# On the Asymptotic Validity of a Fully Sequential Elimination Procedure for Indifference-Zone Ranking and Selection with Tight Bounds on Probability of Correct Selection

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We prove the validity of the sequential elimination IZ procedure proposed by Frazier [] when $\delta$ goes to 0. Specifically, we analyze Algorithm 2, when $B_1 = \cdots = B_k = 1$:

**Algorithm 2: Discrete-time implementation of BIZ, for unknown and/or heterogeneous variances.**

**Require:** $c \in [0, 1-(P^*)^{\frac{1}{k-1}}]$, $\delta > 0$, $P^* \in (1/k, 1)$, $n_0 \geq 0$ an integer, $B_1, \ldots, B_k$ strictly positive integers. Recommended choices are $c = 1-(P^*)^{\frac{1}{k-1}}$, $B_1 = \cdots = B_k = 1$ and $n_0$ between 10 and 30. If the sampling variances $\lambda_x^2$ are known, replace the estimators $\widehat{\lambda}_{tx}^2$ with the true values $\lambda_x^2$, and set $n_0 = 0$. To compute $\widehat{q}_{tx}(A)$, use

$$q'_{t,x}(A) = \exp\left(\gamma\delta Y'_{tx}\right) \Big/ \sum_{x' \in A} \exp\left(\gamma\delta Y'_{tx'}\right) = \exp\left(\frac{\delta}{\lambda_x^2} Y_{n_x(t),x}\right) \Big/ \sum_{x' \in A} \exp\left(\frac{\delta}{\lambda_{x'}^2} Y_{n'_x(t),x'}\right), \tag{1}$$

where $Y_{n_x(t),x}$ is the sum of the first $n_x(t)$ samples.

For each $x$, sample alternative $x$ $n_0$ times and set $n_{0x} \leftarrow n_0$. Let $W_{0x}$ and $\widehat{\lambda}_{0x}^2$ be the sample mean and sample variance respectively of these samples. Let $t \leftarrow 0$.
Let $A \leftarrow \{1, \ldots, k\}$, $P \leftarrow P^*$, $t \leftarrow 1$.
**while** $x \in \max_{x\in A} \widehat{q}_{tx}(A) < P$ **do**
  **while** $\min_{x \in A} \widehat{q}_{tx}(A) \leq c$ **do**
    Let $x \in \arg\min_{x\in A} \widehat{q}_{tx}(A)$.
    Let $P \leftarrow P/(1-\widehat{q}_{tx}(A))$.
    Remove $x$ from $A$.
  **end while**
  Let $z \in \arg\min_{x\in A} n_{tx}/\widehat{\lambda}_{tx}^2$.
  For each $x \in A$, let $n_{t+1,x} = \text{ceil}\left(\widehat{\lambda}_{tx}^2 (n_{tz} + B_z)/\widehat{\lambda}_{tz}^2\right)$.
  For each $x \in A$, if $n_{t+1,x} > n_{tx}$, take $n_{t+1,x} - n_{tx}$ additional samples from alternative $x$. Let $W_{t+1,x}$ and $\widehat{\lambda}_{t+1,x}^2$ be the sample mean and sample variance respectively of all samples from alternative $x$ thus far.
  Increment $t$.
**end while**
Select $\hat{x} \in \arg\max_{x\in A} W_{tx}/n_{tx}$ as our estimate of the best.

# 1 Introduction

This paper is organized as follows: In §2, we present the proof of the validity of the algorithm when the variances are known. In §3, we prove the case when the variances are unknown.

To prove the case when the variances are known, we use a Functional Central Limit Thoerem that shows how to standardize the output data to make them behave like Brownian motion processes in the limit. We also use an extension of the Continuous Mapping Theorem (Theorem 5.5 of Billingsley 1968) to see that the algorithm behaves like a sequential elimination IZ procedure with a Brownian motion process instead of the standardize of the sum of the output data in the limit. Finally, we use the results of the paper of Frazier [] to prove the validity of this algorithm in the limit.

# 2 Asymptotic Validity when the Variances are Known

Without loss of generality, suppose that the true means of the systems are indexed so that $\mu_k \geq \mu_{k-1} \geq \cdots \geq \mu_1$. We suppose that samples from system $x \in \{1 \ldots, k\}$ are identically distributed and independent, over time and across alternatives. We also define $\lambda_z^2 := \max_{i\in\{1\ldots,k\}} \lambda_i^2$. We suppose that $\min_{i\in\{1\ldots,k\}}\lambda_i^2 > 0$.

Now we are going to see that the standardize of the sum of the output data converges to a Brownian motion in $D[0,\infty)$, which is the set of functions from $[0,\infty)$ to $\mathbb{R}$ that are right-continuous and have left-hand limits, with the Skorohod topology. The Skorohod metric $d_t$ on $D\left[0,t\right]$ is:

$$d_t\left(X,Y\right) = \inf_{\lambda\in\Lambda_t}\left\{\|\lambda - I\| \vee \|X - Y\circ\lambda\|\right\}$$

where $\Lambda_t$ is the set of strictly increasing, continuous mappings of $[0,t]$ onto itself, and $\|\cdot\|$ is the uniform norm, and $I$ is the identity map.

For $X \in D\left[0,\infty\right)$, let $X^m$ be the element of $D[0,\infty)$ defined by

$$X^m\left(t\right) = g_m\left(t\right)X\left(t\right)$$

where

$$g_m\left(t\right) = \begin{cases} 1 & \text{if } t \leq m-1, \\ m-t & \text{if } m-1 \leq t \leq m, \\ 0 & \text{if } t \geq m. \end{cases}$$

And now take

$$d_\infty\left(X,Y\right) = \sum_{m=1}^{\infty} 2^{-m}\left(1 \wedge d_m\left(X,Y\right)\right).$$

**Lemma 1.** If $x \in \{1\ldots,k\}$, then

$$C_x\left(\delta,\cdot\right) := \frac{Y_{\mathrm{ceil}\left(\frac{\lambda_x^2}{\lambda_z^2}\left(n_0+\cdot\frac{1}{\delta^2}\right)\right),x} - \frac{\lambda_x^2}{\lambda_z^2}\left(n_0+\cdot\frac{1}{\delta^2}\right)\mu_x}{\frac{\lambda_x^2}{\lambda_z}\sqrt{\frac{1}{\delta^2}}} \Rightarrow W_x\left(\cdot\right)$$

in the sense of $D[0,\infty)$, where $Y_{n,x}$ is the sum of the first $n$ samples and $W_x$ is a standard Brownian motion.

