On the Asymptotic Validity of a Fully Sequential Elimination Procedure for Indifference-Zone Ranking and Selection with Tight Bounds on Probability of Correct Selection

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We prove the validity of the sequential elimination IZ procedure proposed by Frazier [] when δ goes to 0. Specifically, we analyze Algorithm 2, when $B_1 = \cdots = B_k = 1$:

Algorithm 2: Discrete-time implementation of BIZ, for unknown and/or heterogeneous variances.

Require: $c \in [0, 1-(P^*)^{\frac{1}{k-1}}]$, $\delta > 0$, $P^* \in (1/k, 1)$, $n_0 \geq 0$ an integer, $B_1, \ldots, B_k$ strictly positive integers. Recommended choices are $c = 1-(P^*)^{\frac{1}{k-1}}$, $B_1 = \cdots = B_k = 1$ and n_0 between 10 and 30. If the sampling variances λ_x^2 are known, replace the estimators $\widehat{\lambda}_{tx}^2$ with the true values λ_x^2, and set $n_0 = 0$. To compute $\widehat{q}_{tx}(A)$, use

$$q'_{t,x}(A) = \exp\left(\gamma\delta Y'_{tx}\right) \Big/ \sum_{x' \in A} \exp\left(\gamma\delta Y'_{tx'}\right) = \exp\left(\frac{\delta}{\lambda_x^2} Y_{n_x(t),x}\right) \Big/ \sum_{x' \in A} \exp\left(\frac{\delta}{\lambda_{x'}^2} Y_{n'_x(t),x'}\right), \tag{1}$$

where $Y_{n_x(t),x}$ is the sum of the first $n_x(t)$ samples.

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For each x, sample alternative x n_0 times and set n_{0x} ← n_0. Let W_{0x} and \hat{λ}^2_{0x} be the sample mean and sample variance respectively of these samples. Let t ← 0.
Let A ← {1, ..., k}, P ← P*, t ← 1.
while x ∈ max_{x∈A} \hat{q}_{tx}(A) < P do
  while min_{x∈A} \hat{q}_{tx}(A) ≤ c do
    Let x ∈ arg min_{x∈A} \hat{q}_{tx}(A).
    Let P ← P/(1 − \hat{q}_{tx}(A)).
    Remove x from A.
  end while
  Let z ∈ arg min_{x∈A} n_{tx}/\hat{λ}^2_{tx}.
  For each x ∈ A, let n_{t+1,x} = ceil(\hat{λ}^2_{tx}(n_{tz} + B_z)/\hat{λ}^2_{tz}).
  For each x ∈ A, if n_{t+1,x} > n_{tx}, take n_{t+1,x} − n_{tx} additional samples from alternative x. Let W_{t+1,x} and \hat{λ}^2_{t+1,x} be the sample mean and sample variance respectively of all samples from alternative x thus far.
  Increment t.
end while
Select \hat{x} ∈ arg max_{x∈A} W_{tx}/n_{tx} as our estimate of the best.
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1 Introduction

This paper is organized as follows: In §2, we present the proof of the validity of the algorithm when the variances are known. In §3, we prove the case when the variances are unknown.

To prove the case when the variances are known, we use a Functional Central Limit Thoerem that shows how to standardize the output data to make them behave like Brownian motion processes in the limit. We also use an extension of the Continuous Mapping Theorem (Theorem 5.5 of Billingsley 1968) to see that the algorithm behaves like a sequential elimination IZ procedure with a Brownian motion process instead of the standardize of the sum of the output data in the limit. Finally, we use the results of the paper of Frazier [] to prove the validity of this algorithm in the limit.

2 Asymptotic Validity when the Variances are Known

Without loss of generality, suppose that the true means of the systems are indexed so that $\mu_k \geq \mu_{k-1} \geq \cdots \geq \mu_1$. We suppose that samples from system $x \in \{1 \ldots, k\}$ are identically distributed and independent, over time and across alternatives. The algorithm ends in $R(\delta) \in \mathbb{N}$ iterations where $R(\delta)$ is a random variable , and $R(\delta) \to \infty$ as $\delta \to 0$ with probability 1. We know that $R(\delta) < \infty$ almost surely (Lemma 4 of Frazier 2011). To simplify notation, we write R in place of $R(\delta)$, and let the dependence on δ be implicit. Furthermore, we suppose $R^{1/2}\delta$ converges to a random variable Δ with probability 1 where $\infty > \Delta > 0$ with probability 1. We also define $\lambda_z^2 := \max_{i \in \{1 \ldots, k\}} \lambda_i^2$. We suppose that $\min_{i \in \{1 \ldots, k\}} \lambda_i^2 > 0$.

