On the Asymptotic Validity of a Fully Sequential Elimination Procedure for Indifference-Zone Ranking and Selection with Tight Bounds on Probability of Correct Selection

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We prove the validity of the sequential elimination IZ procedure proposed by Frazier [3] when δ goes to 0. Specifically, if k is the number of systems, we analyze Algorithm 2, when $B_1 = \cdots = B_k = 1$:

Algorithm 2: Discrete-time implementation of BIZ, for unknown and/or heterogeneous variances.

Require: $c \in [0, 1-(P^*)^{\frac{1}{k-1}}]$, $\delta > 0$, $P^* \in (1/k, 1)$, $n_0 \geq 0$ an integer, $B_1, \ldots, B_k$ strictly positive integers. Recommended choices are $c = 1-(P^*)^{\frac{1}{k-1}}$, $B_1 = \cdots = B_k = 1$ and n_0 between 10 and 30. If the sampling variances λ_x^2 are known, replace the estimators $\widehat{\lambda}_{tx}^2$ with the true values λ_x^2, and set $n_0 = 0$. We define $q_{tx}(A)$ where $A = \{1, \ldots, k\}$ as

$$q_{tx}(A) = \exp\left(\frac{\delta}{\lambda_x^2} Y_{n_x(t),x}\right) \Bigg/ \sum_{x' \in A} \exp\left(\frac{\delta}{\lambda_{x'}^2} Y_{n'_x(t),x'}\right), \tag{1}$$

where $Y_{n_x(t),x}$ is the sum of the first $n_x(t)$ samples.

For each x, sample alternative x n_0 times and set $n_{0x} \leftarrow n_0$. Let W_{0x} and $\widehat{\lambda}_{0x}^2$ be the sample mean and sample variance respectively of these samples. Let $t \leftarrow 0$.

Let $A \leftarrow \{1, \ldots, k\}$, $P \leftarrow P^*$, $t \leftarrow 1$.

while $x \in \max_{x \in A} q_{tx}(A) < P$ **do**

  while $\min_{x \in A} q_{tx}(A) \leq c$ **do**

    Let $x \in \arg\min_{x \in A} q_{tx}(A)$.

    Let $P \leftarrow P/(1 - q_{tx}(A))$.

    Remove x from A.

  end while

  Let $z \in \arg\min_{x \in A} n_{tx}/\widehat{\lambda}_{tx}^2$.

  For each $x \in A$, let $n_{t+1,x} = \text{ceil}\left(\widehat{\lambda}_{tx}^2 (n_{tz} + B_z)/\widehat{\lambda}_{tz}^2\right)$.

  For each $x \in A$, if $n_{t+1,x} > n_{tx}$, take $n_{t+1,x} - n_{tx}$ additional samples from alternative x. Let $W_{t+1,x}$ and $\widehat{\lambda}_{t+1,x}^2$ be the sample mean and sample variance respectively of all samples from alternative x thus far.

  Increment t.

end while

Select $\hat{x} \in \arg\max_{x \in A} W_{tx}/n_{tx}$ as our estimate of the best.

1 Introduction

This paper is organized as follows: In §2, we present the proof of the validity of the algorithm when the variances are known. In §3, we prove the case when the variances are unknown.

To prove the case when the variances are known, we use a theorem for Ergodic processes that shows how to standardize the output data to make them behave like Brownian motion processes in the limit. We also use an extension of the Continuous Mapping Theorem (Theorem 5.5 of Billingsley 1968) to see that the algorithm behaves like a sequential elimination IZ procedure with a Brownian motion process instead of the standardized sum of the output data in the limit. Finally, we use the results of the paper of Frazier [3] to prove the validity of this algorithm in the limit.

2 Asymptotic Validity when the Variances are Known

Without loss of generality, suppose that the true means of the systems are indexed so that $\mu_k > \mu_{k-1} > \cdots > \mu_1$. We suppose that samples from system $x \in \{1\ldots,k\}$ are identically distributed and independent, over time and across alternatives. We also define $\lambda_z^2 := \max_{i\in\{1\ldots,k\}} \lambda_i^2$. We suppose that $\min_{i\in\{1\ldots,k\}}\lambda_i^2 > 0$ and $c \neq 1/k$. First we are going to suppose that $\delta > 0$ and $\mu = \delta a$ for some a such that $a_k - a_i \geq 1$ for all i.