**Proof.** By the Theorem 19.1 of Billingsley 1999 and the sandwich theorem,

$$\frac{Y_{\text{ceil}\left(\frac{\lambda_x^2}{\lambda_z^2}\left(\cdot\frac{1}{\delta^2}\right)\right),x}-\text{ceil}\left(\frac{\lambda_x^2}{\lambda_z^2}\left(\cdot\frac{1}{\delta^2}\right)\right)\mu_x}{\frac{\lambda_x^2}{\lambda_z}\sqrt{\frac{1}{\delta^2}}}\Rightarrow W_x\left(\cdot\right).$$

Since $\frac{\frac{\lambda_x^2}{\lambda_z^2}t\frac{1}{\delta^2}-ceil\left(\frac{\lambda_x^2}{\lambda_z^2}t\frac{1}{\delta^2}\right)}{\frac{\lambda_x^2}{\lambda_z}\sqrt{\frac{1}{\delta^2}}}\to 0$ uniformly on $[0,r]$ for every $r$, then it also converges to 0 on $D[0,\infty)$ and so

$$\frac{Y_{\text{ceil}\left(\frac{\lambda_x^2}{\lambda_z^2}\left(\cdot\frac{1}{\delta^2}\right)\right),x}-\left(\frac{\lambda_x^2}{\lambda_z^2}\left(\cdot\frac{1}{\delta^2}\right)\right)\mu_x}{\frac{\lambda_x^2}{\lambda_z}\sqrt{\frac{1}{\delta^2}}}\Rightarrow W_x\left(\cdot\right).$$

Observe that for $\epsilon>0$ and $\delta$ sufficiently small

$$\left|\frac{-Y_{\text{ceil}\left(\frac{\lambda_x^2}{\lambda_z^2}t\frac{1}{\delta^2}\right),x}+Y_{\text{ceil}\left(n_0\frac{\lambda_x^2}{\lambda_z^2}+\frac{\lambda_x^2}{\lambda_z^2}t\frac{1}{\delta^2}\right),x}}{\frac{\lambda_x^2}{\lambda_z}\sqrt{\frac{1}{\delta^2}}}\right|<\epsilon\left(n_0\frac{\lambda_x^2}{\lambda_z^2}+2\right)$$

and then

$$C_x\left(\delta,\cdot\right)\Rightarrow W_x\left(\cdot\right).$$

■

Now we are going to define new algorithms that are almost the same than the one proposed by Frazier, but instead of $q_{tx}^{\delta}\left(A\right)$, these algorithms use new functions $q_{tx}^{Y,\delta}\left(A\right)$ which depend on a function $Y$ that is in $D\left[0,\infty\right)^k$.

First we are going to suppose that $\delta>0$ and $\mu_k=\delta$,$\mu_{k-1}=\cdots=\mu_1=0$.

Let $Y\in D\left[0,\infty\right)^k$ , we define

$$q_{tx}^{Y,\delta}\left(A\right):\ =\ \exp\left(\frac{Y_x\left(t\right)}{\lambda_z}+\delta^2\beta_{t\frac{1}{\delta^2}}I_{\{x=k\}}\right)/\sum_{x'\in A}\exp\left(\frac{Y_{x'}\left(t\right)}{\lambda_z}+\delta^2\beta_{t\frac{1}{\delta^2}}I_{\left\{x'=k\right\}}\right)$$

where $\beta_{t\frac{1}{\delta^2}}=\frac{\left(n_0+t\frac{1}{\delta^2}\right)}{\lambda_z^2}$.

We define

$$
\begin{aligned}
T_{Y,\delta}^{0} &= 0 \\
A_{0}^{Y,\delta} &= \{1,\ldots,k\} \\
P_{0}^{Y,\delta} &= P^{*} \\
T_{Y,\delta}^{n+1}\left(P_{n}^{Y,\delta}\right) &= \inf\left\{t \geq T_{Y,\delta}^{n}:\ \min_{x\in A_{n}^{Y,\delta}} q_{tx}^{Y,\delta}\left(A_{n}^{Y,\delta}\right) \leq c \text{ or } \max_{x\in A_{n}^{Y,\delta}} q_{tx}^{Y,\delta}\left(A_{n}^{Y,\delta}\right) \geq P_{n}^{Y,\delta}\right\} \\
Z_{n+1}^{Y,\delta} &\in \arg\min_{x\in A_{n}^{Y,\delta}} q_{T_{Y,\delta}^{n+1},x}^{Y,\delta}\left(A_{n}^{Y,\delta}\right) \\
A_{n+1}^{Y,\delta} &= A_{n} - \left\{Z_{n+1}^{Y,\delta}\right\} \\
P_{n+1}^{Y,\delta} &= P_{n}^{Y,\delta}/\left(1-\min_{x\in A_{n}^{Y,\delta}} q_{T_{Y,\delta}^{n+1}x}^{Y,\delta}\left(A_{n}^{Y,\delta}\right)\right).
\end{aligned}
$$

Now, let

$$M_{Y,\delta} = \inf\left\{n=1,\ldots,k-1 : \max_{x\in A_{n-1}^{Y,\delta}} q_{T_{Y,\delta}^{n},x}\left(A_{n-1}^{Y,\delta}\right) \geq P_{n-1}^{Y,\delta}\right\}$$

and

$$f\left(Y,\delta\right) = \begin{cases} 1 & \text{if } k \text{ is chosen} \\ 0 & \text{otherwise} \end{cases}.$$