Lemma 1. If $x \in \{1 \ldots, k\}$, then

$$C_x(\delta, \cdot) := \frac{Y_{\text{ceil}\left(\frac{\lambda_x^2}{\lambda_z^2}(n_0 + tR(\delta))\right),x} - \frac{\lambda_x^2}{\lambda_z^2}(n_0 + tR(\delta))\mu_x}{\frac{\lambda_x^2}{\lambda_z}\sqrt{R(\delta)}} \Rightarrow W_x(\cdot)$$

where $Y_{n,x}$ is the sum of the first n samples and W_x is a standard Brownian motion.

Proof. We define

$$C_{x\delta}(t) := \frac{Y_{\text{ceil}\left(\frac{\lambda_x^2}{\lambda_z^2}tR\right),x} - \text{ceil}\left(\frac{\lambda_x^2}{\lambda_z^2}tR\right)\mu_x}{\frac{\lambda_x^2}{\lambda_z}\sqrt{R}}$$

where $t \in [0, 1]$. Since $\frac{\lambda_x^2}{\lambda_z^2}R(\delta)\delta^2 \xrightarrow{P} \frac{\lambda_x^2}{\lambda_z^2}\Delta^2$ in probability, then by the Theorem 17.2 of Billingsley 1968,

$$C_{x\delta}(\cdot) \Rightarrow W(\cdot)$$

as $\delta \to 0$.

We now define

$$\tilde{C}_x(\delta, t) := \frac{Y_{\text{ceil}\left(\frac{\lambda_x^2}{\lambda_z^2}tR\right),x} - \frac{\lambda_x^2}{\lambda_z^2}tR\mu_x}{\frac{\lambda_x^2}{\lambda_z}\sqrt{R}}.$$

Consequently, as $\delta \to 0$,

$$\tilde{C}_x(\delta, \cdot) \Rightarrow W(\cdot)$$

because $\frac{\frac{\lambda_x^2}{\lambda_z^2}tR - ceil\left(\frac{\lambda_x^2}{\lambda_z^2}tR\right)}{\frac{\lambda_x^2}{\lambda_z}\sqrt{R}} \to 0$ (see proof of Theorem 9.1 of Billingsley 1968).

Observe that for $\epsilon > 0$ and δ sufficiently small

$$\left| \frac{-Y_{\mathrm{ceil}\left(\frac{\lambda_x^2}{\lambda_z^2}tR\right),x} + Y_{\mathrm{ceil}\left(n_0\frac{\lambda_x^2}{\lambda_z^2}+\frac{\lambda_x^2}{\lambda_z^2}tR\right),x}}{\frac{\lambda_x^2}{\lambda_z}\sqrt{R}} \right| < \epsilon\left(n_0\frac{\lambda_x^2}{\lambda_z^2}+2\right)$$

and then

$$C_x(\delta,\cdot) \Rightarrow W(\cdot).$$

■

Now we are going to define new algorithms that are almost the same than the one proposed by Frazier, but instead of $q_{tx}^{\delta}(A)$, these algorithms use new functions $q_{tx}^{Y,\delta}(A)$ which depend on a function Y that is in $D[0,1]^k$ and $D[0,1]^k$ is the set of functions from $[0,1]^k$ to $\mathbb{R}$ that are right-continuous and have left-hand limits. We'll use the Skorohod metric d on $D[0,1]^k$:

$$d(X,Y) = \inf_{\lambda\in\Lambda}\{\|\lambda - I\| \vee \|X - Y\circ\lambda\|\}$$

where Λ is the set of strictly increasing, continuous mappings of $[0,1]$ onto itself, and $\|\cdot\|$ is the uniform norm, and I is the identity map.