Now we are going to see that the standardized sum of the output data converges to a Brownian motion in $D[0,\infty)$, which is the set of functions from $[0,\infty)$ to $\mathbb{R}$ that are right-continuous and have left-hand limits, with the Skorohod topology. We are going to define the Skorohod topology on $D[0,\infty)$ by defining a metric on the space. The Skorohod metricric d_t on $D\left[0,t\right]$ is:

$$d_t\left(X,Y\right) = \inf_{\lambda\in\Lambda_t}\left\{\|\lambda - I\| \vee \|X - Y\circ\lambda\|\right\}$$

where Λ_t is the set of strictly increasing, continuous mappings of $[0,t]$ onto itself, and $\|\cdot\|$ is the uniform norm, and I is the identity map. Note that uniform convergence on $[0,t]$ implies Skorohod convergence.

We define the Skorohod topology on $D\left[0,\infty\right)$. For $X \in D\left[0,\infty\right)$, let X^m be the element of $D_\infty := D[0,\infty)$ defined by

$$X^m\left(t\right) = g_m\left(t\right)X\left(t\right)$$

where

$$g_m\left(t\right) = \begin{cases} 1 & \text{if } t \leq m-1, \\ m-t & \text{if } m-1 \leq t \leq m, \\ 0 & \text{if } t \geq m. \end{cases}$$

And now take

$$d_\infty\left(X,Y\right) = \sum_{m=1}^{\infty} 2^{-m}\left(1 \wedge d_m\left(X^m,Y^m\right)\right)$$

which is the Skorohod metric on $D\left[0,\infty\right)$. By Theorem 16.2 of Billingsley1999, there is convergence $d_\infty\left(x_n,x\right) \to 0$ in D_∞ if and only if $d_t\left(x_n,x\right) \to 0$ for each continuity point t of x.

The following lemma shows that the standardized sum of the output data converges to a Brownian motion in D_∞.

Lemma 1. If $x \in \{1\ldots,k\}$, then

$$C_x(\delta,\cdot) := \frac{Y_{\mathrm{ceil}\left(\frac{\lambda_x^2}{\lambda_z^2}\left(n_0+\cdot\frac{1}{\delta^2}\right)\right),x} - \frac{\lambda_x^2}{\lambda_z^2}\left(n_0+\cdot\frac{1}{\delta^2}\right)\mu_x}{\frac{\lambda_x^2}{\lambda_z\delta}} \Rightarrow W_x(\cdot)$$

in the sense of $D[0,\infty)$, where $Y_{n,x}$ is the sum of the first n samples and W_x is a standard Brownian motion.

Proof. By the Theorem 19.1 of Billingsley 1999 and the sandwich theorem,

$$\frac{Y_{\mathrm{ceil}\left(\frac{\lambda_x^2}{\lambda_z^2}\left(\cdot\frac{1}{\delta^2}\right)\right),x} - \mathrm{ceil}\left(\frac{\lambda_x^2}{\lambda_z^2}\left(\cdot\frac{1}{\delta^2}\right)\right)\mu_x}{\frac{\lambda_x^2}{\lambda_z}\sqrt{\frac{1}{\delta^2}}} \Rightarrow W_x(\cdot).$$

Since $\frac{\frac{\lambda_x^2}{\lambda_z^2}t\frac{1}{\delta^2} - ceil\left(\frac{\lambda_x^2}{\lambda_z^2}t\frac{1}{\delta^2}\right)}{\frac{\lambda_x^2}{\lambda_z}\sqrt{\frac{1}{\delta^2}}} \to 0$ uniformly on $[0,r]$ for every r, then it also converges to 0 on $D[0,\infty)$ and so

$$\frac{Y_{\mathrm{ceil}\left(\frac{\lambda_x^2}{\lambda_z^2}\left(\cdot\frac{1}{\delta^2}\right)\right),x} - \left(\frac{\lambda_x^2}{\lambda_z^2}\left(\cdot\frac{1}{\delta^2}\right)\right)\mu_x}{\frac{\lambda_x^2}{\lambda_z}\sqrt{\frac{1}{\delta^2}}} \Rightarrow W_x(\cdot).$$

