topology via logic

chapter 2: affirmative and refutative assertions

An observation must be made in finite time after a finite amount of work.

An assertion is affirmative iff it is true precisely in the circumstances when it can be affirmed.

- some babies have tartan eyes

An assertion is refutative iff it is false precisely in the circumstances when it can be refuted.

- all ravens are black
- Smith is exactly six feet tall

boundary case: Jones is between five foot eleven inches and six foot one; if Jones is exactly five foot eleven, we shall never be able to affirm or refute because it is unclear if 'between' is inclusive or exclusive

Logic:

- negation: transforms affirmative to refutative and vice versa (so wlog assume assertions are assertive)
- disjunction (or): if P and Q are both affirmative, then so is $P \vee Q$, can infinite because you only need to affirm one (finite work)
- conjunction (and): if P and Q are both affirmative, then so is $P \wedge Q$, cannnot infinite because you need to affirm all of them (infinite work)
- true and false: empty conjunction and empty disjunction respectively; to affirm true, you don't need to do anything, so it is affirmative; to affirm false, you must find something contradictory, so you can never affirm it; it is never true and never affirmable, and so it is an affirmative assertion.
- implication: can be refuted by affirming P and refuting Q; however if they are both affirmative, it is unclear. Therefore, the logic of affirmable assertions must not include implication.
- distributivity: both holds

Therefore, the *logic of affirmative assertions* has arbitrary disjunctions, finite conjunctions, two distributive laws; does not include negation, implication or infinite conjunctions. (technically known as propositional geometric logic)

chapter 3: frames

3.1: Algebraicizing logic

Identify formulae that are logically equivalent: the logical connectives then become *algebraic operators subject to laws.*

This is similar to the *Lindenbaum algebra*: take a countable set of variables, representing propositions, form all possible formulae, then take the equivalence classes for logical equivalence. In this algebra, the logical connectives and the axioms of logic become the operators and laws of Boolean algebra; it is a complete system of propositions, closed under logic. Classical logic has Boolean algebras, intuitionistic logic has Heyting algebras, and geometric logic has the *frames.*

The Lindenbaum algebra is *free* on its variables, which means the only equalities that hold in the algebra are those that derive purely from the logic. We shall want to incorporate assumptions about the real world, such as 2 propositions being inconsistent.

3.2 Posets

Def 3.2.1 A poset is a set P with binary relation $\leq$ that satisfies:

- reflexivity
- transitivity
- antisymmetry

A function is *monotone* if it preserves order.

We shall think of the elements as being *propositions*, and of $\leq$ as meaning $\Rightarrow$, or ‘entails’, or ‘is logically stronger than’. Then antisymmetry says is 2 propositions are equal iff each entails the other.

Some elements are also *incomparable*.

Examples 3.2.2
- $\mathcal{P}(X)$ is a poset if you take $\leq$ to mean $\subseteq$; antisymmetry corresponds to the extensional definition of set equality
- Let P be any poset. The *opposite* poset, P^{op} has the same element but reverse ordering

Def 3.2.3 A *preorder* is a set P equipped with a binary relation $\leq$ that is reflexive and transitive.

Proposition 3.2.4 Let P be a preorder. We define a binary relation $=$ on P by $a = b$ iff $a \leq b$ and $b \leq a$. Then $=$ is an equivalence relation, and the equivalence classes $[a]$ form a poset $P/=$, with $[a] \leq [b]$ iff $a \leq b$.

3.3 Meets and joins

Def 3.3.1 Let P be a poset, $X \subseteq P$ and $y \in P$. Then y is a *meet* (or *glb*) for X ($y = \bigwedge X$) iff
- y is a *lower bound* for X: if $x \in X$ then $y \leq x$
- if x is any other lower bound for X then $x \leq y$

Proposition 3.3.2 Let P be a poset and $X \subseteq P$. Then X can have at most 1 meet.

Let y and y' be 2 meets of X. Since y is a meet and y' is a lower bound, $y' \leq y$. Similarly, $y \leq y'$. By antisymmetry, $y = y'$.

Def 3.3.3 Let P be a poset, $X \subseteq P$ and $y \in P$. Then y is a *join* (or *lub*) for X ($y = \bigvee X$) iff
- y is a *upper bound* for X: if $x \in X$ then $y \geq x$
- if x is any other upper bound for X then $x \geq y$

Proposition 3.3.4 Let P be a poset, $X \subseteq P$ and $y \in P$. Then y is a join for X iff y is a meet for X in P^{op}.

Proposition 3.3.5 Let P be a poset. Then for all $x, y \in P$,

$$x = \bigwedge\{x, y\} \Leftrightarrow x \leq y \Leftrightarrow y = \bigvee\{x, y\}$$

If $x = \bigwedge\{x, y\}$, then x is a lower bound for $\{x, y\}$ and hence $x \leq y$. Conversely, if $x \leq y$ then x is a lower bound for $\{x, y\}$ and is greater than any other lower bound.

The significance of this proposition is that if we define $\bigwedge$ or $\bigvee$ independently, we have to define $\leq$ to have any hope of recovering them as actual meets and joins.

Proposition 3.3.6 Let P be a poset. Then for all $y \in P$,
- y is the empty meet ($\top$) iff it is a top element
- y is the empty join ($\bot$) iff it is a bottom element

Empty meets and joins need not exist. There may be multiple *minimal* elements, but no *least* element.

Example 3.3.7 In logic, $\leq$ means $\Rightarrow$, $P \wedge Q \Rightarrow P$ and $P \wedge Q \Rightarrow Q$; these are often seen as *elimination rules* for $\wedge$, ways of deribing formulae without it.

Next, if $R \Rightarrow P$ and $R \Rightarrow Q$, then $R \Rightarrow P \wedge Q$. This is an *introduction rule* for $\wedge$. **true** is a top element.

For joins, $P \Rightarrow P \vee Q$ and $Q \Rightarrow P \vee Q$ (introduction rules) and if $P \Rightarrow R$ and $Q \Rightarrow R$ then $P \vee Q \Rightarrow R$ (elimination rule). Moreover, **false** is bottom because of the standard logical idea that if you assume a contradiction then you can prove anything.

Example 3.3.8 In set theory, meets are intersections and joins are unions, We work with subsets of a universe U and $\leq$ is set inclusion $\subseteq$.

Def 3.3.9 Let P and Q be posets, and $f : P \to Q$ be a function. f *preserves meets* iff whenever $X \subseteq P$ has a meet y, then $f(y)$ is a meet for $\{f(x) : x \in X\}$.

3.4 Lattices

Def 3.4.1 A poset P is a *lattice* iff every finite subset has both a meet and a join. A function between 2 lattices is a *lattice homomorphism* iff it preserves all finite meets and joins.

Note that this is *self-dual*: the opposite of a lattice is a lattice.

Prop 3.4.2 A poset P is a lattice iff $\emptyset$ and all two-element subsets have meets and joins.

By induction on number of elements, a finite subset if P must have a meet and a join.

$\bigwedge\{x_1, x_2, ..., x_n\} = x_1 \wedge x_2 \wedge ... \wedge x_n$

Algebraic properties of $\wedge$ and $\vee$: commutativity, associatibity, unit laws, idempotence, absorption

Also, $x \leq y \Leftrightarrow x \wedge y = x \Leftrightarrow x \vee y = y$

Def 3.4.3 A lattice P is *distributive* iff for every $x, y, z \in P$ we have

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$

Prop 3.4.4 In a distributive lattice P, $\vee$ also distributes over $\wedge$.

$$(x \vee y) \wedge (x \vee z) = ((x \vee y) \wedge x) \vee ((x \vee y) \wedge z) = x \vee ((z \wedge x) \vee (z \wedge y)) = (x \vee (z \wedge x)) \vee (y \wedge z) = x \vee (y \wedge z)$$

Ex 3.4.5 If U is a set, then $\mathcal{P}(U)$ is a distributive lattice.

Ex 3.4.6 A poset P is *linearly ordered* if any 2 elements are comparable. If P has a top and a bottom then it is a distributive lattice.

It is easy to prove that for any lattice $x \wedge (y \vee z) \geq (x \wedge y) \vee (x \wedge z)$.

$x \wedge y \leq x$ and $x \wedge y \leq y \leq y \vee z$ hence $x \wedge y \leq x \wedge (y \vee z)$. Similarly, $x \wedge z \leq x \wedge (y \vee z)$. Therefore, $(x \wedge y) \vee (x \wedge z) \leq x \wedge (y \vee z)$.

For the reverse inequality, $y \vee z$ mut be either y or z, so $x \wedge (y \vee z)$ mut be either $x \wedge y$ or $x \wedge z$, both of which are less than $(x \wedge y) \vee (x \wedge z)$.

Ex 3.4.7 If P is a distributive lattice then so is its opposite P^{op}.

Ex 3.4.8 Let U be an infinite set, and let $\mathcal{P}_{\text{fin}(U)}$ be the set of all finite subsets of U. This is almost a distributive lattive because it lacks the nullary meet U. Hence $\mathcal{P}_{\text{fin}(U)} \cup U$ is a distributive lattice.

Ex 3.4.9 Two non-examples of distributive lattices: the pentagon and the diamond.

The pentagon is not distributive because $a \wedge (b \vee c) = a \wedge d = a$ but $(a \wedge b) \vee (a \wedge c) = b \vee e = b$.