First we are going to suppose that $\delta > 0$ and $\mu_k = \delta$,$\mu_{k-1} = \cdots = \mu_1 = 0$.

Suppose Δ is given. Let $Y \in D[0,1]^k$ and $t \in [0,1]$, we define

$$q_{tx}^{Y,\delta}(A) := \exp\left(\delta\sqrt{R}\frac{Y_x(t)}{\lambda_z} + \delta\beta_{tR}\mu_x\right) / \sum_{x'\in A}\exp\left(\delta\sqrt{R}\frac{Y_{x'}(t)}{\lambda_z} + \delta\beta_{tR}\mu_{x'}\right)$$

where $\beta_{tR} = \frac{(n_0+tR)}{\lambda_z^2}$.

We then define for $0 < a < P^* - \frac{1}{k}$

$$\begin{aligned}
T_{Y,\delta}^0 &= 0\\
A_0^{Y,\delta} &= \{1,\ldots,k\}\\
P_0^{Y,\delta} &= P^* - a\\
T_{Y,\delta}^{n+1} &= \inf\left\{t\in\left[T_{Y,\delta}^n,1\right] : \min_{x\in A_n^{Y,\delta}} q_{tx}^{Y,\delta}\left(A_n^{Y,\delta}\right) \le c \text{ or } \max_{x\in A_n^{Y,\delta}} q_{tx}^{Y,\delta}\left(A_n^{Y,\delta}\right) \ge P_n^{Y,\delta}\right\}\\
Z_{n+1}^{Y,\delta} &\in \arg\min_{x\in A_n^{Y,\delta}} q_{T_{Y,\delta}^{n+1},x}^{Y,\delta}\left(A_n^{Y,\delta}\right)\\
A_{n+1}^{Y,\delta} &= A_n - \left\{Z_{n+1}^{Y,\delta}\right\}\\
P_{n+1}^{Y,\delta} &= P_n^{Y,\delta}/\left(1-\min_{x\in A_n^{Y,\delta}} q_{T_{Y,\delta}^{n+1}x}^{Y,\delta}\left(A_n^{Y,\delta}\right)\right).
\end{aligned}$$

Now, let

$$M_{Y,\delta} = \inf\left\{n = 1,\ldots,k-1 : \max_{x\in A_{n-1}^{Y,\delta}} q_{T_{Y,\delta}^n,x}\left(A_{n-1}^{Y,\delta}\right) \ge P_{n-1}^{Y,\delta}\right\}$$

and

$$f(Y,\delta) = \begin{cases} 1 & \text{if k} \in A^{Y,\delta}_{M-1} \text{ and } \lambda_k^2\left(Y_k\left(T_{Y,\delta}^{M_{Y,\delta}}\right)\right) + \delta\frac{\lambda_k^2}{\lambda_z\sqrt{R}}\left(n_0 + T_{Y,\delta}^{M_{Y,\delta}}R\right) \geq \lambda_x^2\left(Y_x\left(T_{Y,\delta}^{M_{Y,\delta}}\right)\right)\ \forall x \in A^{Y,\delta}_{M-1} \\ 0 & \text{otherwise} \end{cases}.$$

Now, we also define

$$q_{tx}^Y(A) := \exp\left(\Delta\frac{Y_x(t)}{\lambda_z} + \frac{1}{\lambda_z^2}\Delta^2 t I_{\{x=k\}}\right) / \sum_{x' \in A} \exp\left(\Delta\frac{Y_{x'}(t)}{\lambda_z} + \frac{1}{\lambda_z^2}\Delta^2 t I_{\{x'=k\}}\right)$$

$$\begin{aligned}
T_Y^0 &= 0 \\
A_0^Y &= \{1,\dots,k\} \\
P_0^Y &= P^* \\
T_Y^{n+1} &= \inf\left\{t \in [T_Y^n, 1] :\ \min_{x\in A_n^Y} q_{tx}^Y\left(A_n^Y\right) \leq c \text{ or } \max_{x\in A_n^Y} q_{tx}^Y\left(A_n^Y\right) \geq P_n^Y\right\} \\
Z_{n+1}^Y &\in \arg\min_{x\in A_n^Y} q^Y_{T_Y^{n+1},x}\left(A_n^Y\right) \\
A_{n+1}^Y &= A_n^Y - \left\{Z_{n+1}^Y\right\} \\
P_{n+1}^Y &= P_n^{Y,\delta} / \left(1 - \min_{x\in A_n^Y,} q^Y_{T_Y^{n+1}x}\left(A_n^Y\right)\right).
\end{aligned}$$