Observe that for $\epsilon > 0$ and δ sufficiently small

$$\left|\frac{-Y_{\mathrm{ceil}\left(\frac{\lambda_x^2}{\lambda_z^2}t\frac{1}{\delta^2}\right),x} + Y_{\mathrm{ceil}\left(n_0\frac{\lambda_x^2}{\lambda_z^2}+\frac{\lambda_x^2}{\lambda_z^2}t\frac{1}{\delta^2}\right),x}}{\frac{\lambda_x^2}{\lambda_z}\sqrt{\frac{1}{\delta^2}}}\right| < \epsilon\left(n_0\frac{\lambda_x^2}{\lambda_z^2}+2\right)$$

and then

$$C_x(\delta,\cdot) \Rightarrow W_x(\cdot).$$

■

Now we are going to define new algorithms that are almost the same as the continuous-time procedure proposed by Frazier, but these algorithms use new functions $q_{tx}^{Y,\delta}(A)$ which depend on δ and a function C that is in $D[0,\infty)^k$. More explicity, if we define

$$q'_{tx}(A) = \exp\left(\frac{\delta}{\lambda_x^2}Y'_{n_x(t),x}\right)\Big/\sum_{x'\in A}\exp\left(\frac{\delta}{\lambda_{x'}^2}Y'_{n'_x(t),x'}\right)$$

where Y'_{tx} is a Brownian motion under $\mathbb{P}_{\mu,\lambda}$ starting from 0, with drift μ_x, volatility λ_x, and independence across x, the procedure proposed by Frazier is defined by first setting

$$\tau_0 = 0, P_0 = P^*, A_0 = \{1,\ldots,k\}$$

then defining recursively, for $n = 0, 1, \ldots, k-2$,

$$\begin{aligned}
\tau_{n+1} &= \inf\left\{t \geq \tau_n : \min_{x \in A_n} q^{'}_{tx}(A_n) \leq c \text{ or } \max_{x \in A_n} q^{'}_{tx}(A_n) \geq P_n\right\} \\
Z_{n+1} &\in \arg\min_{x \in A_n} q^{'}_{\tau_{n+1},x}(A_n) \\
A_{n+1} &= A_n \setminus \{Z_{n+1}\} \\
P_{n+1} &= P_n \Big/ \left(1 - \min_{x \in A_n} q^{'}_{\tau_{n+1},x}(A_n)\right)
\end{aligned}$$

and finally letting the selected alternative $\hat{x}$ be the single entry in A_{k-1}. We also define

$$M = \inf\left\{n = 1, \ldots, k-1 : \max_{x \in A_n} q^{'}_{\tau_n,x}(A_{n-1}) \geq P_{n-1}\right\}.$$

If $C \in D[0,\infty)^k$, we define new functions $q^{C,\delta}_{tx}(A)$ by

$$q^{C,\delta}_{tx}(A) = \exp\left(\frac{C_x(t)}{\lambda_z} + \delta^2 \frac{\left(n_0 + t\frac{1}{\delta^2}\right)}{\lambda_z^2} a_x\right) / \sum_{x' \in A} \exp\left(\frac{C_{x'}(t)}{\lambda_z} + \delta^2 \frac{\left(n_0 + t\frac{1}{\delta^2}\right)}{\lambda_z^2} a_{x'}\right).$$

The reason of this definition is because $C(\delta, \cdot) = (C_x(\delta, \cdot))_{x \in A} \in D[0,\infty)^k$ and

$$\frac{\delta}{\lambda_x^2} Y_{n_x(\cdot),x} = \frac{C_x(\delta, \cdot)}{\lambda_z} + \delta^2 \frac{\left(n_0 + \cdot \frac{1}{\delta^2}\right)}{\lambda_z^2} a_x \Rightarrow W^{'}_x(\cdot) := \frac{W_x(\cdot)}{\lambda_z} + \frac{\cdot}{\lambda_z^2} a_x$$

where $W^{'}_x$ is a Brownian motion starting from 0, with drift $\frac{\mu_x}{\lambda_z^2}$ and volatility $\frac{1}{\lambda_z}$.

Note that the functions $q^{C,\delta}_{tx}(A)$ define a new algorithm for each $C \in D[0,\infty)^k$and $\delta > 0$.

We also define a new function

$$f(C, \delta) = \begin{cases} 1 & \text{if } k \text{ is chosen} \\ 0 & \text{otherwise} \end{cases}$$

using the algorithm that (C, δ) induces.

Now, we also define new functions $q^{C}_{tx}(A)$ to analyze the limit of $f(C, \delta)$ when δ goes to zero:

$$q^{C}_{tx}(A) := \exp\left(\frac{C_x(t)}{\lambda_z} + \frac{1}{\lambda_z^2} t a_x\right) / \sum_{x' \in A} \exp\left(\frac{C_{x'}(t)}{\lambda_z} + \frac{1}{\lambda_z^2} t a_{x'}\right)$$

and we define the functions

$$g(Y) = \begin{cases} 1 & \text{if } k \text{ is chosen} \\ 0 & \text{otherwise} \end{cases}.$$

We want to prove that

$$f(C(\delta, \cdot), \delta) \Rightarrow g(W).$$

In order to prove this, we will prove the Lemma 2 which will allow us to use the Theorem 5.5 of Billingsley 1968 that implies the desired result.