Now, we also define

$$q_{tx}^{Y}\left(A\right) := \exp\left(\frac{Y_{x}\left(t\right)}{\lambda_{z}} + \frac{1}{\lambda_{z}^{2}} t I_{\{x=k\}}\right) / \sum_{x'\in A} \exp\left(\frac{Y_{x'}\left(t\right)}{\lambda_{z}} + \frac{1}{\lambda_{z}^{2}} t I_{\{x'=k\}}\right)$$

$$
\begin{aligned}
T_{Y}^{0} &= 0 \\
A_{0}^{Y} &= \{1,\ldots,k\} \\
P_{0}^{Y} &= P^{*} \\
T_{Y}^{n+1}\left(P_{n}^{Y}\right) &= \inf\left\{t \geq T_{Y}^{n}:\ \min_{x\in A_{n}^{Y}} q_{tx}^{Y}\left(A_{n}^{Y}\right) \leq c \text{ or } \max_{x\in A_{n}^{Y}} q_{tx}^{Y}\left(A_{n}^{Y}\right) \geq P_{n}^{Y}\right\} \\
Z_{n+1}^{Y} &\in \arg\min_{x\in A_{n}^{Y}} q_{T_{Y}^{n+1},x}^{Y}\left(A_{n}^{Y}\right) \\
A_{n+1}^{Y} &= A_{n}^{Y} - \left\{Z_{n+1}^{Y}\right\} \\
P_{n+1}^{Y} &= P_{n}^{Y,\delta}/\left(1-\min_{x\in A_{n}^{Y,}} q_{T_{Y}^{n+1}x}^{Y}\left(A_{n}^{Y}\right)\right).
\end{aligned}
$$

Now, let

$$M_{Y} = \inf\left\{n=1,\ldots,k-1 : \max_{x\in A_{n-1}^{Y}} q_{T_{Y}^{n},x}\left(A_{n-1}^{Y}\right) \geq P_{n-1}^{Y}\right\}$$

and

$$g\left(Y\right) = \begin{cases} 1 & \text{if } k \text{ is chosen} \\ 0 & \text{otherwise} \end{cases}.$$

We define for $x \in A_{M-1}^{Y,\delta}$

$$H_x\left(C\left(\delta,\cdot\right)\right) = \lambda_k^2\left(C_k\left(\delta, T_{C(\delta,\cdot),\delta}^{M_{C(\delta,\cdot),\delta}}\right)\right) + \delta\frac{\lambda_k^2}{\lambda_z\sqrt{R}}\left(n_0 + T_{C(\delta,\cdot),\delta}^{M_{C(\delta,\cdot),\delta}} R\right) - \lambda_x^2\left(Y_x\left(T_{C(\delta,\cdot),\delta}^{M_{C(\delta,\cdot),\delta}}\right)\right)$$

and for $x \in A_{M-1}^{Y}$

$$S_x\left(W\left(\cdot\right), \Delta\right) = W_k\left(T_W^{M_W}\right) + \Delta\frac{1}{\lambda_z}T_W^{M_W} - \frac{\lambda_x^2}{\lambda_k^2}\left(W_x\left(T_W^{M_W}\right)\right)$$

Now, we want to prove that for $\delta$ sufficiently small $A_{M-1}^{Y,\delta} \subset A_{M-1}^{Y}$ and if $x \in A_{M-1}^{Y,\delta}$ for $\delta$ sufficiently small

$$H_x\left(C\left(\delta,\cdot\right)\right) \Rightarrow S_x\left(W\left(\cdot\right), \Delta\right)$$

as $\delta \to 0$.

We want to prove that

$$f\left(C\left(\delta,\cdot\right), \delta\right) \Rightarrow g\left(W\right).$$

In order to prove this, we will prove the following lemma which will allow us to use the Theorem 5.5 of Billingsley 1968 that implies the desired result.

**Lemma 2.** Let $\{\delta_n\} \subset (0,\infty)$ such that $\delta_n \to 0$. If $D_s \equiv \{x \in D\left[0,\infty\right)^k :$ for all sequences $\{x_n\} \subset D\left[0,\infty\right)^k$, such that $lim_n d\left(x_n, x\right) = 0$ the sequence $\{f\left(x_n, \delta_n\right)\}$ converges to $\{g\left(x\right)\}\}$, then $\mathbb{P}\left(W \in D_s\right) = 1$.

**Proof.** We will use the following property: if $T$ is a stopping time, then by the local version of the law of the iterated logarithm for Brownian motion

$$\lim\ \sup_{u\to 0^+}\frac{\left[W_x\left(T+u\right) - W_x\left(T\right)\right]}{\sqrt{2u\ln\left[\ln\left(1/u\right)\right]}} = 1 \tag{2}$$

almost surely for each system $x$. Furthermore, note that $W$ is almost surely continuous on $\left[0,t\right]^k$ if $t>0$ and so $W$ is also almost surely uniformly continuous on $\left[0,t\right]^k$.

Let $\{Z_n\} \subset D\left[0,\infty\right)^k$ such that $Z_n \to W$. Furthermore, $\delta_n^2\frac{\left(n_0 + t\frac{1}{\delta_n^2}\right)}{\lambda_z^2} \to \frac{t}{\lambda_z^2}$ in $D\left[0,\infty\right)$ and $\frac{\delta_n\sqrt{\frac{1}{\delta_n^2}}}{\lambda_z} \to \frac{1}{\lambda_z}$ in $D\left[0,\infty\right)$ because uniformly convergence implies convergence in the Skorohod topology. Consequently, for each $s>0$ there exist functions $\lambda_s^n$ in $\Lambda$ such that

$$\lim_n Z_n\left(\lambda_s^n t\right) = W\left(t\right)$$

uniformly in $t$ and

$$\lim_n \lambda_s^n t = t$$

uniformly in $t$. Then

$$\lim_n \frac{\delta_n\sqrt{\frac{1}{\delta_n^2}}}{\lambda_z} Z_n\left(\lambda_s^n t\right) + \delta_n^2 \frac{\left(n_0 + \lambda_s^n\left(t\right)\frac{1}{\delta_n^2}\right)}{\lambda_z^2} = W\left(t\right)\frac{1}{\lambda_z} + \frac{t}{\lambda_z^2}$$