Now, let

$$M_Y = \inf\left\{n = 1,\dots,k-1 : \max_{x\in A_{n-1}^Y} q_{T_Y^n,x}\left(A_{n-1}^Y\right) \geq P_{n-1}^Y\right\}$$

and

$$g(Y) = \begin{cases} 1 & \text{if k} \in A^Y_{M-1} \text{ and } Y_k\left(T_Y^{M_Y}\right) + \Delta\frac{1}{\lambda_z}T_Y^{M_Y} \geq \frac{\lambda_x^2}{\lambda_k^2}\left(Y_x\left(T_Y^{M_Y}\right)\right)\ \forall x \in A^Y_{M-1} \\ 0 & \text{otherwise} \end{cases}.$$

Now, we want to prove that

$$f(C(\delta,\cdot)) \Rightarrow g(W)$$

as $\delta \to 0$. In order to do this, we will prove the following lemma which will allow us to use the Theorem 5.5 of Billingsley 1968 that implies the desired result.

Lemma 2. Let $D_s \equiv \{x \in D[0,1]^k :$ for some sequence $\{x_n\} \subset D[0,1]^k$ with $lim_n d(x_n, x) = 0$, the sequence $\{f_n(x_n)\}$ converges to $\{g(x)\}\}$, then $\mathbb{P}(W \in D_s) = 1$.

Proof. We will use the following property: if $0 < T < 1$ where T is a stopping time, then by the local version of the law of the iterated logarithm for Brownian motion

$$\lim\sup_{u\to 0^+} \frac{[W_x(T+u) - W_x(T)]}{\sqrt{2u\ln[\ln(1/u)]}} = 1 \tag{2}$$

almost surely for each system x. Furthermore, note that W is continuous on $[0,1]^k$ and so W is also uniformly continuous on $[0,1]^k$.

Let $\{Z_n\} \subset D[0,1]^k$ such that $Z_n \to W$. Furthermore, $\delta_n^2 \frac{(n_0+tR(\delta_n))}{\lambda_z^2} \to \frac{\Delta^2 t}{\lambda_z^2}$ in $D[0,1]$ and $\frac{\delta_n\sqrt{R(\delta_n)}}{\lambda_z} \to \frac{\Delta}{\lambda_z}$ in $D[0,1]$ because uniformly convergence implies convergence in the Skorohod topology. Consequently, there exist functions λ_n in Λ such that

$$\lim_n Z_n(\lambda_n t) = W(t)$$

uniformly in t and

$$\lim_n \lambda_n t = t$$

uniformly in t. Then

$$\lim_n \frac{\delta_n\sqrt{R(\delta_n)}}{\lambda_z} Z_n(\lambda_n t) + \delta_n^2 \frac{(n_0 + \lambda_n(t) R(\delta_n))}{\lambda_z^2} = W(t)\frac{\Delta}{\lambda_z} + \frac{\Delta^2 t}{\lambda_z^2}$$

uniformly in t, and so

$$\lim_n \exp\left(\frac{\delta_n\sqrt{R(\delta_n)}}{\lambda_z} Z_n(\lambda_n t) + \delta_n^2 \frac{(n_0 + \lambda_n(t) R(\delta_n))}{\lambda_z^2}\right) = \exp\left(W(t)\frac{\Delta}{\lambda_z} + \frac{\Delta^2 t}{\lambda_z^2}\right)$$