The diamond is not distributive because $a \wedge (b \vee c) = a \wedge d = a$ but $(a \wedge b) \vee (a \wedge c) = e \vee e = e$

3.5 Frames

A frame is supposed to consist of the possible finiteobservations for some system, with equivalent observations identified, and the logic of finite observations built in as joins and meets.

Def 3.5.1 A poset A is a *frame* iff

- every subset has a join
- every finite subset has a meet
- binary meets distribute over joins

$$x \wedge \bigvee Y = \bigvee \{x \wedge y : y \in Y\}$$

We write **true** for the empty meet and **false** for the empty join. A function between 2 frames is a *frame homomorphism* iff it preserves all joins and finite meets.

Some authors use the terms *locale* or *complete Heyting algebra* for what we have called a frame.

A frame is clearly a distributive lattice. The second point in the definition is also unnecessary:

Prop 3.5.2 Let P be a poset in which every subset has a join. Then every subset has a meet.

Let $S \subseteq P$ and L be the set of its lower bounds. If a meet of S exists, then it must be $\bigvee L$.

Such a poset, with all joins and hence all meets, is called a *complete lattice*.

Ex 3.5.3

- Any finite distributive lattice is a frame.
- If U is a set, then $\mathcal{P}(U)$ is a frame. In fact, $(\mathcal{P}(U))^{\mathrm{op}}$ is also a frame.
- In general, if A is a frame, A^{op} is not a frame.
- $\mathbb{1} = \cdot$, the *inconsistent frame* (**true** = **false**)
- $\mathbb{2} = \{\text{false} \leq \text{true}\}$, sometimes called the *Sierpinski* frame

3.6 Topological spaces

Let X be a set, and A any *subframe* of $\mathcal{P}(X)$:

- If $S \subseteq A$ then $\bigcup S \in A$
- If $S \subseteq A$ is finite then $\bigcap S \in A$

This is well known as a *topology* on X.

Def 3.6.1 A *topological space* is a set X equipped with a topology ΩX on it. The elements of ΩX are known as the *open* subsets of the space.

Ex 3.6.1

- $\Omega X = \mathcal{P}(X)$ This is the *discrete* topology on X. It is the *finest* topology on X.
- $\Omega X = \{\emptyset, X\}$ This is the *indiscrete* topology on X. It is its *coarsest* topology.
- Let P be a poset. A subset $S \subseteq P$ is called *upper closed* iff for all $x, y \in P$, $y \geq x \in S \Rightarrow y \in S$. The upper closed sets form a subframe of $\mathcal{P}(P)$, and is known as the *Alexandrov* topology on P. By duality, the *lower closed* subsets of P, the upper closed subsets of P^{op}, also form a topology.

Def 3.6.3 Let P be a poset. If $x \in P$ and $S \subseteq P$, we write

- $\uparrow x = \{y \in P : y \geq x\}$, the *upper closure* of x
- $\uparrow S = \bigcup \{\uparrow x : x \in S\}$, the upper closure of S

and $\downarrow x, \downarrow S$ similarly.

Def 3.6.4 Let X be a topological space and $S \subseteq X$.

$$\mathrm{Int}(S) = \bigcup\{U \in \Omega X : U \subseteq S\}$$

A subset $F \subseteq X$ is *closed* iff its complement is open. They are preserved under arbitrary intersections and finite unions. A subset is *clopen* iff it is both open and closed.

The *topological closure* of a subset $S \subseteq X$ is

$$\mathrm{Cl}(S) = (\mathrm{Int}(S^c))^c = \bigcap\{F : S \subseteq F, F \text{ is closed}\}$$

If $x \in N \subseteq X$, then N is a *neighbourhood* of x iff there is some open set U with $x \in U \subseteq N$.

3.7 Some examples from computer science

Finite observations on bit streams

For each natural number $n \geq 1$ we have 2 *subbasic* observations:

- the n-th bit has been read as a zero: $'s_n = 0'$
- the n-th bit has been read as a one: $'s_n = 1'$

$$'s_n = 0' \wedge 's_n = 1' = \text{false}$$

We take $'s_n = 0' \vee 's_n = 1'$ to mean that the n-th bit has now been read, but we're not saying what value it was.

$$'s_{n+1} = 0' \vee 's_{n+1} = 1' \leq 's_n = 0' \vee 's_n = 1'$$

Now that we have subbasic observations, follow these steps to build them:

1. Finite meets of subbasics
2. Joins of finite meets of subbasics
3. Finite meets of joins of finite meets of subbasics. But if each C_i, D_j is a finite meet of subbasics,

$$\left(\bigvee_i C_i\right) \wedge \left(\bigvee_j D_j\right) = \bigvee_{i,j} C_i \wedge D_j$$

so step 3 and beyond doesn't give us anything new.

$$\begin{aligned}'s_2 = 0' &= 's_2 = 0' \wedge ('s_1 = 0' \vee 's_1 = 1') \\ &= ('s_1 = 0' \wedge 's_2 = 0') \vee ('s_1 = 1' \wedge 's_2 = 0')\end{aligned}$$

We can do this with finite meets of subbasics, as long as they're consistent.

$$'s_1 = 0' \wedge 's_2 = 1' \wedge 's_5 = 1' = \text{starts } 01001 \vee \text{starts } 01011 \vee \text{starts } 01101 \vee \text{starts } 01111$$

every observation is a disjunction of observations starts l *where* l *is a finite list of bits.*

Def 3.7.1 Let l and m be 2 finite lists. Then l *prefixes* m (denoted $l \subseteq m$) iff for some list x, m is the concatenation of l and x.

Lem 3.7.2 If $l \subseteq m$ then starts $l \geq$ starts m because m is a meet of subbasics of starts l and some more.

The converse is also true: Suppose starts $l \geq$ starts m. We can affirm starts m, and hence starts l, so l must be a prefix of m.

Prop 3.7.3 Let L, M be sets of finite lists of bits, and suppose $\forall l \in L \exists m \in M$ s.t. $m \subseteq l$. Then

$$\bigvee_{i \in L} \text{starts } l \leq \bigvee_{m \in M} \text{starts } m$$

Proof If $l \in L$ then there is $m \in M$ s.t. $m \subseteq l$.

$$\text{starts } l \leq \text{starts } m \leq \bigvee_{m \in M} \text{starts } m$$

The conclusion follows from defining property of joins. ∎

Suppose L is a set of finite lists of bits. Then $\bigvee_{l \in L}$ starts l is unchanged if we add to L lists with prefixes already in it. Adding all such lists, we obtain the upper closure of L. Thus:

Prop 3.7.4 Every observation is a disjunction of the form $\bigvee_{l \in L}$ starts l, where L is an upper closed set of finite lists of bits.

Is this representation unique? Is there some distinct upper closed sets L, M s.t. $\bigvee_{l \in L} \text{starts } l = \bigvee_{m \in M} \text{starts } m$?

From Ex 3.6.2, the upper closed sets of finite lists form a frame A, the Alexandrov topology, ordered by $\subseteq$. We can deduce that the 2 axioms are valid in A.

Since A is a frame satisfying the axioms, and our reasoning uses only the frame laws and these axioms, if we can deduce that if $\bigvee_{l \in L} \text{starts } l = \bigvee_{m \in M} \text{starts } m$ with L, M upper closed, then this equality must also hold in A, so $L = M$.

We have shown in a particular case how to reason from a *presentation* of a frame to a concrete *definition*. In chapter 4 we shall see how to do this in full generality.

Def 3.7.5 We write $\Omega 2^{*\omega}$ for the frame of Alexandrov opens in the set of finite lists of bits under the prefix ordering.

Different physical assumptions

We now explore different assumptions, which have the same subbasic observations, but have different axioms.

1. Different bits are read independently.

$$'s_2 = 0' \neq ('s_1 = 0' \wedge 's_2 = 0') \vee ('s_1 = 1' \wedge 's_2 = 0')$$

The frame constructed will have fewer equalities holding between the possible expressions than the first frame, so it has more elements.

2. Time is not important.

$$'s_n = 0' \vee 's_n = 1' = \text{true}$$

We can also reduce every element to a join of elements starts l like the main example. However, there are more equalities, so the frame is smaller.

Def 3.7.6 The subbasics $'s_n = 0'$ and $'s_n = 1'$ with axioms $'s_n = 0' \wedge 's_n = 1' = \text{false}$ and $'s_n = 0' \vee 's_n = 1' = \text{true}$ present a frame. We shall call it $\Omega 2^{\omega}$.

Flat domains

If X is any set, then $\mathcal{P}(X)$ is a frame. We can view the subsets as being observations on some object that has value in X. $S \subseteq X$ is interpreted as

$$\bigvee (x \in S) \text{ it is } x$$

This frame has propositions it is x for each $x \in X$, and satisfies axioms

$$\text{it is } x \wedge \text{it is } y = \text{false } (x \neq y)$$

$$\bigvee_{x \in X} \text{it is } x = \text{true}$$

The frame now has the subsets of X together with a new true, bigger than any subset. This can also be described as an Alexandrov topology. Construct a poset $X_\perp = X \cup \{\perp\}$ (X *lifted*, or the *flat domain* on X) where $\perp$ represents a computation that will never finish. Define $\sqsubseteq$ by

$$\begin{aligned}
&\perp \sqsubseteq \perp \\
&\perp \sqsubseteq x (x \in X) \\
&x \sqsubseteq y \Leftrightarrow x = y (x, y \in X)
\end{aligned}$$

The presence of an Alexandrov topology is rather typical of our computer science applications, but it is only appropriate when the elements of its poset P represent finite pieces of information (they are *compact*). If P is also to include infinite elements, then the more complicated Scott topology is called for.