uniformly in $t$, and so

$$\lim_n \exp\left(\frac{\delta_n\sqrt{\frac{1}{\delta_n^2}}}{\lambda_z} Z_n\left(\lambda_n t\right) + \delta_n^2 \frac{\left(n_0 + \lambda_s^n\left(t\right)\frac{1}{\delta_n^2}\right)}{\lambda_z^2}\right) = \exp\left(W\left(t\right)\frac{1}{\lambda_z} + \frac{t}{\lambda_z^2}\right)$$

uniformly in $t$ since exp is uniformly continuous in $[0,s]$. Consequently,

$$q_{\lambda_s^n(t)x}^{Z_n,\delta_n}\left(A\right) \to q_{tx}^{W}\left(A\right)$$

uniformly in $t \in [0,s]$. Thus, $q_{\cdot x}^{Z_n,\delta_n}\left(A\right) \to q_{\cdot x}^{W}\left(A\right)$ in $D\left[0,s\right]$ for any set $A \subset \{1,\ldots,k\}$ and $s \geq 0$. Consequently, $q_{\cdot x}^{Z_n,\delta_n}\left(A\right) \to q_{\cdot x}^{W}\left(A\right)$ in $D\left[0,\infty\right)$. Then for each $m \geq 0$ there exists $\lambda_{n,A} \in \Lambda_\infty$ such that $\sup_{t<\infty}\left\|\lambda_{n,A}\left(t\right) - t\right\| \leq d\left(q_{\cdot x}^{Z_n,\delta_n}\left(A\right), q_{\cdot x}^{W}\left(A\right)\right) + \frac{1}{n}$ and $\sup_{t\leq m}\left\|q_{tx}^{Z_n,\delta_n}\left(A\right) - q_{\lambda_{n,A}(t)x}^{W}\left(A\right)\right\| \leq d\left(q_{\cdot x}^{Z_n,\delta_n}\left(A\right), q_{\cdot x}^{W}\left(A\right)\right) + \frac{1}{n}$. Taking $g_n^A \equiv \sup_{t\leq m}\left\|q_{tx}^{W}\left(A\right) - q_{\lambda_n(t)x}^{W}\left(A\right)\right\|$, we see from the uniform continuity of $W$ on $[0,m]^k$ and the definition of $g_n^A$ that $\lim_{n\to\infty} g_n^A = 0$. Moreover, if we take $\epsilon_n^A = 3n^{-1} + 3\sup\left\{d\left(q_{\cdot x}^{Z_l,\delta_l}\left(A\right), q_{\cdot x}^{W}\left(A\right)\right) + g_l^A : l = n, n+1, \ldots\right\}$, then $\left\{\epsilon_n^A\right\}$ is a monotonically decreasing sequence of positive numbers with limit zero.

From the definition of $\epsilon_n$ we have $d\left(q_{\cdot x}^{Z_n,\delta_n}\left(A\right), q_{\cdot x}^{W}\left(A\right)\right) < \epsilon_n/2$ and $g_n^A < \epsilon_n/2$ for $n = 1,2,\ldots$ Consequently, we have

$$\begin{aligned}
\left\|q_{tx}^{Z_n,\delta_n}\left(A\right) - q_{tx}^{W}\left(A\right)\right\| &\leq \left\|q_{tx}^{Z_n,\delta_n}\left(A\right) - q_{\lambda_n(t)x}^{W}\left(A\right)\right\| + \left\|q_{\lambda_n(t)x}^{W}\left(A\right) - q_{tx}^{W}\left(A\right)\right\| \\
&< \epsilon_n^A
\end{aligned}$$

for all $t \in [0,m]$ and $x \in A$.

We will show that $\mathbb{P}\left(W \in D_s \mid M_W = i\right) = 1$ for $i \in \{1,\ldots,k-1\}$ so that the desired conclusion follows.

Suppose first that $M_W = 1$. Let's prove that $T_{Z_n,\delta_n}^1\left(P\right) \to T_W^1\left(P\right)$ as $n \to \infty$. Since $M_W = 1$, then $\max_{x\in A} q_{T_W^1 x}^{W}\left(A\right) = P$ almost surely because $W$ is continuous almost surely. Let $q_x^* = \min_{t\in\left[0,T_W^1(P)\right]} q_{tx}^W$ (the minimum exists because $q_{tx}^W$ is continuous) and let $q^* = \min_x q_x^*$. Let $N$ such that if $n > N$ then $\epsilon_n^A < q^* - c$ (note that in the last equation $m = T_W^1\left(P\right)$ and $q^* - c > 0$ because $M_W = 1$ and $N$ is a random variable). If $n > N$ and $t \leq T_W^1\left(P\right)$ then

$$\begin{aligned}
q_{tx}^W &< q_{tx}^{Z_n,\delta_n} + q^* - c \\
\Rightarrow c &< q_{tx}^{Z_n,\delta_n} + q^* - q_{tx}^W \leq q_{tx}^{Z_n,\delta_n}.
\end{aligned}$$

Suppose $T_W^1\left(P\right) > \epsilon > 0$, and let $q_x^+ = \max_{t\in\left[0,T_W^1-\epsilon\right]} q_{tx}^W$ and $q^+ = \max_x q_x^*$. Let $N_2^*$ such that if $n > N_2^*$, then $\delta_n < r < P - q^+$ where $r \in \mathbb{Q}$. Let $N_2^{**}$ such that if $n > N_2^{**}$, then $\epsilon_n^A < P - q^+ - r$. Thus, if $n > N_2 := \max\left\{N_2^*, N_2^{**}\right\}$ and $t \leq T_W^1\left(P\right) - \epsilon$, then

$$\begin{aligned}
q_{tx}^{Z_n,\delta_n}\left(A\right) &< q_{tx}^W\left(A\right) + \epsilon_n^A \\
&< q_{tx}^W\left(A\right) + P - q^+ - r \\
&\leq P - r \leq P - \delta_n
\end{aligned}$$

and so $T^1_{Z_n,\delta_n}(P) > T^1_W(P) - \epsilon$ if $n > N_3 := \max\{N_1, N_2\}$.

