[Note]
Delensing bias

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1 Algorithm of lensing reconstruction and delensing

1.1 Quadratic lensing reconstruction

The distortion effect of lensing on the primary polarization anisotropies is expressed by a remapping of the primary anisotropies. Denoting the primary polarization anisotropies at position $\hat{\boldsymbol{n}}$ on the last scattering surface as $[Q \pm \mathrm{i}U](\hat{\boldsymbol{n}})$, the lensed anisotropies in a direction $\hat{\boldsymbol{n}}$, are given by (e.g., [9])

$$\begin{aligned}[Q \pm \mathrm{i}U](\hat{\boldsymbol{n}}) &= [Q \pm \mathrm{i}U](\hat{\boldsymbol{n}} + \boldsymbol{\nabla}\phi(\hat{\boldsymbol{n}})) \\ &= [Q \pm \mathrm{i}U](\hat{\boldsymbol{n}}) + \boldsymbol{\nabla}\phi(\hat{\boldsymbol{n}}) \cdot \boldsymbol{\nabla}[Q \pm \mathrm{i}U](\hat{\boldsymbol{n}}) + \mathcal{O}(|\boldsymbol{\nabla}\phi|^2) \,. \end{aligned} \tag{1}$$

The two-dimensional vector $\boldsymbol{\nabla}\phi(\hat{\boldsymbol{n}})$ is the deflection angle, and a scalar quantity ϕ is the lensing potential. Instead of the spin-2 quantity, the polarization anisotropies are usually decomposed into the rotationally invariant combination, i.e., the E and B mode polarizations (e.g., [9]). In harmonics space, the E and B-modes are defined with the spin-2 spherical harmonics ${}_{\pm 2}Y_{L,M}$ [10]) :

$$[E \pm \mathrm{i}B]_{L,M} = \int \mathrm{d}\hat{\boldsymbol{n}}\ {}_{\pm 2}Y^*_{L,M}(\hat{\boldsymbol{n}})[Q \pm \mathrm{i}U](\hat{\boldsymbol{n}})\,. \tag{2}$$

Similarly, the harmonic coefficients of the scalar quantity ϕ is given by

$$\phi_{\ell,m} = \int \mathrm{d}\hat{\boldsymbol{n}}\ {}_0Y^*_{\ell,m}(\hat{\boldsymbol{n}})\phi(\hat{\boldsymbol{n}})\,, \tag{3}$$

where ${}_0Y_{\ell,m}$ is the spin-0 spherical harmonics. With Eq. (1), the lensed E and B modes are then given by (e.g., [10])

$$\tilde{E}_{L,M} = E_{L,M} + \sum_{L',M'}\sum_{\ell,m}\begin{pmatrix} L & L' & \ell \\ M & M' & m \end{pmatrix}\{\mathcal{S}^{(+)}_{L,L',\ell}E^*_{L',M'} + \mathrm{i}\mathcal{S}^{(-)}_{L,L',\ell}B^*_{L',M'}\}\phi^*_{\ell,m}\,, \tag{4}$$

$$\tilde{B}_{L,M} = B_{L,M} + \sum_{L',M'}\sum_{\ell,m}\begin{pmatrix} L & L' & \ell \\ M & M' & m \end{pmatrix}\{\mathcal{S}^{(+)}_{L,L',\ell}B^*_{L',M'} - \mathrm{i}\mathcal{S}^{(-)}_{L,L',\ell}E^*_{L',M'}\}\phi^*_{\ell,m}\,, \tag{5}$$

where the quantities, $\mathcal{S}^{(\pm)}_{L,L',\ell}$ is given by [1]

$$\mathcal{S}^{(\pm)}_{L,L',\ell} = \frac{1 \pm (-1)^{L+\ell+L'}}{2}\,{}_2\mathcal{S}_{L,L',\ell}\,;$$
$${}_2\mathcal{S}_{L,L',\ell} = \sqrt{\frac{(2\ell+1)(2L'+1)(2L+1)}{16\pi}}[\ell(\ell+1) + L'(L'+1) - L(L+1)]\begin{pmatrix} L & L' & \ell \\ 2 & -2 & 0 \end{pmatrix}\,. \tag{6}$$