uniformly in t since exp is uniformly continuous in $[0,1]$. Consequently,

$$q_{\lambda_n(t)x}^{Z_n,\delta_n}(A) \to q_{tx}^{W}(A)$$

uniformly in t. Thus, $q_{\cdot x}^{Z_n,\delta_n}(A) \to q_{\cdot x}^{W}(A)$ in $D[0,1]$ for any set $A \subset \{1,\ldots,k\}$. Then there exists $\lambda_{n,A} \in \Lambda$ such that $\sup_{t\in[0,1]} \|\lambda_{n,A}(t) - t\| \leq d\left(q_{\cdot x}^{Z_n,\delta_n}(A), q_{\cdot x}^{W}(A)\right) + \frac{1}{n}$ and $\sup_{t\in[0,1]} \left\|q_{tx}^{Z_n,\delta_n}(A) - q_{\lambda_{n,A}(t)x}^{W}(A)\right\| \leq d\left(q_{\cdot x}^{Z_n,\delta_n}(A), q_{\cdot x}^{W}(A)\right) + \frac{1}{n}$. Taking $g_n^A \equiv \sup_{t\in[0,1]}\left\|q_{tx}^{W}(A) - q_{\lambda_n(t)x}^{W}(A)\right\|$, we see from the uniform continuity of W on $[0,1]^k$ and the definition of g_n^A that $\lim_{n\to\infty} g_n^A = 0$. Moreover, if we take $\epsilon_n^A = 3n^{-1} + 3\sup\left\{d\left(q_{\cdot x}^{Z_l,\delta_l}(A), q_{\cdot x}^{W}(A)\right) + g_l^A : l = n, n+1, \ldots\right\}$, then $\{\epsilon_n^A\}$ is a monotonically decreasing sequence of positive numbers with limit zero.

From the definition of ϵ_n we have $d\left(q_{\cdot x}^{Z_n,\delta_n}(A), q_{\cdot x}^{W}(A)\right) < \epsilon_n/2$ and $g_n^A < \epsilon_n/2$ for $n = 1, 2, \ldots$ Consequently, we have

$$\begin{aligned}\left\|q_{tx}^{Z_n,\delta_n}(A) - q_{tx}^{W}(A)\right\| &\leq \left\|q_{tx}^{Z_n,\delta_n}(A) - q_{\lambda_n(t)x}^{W}(A)\right\| + \left\|q_{\lambda_n(t)x}^{W}(A) - q_{tx}^{W}(A)\right\| \\ &< \epsilon_n^A\end{aligned}$$

for all $t \in [0,1]$ and $x \in A$.

We will show that $\mathbb{P}\left(W \in \mathrm{D}[0,1]^k\text{-}D_s \mid M_W = i\right) = 1$ for $i \in \{1,\ldots,k-1\}$ so that the desired conclusion follows.

Suppose first that $M_W = 1$. Let's prove that $T_{Z_n,\delta_n}^1 \to T_W^1$ as $n \to \infty$. Since $M_W = 1$, then $\max_{x\in A} q_{T_W^1 x}^{W}(A) = P$ almost surely because W is continuous almost surely. Let $q_x^* = \min_{t\in[0,T_W^1]} q_{tx}^{W}$ (the minimum exists because q_{tx}^{W} is continuous) and let $q^* = \min_x q_x^*$. Let N such that if $n > N$ then $\epsilon_n^A < q^* - c$ (note that $q^* - c > 0$ because

$M_W = 1$ and N is a random variable). If $n > N$ and $t \leq T_W^1$ then

$$\begin{aligned} q_{tx}^{W} &< q_{tx}^{Z_n,\delta_n} + q^* - c \\ \Rightarrow c &< q_{tx}^{Z_n,\delta_n} + q^* - q_{tx}^{W} \leq q_{tx}^{Z_n,\delta_n}. \end{aligned}$$

Let $T_W^1 > \epsilon > 0$, $q_x^+ = \max_{t\in\left[0,T_W^1-\epsilon\right]} q_{tx}^W$ and $q^+ = \max_x q_x^*$. Let N_2^* such that if $n > N_2^*$, then $\delta_n < r < P - q^+$ where $r \in \mathbb{Q}$. Let N_2^{**} such that if $n > N_2^{**}$, then $\epsilon_n^A < P - q^+ - r$. Thus, if $n > N_2 := \max\left\{N_2^*, N_2^{**}\right\}$ and $t \leq T_W^1 - \epsilon$, then

$$\begin{aligned} q_{tx}^{Z_n,\delta_n}(A) &< q_{tx}^{W}(A) + \epsilon_n^A \\ &< q_{tx}^{W}(A) + P - q^+ - r \\ &\leq P - r \leq P - \delta_n \end{aligned}$$

and so $T_{Z_n,\delta_n}^1 > T_W^1 - \epsilon$ if $n > N_3 := \max\left\{N_1, N_2\right\}$.