Case 1. $T^1_W(P) = 1$, so $\left|T^1_W(P) - T^1_{Z_n,\delta_n}(P)\right| < \epsilon$ if $n > N_3$ by the previous equation.

If $\epsilon_2 \geq T^1_W(P)$=1, then $\left|T^1_W(P) - T^1_{Z_n,\delta_n}(P)\right| < \epsilon_2$ if $n > N_3$.

Case 2. $T^1_W(P) = 0$. Note that $q^W_{0x}(A) = \frac{1}{k} < P$ and so this case is not possible.

Case 3. $0 < T^1_W(P)$.

Note that $T^1_{Z_n,\delta_n}(P) \leq T^1_{Z_n,\delta_n}(P+\delta_n) \leq T^1_W(P+\delta_n+\epsilon_n)$ because if $t < T^1_{Z_n,\delta_n}(P+\delta_n)$ (for the sequence $\{\epsilon_n\}$ we take $m = \sup_n P + \delta_n$)

$$q^W_{tx}(A) - \epsilon_n < q^{Z_n,\delta_n}_{tx}(A) < P + \delta_n.$$

Furthermore,

$$\begin{aligned} T^1_W(P) &\leq \lim\inf_n T^1_W(P+\delta_n+\epsilon_n) \\ &\leq \lim\sup_n T^1_W(P+\delta_n+\epsilon_n). \end{aligned}$$

Now, let $x = \arg\max q^W_{T^1_W x}$. By (2), for any $\varepsilon \in (0,1)$, there exists a monotonically decreasing sequence $\{s_k: \ k = 1,2,\ldots\} \subset \left(T^1_W(P), \infty\right)$ such that $\lim_n s_n = T^1_W(P)$; and for $k = 1,2,\ldots$ we have

$$W_x(s_k) - W_x\left(T^1_W(P)\right) \quad > v_k \equiv \quad (\varepsilon)\sqrt{2\left(s_k - T^1_W(P)\right)\ln\ln\left(1/\left(s_k - T^1_W(P)\right)\right)}.$$

Notice that $v_k > 0$ and $\lim_k v_k = 0$. Let $\alpha > 0$. Let $\delta$ such that if $\delta_3 < \delta$, then $\alpha > -\log(1-\delta_3)$. Pick $\delta_3 < \delta$. Let $K$ such that if $k > K$, $\frac{\sum_{x'\in A}\exp\left(\Delta\frac{W_{x'}\left(T^1_W\right)}{\lambda_z} + \frac{1}{\lambda_z^2}\Delta^2 T^1_W I_{\{x'=k\}}\right)}{\sum_{x'\in A}\exp\left(\Delta\frac{W_{x'}(s_k)}{\lambda_z} + \frac{1}{\lambda_z^2}\Delta^2 s_k I_{\{x'=k\}}\right)} > 1-\delta_3$. Pick $k > K$. Let $N_4$ such that if $n > N_4$, then $\frac{\epsilon_n}{P} < v_k\frac{\Delta}{\lambda_z} + \alpha$ and $\frac{\delta_n}{P} < \left(v_k\frac{\Delta}{\lambda_z} + \alpha\right)^2/2$. Observe that

$$\begin{aligned} q^W_{s_k x} &= \exp\left(\Delta\frac{W_x(s_k)}{\lambda_z} + \frac{1}{\lambda_z^2}\Delta^2 s_k I_{\{x=k\}}\right) / \sum_{x'\in A}\exp\left(\Delta\frac{W_{x'}(s_k)}{\lambda_z} + \frac{1}{\lambda_z^2}\Delta^2 s_k I_{\{x'=k\}}\right) \\ &> \frac{\exp\left(\Delta\frac{W_x\left(T^1_W\right)+v_k}{\lambda_z} + \frac{1}{\lambda_z^2}\Delta^2 T^1_W I_{\{x=k\}}\right)}{\sum_{x'\in A}\exp\left(\Delta\frac{W_{x'}\left(T^1_W\right)}{\lambda_z} + \frac{1}{\lambda_z^2}\Delta^2 T^1_W I_{\{x'=k\}}\right)} \times \frac{\sum_{x'\in A}\exp\left(\Delta\frac{W_{x'}\left(T^1_W\right)}{\lambda_z} + \frac{1}{\lambda_z^2}\Delta^2 T^1_W I_{\{x'=k\}}\right)}{\sum_{x'\in A}\exp\left(\Delta\frac{W_{x'}(s_k)}{\lambda_z} + \frac{1}{\lambda_z^2}\Delta^2 s_k I_{\{x'=k\}}\right)} \\ &\geq q^W_{T^1_W x}\exp\left(v_k\frac{\Delta}{\lambda_z}\right)(1-\delta_3) \geq q^W_{T^1_W x}\exp\left(v_k\frac{\Delta}{\lambda_z}\right)\exp(\alpha) \\ &\geq P\left(1 + v_k\frac{\Delta}{\lambda_z} + \alpha + \frac{\left(v_k\frac{\Delta}{\lambda_z}+\alpha\right)^2}{2}\right) > (P+\epsilon_n+\delta_n). \end{aligned}$$

Consequently,

$$s_k > T^1_W(P+\epsilon_n+\delta_n)$$

if $n > N_4$, and so

$$s_k \geq \lim\sup_n T^1_W(P+\delta_n+\epsilon_n).$$

Since $s_k \to T^1_W$, we must have that

$$\begin{aligned} T^1_W(P) &\geq \lim\sup_n T^1_W(P+\delta_n+\epsilon_n) \\ &\geq \lim\inf_n T^1_W(P+\delta_n+\epsilon_n) \\ &\geq T^1_W(P) \end{aligned}$$