The off-diagonal covariance ($L \neq L'$ or $M \neq M'$) includes the lensing potential as a leading order term:

$$\langle \tilde{X}_{L,M}\tilde{Y}_{L',M'}\rangle_{\mathrm{CMB}} = \sum_{\ell,m}\begin{pmatrix} L & L' & \ell \\ M & M' & m \end{pmatrix} f^{(XY)}_{L,L',\ell}\phi^*_{\ell,m}\,, \tag{7}$$

where $\langle\cdots\rangle_{\mathrm{CMB}}$ denotes the ensemble average over unlensed E or B, with a fixed realization of the lensing potential, and we ignore the higher-order terms of lensing fields. The weight functions for the lensing potential is given by [11, 12]

$$f^{(EE)}_{L,L',\ell} = \mathcal{S}^{(+)}_{L,L',\ell}C^{\mathrm{EE}}_{L'} + \mathcal{S}^{(+)}_{L',L,\ell}C^{\mathrm{EE}}_{L}\,, \tag{8}$$
$$f^{(EB)}_{L,L',\ell} = \mathrm{i}[\mathcal{S}^{(-)}_{L,L',\ell}C^{\mathrm{BB}}_{L'} + \mathcal{S}^{(-)}_{L',L,\ell}C^{\mathrm{EE}}_{L}]\,. \tag{9}$$

Note that, to mitigate the higher-order biases [5], the lensed power spectrum is used rather than the unlensed one [13, 6]: With a quadratic combination of X and Y fluctuations, the lensing estimators are then formed as (e.g., [11]),

$$[\widehat{\phi}^{(XY)}_{\ell,m}]^* = N^{(XY)}_{\ell}\sum_{L,L'}\sum_{M,M'}\begin{pmatrix} L & L' & \ell \\ M & M' & m \end{pmatrix} g^{(XY)}_{L,L',\ell}\widehat{X}^{\mathrm{obs}}_{L,M}\widehat{Y}^{\mathrm{obs}}_{L',M'}\,, \tag{10}$$

[1] Note that the quantity ${}_2\mathcal{S}_{L,L',\ell}$ corresponds to $F^2_{L,\ell,L'}$ defined in Eq. (14) of [11], $F^{-2}_{L,L',\ell}$ given in Eq. (2.7) of [8] and $(-1)^{L+L'+\ell}\mathcal{S}^{\phi}_{L,\ell,L'}$ defined in Eq. (2.12) of [12].

where, with $\Delta^{(EE)}=2$ and $\Delta^{(EB)}=1$, we define

$$
\begin{aligned}
g^{(XY)}_{L,L',\ell} &= \frac{[f^{(XY)}_{L,L',\ell}]^*}{\Delta^{(XY)}\widehat{C}^{XX}_L\widehat{C}^{YY}_{L'}} \\
N^{(XY)}_\ell &= \frac{1}{2\ell+1}\sum_{L,L'} f^{(XY)}_{L,L',\ell}g^{(XY)}_{L,L',\ell}\,.
\end{aligned}
\tag{11}
$$

For the cosmic variance case, the estimated power spectrum reduces to the lensed power spectrum. Through this paper, the lensing reconstruction is performed using the optimal combination of the EE and EB quadratic estimators.

For EB-quadratic estimator, we ignore the B-mode power spectrum contribution in the weight function since it affects on negligible contributions to the lensing estimator [11]. Then we obtain

$$
g^{(EB)}_{L',L,\ell} = -\mathrm{i}\frac{\mathcal{S}^{(-)}_{L,L',\ell}}{\widehat{C}^{\rm BB}_L}\mathcal{W}^E_{L'}\,,
\tag{12}
$$

where we define $\mathcal{W}^E_{L'}=\widetilde{C}^{\rm EE}_{L'}/\widehat{C}^{\rm EE}_{L'}$ and

$$
\begin{aligned}
[\widehat{\phi}^{(EB)}_{\ell,m}]^* &= N^{(EB)}_\ell\sum_{L,L',M,M'}\begin{pmatrix} L & L' & \ell \\ M & M' & m\end{pmatrix}(g^{(EB)}_{L',L,\ell})^*\widehat{B}^{\rm obs}_{L,M}\widehat{E}^{\rm obs}_{L',M'} \\
&= \mathrm{i}N^{(EB)}_\ell\sum_{L,L',M,M'}\begin{pmatrix} L & L' & \ell \\ M & M' & m\end{pmatrix}\frac{\mathcal{S}^{(-)}_{L,L',\ell}\mathcal{W}^E_{L'}}{\widehat{C}^{\rm BB}_L}\widehat{B}^{\rm obs}_{L,M}\widehat{E}^{\rm obs}_{L',M'}\,,
\end{aligned}
\tag{13}
$$

where we use $(-1)^{L+L'+\ell}\mathcal{S}^{(-)}_{L,L',\ell}=-\mathcal{S}^{(-)}_{L,L',\ell}$.