Case 1. $T_W^1 = 1$, so $\left|T_W^1 - T_{Z_n,\delta_n}^1\right| < \epsilon$ if $n > N_3$.

If $\epsilon_2 \geq T_W^1 = 1$, then $\left|T_W^1 - T_{Z_n,\delta_n}^1\right| < \epsilon_2$ if $n > N_3$.

Case 2. $T_W^1 = 0$. Note that $q_{0x}^W(A) = \frac{1}{k} < P$ and so this case is not possible.

Case 3. $0 < T_W^1 < 1$.

Note that $T_{Z_n,\delta_n}^1 \leq T_{Z_n,\delta_n}^1(P+\delta_n) \leq T_W^1(P+\delta_n+\epsilon_n)$ because if $t < T_{Z_n,\delta_n}^1(P+\delta_n)$

$$q_{tx}^{W}(A) - \epsilon_n < q_{tx}^{Z_n,\delta_n}(A) < P + \delta_n.$$

Furthermore,

$$\begin{aligned} T_W^1(A) &\leq \liminf_n T_W^1(P+\delta_n+\epsilon_n) \\ &\leq \limsup_n T_W^1(P+\delta_n+\epsilon_n). \end{aligned}$$

Now, let $x = \arg\max q_{T_W^1 x}^W$. By (2), for any $\varepsilon \in (0,1)$, there exists a monotonically decreasing sequence $\{s_k : \ k = 1, 2, \ldots\} \subset \left(T_W^1, 1\right)$ such that $\lim_n s_n = T_W^1$; and for $k = 1, 2, \ldots$ we have

$$W_x(s_k) - W_x\left(T_W^1\right) \quad > v_k \equiv \quad (\varepsilon)\sqrt{2\left(s_k - T_W^1\right)\ln\ln\left(1/\left(s_k - T_W^1\right)\right)}.$$

Notice that $v_k > 0$ and $\lim_k v_k = 0$. Pick k arbitrarily. Let N_1 such that if $n > N_4$, then $\frac{\epsilon_n}{P} < v_k \frac{\Delta}{\lambda_z}$ and

$\frac{\delta_n}{P} < \left(v_k \frac{\Delta}{\lambda_z}\right)^2 /2$. Observe that

$$\begin{aligned}
q^W_{s_k x} &= \exp\left(\Delta \frac{W_x\left(s_k\right)}{\lambda_z} + \frac{1}{\lambda_z^2}\Delta^2 s_k I_{\{x=k\}}\right) / \sum_{x' \in A} \exp\left(\Delta \frac{W_{x'}\left(s_k\right)}{\lambda_z} + \frac{1}{\lambda_z^2}\Delta^2 s_k I_{\{x'=k\}}\right) \\
&> \frac{\exp\left(\Delta \frac{W_x\left(T^1_W\right)+v_k}{\lambda_z} + \frac{1}{\lambda_z^2}\Delta^2 T^1_W I_{\{x=k\}}\right)}{\sum_{x' \in A} \exp\left(\Delta \frac{W_{x'}\left(T^1_W\right)}{\lambda_z} + \frac{1}{\lambda_z^2}\Delta^2 T^1_W I_{\left\{x'=k\right\}}\right)} \times \frac{\sum_{x' \in A} \exp\left(\Delta \frac{W_{x'}\left(T^1_W\right)}{\lambda_z} + \frac{1}{\lambda_z^2}\Delta^2 T^1_W I_{\left\{x'=k\right\}}\right)}{\sum_{x' \in A} \exp\left(\Delta \frac{W_{x'}\left(s_k\right)}{\lambda_z} + \frac{1}{\lambda_z^2}\Delta^2 s_k I_{\{x'=k\}}\right)} \\
&\geq q^W_{T^1_W x} \exp\left(v_k \frac{\Delta}{\lambda_z}\right) \geq P\left(1 + v_k \frac{\Delta}{\lambda_z} + \frac{\left(v_k \frac{\Delta}{\lambda_z}\right)^2}{2}\right) > P + \epsilon_n + \delta_n.
\end{aligned}$$