Consequently,

$$T^1_W(P) = \lim_n T^1_W(P+\delta_n+\epsilon_n).$$

Now,

$$T^1_{Z_n,\delta_n}(P) \geq T^1_{Z_n,\delta_n}(P-\delta_n) \geq T^1_W(P-\delta_n-\epsilon_n)$$

because if $t < T^1_W(P-\delta_n-\epsilon_n)$

$$q^{Z_n,\delta_n}_{tx}(A) - \epsilon_n < q^W_{tx}(A) < P-\delta_n-\epsilon_n.$$

Similarly we can prove that

$$\lim_n T^1_W(P-\delta_n-\epsilon_n) = T^1_W(P).$$

Consequently,

$$T^1_W(P) = \lim_n T^1_{Z_n,\delta_n}(P).$$

Let $x = \arg\max_x q^W_{T^1_W,x}$. Let $\epsilon > 0$. Let $N$ such that if $n > N$, then $2\epsilon_n + 2\epsilon < -\max_{y\in A-\{x\}} q^W_{T^1_{W(P)},y}(A) + q^W_{T^1_W(P),x}(A)$

$$\left| q^W_{T^1_W(P),x}(A) - q^{Z_n,\delta_n}_{T^1_{Z_n,\delta_n}(P),x}(A) \right| < \epsilon + \epsilon_n$$

for all $x \in A$, and so if $z \in A - \{x\}$,

$$\begin{aligned} q^{Z_n,\delta_n}_{T^1_{Z_n,\delta_n}(P),z}(A) &< \epsilon + \epsilon_n + q^W_{T^1_W(P),z}(A) \\ &< -\epsilon - \epsilon_n + q^W_{T^1_W(P),x}(A) \\ &< q^{Z_n,\delta_n}_{T^1_{Z_n,\delta_n}(P),x}(A) \end{aligned}$$

and so $x = \arg\max_x q^{Z_n,\delta_n}_{T^1_{Z_n,\delta_n}(P),x}(A)$.

Now, we suppose that $M_W = 2$. Let's prove that $T^1_W(A) = \lim_n T^1_{Z_n,\delta_n}$.

Case 1. $0 < T^1_W(P) < 1$. Like in the previos proof, we conclude that

$$T^1_W(P) = \lim_n T^1_{Z_n,\delta_n}(P).$$

Furthermore, if $x = \arg\min_{x\in A^W_1} q^W_{T^1_W,x}(A)$. Let $N$ such that if $n > N$, then $2\epsilon_n + 2\epsilon < \min_{y\in A^W_1-\{x\}} q^W_{T^1_W,y}(A) - q^W_{T^1_W,x}(A)$ for some $\epsilon > 0$ and

$$\left| q^{Z_n,\delta_n}_{T^1_{Z_n,\delta_n z}} - q^W_{T^1_W,z} \right| < \epsilon$$

for all $z \in A - \{x\}$.

Then if $z \in A - \{x\}$,

$$\begin{aligned} q^{Z_n,\delta_n}_{T^1_{Z_n,\delta_n}z}(A) &> -\epsilon_n - \epsilon + q^W_{T^1_W,z}(A) > q^W_{T^1_W,x}(A) + \epsilon_n + \epsilon \\ &> q^{Z_n,\delta_n}_{T^1_{Z_n,\delta_n}x}(A) \end{aligned}$$

for $n$ sufficiently large and so $x = \arg\min_{x\in A^Y_n} q^{Z_n,\delta_n}_{T^1_{Z_n,\delta_n},x}(A)$. Consequently, $A^{Z_n,\delta_n}_1 = A^W_1$ for $n$ sufficiently large. Furthermore,

Case 2. $T^1_W(P) = 0$, then $\min_{x\in A^Y_n} q^W_{0x}(A^W_n) = 1/k \leq c$. Suppose $c > \frac{1}{k}$. Let $N$ such that if $n > N$, then $\epsilon_n < c - \frac{1}{k}$. Thus if $n > N$,

$$q^{Z_n,\delta_n}_{0x} < \epsilon_n + \frac{1}{k} < c$$

and so $T^1_{Z_n,\delta_n} = 0$. If $c = \frac{1}{k}$,...... (suppose $c \neq 1/k$). We also can see that $A^{Z_n,\delta_n}_1 = A^W_1$ for $n$ sufficiently large.

Case 3. $T^1_W(P) = 1$. It's almost the same than the previous cases.

Now, let's prove that $T^2_W(P^W_1) = \lim_n T^2_{Z_n,\delta_n}\left(P^{Z_n,\delta_n}_1\right)$. By the above argument, we kno that there exists $N$ such that if $n > N$, then $A^{Z_n,\delta_n}_1 = A^W_1$ and

$$P^{Z_n,\delta_n}_1 < P^W_1.$$

Now, suppose $T^2_W(P^W_1) - T^1_W(P) > \epsilon > 0$, and let $q^+_x = \max_{t\in\left[T^1_W(P),T^2_W(P^W_1)-\epsilon\right]} q^W_{tx}$ and $q^+ = \max_x q^*_x$. Let $N^*_2$ such that if $n > N^*_2$, then $\delta_n < r < P^W_1 - q^+$ where $r \in \mathbb{Q}$. Let $N^{**}_2$ such that if $n > N^{**}_2$, then $\epsilon^{A^W_1}_n < P^W_1 - q^+ - r$. Thus, if $n > N_2 := \max\{N^*_2, N^{**}_2\}$ and $T^1_W(P) \leq t \leq T^2_W(P^W_1) - \epsilon$, then

$$\begin{aligned} q^{Z_n,\delta_n}_{tx}(A^W_1) &< q^W_{tx}(A^W_1) + \epsilon^{A^W_1}_n \\ &< q^W_{tx}(A^W_1) + P^W_1 - q^+ - r \\ &\leq P^W_1 - r \leq P^W_1 - \delta_n \end{aligned}$$

and so $T^2_{Z_n,\delta_n}(P) > T^2_W(P) - \epsilon$ if $n > N_3 := \max\{N_1, N_2\}$.

Case 1. $T^2_W(P^W_1) = 1$, so $\left|T^2_W(P) - T^1_{Z_n,\delta_n}(P)\right| < \epsilon$ if $n > N_3$.

If $\epsilon_2 \geq T^1_W(P)$=1, then $\left|T^1_W(P) - T^1_{Z_n,\delta_n}(P)\right| < \epsilon_2$ if $n > N_3$.

Case 2. $T^2_W(P^W_1) = 0 = T^1_W(P)$. This is impossible because $P^W_1 \leq \frac{1}{k} \leq c$ and $P^W_1 \geq P^* > \frac{1}{k}$.