1.2 Delensing

Once we obtain the lensing potential, based on Eq. (5), we can estimate the contribution of the lensing to the B-mode polarization [8]

$$
\begin{aligned}
\widehat{B}^{\rm lens}_{L,M} &= -\mathrm{i}\sum_{L',M'}\sum_{\ell,m}\begin{pmatrix} L & L' & \ell \\ M & M' & m\end{pmatrix}\mathcal{S}^{(-)}_{L,L',\ell}\mathcal{W}^E_{L'}\mathcal{W}^\phi_\ell(\widehat{E}^{\rm obs}_{L',M'}\widehat{\phi}_{\ell,m})^* \\
&= \sum_{L',M'}\sum_{\ell,m}\begin{pmatrix} L & L' & \ell \\ M & M' & m\end{pmatrix}h_{L',L,\ell}(\widehat{E}^{\rm obs}_{L',M'}\widehat{\phi}_{\ell,m})^*\,,
\end{aligned}
\tag{14}
$$

where $\mathcal{W}^\phi_\ell = C^{\phi\phi}_\ell/(C^{\phi\phi}_\ell+N^{(XY)}_\ell)$ is the winer-filtered multipoles of the estimated lensing potential [14, 8], and we define $h_{L',L,\ell}=-\mathrm{i}\mathcal{S}^{(-)}_{L,L',\ell}\mathcal{W}^E_{L'}\mathcal{W}^\phi_\ell$. The residual B-mode polarization is then simply given by

$$
\widehat{B}^{\rm res}_{L,M} = \widehat{B}^{\rm obs}_{L,M} - \widehat{B}^{\rm lens}_{L,M}\,.
\tag{15}
$$

2 Delensing bias

Here we derive explicit expressions for the delensing bias on the residual B-mode power spectrum and discuss the terms which have non-negligible contributions to the residual B-mode. Here we assume that the observed B-mode polarization from LiteBIRD, $\widehat{B}^{\rm LB}$, is sum of the primary B-mode $B_{L,M}$, lensed B-mode $\widetilde{B}_{L,M}$, and instrumental noise $n^B_{L,M}$. The lensing potential is estimated from a ground-based experiment, while the E and B modes used in the delensing are obtained from LiteBIRD. For this reason, we distinguish the noise of LiteBIRD from that of ground-based experiment by, e.g., $n^{\rm LB}_{L,M}$ and $n^{\rm G}_{L,M}$.

The LiteBIRD B-mode is described as

$$\langle|\widehat{B}^{\rm LB}_{L,M}|^2\rangle = C^{\rm BB}_L + \widetilde{C}^{\rm BB}_L + N^{BB,\rm LB}_L = C^{\rm BB,tot}_L + N^{BB,\rm LB}_L = \widehat{C}^{\rm BB,LB}_L \,. \tag{16}$$

The residual B-mode power spectrum given in Eq. (15) is decomposed into the following three terms:

$$\begin{aligned}\langle|\widehat{B}^{\rm res}_{L,M}|^2\rangle &= \langle(\widehat{B}^{\rm LB}_{L,M} - \widehat{B}^{\rm lens}_{L,M})(\widehat{B}^{\rm LB}_{L,M} - \widehat{B}^{\rm lens}_{L,M})^*\rangle \\ &= \langle|\widehat{B}^{\rm LB}_{L,M}|^2\rangle - 2\Re\langle\widehat{B}^{\rm lens}_{L,M}(\widehat{B}^{\rm LB}_{L,M})^*\rangle + \langle|\widehat{B}^{\rm lens}_{L,M}|^2\rangle \,. \end{aligned}\tag{17}$$