Function spaces

A subbasic observation on a function f is then $[x \to a]$, meaning "we have manufactured x, fed it to f, and observed a for $f(x)$.

The order to do observation doesn't matter (imagine a reset button on the black box), and we also make a *physical assumption*, that the observations have no effect on what is observed.

Consider an f for which both the argument and result have values in $X = \{\mathbb{t}, \mathbb{f}\}$. Naturally, we want to make observations appropriate to the flat domain $X_\perp$.

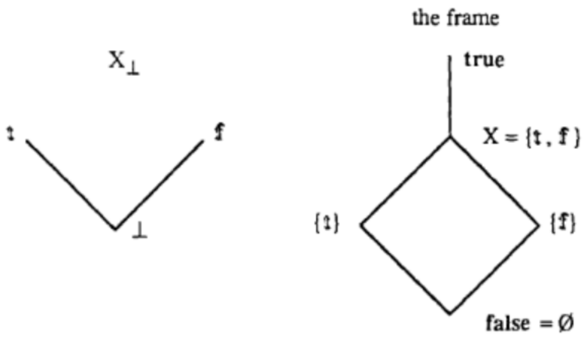

Figure 1: the domain and range

We now have 15 possible observations on the function, namely $[x \to a]$ where $x \in X_\perp$ and $a \in A$.

First note that for all $x \in X_\perp$ and $a, b \in A$,

$$a \leq b \Rightarrow [x \to a] \leq [x \to b]$$

Second, for all $x \in X_\perp$ and $a \in A$,

$$[\perp \to a] \leq [x \to a]$$

If the function allows us to observe a on the basis of no input at all, it is not allowed to retract that if we put in a more solid x.

Third, for all $x \in X_\perp$ and $S \subseteq A$,

$$\left[x \to \bigvee S\right] = \bigvee_{a \in S} [x \to a]$$

This says that to observe $\bigvee S$ for input x, we have to observe some $a \in S$. A corollary of this is that $[x \to \text{false}] = \text{false}$, so we have reduced our fifteen original observations to thirteen.

Fourth, for all $x \in X_\perp$ and finite $S \subseteq A$,

$$\left[x \to \bigwedge S\right] = \bigwedge_{a \in S} [x \to a]$$

For binary meets, we want to know if we observe a for input x, and b for input x, then we can observe $a \wedge b$. The first 2 observations may have been on different runs of the function, and the above is tantamount to another physical assumption, that different runs with equal arguments give equal results, the function is deterministic.

3.8 Bases and subbases

If any assertion is a disjunction of subbasics, then the subbasis is called a basis, and its elements are basic. For example, given any subbasis, the finite conjunctions of subbasics form a basis.

The frames $\Omega 2^{*\omega}$ and $\Omega 2^{\omega}$, the assertions starts l form a basis.

3.9 The real line

We now describe a frame of assertions for real numbers. We write a subbasic assertion as $(q \pm \varepsilon)$ where q, a rational number, is the result, and ε, a positive rational number, is the possible error. If $x \in \mathbb{R}$, we write $x \vDash (q \pm \varepsilon)$ iff $q - \varepsilon < x < q + \varepsilon$.

A finite conjunction of these subbasic measurements is either false or another such measurement. Thus the subbasics form a basis. Disjunctions give us many new assertions, such as

$$'x > 0' = \bigvee \{(n \pm 1) : n \in \mathbb{N}^+\}$$

$$'x \neq 0' = 'x > 0' \vee 'x < 0'$$

$$'x > \sqrt{2}'$$

Prop 3.9.1 A set U is a union of basic assertions iff

$$\forall x \in U, \exists \delta \in \mathbb{R}, (\delta > 0 \wedge (x - \delta) < y < x + \delta \to y \in U)$$

Proof Intuitively, it makes sense but x and δ might be irrational.

$\Rightarrow$: Suppose $q - \varepsilon < x < q + \varepsilon$. Choose $\delta = \min(x - (q - \varepsilon, (q + \varepsilon - x)))$. Then, $x - \delta < y < x + \delta \Rightarrow q - \varepsilon < y < q + \varepsilon \Rightarrow y \in U$

If U is a join of basics, instead of a basic, we can still find one of the basic disjuncts containing x and choose δ for that.

$\Leftarrow$: Given $x \in U$, we look for rationals q_x and ε_x such that $x \in (q_x \pm \varepsilon_x) \subseteq U$, then U is the join of the $(q_x \pm \varepsilon_x)$s and hence is affirmative. We shall use the important fact that there is a rational between every pair of distinct real numbers.

Choose a rational q with $x - \delta/2 < q < x$ and a rational ε with $x - q < \varepsilon < q - (x - \delta)$.

Then

$$x \in (q \pm \varepsilon) \subseteq (x \pm \delta) \subseteq U \blacksquare$$

3.10 Complete Heyting algebras

We carefully ruled out infinite conjunctions, but frames have operations to support this. The reason for ignoring this is not that it dont exist, but that they are less well-behaved.

In a topology, the meet can be expressed in terms of interiors by

$$\bigwedge S = \mathrm{Int}\left(\bigcap S\right)$$

Def 3.10.1 Let A be a lattice. A is a *Heyting algebra* iff for every $a, b \in A$ there is an element $a \to b$ satisfying

$$c \leq (a \to b) \Leftrightarrow (c \wedge a) \leq b$$

A *Heyting algebra homomorphism* is a function f between Heyting algebras that preserves finite meets and joins, and the $\to$ operation.

A Heyting algebra A is a *complete Heyting algebra* (cHa) iff it is a complete lattice.

A *cHa homomorphism* is a function between cHas that is a Heyting algebra homomorphism and moreover preserves all joins.

Prop 3.10.2 Let A be a lattice. Then A is a frame iff it is a cHa.

To define $\to$ in a frame, we put

$$a \to b = \bigvee \{c : c \wedge a \leq b\}$$

To show that a cHa is a frame, we just need to prove the frame distributivity law. Let $S \subseteq A, a \in A$,

$$\begin{aligned} b' \in S \Rightarrow (a \wedge b') \leq \bigvee \{a \wedge b : b \in S\} &\Rightarrow b' \leq \bigvee \left\{c : c \wedge a \leq \bigvee \{a \wedge b : b \in S\}\right\} \\ &\Rightarrow b' \leq \left(a \to \bigvee \{a \wedge b : b \in S\}\right) \end{aligned}$$

Therefore

$$\bigvee S \leq a \to \bigvee \{a \wedge b : b \in S\} \Rightarrow a \wedge \bigvee S \leq \bigvee \{a \wedge b : b \in S\}$$

For the reverse inequality, for a $b \in S$,

$$a \wedge b \leq a \wedge \bigvee S$$

so

$$\bigvee \{a \wedge b : b \in S\} \leq a \wedge \bigvee S$$

In themselves, frames and cHa are the same thing. However, the notions become different when we consider homomorphisms. A frame homomorphism need not preserve the $\to$ operation, and hence need not be a cHa homomorphism.

Define negation by $\neg a = a \to \text{false}$ then any Heyting algebra is a model for propositional intuitionist logic. This is the same as classical logic except that the law of excluded middle may fail. This means that any frame, and in particular that of open sets of a topological space, forms such a model.

Exercises

1. Prove directly that the subsets of $\mathbb{R}$ that satisfy the second condition in Proposition 3.9.1 are closed under finite intersections and arbitrary unions and hence form a frame (a subframe of $\mathcal{P}(\mathbb{R})$).

$$\forall x \in U, \exists \delta_U \in \mathbb{R}^+, x - \delta_U < y < x + \delta_U \to y \in U$$

$$\forall x \in V, \exists \delta_V \in \mathbb{R}^+, x - \delta_V < y < x + \delta_V \to y \in V$$

For $U \cap V$, choose δ to be $\min(\delta_U, \delta_V)$; then $x - \delta_U < x - \delta_V < y < x + \delta_V < x + \delta_U$ (or vice versa) so $y \in U \cap V$.

For $\bigcup \mathcal{U}$, find a set $U \in \mathcal{U}$ s.t. $x \in U$; then just use δ_U so as a result $y \in U \subseteq \bigcup \mathcal{U}$. ■

2. If x is real, show that $\{x\}$ is not open. Thus our assertions do not allow us to observe that a real number is exactly equal to another one.