Lemma 2. Let $\{\delta_n\} \subset (0,\infty)$ such that $\delta_n \to 0$. If $D_s \equiv \{x \in D[0,\infty)^k$: for all sequences $\{x_n\} \subset D[0,\infty)^k$ such that $\lim_n d(x_n, x) = 0$ the sequence $\{f(x_n,\delta_n)\}$ converges to $\{g(x)\}\}$, then $\mathbb{P}(W \in D_s) = 1$.

First, we are going to prove the following two propositions.

Proposition 1. Suppose $\{f_n\}$ and $\{g_n\}$ are two sequences of functions on D_∞ such that $f_n \to f$ and $g_n \to g$ in the sense of D_∞, then

$$\min(f_n, g_n) \to \min(f, g)$$

in the sense of D_∞.

Proof. Since $f_n \to f$ and $g_n \to g$, then there exist elements λ_n, β_n of Λ_∞ such that

$$\begin{aligned} \sup\nolimits_{t<\infty} |\lambda_n t - t| &\to 0 \\ \sup\nolimits_{t<\infty} |\beta_n t - t| &\to 0 \end{aligned}$$

and, for each m,

$$\begin{aligned} \sup\nolimits_{t\leq m} |f_n(\lambda_n t) - f(t)| &\to 0 \\ \sup\nolimits_{t\leq m} |g_n(\beta_n t) - g(t)| &\to 0. \end{aligned}$$

Define

Proposition 2. Suppose $\{f_n\}$ is a sequence of functions on D_∞ such that $f_n \to f$ in the sense of D_∞ , f is continuous, and $\{T_n\} \subset [0,\infty)$ is a sequence such that $T_n \to T$. We define $T(a) := \inf\{t \geq T : f(t) \geq a\}$ for each $a \in \mathbb{R}$. Suppose $T(0) \in \mathbb{R}$. Suppose there exists $\{\epsilon_n\} \subset (0,\infty)$ such that $\epsilon_n \to 0$, ed$\epsilon_n \geq \epsilon_{n+1}$, and

$$\left\| f_n^{\delta_n}(t) - f(t) \right\|_2 < \epsilon_n$$

for all $t \in [0, T(0)]$. We also suppose that $\lim\sup_n T(\epsilon_n) \leq T(0)$. Thus we have that

$$\inf\left\{t \geq T_n : f_n^{\delta_n}(t) \geq 0\right\} \to \inf\{t \geq T : f(t) \geq 0\}$$

if $T(0) > T$ or $T_n = T$.

Proof. We are going to suppose $T(0) > T$, and the case when $T_n = T$ can be proved using almost the same ideas. We introduce the notation $T_n(a) : \inf\{t \geq T_n : f_n(t) \geq a\}$ $a \in \mathbb{R}$. Let N such that if $n > N$, $T_n < T + T(0) - T = T(0)$ and $\epsilon > \epsilon_n$ where $\epsilon := sup_n \epsilon_n$. Let $n > N$. Note that $T_n(0) \leq T_n(\epsilon_n)$.

We also have that

$$\begin{aligned} T(0) &\leq \lim\inf\nolimits_n T(\epsilon_n) \\ &\leq \lim\sup\nolimits_n T(\epsilon_n) \end{aligned} \tag{2}$$

Now, since $\lim\ \sup_n T(\epsilon_n) \le T(0)$,

$$\begin{aligned} T(0) &\ge \lim\ \sup_n T(\epsilon_n) \\ &\ge \lim\ \inf_n T(\epsilon_n) \\ &\ge T(0) \end{aligned}$$

and so

$$\lim\ \inf_n T_n(0) \le \lim\ \sup_n T_n(0) \le \lim_n T(\epsilon_n) = T(0)$$

Now, let's prove that $\lim\ \inf_n T_n(0) \ge T(0)$. Let M such that $T(0) - \frac{1}{m} \ge T$ if $m > M$, and let $t_m = T(0) - \frac{1}{m}$ and $\alpha_m = \max\{f(t) : t \in [T, t_m]\}$. Note that $\alpha_m < 0$ because $t_m < T(0)$. Let N such that if $n > N$, then

$$\epsilon_n \le -\alpha_m$$

Thus, if $t \in [T, t_m]$

$$f_n^{\delta_n}(t) < f(t) + \epsilon_n \le 0$$

If $T \le T_n(0)$, then $T_n(0) \ge t_m$ and so $\lim\ \inf_n T_n(0) \ge t_m$. Consequently, $\lim\ \inf_n T_n(0) \ge T(0)$.