In the following calculations, we discuss expression for the second and third terms. Note that we frequently use the orthogonality relation [1]

$$\sum_{m_2,m_3}\begin{pmatrix}\ell_1 & \ell_2 & \ell_3\\ m_1 & m_2 & m_3\end{pmatrix}\begin{pmatrix}\ell_1 & \ell_2 & \ell_3'\\ m_1 & m_2 & m_3'\end{pmatrix} = \frac{\delta_{\ell_3\ell_3'}\delta_{m_3m_3'}}{2\ell_3+1}\,, \tag{18}$$

and the symmetric property of the Wigner 3-j symbols [1]:

$$\begin{pmatrix}\ell_1 & \ell_2 & \ell_3\\ m_1 & m_2 & m_3\end{pmatrix} = \begin{pmatrix}\ell_2 & \ell_3 & \ell_1\\ m_2 & m_3 & m_1\end{pmatrix}\,, \tag{19}$$

$$\begin{pmatrix}\ell_1 & \ell_2 & \ell_3\\ m_1 & m_2 & m_3\end{pmatrix} = (-1)^{\ell_1+\ell_2+\ell_3}\begin{pmatrix}\ell_2 & \ell_1 & \ell_3\\ m_2 & m_1 & m_3\end{pmatrix}\,. \tag{20}$$

2.1 EB quadratic estimator

Let us consider if we use the EB-estimator for the lensing reconstruction.

2.1.1 Cross correlation

With Eq. (14), the ensemble average involved in the second term of Eq. (43) becomes

$$\langle\widehat{B}^{\rm lens}_{L,M}(\widehat{B}^{\rm LB}_{L,M})^*\rangle = \sum_{\ell,L'}\mathcal{W}^\phi_\ell\frac{(f^{(EB)}_{L',L,\ell})^*}{\widehat{C}^{EE,\rm LB}_{L'}}\sum_{m,M'}\begin{pmatrix}L & L' & \ell\\ M & M' & m\end{pmatrix}\langle(\widehat{B}^{\rm LB}_{L,M}\widehat{E}^{\rm LB}_{L',M'}\widehat{\phi}_{\ell,m})^*\rangle\,. \tag{21}$$

Using the expression of the EB-estimator (13), the above equation becomes

$$\begin{aligned}\langle\widehat{B}^{\rm lens}_{L,M}(\widehat{B}^{\rm LB}_{L,M})^*\rangle &= \sum_{\ell,L'}\mathcal{W}^\phi_\ell N^{(EB),\rm G}_\ell\frac{(f^{(EB)}_{L',L,\ell})^*}{\widehat{C}^{EE,\rm LB}_{L'}}\sum_{m,M'}\begin{pmatrix}L & L' & \ell\\ M & M' & m\end{pmatrix}\\ &\times\sum_{L_1,L_1'}\sum_{M_1,M_1'}\begin{pmatrix}L_1 & L_1' & \ell\\ M_1 & M_1' & m\end{pmatrix}(g^{(EB),\rm G}_{L_1',L_1,\ell})^*\langle(\widehat{B}^{\rm LB}_{L,M}\widehat{E}^{\rm LB}_{L',M'})^*\widehat{B}^{\rm G}_{L_1,M_1}\widehat{E}^{\rm G}_{L_1',M_1'}\rangle\,.\end{aligned}\tag{22}$$

The above four-point correlation is decomposed into the disconnected and connected parts.

The disconnected part is given by

$$\langle(\widehat{B}^{\rm LB}_{L,M}\widehat{E}^{\rm LB}_{L',M'})^*\widehat{B}^{\rm G}_{L_1,M_1}\widehat{E}^{\rm G}_{L_1',M_1'}\rangle_{\rm d} = \delta_{L,L_1}\delta_{M,M_1}\delta_{L',L_1'}\delta_{M',M_1'}C^{\rm BB,tot}_L\widetilde{C}^{\rm EE}_{L'}\,, \tag{23}$$

where we assume that the noise fluctuations of LiteBIRD and a ground-based experiment are uncorrelated. Substituting the above equation into Eq. (45), we obtain

$$\begin{aligned}
N^{(c,0)}_L &\equiv \langle\widehat{B}^{\rm lens}_{L,M}(\widehat{B}^{\rm LB}_{L,M})^*\rangle_{\rm d} \\
&= \sum_{\ell,L'}\mathcal{W}^\phi_\ell N^{(EB),{\rm G}}_\ell\sum_{m,M'}\begin{pmatrix} L & L' & \ell \\ M & M' & m\end{pmatrix}\begin{pmatrix} L & L' & \ell \\ M & M' & m\end{pmatrix}(f^{(EB)}_{L',L,\ell}g^{(EB),{\rm G}}_{L',L,\ell})^*C^{\rm BB,tot}_L\mathcal{W}^{E,{\rm LB}}_{L'} \\
&= \frac{1}{2L+1}\sum_{\ell,L'}\mathcal{W}^\phi_\ell N^{(EB),{\rm G}}_\ell(f^{(EB)}_{L',L,\ell}g^{(EB),{\rm G}}_{L',L,\ell})^*C^{\rm BB,tot}_L\mathcal{W}^{E,{\rm LB}}_{L'} \\
&= \frac{\mathcal{W}^{\rm B,G}_L}{2L+1}\sum_{\ell,L'}(\mathcal{S}^{(-)}_{L,L',\ell})^2[\mathcal{W}^{\rm E,LB}_{L'}C^{\rm EE}_{L'}\mathcal{W}^{\rm E,G}_{L'}][\mathcal{W}^\phi_\ell N^{(EB),{\rm G}}_\ell] \\
&= \mathcal{W}^{\rm B,G}_L\Xi_L[\mathcal{W}^{\rm E,LB}C^{\rm EE}\mathcal{W}^{\rm E,G},\mathcal{W}^\phi N^{(EB),{\rm G}}]\,,
\end{aligned} \tag{24}$$

