1$ and n_0 between 10 and 30. If the sampling variances λ^2_x are known, replace the estimators $\widehat{\lambda}^2_{tx}$ with the true values λ^2_x, and set $n_0 = 0$.

For each x, sample alternative x n_0 times and set $n_{0x} \leftarrow n_0$. Let W_{0x} and $\widehat{\lambda}^2_{0x}$ be the sample mean and sample variance respectively of these samples. Let $t \leftarrow 0$.

Let $A \leftarrow \{1,\ldots,k\}$, $P \leftarrow P^*$.
while $x \in \max_{x\in A} q_{tx}(A) < P$ **do**
  while $\min_{x\in A} q_{tx}(A) \le c$ **do**
    Let $x \in \arg\min_{x\in A} q_{tx}(A)$.
    Let $P \leftarrow P/(1-q_{tx}(A))$.
    Remove x from A.
  end while
  Let $z \in \arg\min_{x\in A} n_{tx}/\widehat{\lambda}^2_{tx}$.
  For each $x \in A$, let $n_{t+1,x} = \text{ceil}\left(\widehat{\lambda}^2_{tx}(n_{tz}+B_z)/\widehat{\lambda}^2_{tz}\right)$.
  For each $x \in A$, if $n_{t+1,x} > n_{tx}$, take $n_{t+1,x} - n_{tx}$ additional samples from alternative x. Let $W_{t+1,x}$ and $\widehat{\lambda}^2_{t+1,x}$ be the sample mean and sample variance respectively of all samples from alternative x thus far.
  Increment t.
end while
Select $\hat{x} \in \arg\max_{x\in A} W_{tx}/n_{tx}$ as our estimate of the best.

B.3 Generalization to Continuous Time

We let $(Y_{tx} : t \in \mathbb{R}_+)$ be a Brownian motion under $\mathbb{P}_{\mu,\lambda}$ starting from 0, with drift μ_x, volatility λ_x and independence across x.

B.3.1 Generalization to Continuous Time: The BIZ Procedure with Common Variance

In this section, we assume $\lambda_x^2 = \sigma^2$ for all x, with σ^2 known. To define this procedure, we recursively define the stopping times $0 = \tau_1 \le \tau_2 \le \cdots \le \tau_{k-1} \le \infty$, random variables $Z_1,\ldots,Z_{k-1}$, and $P_0,\ldots,P_{k-1}$, and random sets $A_0, A_1, \ldots, A_{k-1}$. We first define τ_0, A_0 and P_0 as

$$\tau_0 = 0,\ \ A_0 = \{1,\ldots,k\}\,,\ \ , P_0 = P^*.$$

Then, for each $n = 0, 1, \ldots, k-2$, we define

$$\begin{aligned}
\tau_{n+1} &= \inf\left\{t \in \mathbb{R}_+ \bigcap [\tau_n,\infty) : \min_{x\in A_n}\hat{q}_{tx}(A_n) \le c \text{ or} \max_{x\in A_n}\hat{q}_{tx}(A_n) \ge P\right\} \\
Z_{n+1} &\in \arg\min_{x\in A_n}\hat{q}_{\tau_{n+1},x}(A_n) \\
A_{n+1} &= A_n \setminus \{Z_{n+1}\} \\
P_{n+1} &= P_n \big/ \left(1 - \min_{x\in A_n}\hat{q}_{\tau_{n+1},x}(A_n)\right)
\end{aligned}$$

Finally, $\hat{x}$ is the single alternative in A_{k-1}.

We also define

$$M = \inf\left\{n = 1,\ldots,k-1 : \max_{x\in A_{n-1}}\hat{q}_{\tau_n,x}(A_{n-1}) \ge P_{n-1}\right\}.$$

Frazier (2014) showed that $\tau_M = \tau_{M+1} = \cdots = \tau_{k-1}$, and that the one alternative remaining in A_{k-1} is the one whose $\hat{q}_{tx}(A_n)$ satisfied $\hat{q}_{tx}(A_n) \ge P_n$ at time τ_M. He also showed that $\tau_M < \infty$ almost surely under any $\mathbb{P}_{\mu,\sigma}$.

B.3.2 Generalization to Continuous Time: The BIZ Procedure for the Heterogeneous Variance Setting

This procedure is similar than the one given in 3.4. For each x let $n_x(t) = \gamma\lambda_x^2 t$. Now, we define a stochastic process $(Y'_{tx} : t \geq 0)$ as $Y'_{tx} = Y_{n_x(t),x}/\gamma\lambda_x^2$. We have that $(Y'_{tx} : t \geq 0)$ is a Brownian motion with drift μ_x and volatility $1/\gamma$.

The procedure is defined by first defining τ_0, A_0 and P_0 as

$$\tau_0 = 0,\ A_0 = \{1, \ldots, k\},\ ,P_0 = P^*,$$

then defining recursively, for $n = 0, 1, \ldots, k-2$,

$$\begin{aligned}
\tau_{n+1} &= \inf\left\{t \in \mathbb{R}_+ \bigcap [\tau_n, \infty) : \min_{x \in A_n} q'_{tx}(A_n) \leq c \text{ or} \max_{x \in A_n} q'_{tx}(A_n) \geq P\right\} \\
Z_{n+1} &\in \arg\min_{x \in A_n} q'_{\tau_{n+1},x}(A_n) \\
A_{n+1} &= A_n \setminus \{Z_{n+1}\} \\
P_{n+1} &= P_n \Big/ \left(1 - \min_{x \in A_n} q'_{\tau_{n+1},x}(A_n)\right)
\end{aligned}$$

where $q'_{tx}(A)$ is defined by

$$q'_{tx}(A) = \exp\left(\gamma\delta Y'_{tx}\right) \Big/ \sum_{x' \in A} \exp\left(\gamma\delta Y'_{tx'}\right) = \exp\left(\frac{\delta}{\lambda_x^2} Y_{n_x(t),x}\right) \Big/ \sum_{x' \in A} \exp\left(\frac{\delta}{\lambda_{x'}^2} Y_{n_{x'}(t),x'}\right).$$

Frazier (2014) showed that this procedure satisfies the IZ guarantee, and and its worst-case preference-zone PCS bound is tight in continuous time.

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