625.714 SDE Course Project

SDE Modeling and Control of Systemic Risk in Interbank Lending

Toshiaki Yamanaka

August 5, 2025

1 Introduction

In this work, we study the systemic risk faced by banks arising from interbank lending and borrowing. To this end, we introduce the frameworks of both Carmona, Fouque, and Sun [1] and Cont, Guo, and Xu [2]. We provide our own proofs for certain parts of these papers, with reference to Øksendal [3]. Specifically, we examine a) the uniqueness of X_t within the framework of Carmona, Fouque, and Sun [1] in Section 2, and b) the HJB (Hamilton-Jacobi-Bellman) variational inequality for the model of Cont, Guo, and Xu [2] in Section 3. We present our original simulation result in Section 4.

2 Carmona, Fouque, and Sun (2015)

The model of Carmona, Fouque, and Sun [1] is summarized in Carmona and Delarue [4] as follows. They describe the model as a network of N banks and denote by X_t^i the logarithm of the cash reserves of bank $i \in \{1, \ldots, N\}$ at time t. For independent Wiener processes $W^i = (W_t^i)_{0 \le t \le T}$ for $i = 0, 1, \ldots, N$ and a constant $\sigma > 0$,

$$\mathrm{d}X_t^i = \frac{a}{N} \sum_{j=1}^{N} \big(X_t^j - X_t^i\big)\, \mathrm{d}t + \sigma\, \mathrm{d}B_t^i = a(\bar{X}_t - X_t^i)\mathrm{d}t + \sigma \mathrm{d}B_t^i, \qquad i = 1, \ldots, N, \tag{1}$$

where $\mathrm{d}B_t^i = \sqrt{1-\rho^2}\, \mathrm{d}W_t^i + \rho\, \mathrm{d}W_t^0$, for some correlation parameter $\rho \in [-1, 1]$. This is referred to as "an explicitly solvable toy model of systemic risk."

To the best of our knowledge, a proof of the existence of a unique strong solution to this SDE has not been provided in the literature. Thus, we offer the following.

Theorem 2.1 (Existence and strong uniqueness)**.** Let $\big(W^0, W^1, \ldots, W^N\big)$ be independent Brownian motions. Fix constants $a \ge 0,\ \sigma > 0,\ \rho \in [-1, 1]$ and define

$$\mathrm{d}B_t^i \;=\; \sqrt{1-\rho^2}\, \mathrm{d}W_t^i + \rho\, \mathrm{d}W_t^0, \qquad i = 1, \ldots, N. \tag{2}$$

We have $X_0^i \in \mathbb{R}$. We consider $T > 0$. We are given the following SDE.

$$\mathrm{d}X_t^i \;=\; a\Big(\bar{X}_t - X_t^i\Big)\mathrm{d}t + \sigma\, \mathrm{d}B_t^i, \tag{3}$$

where $\bar{X}_t := \frac{1}{N} \sum_{j=1}^{N} X_t^j, \qquad i = 1, \ldots, N.$

Then, Eq. (3) admits a *unique strong* solution $(X_t^1, \ldots, X_t^N)_{0 \le t \le T}$.

Proof. **1. Lipschitz property.** For $x = (x_1, \ldots, x_N)$, $y = (y_1, \ldots, y_N) \in \mathbb{R}^N$, let

$$f_i(x) \ := \ a\Big(\tfrac{1}{N}\textstyle\sum_{j=1}^{N} x_j - x_i\Big), \qquad i = 1, \ldots, N.$$

Then, $f_i(x) - f_i(y) \ = \ a\Big(\frac{1}{N}\sum_{j=1}^{N}(x_j - y_j) - (x_i - y_i)\Big)$. By the Cauchy–Schwarz inequality,

$$|f_i(x) - f_i(y)| \ \leq \ a\Big(\frac{1}{N}\sum_{j=1}^{N}|x_j - y_j| + |x_i - y_i|\Big) \ \leq \ 2a\,\|x - y\|,$$

where $\|\cdot\|$ is the Euclidean norm. Consequently, $\|f(x) - f(y)\| \leq 2a\,\|x - y\|$, *i.e.*, the drift is Lipschitz continuous (Øksendal [3], Definition 7.1.1). Since the diffusion coefficient σ in Eq. (3) is a constant, it is also Lipschitz continuous.