Consequently,

$$s_k > T^1_W\left(P + \epsilon_n + \delta_n\right)$$

if $n > N_4$, and so

$$s_k \geq \lim \sup_n T^1_W\left(P + \delta_n + \epsilon_n\right).$$

Since $s_k \to T^1_W$, we must have that

$$\begin{aligned}
T^1_W(A) &\geq \lim \sup_n T^1_W\left(P + \delta_n + \epsilon_n\right) \\
&\geq \lim \inf_n T^1_W\left(P + \delta_n + \epsilon_n\right) \\
&\geq T^1_W(A)
\end{aligned}$$

Consequently,

$$T^1_W(A) = \lim_n T^1_W\left(P + \delta_n + \epsilon_n\right).$$

Now,

$$T^1_{Z_n,\delta_n} \geq T^1_{Z_n,\delta_n}\left(P - \delta_n\right) \geq T^1_W\left(P - \delta_n - \epsilon_n\right)$$

because if $t < T^1_W\left(P - \delta_n - \epsilon_n\right)$

$$q^{Z_n,\delta_n}_{tx}(A) - \epsilon_n < q^W_{tx}(A) < P - \delta_n - \epsilon_n.$$

Similarly we can prove that

$$\lim_n T^1_W\left(P - \delta_n - \epsilon_n\right) = T^1_W(A).$$

Consequently,

$$T^1_W(A) = \lim_n T^1_{Z_n,\delta_n}.$$

The following is wrong

We will prove by induction that

$$\begin{aligned}
T^m_{x_n,\delta_n} &\to T^m_W \\
Z^{x_n,\delta_n}_m &\to Z^W_m \\
A^{x_n,\delta_n}_m &\to A^W_m \\
P^{x_n,\delta_n}_m &\to P^W_m
\end{aligned}$$

Suppose this is true for m. Observe that

$$T^m_{W-\epsilon_n e,\delta_n} \leq T^m_{x_n,\delta_n} \leq T^m_{W+\epsilon_n e,\delta_n}$$

We have that if $\hat{\tau}_M$ is the continuous version of $\frac{\tau_M}{R}$, then

$$f\left(C\left(\delta,t\right),\delta\right) = \begin{cases} 1 & \text{if k} \in \text{arg max}_{x \in A^{Y,\delta}_{M-1}} \frac{\lambda_x^2}{\lambda_z}\sqrt{R}\left(C_x\left(\delta,\hat{\tau}_M\right) + \frac{\lambda_x^2}{\lambda_z^2}\left(n_0 + \hat{\tau}_M R\right)\mu_x\right) \\ 0 & \text{otherwise} \end{cases}.$$

By lemma 1,

$$C\left(\delta,t\right) \Rightarrow W\left(t\right).$$

Now,

$$\begin{aligned} \left|\frac{\lambda_x^2}{\lambda_z n_{tR,x}}\sqrt{R}\delta\beta_{tR}x_{n_x}\left(t\right) - \frac{\Delta}{\lambda_z}W_x\left(t\right)\right| &\leq \left|\frac{\lambda_x^2}{\lambda_z n_{tR,x}}\sqrt{R}\delta\beta_{tR}x_{n_x}\left(t\right) - \frac{\Delta}{\lambda_z}x_{n_x}\left(t\right)\right| \\ &\quad + \left|\frac{\Delta}{\lambda_z}W_x\left(t\right) - \frac{\Delta}{\lambda_z}x_{n_x}\left(t\right)\right| \\ &\leq \frac{\Delta}{\lambda_z}\epsilon_n + \epsilon\left|x_{n_x}\left(t\right)\right|, \end{aligned}$$