Let's prove that $T \le T_n(0)$ for n large. Let $s > 0$ be any number. Let N such that if $n > N$, then $\epsilon_n < s$. Suppose that for every $\epsilon > 0$ there exists $n > N$ such that $T - \epsilon < T_n(0) < T$. Note that

$$\begin{aligned} f(T_n(0)) &\ge f_n(T_n(0)) - \epsilon_n \\ &\ge -\epsilon_n > -s \end{aligned}$$

Let $a > 0$, since f is continuous there exists $\epsilon > 0$ such that $f(x) < f(T) + a$ if $|x - T| < \epsilon$. Thus, there exists $n > N$ such that $T - \epsilon < T_n(0) < T$ and so

$$-s < f(T_n(0)) < f(T) + a$$

Since s and a are arbitrary,

$$f(T) \ge 0$$

which is a contradiction. Consequently, $T \le T_n(0)$ and so

$$\lim_n T_n(0) = T(0).$$

■

Proof of Lemma 2. We will use the following property: if T is a stopping time, then by the local version of the law of the iterated logarithm for Brownian motion

$$\lim\ \sup_{u \to 0^+} \frac{[W_x(T+u) - W_x(T)]}{\sqrt{2u \ln[\ln(1/u)]}} = 1 \tag{3}$$

almost surely for each system x. Furthermore, note that W is almost surely continuous on $[0,t]^k$ if $t > 0$ and so W is also almost surely uniformly continuous on $[0,t]^k$ because $[0,t]^k$ is compact. We will only consider the event $\mathcal{S}$ such that the previous properties hold which has probability one.

Let $\{Z_n\} \subset D[0,\infty)^k$ such that $Z_n \to W$. Note that $\delta_n^2 \frac{\left(n_0 + t\frac{1}{\delta_n^2}\right)}{\lambda_z^2} a_x \to \frac{t}{\lambda_z^2} a_x$ in $D[0,\infty)$ and $\frac{\delta_n \sqrt{\frac{1}{\delta_n^2}}}{\lambda_z} \to \frac{1}{\lambda_z}$ in $D[0,\infty)$ because uniformly convergence implies convergence in the Skorohod topology. Consequently, for each $s > 0$ there exist functions λ_s^n in Λ_s such that

$$\lim_n Z_n\left(\lambda_s^n t\right) = W(t)$$

uniformly in t and

$$\lim_n \lambda_s^n t = t$$

uniformly in t. Then

$$\lim_n \frac{\delta_n \sqrt{\frac{1}{\delta_n^2}}}{\lambda_z} Z_n\left(\lambda_s^n t\right) + \delta_n^2 \frac{\left(n_0 + \lambda_s^n(t)\frac{1}{\delta_n^2}\right)}{\lambda_z^2} a_x = W(t)\frac{1}{\lambda_z} + \frac{t}{\lambda_z^2} a_x$$

uniformly in t, and so

$$\lim_n \exp\left(\frac{\delta_n \sqrt{\frac{1}{\delta_n^2}}}{\lambda_z} Z_n\left(\lambda_n t\right) + \delta_n^2 \frac{\left(n_0 + \lambda_s^n(t)\frac{1}{\delta_n^2}\right)}{\lambda_z^2} a_x\right) = \exp\left(W(t)\frac{1}{\lambda_z} + \frac{t}{\lambda_z^2} a_x\right)$$

uniformly in t since exp is uniformly continuous in $[0,s]$. Consequently,

$$q_{\lambda_s^n(t)x}^{Z_n,\delta_n}(A) \to q_{tx}^{W}(A)$$

uniformly in $t \in [0,s]$. Thus, $q_{\cdot x}^{Z_n,\delta_n}(A) \to q_{\cdot x}^{W}(A)$ in $D[0,s]$ for any set $A \subset \{1,\ldots,k\}$ and $s \geq 0$. Consequently, $q_{\cdot x}^{Z_n,\delta_n}(A) \to q_{\cdot x}^{W}(A)$ in $D[0,\infty)$. By Proposition 1, $\min_{x\in A} q_{\cdot x}^{Z_n,\delta_n}(A) \to \min_{x\in A} q_{\cdot x}^{W}(A)$ and $\max_{x\in A} q_{\cdot x}^{Z_n,\delta_n}(A) \to \max_{x\in A} q_{\cdot x}^{W}(A)$, and so

$$f_{Z_n} := \max\left\{c - \min_{x\in A} q_{\cdot x}^{Z_n,\delta_n}(A), \max_{x\in A} q_{\cdot x}^{Z_n,\delta_n}(A) - P_s\right\} \to f_W := \max\left\{c - \min_{x\in A} q_{\cdot x}^{W}(A), \max_{x\in A} q_{\cdot x}^{W}(A) - P_s\right\}$$

in $D[0,\infty)$ for any P_s of the algorithm.

