Without loss of generality, suppose that the true means of the systems are indexed so that $\mu_k \geq \mu_{k-1} \geq \cdots \geq \mu_1$. Assume $c > 0$.

Assumption 1 There exist finite constants μ_i and v_i^2 such that the probability of distribution of $C_i(t,r) \equiv \frac{\sum_{j=1}^{floor(rt)} X_{ij} - rt\mu_i}{v_i\sqrt{r}}$ over $D[0,1]$ converges to that of a standard Brownian motion process, $W(t)$, for t on the unit interval, as r increases; i.e.

$$C_i(,r) \Longrightarrow W()$$

as $r \to \infty$.

Note: $D[0,1]$ is the Skorohod space, i.e. it's the space of real-valued functions on $[0,1]$ that are right-continuous and have left-hand limits.

Under Assumption 1, μ_i is the mean, and $v_i^2 = \lim_{r\to\infty} r\mathrm{Var}\left(\bar{X}_i(r)\right)$ where $\bar{X}_i(r)$ is the sample mean of the first r observations from system i.

Lemma 1. For $i \neq l$, if $\mathbf{X}_i$ and $\mathbf{X}_l$ satisfy Assumption 1 and are independent, then there exists a constant v_{il}^2 such that

$$\frac{t_l\left(Y_{\text{floor}(rt_i),i} - rt_i\mu_i\right) - t_i\left(Y_{\text{floor}(rt_l),l} - rt_l\mu_l\right)}{v_{il}\sqrt{r}} \Longrightarrow t_l W_1(t_i) - t_i W_2(t_l)$$

as $r \to \infty$.

Proof. First, note that

$$v_{il}^2 := \lim_{r,s\to\infty} \mathrm{Var}\left(\sqrt{r}\bar{X}_i(r) + \sqrt{s}\bar{X}_l(s)\right) = v_i^2 + v_l^2$$

because of the independence of $\mathbf{X}_i$ and $\mathbf{X}_l$. Now,

$$\begin{aligned}
\frac{t_l\left(Y_{\text{floor}(rt_i),i} - rt_i\mu_i\right) - t_i\left(Y_{\text{floor}(rt_l),l} - rt_l\mu_l\right)}{v_{il}\sqrt{r}} &= \\
t_l\frac{\sum_{j=1}^{\text{floor}(rt_i)} X_{ij} - rt_i\mu_i}{v_{il}\sqrt{r}} - t_i\frac{\sum_{j=1}^{\text{floor}(rt)} X_{lj} - rt_l\mu_l}{v_{il}\sqrt{r}} &= \\
t_l\left(\frac{v_i}{v_{il}}\right)C_i\left(t,r\right) - t_i\left(\frac{v_l}{v_{il}}\right)C_l\left(t,r\right).
\end{aligned}$$

Because we assume that $\mathbf{X}_i$ and $\mathbf{X}_l$ are independent, so are $C_i\left(t,r\right)$ and $C_l\left(t,r\right)$. Assumption 1 implies $C_i\left(,r\right) \Rightarrow W_i\left(\right)$ and $C_l\left(,r\right) \Rightarrow W_l\left(\right)$ where W_i and W_l are independent standard Brownian motion processes. By Theorem 3.2 of Billingsley, $\left(C_i\left(,r\right), C_l\left(,r\right)\right) \Rightarrow \left(W_i\left(\right), W_l\left(\right)\right)$. By the Continuous Mapping Theorem,

$$t_l\left(\frac{v_i}{v_{il}}\right)C_i\left(t,r\right) - t_i\left(\frac{v_l}{v_{il}}\right)C_l\left(t,s\right) \Rightarrow t_l\left(\frac{v_i}{v_{il}}\right)W_i\left(t\right) - t_i\left(\frac{v_l}{v_{il}}\right)W_l\left(t\right).$$

Lemma 2 (Fabian 1974). Let $W(t,\Delta)$ be a Brownian motion process on $[0,\infty)$, with $E\left[W(t,\Delta)\right] = \Delta t$ and $Var\left[W\left(t,\Delta\right)\right] = t$, where $\Delta > 0$. Let

$$\begin{aligned}
L &= -B \\
U &= B
\end{aligned}$$

for some $B > 0$. Let $R = (L, U)$ and let T^* be the first time that $W(t, \Delta) \notin R$. Finally, let A be the event that $W(T^*, \Delta) \leq -B$. Then,

$$\mathbb{P}\{\mathsf{A}\} = \frac{e^{-2B\Delta}}{1 + e^{-2B\Delta}}.$$

Theorem. If samples from system $x \in \{1 \ldots, k\}$ are normally distributed and independent, over time and across alternatives, then $\lim_{\delta \to 0} Pr\{\text{BIZ selects } k\} \geq P*$ provided $\mu_k \geq \mu_{k-1} + \delta$. We also suppose that the algorithm ends in at most $R(\delta) \in \mathbb{N}$ iterations, and $R(\delta) \to \infty$ as $\delta \to 0$ with probability 1. Furthermore, $\sqrt{R}\delta \to \Delta$ with probability 1 where $\infty > \Delta > 0$ with probability 1. Suppose $B_1 = \cdots = B_k = 1$.

Proof. Suppose the variances are known. We begin by considering the case of only two systems, denoted k and i, with $\mu_k \geq \mu_i + \delta*$. Let $A = \{k, i\}$. Let

$$T(\delta) = \min\left\{t \leq R, t \in \mathbb{N} : \min_{x \in A} \hat{q}_{tx}(A) \leq c \text{ or } \max_{x \in A} \hat{q}_{tx}(A) \geq P*\right\}.$$

Thus $T(\delta)$ is the stage at which the procedure terminates.

Now, let $a_t, b_t \in A$ such that $\exp\left(\delta\beta_t \frac{W_{ta_t}}{n_{ta_t}}\right) \geq \exp\left(\delta\beta_t \frac{W_{tb_t}}{n_{tb_t}}\right)$, so

$$\begin{aligned}
\min_{x\in A}\hat{q}_{tx}\left(A\right) &\leq c \\
\Leftrightarrow \min_{x\in A}\exp\left(\delta\beta_t\frac{W_{tx}}{n_{tx}}\right) &\leq c\sum_{x'\in A}\exp\left(\delta\beta_t\frac{W_{tx'}}{n_{tx'}}\right) \\
\Leftrightarrow \exp\left(\delta\beta_t\frac{W_{tb_t}}{n_{tb_t}}\right)(1-c) &\leq c\exp\left(\delta\beta_t\frac{W_{ta_t}}{n_{ta_t}}\right) \\
\Leftrightarrow \exp\left(\delta\beta_t\left(\frac{W_{tb_t}}{n_{tb_t}}-\frac{W_{ta_t}}{n_{ta_t}}\right)\right) &\leq \frac{c}{1-c}
\end{aligned}$$

an

$$\begin{aligned}
\max_{x\in A}\hat{q}_{tx}\left(A\right) &\geq P* \\
\Leftrightarrow \max_{x\in A}\exp\left(\delta\beta_t\frac{W_{tx}}{n_{tx}}\right) &\geq P*\sum_{x'\in A}\exp\left(\delta\beta_t\frac{W_{tx'}}{n_{tx'}}\right) \\
\Leftrightarrow \exp\left(\delta\beta_t\frac{W_{ta_t}}{n_{ta_t}}\right)(1-P^*) &\geq P^*\exp\left(\delta\beta_t\frac{W_{tb_t}}{n_{tb_t}}\right) \\
\Leftrightarrow \exp\left(\delta\beta_t\left(\frac{W_{ta_t}}{n_{ta_t}}-\frac{W_{tb_t}}{n_{tb_t}}\right)\right) &\geq \frac{P^*}{1-P^*},
\end{aligned}$$

thus

$$\begin{aligned}
T\left(\delta\right) &= \min\left\{t\leq R, t\in\mathbb{N}:\exp\left(\delta\beta_t\left|\frac{W_{tk}}{n_{tk}}-\frac{W_{ti}}{n_{ti}}\right|\right)\geq\min\left\{\frac{P^*}{1-P^*},\frac{1-c}{c}\right\}\right\} \\
&= \min\left\{t\leq R, t\in\mathbb{N}:\left|\frac{W_{tk}}{n_{tk}}-\frac{W_{ti}}{n_{ti}}\right|\geq\frac{\log\left[\min\left\{\frac{P^*}{1-P^*},\frac{1-c}{c}\right\}\right]}{\delta\beta_t}\right\}
\end{aligned}$$