furthermore

$$\begin{aligned} \left|\frac{\lambda_x^2}{\lambda_z n_{tR,x}}\sqrt{R}\delta\beta_{tR}x_{n_x}\left(t\right) - \delta_n^2\beta_{tR_n} - \frac{\Delta}{\lambda_z}W_x\left(t\right) + A\left(t\right)\right| &\leq \left|\frac{\lambda_x^2}{\lambda_z n_{tR,x}}\sqrt{R}\delta\beta_{tR}x_{n_x}\left(t\right) - \frac{\Delta}{\lambda_z}x_{n_x}\left(t\right)\right| \\ &\quad + \left|\frac{\Delta}{\lambda_z}W_x\left(t\right) - \frac{\Delta}{\lambda_z}x_{n_x}\left(t\right)\right| \\ &\quad + \left|A\left(t\right) - \delta_n^2\beta_{tR_n}\right| \\ &\leq \frac{\Delta}{\lambda_z}\epsilon_n + \epsilon\left|x_{n_x}\left(t\right)\right| + \epsilon, \end{aligned}$$

Now consider T_{x_n,δ_n}. Since $x_{n_i}\left(t\right) - W_i\left(t\right) < \epsilon_n$ and $W_i\left(t\right) - x_{n_i}\left(t\right) < \epsilon_n$, consequently

$$T_{W-\epsilon_n e,\delta_n} \leq T_{x_n,\delta_n} \leq T_{W+\epsilon_n e,\delta_n}$$

Observe that

$$t^* = \lim\ \inf_n T_{W-\epsilon_n e,\delta_n} \geq T_W$$

and

$$t_* = \lim\ \sup_n T_{W+\epsilon_n e,\delta_n} \leq T_W.$$

Then

$$t_* \leq T_W \leq t^* \leq \lim\ \inf_n T_{W+\epsilon_n e,\delta_n} \leq t_*$$

thus

$$t_* = t^* = T_W = \lim_n T_{W-\epsilon_n e,\delta_n} = \lim_n T_{W+\epsilon_n e,\delta_n}.$$

Then

$$\lim_n T_{x_n,\delta_n} = T_W$$

and so

$$\lim_n x_{n_i}\left(T_{x_n,\delta_n}\right) = \lim_n W_i\left(T_{x_n,\delta_n}\right) = W_i\left(T_W\right)$$

by the continuity of W_i.

Therefore

$$\lim_n f_n\left(x_n\right) = \lim_n f\left(x_n,\delta_n\right) = g\left(W\right).$$

$\blacksquare$

By the extension of the CMT (Theorem 5.5 of Billingsley 1968), we have the following corollary.

Corollary 1. We have that if $t \in [0,1]$,

$$f\left(C\left(\delta,t\right),\delta\right) \Rightarrow g\left(W\left(t\right)\right)$$

as $\delta \to 0$.

Theorem. If samples from system $x \in \{1\ldots,k\}$ are normally distributed and independent, over time and across alternatives, then $\lim_{\delta\to 0} Pr\left\{\text{BIZ selects } k\right\} \geq P*$ provided $\mu_k \geq \mu_{k-1}+\delta$. We also suppose that the algorithm ends in at most $R\left(\delta\right) \in \mathbb{N}$ iterations, and $R(\delta) \to \infty$ as $\delta \to 0$ with probability 1. Furthermore, $\sqrt{R}\delta \to \Delta$ with probability 1 where $\infty > \Delta > 0$ with probability 1. We also suppose $B_1 = \cdots = B_k = 1$.

Proof.

$$\begin{aligned}
\underline{lim}_{\delta\to 0}\mathbb{P}\left(CS\right) &\geq \underline{lim}_{\delta\to 0}\mathbb{P}\left(f\left(C\left(\delta,t\right),\delta\right)=1\right)\\
&= \underline{lim}_{\delta\to 0}E\left(\mathbb{P}\left(f\left(C\left(\delta,t\right),\delta\right)=1 \mid \Delta\right)\right)\\
&= E\left(\underline{lim}_{\delta\to 0}\mathbb{P}\left(f\left(C\left(\delta,t\right),\delta\right)=1 \mid \Delta\right)\right)\\
&= E\left(\mathbb{P}\left(g\left(W\right)=1 \mid \Delta\right)\right)\\
&\geq P^*
\end{aligned}$$