Then for each $m \geq 0$ there exists $\lambda_n \in \Lambda_\infty$ such that $\sup_{t\leq m} \|\lambda_n(t) - t\|_2 \leq d(f_{Z_n}, f_W) + \frac{1}{n}$ and $\sup_{t\leq m} \|f_{Z_n}(t) - f_W(\lambda$ $d(f_{Z_n}, f_W) + \frac{1}{n}$. Taking $g_n \equiv \sup_{t\leq m} \|f_W(t) - f_W(\lambda_n t)\|_2$, we see from the uniform continuity of f_W on $[0,m]^k$ (f_W is uniformly continuous because it's continuos in a compact set) and the definition of g_n that $\lim_{n\to\infty} g_n = 0$. Moreover, if we take $\epsilon_n = 2n^{-1} + 2\sup\{d(f_{Z_l}, f_W) + g_l : l = n, n+1, \ldots\}$, then $\{\epsilon_n\}$ is a monotonically decreasing sequence of positive numbers with limit zero.

From the definition of ϵ_n we have $d(f_{Z_n}, f_W) < \epsilon_n/2$ and $g_n < \epsilon_n/2$ for $n = 1, 2, \ldots$ Consequently, we have

$$\begin{aligned} \|f_{Z_n}(t) - f_W(t)\| &\leq \|f_{Z_n}(t) - f_W(\lambda_n t)\| + \|f_W(\lambda_n t) - f_W(t)\| \\ &< \epsilon_n \end{aligned}$$

for all $t \in [0,m]$.

We will show that $\mathbb{P}(W \in D_s \mid M_W = i) = 1$ for $i \in \{1,\ldots,k-1\}$ so that the desired conclusion follows.

Suppose first that $M_W = 1$. Let's prove that $T_{Z_n,\delta_n}^1(P) \to T_W^1(P)$ as $n \to \infty$.

First, we are going to prove that $T_W^1(P) = \lim_n T_W^1(P + \epsilon_n)$. Suppose that $k = 2$ and let $x = \arg\max q_{T_W^1 x}^{W}(A)$.

By the local version of the law of the iterated logarithm for Brownian motion, for any $\epsilon \in (0,1)$ there exists a monotonically decreasing sequence $\{t_k : k = 1, 2, \ldots\} \subset \left(T_W^1(P), \infty\right)$ such that

$$W_x(t_k) - W_x\left(T_W^1(P)\right) > \epsilon \sqrt{2\left(t_k - T_W^1(P)\right) \ln\left(\ln\left(\frac{1}{t_k - T_W^1(P)}\right)\right)}$$

Let N such that if $m > N$, then

$$\frac{\epsilon}{\lambda_z} \sqrt{2\left(t_m - T_W^1(P)\right)^{-1} \ln\left(\ln\left(\frac{1}{t_m - T_W^1(P)}\right)\right)} > \frac{a_{x'} - a_x}{\lambda_z^2} - \frac{1}{\lambda_z}$$

Since $\underline{\lim}_{h \to 0^+} \frac{W_{x'}\left(T_W^1(P)+h\right) - W_{x'}\left(T_W^1(P)\right)}{h} = -\infty$, then there exists N_2 such that if $n > N_2$, then there exists $m \geq n$ such that

$$W_{x'}(t_m) - W_{x'}\left(T_W^1(P)\right) < \left(-t_m + T_W^1(P)\right)$$

Let $m > \max\{N, N_2\}$, then

$$\begin{aligned}
& \frac{W_{x'}\left(T_W^1(P)\right) - W_x\left(T_W^1(P)\right) - W_{x'}(t_m) + W_x(t_m)}{\lambda_z} + \frac{1}{\lambda_z^2}\left(t_m - T_W^1(P)\right)\left(a_x - a_{x'}\right) \\
> \; & \frac{-T_W^1(P) + t_m + \epsilon\sqrt{2\left(t_m - T_W^1(P)\right)\ln\left(\ln\left(\frac{1}{t_k - T_W^1(P)}\right)\right)}}{\lambda_z} + \frac{1}{\lambda_z^2}\left(t_m - T_W^1(P)\right)\left(a_x - a_{x'}\right) \\
> \; & \frac{-T_W^1(P) + t_m}{\lambda_z} + \left(t_m - T_W^1(P)\right)\left(\frac{a_{x'} - a_x}{\lambda_z^2} - \frac{1}{\lambda_z}\right) + \frac{1}{\lambda_z^2}\left(t_m - T_W^1(P)\right)\left(a_x - a_{x'}\right) \\
= \; & 0
\end{aligned}$$

