Three-Photon Annihilation of an Electron-Positron Pair

A. ORE* AND J. L. POWELL*
Institute for Nuclear Studies, University of Chicago, Chicago, Illinois
(Received January 12, 1949)

Annihilation of an electron-positron pair accompanied by the emission of three photons is discussed for the case of small relative velocity of the two particles. The energy spectrum of the photons is derived and the cross section for the process is calculated. The ratio of this cross section to that of ordinary two-photon annihilation is found to be 1:370. When the result is applied to the ^3S ground state of the positronium atom, for which two-photon annihilation is forbidden, one finds a lifetime 1.4×10^{-7} sec.

INTRODUCTION

TWO-PHOTON annihilation[1,2] of an electron and a positron is forbidden for states of the electron-positron pair for which the wave function is symmetric in the space and spin coordinates of the two particles. However, annihilation resulting in the simultaneous creation of *three* photons can occur.

In particular, if the electron-positron pair forms a positronium atom in the ground (S) state, two-quantum or three-quantum annihilation is allowed according as the state is a singlet or triplet. Because of the multiple character of the three-quantum process, the triplet state will have a longer lifetime than the singlet.

The lifetime of the singlet state is [1,2]

$$^1\tau=2(\hbar c/e^2)^4(\hbar^2/me^2c)=1.25\times10^{-10}\text{ sec.} \quad (1)$$

This value is obtained from the cross section for two-quantum annihilation of an electron and a positron in free relative motion, as given by Dirac.[3] The Coulomb binding in the positronium atom has a negligible effect upon the decay probability (see below).

In this paper, we present a calculation, by the method of time-dependent perturbation theory, of the cross section for three-photon annihilation in the limit of small velocities, and of the energy spectrum of the photons. We also obtain the lifetime of the positronium atom in the triplet state.

CALCULATIONS

The annihilation of an electron-positron pair by emission of three photons takes place via two intermediate states, I and II. The probability per unit time for the process is given, in natural units ($\hbar=c=1$), by

$$P=2\pi|H_{FA}|^2\rho_F; \qquad H_{FA}=\sum_{I,II}\frac{H_{FII}H_{II\ I}H_{IA}}{(E_A-E_I)(E_A-E_{II})}, \quad (2)$$

where A refers to the initial state consisting of the electron-positron pair, F to the final state composed of three light quanta, and ρ_F is the density of final states per unit energy interval. The matrix elements H_{IA}, etc., are associated with the emission of the quanta in the various transitions and have the general form

$$H_{IA}=e(2\pi/kL^3)^{\frac{1}{2}}(u_I,\ \boldsymbol{\alpha}\cdot\mathbf{a}u_A),$$

where e is the electronic charge, $\mathbf{k}$ and $\mathbf{a}$ are the momentum and polarization vectors of the quantum, $\boldsymbol{\alpha}$ is the Dirac matrix, u_A and u_I are the Dirac amplitudes of the electron wave functions in the states I and A, and L^3 is the normalization volume. The total momentum is conserved in each transition.[4]

The various transitions which can lead from A to F by the emission of three quanta are illustrated in Fig. 1, in which the symbols $\pm$, k refer to the presence of a pair and a quantum, respectively.

The transitions through the intermediate states denoted by b are those which do not involve the formation of a virtual pair. To these transitions one can apply arguments similar to those used by Pirenne[5] in the two-quantum problem and show, firstly, that only an initially symmetric state will give a non-vanishing contribution to H_{FA}, and secondly, that the Coulomb binding may be neglected, its effect being to change the matrix elements by a fractional amount of the order 1/137. In the present problem, we shall assume that the binding can also be neglected for the remaining transitions. Therefore, in the calculation of the decay probability, we use simple plane wave functions. With this approximation, the lifetime of the bound state

* Post-Doctoral Fellow.
[1] J. Pirenne, Thesis, University of Paris (1944), Chapter III. (See also Archives des Sciences physiques et naturelles **28**, 273 (1946) and **29**, 121 (1947).)
[2] J. A. Wheeler, Ann. New York Acad. Sci. **48**, 219 (1946).
[3] P. A. M. Dirac, Proc. Camb. Phil. Soc. **26**, 361 (1930).
[4] W. Heitler, *The Quantum Theory of Radiation* (Oxford University Press, London, 1944), second edition, p. 96.
[5] See reference 1, pp. 124, 129.

is obtained by introducing a normalization which corresponds to the correct particle density at coincidence for the positronium atom.

For non-relativistic velocities, the cross section for three-photon annihilation obeys the $1/v$ law. That is, the rate at which annihilation takes place is determined by the density of electrons at the position of the positron. As a consequence, one can obtain the cross section for small velocities from the limiting case of zero velocities. The assumption of zero velocities leads to considerable mathematical simplification, and we shall here carry out the calculation for this case only, although the following discussion of matrix elements and of methods of calculation applies to the general case of arbitrary relative velocity of the electron and positron.