Now, let's prove that $n_{ti}\to\infty, n_{tk}\to\infty$ as $t\to\infty$. For each t, $n_{t+1,x} =$

$n_{tx} + B_x$ for some $x \in A$. If $n_{t+1,i} = n_{ti} + B_i$ for a finite number of $t's$, then $n_{tk} \to \infty$ as $t \to \infty$. So, there exists l_0 such that $n_{li}/\hat{\lambda}_{li}^2 > n_{lk}/\hat{\lambda}_{lk}^2$ if $l > l_0$. Since $\hat{\lambda}_{ti}^2 \to \sigma_i^2$ and $\hat{\lambda}_{tk}^2 \to \sigma_k^2$ as $t \to \infty$, then $n_{ti} \to \infty$ as $t \to \infty$. Similarly, we get the same result in the other cases. Thus, $n_{ti} \to \infty, n_{tk} \to \infty$ as $t \to \infty$.

Let $\lambda_r^2 = \min\{\lambda_i^2, \lambda_k^2\}$ and $\lambda_s^2 = \max\{\lambda_i^2, \lambda_k^2\}$. Then $n_{1,s} = n_0 + 1$, $n_{1r} = \max\left\{n_0, ceil\left(\frac{\lambda_r^2}{\lambda_s^2}(n_0+1)\right)\right\}$. Note that $\frac{n_{ts}}{\lambda_s^2} \leq \frac{n_{tr}}{\lambda_r^2}$ for all t, because by induction

$$\frac{n_{t+1,r}}{\lambda_r^2} \geq \frac{\frac{\lambda_r^2}{\lambda_s^2}(n_{ts}+1)}{\lambda_r^2} = \frac{n_{ts}+1}{\lambda_s^2}.$$

So, $n_{tr} = \max\left\{n_{t-1,r}, ceil\left(\frac{\lambda_r^2}{\lambda_s^2}(n_{t-1,s}+1)\right)\right\} = \max\left\{n_{t-1,r}, ceil\left(\frac{\lambda_r^2}{\lambda_s^2}(n_0+t)\right)\right\}$. Thus there exists t_0 such that $n_{t_0,r} = ceil\left(\frac{\lambda_r^2}{\lambda_s^2}(n_0+t_0)\right)$ and $n_{tr} = n_0$ if $t < t_0$. Observe that if $t > t_0$, $n_{tr} = ceil\left(\frac{\lambda_r^2}{\lambda_s^2}(n_0+t)\right)$. Furthermore, $n_{ts} = n_0 + t$.

Suppose $i = s$ or $i = r$ and $t_0 = 1$. If $0 \leq t \leq 1$, let

$$\begin{aligned}
D_i(t,R) &= \frac{\sum_{j=1}^{n_{floor(tR),i}} X_{ij} - (n_0 + (tR))\mu_i}{v_i\sqrt{R}} \\
&= \frac{\sum_{j=1}^{n_0+floor(tR)} X_{ij} - (n_0 + (tR))\mu_i}{v_i\sqrt{R}} \\
&= \frac{\sum_{j=1}^{n_0} X_{ij} - n_0\mu_i}{v_i\sqrt{R}} + \frac{\sum_{j=1}^{floor(tR)} X_{i,n_0+j} - (tR)\mu_i}{v_i\sqrt{R}} \\
&\Rightarrow W_i().
\end{aligned}$$

Suppose $\lambda_i^2 \leq \lambda_k^2$. Assume $t_0 = 1$. Let

$$\begin{aligned}
D_{ik}(t,R) &= \frac{n_{tR,i}\left(\sum_{j=1}^{n_{floor(tR),k}} X_{kj} - (n_0 + (tR))\,\mu_k\right)}{v_{ik}R\sqrt{R}} \\
&\quad - \frac{n_{tR,k}\left(\sum_{j=1}^{n_{floor(tR),i}} X_{kj} - ceil\left(\frac{\lambda_i^2}{\lambda_k^2}(n_0 + (tR))\right)\mu_i\right)}{v_{ik}R\sqrt{R}} \\
&= \frac{ceil\left(\frac{\lambda_i^2}{\lambda_k^2}(n_0 + tR)\right)\left(\sum_{j=1}^{(n_0 + floor(tR))} X_{kj} - (n_0 + (tR))\,\mu_k\right)}{v_{ik}R\sqrt{R}} \\
&\quad - \frac{(n_0 + tR)\left(\sum_{j=1}^{ceil\left(\frac{\lambda_i^2}{\lambda_k^2}(n_0 + floor(tR))\right)} X_{kj} - ceil\left(\frac{\lambda_i^2}{\lambda_k^2}(n_0 + (tR))\,\mu_i\right)\right)}{v_{ik}R\sqrt{R}} \\
&\Rightarrow \frac{v_k}{v_{ik}}\frac{\lambda_i^2}{\lambda_k^2} tW_k(t) - \frac{v_i}{v_{ik}} tW_i(t) = tW().
\end{aligned}$$

Now, let $N = \max\{n_{Rk}, n_{Ri}\}$, then

$$\begin{aligned}
\frac{W_{T(\delta)k}}{n_{T(\delta)k}} &< \frac{W_{T(\delta)i}}{n_{T(\delta)i}} \\
\Leftrightarrow \frac{\mu_k - \mu_i}{R^{3/2}v_{ik}}\left(n_{T(\delta)k}n_{T(\delta)i}\right) &+ \frac{n_{T(\delta)i}\left(\sum_{j=1}^{n_{T(\delta)k}} X_{kj} - n_{T(\delta)k}\mu_k\right) - n_{T(\delta)k}\left(\sum_{j=1}^{n_{T(\delta)i}} X_{ij} - n_{T(\delta)i}\mu_i\right)}{R^{3/2}v_{ik}} < 0.
\end{aligned}$$

Then

$$\begin{aligned}
\mathbb{P}\left\{\frac{W_{T(\delta)k}}{n_{T(\delta)k}} < \frac{W_{T(\delta)i}}{n_{T(\delta)i}}\right\} &\leq \mathbb{P}\left\{\frac{n_{T(\delta)i}\left(\sum_{j=1}^{n_{T(\delta)k}} X_{kj} - n_{T(\delta)k}\mu_k\right) - n_{T(\delta)k}\left(\sum_{j=1}^{n_{T(\delta)i}} X_{ij} - n_{T(\delta)i}\mu_i\right)}{R^{3/2}v_{ik}}\right. \\
&\quad \left. + \frac{\delta}{R^{3/2}v_{ik}}\left(n_{T(\delta)k}n_{T(\delta)i}\right) < 0\right\} \\
&= E\left[\mathbb{P}\left\{\frac{n_{T(\delta)i}\left(\sum_{j=1}^{n_{T(\delta)k}} X_{kj} - n_{T(\delta)k}\mu_k\right) - n_{T(\delta)k}\left(\sum_{j=1}^{n_{T(\delta)i}} X_{ij} - n_{T(\delta)i}\mu_i\right)}{R^{3/2}v_{ik}}\right.\right. \\
&\quad \left.\left. + \frac{\delta}{R^{3/2}v_{ik}}\left(n_{T(\delta)k}n_{T(\delta)i}\right) < 0\right\} \mid \Delta\right]
\end{aligned}$$

Define

$$\begin{aligned}
\hat{T}(\delta) &= \min\left\{t \in \left\{\frac{1}{R}, \ldots, 1\right\} : \left|D_{ik}(t,\delta) + \frac{ceil\left(\frac{\lambda_i^2}{\lambda_k^2}(n_0+tR)\right)(n_0+tR)\delta}{R^{3/2}v_{ik}}\right|\right. \\
&\quad \left. \geq \frac{ceil\left(\frac{\lambda_i^2}{\lambda_k^2}(n_0+tR)\right)(n_0+tR)\log\left[\min\left\{\frac{P^*}{1-P^*}, \frac{1-c}{c}\right\}\right]}{R^{3/2}v_{ik}\beta_{floor(tR)}\delta}\right\}
\end{aligned}$$

where $\beta_{tR} = \frac{n_0+tR+ceil\left(\frac{\lambda_i^2}{\lambda_k^2}(n_0+tR)\right)}{\lambda_i^2+\lambda_k^2}$.