If $\{x\}$ is open, then every subset of $\mathbb{R}$ will be open, which is not equal to our $\Omega\mathbb{R}$. ■

3. Show that in $\Omega\mathbb{R}$, $\vee$ does not distribute over $\bigwedge$. Thus $(\Omega\mathbb{R})^{\mathrm{op}}$ is not a frame.

$$(0 \pm 1) \vee \bigwedge\left\{\left(0 \pm \frac{1}{n}\right) : n \in \mathbb{N}\right\} = (0 \pm 1) \vee \text{not an assertion} = \text{not an assertion}$$

$$\bigwedge\left\{(0 \pm 1) \vee \left(0 \pm \frac{1}{n}\right) : n \in \mathbb{N}\right\} = \bigwedge\{(0 \pm 1)\} = (0 \pm 1) ■$$

4. Show that in a topological space, $U \to V = \mathrm{Int}(U^c \cup V)$ and $\neg\neg U = \mathrm{Int}(\mathrm{Cl}\ U)$.

$$(C \cap A) \subseteq B \Leftrightarrow C \cap (A \setminus B) = \emptyset \Leftrightarrow C \subseteq (A \setminus B)^c \Leftrightarrow C \subseteq (A \cap B^c)^c \Leftrightarrow C \subseteq (A^c \cup B)$$

Since $A \to B$ has to be an element in the topology, it is $\mathrm{Int}(A^c \cup B)$.

$$\neg\neg U = (U \to \text{false}) \to \text{false} = \mathrm{Int}\left((\mathrm{Int}(U^c \cup \emptyset))^c \cup \emptyset\right) = \mathrm{Int}(\mathrm{Cl}\ U) ■$$

5. For negation $\neg x$ in a frame, show that $x \wedge \neg x = \text{false}$ but find an example in $\Omega\mathbb{R}$ for which $x \vee \neg x \neq \text{true}$.

$$x \wedge \neg x = x \wedge (x \to \text{false}) = x \wedge \bigvee\{c : c \wedge x \leq \text{false}\} = \bigvee\{x \wedge c : c \wedge x \leq \text{false}\} = \text{false}$$

If U is open, $U \cup \mathrm{Int}(U^c) = \mathbb{R} \setminus \mathrm{fr}\ U \neq \text{true}$. ■

6. *Metric spaces.* These are the prototype topological spaces in most approaches. Define the distance function $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by $d(x, y) = |x - y|$. Then
   (i) $d(x, y) \geq 0$ with equality iff $x = y$
   (ii) $d(x, y) = d(y, x)$
   (iii) $d(x, z) \leq d(x, y) + d(y, z)$

A *metric space* is a set equipped with a function $d : X \times X \to \mathbb{R}$ (the metric) satisfying these 3 axioms. Show that the open balls $B_{\varepsilon(x)} = \{y \in X : d(x, y) < \varepsilon\}$ form a base of open sets for a topology on X.

An *ultrametric space* is a metric space X in which d satisfies the stronger *ultrametric inequality*:

$$d(x, z) \leq \max\{d(x, y), d(y, z)\}$$

Let $A_i = \bigcup\left\{B_{\varepsilon_k}(x_k)\right\}$. Then an arbitrary union of A_is is just a union of open balls so is open.

$$A_i \cap A_j = \bigcup \{ B_{\varepsilon_{i,k}}(x_{i,k}) \} \cap \bigcup \{ B_{\varepsilon_{j,k}}(x_{j,k}) \}$$

For every point x in $A_i \cap A_j$, $\exists B_{\varepsilon_{i,k}}(x_{i,k}) \ni x$ and $\exists B_{\varepsilon_{j,k}}(x_{j,k}) \ni x$. Select a suitably sized ball such that it contains x and is included in the intersection of the 2 balls. Then union of all these balls for all x is equal to $A_i \cap A_j$. Since it is a union of bases, it is an open set.

Therefore, arbitrary unions and finite intersections are preserved. ■

7. Show that in a Heyting algebra (whether complete or not), meets distribute over all joins that exist.

https://math.stackexchange.com/questions/1988777/a-lattice-that-is-not-a-complete-lattice

We just need to prove that the various joins exist in the proof of Proposition 3.10.2.

If $\bigvee S$ exists, then $\bigvee \{a \wedge b : b \in S\}$ exists because $\bigvee \{a \wedge b : b \in S\} \leq a \wedge \bigvee S$.

Similarly, $\bigvee \{c : c \wedge a \leq x\} = \bigvee \{c : c \wedge a \wedge x = c \wedge a\} = \bigvee \{c : c \wedge x = c\} \leq x$ so $\bigvee \{c : c \wedge a \leq \bigvee \{a \wedge b : b \in S\}\}$ exists. ■

8. Let A be a finite, linearly ordered poset (hence a frame). Show that $\to$ in A is defined by

$$a \to b = \begin{cases} \text{true if } a \leq b \\ b \text{ if } a > b \end{cases}$$

If B is another frame, show that a function $f : A \to B$ is a frame homomorphism iff it is monotone and perserves true and false. Give an example of a frame homomorphism that is not a Heyting algebra homomorphism.

If $a \leq b$,

$$a \to b = \bigvee \{c : c \wedge a \leq b\} = \bigvee A = \text{true}$$

If $a > b$,

$$a \to b = \bigvee \{c : c \wedge a \leq b\} = \bigvee \{c : c \leq b\} = b$$

$\Rightarrow$: Let $f : A \to B$ be a frame homomorphism.

chapter 4: frames as algebras

4.1 Semilattices

Def 4.1.1 A poset S is a *semilattice* iff every finite subset has a meet. A *semilattice homomorphism* is a function between semilattices that preserves finite meets.

More precisely, these are *meet-semilattices*; their opposites are *join-semilattices*. Just as for lattices, it suffices to find a top element true and meets $x \wedge y$ for all pairs $\{x, y\}$. Note that a semilattice homomorphism f is monotone:

$$x \leq y \Rightarrow x = x \wedge y \Rightarrow f(x) = f(x \wedge y) \Rightarrow f(x) = f(x) \wedge f(y) \Rightarrow f(x) \leq f(y)$$

An important property of semilattices is that they are purely algebraic. Unlike posets, where the inequality $\leq$, a non-algebraic idea, is essential, semilattices can be defined in terms of $\wedge$ and equality: $x \leq y \Leftrightarrow x \wedge y = x$.

Prop 4.1.2 Semilattices are equivalent to algebras equipped with 2 operators, true (nullary) and $\wedge$ (binary), subject to the following *laws*:

- commutativity: $x \wedge y = y \wedge x$
- associativity: $x \wedge (y \wedge z) = (x \wedge y) \wedge z$
- unit law: $x \wedge \text{true} = x$
- idempotence: $x \wedge x = x$

Then $x \leq y$ iff $x \wedge y = x$.

A function $f : A \to B$ between semilattices is a homomorphism iff $f(\text{true}) = \text{true}$ and $f(x \wedge y) = f(x) \wedge f(y)$ for all $x, y \in A$.

Proof We have already seen in lattices how the meets give rise to true and $\wedge$. The interesting case here is to go from an algebra S to a semilattice. By Proposition 3.3.5, we define $\leq$ as $x \wedge y = x$. Then $\leq$ is reflexive by idempotence, transitive because if $x \leq y$ and $y \leq z$ then

$$x \wedge z = (x \wedge y) \wedge z = x \wedge (y \wedge z) = x \wedge y = x$$

and antisymmetric because if $x \leq y \leq x$ then $x = x \wedge y = y \wedge x = y$. Hence S is a poset under $\leq$.

The unit law says that true is top, and it remains to show that $x \wedge y$ is the meet of $\{x, y\}$.

$$(x \wedge y) \wedge y = x \wedge (y \wedge y) = x \wedge y \Rightarrow x \wedge y \leq y \text{ Similarly } x \wedge y = y \wedge x \leq x$$

Let z be any other lower bound. Then

$$z \wedge (x \wedge y) = (z \wedge x) \wedge y = z \wedge y = z \Rightarrow z \leq x \wedge y$$

The second statement follows by indunction on the number of conjuncts in the finite meet. ■

A combination like this of operators and equational laws makes up an *algebraic theory* and puts us in the realms of universal algebra. For completeness, note that lattices, distributive lattices and Heyting algebras can also be described algebraically in this way. Frames can too with a bit of work.

4.2 Generators and relations

Suppose we want to describe a frame. One method, which we have already seen informally in section 3.7 *presents* a frame in 4 steps.

1. Specify some subbasic elements (*generators*)
2. Derive from these all possible joins of meets of subbasics

3. Specify certain axiomatic *relations* to hold between expressions of step 2. They can be of the form $e_1 \leq e_2$ (*inequations*) or $e_1 = e_2$ (*equations*). It doesnt matter which you use, because, as in Proposition 4.1.2, the 2 forms are interconvertible.
4. Deduce, just from the relations and the frame laws, when any two given expressions must be equal. This, then, is an equivalence relation on the Step 2 expressions.

The Step 4 equivalence relation is supposed to mean equal in the frame were defining, so formally the frame is the set of equivalence classes. We can deduce that it actually is a frame from the fact that all consequences of the frame laws have been build in to Step 4.

From one point of view, this is a method of logic. The generators are the primitive propositional symbols, the Step 2 expressions are well-formed formulae, the Step 3 relations (written as inequations) are the axioms, and the Step 4 equivalence is mutual entailment.

However, it also represents a very general method of Univeral Algebra, enabling us in a wide class of algebraic theories to present an algebra by writing down generators and relations for it. Unfortunately, for the theory of framse the infinite joins give rise to obstacles to formalizing the general argument. In Section 4.4 we shall see that these can be overcome, but meanwhile we shall concentrate on developing the practical intuition.