The transitions through the intermediate states denoted by a and c in Fig. 1 involve the formation of a virtual pair. In the states c, in which two pairs are present simultaneously, one must exclude those intermediate states which would violate the Pauli exclusion principle. However, if one calculates the total contribution of such states, one finds that it is zero, so that the requirements of the Pauli principle are automatically fulfilled. This may be verified by consideration of the energy denominators and signs of the matrix elements involved.

With regard to the signs of the matrix elements, it should be mentioned that in transitions involving the annihilation of the members of the virtual pair by those of the original, a change in sign is necessary, corresponding to an exchange of two negative-energy electrons in the final state. These effects of the exclusion principle are brought in automatically if the problem is treated in terms of the Jordan-Wigner operators,[6] but the essential facts are contained in the above discussion.

Among the transitions through the states a and c are some which are of a character which leads to divergence, namely, those in which a virtual pair is first created and subsequently annihilates itself. The reason for the divergence is that the condition of conservation of momentum only requires that the sum of the momenta of the members of the virtual pair shall be equal to that of the annihilation quantum, but places no restriction upon the momenta individually. Hence, in the sum over intermediate states one must include an integration over the momentum space for one of the members of the virtual pair. For each of these groups of transitions, this integral is divergent. However, if one considers the *total* contribution of all such transitions, the result is a detailed cancellation of these divergent terms.

Taking the above remarks into account, one finds

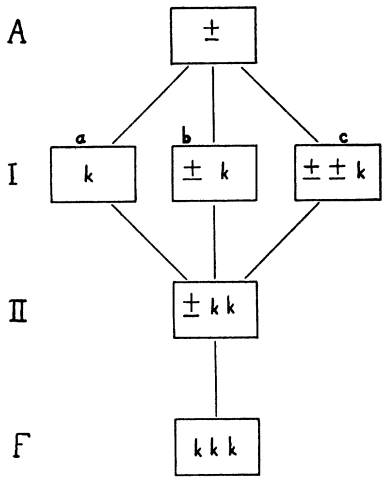

FIG. 1. Diagram illustrating the transitions which result in three-photon annihilation of an electron-positron pair.

that the final expression for H_{FA} is the same as the expression which one obtains if one considers the problem of the transition of an electron, by emission of three quanta, from a given state of positive energy to a given state of negative energy, all negative energy states being initially unoccupied, namely,

$$H_{FA}=e^3((2\pi)^3/k_1k_2k_3L^3)^{\frac{1}{2}}$$
$$\times\sum_k\sum_{I,II}\frac{(u_F,\ \boldsymbol{\alpha}\cdot\mathbf{a}_3u_{II})(u_{II},\ \boldsymbol{\alpha}\cdot\mathbf{a}_2u_I)(u_I,\ \boldsymbol{\alpha}\cdot\mathbf{a}_1u_A)}{(E_1-k_1-E')(k_3-E_2-E'')}$$

where E_1 and E_2 are the initial energies of the electron and positron, respectively, $\mathbf{k}_1$, $\mathbf{k}_2$, $\mathbf{k}_3$ are the momenta of the quanta and E' and E'' are the energies of the electron in the first and second intermediate states, respectively. $\sum_k$ denotes summation over all terms obtained from the above by permutation of $\mathbf{k}_1$, $\mathbf{k}_2$, and $\mathbf{k}_3$. Taking $E_1=E_2=m$, the conservation of momentum and energy is expressed by

$$\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3=0;\quad k_1+k_2+k_3=2m. \tag{3}$$

In order to pass from the case of free particles to that of the positronium atom, the normalization factor $L^{-\frac{3}{2}}$ must be replaced by $(\kappa^3/\pi)^{\frac{1}{2}}$, where $1/\kappa$ is equal to two times the Bohr radius. The sum over intermediate states, which must include both positive and negative energies, and both spin directions of the electron, may be carried out by the method of Casimir.[7] The result of these transformations is

$$H_{FA}=-(2\pi e^3/m^2L^3)(\kappa^3/k_1k_2k_3)^{\frac{1}{2}}(\mathbf{t}_1+\mathbf{t}_2+\mathbf{t}_3)\cdot\mathbf{u}$$

where $\mathbf{t}_1$ is the following function of the polarization

[6] See G. Wentzel, *Quantentheorie d. Wellenfelder* (Franz Deuticke, Vienna, 1943), p. 183.

[7] See reference 4, p. 149.

vectors:

$$\mathbf{t}_1=\mathbf{a}_1(\mathbf{a}_2\cdot\mathbf{a}_3)-\mathbf{a}_2(\mathbf{a}_3\cdot\mathbf{a}_1)-\mathbf{a}_3(\mathbf{a}_1\cdot\mathbf{a}_2)$$
$$+\mathbf{a}_1(\mathbf{a}_2'\cdot\mathbf{a}_3')-\mathbf{a}_2'(\mathbf{a}_3'\cdot\mathbf{a}_1)-\mathbf{a}_3'(\mathbf{a}_1\cdot\mathbf{a}_2')$$
$$\mathbf{a}_1'=\mathbf{a}_1\times(\mathbf{k}_1/|\mathbf{k}_1|)$$

and $\mathbf{t}_2$ and $\mathbf{t}_3$ are obtained by cyclic permutation of the subscripts. $\mathbf{u}$ is a complex unit vector which characterizes the initial triplet state of the positronium atom.[8] Three independent triplet states are represented by three mutually orthogonal unit vectors $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$. For an initial singlet state H_{FA} vanishes.

Finally, taking the average of $|H_{FA}|^2$ over the three independent triplet states, and summing the result over the directions of polarization of the three quanta, one finds

$$\sum_{\text{polarizations}}\langle|H_{FA}|^2\rangle_{Av}=(32\pi^2e^6/3L^3)(\kappa^3/k_1k_2k_3)$$
$$\times\{[1-\cos(\mathbf{k}_2\mathbf{k}_3)]^2+[1-\cos(\mathbf{k}_3\mathbf{k}_1)]^2$$
$$+[1-\cos(\mathbf{k}_1\mathbf{k}_2)]^2\} \quad (4)$$

where $(\mathbf{k}_2\mathbf{k}_3)$, etc. are the angles between the vectors indicated.

The density of final states is calculated in the usual way, by making use of the momentum conservation law. One obtains

$$\rho_F=(2\pi)^{-5}L^6k_1k_2k_3dk_2dk_1d\Omega_1 \quad (5)$$

representing the number of final states per unit energy in which k_1 is in the interval (k_1, k_1+dk_1) and its direction within the element of solid angle $d\Omega_1$, the magnitude of k_2 being at the same time in the interval (k_2, k_2+dk_2).

Combining the Eqs. (2), (4) and (5) and making use of (3), one finds, after integration over k_1 and k_2

$$1/^3\tau=(16e^6\kappa^3/9\pi m^3)\int_0^m F(k_1)dk_1$$
$$=(16/9\pi)(\pi^2-9)(e^6\kappa^3/m^2)$$

where

$$F(k_1)=\int_{m-k_1}^m\left\{\frac{m^2(m-k_1)^2}{k_2^2k_3^2}+\frac{m^2(m-k_2)^2}{k_3^2k_1^2}\right.$$
$$\left.+\frac{m^2(m-k_3)^2}{k_1^2k_2^2}\right\}\frac{dk_2}{m}=2\left\{\frac{k_1(m-k_1)}{(2m-k_1)^2}\right.$$
$$-\frac{2m(m-k_1)^2}{(2m-k_1)^3}\ln\frac{m-k_1}{m}+\frac{2m-k_1}{k_1}$$
$$\left.+\frac{2m(m-k_1)}{k_1^2}\ln\frac{m-k_1}{m}\right\}.$$

A factor $\frac{1}{6}$ has also been included in the above formula, since the 6 permutations of the three quanta all correspond to the same final state.

Restoring $\hbar$ and c, and using the relation (1), our result may be conveniently written in the form

$$^1\tau/^3\tau=(4/9\pi)(\pi^2-9)(e^2/\hbar c)\approx(1/8)(e^2/\hbar c)$$
$$^3\tau=1.4\times10^{-7}\text{ sec.}$$

This may also be expressed in terms of the cross section for annihilation of *free* particles with small relative velocity. We have

$$\sigma_{3k}/\sigma_{2k}=3(^1\tau/^3\tau)\approx1/370$$

where σ_{2k} and σ_{3k} are respectively the cross sections for two- and three-photon annihilation of a free electron and positron with arbitrary spins. The statistical factor 3 enters because only singlet states contribute to σ_{2k}, and only triplets to σ_{3k}.

The function $F(k_1)$, which is proportional to the total probability that one of the photons has energy in the interval (k_1, k_1+dk_1), represents the energy spectrum of the quanta. A graph of this function is shown in Fig. 2 (solid curve). The dashed curve represents the function

$$\int_{m-k_1}^m\rho_Fdk_2\sim k_1^2(6m^2-6mk_1+k_1^2),$$

i.e., the spectrum which one would obtain from a statistical estimate based upon a consideration of the available volume in momentum space.