Clearly, $\hat{T}(\delta) = T(\delta)/R$. Also, define the stopping time of the corresponding continuous-time process as

$$\begin{aligned}
\tilde{T}(\delta) &= \min\left\{1 \geq t \geq \frac{1}{R} : \left|D_{ik}(t,\delta) + \frac{ceil\left(\frac{\lambda_i^2}{\lambda_k^2}(n_0+tR)\right)(n_0+tR)\delta}{R^{3/2}v_{ik}}\right|\right. \\
&\quad \left. \geq \frac{ceil\left(\frac{\lambda_i^2}{\lambda_k^2}(n_0+tR)\right)(n_0+tR)\log\left[\min\left\{\frac{P^*}{1-P^*}, \frac{1-c}{c}\right\}\right]}{R^{3/2}v_{ik}\beta_{floor(tR)}\delta}\right\}
\end{aligned}$$

Note that for fixed δ, $D_{ik}\left(\hat{T}(\delta),\delta\right)$ corresponds to the right-hand limit of a point of discontinuity of $D_{ik}(,\delta)$. We can show that $\hat{T}(\delta) \to \tilde{T}(\delta)$ with probability 1 as $\delta \to 0$, making use of the fact that $1/R \to 0$ with probability 1. Thus, in the limit, we can focus on $D_{ik}\left(\tilde{T}(\delta),\delta\right)$.

Now, condition on Δ. By Assumption 1, Lemma 1, and the CMT we have that

$$
\begin{aligned}
D_{ik}(t,\delta) + \frac{ceil\left(\frac{\lambda_i^2}{\lambda_k^2}(n_0+tR)\right)(n_0+tR)\delta}{R^{3/2}v_{ik}} &\Rightarrow tW(t) + t^2\frac{\lambda_i^2}{\lambda_k^2}\Delta. \\
&= t\left(W(t) + t\frac{\lambda_i^2}{\lambda_k^2}\Delta\right)
\end{aligned}
$$

Let

$$
\begin{aligned}
A(\delta) &= \frac{n_0 ceil\left(\frac{\lambda_i^2}{\lambda_k^2}(n_0+tR)\right)\log\left[\min\left\{\frac{P^*}{1-P^*},\frac{1-c}{c}\right\}\right]}{R^{3/2}v_{ik}\beta_{tR}\delta} \\
&= \frac{\left(\lambda_i^2+\lambda_k^2\right)n_0 ceil\left(\frac{\lambda_i^2}{\lambda_k^2}(n_0+tR)\right)\log\left[\min\left\{\frac{P^*}{1-P^*},\frac{1-c}{c}\right\}\right]}{\left(n_0+tR+ceil\left(\frac{\lambda_i^2}{\lambda_k^2}(n_0+tR)\right)\right)R^{3/2}v_{ik}\delta} \xrightarrow{\delta\to 0} 0
\end{aligned}
$$

and

$$
\begin{aligned}
tB(\delta) &= \frac{(\lambda_i^2+\lambda_k^2)\, ceil\left(\frac{\lambda_i^2}{\lambda_k^2}(n_0+tR)\right)(tR)\log\left[\min\left\{\frac{P^*}{1-P^*},\frac{1-c}{c}\right\}\right]}{\left(n_0+tR+ceil\left(\frac{\lambda_i^2}{\lambda_k^2}(n_0+tR)\right)\right)R^{3/2}v_{ik}\delta} \\
&= \frac{(\lambda_i^2+\lambda_k^2)\, ceil\left(\frac{\lambda_i^2}{\lambda_k^2}(n_0+tR)\right)(t)\log\left[\min\left\{\frac{P^*}{1-P^*},\frac{1-c}{c}\right\}\right]}{\left(n_0+tR+ceil\left(\frac{\lambda_i^2}{\lambda_k^2}(n_0+tR)\right)\right)R^{1/2}v_{ik}\delta} \\
&\xrightarrow[\delta\to 0]{} \frac{(\lambda_i^2+\lambda_k^2)\frac{\lambda_i^2}{\lambda_k^2}(t)\log\left[\min\left\{\frac{P^*}{1-P^*},\frac{1-c}{c}\right\}\right]}{\left(1+\frac{\lambda_i^2}{\lambda_k^2}\right)v_{ik}\Delta} = Bt.
\end{aligned}
$$

Note that the stopping time $\tilde{T}(\delta)$ is the first time t at which the event

$$
\left\{\left|D_{ik}(t,\delta)+\frac{ceil\left(\frac{\lambda_i^2}{\lambda_k^2}(n_0+tR)\right)(n_0+tR)\,\delta}{R^{3/2}v_{ik}}\right|-A(\delta)-tB(\delta)\geq 0\right\}
$$

occurs. Define the mapping $s_\delta : D[0,1]\to\mathbb{R}$ such that $s_\delta(Y)=Y(T_{Y,\delta})$, where

$$
T_{Y,\delta}=\inf\{t:|Y(t)|-A(\delta)-B(\delta)\,t\geq 0\}
$$

for every $Y\in D[0,1]$ and $\delta>0$. Similarly, define $s(Y)=Y(T_Y)$, where

$$
T_Y=\inf\{t>0:|Y(t)|-Bt\geq 0\}
$$

for every $Y \in D[0,1]$. Note that

$$s_\delta\left(D_{ik}(t,\delta)+\frac{ceil\left(\frac{\lambda_i^2}{\lambda_k^2}(n_0+tR)\right)(n_0+tR)\delta}{R^{3/2}v_{ik}}\right) = D_{ik}\left(\tilde{T}(\delta),\delta\right) + \frac{ceil\left(\frac{\lambda_i^2}{\lambda_k^2}\left(n_0+\tilde{T}(\delta)R\right)\right)\left(n_0+\tilde{T}(\delta)R\right)\delta}{R^{3/2}v_{ik}},$$

$s(tW(t,\Delta)) = T_{\mathfrak{W}()}W\left(T_{\mathfrak{W}()},\Delta\right)$ where $\mathfrak{W}(t) = tW(t,\Delta)$.

We need to show that

$$s_\delta(G_{ik}(t,\delta)) \Rightarrow s(tW(t,\Delta))$$

as $\delta \to 0$, where

$$G_{ik}(t,\delta) \equiv D_{ik}(t,\delta)+\frac{ceil\left(\frac{\lambda_i^2}{\lambda_k^2}(n_0+tR)\right)(n_0+tR)\delta}{R^{3/2}v_{ik}}$$

for $t \in [0,1]$ and $\delta > 0$. If $\mathbb{P}(tW(t,\Delta)\in \mathrm{D}[0,1]\text{-}D_s)) = 1$ where $D_s \equiv \{x \in D[0,1] :$ for some sequence $\{x_n\} \subset D[0,1]$ with $lim_n d(x_n,x) = 0,$ the sequence $\{s_{\delta_n}(x_n)\}$ does not converge to $s(x)\}$, and $d(X,Y)$ is the infimum of those positive w for which there exists $\lambda \in \Lambda$ such that $sup_{t\in[0,1]}|X(t)-Y(\lambda(t))| \leq w$ and $sup_{t\in[0,1]}|\lambda(t)-t| \leq w$ (Λ is the class of strictly increasing, continuous mappings of $[0,1]$ onto itself such that for every $\lambda \in \Lambda$, we have $\lambda(0) = 0$ and $\lambda(1) = 1$), then by Theorem 5.5 of Billingsley 1968 we conclude that this is true. By Kim et al. (2005), we know that $\mathbb{P}(W(t,\Delta)\in \mathrm{D}[0,1]\text{-}D_s)) = 1$, thus it follows this is true (prove).