Assuming that the method does indeed define a frame, we write it as

$$\text{Fr } \langle \text{generators} \mid \text{relations} \rangle$$

and call this is a *presentation* of the frame.

Ex 4.2.1 Fr $\langle | \rangle$. The only elements we can generate are true and false. Our frame is therefore $\mathbb{2}$.

Ex 4.2.2 Fr $\langle a, b \mid \rangle$. The finite meets of the generates are true, a, b and $a \wedge b$, and the joins of these are false, a, b and $a \vee b$.

We obtain this frame:

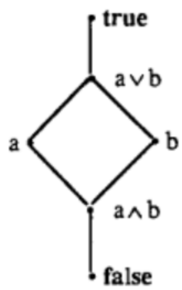

Figure 2: Example 4.2.2

How do we know that this incorporates all the deductions that we were supposed to be making? If we could deduce, just from frame laws and the relations, that 2 of the 6 expressions had to be equal, then they would be equal in our diagrams as well, and they are not. Therefore the diagram illustrates precisely the frame we were trying to construct.

Def 4.2.3 A presentation with generators only, no relations, is called *free*. By extension, the algebra it presents is called free (on those generators).

Ex 4.2.4 Fr $\langle a, b \mid a \leq b \rangle$. This is the same as Fr $\langle a, b \mid a \wedge b = a \rangle$; but there is no reason why we shoud not think in terms of the inequality during our reasoning.

We start of with the frame of Example 4.2.2, and look for where the diagram collapses because of the new equalities. We get $a \wedge b = a$ and $a \vee b = b$, leaving:

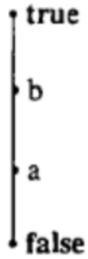

Figure 3: Example 4.2.4

The ordering is linear, so we can use Example 3.4.6 to check this is a frame.

In general if $\mathbb{T}$ is any algebraic theory (described by operators and equational laws) for which this method works, then we write

$$\mathbb{T}\langle \text{generators} \mid \text{relations} \rangle$$

for the $\mathbb{T}$-algebra presented by the given generators and relations.

4.3 The universal characterization of presentations

Def 4.3.1 Let $\mathbb{T}$ be an algebraic theory. It has some *operators*, each with an *arity*, and some *laws*, each of the form $e_1 = e_2$. We define A to be a $\mathbb{T}$-algebra iff
1. A is a set, known as the *carrier* of the algebra
2. for each n-ary operator ω of $\mathbb{T}$, A is equipped with a corresponding *operations*, a function $\omega : A^n \to A$.

Now suppose e is an expression formed using variables X_i ($i \in m$) and operators of $\mathbb{T}$. If we substitute an m-tuple of elements from A for the X_is, we can evaluate the expression in A. This gives us a corresponding function $e : A^m \to A$.
3. If $e_1 = e_2$ is a law for $\mathbb{T}$, then the 2 functions $e_1, e_2 : A^m \to A$ must be equal.

Let A and B be 2 $\mathbb{T}$-algebras. A $\mathbb{T}$-homomorphism from A to B is a function $f : A \to B$ s.t. if ω is any n-ary $\mathbb{T}$-operator, then if n is finite,

$$\omega(f(a_1), ..., f(a_n)) = f(\omega(a_1, ..., a_n))$$

Ex 4.3.2
1. Semilattices. The theory has the 2 operators and 4 laws.
2. Frames. The theory has a proper class of operators, true, $\wedge$, and $\bigvee_I$, the I-ary join for each set I, and a proper class of laws: the 4 semilattice laws, and

$$x_j \wedge \bigvee_I (x_i : i \in I) = x_j \text{ If } j \in I$$

$$y \wedge \bigvee_I (x_i : i \in I) = \bigvee_I (y \wedge x_i : i \in I)$$

Let us not be more precise about what it means for an algebra A to be presented as $\mathbb{T}\langle G \mid R \rangle$,

Def 4.3.3 Let $\mathbb{T}\langle G \mid R \rangle$ be a presentation. A *model* for the presentation is an A satisfying the following conditions.
- A is a $\mathbb{T}$-algebra
- A is equipped with a function $[-] : G \to A, g \mapsto [g]$

This function can be extended to apply to any expression e built up from the generators and the $\mathbb{T}$-operators: replace the generators g by their interpretations $[g]$, and evaluate the expression in A to give $[e] \in A$.
- If $e_1 = e_2$ is a relation in R, then it must hold in A that $[e_1] = [e_2]$.

In a law, an equation contains variables, and the equation must always hold, whatever values from an algebra are substituted for the variables. In an relation, the equation contains generators, and the equation must hold when the generators are given their particular values in a model.

Def 4.3.4 Let $\mathbb{T}$ be an algebraic theory. A $\mathbb{T}$-algebra A is *presented* by a presentation $\mathbb{T}\langle \text{generators} \mid \text{relations} \rangle$ iff
- it is a model for the presentation
- if B is any other model, then there is a unique homomorphism $\theta : A \to B$ such that $\theta(g_A) = g_B$ for every generator g.

Because the second condition applies to all models, it is called a *universal* property. We have been trying to define *the* algebra presented by generators and relations, so we must ask what happens if 2 different algebras, A and A', both fit the definition. Letting A' play the role of B in the definition, we find a unique homomorphism $\theta : A \to A'$ mapping each $g_A \mapsto g_{A'}$, and similarly we find $\theta' : A' \to A$ mapping each $g_{A'} \mapsto g_A$. Therefore, $\theta; \theta'$ maps each g_A to itself and is the unique such by definition. However, the identity homomorphism Id_A also does this.

As mutually inverse homomorphisms (*isomorphisms*), θ and θ' are structure-preserving bijections. Thus A and A', although not strictly equal, are structureally equivalent; they are *isomorphic*. Hence,

Prop 4.3.5 The algebra presented by generators and relations, if it exists at all, is defined up to isomorphism by 4.3.4.

Theo 4.3.6 Let $\mathbb{T}$ be a *finitary* algebraic theory, which means one in which
- the operators form a set (not a proper class)
- each operator has finite arity.

Then every presentation $\mathbb{T}\langle G \mid R \rangle$ presents a $\mathbb{T}$-algebra.

Proof See Manes for a full proof. We use 4 steps as below.
1. Take the generators G.
2. Derive from these all possible well-formed expressions using the generators and the operators of $\mathbb{T}$. $\mathbb{T}$'s finiteness ensures that these expressions from a set and not a proper class.
3. Take the relation R. For a general theory $\mathbb{T}$, these have to be equations, because the operators of $\mathbb{T}$ may not allow us to define the partial ordering we have for semilattices.
4. Deduce, just from the relations and the laws of $\mathbb{T}$, when any 2 given expressions must be equal. This, then, is an equivalence relation on the step 2 expressions.

The equivalence classes of expressions then form the algebra we are trying to present. ■

Prop 4.3.7 Let $\mathbb{T}$ be a finitary algebraic theory, and let A be a $\mathbb{T}$-algebra. Then A has a presentation.

Proof For each element a of A, let there be a generator a'. For each n-ary operator ω, and each tuple $(a_i) \in A^n$, let there be a relation

$$\omega(a_{1'}, ..., a_{n'}) = \omega(a_1, ..., a_n)'$$

A model B for this has a function from A to B, and the fact that this respects the relations says precisely that the function is a homomorphism. Thus the models are the algebras equipped with homomorphisms from A. This says precisely that the presentation presents A. ■

For instance, let S be a semilattice. Then we can construct a *universal distributive lattice* L over S. This means that L is a distributive lattice, there is a semilattice homomorphism $f : S \to L$, and that if $g : S \to K$ is any other semilattice homomorphism $h : L \to K$ such that $g = f; h$. This is a universal property of L, characterizing it up to isomorphism. It is not hard to see that this universal property is satisfied by the distributive lattice presented as

$$\text{DL } \langle f(a) : a \in S \mid f(\text{true}) = \text{true}, f(a \wedge b) = f(a) \wedge f(b) : a, b \in S$$

With the power of the method of generators and relations comes a disadvantage. Although it tells us that certain algebras with certain universal properties exist, it does not tell us much about their structure. For instance, when we construct a universal distributibe lattice as above, it is a fact that the homomorphism f always $1 - 1$; but it takes a little work to prove this. (L can be constructed concretely as the set of lower closed subsets $\downarrow x_1 \cup \ldots \cup \downarrow x_n$ of S.)

4.4 Generators and relations for frames

For infinitary theories, such as that of frames, there is a hitch. Step 2 of Theorem 4.3.6 tells us to form the set of all possible expressions using the operators, and at this stage the general theory doesn't use the algebraic laws to make any identification between expressions. This is fine for the finitary algebraic theories. However, for frames, we can make new expressions by forming joins of arbitrary sets of older expressions, and this can't be done in set theory. Technically, the "set" of all possible expressions would be a proper class. This is a genuine problem. There are infinitary theories such as that of complete Boolean algebras where this is insuperable and presentations simply don't present algebras. For frames, fortunately, presentations do present, but we have to argue slightly carefully to show this. The trick is to import our knowledge of the frame laws into Step 2 and that all possible expressions are joins of finite meets of generators, which do form a set.

Def 4.4.1 Let S be a semilattice. A *cover relation* in S is a formula $U \mathrel{-\!:} a$ (U *covers* A), where $a \in S$ and $U \subseteq \downarrow a$. This is intended to mean $\bigvee U = a$. A function f from S into a lattice is said to *transform the cover to a join* iff $\bigvee \{f(u) : u \in U\} = f(a)$.