and so

$$\frac{W_{x'}\left(T_W^1(P)\right) - W_x\left(T_W^1(P)\right)}{\lambda_z} + \frac{1}{\lambda_z^2}\left(T_W^1(P)\right)\left(a_{x'} - a_x\right) > \frac{W_{x'}(t_m) - W_x(t_m)}{\lambda_z} + \frac{1}{\lambda_z^2}(t_m)\left(a_{x'} - a_x\right)$$

Since $q_{t,x}^W(A) = \left(1 + \sum_{x' \in A - \{x\}} \frac{q_{t,x'}^W(A)}{q_{t,x}^W(A)}\right)^{-1}$, we have that

$$q_{t_m,x}^W(A) > q_{T_W^1,x}^W(A) = P$$

Since $\lim_n T_W^1(P + \epsilon_n) = \inf_n T_W^1(P + \epsilon_n) \geq T_W^1(P)$, then $T_W^1(P) = \lim_n T_W^1(P + \epsilon_n)$. Suppose that we have $k+1$ systems and $x = \arg\max q_{T_W^1 x}^W(A)$. Suppose that

$$\sum_{x' \in A - \{x\} - \{y\}} \frac{q_{t,x'}^W(A)}{q_{t,x}^W(A)}$$

has no a minimum local in $T_W^1(P)$. By the previous proof $\frac{q_{t,y}^W(A)}{q_{t,x}^W(A)}$ has no a minimum local in $T_W^1(P)$. Consequently,

Suppose that there exists a sequence $\{t_m\} \subset \left(T_W^1(P), \infty\right)$ such that $t_m \downarrow T_W^1(P)$ and there exists N_1 such that if $m > N_1$, then

$$\sum_{x' \in A-\{x\}-\{y\}} \frac{q^W_{t_m,x'}(A)}{q^W_{t_m,x}(A)} < \sum_{x' \in A-\{x\}-\{y\}} \frac{q^W_{T^1_W,x'}(A)}{q^W_{T^1_W,x}(A)}$$

By the previous proof, there exists a sequence $\{s_m\} \subset \left(T^1_W(P), \infty\right)$ such that $s_m \downarrow T^1_W(P)$ and there exists N_2 such that if $m > N_2$, then

$$\frac{q^W_{s_m,y}(A)}{q^W_{s_m,x}(A)} < \frac{q^W_{T^1_W,y}(A)}{q^W_{T^1_W,x}(A)}$$

Since $\frac{q^W_{s_m,y}(A)}{q^W_{s_m,x}(A)}$ is uniformly continuous on $[T^1_W(P), T^1_W(P)+1]$ there exists N_3 such that if $m > N_3$ then

$$\frac{q^W_{t_m,y}(A)}{q^W_{t_m,x}(A)} < \frac{q^W_{s_m,y}(A)}{q^W_{s_m,x}(A)} + a$$

Consequently, if $m > N = \max\{N_1, N_2, N_3\}$, then

$$\sum_{x' \in A-\{x\}} \frac{q^W_{t_m,x'}(A)}{q^W_{t_m,x}(A)} < \sum_{x' \in A-\{x\}} \frac{q^W_{T^1_W,x'}(A)}{q^W_{T^1_W,x}(A)} + a$$

Thus, by induction for all $a > 0$, there exists N such that if $m > N$, then

$$\sum_{x' \in A-\{x\}} \frac{q^W_{t_m,x'}(A)}{q^W_{t_m,x}(A)} < \sum_{x' \in A-\{x\}} \frac{q^W_{T^1_W,x'}(A)}{q^W_{T^1_W,x}(A)} + a$$

Thus, $T^1_W(P) = \lim_n T^1_W(P+\epsilon_n)$.