2. Strong uniqueness. Let $X, \tilde{X}$ be two solutions of Eq. (3) produced by the same Brownian vector $(B^1, \ldots, B^N)$, with the same X_0. Let $\Delta_t := X_t - \tilde{X}_t$ in vector notation. Taking the difference of the two SDEs removes the stochastic term $\mathrm{d}B_t^i$:

$$\mathrm{d}\Delta_t^i \ = \ a\Big(\bar{X}_t - \bar{\tilde{X}}_t - \big(X_t^i - \tilde{X}_t^i\big)\Big)\mathrm{d}t, \quad \Delta_0^i = 0.$$

Since $\bar{X}_t - \bar{\tilde{X}}_t = \frac{1}{N}\sum_{k=1}^{N}\Delta_t^k$, we have $|\mathrm{d}\Delta_t^i| \leq 2a\,\|\Delta_t\|\,\mathrm{d}t$. Therefore,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta_t\|^2 \ = \ 2\sum_{i=1}^{N}\Delta_t^i\mathrm{d}\Delta_t^i \ \leq \ 4a\,\|\Delta_t\|^2.$$

Since $\|\Delta_0\| = 0$, the Gronwall inequality (Øksendal [3], Exercise 5.17) yields $\|\Delta_t\|^2 = 0$ for every $t \leq T$. Thus, $X_t = \tilde{X}_t$ almost surely. The strong uniqueness is proved.

3. Existence. Define

$$X_t^{(0)} := X_0, \qquad X_t^{(n+1),i} := X_0^i + \int_0^t f_i\big(X_s^{(n)}\big)\mathrm{d}s + \sigma B_t^i.$$

Using the successive iteration (Øksendal [3], Proof of Theorem 5.2.1), applying the Jensen's inequality (Øksendal [3], Theorem B.4) to the time integral, and by both the Ito isometry (Øksendal [3], Lemma 3.1.5) and the Lipschitz condition (Kuo [5], Section 10.3),

$$E\|X_t^{(n+1)} - X_t^{(n)}\|^2 \ \leq \ t(2a)^2\int_0^t E\|X_s^{(n)} - X_s^{(n-1)}\|^2\mathrm{d}s.$$

The Gronwall inequality shows that $(X^{(n)})_{n\geq 0}$ forms a Cauchy sequence in $L^2(\Omega; C([0,T];\mathbb{R}^N))$. Thus, it converges to a process X that satisfies Eq. (3). $\square$

3 Cont, Guo, and Xu (2021)

The model of Cont, Guo, and Xu [2] is as follows. They study a class of N-player stochastic differential games in which each player $i = 1, \ldots, N$ controls a diffusive process X_t^i through $\xi^i := (\xi^{i,+}, \xi^{i,-})$ additive control terms:

$$\mathrm{d}X_t^i = \mu^i\,\mathrm{d}t + \sigma^i\,\mathrm{d}B_t + \mathrm{d}\xi_t^{i,+} - \mathrm{d}\xi_t^{i,-}, \qquad X_{0-}^i = x_i. \tag{4}$$

Each player i seeks to minimize the sum of a discounted running cost and a proportional cost of intervention:

$$J^i(x;\xi) \;=\; E\left[\int_0^\infty e^{-\rho t}\Big(h^i\big(X_t\big)\,\mathrm{d}t + K_i^+\,\mathrm{d}\xi_t^{i,+} + K_i^-\,\mathrm{d}\xi_t^{i,-}\Big)\;\middle|\; X_{0-}=x_i\right]. \tag{5}$$

The first two terms in Eq. (4) correspond to the baseline (uncontrolled) diffusion dynamics, and the last two terms represent the control $\xi^i = (\xi^{i,+}, \xi^{i,-})$, modeled as a pair of non-decreasing càdlàg processes[1]. The controls $\xi^{i,+}, \xi^{i,-}$ have possible jumps and continuous adjustments to the rates. Such controls are called *singular controls* (Karatzas [7]) and used for analyzing optimal investment policy with transaction costs (*e.g.*, Kallsen and Muhle-Karbe [8]).

We now take a different step from Cont, Guo, and Xu [2]. From Eqs. (4) and (5), we derive the HJB variational inequality[2] for N-player control problems as a necessary condition for optimality.