A *coverage* on S is a set C of cover relations such that if $U \mathrel{-\!:} a$ is in C, and $b \leq a$, then C also contains $\{b \wedge u : u \in U\} \mathrel{-\!:} b$. This corresponds to frame distributivity (below formula is just illustrating the case where a and b is in a frame):

$$\text{if} \bigvee U = a \text{ then} \bigvee \{b \wedge u : u \in U\} = b \wedge a = b$$

Clearly any set C' of cover relations generates a coverage given by the cover relations

$$\{b \wedge u : u \in U\} \mathrel{-\!:} b \text{ for some } U \mathrel{-\!:} a \text{ in } C', b \leq a$$

Suppose we want to present a frame with S as a basis, and a coverage C to define certain joins. Any element a can be expressed as a join of elements of S that are less than a. We can therefore identify the elements of the frame with certain subsets of S, namly the *C-ideals*:

If C is a coverage on S, then a *C-ideal* in S is a subset I satisfying
- I is lower closed
- If $U \mathrel{-\!:} a$ is a cover relation in C, and $U \subseteq I$, then $a \in I$.

We write C-Idl(S) for the set of C-ideals in S. A *site* is a semilattice equipped with a coverage C.

Theo 4.4.2 Let S be a semilattice, C' a set of cover relations and C the coverage it generates.
1. C-Idl(S) is a frame under the subset ordering ($I \leq J$ iff $I \subseteq J$).
2. There is a function $f : S \to$ C-Idl(S) that preserves finite meets and transforms the covers of C' to joins.

3. Any other such function from S to a frame factors uniquely as f followed by a frame homomorphism.

Proof

1. First, we note that any intersections of C-ideals is still a C-ideal.

The intersection of lower closed sets are also lower closed: If there exists $y \leq x \in \bigcap \mathcal{I} = \bigcap \{I_1, ...\}$ that is not in $\bigcap \mathcal{I}$, then there exist an I where $x \in I$ and $y \notin I$, then I is not lower closed.

If $(U -: a) \in C$ and $U \subseteq \bigcap \mathcal{I}$, then $\forall I \in \mathcal{I}, U \subseteq I \Rightarrow a \in I$ so $a \in \bigcap \mathcal{I}$.

Hence intersection is meet and C-Idl(S) is a complete lattice *(semilattice?)*.

A union of C-ideals is not necessarily a C-ideal; but any subset X of S generates a C-ideal, namely the intersection of all the C-ideals that contain it. Let us write this as $C{-}{\downarrow} X$. The join in C-Idl(S) is then $\bigvee_i I_i = C{-}{\downarrow} \bigcup_i I_i$. It remains for us to prove distributivity.

Def 4.4.2.1 Let X and Y be arbitrary subsets of S. Then we write

$$X \wedge Y = \{x \wedge y : x \in X, y \in Y\}$$

$$X/Y = \{z \in S : y \wedge z \in X \text{ for all } y \in Y\}$$

Prop 4.4.2.2 For any subsets $X, Y, Z \subseteq S$,
1. $Z \subseteq X/Y \Leftrightarrow Y \wedge Z \subseteq X \Leftrightarrow Y \subseteq X/Z$
2. If X and Y are C-ideals, then $X \wedge Y = X \cap Y$
3. If X is a C-ideal, then so is X/Y

$$Z \subseteq X/Y \Leftrightarrow \forall z \in Z, \forall y \in Y, y \wedge z \in X \Leftrightarrow Y \wedge Z \subseteq X$$

$X \wedge Y \subseteq X \cap Y$ because X and Y are lower closed. If $z \in X \cap Y$, then $z = z \wedge z \in X \wedge Y$.

If X is a C-ideal, $a \in X/Y$ and $b \leq a$, then $\forall y \in Y, y \wedge a \in X$ so $y \wedge b \in X$ (X is lower closed) so $b \in X/Y$ hence X/Y is lower closed.

If $(U -: a) \in C$ and $U \subseteq X/Y$, then $Y \wedge U \subseteq X$. Since $(\{(a \wedge y) \wedge u : u \in U\} -: (a \wedge y)) \in C$ and $\{(a \wedge y) \wedge u : u \in U\} \subseteq X$ (X is lower closed), so $a \wedge y \in X$ for all y. This is precisely the definition for a to be in X/Y. Therefore, X/Y is a C-ideal.

If we are given C-ideals I and J_i then we want the following expression (the reverse inequality follows from $I \wedge \bigvee_i J_i \geq I \wedge J_i$):

$$I \wedge \bigvee_i J_i \leq \bigvee_i (I \wedge J_i) = K$$

i.e. $\bigvee_i J_i \leq K/I$ (remember $\leq$ in C-Idl(S) is $\subseteq$)

i.e. $\bigcup J_i \subseteq K/I$ (definition of join in C-Idl(S))

i.e. $J_i \leq K/I$ for all i

i.e. $I \wedge J_i \leq K$ for all i, and this is obvious.

Reading upwards, we arrive at the distributive law.

2. We map into C-Idl(S) by $f(x) = C{-}{\downarrow} \{x\}$.

Clearly $f(\text{true}) = S$, the top C-ideal. Now take $x, y \in S$. We want

$$f(x) \wedge f(y) = f(x \wedge y)$$

i.e. $f(x) \wedge f(y) \leq f(x \wedge y)$ ($f(x) \wedge f(y)$ is a C-ideal and by definition of $f(x \wedge y)$, $f(x) \wedge f(y) \supseteq f(x \wedge y)$)

i.e. $f(x) \leq f(x \wedge y)/f(y)$

i.e. $\{x\} \subseteq f(x \wedge y)/f(y)$ ($f(x \wedge y)/f(y)$ is a C-ideal)

i.e. $f(y) \leq f(x \wedge y)/\{x\}$

i.e. $\{y\} \leq f(x \wedge y)/\{x\}$

i.e. $\{x\} \wedge \{y\} = \{x \wedge y\} \subseteq f(x \wedge y)$, which is clear.

Therefore, f preserves finite meets.

Finally, let $U \mathrel{-\!\!:} a$ be a cover relation in C'. We want $f(a) \leq \bigvee\{f(u) : u \in U\}$, which comes from the definition of C-ideals. The reverse inequality is obvious: $f(a) = C\text{--}\!\downarrow \{a\}$ is a superset of all $f(u)$ because $a \geq u$. f also transforms all the covers in C into joins.

3. Let $g : S \to A$ be another such function into a frame. We want there to be a unique frame homomorphism $h : \text{C-Idl}(S) \to A$ such that $f; h = g$. Uniqueness is easy:

$$h(I) = h\left(\bigvee\{f(x) : x \in I\}\right) = \bigvee\{h(f(x)) : x \in I\} = \bigvee\{g(x) : x \in I\}$$

It remains to show that this does indeed define a frame homomorphism, and that $f; h = g$.

$$h(\text{true}) = \bigvee\{g(x) : x \in S\} = g(\text{true}) = \text{true}$$

$$\begin{aligned} h(I) \wedge h(J) &= \bigvee\{g(x) : x \in I\} \wedge \bigvee\{g(y) : y \in J\} \\ &= \bigvee\{g(x) \wedge g(y) : x \in I, y \in J\} \\ &= \bigvee\{g(x \wedge y) : x \in I, y \in J\} \\ &= \bigvee\{g(z) : z \in I \wedge J\} = h(I \wedge J) \end{aligned}$$

$$\bigvee_i h(I_i) = \bigvee_i \bigvee\{g(x) : x \in I_i\} = \bigvee\left\{g(x) : x \in \bigcup_i I_i\right\} = a$$

Let $(U \mathrel{-\!\!:} b) \in C$ s.t. $\forall u \in U, g(u) \leq a$. Then $g(b) = \bigvee\{g(u) : u \in U\} \leq \bigvee\{a : u \in U\} = a$ because g transforms covers to joins. Therefore, $\{x \in S : g(x) \leq a\}$ is a C-ideal. Hence it contains $\bigvee_i I_i = C\text{--}\!\downarrow \bigcup I_i$.

$$h\left(\bigvee_i I_i\right) \leq h(\{x \in S : g(x) \leq a\}) = \bigvee\{g(x) : g(x) \leq a\} = a = \bigvee_i h(I_i)$$

However, $h(I_i) \leq h\left(\bigvee_i I_i\right)$ so $\bigvee_i h(I_I) = h\left(\bigvee_i I_i\right)$.

To show that $f; h = g$, let $x \in S$. We must show

$$g(x) = \bigvee\{g(y) : y \in C\text{--}\!\downarrow \{x\}\}$$

$\leq$ is obvious, so it remains to show that if $y \in C\text{--}\!\downarrow \{x\}$ then $g(y) \leq g(x)$. Define

$$I = \{y \in S : g(y) \leq g(x)\}$$

By the same argument as above, I is a C-ideal containing x, and hence $C\text{--}\!\downarrow \{x\} \subseteq I$. ■

Theo 4.4.3 In the theory of frames, any presentation by generators and relations presents a frame.

Proof First, rewrite the relations in equational form $e_1 = e_2$, where each expression e_i is a join of finite meets of generators.