By Proposition 2,

$$T^1_W(P) = \lim_n T^1_{Z_n,\delta_n}(P).$$

Let $x = \arg\max_x q^W_{T^1_W(P),x}(A)$. Let $\epsilon > 0$. Let N such that if $n > N$, then $2\epsilon_n + 2\epsilon < -\max_{y \in A-\{x\}} q^W_{T^1_{W(P)},y}(A) + q^W_{T^1_W(P),x}(A)$

$$\left| q^W_{T^1_W(P),x}(A) - q^{Z_n,\delta_n}_{T^1_{Z_n,\delta_n}(P),x}(A) \right| < \epsilon + \epsilon_n$$

for all $x \in A$, and so if $z \in A - \{x\}$,

$$\begin{aligned} q^{Z_n,\delta_n}_{T^1_{Z_n,\delta_n}(P),z}(A) &< \epsilon + \epsilon_n + q^W_{T^1_W(P),z}(A) \\ &< -\epsilon - \epsilon_n + q^W_{T^1_W(P),x}(A) \\ &< q^{Z_n,\delta_n}_{T^1_{Z_n,\delta_n}(P),x}(A) \end{aligned}$$

and so $x = \arg\max_x q^{Z_n,\delta_n}_{T^1_{Z_n,\delta_n}(P),x}(A)$.

Now, we suppose that $M_W = 2$. By a similar argument than the previous one, we should have that $T^1_W(A) = \lim_n T^1_{Z_n,\delta_n}$ and $T^2_W\left(P^W_1\right) = \lim_n T^2_{Z_n,\delta_n}\left(P^{Z_n,\delta_n}_1\right)$. By a similar argument than before, we can see that

$\arg\max_x q^{Z_n,\delta_n}_{T^2_{Z_n,\delta_n}(P^W_1),x}\left(A^W_1\right) = \arg\max_x q^{W}_{T^2_W(P^W_1),x}\left(A^W_1\right)$ for n sufficiently large.

The cases $M_W = i$ for $k-1 \geq i \geq 3$ can be proved in a similar way.

Since almost surely $M_Y \in \{1,\ldots,k-1\}$ by Frazier [3], we conclude that

$$\mathbb{P}\left(W \in \mathrm{D}[0,1]^k\text{-}D_s\right) = 1.$$

■

By the extension of the CMT (Theorem 5.5 of Billingsley 1968), we have the following corollary.

Corollary 1. We have that

$$f\left(C\left(\delta,t\right),\delta\right) \Rightarrow g\left(W\left(t\right)\right)$$

in distribution as $\delta \to 0$.

Theorem 1. If samples from system $x \in \{1\ldots,k\}$ are identically distributed and independent, over time and across alternatives, then $\lim_{\delta\to 0} Pr\left\{\text{BIZ selects } k\right\} \geq P*$ provided $\mu_k = a_k\delta, \mu_{k-1} = a_{k-1}\delta, \ldots, \mu_1 = a_1\delta$. We also suppose $B_1 = \cdots = B_k = 1$.

Proof. Let

$$\hat{T}_n\left(\delta\right) = \min\left\{t \in \left\{0,\delta^2,2\delta^2,\ldots\right\} : \ \min_{x\in A^{Y,\delta}_n} q^{C(\delta,\cdot),\delta}_{tx}\left(A^{C(\delta,\cdot),\delta}_n\right) \leq c \text{ or } \max_{x\in A^{Y,\delta}_n} q^{C(\delta,\cdot),\delta}_{tx}\left(A^{C(\delta,\cdot),\delta}_n\right) \geq P^{C(\delta,\cdot),\delta}_n\right\}$$

and $T_n\left(\delta\right)$ the usual stopping times of the algorithm. Then $T_n\left(\delta\right) = \hat{T}_n\left(\delta\right)/\delta^2$. Now, we can prove that $\hat{T}_n\left(\delta\right) - T^n_{C(\delta,\cdot),\delta}\left(P^{C(\delta,\cdot),\delta}_n\right) \to 0$ with probability 1 as $\delta \to 0$ using that $C\left(\delta,\cdot\right)$ is right-continuous and $\delta^2 \to 0$. Consequently, we can use $C\left(\delta, T^n_{C(\delta,\cdot),\delta}\left(P^{C(\delta,\cdot),\delta}_n\right)\right)$ instead of $C\left(\delta,\hat{T}_n\left(\delta\right)\right)$.

Let CS_δ be the event of doing a correct selection given the configuration $\mu_k = a_k\delta, \mu_{k-1}, \mu_{k-1} = a_{k-1}\delta, \ldots, \mu_1 = a_1\delta$. Then

$$\begin{aligned}
\underline{lim}_{\delta\to 0}\mathbb{P}\left(CS_\delta\right) &= \underline{lim}_{\delta\to 0}\mathbb{P}\left(f\left(C\left(\delta,t\right),\delta\right) = 1\right) \\
&= \mathbb{P}\left(g\left(W\right) = 1\right) \\
&\geq P^*
\end{aligned}$$

where the last inequality follows from the paper of Frazier [3].

■

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