Now, unconditioning on Δ gives

$$\begin{aligned}
\lim sup_{\delta\to 0}\mathbb{P}\left(ICS\right) &\leq E\left[\mathbb{P}\left[\tilde{T}\left(\delta\right)W\left(\tilde{T}\left(\delta\right),\frac{\lambda_i^2}{\lambda_k^2}\Delta\right)<0\mid\Delta\right]\right]\\
&= E\left[\frac{e^{-2B\Delta\frac{\lambda_i^2}{\lambda_k^2}}}{1+e^{-2B\Delta\frac{\lambda_i^2}{\lambda_k^2}}}\right]\\
&= E\left[\frac{e^{-\frac{2\frac{\lambda_i^4}{\lambda_k^2}\log\left[\min\left\{\frac{P^*}{1-P^*},\frac{1-c}{c}\right\}\right]}{v_{ik}}}}{1+e^{-\frac{2\frac{\lambda_i^4}{\lambda_k^2}\log\left[\min\left\{\frac{P^*}{1-P^*},\frac{1-c}{c}\right\}\right]}{v_{ik}}}}\right]\\
&= E\left[\frac{1}{1+e^{\frac{2\frac{\lambda_i^4}{\lambda_k^2}\log\left[\min\left\{\frac{P^*}{1-P^*},\frac{1-c}{c}\right\}\right]}{v_{ik}}}}\right]\\
&= E\left[\frac{1}{1+\left(\min\left\{\frac{P^*}{1-P^*},\frac{1-c}{c}\right\}\right)^{2\frac{\lambda_i^4}{\lambda_k^2 v_{ik}}}}\right]\\
&\leq \frac{1}{1+\left(\frac{P^*}{1-P^*}\right)^{2\frac{\lambda_i^4}{\lambda_k^2 v_{ik}}}}\leq\left(1-P^*\right)\leq\left(1-P^*\right)^{1/(k-1)}
\end{aligned}$$

the last equality follows because, if $a = 2\frac{\lambda_i^4}{\lambda_k^2 v_{ik}}$,

$$\begin{aligned}
\frac{1}{1+\left(\frac{P^*}{1-P^*}\right)^a} &\leq 1-P^* \\
\Leftrightarrow \frac{(1-P^*)^a}{(1-P^*)^a + P^{*a}} &\leq 1-P^* \\
\Leftrightarrow (1-P^*)^a &\leq (1-P^*)^a + P^{*a} \\
\Leftrightarrow (1-P^*)^a (1-1+P^*) &\leq P^{*a} \\
\Leftrightarrow (1-P^*)^{a-1} &\leq P^{*(a-1)} \\
\Leftrightarrow 1 &\leq 2P^*.
\end{aligned}$$

PROOF OF THE GENERAL CASE. First, suppose $c = 0$, $\lambda_k = max_i \{\lambda_i\}$.

Let

$$\begin{aligned}
C_x(\delta, t) &= \frac{Y_{floor(t/\delta),x} - t\frac{1}{\delta}\mu_x}{\sqrt{1/\delta}\lambda_x} \\
&= \frac{\delta Y_{floor(t/\delta),x} - t\mu_x}{\sqrt{\delta}\lambda_x} \\
&\Rightarrow W_x(t)
\end{aligned}$$

and

$$\mathfrak{C}(\delta) = \left(C_x(\delta, t) : t \in [0, R(\delta)]; x \in A\right),$$

then

$$\mathfrak{C}(\delta) \Rightarrow (W_x(t) : t \in [0, R(\delta)]; x \in A)$$

Let

$$f\left(\mathfrak{C}\left(\delta\right)\right)=\begin{cases}1 & \text{correct selection}\\ 0 & \text{otherwise}\end{cases}.$$

We would like to prove that

$$\mathbb{P}\left(f\left(\mathfrak{C}\left(\delta\right)\right)=1\right)\rightarrow\mathbb{P}\left(f\left(W\right)=1\right)$$

as δ goes to 0 and W is a standard Brownian motion process in $\mathbb{R}^k$.

Define the mapping $s_\delta : D\left[0,1\right]\rightarrow\mathbb{R}$ such that $s_\delta\left(Y\right)=Y\left(T_{Y,\delta}\right)$, where

$$T_{Y,\delta}=\inf\left\{t:\left|Y\left(t\right)\right|-A\left(\delta\right)-B\left(\delta\right)t\geq 0\right\}$$

for every $Y\in D\left[0,1\right]$ and $\delta>0$.

Define the mapping $g_\delta\left(Y\right))=exp\left(\left(Y\left(t\delta\right)\sqrt{1/\delta}\lambda_x^2+t\mu_x\right)\beta_t/n_{tx}\right)/\sum_{x'\in A}exp\left(\left(Y\left(t\delta\right)\sqrt{1/\delta}\lambda\right.\right.$

$$g_\delta\left(C_x\left(\delta,t\right)\right)=C_x\left(\delta,t\delta\right)$$

$$\begin{aligned}\hat{q}_{tx}\left(A\right) &= \exp\left(\delta\beta_t\frac{W_{tx}}{n_{tx}}\right)/\sum_{x'\in A}exp\left(\delta\beta_t\frac{W_{tx'}}{n_{tx'}}\right)\\ &= exp\left(\left(C_x\left(\delta,t\delta\right)\sqrt{1/\delta}\lambda_x^2+t\mu_x\right)\beta_t/n_{tx}\right)/\sum_{x'\in A}exp\left(\left(C_{x'}\left(\delta,t\delta\right)\sqrt{1/\delta}\lambda_{x'}^2+t\mu_{x'}\right)\beta_t/n_{tx'}\right.\\ &= g_\delta\left(C_x\left(\delta,t\right)\right)\end{aligned}$$

Let

$$
\begin{aligned}
\hat{T}(\delta) &= \min\left\{t \in \left\{\frac{1}{R},\ldots,1\right\} : max_x q_{tR,x}(A) \geq P\right\}. \\
&= \min\left\{t \in \left\{\frac{1}{R},\ldots,1\right\} : max_x exp\left(\delta\beta_{tR}\frac{W_{tR,x}}{n_{tR,x}}\right) \geq P\sum_{x'\in A} exp\left(\delta\beta_{tR}\frac{W_{tR,x'}}{n_{tR,x'}}\right)\right\} \\
&= \min\left\{t \in \left\{\frac{1}{R},\ldots,1\right\} : \exists x \text{ s.t. } \left(\delta\beta_{tR}\frac{Y_{n_{tR,x}}}{n_{tR,x}}\right) \geq log\left(P\sum_{x'\in A} exp\left(\delta\beta_{tR}\frac{W_{n_{tR,x'}}}{n_{tR,x'}}\right)\right)\right\} \\
&= \min\left\{t \in \left\{\frac{1}{R},\ldots,1\right\} : \exists x \text{ s.t. } \left(\frac{Y_{n_{tR,x}-\left(n_{tR,x}\right)\mu_X}}{\lambda_x\sqrt{R}}\right) \geq \right. \\
& \frac{n_{tR,x}}{\lambda_x\sqrt{R}} \\
& \left(log\left(P\sum_{x'\in A} exp\left(\left(\frac{\lambda_{x'}\sqrt{R}}{n_{tR,x'}}\left(\delta\beta_{tR}\frac{Y_{n_{tR,x'}-\left(n_{tR,x'}\right)\mu_{X'}}}{\lambda_{x'}\sqrt{R}}\right) + \delta\beta_{tR}\mu_{x'}\right)\right)\right) - \mu_x\delta\beta_{tR}\right) / (\delta\beta_{t
\end{aligned}
$$

and

$$T(\delta) = \hat{T}(\delta)R$$

Now, we know that

$$\frac{Y_{n_{tR,x}-\left(n_{tR,x}\right)\mu_X}}{\lambda_x\sqrt{R}} \Rightarrow W_x(t)$$

as $\delta \to 0$.