Second, for each relation $e_1 = e_2$ introduce a new generator x and replace the relation by $e_1 = x$ and $e_2 = x$.

Third, present a semilattice S as follows: its generators are those of the frame (including ones previously introduced), and for each frame relation $\bigvee_i \bigwedge_j y_{ij} = x$ it has relations $\bigwedge_j y_{ij} \leq x$.

Fourth, generate a coverage C on S from the cover relations $\left\{\bigwedge_j y_{ij} : i\right\} -\!: x$ and take the frame C-idl(S).

It is not hard to see that frame homomorphisms from C-Idl(S) to a frame A correspond to semilattice homomorphisms from S to A that transforms covers in C to joins, and that these correspond to functions from the generators to A that validate all the equations. This shows that C-Idl(S) is the frame we wished to present. ■

Prop 4.4.4 Let A be a frame. Then A has a presentation.

Proof For each $a \in A$, let there be a generator a'. Take as relations

$$\begin{aligned} \text{true} &= \text{true}' \\ a' \wedge b' &= (a \wedge b)' \qquad (a, b \in A) \\ \bigvee_S (a' : a \in S) &= \left(\bigvee S\right)' \qquad (S \subseteq A) \qquad ■ \end{aligned}$$

Exercises

1. Show that lattices can be described as algebras for a theory with 4 operators for binary and nullary meets and joins, the semilattice laws of Proposition 4.1.2 for both meets and joins, and the absorptive laws: $x \wedge (x \vee y) = x$ and $x \vee (x \wedge y) = x$. Clearly, distributive lattices are described using an extra algebraic law, distributivity.

A lattice is a poset where $\emptyset$ and every pair of elements have a meet and a join (Proposition 3.4.2). By the definitions of the 4 operators, this holds. It remains to prove that it is a poset.

Proposition 3.3.5 states that:

$$x = \bigwedge \{x, y\} \Leftrightarrow x \leq y \Leftrightarrow y = \bigvee \{x, y\}$$

Using the absorptive laws, $x = x \wedge y \Leftrightarrow x \vee y = (x \wedge y) \vee y \Leftrightarrow x \vee y = y$.

Proving that it is a poset:

$$\begin{gathered} x = x \wedge x = x \vee x \\ x = x \wedge y,\ y = y \wedge z \Rightarrow x = x \wedge (y \wedge z) = (x \wedge y) \wedge z = x \wedge z \\ x = x \wedge y,\ y = x \wedge y \Rightarrow x = y \quad ■ \end{gathered}$$

2. The free semilattice on a set U is $\mathcal{P}_{\text{fin}}(U)$, the set of all finite subsets of U, under the superset ordering: $X \leq Y$ iff $X \supseteq Y$.

From Definition 4.2.3, a presentation with generators only, no relations, is free.

By Theorem 4.3.6, Semilattice$\langle U \mid \rangle$ presents a semilattice algebra S, with the 2 operators and 4 laws.

$$x \leq y \Leftrightarrow x \wedge y = x \Leftrightarrow x \cup y = x \Leftrightarrow x \supseteq y \quad ■$$

3. Show that Heyting algebras are described using the operators and laws for lattices, together with an extra operator $\to$ and laws $a \land (a \to b) = a \land b$ and $c = c \land (a \to ((a \land c) \lor b))$. From Exercise 7 of Chapter 3 Heyting algebras are distributive; prove this directly from the algebraic laws.

From Definition 3.10.1, A Heyting algebra is a lattice equipped with $\to$ satisfying

$$c \le (a \to b) \Leftrightarrow c \land a \le b$$

If $c = c \land (a \to ((a \land c) \lor b))$, it is clear that $(c \land a) \lor b = b \Rightarrow c \land (a \to b) = c$.

On the other hand, if $a \land (a \to b) = a \land b$, then

$$\begin{aligned} c &\le (a \to b) \\ c \land (a \to b) &= c \\ c \land a \land (a \to b) &= c \land a \\ c \land a \land b &= c \land a \\ c \land a &\le b \end{aligned}$$

So both directions work (the reverse for both laws doesn't work).

Using the identity we just proved,

$$\begin{aligned} b' &\in S \\ a \land b' &\le \bigvee \{a \land b : b \in S\} \\ b' &\le \left(a \to \bigvee \{a \land b : b \in S\}\right) \\ \bigvee S &\le \left(a \to \bigvee \{a \land b : b \in S\}\right) \\ a \land \bigvee S &\le \bigvee \{a \land b : b \in S\} \end{aligned}$$

The reverse inequality is trivial. ■

4. Consider the following 2 presentations of $\Omega 2^{*\omega}$:

$$A = \mathrm{Fr}\langle {'s_n = 0'}, {'s_n = 1'} : n \in \mathbb{N}, n \ge 1 \mid \\ {'s_n = 0'} \land {'s_n = 1'} = \text{false}, {'s_{n+1} = 0'} \lor {'s_{n+1} = 1'} \le {'s_n = 0'} \lor {'s_n = 1'} \rangle$$

$$B = \mathrm{Fr}\langle \text{starts } l : l \text{ a finite list of bits} \mid \\ \text{starts } l \le \text{starts } m \text{ if } l \supseteq m, \text{starts } l \land \text{starts } m = \text{false if } l \not\subseteq m \text{ and } m \not\subseteq l \rangle$$

Just using Definition 4.3.4, construct mutually inverse isomorphisms between A and B.

We must find homomorphisms $\theta_{AB} : A \to B$ and $\theta_{BA} : B \to A$ such that $\theta_{AB}; \theta_{BA} = \mathrm{Id}_A$ and $\theta_{BA}; \theta_{AB} = \mathrm{Id}_B$.

$$\theta_{AB} : \bigvee_i \left\{ {'s_{n_{i,1}}} = b_{i,1}{'} \land \ldots \land {'s_{n_{i,k_i}}} = b_{i,k_i}{'}; i \in I, n_{i,k} \text{ decreasing} \right\} \mapsto \\ \bigvee_i \left\{ \bigvee \{\text{all } 2^{n_{i,1} - k_i + 1} \text{ lists that fits above conditions}\} : i \in I \right\}, \text{false} \mapsto \text{false}$$

$$\theta_{BA} : \bigvee_i \left\{ \bigwedge \{\text{starts } l_{i,j} : j \in J_i\} : i \in I \right\} = \bigvee_i \{\text{starts } l_i : i \in I\} \mapsto \\ \bigvee_i \left\{ {'s_1 = l_i[1]'} \land \ldots \land {'s_{k_i} = l_i[k]'} : i \in I \right\}, \text{false} \mapsto \text{false} \quad \blacksquare$$

5. Let P be a poset, ordered by $\sqsubseteq$. Show that its Alexandrov topology can be presented as

$$\mathrm{Fr}\langle \uparrow x : x \in P \mid \text{true} = \bigvee\{\uparrow x : x \in P\}, \uparrow x \wedge \uparrow y = \bigvee\{\uparrow z : z \in P, z \sqsupseteq x, z \sqsupseteq y\}\rangle$$

Because of the second relation, all possible joins of meets of the frame is of the form

$$\bigvee\{\uparrow x : x \in X \subseteq P\}$$

Let f be a homomorphism from the presented frame to the Alexandrov topology.

$$f : \bigvee\{\uparrow x : x \in X\} \mapsto \bigcup\{\uparrow x : x \in X\}$$

Obviously, f is injective; now we need to show that the set of all $\bigcup\{\uparrow x : x \in X\}$ is precisely the upper closed sets of P, or that $\{\text{upper closed sets of } P\} \subseteq \{\bigcup\{\uparrow x : x \in X\} : X \subseteq P\}$.

Let U be an upper closed subset of P.

$$U = \bigcup\{\uparrow x : x \in U\}$$

Therefore f is bijective, and the presentation and Alexandrov topology are isomorphic. ■

chapter 5: topology: the definitions

5.1: Topological systems

We have seen frames, as systems of finite observations, but with no formalization of what they might be observations of.

Def 5.1.1 Let A be a frame; we call its elements *opens*. Let X be a set; we call its elements *points*. Finally, let $\vDash$ be a subset of $X \times A$, and say x *satisfies* a if it is in the relation.

X and A, equipped with $\vDash$, form a *topological system* iff $\vDash$ matches the logic of finite observations.
- If S is a finite subset of A, then $x \vDash \bigwedge S \Leftrightarrow x \vDash a$ for all $a \in S$
- If S is any subset of A, then $x \vDash \bigvee S \Leftrightarrow x \vDash a$ for some $a \in S$

If $D = (X, A)$ is a topological system, we write $D = (X, A) = (\mathrm{pt}D, \Omega D)$.

We might also think of the points and opens as being *subjects* and *predicates*. It is easy to deduce that
- $x \vDash$ true for all x
- $x \vDash$ false for no x
- if $x \vDash a \leq b$ then $x \vDash b$

Ex 5.1.2 $X =$ some set of streams, $A = \Omega 2^{*\omega}$.