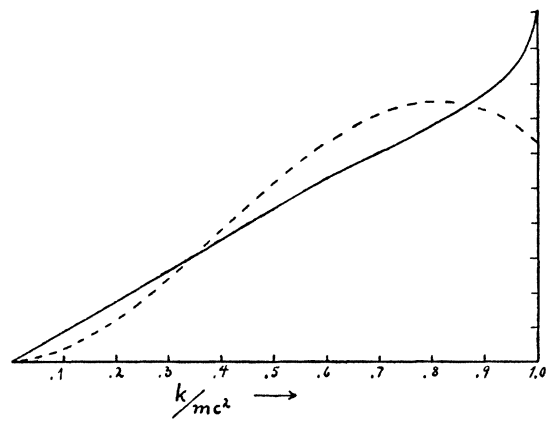

FIG. 2. Energy spectrum of photons resulting from three-photon annihilation of an electron and a positron. The abscissa is the photon energy in units mc^2 and the ordinate is proportional to the number of photons per unit energy interval. (Dashed curve: statistical estimate.)

DISCUSSION

In the ground state of the positronium atom, the fine-structure separation between the triplet and singlet states is approximately 8×10^{-4} ev.[1] The

[8] See reference 1, p. 108.

former, relatively long-lived state lies above the short-lived singlet state. However, radiative transitions from triplet to singlet state are forbidden. Provided that transitions resulting from thermal collisions between the positronium atom and, say, gas molecules have a small probability of occurrence, the long-lived triplet state should manifest itself.

For S states in general, the lifetime with respect to annihilation increases as the cube of the principle quantum number. For the other excited states (P, D, etc.) this lifetime will be long, since the absolute value of the wave function goes to zero as the electron and positron approach coincidence.

The existence of excited states of the positronium atom can have a decisive effect upon the time required for such an atom to undergo annihilation, even though annihilation in an excited state is itself unlikely. The lifetimes of the excited states with respect to spontaneous radiative transitions are twice the natural lifetimes of the corresponding excited states of the hydrogen atom. They are, therefore, of the order 10^{-7} to 10^{-9} sec., except in the case of the $2S$ state, which is metastable in this respect.[9] Hence, if a free positronium atom has been formed in an excited state, the observed lifetime may be essentially the time required for spontaneous de-excitation.

However, if formed in a gas of normal temperature and pressure, the positronium atom will undergo thermal collisions at a rate of the order 10^{12} sec.$^{-1}$. The time for de-excitation may in this case be negligible as compared to the lifetime of the lowest states against annihilation.

Excitation from the ground state, due to thermal collisions, is not to be expected, since the energy of even the first excited state lies about 5 ev. above that of the ground state.

The study of the three-photon annihilation process is of importance quite apart from the relation of this process to the lifetime of positronium. In particular, a comparison of the theoretical predictions with experimental results is of interest since it seems to offer a means of checking the theoretical foundations of the calculation.

The simplest way of testing the results seems to be by observing the relative frequency of the two- and three-photon annihilation radiation from a target irradiated by slow positrons.

Also, three-photon annihilation might be of importance in a detailed study of the mechanism of cosmic ray showers. The present calculation is not applicable in this case, since it has been performed for small relative velocities. However, one will expect that the multiple process becomes relatively more important at high energy.

The energy spectrum (Fig. 2) shows that photons of any energy not exceeding the rest energy of the electron have an appreciable probability of occurrence, except photons of very low energy. The Coulomb force, whether in the positronium atom or in a target bombarded with positrons, will be expected to affect the annihilation process appreciably only when a photon of small momentum is involved. The energy spectrum indicates that such processes are rare. Therefore, the neglect of the Coulomb force should not be a serious limitation to the applicability of the results.

The spectrum is seen to deviate considerably from that estimated statistically. It may be mentioned that Tiomno and Wheeler[10] have obtained a somewhat similar result in the case of electrons resulting from the decay of a μ-meson, under certain assumptions as to the mass of the neutral meson and the ineractions of the particles involved.

The authors are indebted to Professor E. Teller and Dr. C. N. Yang for valuable discussions.

Note added in proof.—The statement that certain transitions are forbidden for the positronium atom is true only in a limited sense. Conversion between triplet and singlet ground states can take place by magnetic dipole radiation. However, the lifetime of the triplet state with respect to this kind of decay is several months, which at normal temperatures is an order of magnitude longer than the corresponding thermally induced transition. Also, $2S$–$1S$ transitions can take place via P states by the emission of two photons. As in the case of the hydrogen atom,** the lifetime is again long (1–0.1 sec.).

Consequently, these effects are not important. However, this may not be true of the effect of direct annihilation of the positron with an electron of an atom or molecule with which the positronium atom collides.

[9] See reference 4, p. 118.

[10] J. Tiomno and J. A. Wheeler, Rev. Mod. Phys. **21**, 144 (1949).

** G. Breit and E. Teller, Astrophys. J. **91**, 215 (1940).