Theorem 3.1 (Multi-player HJB variational inequality)**.** Eqs. (4) and (5) are given. Fix $N\in\mathbb{N}$. For each player $i=1,\dots,N$, assume that, (a) regularity: $v^i \in C^2$ on every inaction region and has polynomial growth, (b) a multi-player DPP (dynamic programming principle)[3] holds, and (c) Ito's formula for finite-variation impulses applies to $e^{-\rho t}v^i(X_t)$. Then, for every $x\in\mathbb{R}^N$ and each $i=1,\dots,N$,

$$\min\Big\{\,\rho v^i(x) - \mathcal{L}v^i(x) - h^i(x),\; K_i^+ + \partial_{x\ i} v^i(x),\; K_i^- - \partial_{x\ i} v^i(x)\Big\} = 0,$$

where $\mathcal{L}$ is the second-order operator:

$$\mathcal{L} = \sum_{k=1}^{N}\big(\mu^k\partial_{x^k} + \tfrac{1}{2}(\sigma^k)^2\partial^2_{x^k x^k}\big). \tag{6}$$

Proof. Detailed proofs for Steps 2 and 4 are provided in Appendices A and B, respectively.
1. Controlled state. Fix $i\in\{1,\dots,N\}$. The controlled state satisfies Eq. (4), where each $\xi^{i,\pm}$ is right-continuous, non-decreasing, and of finite variation.
2. DPP. For any stopping time τ and any admissible control ξ^i for player i, by the impulse-control DPP (Øksendal and Sulem [9], Lemmas 10.3 and 10.4), we have:

$$v^i(x) = \inf_{\xi^i} E^x\Big[\int_0^\tau e^{-\rho t}\,h^i(X_t)\,\mathrm{d}t + \sum_{0\le t<\tau} e^{-\rho t}\left(K_i^+\Delta\xi_t^{i,+} + K_i^-\Delta\xi_t^{i,-}\right) + e^{-\rho\tau}v^i(X_\tau)\Big]. \tag{7}$$

3. Short horizon. Fix $\delta>0$ and set a valid stopping time $\tau = \inf\{t\le\delta : \Delta\xi_t^{i,+}+\Delta\xi_t^{i,-}>0\}$, since ξ^i is an adapted càdlàg process. Player i acts at most once on $[0,\tau)$.
4. Ito expansion Apply Ito's formula to $e^{-\rho t}v^i(X_t)$ on $[0,\tau)$:

$$\begin{aligned} e^{-\rho\tau}v^i(X_\tau) - v^i(x) = &\int_0^\tau e^{-\rho t}\big[-\rho v^i(X_t) + \mathcal{L}v^i(X_t)\big]\mathrm{d}t + \int_0^\tau e^{-\rho t}\nabla v^i(X_t)\cdot\sigma\,\mathrm{d}B_t \\ &+ \sum_{0\le t<\tau} e^{-\rho t}\sum_k \partial_{x^k}v^i(X_{t-})\,\Delta\xi_t^k. \end{aligned}$$

[1]A càdlàg (RCLL) process is a process that is right-continuous on $[0,\infty)$ with finite left-hand limits on $(0,\infty)$ (Karatzas and Shreve [6], 1.7 Exercise).

[2]See Øksendal [3], Section 10.4; Øksendal and Sulem [9], Chapter 7.

[3]See Øksendal and Sulem [9], Lemmas 10.3 and 10.4.

Taking expectations eliminates the martingale term, and after rearranging:

$$v^i(x) = E^x\Big[\int_0^\tau e^{-\rho t}\big(h^i(X_t) + \rho v^i - \mathcal{L}v^i\big)\,dt + e^{-\rho\tau} v^i(X_\tau) + \sum_{0\le t<\tau} e^{-\rho t}\sum_k \partial_{x^k} v^i(X_{t-})\,\Delta\xi_t^k\Big]. \tag{8}$$

5. Waiting inequality. Comparing Eq. (8) with Eq. (7), optimality implies:

$$\rho v^i(x) - \mathcal{L}v^i(x) - h^i(x) \;\ge\; 0. \tag{9}$$

Otherwise, a postponement of the next impulse would lower the cost.
6. Gradient constraints. Let player i perform a single impulse εe_i at $t = 0$. This costs $K_i^+\varepsilon$ and moves the state to $x + \varepsilon e_i$. Therefore, $v^i(x) \le K_i^+\varepsilon + v^i(x + \varepsilon e_i)$. Divide by ε and let ε decreases to zero to have:

$$K_i^+ + \partial_{x^i} v^i(x) \;\ge\; 0. \tag{10}$$

A symmetric argument with $-\varepsilon e_i$ yields:

$$K_i^- - \partial_{x^i} v^i(x) \;\ge\; 0. \tag{11}$$

7. Complementarity. If the gradient constraints Eqs. (10)–(11) are strict, then the player waits. Thus, Eq. (9) binds with equality. Conversely, if the left-hand side of Eq. (9) is strictly positive, delaying the impulse lowers cost, contradicting optimality. Therefore, the three expressions are non-negative and their minimum equals 0, which completes the proof. □