$x \vDash$ starts l iff the first few bits of stream x are those specified in l

$x \vDash \bigvee_{l \in L}$ starts l iff $x \vDash$ starts l for some $l \in L$

Ex 5.1.3 With the same frame A, we can take X to be the set of programs in some language that generate streams. For example,

$$|\overline{\underline{\text{while true do \{output 0; output 1;\}}}}| \vDash \text{starts } 01010$$

It is worth trying to think of this not as an abtract, set theoretic relation, but as a concrete sequence of actions. It says somebody has
1. typed the program
2. pressed go
3. observed initial output 01010

Thus the opens are finite run-time observations. A question of interest in computer science is how we can relate the programmer's analytic logic and the customer's run-time observations. The points here contain more information than can be observed through the frame A. For instance, 2 programs can satisfy exactly the same opens, but one terminate and the other loop infinitely. They are indistinguishable *using the opens that we have described*, which don't include observations of termination.

Ex 5.1.4 $X =$ same set of programs, but this time take the bigger A of section 3.7 in which bits are read independently. This A makes distinctions that cannot be observed in practice.

Ex 5.1.5 $X =$ same set of programs, but $A = \Omega 2^{\omega}$

$$'s_n = 0' \vee 's_n = 1' = \text{true}$$

This is no good, because it assumes the programs all output infinitely many bits.

Ex 5.1.6 Let X be any set, and let A be any topology on X, with

$$x \vDash a \text{ iff } x \in a$$

5.2 Continuous maps

Suppose that (X, A) and (Y, B) are 2 topological systems, and that $f : X \to Y$ is a *computable* function. Consider the following scenario:

1. We take an X-point
2. We put it through the black box f
3. We get to affirm a B-open b

This ought to be a finite observation on X-points, because everything in it was done finitely. Let us write it $\phi(b)$, with

$$x \vDash \phi(b) \text{ iff } f(x) \vDash b$$

Therefore ϕ is a function from B to A. For all points x and subsets S of A,

$$x \vDash \phi\left(\bigvee S\right) \Leftrightarrow f(x) \vDash \bigvee S \Leftrightarrow \exists b \in S, f(x) \vDash b \Leftrightarrow \exists b \in S, x \vDash \phi(b) \Leftrightarrow x \vDash \bigvee\{\phi(b) : b \in S\}$$

Similarly, if S is finite,

$$x \vDash \phi\left(\bigwedge S\right) \Leftrightarrow \bigwedge\{\phi(b) : b \in S\}$$

Def 5.2.1 Let D and E be topological systems. A *continuous map* f from D to E is a pair $(\text{pt}f, \Omega f)$:

$$\begin{aligned} \text{pt}f : \text{pt}D \to \text{pt}E &\text{ is a function} \\ \Omega f : \Omega E \to \Omega D &\text{ is a frame homomorphism} \\ x \vDash \Omega f(b) &\Leftrightarrow \text{pt}f(x) \vDash b \end{aligned}$$

Although not all continuous maps are computable, they are a useful generalization. If we don't want to consider how a function is computed then it is useful just to know that it is continuous.

Ex 5.2.2 Let the points be bit streams, and the opens be those of $\Omega 2^{*\omega}$. We can define functions by structural recursion:

$$\text{complement}(b :: s) = (\text{if } b = 0 \text{ then } 1 \text{ else } 0) :: \text{complement}(s)$$

(:: is analogous to the cons operator of LISP.) This definition is really desifned for infinite bit streams, but let us abuse notation by applying the function complement to finite lists of bits.

$$s \models \Omega\text{complement}(\text{starts } l) \Leftrightarrow \text{complement}(s) \models \text{starts } l \Leftrightarrow s \models \text{starts complement}(l)$$

Therefore,

$$\Omega\text{complement}\left(\bigvee_{l \in L} \text{starts } l\right) = \bigvee_{l \in L} \text{starts complement}(l)$$

We can check that this is well-defined:

$$\bigvee_{l \in L} \text{starts } l = \bigvee_{m \in M} \text{starts } m \Rightarrow \bigvee_{l \in L} \text{starts complement}(l) = \bigvee_{m \in M} \text{starts complement}(m)$$

In computational systems, domain theoretic semantics of programming languages usually assure us that any computer program automatically gives a continuous map. **Def 5.2.3** Let D be a topological system. The *identity* map $\mathrm{Id}_D : D \to D$ is defined by

$$\mathrm{pt}\mathrm{Id}_D = \mathrm{Id}_{\mathrm{pt}D} : \mathrm{pt}D \to \mathrm{pt}D$$
$$\Omega\mathrm{Id}_D = \mathrm{Id}_{\Omega D} : \Omega D \to \Omega D$$

Let D, E, F be topological systems and let $f : D \to E$ and $g : E \to F$ be continuous maps. The *composite* $f; g : D \to F$ is defined by $((\mathrm{pt}f); (\mathrm{pt}g), (\Omega g); (\Omega f))$.

For algebras we saw that it is sometimes useful to consider 2 algebras as being essentially the same if they are isomorphic. The corresponding notion for topological systems is that of being *homeomorphic.*

Def 5.2.4 A continuous map $f : D \to E$ is a *homeomorphism* iff there is another map $g : E \to D$ such that $f; g = \mathrm{Id}_D$ and $g; f = \mathrm{Id}_E$. When there is a homeomorphism from D to E, we say D and E are *homeomorphic* and write $D \cong E$.

This means that the systems are structurally equivalaent:
- $\mathrm{pt}D$ and $\mathrm{pt}E$ are isomorphic sets
- ΩD and ΩE are isomorphic frames
- $\models$ is the same for both

Topological spaces

Prop 5.3.1 Let X and Y be topological spaces. Then continuous maps from X to Y are equivalent to functions $f : X \to Y$ such that if $V \in \Omega Y$ then $f^{-1}(V) \in \Omega X$.

Proof The condition $x \models \Omega f(V) \Leftrightarrow \mathrm{pt}f(x) \models V$ says precisely that $\Omega f(V) = (\mathrm{pt}f)^{-1}(V)$, which must thus be open.

This also tells us that for topological spaces a continuous map is completely determined by its points part.

A topological space is commonly thought of as set X "equipped with" its frame ΩX of open sets. This means that the same symbol X is used both for the unstructured set of points and for the topological space with its structure. When we think of a space as a topological system, we distinguish between the system X, and the set $\mathrm{pt}X$ if points. To regain the standard notation, omit "pt" whever it occurs and rewrite Ωf as f^{-1}.

Theo 5.3.2 *Continuity for real numbers*

Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Then f is the points part of a continuous map from $(\mathbb{R}, \Omega\mathbb{R})$ to itself iff "the graph of f has no jumps".

Proof We concentrate on jumps at 0, and assume for definiteness that $f(0) = 0$. There are blatant jumps like the step function, and more vacillatory jump such as $\sin\left(\frac{1}{x}\right)$. In both cases, however close x gets to 0, $f(x)$ refuses to stay away from 1.

Suppose Ωf exists. Then,

$$\Omega f\left(0 \pm \frac{1}{2}\right) = \bigvee_i (q_i \pm \delta_i)$$

$$f(0) = 0 \vDash \left(0 \pm \frac{1}{2}\right)$$

Therefore,

$$0 \vDash \Omega f\left(0 \pm \frac{1}{2}\right) \Rightarrow \exists i, 0 \vDash (q_i \pm \delta_i)$$

But some x in this region has $f(x)$ close to 1:

$$f(x) \vDash \left(1 \pm \frac{1}{2}\right) \text{ and } x \vDash (q_i \pm \delta_i \leq \Omega f(0 \pm 1.2))$$

Therefore $f(x) \vDash \left(0 \pm \frac{1}{2}\right) \wedge \left(1 \pm \frac{1}{2}\right) =$ false, so such a jump is impossible and if Ωf exists then there are no jumps.

For the converse, suppose that Ωf does not exist, so for some open U, $f^{-1}(U)$ is not open: it contains a boundary point x. $f(x) \in U$, so for some $\delta > 0$,

$$f(x) - \delta < y < f(x) + \delta \Rightarrow y \in U$$

x is a boundary point for $f^-1(U)$, so however closely we look at x, it has neighbouring points outside $f^{-1}(U)$. For such a point w, $f(w) \notin U$, so $f(w) \leq f(x) - \delta$ or $f(w) \geq f(x) + \delta$.

Thus, "However close w gets to x, f(w) refuses to stay within δ of $f(x)$.", so if Ωf doesn't exist then f has a jump.

Spatialization

We stick with topological spaces if we take the view that there is no reason to distinguish between 2 opens when they're satisfied by exactly the same points. This is reasonable for fixed points like the real numbers, but perhaps it is less so when we're not quite sure what the points are, like the streams. If we want to, we can always convert a topological system D into a topological space, by comparing opens entirely by the points that satisfy them.

Def 5.3.3 For each open a, its *extent* in D is $\{x \in \mathrm{pt}D : x \vDash a\}$

By the definition of $\vDash$, joins and finite meets of opens correspond to unions and finite intersections of their extents. Hence the extents of opens form a topology Ω pt$D$ on pt$D$. (Note that it depends on both pt$D$ as well as ΩD)

Theo 5.3.4

1. $(\mathrm{pt}D, \Omega\ \mathrm{pt}D)$ is a topological space, the *spatialization* Spat$D$ of D.
2. There is a natural continuous map $e : \mathrm{Spat}D \to D$ defined by

$$\mathrm{pt}e(x) = x \quad \Omega e(a) = \text{extent of } a \text{ in } D$$

3. Any other continuous map from a topological space into D factors uniquely via e: