On the Asymptotic Validity of a Fully Sequential Elimination Procedure for Indifference-Zone Ranking and Selection with Tight Bounds on Probability of Correct Selection

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We prove the validity of the sequential elimination IZ procedure proposed by Frazier [] when δ goes to 0. Specifically, we analyze Algorithm 2, with known sampling variances, and $B_1 = \cdots = B_k = 1$.

Asymptotic Validity when the Variances are Known

Without loss of generality, suppose that the true means of the systems are indexed so that $\mu_k \geq \mu_{k-1} \geq \cdots \geq \mu_1$. We suppose that samples from system $x \in \{1 \ldots, k\}$ are normally distributed and independent, over time and across alternatives. We also suppose that the algorithm ends in at most $R(\delta) \in \mathbb{N}$ iterations, and $R(\delta) \to \infty$ as $\delta \to 0$ with probability 1. To simplify notation, we write R in place of $R(\delta)$, and let the dependence on δ be implicit. Furthermore, $R^{1/2}\delta$ converges to a random variable Δ with probability 1 where $\infty > \Delta > 0$ with probability 1. We also define $\lambda_z^2 := \max_{i\in\{1\ldots,k\}} \lambda_i^2$.

Lemma 1. If $x \in \{1 \ldots, k\}$ and $t \in [0,1]$, then

$$C_x(\delta, t) := \frac{Y_{ceil\left(\frac{\lambda_x^2}{\lambda_z^2}(n_0+tR)\right),x} - \frac{\lambda_x^2}{\lambda_z^2}(n_0+tR)\,\mu_x}{\frac{\lambda_x^2}{\lambda_z}\sqrt{R}} \Rightarrow W_x(t)$$

where $Y_{n,x}$ is the sum of the first n samples and W_x is a standard Brownian motion.

Proof. Let $(a_n) \subset \mathbb{R}_+$ be a sequence, by the Lindeberg-Lévy central limit theorem and the fact that $\frac{ceil(a_n t)}{a_n} \to t$, as $n \to \infty$,

$$\frac{Y_{ceil(a_n t),x} - ceil\,(a_n t)\,(\mu_x)}{\lambda_x \sqrt{a_n}} \Rightarrow W_x\,(t)\,.$$

Consequently, as $n \to \infty$,

$$\frac{Y_{ceil(a_n t),x} - a_n t\,(\mu_x)}{\lambda_x \sqrt{a_n}} \Rightarrow W_x\,(t)$$

because $\frac{nt - ceil(nt)}{\lambda_x \sqrt{n}} \to 0$. Thus, as $\delta \to 0$,

$$\frac{Y_{ceil\left(\frac{\lambda_x^2}{\lambda_z^2} tR\right),x} - \frac{\lambda_x^2}{\lambda_z^2} tR\,(\mu_x)}{\frac{\lambda_x^2}{\lambda_z}\sqrt{R}} \Rightarrow W_x\,(t)\,,$$

and then

$$\frac{Y_{ceil\left(\frac{\lambda_x^2}{\lambda_z^2}(n_0 + tR)\right),x} - \frac{\lambda_x^2}{\lambda_z^2}\,(n_0 + tR)\,\mu_x}{\frac{\lambda_x^2}{\lambda_z}\sqrt{R}} \Rightarrow W_x\,(t)\,.$$

■

Now, we consider the set $D\,[0,1]^k$ which is the set of functions from $[0,1]$ to $\mathbb{R}^k$ that are right-continuous and have left-hand limits. We'll use the Skorokhod metric d on $D\,[0,1]^k$:

$$d\,(X,Y) = \inf_{\lambda \in \Lambda} \{\|\lambda - I\| \vee \|X - Y \circ \lambda\|\}$$

where Λ is the set of strictly increasing, continuous mappings of $[0,1]$ onto itself, and $\|\cdot\|$ is the uniform norm, and I is the identity map.

Suppose Δ is given. Let $Y \in D\,[0,1]^k$ and $t \in [0,1]$, we define

$$q_{tx}^{Y,\delta}\,(A) : \;=\; \exp\left(\delta \beta_{tR} \lambda_z \sqrt{R} \frac{Y_x\,(t)}{(n_0 + tR)}\right) / \sum_{x' \in A} \exp\left(\delta \beta_{tR} \lambda_z \sqrt{R} \frac{Y_{x'}\,(t)}{(n_0 + tR)} + \beta_{tR} \delta^2 I_{x',x}\right)$$

where $\beta_{tR} = \frac{(n_0 + tR)}{\lambda_z^2}$ and

$$I_{x',x} = \begin{cases} 0 & \text{if } x' = x \\ 1 & \text{if } x' > x\,\cdot \\ -1 & \text{if } x' < x \end{cases}$$

We then define

$$
\begin{aligned}
T^{0}_{Y,\delta} &= 0 \\
A^{Y,\delta}_{0} &= \{1,\ldots,k\} \\
P^{Y,\delta}_{0} &= P^{*} \\
T^{n+1}_{Y,\delta} &= \inf\left\{t \in \left[T^{n}_{Y,\delta},1\right] :\ \min_{x\in A^{Y,\delta}_{n}} q^{Y,\delta}_{tx}\left(A^{Y,\delta}_{n}\right) \leq c \text{ or } \max_{x\in A^{Y,\delta}_{n}} q^{Y,\delta}_{tx}\left(A^{Y,\delta}_{n}\right) \geq P^{Y,\delta}_{n}\right\} \\
Z^{Y,\delta}_{n+1} &\in \arg\min_{x\in A^{Y,\delta}_{n}} q^{Y,\delta}_{T^{n+1}_{Y,\delta},x}\left(A^{Y,\delta}_{n}\right) \\
A^{Y,\delta}_{n+1} &= A_{n} - \left\{Z^{Y,\delta}_{n+1}\right\} \\
P^{Y,\delta}_{n+1} &= P^{Y,\delta}_{n} / \left(1 - \min_{x\in A^{Y,\delta}_{n}} q^{Y,\delta}_{T^{n+1}_{Y,\delta}x}\left(A^{Y,\delta}_{n}\right)\right).
\end{aligned}
$$

Now, let

$$
M_{Y,\delta} = \inf\left\{n = 1,\ldots,k-1 : \max_{x\in A^{Y,\delta}_{n-1}} q_{T^{n}_{Y,\delta},x}\left(A^{Y,\delta}_{n}\right) \geq P^{Y,\delta}_{n-1}\right\}
$$

and

$$
f\left(Y,\delta\right) = \begin{cases} 1 & \text{if k} \in A^{Y,\delta}_{M-1} \text{ and } \frac{\lambda^2_k}{\lambda_z}\left(Y_k\left(T^{M_{Y,\delta}}_{Y,\delta}\right)\right) + \delta\frac{\lambda^2_k}{\lambda^2_z\sqrt{R}}\left(n_0 + T^{M_{Y,\delta}}_{Y,\delta}R\right) \geq \frac{\lambda^2_x}{\lambda_z}\left(Y_x\left(T^{M_{Y,\delta}}_{Y,\delta}\right)\right)\ \forall x \in A^{Y,\delta}_{M-1} \\ 0 & \text{otherwise} \end{cases}.
$$

Now, we also define

$$
q^{Y}_{tx}\left(A\right) := \exp\left(\Delta\frac{Y_x\left(t\right)}{\lambda_z}\right) / \sum_{x'\in A} \exp\left(\Delta\frac{Y_{x'}\left(t\right)}{\lambda_z} + t\frac{\Delta^2}{\lambda^2_z} I_{x',x}\right)
$$

$$
\begin{aligned}
T_Y^0 &= 0 \\
A_0^Y &= \{1, \ldots, k\} \\
P_0^Y &= P^* \\
T_Y^{n+1} &= \inf\left\{t \in [T_Y^n, 1] : \ \min_{x \in A_n^Y} q_{tx}^Y \left(A_n^Y\right) \leq c \text{ or } \max_{x \in A_n^Y} q_{tx}^Y \left(A_n^Y\right) \geq P_n^Y\right\} \\
Z_{n+1}^Y &\in \arg\min_{x \in A_n^Y} q_{T_Y^{n+1},x}^Y \left(A_n^Y\right) \\
A_{n+1}^Y &= A_n^Y - \left\{Z_{n+1}^Y\right\} \\
P_{n+1}^Y &= P_n^{Y,\delta} / \left(1 - \min_{x \in A_n^Y,} q_{T_Y^{n+1}x}^Y \left(A_n^Y\right)\right).
\end{aligned}
$$

Now, let

$$
M_Y = \inf\left\{n = 1, \ldots, k-1 : \max_{x \in A_{n-1}^Y} q_{T_Y^n,x} \left(A_n^Y\right) \geq P_{n-1}^Y\right\}
$$

and

$$
g(Y) = \begin{cases} 1 & \text{if k} \in A_{M-1}^Y \text{ and } Y_k\left(T_Y^{M_Y}\right) + \Delta \frac{1}{\lambda_z} T_Y^{M_Y} \geq \frac{\lambda_x^2}{\lambda_k^2}\left(Y_x\left(T_Y^{M_Y}\right)\right) \ \forall x \in A_{M-1}^Y \\ 0 & \text{otherwise} \end{cases}.
$$

Lemma 2. Let $D_s \equiv \{x \in D[0,1]^k :$ for some sequence $\{x_n\} \subset D[0,1]^k$ with $lim_n d(x_n, x) = 0,$ the sequence $\{f_n(x_n)\}$ does not converge to $\{g(x)\}\}$, then $\mathbb{P}\left(W \in \mathrm{D}[0,1]^k\text{-}D_s\right) = 1$.

Proof. Let $\{x_n\} \subset D[0,1]^k$ such that $x_n \to W$. Then there exists $\lambda_n \in \Lambda$ such that $\sup_{t \in [0,1]} \|\lambda_n(t) - t\| \leq d(x_n, W) + \frac{1}{n}$ and $\sup_{t \in [0,1]} \|x_n(t) - W(\lambda_n(t))\| \leq d(x_n, W) + \frac{1}{n}$. Taking $g_n \equiv \sup_{t \in [0,1]} \|W(t) - W(\lambda_n(t))\|$, we see from the uniform continuity of W on $[0,1]^k$ and the definition of g_n that $\lim_{n \to \infty} g_n = 0$. Moreover, if we take $\epsilon_n = 3n^{-1} + 3\sup\{d(x_l, W) + g_l : l = n, n+1, \ldots\}$, then $\{\epsilon_n\}$ is a monotonically decreasing sequence of positive numbers with limit zero.

From the definition of ϵ_n we have $d(x_n, W) < \epsilon_n/2$ and $g_n < \epsilon_n/2$ for $n = 1, 2, \ldots$ Consequently, we have

$$
\begin{aligned}
\|x_n(t) - W(t)\| &\leq \|x_n(t) - W(\lambda_n(t))\| + \|W(\lambda_n(t)) - W(t)\| \\
&< \epsilon_n
\end{aligned}
$$

Consequently,

$$\left\| W\left(T_{x_n,\delta_n}^{M_{x_n,\delta_n}}\right) - x_n\left(T_{x_n,\delta_n}^{M_{x_n,\delta_n}}\right)\right\| < \epsilon_n.$$

We will prove by induction that

$$
\begin{aligned}
T_{x_n,\delta_n}^m &\to T_W^m \\
Z_m^{x_n,\delta_n} &\to Z_m^W \\
A_m^{x_n,\delta_n} &\to A_m^W \\
P_m^{x_n,\delta_n} &\to P_m^W
\end{aligned}
$$

Suppose this is true for m. Observe that

$$T_{W-\epsilon_n e,\delta_n}^m \leq T_{x_n,\delta_n}^m \leq T_{W+\epsilon_n e,\delta_n}^m$$

We have that if $\hat{\tau}_M$ is the continuous version of $\frac{\tau_M}{R}$, then

$$f\left(C\left(\delta,t\right),\delta\right) = \begin{cases} 1 & \text{if k} \in \text{arg max}_{x \in A_{M-1}^{Y,\delta}} \frac{\lambda_x^2}{\lambda_z}\sqrt{R}\left(C_x\left(\delta,\hat{\tau}_M\right) + \frac{\lambda_x^2}{\lambda_z^2}\left(n_0 + \hat{\tau}_M R\right)\mu_x\right) \\ 0 & \text{otherwise} \end{cases}.$$

By lemma 1,

$$C\left(\delta,t\right) \Rightarrow W\left(t\right).$$

Now,

$$
\begin{aligned}
\left|\frac{\lambda_x^2}{\lambda_z n_{tR,x}}\sqrt{R}\delta\beta_{tR}x_{n_x}(t)-\frac{\Delta}{\lambda_z}W_x(t)\right| &\leq \left|\frac{\lambda_x^2}{\lambda_z n_{tR,x}}\sqrt{R}\delta\beta_{tR}x_{n_x}(t)-\frac{\Delta}{\lambda_z}x_{n_x}(t)\right| \\
&\quad +\left|\frac{\Delta}{\lambda_z}W_x(t)-\frac{\Delta}{\lambda_z}x_{n_x}(t)\right| \\
&\leq \frac{\Delta}{\lambda_z}\epsilon_n+\epsilon\left|x_{n_x}(t)\right|,
\end{aligned}
$$

furthermore

$$
\begin{aligned}
\left|\frac{\lambda_x^2}{\lambda_z n_{tR,x}}\sqrt{R}\delta\beta_{tR}x_{n_x}(t)-\delta_n^2\beta_{tR_n}-\frac{\Delta}{\lambda_z}W_x(t)+A(t)\right| &\leq \left|\frac{\lambda_x^2}{\lambda_z n_{tR,x}}\sqrt{R}\delta\beta_{tR}x_{n_x}(t)-\frac{\Delta}{\lambda_z}x_{n_x}(t)\right| \\
&\quad +\left|\frac{\Delta}{\lambda_z}W_x(t)-\frac{\Delta}{\lambda_z}x_{n_x}(t)\right| \\
&\quad +\left|A(t)-\delta_n^2\beta_{tR_n}\right| \\
&\leq \frac{\Delta}{\lambda_z}\epsilon_n+\epsilon\left|x_{n_x}(t)\right|+\epsilon,
\end{aligned}
$$

Now consider T_{x_n,δ_n}. Since $x_{n_i}(t)-W_i(t)<\epsilon_n$ and $W_i(t)-x_{n_i}(t)<\epsilon_n$, consequently

$$T_{W-\epsilon_n e,\delta_n} \leq T_{x_n,\delta_n} \leq T_{W+\epsilon_n e,\delta_n}$$

Observe that

$$t^* = \lim\ \inf_n T_{W-\epsilon_n e,\delta_n} \geq T_W$$

and

$$t_* = \lim\ \sup_n T_{W+\epsilon_n e,\delta_n} \leq T_W.$$

Then

$$t_* \leq T_W \leq t^* \leq \lim\ \inf_n T_{W+\epsilon_n e,\delta_n} \leq t_*$$

thus

$$t_* = t^* = T_W = \lim_n T_{W-\epsilon_n e,\delta_n} = \lim_n T_{W+\epsilon_n e,\delta_n}.$$

Then

$$\lim_n T_{x_n,\delta_n} = T_W$$

and so

$$\lim_n x_{n_i}\left(T_{x_n,\delta_n}\right) = \lim_n W_i\left(T_{x_n,\delta_n}\right) = W_i\left(T_W\right)$$

by the continuity of W_i.

Therefore

$$\lim_n f_n\left(x_n\right) = \lim_n f\left(x_n, \delta_n\right) = g\left(W\right).$$

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By the extension of the CMT (Theorem 5.5 of Billingsley 1968), we have the following corollary.

Corollary 1. We have that if $t \in [0,1]$,

$$f\left(C\left(\delta, t\right), \delta\right) \Rightarrow g\left(W\left(t\right)\right)$$

as $\delta \to 0$.

Theorem. If samples from system $x \in \{1 \ldots, k\}$ are normally distributed and independent, over time and across alternatives, then $\lim_{\delta\to 0} Pr\left\{\text{BIZ selects } k\right\} \geq P*$ provided $\mu_k \geq \mu_{k-1} + \delta$. We also suppose that the algorithm ends in at most $R\left(\delta\right) \in \mathbb{N}$ iterations, and $R(\delta) \to \infty$ as $\delta \to 0$ with probability 1. Furthermore, $\sqrt{R}\delta \to \Delta$ with probability 1 where $\infty > \Delta > 0$ with probability 1. We also suppose $B_1 = \cdots = B_k = 1$.

Proof.

$$
\begin{aligned}
\underline{lim}_{\delta\to 0}\mathbb{P}\left(CS\right) &\geq \underline{lim}_{\delta\to 0}\mathbb{P}\left(f\left(C\left(\delta,t\right),\delta\right)=1\right)\\
&= \underline{lim}_{\delta\to 0}E\left(\mathbb{P}\left(f\left(C\left(\delta,t\right),\delta\right)=1\mid\Delta\right)\right)\\
&= E\left(\underline{lim}_{\delta\to 0}\mathbb{P}\left(f\left(C\left(\delta,t\right),\delta\right)=1\mid\Delta\right)\right)\\
&= E\left(\mathbb{P}\left(g\left(W\right)=1\mid\Delta\right)\right)\\
&= E\left(\mathbb{P}\left(W_k\left(\hat{\tau}_M\right)+\frac{1}{\lambda_z}\hat{\tau}_M\Delta\geq W_x\left(\hat{\tau}_M\right)\ \forall x\in A^Y_{M-1}\mid\Delta\right)\right)\\
&\geq
\end{aligned}
$$