Case 3. $0 < T^2_W(P^W_1)$.

Note that $T^2_{Z_n,\delta_n}\left(P^{Z_n}_1\right) \leq T^2_{Z_n,\delta_n}\left(P^{Z_n}_1 + \delta_n\right) \leq T^2_W\left(P^{Z_n}_1 + \delta_n + \epsilon_n\right)$ because if $T^1_{Z_n,\delta_n}(P) \leq t < T^2_{Z_n,\delta_n}\left(P^{Z_n}_1 + \delta_n\right)$

$$q^W_{tx}(A^W_1) - \epsilon_n < q^{Z_n,\delta_n}_{tx}(A^W_1) < P^{Z_n}_1 + \delta_n$$

Furthermore, for $n$ sufficiently large and $\delta_2 > 0$ sufficiently small (THIS PARAGRAPH CAN BE REMOVED...)

$$\begin{aligned} T^2_W(P^W_1) - \delta_2 \leq T^2_W\left(P^{Z_n}_1\right) &\leq \liminf_n T^2_W\left(P^{Z_n}_1 + \delta_n + \epsilon_n\right) \\ &\leq \limsup_n T^2_W\left(P^{Z_n}_1 + \delta_n + \epsilon_n\right) \end{aligned}$$

because $P_1^{Z_n} \to P_1^W$ and so for $n$ sufficiently large

$$P_1^W \quad < \quad P_1^W - \max_{t \in \left[T_W^1, T_W^2\left(P_1^W\right) - \delta_2\right]} q_{tx}^W\left(A_1^W\right) + P_1^{Z_n}.$$

Now, let $x = \arg\max q_{T_W^2\left(P_1^W\right)x}^W$. By (2), for any $\varepsilon \in (0,1)$, there exists a monotonically decreasing sequence $\{s_k :\ k = 1, 2, \ldots\} \subset \left(T_W^2\left(P_1^W\right), 1\right)$ such that $\lim_n s_n = T_W^2\left(P_1^W\right)$; and for $k = 1, 2, \ldots$ we have

$$W_x\left(s_k\right) - W_x\left(T_W^2\left(P_1^W\right)\right) \quad > v_k \equiv \quad (\varepsilon)\sqrt{2\left(s_k - T_W^2\left(P_1^W\right)\right) \ln\ln\left(1/\left(s_k - T_W^2\left(P_1^W\right)\right)\right)}.$$

Notice that $v_k > 0$ and $\lim_k v_k = 0$. Let $\alpha > 0$. Let $\delta$ such that if $\delta_3 < \delta$, then $\alpha > -\log\left(1 - \delta_3\right)$. Pick $\delta_3 < \delta$. Let $K$ such that if $k > K$, $\frac{\sum_{x' \in A_1^W} \exp\left(\Delta \frac{W_{x'}\left(T_W^2\left(P_1^W\right)\right)}{\lambda_z} + \frac{1}{\lambda_z^2}\Delta^2 T_W^2\left(P_1^W\right) I_{\{x'=k\}}\right)}{\sum_{x' \in A_1^W} \exp\left(\Delta \frac{W_{x'}(s_k)}{\lambda_z} + \frac{1}{\lambda_z^2}\Delta^2 s_k I_{\{x'=k\}}\right)} > 1 - \delta_3$. Pick $k > K$. Let $N_4$ such that if $n > N_4$, then $\frac{\epsilon_n}{P_1^W} < v_k \frac{\Delta}{\lambda_z} + \alpha$ and $\frac{\delta_n}{P_1^W} < \left(v_k \frac{\Delta}{\lambda_z} + \alpha\right)^2 / 2$. Observe that

$$\begin{aligned}
q_{s_k x}^W\left(A_1^W\right) &= \exp\left(\Delta \frac{W_x\left(s_k\right)}{\lambda_z} + \frac{1}{\lambda_z^2}\Delta^2 s_k I_{\{x=k\}}\right) \Big/ \sum_{x' \in A_1^W} \exp\left(\Delta \frac{W_{x'}\left(s_k\right)}{\lambda_z} + \frac{1}{\lambda_z^2}\Delta^2 s_k I_{\{x'=k\}}\right) \\
&> \frac{\exp\left(\Delta \frac{W_k\left(T_W^2\left(P_1^W\right)\right) + v_k}{\lambda_z} + \frac{1}{\lambda_z^2}\Delta^2 T_W^2\left(P_1^W\right) I_{\{x=k\}}\right)}{\sum_{x' \in A_1^W} \exp\left(\Delta \frac{W_{x'}\left(T_W^2\left(P_1^W\right)\right)}{\lambda_z} + \frac{1}{\lambda_z^2}\Delta^2 T_W^2\left(P_1^W\right) I_{\{x'=k\}}\right)} \\
&\quad \times \frac{\sum_{x' \in A_1^W} \exp\left(\Delta \frac{W_{x'}\left(T_W^2\left(P_1^W\right)\right)}{\lambda_z} + \frac{1}{\lambda_z^2}\Delta^2 T_W^2\left(P_1^W\right) I_{\{x'=k\}}\right)}{\sum_{x' \in A_1^W} \exp\left(\Delta \frac{W_{x'}(s_k)}{\lambda_z} + \frac{1}{\lambda_z^2}\Delta^2 s_k I_{\{x'=k\}}\right)} \\
&\geq q_{T_W^2\left(P_1^W\right)x}^W \exp\left(v_k \frac{\Delta}{\lambda_z}\right)\left(1 - \delta_3\right) \geq q_{T_W^1 x}^W \exp\left(v_k \frac{\Delta}{\lambda_z}\right) \exp\left(\alpha\right) \\
&\geq P_1^W \left(1 + v_k \frac{\Delta}{\lambda_z} + \alpha + \frac{\left(v_k \frac{\Delta}{\lambda_z} + \alpha\right)^2}{2}\right) > P_1^W + \epsilon_n + \delta_n.
\end{aligned}$$

