

# Fast Algorithm of General Quadratic Estimator Normalization

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## 1 Preliminaries

Here I will generalize the fast modal algorithm of [1, 2] to the case of rotation and others. In the followings, for multipoles of the CMB anisotropies, we use small letters (e.g.,  $\ell$ ), while large letters are used for multipoles of the distortion fields (lensing, rotation, etc).

## 1.1 CMB

The CMB temperature fluctuations are denoted as  $\Theta$  and the CMB linear polarization is expressed by the Stokes parameters,  $Q$  and  $U$ . The harmonic coefficients of the temperature anisotropies (and, in general, any scalar quantities  $x$ ) are given by

$$x_{LM} = \int d^2\hat{n} Y_{LM}^*(\hat{n}) x(\hat{n}). \quad (1)$$

where  $Y_{LM}$  is the spin-0 spherical harmonics. On the other hand, the values of  $Q$  and  $U$  are changed by the rotation of the sphere. These Stokes parameters are therefore usually transformed into the rotational invariant quantities, the  $E$  and  $B$  modes, as

$$[E \pm iB]_{\ell m} = \int d^2\hat{n} (Y_{\ell m}^{\pm 2})^*(\hat{n}) [Q \pm iU](\hat{n}). \quad (2)$$

Here,  $Y_{\ell m}^{\pm 2}$  is the spin-2 spherical harmonics. For short notation, we also use

$$\begin{aligned} \Xi^\pm &= E \pm iB, \\ P^\pm &= Q \pm iU \end{aligned} \quad (3)$$

## 1.2 Lensing

The lensing effect on CMB anisotropies is described as remapping of the unlensed CMB anisotropies by the deflection angle,

$$X(\hat{n}) = X(\hat{n} + \mathbf{d}), \quad (4)$$

where  $X$  is  $\Theta$  or  $P^\pm$ .

The deflection angle of the CMB lensing is decomposed into the lensing potential,  $\phi$ , and curl mode,  $\varpi$ , as

$$\mathbf{d} = \nabla\phi + \Delta\varpi, \quad (5)$$

where the operator  $\Delta = \star\nabla$  denotes the derivatives with 90° rotation counterclockwise on the plane perpendicular to the line-of-sight direction and then operation. The harmonic coefficients of  $\phi$  and  $\varpi$  are given by Eq. (1). The remapping of the CMB anisotropies is then given by

$$X(\hat{n}) = X(\hat{n}) + [\nabla\phi + \Delta\varpi] \cdot \nabla X + \mathcal{O}(\phi^2, \varpi^2). \quad (6)$$

## 1.3 Rotation

If the rotation angle is small, the modulation of polarization after rotation by an angle  $\alpha$  is given by (e.g. [3])

$$\delta P^\pm = \pm 2\alpha P^\pm. \quad (7)$$

The harmonic coefficients of  $\alpha$  is given by Eq. (1).

## 1.4 Inhomogeneous Reionization

The inhomogeneities of the reionization could vary the optical depth  $\tau$  across the CMB sky. If the spatial variation of  $\tau$  is very small, this leads to the modulation in CMB temperature and polarization as (e.g. [1, 4])

$$\Theta \rightarrow \Theta + \tau\Theta, P^\pm \rightarrow P^\pm + \tau P^\pm. \quad (8)$$

The harmonic coefficients of  $\tau$  is given by Eq. (1).

## 1.5 Spherical Harmonics and Wigner-3j

The spherical harmonics is related to the Wigner-3j symbols as [5]

$$\int d^2\hat{n} Y_{\ell_1 m_1}^{s_1} Y_{\ell_2 m_2}^{s_2} Y_{\ell_3 m_3}^{s_3} = \sqrt{\frac{(2\ell_1+1)(2\ell_2+1)(2\ell_3+1)}{4\pi}} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ -s_1 & -s_2 & -s_3 \end{pmatrix} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix}. \quad (9)$$

## 1.6 Derivatives of Spherical Harmonics

In general, denoting  $a_\ell^s = \sqrt{(\ell-s)(\ell+s)/2}$ , the derivative of the spherical harmonics is given by

$$\nabla Y_{\ell m}^s = a_\ell^s Y_{\ell m}^{s+1} \mathbf{e}^* - a_\ell^{-s} Y_{\ell m}^{s-1} \mathbf{e}. \quad (10)$$

Here, we introduce the polarization vector  $\mathbf{e}$  which are defined

$$\mathbf{e} = \frac{\mathbf{e}_1 + i\mathbf{e}_2}{\sqrt{2}} \quad (11)$$

with  $\mathbf{e}_i$  denoting the basis vectors orthogonal to the radial vector. The polarization vector satisfies  $\mathbf{e} \cdot \mathbf{e} = 0$ ,  $\mathbf{e} \cdot \mathbf{e}^* = 1$ ,  $\star \mathbf{e} = -i\mathbf{e}$ . In particular, for  $s = 0$ ,

$$\nabla Y_{\ell m} = a_\ell^0 (Y_{\ell m}^1 \mathbf{e}^* - Y_{\ell m}^{-1} \mathbf{e}), \quad (12)$$

and, for  $s = \pm 2$ , denoting  $a_\ell^\pm = a_\ell^{\pm 2}$ ,

$$\begin{aligned} \nabla Y_{\ell m}^2 &= a_\ell^+ Y_{\ell m}^3 \mathbf{e}^* - a_\ell^- Y_{\ell m}^1 \mathbf{e}, \\ \nabla Y_{\ell m}^{-2} &= a_\ell^- Y_{\ell m}^{-1} \mathbf{e}^* - a_\ell^+ Y_{\ell m}^{-3} \mathbf{e}. \end{aligned} \quad (13)$$

## 1.7 Map derivatives

Derivative of scalar quantities such as the CMB temperature fluctuations and lensing potential is

$$\nabla x = \sum_{LM} x_{LM} \nabla Y_{LM} = \sum_{LM} x_{LM} a_L^0 (Y_{LM}^1 \mathbf{e}^* - Y_{LM}^{-1} \mathbf{e}) = x^+ \mathbf{e}^* - x^- \mathbf{e}. \quad (14)$$

where we define

$$x^\pm \equiv \sum_{LM} x_{LM} a_L^0 Y_{LM}^{\pm 1}, \quad (15)$$

and  $(x^+)^* = -x^-$ . The rotation of a pseudo-scalar quantity is given by

$$\Delta \varpi = \sum_{LM} \varpi_{LM} \Delta Y_{LM} = \sum_{LM} \varpi_{LM} a_L^0 i (Y_{LM}^1 \mathbf{e}^* + Y_{LM}^{-1} \mathbf{e}) = i(\varpi^+ \mathbf{e}^* + \varpi^- \mathbf{e}), \quad (16)$$

and  $(\varpi^+)^* = -\varpi^-$ . Spin-2 fields such as the CMB linear polarization is given by

$$\nabla P^+ = \sum_{\ell m} \Xi_{\ell m}^+ \nabla Y_{\ell m}^2 = \sum_{\ell m} \Xi_{\ell m}^+ (a_\ell^+ Y_{\ell m}^3 \mathbf{e}^* - a_\ell^- Y_{\ell m}^1 \mathbf{e}) = \Xi^{++} \mathbf{e}^* - \Xi^{+-} \mathbf{e}, \quad (17)$$

$$\nabla P^- = (\nabla P^+)^* = \sum_{\ell m} \Xi_{\ell m}^- \nabla Y_{\ell m}^{-2} = \sum_{\ell m} \Xi_{\ell m}^- (a_\ell^- Y_{\ell m}^{-1} \mathbf{e}^* - a_\ell^+ Y_{\ell m}^{-3} \mathbf{e}) = \Xi^{-+} \mathbf{e}^* - \Xi^{--} \mathbf{e}. \quad (18)$$

Note that  $(\Xi^{++})^* = -\Xi^{--}$  and  $(\Xi^{+-})^* = -\Xi^{-+}$ .

## 2 Distortion of CMB anisotropies

In the following, we first define useful quantities to compute the distortion effect. The parity symmetry indicator is given by

$$p_{\ell_1 \ell_2 \ell_3}^{\pm} \equiv \frac{1 \pm (-1)^{\ell_1 + \ell_2 + \ell_3}}{2}, \quad (19)$$

$$(20)$$

An even (odd) parity quantity contains  $p^+$  ( $p^-$ ). A multipole factor is defined as

$$\gamma_{\ell_1 \ell_2 \ell_3} \equiv \sqrt{\frac{(2\ell_1 + 1)(2\ell_2 + 1)(2\ell_3 + 1)}{4\pi}}. \quad (21)$$

The convolutuon operator in full sky is defined as

$$\widetilde{\sum}_{LM\ell'm'}^{(\ell m)} \equiv \sum_{LM\ell'm'} (-1)^m \begin{pmatrix} \ell & L & \ell' \\ -m & M & m' \end{pmatrix}. \quad (22)$$

### 2.1 Lensing distortion

The lensing contributions in the position space become

$$\begin{aligned} \delta^\phi \Theta &= \nabla \phi \cdot \nabla \Theta = -\phi^- \Theta^+ - \phi^+ \Theta^-, \\ \delta^\varpi \Theta &= \Delta \varpi \cdot \nabla \Theta = i(\varpi^- \Theta^+ - \varpi^+ \Theta^-), \\ \delta^\phi P^\pm &= \nabla \phi \cdot \nabla P^\pm = -\phi^- \Xi^{\pm+} - \phi^+ \Xi^{\pm-}, \\ \delta^\varpi P^\pm &= \Delta \varpi \cdot \nabla P^\pm = i(\varpi^- \Xi^{\pm+} - \varpi^+ \Xi^{\pm-}). \end{aligned} \quad (23)$$

#### 2.1.1 Lens distortion in harmonic space: Temperature

The harmonics transform of the lensing contributions is

$$\begin{aligned} \delta^\phi \Theta_{\ell m} &= - \int d^2 \hat{n} Y_{\ell m}^* [\phi^- \Theta^+ + \phi^+ \Theta^-] \\ &= - \sum_{LM\ell'm'} \phi_{LM} \Theta_{\ell'm'} a_L^0 a_{\ell'}^0 \int d^2 \hat{n} (-1)^m Y_{\ell, -m} [Y_{LM}^{-1} Y_{\ell'm'}^1 + Y_{LM}^1 Y_{\ell'm'}^{-1}] \\ &= - \sum_{LM\ell'm'} \phi_{LM} \Theta_{\ell'm'} 2a_L^0 a_{\ell'}^0 p_{\ell L \ell'}^+ \gamma_{\ell L \ell'} (-1)^m \begin{pmatrix} \ell & L & \ell' \\ -m & M & m' \end{pmatrix} \begin{pmatrix} \ell & L & \ell' \\ 0 & 1 & -1 \end{pmatrix} \\ &= - \widetilde{\sum}_{LM\ell'm'}^{(\ell m)} \phi_{LM} \Theta_{\ell'm'} 2a_L^0 a_{\ell'}^0 p_{\ell L \ell'}^+ \gamma_{\ell L \ell'} \begin{pmatrix} \ell & L & \ell' \\ 0 & 1 & -1 \end{pmatrix} \\ &= \widetilde{\sum}_{LM\ell'm'}^{(\ell m)} \phi_{LM} \Theta_{\ell'm'} W_{\ell L \ell'}^{\phi, 0}. \end{aligned} \quad (24)$$

Here we introduce coefficients  $c_\phi = 1$  and  $c_\varpi = -i$ , and denote

$$W_{\ell_1 \ell_2 \ell_3}^{\phi, 0} = -2c_\phi a_{\ell_2}^0 a_{\ell_3}^0 p_{\ell_1 \ell_2 \ell_3}^+ \gamma_{\ell_1 \ell_2 \ell_3} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ 0 & 1 & -1 \end{pmatrix}. \quad (25)$$

Note that  $(W_{\ell_1 \ell_2 \ell_3}^{\phi, 0})^* = W_{\ell_1 \ell_2 \ell_3}^{\phi, 0}$ .

On the other hand, for curl mode,

$$\begin{aligned}
\delta^\varpi \Theta_{\ell m} &= i \int d^2 \hat{\mathbf{n}} Y_{\ell m}^* [\varpi^- \Theta^+ - \varpi^+ \Theta^-] \\
&= \sum_{LM\ell'm'} \varpi_{LM} \Theta_{\ell'm'} 2i a_L^0 a_{\ell'}^0 p_{\ell L \ell'}^- \gamma_{\ell L \ell'} (-1)^m \begin{pmatrix} \ell & L & \ell' \\ -m & M & m' \end{pmatrix} \begin{pmatrix} \ell & L & \ell' \\ 0 & 1 & -1 \end{pmatrix} \\
&= \widetilde{\sum_{LM\ell'm'}^{(\ell m)}} \varpi_{LM} \Theta_{\ell'm'} W_{\ell L \ell'}^{\varpi, 0},
\end{aligned} \tag{26}$$

with

$$W_{\ell_1 \ell_2 \ell_3}^{\varpi, 0} = -2c_\varpi a_{\ell_2}^0 a_{\ell_3}^0 p_{\ell_1 \ell_2 \ell_3}^- \gamma_{\ell_1 \ell_2 \ell_3} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ 0 & 1 & -1 \end{pmatrix}. \tag{27}$$

Note that the above quantity is consistent with Ref. [6] and also  $(W_{\ell_1 \ell_2 \ell_3}^{\varpi, 0})^* = (-1)^{\ell_1 + \ell_2 + \ell_3} W_{\ell_1 \ell_2 \ell_3}^{\varpi, 0}$  which is consistent with (27).

### 2.1.2 Lens distortion in harmonic space: Polarization

The lensed anisotropies for polarizations are given by

$$\begin{aligned}
\delta^\phi \Xi_{\ell m}^\pm &= - \int d^2 \hat{\mathbf{n}} (Y_{\ell m}^{\pm 2})^* [\phi^- \Xi^{\pm+} + \phi^+ \Xi^{\pm-}] \\
&= - \sum_{LM\ell'm'} \phi_{LM} \Xi_{\ell'm'}^\pm a_L^0 \int d^2 \hat{\mathbf{n}} (Y_{\ell m}^{\pm 2})^* [a_{\ell'}^+ Y_{LM}^{\mp 1} Y_{\ell'm'}^{\pm 3} + a_{\ell'}^- Y_{LM}^{\pm 1} Y_{\ell'm'}^{\pm 1}] \\
&= - \sum_{LM\ell'm'} (-1)^m \begin{pmatrix} \ell & L & \ell' \\ -m & M & m' \end{pmatrix} \phi_{LM} \Xi_{\ell'm'}^\pm \gamma_{\ell L \ell'} a_L^0 \left[ a_{\ell'}^+ \begin{pmatrix} \ell & L & \ell' \\ \mp 2 & \mp 1 & \pm 3 \end{pmatrix} + a_{\ell'}^- \begin{pmatrix} \ell & L & \ell' \\ \mp 2 & \pm 1 & \pm 1 \end{pmatrix} \right] \\
&= \widetilde{\sum_{LM\ell'm'}^{(\ell m)}} \phi_{LM} \Xi_{\ell'm'}^\pm W_{\ell L \ell'}^{\phi, \pm 2},
\end{aligned} \tag{28}$$

with

$$S_{\ell_1 \ell_2 \ell_3}^{\phi, 2} = (-1)^{\ell_1 + \ell_2 + \ell_3} S_{\ell_1 \ell_2 \ell_3}^{\phi, -2} = -c_\phi \gamma_{\ell_1 \ell_2 \ell_3} a_{\ell_2}^0 \left[ a_{\ell_3}^+ \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ -2 & -1 & 3 \end{pmatrix} + a_{\ell_3}^- \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ -2 & 1 & 1 \end{pmatrix} \right]. \tag{29}$$

For curl mode,

$$\begin{aligned}
\delta^\varpi \Xi_{\ell m}^\pm &= i \int d^2 \hat{\mathbf{n}} (Y_{\ell m}^{\pm 2})^* [\varpi^- \Xi^{\pm+} - \varpi^+ \Xi^{\pm-}] \\
&= \pm i \sum_{LM\ell'm'} (-1)^m \begin{pmatrix} \ell & L & \ell' \\ -m & M & m' \end{pmatrix} \varpi_{LM} \Xi_{\ell'm'}^\pm a_L^0 \gamma_{\ell L \ell'} \left[ a_{\ell'}^+ \begin{pmatrix} \ell & L & \ell' \\ \mp 2 & \mp 1 & \pm 3 \end{pmatrix} - a_{\ell'}^- \begin{pmatrix} \ell & L & \ell' \\ \mp 2 & \pm 1 & \pm 1 \end{pmatrix} \right] \\
&= \widetilde{\sum_{LM\ell'm'}^{(\ell m)}} \varpi_{LM} \Xi_{\ell'm'}^\pm W_{\ell L \ell'}^{\varpi, \pm 2},
\end{aligned} \tag{30}$$

with

$$W_{\ell_1 \ell_2 \ell_3}^{\varpi, 2} = -(-1)^{\ell_1 + \ell_2 + \ell_3} W_{\ell_1 \ell_2 \ell_3}^{\varpi, -2} = -c_\varpi \gamma_{\ell_1 \ell_2 \ell_3} a_{\ell_2}^0 \left[ a_{\ell_3}^+ \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ -2 & -1 & 3 \end{pmatrix} - a_{\ell_3}^- \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ -2 & 1 & 1 \end{pmatrix} \right]. \tag{31}$$

Now we consider the lensed E/B modes separately. In general, for  $X^\pm = A \pm iB = (a \pm ib)c^{(\pm)}$ ,

$$A = \frac{X^+ + X^-}{2} = \left( a \frac{c^{(+)} + c^{(-)}}{2} + ib \frac{c^{(+)} - c^{(-)}}{2} \right), \tag{32}$$

$$B = \frac{X^+ - X^-}{2i} = \left( -ai \frac{c^{(+)} - c^{(-)}}{2} + b \frac{c^{(+)} + c^{(-)}}{2} \right) \tag{33}$$

The lensing correction terms for E/B modes are then given by

$$\delta^x E_{\ell m} = \widetilde{\sum_{LM\ell'm'}}^{(\ell m)} \phi_{LM} [W_{\ell L \ell'}^{x,+} E_{\ell' m'} + W_{\ell L \ell'}^{x,-} B_{\ell' m'}] , \quad (34)$$

$$\delta^x B_{\ell m} = \widetilde{\sum_{LM\ell'm'}}^{(\ell m)} \phi_{LM} [-W_{\ell L \ell'}^{x,-} E_{\ell' m'} + W_{\ell L \ell'}^{x,+} B_{\ell' m'}] . \quad (35)$$

Here we define

$$\begin{aligned} W_{\ell_1 \ell_2 \ell_3}^{x,+} &\equiv \frac{W_{\ell_1 \ell_2 \ell_3}^{x,2} + W_{\ell_1 \ell_2 \ell_3}^{x,-2}}{2} = \frac{1 + c_x^2 (-1)^{\ell_1 + \ell_2 + \ell_3}}{2} W_{\ell_1 \ell_2 \ell_3}^{x,2} \\ &= -\wp_{\ell_1 \ell_2 \ell_3}^{x,+} \gamma_{\ell_1 \ell_2 \ell_3} a_{\ell_2}^0 \left[ a_{\ell_3}^+ \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ -2 & -1 & 3 \end{pmatrix} + c_x^2 a_{\ell_3}^- \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ -2 & 1 & 1 \end{pmatrix} \right] , \\ W_{\ell_1 \ell_2 \ell_3}^{x,-} &\equiv i \frac{W_{\ell_1 \ell_2 \ell_3}^{x,2} - W_{\ell_1 \ell_2 \ell_3}^{x,-2}}{2} = i \frac{1 - c_x^2 (-1)^{\ell_1 + \ell_2 + \ell_3}}{2} W_{\ell_1 \ell_2 \ell_3}^{x,2} \\ &= -\wp_{\ell_1 \ell_2 \ell_3}^{x,-} \gamma_{\ell_1 \ell_2 \ell_3} a_{\ell_2}^0 \left[ a_{\ell_3}^+ \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ -2 & -1 & 3 \end{pmatrix} + c_x^2 a_{\ell_3}^- \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ -2 & 1 & 1 \end{pmatrix} \right] , \end{aligned} \quad (36)$$

where we reintroduce a parity indicator as

$$\begin{aligned} \wp_{\ell_1 \ell_2 \ell_3}^{x,+} &= c_x \frac{1 + c_x^2 (-1)^{\ell_1 + \ell_2 + \ell_3}}{2} , \\ \wp_{\ell_1 \ell_2 \ell_3}^{x,-} &= i c_x \frac{1 - c_x^2 (-1)^{\ell_1 + \ell_2 + \ell_3}}{2} . \end{aligned} \quad (37)$$

## 2.2 Rotation distortion

The E and B modes after the rotation are given by

$$\begin{aligned} \delta \Xi^\pm &= \pm 2 \int d^2 \hat{n} (Y_{\ell m}^{\pm 2})^* \alpha P^\pm \\ &= \pm 2 \sum_{LM\ell'm'} \alpha_{LM} \Xi_{\ell'm'}^\pm \int d^2 \hat{n} (Y_{\ell m}^{\pm 2})^* Y_{LM} Y_{\ell'm'}^{\pm 2} \\ &= \pm 2 \sum_{LM\ell'm'} (-1)^m \begin{pmatrix} \ell & L & \ell' \\ -m & M & m' \end{pmatrix} \alpha_{LM} \Xi_{\ell'm'}^\pm \gamma_{\ell L \ell'} \begin{pmatrix} \ell & L & \ell' \\ \mp 2 & 0 & \pm 2 \end{pmatrix} \\ &= \widetilde{\sum_{LM\ell'm'}}^{(\ell m)} \alpha_{LM} \Xi_{\ell'm'}^\pm W_{\ell L \ell'}^{\alpha, \pm 2} , \end{aligned} \quad (38)$$

with

$$W_{\ell_1 \ell_2 \ell_3}^{\alpha, \pm 2} = \pm 2 \gamma_{\ell_1 \ell_2 \ell_3} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ \mp 2 & 0 & \pm 2 \end{pmatrix} . \quad (39)$$

The distorted E and B modes are then described as

$$\delta E_{\ell m} = \widetilde{\sum_{LM\ell'm'}}^{(\ell m)} \alpha_{LM} (E_{\ell'm'} W_{\ell L \ell'}^{\alpha,+} + B_{\ell'm'} W_{\ell L \ell'}^{\alpha,-}) , \quad (40)$$

$$\delta B_{\ell m} = \widetilde{\sum_{LM\ell'm'}}^{(\ell m)} \alpha_{LM} (-E_{\ell'm'} W_{\ell L \ell'}^{\alpha,-} + B_{\ell'm'} W_{\ell L \ell'}^{\alpha,+}) \quad (41)$$

where we define  $c_\alpha = 1$  and

$$W_{\ell_1 \ell_2 \ell_3}^{\alpha, \pm} = 2 \wp_{\ell_1 \ell_2 \ell_3}^{\alpha, \mp} \gamma_{\ell_1 \ell_2 \ell_3} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ -2 & 0 & 2 \end{pmatrix} \quad (42)$$

## 2.3 Tau distortion

The harmonics transform of  $\tau(\hat{\mathbf{n}})\Theta(\hat{\mathbf{n}})$  is

$$\begin{aligned}
\delta\Theta_{\ell m} &= \int d^2\hat{\mathbf{n}} Y_{\ell m}^* \tau(\hat{\mathbf{n}})\Theta(\hat{\mathbf{n}}) \\
&= \sum_{LM\ell'm'} \tau_{LM}\Theta_{\ell'm'} \int d^2\hat{\mathbf{n}} Y_{\ell m}^* Y_{LM} Y_{\ell'm'} \\
&= \sum_{LM\ell'm'} \tau_{LM}\Theta_{\ell'm'} p_{\ell L\ell'}^+ \gamma_{\ell L\ell'} (-1)^m \begin{pmatrix} \ell & L & \ell' \\ -m & M & m' \end{pmatrix} \begin{pmatrix} \ell & L & \ell' \\ 0 & 0 & 0 \end{pmatrix} \\
&= \widetilde{\sum_{LM\ell'm'}}^{(\ell m)} \tau_{LM}\Theta_{\ell'm'} W_{\ell L\ell'}^{\tau,0}, \tag{43}
\end{aligned}$$

where

$$W_{\ell L\ell'}^{\tau,0} = p_{\ell L\ell'}^+ \gamma_{\ell L\ell'} \begin{pmatrix} \ell & L & \ell' \\ 0 & 0 & 0 \end{pmatrix}. \tag{44}$$

## 2.4 Summary

The above all distortions are described in the following form:

$$\delta\Theta_{\ell m} = \widetilde{\sum_{LM\ell'm'}}^{(\ell m)} x_{LM}\Theta_{\ell'm'} W_{\ell L\ell'}^{x,0}, \tag{45}$$

$$\delta E_{\ell m} = \widetilde{\sum_{LM\ell'm'}}^{(\ell m)} x_{LM} (E_{\ell'm'} W_{\ell L\ell'}^{x,+} + B_{\ell'm'} W_{\ell L\ell'}^{x,-}), \tag{46}$$

$$\delta B_{\ell m} = \widetilde{\sum_{LM\ell'm'}}^{(\ell m)} x_{LM} (-E_{\ell'm'} W_{\ell L\ell'}^{x,-} + B_{\ell'm'} W_{\ell L\ell'}^{x,+}) \tag{47}$$

where  $x$  is a distortion field. The functional form of  $W$  is given above.

### 3 Quadratic estimator

#### 3.1 Distortion induced anisotropies

The distortion fields  $x$  described above induce the off-diagonal elements of the covariance ( $\ell \neq \ell'$  or  $m \neq m'$ ),

$$\langle \tilde{X}_{\ell m} \tilde{Y}_{\ell' m'} \rangle_{\text{CMB}} = \sum_{LM} \begin{pmatrix} \ell & \ell' & L \\ m & m' & M \end{pmatrix} f_{\ell L \ell'}^{x, XY} x_{LM}^*, \quad (48)$$

where  $\langle \cdots \rangle_{\text{CMB}}$  denotes the ensemble average over the primary CMB anisotropies with a fixed realization of the distortion fields. We ignore the higher-order terms of the distortion fields. The functional form of the weight functions  $f$  are discussed later.

With a quadratic combination of observed CMB anisotropies,  $\hat{X}$  and  $\hat{Y}$ , the general quadratic estimators are formed as (e.g., [7]),

$$[\hat{x}_{LM}^{XY}]^* = A_L^{x, XY} \sum_{\ell \ell' m m'} \begin{pmatrix} \ell & \ell' & L \\ m & m' & M \end{pmatrix} g_{\ell L \ell'}^{x, XY} \hat{X}_{\ell m} \hat{Y}_{\ell' m'}. \quad (49)$$

Here we define

$$g_{\ell L \ell'}^{x, XY} = \frac{[f_{\ell L \ell'}^{x, XY}]^*}{\Delta^{XY} \hat{C}_{\ell}^{XX} \hat{C}_{\ell'}^{YY}} \quad (50)$$

$$A_L^{x, XY} = \frac{1}{2L+1} \sum_{\ell \ell'} f_{\ell L \ell'}^{x, XY} g_{\ell L \ell'}^{x, XY}, \quad (51)$$

where  $\Delta^{XX} = 2$ ,  $\Delta^{EB} = \Delta^{TB} = 1$ , and  $\hat{C}_{\ell}^{XX}$  ( $\hat{C}_{\ell}^{YY}$ ) is the observed power spectrum.

#### 3.2 Weight Function

The weight functions are, in general, given as

$$f_{\ell L \ell'}^{x, (\Theta\Theta)} = W_{\ell L \ell'}^{x, 0} C_{\ell'}^{\Theta\Theta} + p_x W_{\ell' L \ell}^{x, 0} C_{\ell}^{\Theta\Theta}, \quad (52)$$

$$f_{\ell L \ell'}^{x, (\Theta E)} = W_{\ell L \ell'}^{x, 0} C_{\ell'}^{\Theta E} + p_x W_{\ell' L \ell}^{x, +} C_{\ell}^{\Theta E}, \quad (53)$$

$$f_{\ell L \ell'}^{x, (\Theta B)} = p_x W_{\ell' L \ell}^{x, -} C_{\ell}^{\Theta E}, \quad (54)$$

$$f_{\ell L \ell'}^{x, (EE)} = W_{\ell L \ell'}^{x, +} C_{\ell'}^{EE} + p_x W_{\ell' L \ell}^{x, +} C_{\ell}^{EE}, \quad (55)$$

$$f_{\ell L \ell'}^{x, (EB)} = W_{\ell L \ell'}^{x, -} C_{\ell'}^{BB} + p_x W_{\ell' L \ell}^{x, -} C_{\ell}^{EE}, \quad (56)$$

$$f_{\ell L \ell'}^{x, (BB)} = W_{\ell L \ell'}^{x, +} C_{\ell'}^{BB} + p_x W_{\ell' L \ell}^{x, +} C_{\ell}^{BB}. \quad (57)$$

Here, the parity index is  $p_{\phi} = p_{\epsilon} = 1$  and  $p_{\varpi} = p_{\alpha} = -1$ . Strictly speaking,  $p_x$  should be  $(-1)^{\ell' + L + \ell}$ . However,  $W$  is only non-zero when  $\ell' + L + \ell$  is even, and vice versa. The parity even quantities are  $x = \phi$  and  $\epsilon$ . The odd parity quantities are  $x = \varpi$  and  $\alpha$ . Note that the above weight functions are consistent with Ref. [6] ( $W_{\ell L \ell'}^{x, -} = -_{\Theta} S_{\ell L \ell'}^x$ ) for the lensing case.

#### 3.3 Weight Function: Derivations

Let us first consider the temperature case. There are two contributions to the temperature quadratic estimator, and the one is given as

$$\begin{aligned} \langle \Theta_{\ell'' m''} \delta \Theta_{\ell m} \rangle &= \sum_{LM \ell' m'}^{(\ell m)} x_{LM} \langle \Theta_{\ell'' m''} \Theta_{\ell' m'} \rangle W_{\ell L \ell'}^{x, 0} \\ &= \sum_{LM \ell' m'}^{(\ell m)} x_{LM} \delta_{\ell'' \ell'} \delta_{m'', -m'} (-1)^{m'} C_{\ell'}^{\Theta\Theta} W_{\ell L \ell'}^{x, 0} \\ &= \sum_{LM} \begin{pmatrix} \ell & \ell'' & L \\ m & m'' & M \end{pmatrix} x_{LM}^* C_{\ell''}^{\Theta\Theta} W_{\ell L \ell''}^{x, 0}. \end{aligned} \quad (58)$$



Here, we use

$$\begin{aligned}
\widetilde{\sum_{LM\ell'm'}}^{(\ell m)} \delta_{\ell''\ell'} \delta_{m'',-m'} (-1)^{m'} x_{LM} &= \sum_{LM} (-1)^{m+m'} \begin{pmatrix} \ell & L & \ell'' \\ -m & M & -m'' \end{pmatrix} x_{LM} \\
&= \sum_{LM} (-1)^{-M} \begin{pmatrix} \ell & \ell'' & L \\ m & m'' & -M \end{pmatrix} x_{LM} \\
&= \sum_{LM} \begin{pmatrix} \ell & \ell'' & L \\ m & m'' & M \end{pmatrix} x_{LM}^* \tag{59}
\end{aligned}$$

The other term is obtained by exchanging  $(\ell'', m'') \leftrightarrow (\ell, m)$  and is given by

$$\langle \Theta_{\ell m} \delta \Theta_{\ell'' m''} \rangle = \sum_{LM} (-1)^{\ell+\ell''+L} \begin{pmatrix} \ell & \ell'' & L \\ m & m'' & M \end{pmatrix} x_{LM}^* C_{\ell}^{\Theta\Theta} W_{\ell' L \ell}^{x,0}. \tag{60}$$

The sign  $(-1)^{\ell+\ell''+L}$  depends on the parity of  $W$ .

For  $EB$  estimator, the two contributions are

$$\begin{aligned}
E_{\ell'' m''} \delta B_{\ell m} &= - \widetilde{\sum_{LM\ell'm'}}^{(\ell m)} x_{LM} \langle E_{\ell'' m''} E_{\ell' m'} \rangle W_{\ell L \ell'}^{x,-} \\
&= - \widetilde{\sum_{LM\ell'm'}}^{(\ell m)} x_{LM} (-1)^{m''} \delta_{\ell''\ell'} \delta_{m'',-m'} C_{\ell''}^{\text{EE}} W_{\ell L \ell'}^{x,-} \\
&= - \sum_{LM} \begin{pmatrix} \ell & \ell'' & L \\ m & m'' & M \end{pmatrix} x_{LM}^* C_{\ell''}^{\text{EE}} W_{\ell L \ell'}^{x,-}, \tag{61}
\end{aligned}$$

and

$$\begin{aligned}
B_{\ell'' m''} \delta E_{\ell m} &= \widetilde{\sum_{LM\ell'm'}}^{(\ell m)} x_{LM} \langle E_{\ell'' m''} E_{\ell' m'} \rangle W_{\ell L \ell'}^{x,-} \\
&= \sum_{LM} (-1)^{\ell+\ell''+L} \begin{pmatrix} \ell & \ell'' & L \\ m & m'' & M \end{pmatrix} x_{LM}^* C_{\ell}^{\text{EE}} W_{\ell' L \ell}^{x,-}. \tag{62}
\end{aligned}$$

## 4 General Fast formalism of Normalization

### 4.1 Normalization and Kernel function

The normalization of the  $\Theta\Theta$  quadratic estimator is

$$\begin{aligned} \frac{1}{A_L^{x,(\Theta\Theta)}} &= \frac{1}{2L+1} \sum_{\ell\ell'} \frac{[W_{\ell L\ell'}^{x,0} C_{\ell'}^{\Theta\Theta} + p_x W_{\ell' L\ell}^{x,0} C_{\ell}^{\Theta\Theta}]^2}{2\widehat{C}_{\ell}^{\Theta\Theta} \widehat{C}_{\ell'}^{\Theta\Theta}} \\ &= \frac{1}{2} \Sigma_L^{(0),x} \left[ \frac{1}{\widehat{C}^{\Theta\Theta}}, \frac{(C^{\Theta\Theta})^2}{\widehat{C}^{\Theta\Theta}} \right] + p_x \Gamma_L^{(0),x} \left[ \frac{C^{\Theta\Theta}}{\widehat{C}^{\Theta\Theta}}, \frac{C^{\Theta\Theta}}{\widehat{C}^{\Theta\Theta}} \right] + \frac{1}{2} \Sigma_L^{(0),x} \left[ \frac{(C^{\Theta\Theta})^2}{\widehat{C}^{\Theta\Theta}}, \frac{1}{\widehat{C}^{\Theta\Theta}} \right], \end{aligned} \quad (63)$$

where we define kernel functions as

$$\Sigma_L^{(0),x}[A, B] = \frac{1}{2L+1} \sum_{\ell\ell'} (W_{\ell L\ell'}^{x,0})^2 A_{\ell} B_{\ell'}, \quad (64)$$

$$\Gamma_L^{(0),x}[A, B] = \frac{1}{2L+1} \sum_{\ell\ell'} W_{\ell L\ell'}^{x,0} W_{\ell' L\ell}^{x,0} A_{\ell} B_{\ell'}. \quad (65)$$

For  $\Theta E$ ,

$$\begin{aligned} \frac{1}{A_L^{x,(\Theta E)}} &= \frac{1}{2L+1} \sum_{\ell\ell'} \frac{|W_{\ell L\ell'}^{x,0} C_{\ell'}^{\Theta E} + p_x W_{\ell' L\ell}^{x,+} C_{\ell}^{\Theta E}|^2}{\widehat{C}_{\ell}^{\Theta\Theta} \widehat{C}_{\ell'}^{\Theta E}} \\ &= \frac{1}{2L+1} \sum_{\ell\ell'} \left[ (W_{\ell L\ell'}^{x,0})^2 \frac{(C_{\ell'}^{\Theta E})^2}{\widehat{C}_{\ell}^{\Theta\Theta} \widehat{C}_{\ell'}^{\Theta E}} + 2p_x W_{\ell L\ell'}^{x,0} W_{\ell' L\ell}^{x,+} \frac{C_{\ell'}^{\Theta E} C_{\ell}^{\Theta E}}{\widehat{C}_{\ell}^{\Theta\Theta} \widehat{C}_{\ell'}^{\Theta E}} + (W_{\ell' L\ell}^{x,+})^2 \frac{(C_{\ell}^{\Theta E})^2}{\widehat{C}_{\ell}^{\Theta\Theta} \widehat{C}_{\ell'}^{\Theta E}} \right] \\ &= \Sigma_L^{(0),x} \left[ \frac{1}{\widehat{C}^{\Theta\Theta}}, \frac{(C^{\Theta E})^2}{\widehat{C}^{\Theta E}} \right] + 2p_x \Gamma_L^{(\times),x} \left[ \frac{C^{\Theta E}}{\widehat{C}^{\Theta\Theta}}, \frac{C^{\Theta E}}{\widehat{C}^{\Theta E}} \right] + \Sigma_L^{(+),x} \left[ \frac{1}{\widehat{C}^{\Theta E}}, \frac{(C^{\Theta E})^2}{\widehat{C}^{\Theta\Theta}} \right], \end{aligned} \quad (66)$$

where kernel functions are defined as

$$\begin{aligned} \Gamma_L^{(\times),x}[A, B] &= \frac{1}{2L+1} \sum_{\ell\ell'} W_{\ell L\ell'}^{x,0} W_{\ell' L\ell}^{x,+} A_{\ell} B_{\ell'}, \\ \Sigma_L^{(+),x}[A, B] &= \frac{1}{2L+1} \sum_{\ell\ell'} (W_{\ell L\ell'}^{x,+})^2 A_{\ell} B_{\ell'}. \end{aligned} \quad (67)$$

For  $\Theta B$ ,

$$\begin{aligned} \frac{1}{A_L^{x,(\Theta B)}} &= \frac{1}{2L+1} \sum_{\ell\ell'} \frac{|W_{\ell' L\ell}^{x,-} C_{\ell}^{\Theta E}|^2}{\widehat{C}_{\ell}^{\Theta\Theta} \widehat{C}_{\ell'}^{\Theta B}} \\ &= \Sigma_L^{(-),x} \left[ \frac{1}{\widehat{C}^{\Theta B}}, \frac{(C^{\Theta E})^2}{\widehat{C}^{\Theta\Theta}} \right], \end{aligned} \quad (68)$$

where

$$\Sigma_L^{(-),x}[A, B] = \frac{1}{2L+1} \sum_{\ell\ell'} |W_{\ell L\ell'}^{x,-}|^2 A_{\ell} B_{\ell'}. \quad (69)$$

For EE (and for BB by replacing  $EE \rightarrow BB$ ),

$$\begin{aligned} \frac{1}{A_L^x} &= \frac{1}{2L+1} \sum_{\ell\ell'} \frac{|W_{\ell L\ell'}^{x,+} C_{\ell'}^{\Theta E} + p_x W_{\ell' L\ell}^{x,+} C_{\ell}^{\Theta E}|^2}{2\widehat{C}_{\ell}^{\Theta E} \widehat{C}_{\ell'}^{\Theta E}} \\ &= \Sigma_L^{(+),x} \left[ \frac{1}{\widehat{C}^{\Theta E}}, \frac{(C^{\Theta E})^2}{\widehat{C}^{\Theta E}} \right] + p_x \Gamma_L^{(+),x} \left[ \frac{C^{\Theta E}}{\widehat{C}^{\Theta E}}, \frac{C^{\Theta E}}{\widehat{C}^{\Theta E}} \right], \end{aligned} \quad (70)$$

where

$$\Gamma_L^{(+),x}[A, B] = \frac{1}{2L+1} \sum_{\ell\ell'} W_{\ell'L\ell}^{x,+} W_{\ell L\ell'}^{x,+} A_\ell B_{\ell'} = \Gamma_L^{(+),x}[B, A]. \quad (71)$$

For  $EB$ ,

$$\begin{aligned} \frac{1}{A_L^{x,(EB)}} &= \frac{1}{2L+1} \sum_{\ell\ell'} \frac{|W_{\ell L\ell'}^{x,-} C_{\ell'}^{\text{BB}} + p_x W_{\ell'L\ell}^{x,-} C_\ell^{\text{EE}}|^2}{\widehat{C}_\ell^{\text{EE}} \widehat{C}_{\ell'}^{\text{BB}}} \\ &= \Sigma_L^{(-),x} \left[ \frac{1}{\widehat{C}^{\text{EE}}}, \frac{(C^{\text{BB}})^2}{\widehat{C}^{\text{BB}}} \right] + 2p_x \Gamma_L^{(-),x} \left[ \frac{C^{\text{EE}}}{\widehat{C}^{\text{EE}}}, \frac{C^{\text{BB}}}{\widehat{C}^{\text{BB}}} \right] + \Sigma_L^{(-),x} \left[ \frac{1}{\widehat{C}^{\text{BB}}}, \frac{(C^{\text{EE}})^2}{\widehat{C}^{\text{EE}}} \right], \end{aligned} \quad (72)$$

where

$$\Gamma_L^{(-),x}[A, B] = \frac{1}{2L+1} \sum_{\ell\ell'} [W_{\ell L\ell'}^{x,-}]^* W_{\ell'L\ell}^{x,-} A_\ell B_{\ell'} = \Gamma_L^{(-),x}[B, A]. \quad (73)$$

## 4.2 Noise covariance and kernel function

For  $\Theta\Theta\Theta E$ ,

$$\begin{aligned} \frac{A_L^{x,(\Theta\Theta)} A_L^{x,(\Theta E)}}{N_L^{x,(\Theta\Theta\Theta E)}} &= \frac{1}{2L+1} \sum_{\ell\ell'} \left[ \frac{W_{\ell L\ell'}^{x,0} C_{\ell'}^{\Theta\Theta}}{2\widehat{C}_\ell^{\Theta\Theta} \widehat{C}_{\ell'}^{\Theta\Theta}} + p_x(\ell \leftrightarrow \ell') \right] \left[ \frac{(W_{\ell L\ell'}^{x,0} C_{\ell'}^{\Theta E} + p_x W_{\ell'L\ell}^{x,+} C_\ell^{\Theta E}) \widehat{C}_{\ell'}^{\Theta E}}{\widehat{C}_\ell^{\text{EE}}} + p_x(\ell \leftrightarrow \ell') \right] \\ &= \frac{1}{2L+1} \sum_{\ell\ell'} \left[ \frac{W_{\ell L\ell'}^{x,0} C_{\ell'}^{\Theta\Theta}}{\widehat{C}_\ell^{\Theta\Theta} \widehat{C}_{\ell'}^{\Theta\Theta}} \frac{(W_{\ell L\ell'}^{x,0} C_{\ell'}^{\Theta E} + p_x W_{\ell'L\ell}^{x,+} C_\ell^{\Theta E}) \widehat{C}_{\ell'}^{\Theta E}}{\widehat{C}_\ell^{\text{EE}}} \right. \\ &\quad \left. + p_x \frac{W_{\ell L\ell'}^{x,0} C_{\ell'}^{\Theta\Theta}}{\widehat{C}_\ell^{\Theta\Theta} \widehat{C}_{\ell'}^{\Theta\Theta}} \frac{(W_{\ell'L\ell}^{x,0} C_\ell^{\Theta E} + p_x W_{\ell L\ell'}^{x,+} C_{\ell'}^{\Theta E}) \widehat{C}_\ell^{\Theta E}}{\widehat{C}_{\ell'}^{\text{EE}}} \right] \\ &= \Sigma_L^{(0),x} \left[ \frac{1}{\widehat{C}^{\Theta\Theta}}, \frac{C^{\Theta\Theta} C^{\Theta E} \widehat{C}^{\Theta E}}{\widehat{C}^{\Theta\Theta} \widehat{C}^{\text{EE}}} \right] + p_x \Gamma_L^{(\times),x} \left[ \frac{C^{\Theta E}}{\widehat{C}^{\Theta\Theta}}, \frac{C^{\Theta\Theta} \widehat{C}^{\Theta E}}{\widehat{C}^{\Theta\Theta} \widehat{C}^{\text{EE}}} \right] \\ &\quad + p_x \Gamma_L^{(0),x} \left[ \frac{C^{\Theta E} \widehat{C}^{\Theta E}}{\widehat{C}^{\Theta\Theta} \widehat{C}^{\text{EE}}}, \frac{C^{\Theta\Theta}}{\widehat{C}^{\Theta\Theta}} \right] + \Sigma_L^{(\times),x} \left[ \frac{\widehat{C}^{\Theta E}}{\widehat{C}^{\Theta\Theta} \widehat{C}^{\text{EE}}}, \frac{C^{\Theta E} C^{\Theta\Theta}}{\widehat{C}^{\Theta\Theta}} \right], \end{aligned} \quad (74)$$

where

$$\Sigma_L^{(\times),x}[A, B] = \frac{1}{2L+1} \sum_{\ell\ell'} W_{\ell L\ell'}^{x,0} W_{\ell L\ell'}^{x,+} A_\ell B_{\ell'}. \quad (75)$$

For  $\Theta\Theta EE$ ,

$$\begin{aligned} \frac{A_L^{x,(\Theta\Theta)} A_L^{x,(EE)}}{N_L^{x,(\Theta\Theta EE)}} &= \frac{1}{2L+1} \sum_{\ell\ell'} \left[ \frac{W_{\ell L\ell'}^{x,0} C_{\ell'}^{\Theta\Theta}}{2\widehat{C}_\ell^{\Theta\Theta} \widehat{C}_{\ell'}^{\Theta\Theta}} + p_x(\ell \leftrightarrow \ell') \right] \left[ \frac{(W_{\ell L\ell'}^{x,+} C_{\ell'}^{\text{EE}} + p_x W_{\ell'L\ell}^{x,+} C_\ell^{\text{EE}}) \widehat{C}_\ell^{\Theta E} \widehat{C}_{\ell'}^{\Theta E}}{2\widehat{C}_\ell^{\text{EE}} \widehat{C}_{\ell'}^{\text{EE}}} + p_x(\ell \leftrightarrow \ell') \right] \\ &= \frac{1}{2L+1} \sum_{\ell\ell'} \frac{W_{\ell L\ell'}^{x,0} C_{\ell'}^{\Theta\Theta}}{\widehat{C}_\ell^{\Theta\Theta} \widehat{C}_{\ell'}^{\Theta\Theta}} \left[ \frac{(W_{\ell L\ell'}^{x,+} C_{\ell'}^{\text{EE}} + p_x W_{\ell'L\ell}^{x,+} C_\ell^{\text{EE}}) \widehat{C}_\ell^{\Theta E} \widehat{C}_{\ell'}^{\Theta E}}{2\widehat{C}_\ell^{\text{EE}} \widehat{C}_{\ell'}^{\text{EE}}} + p_x(\ell \leftrightarrow \ell') \right] \\ &= \frac{1}{2L+1} \sum_{\ell\ell'} \frac{W_{\ell L\ell'}^{x,0} C_{\ell'}^{\Theta\Theta}}{\widehat{C}_\ell^{\Theta\Theta} \widehat{C}_{\ell'}^{\Theta\Theta}} \frac{(W_{\ell L\ell'}^{x,+} C_{\ell'}^{\text{EE}} + p_x W_{\ell'L\ell}^{x,+} C_\ell^{\text{EE}}) \widehat{C}_\ell^{\Theta E} \widehat{C}_{\ell'}^{\Theta E}}{\widehat{C}_\ell^{\text{EE}} \widehat{C}_{\ell'}^{\text{EE}}} \\ &= \Sigma_L^{(0),x} \left[ \frac{\widehat{C}^{\Theta E}}{\widehat{C}^{\Theta\Theta} \widehat{C}^{\text{EE}}}, \frac{C^{\Theta\Theta} C^{\text{EE}} \widehat{C}^{\Theta E}}{\widehat{C}^{\Theta\Theta} \widehat{C}^{\text{EE}}} \right] + p_x \Gamma_L^{(\times),x} \left[ \frac{\widehat{C}^{\Theta E} C^{\text{EE}}}{\widehat{C}^{\Theta\Theta} \widehat{C}^{\text{EE}}}, \frac{C^{\Theta\Theta} \widehat{C}^{\Theta E}}{\widehat{C}^{\Theta\Theta} \widehat{C}^{\text{EE}}} \right]. \end{aligned} \quad (76)$$

For  $\Theta EEE$ ,

$$\begin{aligned}
\frac{A_L^{x,(\Theta E)} A_L^{x,(EE)}}{N_L^{x,(\Theta EEE)}} &= \frac{1}{2L+1} \sum_{\ell\ell'} \left[ \frac{W_{\ell L\ell'}^{x,+} C_{\ell'}^{EE}}{2\hat{C}_{\ell}^{EE} \hat{C}_{\ell'}^{EE}} + p_x(\ell \leftrightarrow \ell') \right] \left[ \frac{(W_{\ell L\ell'}^{x,0} C_{\ell'}^{\Theta E} + p_x W_{\ell' L\ell}^{x,+} C_{\ell}^{\Theta E}) \hat{C}_{\ell}^{\Theta E}}{\hat{C}_{\ell}^{\Theta\Theta}} + p_x(\ell \leftrightarrow \ell') \right] \\
&= \frac{1}{2L+1} \sum_{\ell\ell'} \left[ \frac{W_{\ell L\ell'}^{x,+} C_{\ell'}^{EE}}{\hat{C}_{\ell}^{EE} \hat{C}_{\ell'}^{EE}} + p_x \frac{W_{\ell' L\ell}^{x,+} C_{\ell}^{EE}}{\hat{C}_{\ell}^{EE} \hat{C}_{\ell'}^{EE}} \right] \left[ \frac{(W_{\ell L\ell'}^{x,0} C_{\ell'}^{\Theta E} + p_x W_{\ell' L\ell}^{x,+} C_{\ell}^{\Theta E}) \hat{C}_{\ell}^{\Theta E}}{\hat{C}_{\ell}^{\Theta\Theta}} \right] \\
&= \Sigma_L^{(\times),x} \left[ \frac{\hat{C}^{\Theta E}}{\hat{C}^{\Theta\Theta} \hat{C}^{EE}}, \frac{C^{\Theta E} C^{EE}}{\hat{C}^{EE}} \right] + p_x \Gamma_L^{(+),x} \left[ \frac{C^{\Theta E} \hat{C}^{\Theta E}}{\hat{C}^{\Theta\Theta} \hat{C}^{EE}}, \frac{C^{EE}}{\hat{C}^{EE}} \right] \\
&\quad + p_x \Gamma_L^{(\times),x} \left[ \frac{\hat{C}^{\Theta E} C^{EE}}{\hat{C}^{\Theta\Theta} \hat{C}^{EE}}, \frac{C^{\Theta E}}{\hat{C}^{EE}} \right] + \Sigma_L^{(+),x} \left[ \frac{C^{\Theta E} \hat{C}^{\Theta E} C^{EE}}{\hat{C}^{\Theta\Theta} \hat{C}^{EE}}, \frac{1}{\hat{C}^{EE}} \right]. \tag{77}
\end{aligned}$$

For  $\Theta BEB$ ,

$$\begin{aligned}
\frac{A_L^{x,(\Theta B)} A_L^{x,(EB)}}{N_L^{x,(\Theta BEB)}} &= \frac{1}{2L+1} \sum_{\ell\ell'} \left[ \frac{(W_{\ell L\ell'}^{x,-})^* C_{\ell'}^{BB} - p_x (W_{\ell' L\ell}^{x,-})^* C_{\ell}^{EE}}{\hat{C}_{\ell}^{EE} \hat{C}_{\ell'}^{BB}} \right] \left[ \frac{-p_x W_{\ell' L\ell}^{x,-} C_{\ell}^{\Theta E} \hat{C}_{\ell}^{\Theta E}}{\hat{C}_{\ell}^{\Theta\Theta}} \right] \\
&= -p_x \Gamma_L^{(-),x} \left[ \frac{C^{\Theta E} \hat{C}^{\Theta E}}{\hat{C}^{\Theta\Theta} \hat{C}^{EE}}, \frac{C^{BB}}{\hat{C}^{BB}} \right] + \Sigma_L^{(-),x} \left[ \frac{C^{\Theta E} \hat{C}^{\Theta E} C^{EE}}{\hat{C}^{\Theta\Theta} \hat{C}^{EE}}, \frac{1}{\hat{C}^{BB}} \right]. \tag{78}
\end{aligned}$$

## 5 Explicit Kernel Functions

Here we consider expression for the Kernel functions in terms of the Wigner d-functions. In the following calculations, we frequently use

$$\int_{-1}^1 d\mu d_{s_1, s'_1}^{\ell_1}(\beta) d_{s_2, s'_2}^{\ell_2}(\beta) d_{s_3, s'_3}^{\ell_3}(\beta) = 2 \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ s_1 & s_2 & s_3 \end{pmatrix} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ s'_1 & s'_2 & s'_3 \end{pmatrix}, \quad (79)$$

with  $s_1 + s_2 + s_3 = s'_1 + s'_2 + s'_3 = 0$  and  $\mu = \cos \beta$ , and the symmetric property:

$$d_{mm'}^{\ell}(\beta) = (-1)^{m-m'} d_{-m, -m'}^{\ell}(\beta) = (-1)^{m-m'} d_{m'm}^{\ell}(\beta) \quad (80)$$

$$d_{mm'}^{\ell}(\beta) = (-1)^{\ell+m} d_{m, -m'}^{\ell}(\pi - \beta). \quad (81)$$

We also define

$$X^{p \dots q} = a_{\ell}^p \dots a_{\ell}^q X_{\ell}. \quad (82)$$

### 5.1 Kernel Functions: Lensing

We obtain

$$\begin{aligned} \Sigma_L^{(0),x}[A, B] &= \frac{1}{2L+1} \sum_{\ell\ell'} |W_{\ell L \ell'}^{x,0}|^2 A_{\ell} B_{\ell'} \\ &= \sum_{\ell\ell'} 4\pi L(L+1) \frac{2\ell+1}{4\pi} A_{\ell} \frac{2\ell'+1}{4\pi} B_{\ell'} \ell'(\ell'+1) \frac{1+c_x^2(-1)^{\ell+L+\ell'}}{2} \begin{pmatrix} \ell & L & \ell' \\ 0 & 1 & -1 \end{pmatrix}^2 \\ &= \pi L(L+1) \sum_{\ell\ell'} \frac{2\ell+1}{4\pi} A_{\ell} \frac{2\ell'+1}{4\pi} B_{\ell'} 2\ell'(\ell'+1) \left[ \begin{pmatrix} \ell & L & \ell' \\ 0 & 1 & -1 \end{pmatrix}^2 + c_x^2 \begin{pmatrix} \ell & L & \ell' \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} \ell & L & \ell' \\ 0 & 1 & -1 \end{pmatrix} \right] \\ &= \int_{-1}^1 d\mu \pi L(L+1) \sum_{\ell\ell'} \frac{2\ell+1}{4\pi} A_{\ell} \frac{2\ell'+1}{4\pi} B_{\ell'} \ell'(\ell'+1) [d_{00}^{\ell} d_{11}^L d_{11}^{\ell'} + c_x^2 d_{00}^{\ell} d_{1,-1}^L d_{1,-1}^{\ell'}] \\ &= \int_{-1}^1 d\mu \pi L(L+1) \{ \xi_{00}[A] \xi_{11}[B^{00}] d_{11}^L + c_x^2 \xi_{00}[A] \xi_{1,-1}[B^{00}] d_{1,-1}^L \}, \end{aligned} \quad (83)$$

where

$$\xi_{mm'}[A] = \sum_{\ell} \frac{2\ell+1}{4\pi} A_{\ell} d_{mm'}^{\ell}. \quad (84)$$

The cross-term is

$$\begin{aligned} \Gamma_L^{(0),x}[A, B] &= \frac{1}{2L+1} \sum_{\ell\ell'} (W_{\ell L \ell'}^{x,0})^* W_{\ell' L \ell}^{x,0} A_{\ell} B_{\ell'} \\ &= \sum_{\ell\ell'} 4\pi L(L+1) \frac{2\ell+1}{4\pi} A_{\ell} \frac{2\ell'+1}{4\pi} B_{\ell'} a_{\ell}^0 a_{\ell'}^0 \frac{1+c_x^2(-1)^{\ell+L+\ell'}}{2} \begin{pmatrix} \ell & L & \ell' \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} \ell' & L & \ell \\ 0 & 1 & -1 \end{pmatrix} \\ &= \pi L(L+1) \sum_{\ell\ell'} \frac{2\ell+1}{4\pi} A_{\ell} \frac{2\ell'+1}{4\pi} B_{\ell'}^0 2 \left[ \begin{pmatrix} \ell & L & \ell' \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} \ell & L & \ell' \\ 1 & -1 & 0 \end{pmatrix} + c_x^2 \begin{pmatrix} \ell & L & \ell' \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} \ell & L & \ell' \\ -1 & 1 & 0 \end{pmatrix} \right] \\ &= \int_{-1}^1 d\mu \pi L(L+1) \sum_{\ell\ell'} \frac{2\ell+1}{4\pi} A_{\ell} \frac{2\ell'+1}{4\pi} B_{\ell'}^0 [d_{01}^{\ell} d_{1,-1}^L d_{-1,0}^{\ell'} + c_x^2 d_{0,-1}^{\ell} d_{11}^L d_{-1,0}^{\ell'}] \\ &= - \int_{-1}^1 d\mu \pi L(L+1) \{ \xi_{01}[A^0] \xi_{0,-1}[B^0] d_{1,-1}^L + c_x^2 \xi_{01}[A^0] \xi_{01}[B^0] d_{11}^L \}. \end{aligned} \quad (85)$$

Denoting  $p = \pm$  and  $x = \phi, \varpi$ , the polarization auto kernel is

$$\begin{aligned}
\Sigma_L^{(p),x}[A, B] &= \frac{1}{2L+1} \sum_{\ell\ell'} |W_{\ell L \ell'}^{x,p}|^2 A_\ell B_{\ell'} \\
&= \pi L(L+1) \sum_{\ell\ell'} \frac{2\ell+1}{4\pi} A_\ell \frac{2\ell'+1}{4\pi} B_{\ell'} 2[1 + pc_x^2(-1)^{\ell+L+\ell'}] \left[ a_{\ell'}^+ \begin{pmatrix} \ell & L & \ell' \\ -2 & -1 & 3 \end{pmatrix} + c_x^2 a_{\ell'}^- \begin{pmatrix} \ell & L & \ell' \\ -2 & 1 & 1 \end{pmatrix} \right]^2 \\
&= \pi L(L+1) \sum_{\ell\ell'} \frac{2\ell+1}{4\pi} A_\ell \frac{2\ell'+1}{4\pi} B_{\ell'} [1 + pc_x^2(-1)^{\ell+L+\ell'}] \\
&\quad \times 2 \left[ (a_{\ell'}^+)^2 \begin{pmatrix} \ell & L & \ell' \\ -2 & -1 & 3 \end{pmatrix}^2 + (a_{\ell'}^-)^2 \begin{pmatrix} \ell & L & \ell' \\ -2 & 1 & 1 \end{pmatrix}^2 + 2c_x^2 a_{\ell'}^+ a_{\ell'}^- \begin{pmatrix} \ell & L & \ell' \\ -2 & 1 & 1 \end{pmatrix} \begin{pmatrix} \ell & L & \ell' \\ -2 & -1 & 3 \end{pmatrix} \right] \\
&= \pi L(L+1) \sum_{\ell\ell'} \frac{2\ell+1}{4\pi} A_\ell \frac{2\ell'+1}{4\pi} B_{\ell'} \\
&\quad \times 2 \left[ (a_{\ell'}^+)^2 \begin{pmatrix} \ell & L & \ell' \\ -2 & -1 & 3 \end{pmatrix}^2 + (a_{\ell'}^-)^2 \begin{pmatrix} \ell & L & \ell' \\ -2 & 1 & 1 \end{pmatrix}^2 + 2c_x^2 a_{\ell'}^+ a_{\ell'}^- \begin{pmatrix} \ell & L & \ell' \\ -2 & 1 & 1 \end{pmatrix} \begin{pmatrix} \ell & L & \ell' \\ -2 & -1 & 3 \end{pmatrix} \right. \\
&\quad \left. + pc_x^2 (a_{\ell'}^+)^2 \begin{pmatrix} \ell & L & \ell' \\ -2 & -1 & 3 \end{pmatrix} \begin{pmatrix} \ell & L & \ell' \\ 2 & 1 & -3 \end{pmatrix} + pc_x^2 (a_{\ell'}^-)^2 \begin{pmatrix} \ell & L & \ell' \\ -2 & 1 & 1 \end{pmatrix} \begin{pmatrix} \ell & L & \ell' \\ 2 & -1 & -1 \end{pmatrix} + 2pa_{\ell'}^+ a_{\ell'}^- \begin{pmatrix} \ell & L & \ell' \\ -2 & 1 & 1 \end{pmatrix} \begin{pmatrix} \ell & L & \ell' \\ 2 & 1 & -3 \end{pmatrix} \right] \\
&= \int_{-1}^1 d\mu \pi L(L+1) \sum_{\ell\ell'} \frac{2\ell+1}{4\pi} A_\ell \frac{2\ell'+1}{4\pi} B_{\ell'} [(a_{\ell'}^+)^2 d_{22}^\ell d_{11}^L d_{33}^{\ell'} + (a_{\ell'}^-)^2 d_{22}^\ell d_{11}^L d_{11}^{\ell'} \\
&\quad + 2c_x^2 a_{\ell'}^+ a_{\ell'}^- d_{22}^\ell d_{1,-1}^L d_{13}^{\ell'} + pc_x^2 (a_{\ell'}^+)^2 d_{-2,2}^\ell d_{-1,1}^L d_{3,-3}^{\ell'} + pc_x^2 (a_{\ell'}^-)^2 d_{-2,2}^\ell d_{1,-1}^L d_{1,-1}^{\ell'} + 2pa_{\ell'}^+ a_{\ell'}^- d_{-2,2}^\ell d_{11}^L d_{1,-3}^{\ell'}] \\
&= \int_{-1}^1 d\mu \pi L(L+1) [(\xi_{22}[A]\xi_{33}[B^{++}] + \xi_{22}[A]\xi_{11}[B^{--}] + 2p\xi_{2,-2}[A]\xi_{3,-1}[B^{+-}])d_{11}^L \\
&\quad + c_x^2(p\xi_{2,-2}[A]\xi_{3,-3}[B^{++}] + p\xi_{2,-2}[A]\xi_{1,-1}[B^{--}] + 2\xi_{22}[A]\xi_{31}[B^{+-}])d_{1,-1}^L]. \tag{86}
\end{aligned}$$

The polarization cross kernel is

$$\begin{aligned}
\Gamma_L^{(p),x}[A, B] &= \frac{1}{2L+1} \sum_{\ell\ell'} (W_{\ell L \ell'}^{x,p})^* W_{\ell' L \ell}^{x,p} A_\ell B_{\ell'} \\
&= \pi L(L+1) \sum_{\ell\ell'} \frac{2\ell+1}{4\pi} A_\ell \frac{2\ell'+1}{4\pi} B_{\ell'} 2[1 + pc_x^2(-1)^{\ell+L+\ell'}] \\
&\quad \times \left[ a_{\ell'}^+ \begin{pmatrix} \ell & L & \ell' \\ -2 & -1 & 3 \end{pmatrix} + c_x^2 a_{\ell'}^- \begin{pmatrix} \ell & L & \ell' \\ -2 & 1 & 1 \end{pmatrix} \right] \left[ a_{\ell'}^+ \begin{pmatrix} \ell' & L & \ell \\ -2 & -1 & 3 \end{pmatrix} + c_x^2 a_{\ell'}^- \begin{pmatrix} \ell' & L & \ell \\ -2 & 1 & 1 \end{pmatrix} \right] \\
&= \pi L(L+1) \sum_{\ell\ell'} \frac{2\ell+1}{4\pi} A_\ell \frac{2\ell'+1}{4\pi} B_{\ell'} 2[(-1)^{\ell+L+\ell'} + pc_x^2] \\
&\quad \times \left[ a_{\ell'}^+ \begin{pmatrix} \ell & L & \ell' \\ -2 & -1 & 3 \end{pmatrix} + c_x^2 a_{\ell'}^- \begin{pmatrix} \ell & L & \ell' \\ -2 & 1 & 1 \end{pmatrix} \right] \left[ a_{\ell'}^+ \begin{pmatrix} \ell & L & \ell' \\ 3 & -1 & -2 \end{pmatrix} + c_x^2 a_{\ell'}^- \begin{pmatrix} \ell & L & \ell' \\ 1 & 1 & -2 \end{pmatrix} \right] \\
&= \pi L(L+1) \sum_{\ell\ell'} \frac{2\ell+1}{4\pi} A_\ell \frac{2\ell'+1}{4\pi} B_{\ell'} \\
&\quad \times 2 \left\{ \left[ a_{\ell'}^+ \begin{pmatrix} \ell & L & \ell' \\ 2 & 1 & -3 \end{pmatrix} + c_x^2 a_{\ell'}^- \begin{pmatrix} \ell & L & \ell' \\ 2 & -1 & -1 \end{pmatrix} \right] \left[ a_{\ell'}^+ \begin{pmatrix} \ell & L & \ell' \\ 3 & -1 & -2 \end{pmatrix} + c_x^2 a_{\ell'}^- \begin{pmatrix} \ell & L & \ell' \\ 1 & 1 & -2 \end{pmatrix} \right] \right. \\
&\quad \left. + p \left[ c_x^2 a_{\ell'}^+ \begin{pmatrix} \ell & L & \ell' \\ -2 & -1 & 3 \end{pmatrix} + a_{\ell'}^- \begin{pmatrix} \ell & L & \ell' \\ -2 & 1 & 1 \end{pmatrix} \right] \left[ a_{\ell'}^+ \begin{pmatrix} \ell & L & \ell' \\ 3 & -1 & -2 \end{pmatrix} + c_x^2 a_{\ell'}^- \begin{pmatrix} \ell & L & \ell' \\ 1 & 1 & -2 \end{pmatrix} \right] \right\} \\
&= \int_{-1}^1 d\mu \pi L(L+1) \sum_{\ell\ell'} \frac{2\ell+1}{4\pi} A_\ell \frac{2\ell'+1}{4\pi} B_{\ell'} \\
&\quad \times [a_{\ell'}^+ a_{\ell'}^+ d_{23}^L d_{1,-1}^L d_{-3,-2}^{\ell'} + c_x^2 a_{\ell'}^+ a_{\ell'}^- d_{21}^L d_{11}^L d_{-3,-2}^{\ell'} + c_x^2 a_{\ell'}^- a_{\ell'}^+ d_{23}^L d_{11}^L d_{-1,-2}^{\ell'} + a_{\ell'}^- a_{\ell'}^- d_{21}^L d_{1,-1}^L d_{-1,-2}^{\ell'} \\
&\quad + p(c_x^2 a_{\ell'}^+ a_{\ell'}^+ d_{-2,3}^L d_{11}^L d_{3,-2}^{\ell'} + a_{\ell'}^+ a_{\ell'}^- d_{-2,1}^L d_{1,-1}^L d_{3,-2}^{\ell'} + a_{\ell'}^- a_{\ell'}^+ d_{-2,3}^L d_{1,-1}^L d_{1,-2}^{\ell'} + c_x^2 a_{\ell'}^- a_{\ell'}^- d_{-2,1}^L d_{11}^L d_{1,-2}^{\ell'})] \\
&= \int_{-1}^1 d\mu \pi L(L+1) [-c_x^2 (\xi_{21}[A^-] \xi_{32}[B^+] + \xi_{32}[A^+] \xi_{21}[B^-] + p \xi_{3,-2}[A^+] \xi_{3,-2}[B^+] + p \xi_{2,-1}[A^-] \xi_{2,-1}[B^-]) d_{11}^L \\
&\quad + (\xi_{32}[A^+] \xi_{32}[B^+] + \xi_{21}[A^-] \xi_{21}[B^-] - p \xi_{2,-1}[A^-] \xi_{3,-2}[B^+] - p \xi_{3,-2}[A^+] \xi_{2,-1}[B^-]) d_{1,-1}^L]. \quad (87)
\end{aligned}$$

The temperature-polarization kernel is

$$\begin{aligned}
\Sigma_L^{(\times),x}[A, B] &= \frac{1}{2L+1} \sum_{\ell\ell'} (W_{\ell L \ell'}^{x,0})^* W_{\ell' L \ell}^{x,+} A_\ell B_{\ell'} \\
&= \pi L(L+1) \sum_{\ell\ell'} \frac{2\ell+1}{4\pi} A_\ell \frac{2\ell'+1}{4\pi} B_{\ell'} a_{\ell'}^0 2[1 + c_x^2(-1)^{\ell+L+\ell'}] \\
&\quad \times \begin{pmatrix} \ell & L & \ell' \\ 0 & 1 & -1 \end{pmatrix} \left[ a_{\ell'}^+ \begin{pmatrix} \ell & L & \ell' \\ -2 & -1 & 3 \end{pmatrix} + c_x^2 a_{\ell'}^- \begin{pmatrix} \ell & L & \ell' \\ -2 & 1 & 1 \end{pmatrix} \right] \\
&= \pi L(L+1) \sum_{\ell\ell'} \frac{2\ell+1}{4\pi} A_\ell \frac{2\ell'+1}{4\pi} B_{\ell'} a_{\ell'}^0 \\
&\quad \times 2 \left[ \begin{pmatrix} \ell & L & \ell' \\ 0 & 1 & -1 \end{pmatrix} + c_x^2 \begin{pmatrix} \ell & L & \ell' \\ 0 & -1 & 1 \end{pmatrix} \right] \left[ a_{\ell'}^+ \begin{pmatrix} \ell & L & \ell' \\ -2 & -1 & 3 \end{pmatrix} + c_x^2 a_{\ell'}^- \begin{pmatrix} \ell & L & \ell' \\ -2 & 1 & 1 \end{pmatrix} \right] \\
&= \int_{-1}^1 d\mu \pi L(L+1) \sum_{\ell\ell'} \frac{2\ell+1}{4\pi} A_\ell \frac{2\ell'+1}{4\pi} B_{\ell'} a_{\ell'}^0 \\
&\quad \times \left[ a_{\ell'}^+ d_{0,-2}^L d_{1,-1}^L d_{-1,3}^{\ell'} + c_x^2 a_{\ell'}^- d_{0,-2}^L d_{11}^L d_{-1,1}^{\ell'} + c_x^2 a_{\ell'}^+ d_{0,-2}^L d_{11}^L d_{13}^{\ell'} + a_{\ell'}^- d_{0,-2}^L d_{-1,1}^L d_{11}^{\ell'} \right] \\
&= \int_{-1}^1 d\mu \pi L(L+1) \{ c_x^2 (\xi_{20}[A] \xi_{1,-1}[B^{0-}] + \xi_{20}[A] \xi_{31}[B^{0+}]) d_{11}^L \\
&\quad + (\xi_{20}[A] \xi_{3,-1}[B^{0+}] + \xi_{20}[A] \xi_{11}[B^{0-}]) d_{1,-1}^L \}, \quad (88)
\end{aligned}$$

and

$$\begin{aligned}
 \Gamma_L^{(\times),x}[A, B] &= \frac{1}{2L+1} \sum_{\ell\ell'} (W_{\ell L \ell'}^{x,0})^* W_{\ell' L \ell}^{x,+} A_\ell B_{\ell'} \\
 &= \pi L(L+1) \sum_{\ell\ell'} \frac{2\ell+1}{4\pi} A_\ell \frac{2\ell'+1}{4\pi} B_{\ell'} a_{\ell'}^0 2[1 + c_x^2 (-1)^{\ell+L+\ell'}] \\
 &\quad \times \begin{pmatrix} \ell & L & \ell' \\ 0 & 1 & -1 \end{pmatrix} \left[ a_\ell^+ \begin{pmatrix} \ell' & L & \ell \\ -2 & -1 & 3 \end{pmatrix} + c_x^2 a_\ell^- \begin{pmatrix} \ell' & L & \ell \\ -2 & 1 & 1 \end{pmatrix} \right] \\
 &= \pi L(L+1) \sum_{\ell\ell'} \frac{2\ell+1}{4\pi} A_\ell \frac{2\ell'+1}{4\pi} B_{\ell'} a_{\ell'}^0 \\
 &\quad \times 2 \left[ \begin{pmatrix} \ell & L & \ell' \\ 0 & 1 & -1 \end{pmatrix} + c_x^2 \begin{pmatrix} \ell & L & \ell' \\ 0 & -1 & 1 \end{pmatrix} \right] \left[ a_\ell^+ \begin{pmatrix} \ell & L & \ell' \\ -3 & 1 & 2 \end{pmatrix} + c_x^2 a_\ell^- \begin{pmatrix} \ell & L & \ell' \\ -1 & -1 & 2 \end{pmatrix} \right] \\
 &= \int_{-1}^1 d\mu \pi L(L+1) \sum_{\ell\ell'} \frac{2\ell+1}{4\pi} A_\ell \frac{2\ell'+1}{4\pi} B_{\ell'} a_{\ell'}^0 \\
 &\quad \times \left[ a_\ell^+ d_{0,-3}^\ell d_{11}^L d_{-1,2}^{\ell'} + c_x^2 a_\ell^- d_{0,-1}^\ell d_{1,-1}^L d_{-1,2}^{\ell'} + c_x^2 a_\ell^+ d_{0,-3}^\ell d_{1,-1}^L d_{12}^{\ell'} + a_\ell^- d_{0,-1}^\ell d_{11}^L d_{12}^{\ell'} \right] \\
 &= \int_{-1}^1 d\mu \pi L(L+1) \{ -(\xi_{30}[A^+] \xi_{2,-1}[B^0] + \xi_{10}[A^-] \xi_{12}[B^0]) d_{11}^L \\
 &\quad - c_x^2 (\xi_{10}[A^-] \xi_{2,-1}[B^0] + \xi_{30}[A^0] \xi_{21}[B^-]) d_{1,-1}^L \}.
 \end{aligned} \tag{89}$$

## 5.2 Kernel Functions: Rotation

Next we consider the kernel functions for  $x = \alpha$ . If  $p = -$  and  $x = \alpha$ ,

$$\begin{aligned}
 \Sigma_L^{(-),\alpha}[A, B] &= \pi \sum_{\ell\ell'} \frac{2\ell+1}{4\pi} A_\ell \frac{2\ell'+1}{4\pi} B_{\ell'} 8[1 + (-1)^{\ell+L+\ell'}] \begin{pmatrix} \ell & L & \ell' \\ -2 & 0 & 2 \end{pmatrix}^2 \\
 &= \pi L(L+1) \sum_{\ell\ell'} \frac{2\ell+1}{4\pi} A_\ell \frac{2\ell'+1}{4\pi} B_{\ell'} 8 \left[ \begin{pmatrix} \ell & L & \ell' \\ -2 & 0 & 2 \end{pmatrix}^2 + \begin{pmatrix} \ell & L & \ell' \\ -2 & 0 & 2 \end{pmatrix} \begin{pmatrix} \ell & L & \ell' \\ 2 & 0 & -2 \end{pmatrix} \right] \\
 &= \int_{-1}^1 d\mu \pi L(L+1) \sum_{\ell\ell'} \frac{2\ell+1}{4\pi} A_\ell \frac{2\ell'+1}{4\pi} B_{\ell'} 4(d_{-2,-2}^\ell d_{00}^L d_{22}^{\ell'} + d_{-2,2}^\ell d_{00}^L d_{2,-2}^{\ell'}) \\
 &= \int_{-1}^1 d\mu 4\pi (\xi_{-2,-2}[A] \xi_{22}[B] + \xi_{-2,22}[A] \xi_{22}[B]) d_{00}^L,
 \end{aligned} \tag{90}$$

and

$$\begin{aligned}
 \Gamma_L^{(-),\alpha}[A, B] &= \pi \sum_{\ell\ell'} \frac{2\ell+1}{4\pi} A_\ell \frac{2\ell'+1}{4\pi} B_{\ell'} 8[1 + (-1)^{\ell+L+\ell'}] \begin{pmatrix} \ell & L & \ell' \\ -2 & 0 & 2 \end{pmatrix} \begin{pmatrix} \ell' & L & \ell \\ -2 & 0 & 2 \end{pmatrix} \\
 &= \pi \sum_{\ell\ell'} \frac{2\ell+1}{4\pi} A_\ell \frac{2\ell'+1}{4\pi} B_{\ell'} \\
 &\quad \times 8 \begin{pmatrix} \ell & L & \ell' \\ -2 & 0 & 2 \end{pmatrix} \left[ \begin{pmatrix} \ell & L & \ell' \\ -2 & 0 & 2 \end{pmatrix} + \begin{pmatrix} \ell & L & \ell' \\ 2 & 0 & -2 \end{pmatrix} \right] \\
 &= \int_{-1}^1 d\mu \pi \sum_{\ell\ell'} \frac{2\ell+1}{4\pi} A_\ell \frac{2\ell'+1}{4\pi} B_{\ell'} 4[d_{-2,-2}^\ell d_{00}^L d_{22}^{\ell'} + d_{-2,2}^\ell d_{00}^L d_{2,-2}^{\ell'}] \\
 &= \int_{-1}^1 d\mu 4\pi (\xi_{-2,-2}[A] \xi_{22}[B] + \xi_{-2,2}[A] \xi_{2,-2}[B]) d_{00}^L.
 \end{aligned} \tag{91}$$



### 5.3 Kernel Functions: Tau

$$\begin{aligned}
\Sigma_L^{(0),\tau}[A, B] &= \pi \sum_{\ell\ell'} \frac{2\ell+1}{4\pi} A_\ell \frac{2\ell'+1}{4\pi} B_{\ell'} 2(1 + (-1)^{\ell+L+\ell'}) \begin{pmatrix} \ell & L & \ell' \\ 0 & 0 & 0 \end{pmatrix}^2 \\
&= \int_{-1}^1 d\mu \pi \sum_{\ell\ell'} \frac{2\ell+1}{4\pi} A_\ell \frac{2\ell'+1}{4\pi} B_{\ell'} 2d_{00}^\ell d_{00}^L d_{00}^{\ell'} \\
&= \int_{-1}^1 d\mu 2\pi \zeta_{00}[A] \zeta_{00}[B] d_{00}^L.
\end{aligned} \tag{92}$$

$$\begin{aligned}
\Gamma_L^{(0),\tau}[A, B] &= 4\pi \sum_{\ell\ell'} \frac{2\ell+1}{4\pi} A_\ell \frac{2\ell'+1}{4\pi} B_{\ell'} \frac{1 + (-1)^{\ell+L+\ell'}}{2} \begin{pmatrix} \ell & L & \ell' \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \ell & L & \ell' \\ 0 & 0 & 0 \end{pmatrix} \\
&= \Sigma_L^{(0),\tau}[A, B].
\end{aligned} \tag{93}$$

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