Now, if $\lambda_r^2 = \max_{i\in A-\{k\}}\left\{\lambda_i^2\right\}$, then \lim inf_{\delta\rightarrow0}

$$
\begin{aligned}
\mathbb{P}\left(f\left(\mathfrak{C}\left(\delta\right)\right)=1\right) &= \mathbb{P}\left(\frac{W_{Tk}}{n_{Tk}} \geq \frac{W_{Tx}}{n_{Tx}}\ \forall x \in A\right) \\
&= E\left[\prod_{i\in A-\{k\}} \mathbb{P}\left\{\frac{\left(\sum_{j=1}^{n_{T(\delta)k}} X_{kj} - n_{T(\delta)k}\mu_k\right)}{n_{T(\delta)k}} - \frac{\lambda_i^2\left(\sum_{j=1}^{n_{T(\delta)i}} X_{ij} - n_{T(\delta)i}\mu_i\right)}{\lambda_i^2 n_{T(\delta)i}}\right.\right. \\
&\quad \left.\left. +\delta > 0\right\} \mid \Delta, X_{k1}, \ldots, X_{kn_{Tk}}, T\}\right] \\
&\geq E\left[\prod_{i\in A-\{k\}} \mathbb{P}\left\{\frac{\left(\sum_{j=1}^{n_{T(\delta)k}} dX_{kj} - n_{T(\delta)k}\mu_k\right)}{n_{T(\delta)k}} - \frac{\lambda_r^2\left(\sum_{j=1}^{n_{T(\delta)i}} X_{ij} - n_{T(\delta)i}\mu_i\right)}{\lambda_i^2 n_{T(\delta)i}}\right.\right. \\
&= \left.\left. +\delta > 0\right\} \mid \Delta, X_{k1}, \ldots, X_{kn_{Tk}}, T\}\right] \\
&= E\left[\left(\mathbb{P}\left\{\frac{\left(\sum_{j=1}^{n_{T(\delta)k}} X_{kj} - n_{T(\delta)k}\mu_k\right)}{n_{T(\delta)k}} - \frac{\left(\sum_{j=1}^{n_{T(\delta)r}} X_{rj} - n_{T(\delta)r}\mu_r\right)}{n_{T(\delta)r}}\right.\right.\right. \\
&= \left.\left.\left. +\delta > 0\right\} \mid \Delta, X_{k1}, \ldots, X_{kn_{Tk}}\ , T\}\right)^{k-1}\right] \\
&= E\left[\left(\mathbb{P}\left\{\frac{n_{T(\delta)r}\left(\sum_{j=1}^{n_{T(\delta)k}} X_{kj} - n_{T(\delta)k}\mu_k\right) - n_{T(\delta)k}\left(\sum_{j=1}^{n_{T(\delta)r}} X_{rj} - n_{T(\delta)r}\mu_r\right)}{R^{3/2} v_{rk}}\right.\right.\right. \\
&\quad \left.\left.\left. +\frac{\delta}{R^{3/2} v_{rk}}\left(n_{T(\delta)k} n_{T(\delta)r}\right) > 0\right\} \mid \Delta, X_{k1}, \ldots, X_{kn_{Tk}}, T\right)^{k-1}\right] \\
&\geq E\left[\left(\mathbb{P}\left\{\frac{n_{T(\delta)r}\left(\sum_{j=1}^{n_{T(\delta)k}} X_{kj} - n_{T(\delta)k}\mu_k\right) - n_{T(\delta)k}\left(\sum_{j=1}^{n_{T(\delta)r}} X_{rj} - n_{T(\delta)r}\mu_r\right)}{R^{3/2} v_{rk}}\right.\right.\right. \\
&\quad \left.\left.\left. +\frac{\delta}{R^{3/2} v_{rk}}\left(n_{T(\delta)k} n_{T(\delta)r}\right) > 0\right\} \mid \Delta, X_{k1}, \ldots, X_{kn_{Tk}}, T\right)\right]^{k-1} \\
&= E\left[\left(\mathbb{P}\left\{\frac{n_{T(\delta)r}\left(\sum_{j=1}^{n_{T(\delta)k}} X_{kj} - n_{T(\delta)k}\mu_k\right) - n_{T(\delta)k}\left(\sum_{j=1}^{n_{T(\delta)r}} X_{rj} - n_{T(\delta)r}\mu_r\right)}{R^{3/2} v_{rk}}\right.\right.\right. \\
&\quad \left.\left.\left. +\frac{\delta}{R^{3/2} v_{rk}}\left(n_{T(\delta)k} n_{T(\delta)r}\right) > 0\right\} \mid \Delta\right]^{k-1}\right. \\
&\geq
\end{aligned}
$$

Now,

Suppose $c > 0$.

(I haven't erased the following in case I need it later)

Observe that,

$$\mu_k - \mu_i + \frac{v_k}{\sqrt{n_{T(\delta)k}}} C_k\left(1, n_{T(\delta)k}\right) - \frac{v_i}{\sqrt{n_{T(\delta)i}}} C_i\left(1, n_{T(\delta)i}\right) \Rightarrow \mu_k - \mu_i$$

when $\delta \to 0$ if $n_{T(\delta)j}$ is independent of $C_j(1, n)$ for n sufficiently large and $j = k, i$.

Suppose $i = r$ and $t_0 > 1$. Let $s_0 \leq t \leq 1$ where $floor(s_0 R) \geq t_0$ and $floor(sR) < t_0$ if $s < s_0$. Let

$$\begin{aligned} D_i(t, R) &= \frac{\sum_{j=1}^{n_{floor(tR),i}} X_{ij} - \frac{\lambda_i^2}{\lambda_k^2}(n_0 + (tR))\mu_i}{v_i\sqrt{R}} \\ &= \frac{\sum_{j=1}^{ceil\left(\frac{\lambda_i^2}{\lambda_k^2}(n_0 + floor(tR))\right)} X_{ij} - (n_0 + (tR))\mu_i}{v_i\sqrt{R}} \end{aligned}$$

and if $t < s_0$

$$\begin{aligned} D_i(t, R) &= \frac{\sum_{j=1}^{n_{floor(tR),i}} X_{ij} - n_0\mu_i}{v_i\sqrt{R}} \\ &= \frac{\sum_{j=1}^{n_0} X_{ij} - n_0\mu_i}{v_i\sqrt{R}}, \end{aligned}$$

Suppose $D_i(t, R) \Rightarrow W()$ if $t \geq s_0$.

$$D_{ik}(t,R) = \frac{\sum_{j=1}^{n}}{v_{ik}\sqrt{R}}$$

By Assumption 1, Lemma 1, and the CMT we have that

$$C_{ik}\left(\frac{n_{t,i}}{N}, \frac{n_{t,k}}{N}, \delta\right) + \frac{n_{t,k}n_{t,i}\delta}{N^{1/2}v_{ik}} \Rightarrow n_{ti}W_1(n_{ti}) - n_{tk}W_2(n_{tk}) + \frac{n_{t,k}n_{t,i}\Delta}{v_{ik}}.$$

Let

$$A(\delta) = \frac{\log\left[\min\left\{\frac{P^*}{1-P^*}, \frac{1-c}{c}\right\}\right]}{N^{1/2}v_{ik}\delta} \to \frac{\log\left[\min\left\{\frac{P^*}{1-P^*}, \frac{1-c}{c}\right\}\right]}{N^{1/2}v_{ik}\delta}$$

$$\beta_t = \sum$$

Observe that,

$$\mu_k - \mu_i + \frac{v_k}{\sqrt{n_{T(\delta)k}}}C_k\left(1, n_{T(\delta)k}\right) - \frac{v_i}{\sqrt{n_{T(\delta)i}}}C_i\left(1, n_{T(\delta)i}\right) \Rightarrow \mu_k - \mu_i$$

when $\delta \to 0$ if $n_{T(\delta)j}$ is independent of $C_j(1,n)$ for n sufficiently large and $j = k, i$.

Let ICS denote the even that an incorrect selection is made. Then,

$$\begin{aligned}
\lim\ \inf_{\delta\to 0}\mathbb{P}\left\{ICS\right\} &= \lim\ \inf_{\delta\to 0}\mathbb{P}\left\{\frac{W_{T(\delta)k}}{n_{T(\delta)k}} < \frac{W_{T(\delta)i}}{n_{T(\delta)i}}\right\} \\
&= \mathbb{P}\left\{\mu_k < \mu_i\right\} = 0.
\end{aligned}$$

However, it's very likely $n_{T(\delta)k}$ is not independent of $C_k\left(1,n\right)$. Thus, let's give other proof. First, we'll show that n_{ti} is independent of $C_i(1,n)$ if $n \geq 1$ and $t \geq 0$. Note that

$$\begin{aligned}
C_i\left(1,n\right) &= \frac{\sqrt{n}\sum_{j=1}^{n} X_{ij}}{v_i n} - \frac{\sqrt{n}}{v_i}\mu_i \\
&= \frac{\sqrt{n}\bar{X}_i\left(n\right)}{v_i} - \frac{\sqrt{n}}{v_i}\mu_i.
\end{aligned}$$

Since $n_{0i} = n_0$, then $C_i\left(1,n\right)$ is independent of n_{0i}. Note that

$$n_{1,i} = \text{ceil}\left(\hat{\lambda^2_{0i}}\left(n_0 + B_z\right)/\hat{\lambda^2_{0z}}\right),$$

if $z = i$, then $n_{1,i}$ is independent of $C_i\left(1,n\right)$. Suppose $z = k$. Note that

$$\begin{aligned}
\hat{\lambda_{1i}^2} &= \frac{1}{n-1}\sum_{j=1}^{n_{1i}}\left(X_{ij}-\bar{X}_i\left(n_{1i}\right)\right)^2 \\
&= \frac{1}{n-1}\left(\left(X_{i1}-\bar{X}_i\left(n_{1i}\right)\right)^2+\sum_{j=2}^{n_{ti}}\left(X_{ij}-\bar{X}_i\left(n_{1i}\right)\right)^2\right) \\
&= \frac{1}{n-1}\left(\left[\sum_{j=2}^{n_{1i}}\left(X_{ij}-\bar{X}_i\left(n_{1i}\right)\right)^2\right]+\sum_{j=2}^{n_{1i}}\left(X_{ij}-\bar{X}_i\left(n_{1i}\right)\right)^2\right).
\end{aligned}$$

Then $\hat{\lambda}_{1i}^2$ can be written as a function only of $\left(X_{i2}-\bar{X}_i\left(n_{0i}\right), \ldots, X_{in_{ti}}-\bar{X}_i\left(n_{ti}\right), n_{ti}\right)$.

Suppose $C_i\left(1,n\right)$ is independent of n_{ti}. Observe that

$$n_{t+1,i} = \text{ceil}\left(\hat{\lambda_{ti}^2}\left(n_{tz}+B_z\right)/\hat{\lambda_{tz}^2}\right),$$

if $z=i$, then $n_{t+1,i}=\text{ceil}\left(\left(n_{tz}+B_z\right)\right)$ and $C_i\left(1,n\right)$ are independent random variables. Now suppose $z=k$. Note that

$$\begin{aligned}
\hat{\lambda_{ti}^2} &= \frac{1}{n-1}\sum_{j=1}^{n_{ti}}\left(X_{ij}-\bar{X}_i\left(n_{ti}\right)\right)^2 \\
&= \frac{1}{n-1}\left(\left(X_{i1}-\bar{X}_i\left(n_{ti}\right)\right)^2+\sum_{j=2}^{n_{ti}}\left(X_{ij}-\bar{X}_i\left(n_{ti}\right)\right)^2\right) \\
&= \frac{1}{n-1}\left(\left[\sum_{j=2}^{n_{ti}}\left(X_{ij}-\bar{X}_i\left(n_{ti}\right)\right)\right]^2+\sum_{j=2}^{n_{ti}}\left(X_{ij}-\bar{X}_i\left(n_{ti}\right)\right)^2\right).
\end{aligned}$$

Then $\hat{\lambda_{ti}^2}$ can be written as a function only of $\left(X_{i2} - \bar{X}_i\left(n_{ti}\right), \ldots, X_{in_{ti}} - \bar{X}_i\left(n_{ti}\right), n_{ti}\right)$. Suppose n_{ti} is given. The joint pdf of the sample $X_{i1}, \ldots, X_{in_{ti}}$ is given by

$$f\left(x_1, \ldots, x_{n_{ti}}\right) = \frac{1}{\left(2\pi\right)^{n_{ti}/2}} e^{-(1/2)\sum_{i=1}^{n_{ti}} x_i^2}.$$

Make the transformation $y_1 = \bar{x}\left(n_{ti}\right), y_2 = x_2 - \bar{x}\left(n_{ti}\right), \ldots, y_{n_{ti}} = x_{n_{ti}} - \bar{x}\left(n_{ti}\right)$. This is a linear transformation with a Jacobian equal to $1/n_{ti}$. We have

$$\begin{aligned} f\left(y_1, \ldots, y_{n_{ti}}\right) &= \frac{n_{ti}}{\left(2\pi\right)^{n_{ti}/2}} e^{-(1/2)\left(y_1 - \sum_{i=2}^{n_{ti}} y_i\right)^2} e^{-(1/2)\sum_{i=2}^{n_{ti}}(y_i + y_1)^2} \\ &= \left[\left(\frac{n_{ti}}{2\pi}\right)^{1/2} e^{\left(-n_{ti} y_1^2\right)/2}\right] \left[\frac{n_{ti}^{1/2}}{\left(2\pi\right)^{(n_{ti}-1)/2}} e^{-(1/2)\left[\sum_{i=2}^{n_{ti}} y_i^2 + \left(\sum_{i=2}^{n_{ti}} y_i\right)^2\right]}\right]. \end{aligned}$$

Then y_1 is independent of $\hat{\lambda_{ti}^2}$ given n_{ti}. If $n_{ti} \geq n$, then $\bar{X}_i\left(n\right)$ is independent of $\hat{\lambda_{ti}^2}$ because X_{ij} is independent of $X_{ij'}$ for $j' < j$. If $n_{ti} < n$,

$$\begin{aligned} \bar{X}_i\left(n\right) &= \frac{\sum_{j=1}^{n_{ti}} X_{ij}}{n_{ti}} \frac{n_{ti}}{n} + \frac{\sum_{j=n_{ti+1}}^{n} X_{ij}}{n} \\ &= Y_1 \frac{n_{ti}}{n} + \frac{\sum_{j=n_{ti+1}}^{n} X_{ij}}{n} \end{aligned}$$

and then $\bar{X}_i\left(n\right)$ is independent of $\hat{\lambda_{ti}^2}$. Thus, $C_i\left(1, n\right)$ is independent of $\hat{\lambda_{ti}^2}$ given n_{ti}. So, by induction,

$$\begin{aligned} f\left(C_i\left(1, n\right), \hat{\lambda_{ti}^2}\right) &= \int f\left(C_i\left(1, n\right), \hat{\lambda_{ti}^2} \mid n_{ti}\right) f\left(n_{ti}\right) dn_{ti} \\ &= f\left(C_i\left(1, n\right)\right) f\left(\hat{\lambda_{ti}^2}\right), \end{aligned}$$

then $C_i(1,n)$ is independent of $\hat{\lambda}^2_{ti}$. Consequently, $C_i(1,n)$ is independent of $n_{t,i+1}$. Thus

$$\mu_k - \mu_i + \frac{v_k}{\sqrt{n_{tk}}} C_k(1,n_{tk}) - \frac{v_i}{\sqrt{n_{ti}}} C_i(1,n_{ti}) \Rightarrow \mu_k - \mu_i$$

as $t \to \infty$. Let $Y_t = C_k\left(1, n_{T(1/t)k}\right)$ and $X_t = C_k(1,n_{tk})$, if $d(Y_t, X_t) \to 0$ in probability, then $Y_t \Rightarrow W()$. Thus the result would be proved. Let's show that $d(Y_t, X_t) \to 0$. ???Let $\epsilon > 0$.

$$\mu_k - \mu_i + \frac{v_k}{\sqrt{n_{T(\delta)k}}} C_k(1,n_{tk}) - \frac{v_i}{\sqrt{n_{T(\delta)i}}} C_i\left(1,n_{T(\delta)i}\right) \Rightarrow \mu_k - \mu_i$$

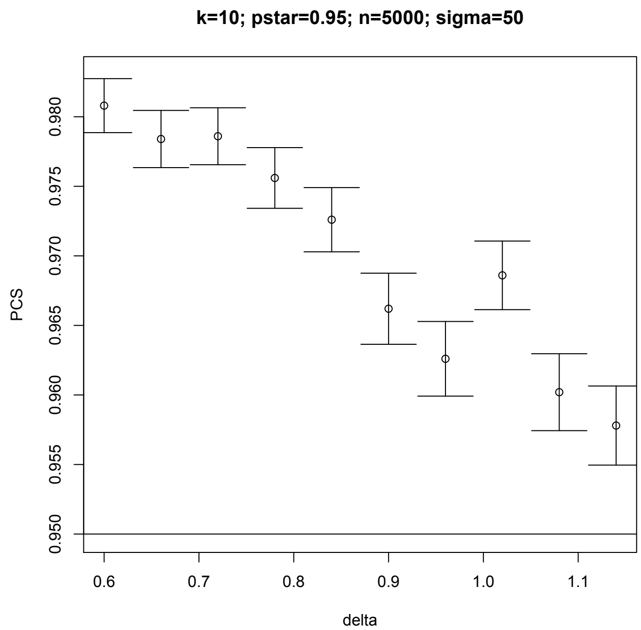
$$\mathbb{P}\left\{\frac{W_{T(\delta)k}}{n_{T(\delta)k}} < \frac{W_{T(\delta)i}}{n_{T(\delta)i}}\right\} = \mathbb{P}\left\{\frac{W_{T(\delta)k}}{n_{T(\delta)k}} < \frac{W_{T(\delta)i}}{n_{T(\delta)i}}\right\}$$

Now consider $k \geq 2$ systems and let CS be the event that k is selected and let ICS_i be the event that an incorrect selection is made when systems

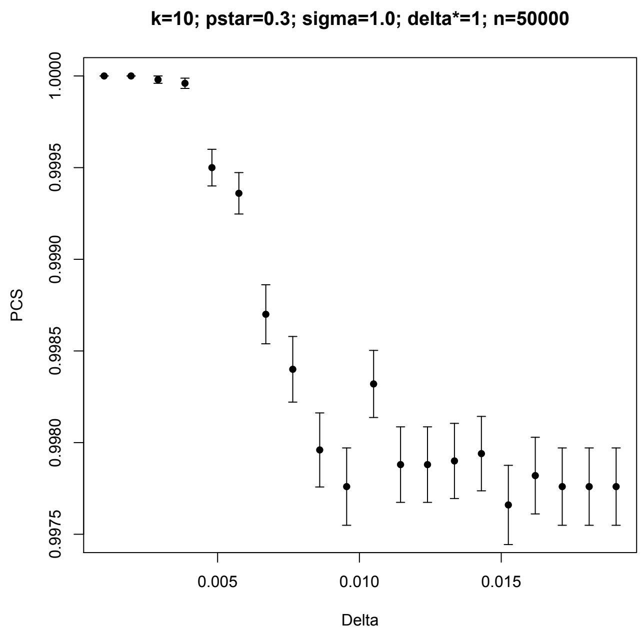
k and i are considered in isolation.

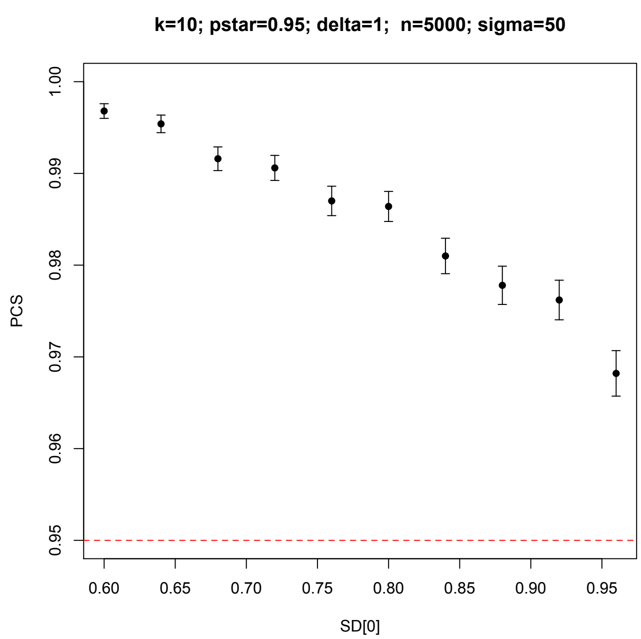
Theorem. If samples from system $x \in \{1 \ldots, k\}$ are normally distributed and independent, over time and across alternatives, then $\lim_{\min_x \sigma_x^2 \to \infty} Pr\{\text{BIZ selects } k\} \geq P*$ provided $\mu_k \geq \mu_{k-1} + \delta*$ for some $\delta* > 0$.

Graphs Delta goes to 0

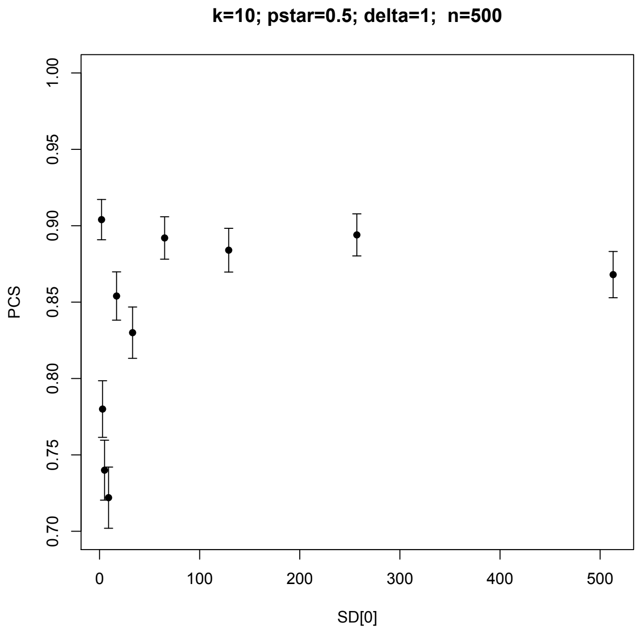


Delta goes to 0.

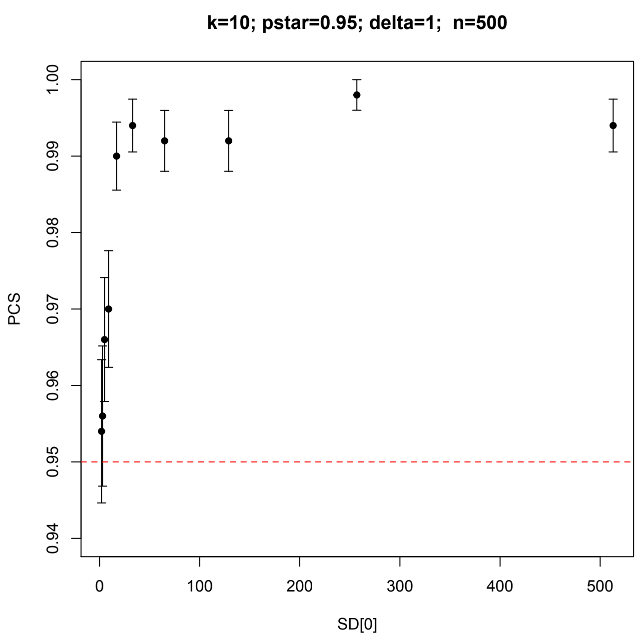




Variances go to infinity.



k=10; pstar=0.5; delta=1; n=500
PCS
SD[0]



k=10; pstar=0.95; delta=1; n=500
PCS
SD[0]

Theorem. If samples from system $x \in \{1 \ldots, k\}$ are normally distributed and independent, over time and across alternatives, then $\lim_{\delta \to 0} Pr\left\{\text{BIZ selects } k\right\} \geq P*$ (or $=1$??) provided $\mu_k \geq \mu_{k-1} + \delta^*$.

Prooof. We begin by considering the case of only two systems, denoted k and i, with $\mu_k \geq \mu_i + \delta *$. Let $A = \{k, i\}$. For $\delta \leq \delta *$, let

$$T(\delta) = \min\left\{t \in \mathbb{N} : \min_{x \in A} \hat{q}_{tx}(A) \leq c \text{ or } \max_{x \in A} \hat{q}_{tx}(A) \geq P*\right\}.$$

Thus $T(\delta)$ is the stage at which the procedure terminates.

Now,

$$\begin{aligned}
\min_{x \in A} \hat{q}_{tx}(A) &\leq c \\
\Leftrightarrow \min_{x \in A} \exp\left(\delta \beta_t \frac{W_{tx}}{n_{tx}}\right) &\leq c \sum_{x' \in A} \exp\left(\delta \beta_t \frac{W_{tx'}}{n_{tx'}}\right) \\
\Leftrightarrow \min_{x \in A} \frac{W_{tx}}{n_{tx}} &\leq \log\left[c \sum_{x' \in A} \exp\left(\delta \beta_t \frac{W_{tx'}}{n_{tx'}}\right)\right] / (\delta \beta_t) \\
\Leftrightarrow \left(\frac{W_{tk}}{n_{tk}} + \frac{W_{ti}}{n_{ti}} - \left|\frac{W_{tk}}{n_{tk}} - \frac{W_{ti}}{n_{ti}}\right|\right) &\leq 2 \log\left[c \sum_{x' \in A} \exp\left(\delta \beta_t \frac{W_{tx'}}{n_{tx'}}\right)\right] / (\delta \beta_t)
\end{aligned}$$

and

$$\begin{aligned}
\max_{x\in A}\hat{q}_{tx}\left(A\right) &\geq P* \\
\Leftrightarrow \max_{x\in A}\exp\left(\delta\beta_t\frac{W_{tx}}{n_{tx}}\right) &\geq P*\sum_{x'\in A}\exp\left(\delta\beta_t\frac{W_{tx'}}{n_{tx'}}\right) \\
\Leftrightarrow \max_{x\in A}\frac{W_{tx}}{n_{tx}} &\geq \log\left[P*\sum_{x'\in A}\exp\left(\delta\beta_t\frac{W_{tx'}}{n_{tx'}}\right)\right]/\left(\delta\beta_t\right) \\
\Leftrightarrow \left(\frac{W_{tk}}{n_{tk}}+\frac{W_{ti}}{n_{ti}}+\left|\frac{W_{tk}}{n_{tk}}-\frac{W_{ti}}{n_{ti}}\right|\right) &\geq 2\log\left[P*\sum_{x'\in A}\exp\left(\delta\beta_t\frac{W_{tx'}}{n_{tx'}}\right)\right]/\left(\delta\beta_t\right).
\end{aligned}$$

Let $d\left(\delta\right) := 2\log\left(\sum_{x'\in A}\exp\left(\delta\beta_t\frac{W_{tx'}}{n_{tx'}}\right)\right)/\left(\delta\beta_t\right)$, then

$$\begin{aligned}
T\left(\delta\right) = \min\Bigg\{ t\in\mathbb{N} : &\left(\frac{W_{tk}}{n_{tk}}+\frac{W_{ti}}{n_{ti}}+\left|\frac{W_{tk}}{n_{tk}}-\frac{W_{ti}}{n_{ti}}\right|\right)\geq 2\frac{\log\left(P*\right)}{\delta\beta_t}+d\left(\delta\right) \text{ or} \\
&\left(\frac{W_{tk}}{n_{tk}}+\frac{W_{ti}}{n_{ti}}-\left|\frac{W_{tk}}{n_{tk}}-\frac{W_{ti}}{n_{ti}}\right|\right)\leq 2\frac{\log c}{\delta\beta_t}+d\left(\delta\right) \Bigg\}.
\end{aligned}$$

Now, let's prove that $n_{ti}\to\infty, n_{tk}\to\infty$ as $t\to\infty$. For each t, $n_{t+1,x} = n_{tx}+B_x$ for some $x\in A$. If $n_{t+1,i}=n_{ti}+B_i$ for a finite number of $t's$, then $n_{tk}\to\infty$ as $t\to\infty$. So, there exists l_0 such that $n_{li}/\hat{\lambda}_{li}^2 > n_{lk}/\hat{\lambda}_{lk}^2$ if $l>l_0$. Since $\hat{\lambda}_{ti}^2\to\sigma_i^2$ and $\hat{\lambda}_{tk}^2\to\sigma_k^2$ as $t\to\infty$, then $n_{ti}\to\infty$ as $t\to\infty$. Similarly, we get the same result in the other cases. Thus, $n_{ti}\to\infty, n_{tk}\to\infty$ as $t\to\infty$. Furthermore, it's easy to see that $T\left(\delta\right)\to\infty$ as $\delta\to 0$.

Now, we'll show that $T\left(\delta\right)$ is finite. Observe that there exists t_0 such that if $t>t_0$

$$W_{ti}/n_{ti} < W_{tk}/n_{tk},$$

otherwise there would exists t such that

$$\mu_k - \delta/2 < W_{tk}/n_{tk} \leq W_{ti}/n_{ti} < \delta/2 + \mu_i$$

which is a contradiction. Moreover, there exists $t_1 > t_0$ such that if $t > t_1$

$$\beta_t \left(W_{tk}/n_{tk} - W_{ti}/n_{ti} \right) > \log \left(P * / \left(1 - P* \right) \right) / \delta$$

since $\beta_t \to \infty$ as $t \to \infty$ and $P* > \frac{1}{2}$. Consequently, if $t > t_1$

$$\begin{aligned}
\exp\left(\delta\beta_t \frac{W_{tk}}{n_{tk}}\right) / \exp\left(\delta\beta_t \frac{W_{ti}}{n_{ti}}\right) &> \frac{P*}{1-P*} \\
\Rightarrow \exp\left(\delta\beta_t \frac{W_{tk}}{n_{tk}}\right) &> P * \exp\left(\delta\beta_t \frac{W_{ti}}{n_{ti}}\right) \\
&\quad + P * \exp\left(\delta\beta_t \frac{W_{tk}}{n_{tk}}\right)
\end{aligned}$$

and so $T(\delta) \leq t_1(\delta) < \infty$.

By the strong law of large numbers, almost surely

$$\frac{W_{T(\delta)k}}{n_{T(\delta)k}} - \frac{W_{T(\delta)i}}{n_{T(\delta)i}} \to \mu_k - \mu_i$$

as $\delta \to 0$. Thus,

$$\frac{W_{T(\delta)k}}{n_{T(\delta)k}} - \frac{W_{T(\delta)i}}{n_{T(\delta)i}} \Rightarrow \mu_k - \mu_i$$

as $\delta \to 0$.

Let ICS denote the even that an incorrect selection is made. Then,

$$\begin{aligned} \lim\,\inf_{\delta\to 0}\mathbb{P}\left\{ICS\right\} &= \lim\,\inf_{\delta\to 0}\mathbb{P}\left\{\frac{W_{T(\delta)k}}{n_{T(\delta)k}} < \frac{W_{T(\delta)i}}{n_{T(\delta)i}}\right\} \\ &= \mathbb{P}\left\{\mu_k < \mu_i\right\} = 0. \end{aligned}$$