Consequently,

$$s_k > T_W^2\left(P_1^W + \epsilon_n + \delta_n\right)$$

if $n > N_4$, and so

$$s_k \geq \lim\sup_n T_W^2\left(P_1^W + \delta_n + \epsilon_n\right).$$

Since $s_k \to T_W^1$, we must have that

$$\begin{aligned} T_W^2\left(P_1^W\right) &\geq \lim\,\sup_n T_W^2\left(P_1^W+\delta_n+\epsilon_n\right) \\ &\geq \lim\,\inf_n T_W^2\left(P_1^W+\delta_n+\epsilon_n\right) \\ &\geq T_W^2\left(P_1^W\right) \end{aligned}$$

Consequently,

$$T_W^2\left(P_1^W\right)=\lim_n T_W^2\left(P_1^W+\delta_n+\epsilon_n\right).$$

Now,

$$T^2_{Z_n,\delta_n}\left(P_1^{Z_n}\right)\geq T^2_{Z_n,\delta_n}\left(P_1^{Z_n}-\delta_n\right)\geq T_W^2\left(P_1^{Z_n}-\delta_n-\epsilon_n\right)$$

because if $t<T_W^2\left(P_1^{Z_n}-\delta_n-\epsilon_n\right)$

$$q_{tx}^{Z_n,\delta_n}\left(A_1^W\right)-\epsilon_n<q_{tx}^W\left(A_1^W\right)<P_1^{Z_n}-\delta_n-\epsilon_n.$$

Similarly we can prove that

$$\lim_n T_W^2\left(P_1^W-\delta_n-\epsilon_n\right)=T_W^2\left(P_1^W\right).$$

Consequently,

$$T_W^2\left(P_1^W\right)=\lim_n T^2_{Z_n,\delta_n}\left(P_1^W\right).$$

By a similar argument than before, we can see that $\arg\max_x q^{Z_n,\delta_n}_{T^2_{Z_n,\delta_n}\left(P_1^W\right),x}\left(A_1^W\right)=\arg\max_x q^W_{T^2_W\left(P_1^W\right),x}\left(A_1^W\right)$ for $n$ sufficiently large.

The cases $M_W=i$ for $k-1\geq i\geq 3$ can be proved in a similar way.

Note that almost surely $M_Y<\infty$ by Frazier[].

Thus, we conclude that

$$\mathbb{P}\left(W\in\text{D}[0,1]^k\text{-}D_s\right)=1.$$

■

By the extension of the CMT (Theorem 5.5 of Billingsley 1968), we have the following corollary.

**Corollary 1.** We have that

$$f\left(C\left(\delta,t\right),\delta\right)\Rightarrow g\left(W\left(t\right)\right)$$

in distribution as $\delta\to 0$.

**Theorem 1.** If samples from system $x\in\{1\ldots,k\}$ are identically distributed and independent, over time and across alternatives, then $\lim_{\delta\to 0}Pr\left\{\text{BIZ selects } k\right\}\geq P*$ provided $\mu_k=\delta,\mu_{k-1}=\cdots=\mu_1=0$. We also suppose $B_1=\cdots=B_k=1$ and $c\neq\frac{1}{k}$.

**Proof.** Let

$$\hat{T}_n(\delta) = \min\left\{t \in \left\{0, \delta^2, 2\delta^2, \dots\right\} : \min_{x\in A_n^{Y,\delta}} q_{tx}^{C(\delta,\cdot),\delta}\left(A_n^{C(\delta,\cdot),\delta}\right) \leq c \text{ or } \max_{x\in A_n^{Y,\delta}} q_{tx}^{C(\delta,\cdot),\delta}\left(A_n^{C(\delta,\cdot),\delta}\right) \geq P_n^{C(\delta,\cdot),\delta}\right\}$$

and $T_n(\delta)$ the usual stopping times of the algorithm. Then $T_n(\delta) = \hat{T}_n(\delta)/\delta^2$. Now, we can prove that $\hat{T}_n(\delta) - T^n_{C(\delta,\cdot),\delta}\left(P_n^{C(\delta,\cdot),\delta}\right) \to 0$ with probability 1 as $\delta \to 0$ using that $C(\delta,\cdot)$ is right-continuous and $\delta^2 \to 0$. Consequently, we can use $C\left(\delta, T^n_{C(\delta,\cdot),\delta}\left(P_n^{C(\delta,\cdot),\delta}\right)\right)$ instead of $C\left(\delta, \hat{T}_n(\delta)\right)$.

Let $CS_\delta$ be the event of doing a correct selection given the configuration $\mu_k = \delta, \mu_{k-1} = \cdots = \mu_1 = 0$. Then

$$\begin{aligned}
\underline{lim}_{\delta\to 0}\mathbb{P}(CS_\delta) &= \underline{lim}_{\delta\to 0}\mathbb{P}(f(C(\delta,t),\delta) = 1) \\
&= \mathbb{P}(g(W) = 1) \\
&\geq P^*
\end{aligned}$$

where the last inequallity follows from the paper of Frazier [].

■

**Theorem 2.** If samples from system $x \in \{1\dots,k\}$ are indetically distributed and independent, over time and across alternatives, then $\lim_{\delta\to 0}\mathbb{P}(CS_\delta) \geq P*$ provided $\mu_k - \mu_{k-1} \geq \delta$. We suppose $B_1 = \cdots = B_k = 1$ and $c \neq \frac{1}{k}$.

**Proof.** Suppose $X_1, \dots, X_k$ are the observations of the systems $1, \dots, k$, respectively. Consider, $\hat{X}_i = X_i - \mu_i e$ if $i \neq k$ and $\hat{X}_k = X_k - \mu_k e + \delta e$. Then

$$\begin{aligned}
\underline{lim}_{\delta\to 0}\mathbb{P}_\mu(CS_\delta \mid \mathbf{X}) &= \underline{lim}_{\delta\to 0}\mathbb{P}_{(0,\dots,0,\delta)}\left(CS_\delta \mid \hat{\mathbf{X}}\right) \\
&= \underline{lim}_{\delta\to 0}\mathbb{P}(g(W) = 1) \\
&\geq P^*.
\end{aligned}$$

■