4 Simulation

4.1 Formulation by Cont, Guo, and Xu (2021)

Cont, Guo, and Xu [2] derive a value function for the regulator, which is a *viscosity solution* to a certain HJB equation[4]. The value function and the associated HJB equation are formulated as follows.

N-players share an aggregate cost:

$$\begin{aligned} J(\mathbf{x};\xi) &= \sum_{i=1}^N L_i J^i(\mathbf{x},\xi) \\ &= E^x \int_0^\infty e^{-\rho t}\left[H(X_t)dt + \sum_{i=1}^N L_i K_i^+ d\xi_t^{i,+} + \sum_{i=1}^N L_i K_i^- d\xi_t^{i,-}\right], \end{aligned} \tag{12}$$

where the dynamics of X_t is given by Eq. (4), and

$$H(\mathbf{x}) := \sum_{i=1}^N L_i h^i(\mathbf{x}), \quad \text{with} \quad L_i > 0 \quad \text{and} \quad \sum_{i=1}^N L_i = 1. \tag{13}$$

The regulator seeking to optimize the aggregated cost Eq. (12) faces the following stochastic control problem:

$$v(\mathbf{x}) = \min_{\xi\in\mathcal{V}_N} J(x;\xi). \tag{14}$$

[4]For viscosity solutions, see Øksendal and Sulem [9], Chapter 12.

Eq. (14) is a viscosity solution to the following HJB:

$$\max\bigl\{\rho u - \mathcal{L}u - H(\mathbf{x}),\ \beta(\nabla u) - 1\bigr\} \ = \ 0, \tag{15}$$

with the operator

$$\mathcal{L} \ = \ \tfrac{1}{2}\sum_{i,j=1}^{N} \sigma^i \sigma^j \partial^2_{x^i x^j} + \sum_{i=1}^{N} \mu^i \partial_{x^i},$$

and

$$\beta(\boldsymbol{q}) = \max_{1\le i\le N}\left[\left(\frac{q^i}{L_i K_i^-}\right)^+ \vee \left(\frac{q^i}{L_i K_i^+}\right)^-\right],$$

where $\boldsymbol{q} := (q^1, \dots, q^N)$, $(a)^+ = \max\{0, a\}$, and $(a)^- = \max\{0, -a\}$ for any $a \in \mathbb{R}$.

4.2 Simulation Method and Result

1. Simulation Method For simplicity, we set $N = 2$ and simulate two-player games. We first solved the HJB variational inequality on a 41×41 uniform grid using central finite differences with Neumann (zero–flux) boundary conditions, obtaining an approximate value surface $u(x_1, x_2)$. From the gradient, we computed the inaction set defined by the threshold $\beta(\nabla V) \le c$. Since the default value $c = 1$ in Eq. (15) produced too small an inaction region, we arbitrary set $c = 5$, thereby enlarging the domain in which no intervention is needed.

Next, we simulated 1,000,000 sample paths each with 400 time steps. Brownian shocks were generated by LHS (Latin Hypercube Sampling) (McKay, Beckman, and Conover [10]), and scaled by $\sqrt{h}$ to discretize Brownian motion. Whenever a coordinate exited the inaction region, the state was reflected back onto the boundary and the corresponding intervention cost was added.

Our simulation procedure is summarized in Algorithm 1.

Algorithm 1 Simulation for 2-player stochastic differential games

Input: drift term μ, volatility σ, discount rate ρ
Build finite difference operators A_1, A_2
Form Generator $L = \frac{1}{2}\sigma_x^2 A_2 \otimes I + \frac{1}{2}\sigma_y^2 I \otimes A_2 + \mu_x A_1 \otimes I + \mu_y I \otimes A_1$
Solve $\rho u - \mathcal{L}u = H$ for u ▷ Numerical approximation to u in the inaction region
Compute ∇u on grid
Derive inaction set $I_c = \{\beta(\nabla u) \le c\}$ ▷ $c = 1$ in Eq. (15)
Generate LH samples $Z_{p,t,i}$ for $p = 1{:}P$
for $p = 1$ **to** P **do**
    Initialize state $X_0 = 0$, costs $C_{\text{run}} = C_{\text{int}} = 0$
    for $t = 0$ **to** $T - 1$ **do**
        Compute $X_{t+1} = X_t + \mu h + \sigma\sqrt{h}\, Z_{p,t}$ from X_t
        if $\beta(X_{t+1}) > c$ **then** ▷ Outside I_c, we need intervention.
            Project X_{t+1} ▷ Nearest point in I_c
            $C_{\text{int}} \mathrel{+}= L_w K\, \|X_{t+1} - \text{pre-jump}\|_1$ ▷ Adds intervention cost (jump size × unit cost)
        $C_{\text{run}} \mathrel{+}= e^{-\rho t h}\, H(X_t)\, h$ ▷ Running cost
    Record $C_p = C_{\text{run}} + C_{\text{int}}$ ▷ Total cost
Output: distribution $\{C_p\}_{p=1}^{P}$

2. Simulation Result Across 1,000,000 LHS sample paths, the empirical statistics are:

$$\text{mean total cost} \;=\; 0.1604, \qquad \text{standard deviation} \;=\; 0.1596,$$
$$\text{mean running cost} \;=\; 0.0373, \qquad \text{mean intervention cost} \;=\; 0.1231.$$

With $n = 1,000,000$ paths, the 95% confidence interval for the expected total cost is:

$$0.1604 \;\pm\; 1.96\,\frac{0.1596}{\sqrt{1,000,000}} \;=\; [0.1601,\; 0.1607].$$

The holding (running) component is small and tightly clustered, whereas the intervention component generates almost all of the dispersion. Roughly one quarter of the paths never exit the inaction region (*i.e.*, their intervention cost is zero). The remaining paths experience discrete control actions, producing a long right tail in the cost distribution. We simulated with an intervention threshold $c = 5$, which is looser than the optimal barrier. Thus, interventions are relatively infrequent, and the observed total cost can be understood as an upper bound on the true optimal cost. Tightening the band would increase expected adjustment costs, decrease holding costs, and shift the distribution rightward.

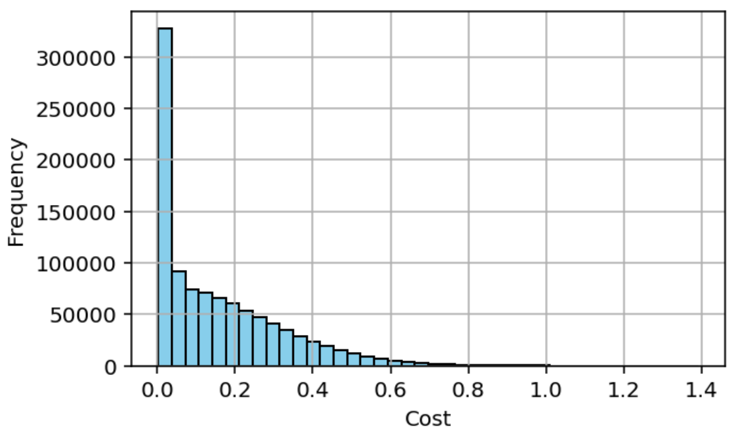

Figure 1: Histogram of total discounted costs across 1,000,000 sample paths.

5 Possibility for Further Research

In addition to the two main articles analyzed in this paper, we will closely examine several related studies, including Amini, Cont, and Minca [11], Bo and Capponi [12], Cuchiero, Reisinger, and Rigger [13], Fouque and Ichiba [14], Minca and Sulem [15], and Sun [16]. These works will play a central role in shaping the next stage of our research.

Appendix A: DPP in Step 2

The DPP relates the value function at the current time to the optimal value at a later stopping time. An optimal control must remain optimal no matter when we re-evaluate decisions. This yields the variational inequality.

Fix a player $i \in \{1, \ldots, N\}$ and an initial state $x \in \mathbb{R}^N$. Let τ be a bounded $(\mathcal{F}_t)_{t\geq 0}$-stopping time, *i.e.*, $\tau \leq T$ for some $T > 0$. Denote the cost functional by $J^i(x;\xi)$ and let $v^i(x) = \inf_{\xi\in\mathcal{A}^i} J^i(x;\xi)$. For every bounded stopping time τ,

$$v^i(x) = \inf_{\xi\in\mathcal{A}^i} E\Big[\int_{[0,\tau]} e^{-\rho t}\big(h^i(X_t)\,dt + K_i^+\,d\xi_t^{i,+} + K_i^-\,d\xi_t^{i,-}\big) + e^{-\rho\tau}\,v^i(X_\tau)\;\Big|\;X_{0-} = x\Big], \tag{16}$$

where $d\xi_t^{i,\pm}$ denotes the Lebesgue-Stieltjes measure associated with $\xi^{i,\pm}$.

Sub-optimality. For any admissible ξ and any τ, splitting J^i at τ and using the law of iterated expectations (Øksendal [3], Theorem B.3) gives the right-hand side of Eq. (16). Taking $\inf_{\xi\in\mathcal{A}^i}$ yields $v^i(x) \leq \text{RHS}$.

Optimal continuation. Fix $\varepsilon > 0$ and pick an ε-optimal control $\hat{\xi}$. Construct ξ^* that follows $\hat{\xi}$ up to τ and, conditional on $\mathcal{F}_\tau$, switches to an ε-optimal control for X_τ. Conditional optimality and the law of iterated expectations give $v^i(x) \geq \text{RHS} - \varepsilon$. Letting $\varepsilon \downarrow 0$ completes the proof.

Consequence for the HJB. Since Eq. (16) holds for all bounded τ, applying Ito's formula to $e^{-\rho t}v^i(X_t)$ with $v^i \in C^2$ on each inaction region and taking expectations under an optimal control yields the multi-player HJB variational inequality:

$$\min\Big\{\rho v^i(x) - \mathcal{L}v^i(x) - h^i(x),\ K_i^+ + \partial_{x^i}v^i(x),\ K_i^- - \partial_{x^i}v^i(x)\Big\} = 0,$$

where $\mathcal{L}$ is given in Eq. (6), which completes Step 2 of the proof.

Appendix B: Ito expansion in Step 4

Extended Ito formula. Let $Y_t = f(X_t, t)$ where X is an $\mathbb{R}^N$-valued semimartingale with càdlàg paths and jump process $\Delta X_t := X_t - X_{t-}$. If $f \in C^2$ in x and C^1 in t, and X has finite-variation jumps, then,

$$f(X_t, t) = f(X_0, 0) + \int_0^t \big(\partial_s f + \mathcal{L}f\big)(X_{s-}, s)\,ds + \int_0^t \nabla_x f(X_{s-}, s)\,\sigma\,dB_s + \sum_{0\leq s\leq t} \nabla_x f(X_{s-}, s)\,\Delta X_s,$$

where $\mathcal{L}$ is the second-order generator introduced in Eq. (6).

Application to $Y_t = e^{-\rho t}v^i(X_t)$. Set $f(x,t) = e^{-\rho t}v^i(x)$ with $v^i \in C^2$ on each inaction region. Since f is time-dependent only through the discount factor, we have:

$$\partial_t f(x,t) = -\rho e^{-\rho t}v^i(x), \quad \nabla_x f(x,t) = e^{-\rho t}\nabla v^i(x), \quad \mathcal{L}f(x,t) = e^{-\rho t}\mathcal{L}v^i(x).$$

Applying the formula up to a bounded stopping time τ gives:

$$\begin{aligned} e^{-\rho\tau}v^i(X_\tau) - v^i(x) = &\int_0^\tau e^{-\rho t}\big[-\rho v^i(X_t) + \mathcal{L}v^i(X_t)\big]\,dt + \int_0^\tau e^{-\rho t}\nabla v^i(X_t)\,\sigma\,dB_t \\ &+ \sum_{0\leq t<\tau} e^{-\rho t}\nabla v^i(X_{t-})\cdot\Delta\xi_t, \end{aligned}$$

since, under the dynamics, the only jumps in X come from $\Delta\xi_t$.

Taking expectations. The stochastic integral is a martingale with zero mean. Thus, taking $E^x[\,\cdot\,]$ yields Eq. (8) in the proof.

Remark. The jump sum contains only the impulse terms, as the Brownian component of X is continuous. If v^i were not twice differentiable, the expansion would instead be interpreted in the viscosity sense, in which case no modification to the variational inequality is required. The identity above, together with the DPP from Appendix A, yields the Waiting inequality in Eq. (9).

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