with $\mathcal{W}^{\rm B,G}_L = C^{\rm BB,tot}_L/\widehat{C}^{\rm BB,G}_L$ and define a convolution operator [2]

$$\Xi_L[A,B] = \frac{1}{2L+1}\sum_{L',\ell}(\mathcal{S}^{(-)}_{L,L',\ell})^2A_{L'}B_\ell \tag{25}$$

Note however that, if we do not use the B-mode polarization at $L < L_{\rm min}$ in the lensing reconstruction, the B-mode power spectrum in Eq. (47) becomes zero at $L < L_{\rm min}$ and this term vanishes in the residual B-mode power spectrum. In other words, this term generates the discontinuity in the residual B-mode power spectrum. This term cause the partial subtraction of primary B-mode signal [2], but the term exists even in the absence of primordial B-mode if we use the delensing procedure given in Eq. (15).

On the other hand, the connected part becomes (e.g., [3])

$$\begin{aligned}
\langle(\widehat{B}^{\rm LB}_{L,M}\widehat{E}^{\rm LB}_{L',M'})^*\widehat{B}^{\rm G}_{L_1,M_1}\widehat{E}^{\rm G}_{L_1',M_1'}\rangle_{\rm c} &\simeq \langle\langle(\widehat{B}^{\rm LB}_{L,M}\widehat{E}^{\rm LB}_{L',M'})^*\rangle_{\rm CMB}\langle\widehat{B}^{\rm G}_{L_1,M_1}\widehat{E}^{\rm G}_{L_1',M_1'}\rangle_{\rm CMB}\rangle \\
&\quad + (\text{other secondary contractions}) + \mathcal{O}(\phi^4) \\
&\simeq \sum_{\ell',m'}\begin{pmatrix} L & L' & \ell' \\ M & M' & m'\end{pmatrix}\begin{pmatrix} L_1 & L_1' & \ell' \\ M_1 & M_1' & m'\end{pmatrix}(f^{(EB)}_{L',L,\ell'})^*f^{(EB)}_{L_1',L_1,\ell'}C^{\phi\phi}_{\ell'}\,.
\end{aligned} \tag{26}$$

Here, in the last line, we ignore other secondary contractions [4] and the higher order terms

[2]If we choose $A_\ell = C^{\rm EE}_\ell$ and $B_\ell = C^{\phi\phi}_\ell$, the quantity $\Xi_L[A,B]$ coincides with the lensed B-mode power spectrum $\widetilde{C}^{\rm BB}_L$ but ignoring the higher order terms $\mathcal{O}[(C^{\phi\phi}_\ell)^2]$.

$\mathcal{O}(\phi^4)$ [5, 6, 7]. This leads to

$$\begin{aligned}\langle \widehat{B}^{\rm lens}_{L,M}(\widehat{B}^{\rm LB}_{L,M})^*\rangle &= \sum_{\ell,m}\mathcal{W}^{\phi}_{\ell}N^{(EB),{\rm G}}_{\ell}\sum_{L',M'}\begin{pmatrix} L & L' & \ell \\ M & M' & m\end{pmatrix}\frac{(f^{(EB)}_{L',L,\ell})^*}{\widehat{C}^{EE,{\rm LB}}_{L'}} \\ &\quad\times\frac{1}{2\ell+1}\sum_{L_1,L_1'}(g^{(EB),{\rm G}}_{L_1',L_1,\ell})^*\begin{pmatrix} L & L' & \ell \\ M & M' & m\end{pmatrix}(f^{(EB)}_{L',L,\ell})^* f^{(EB)}_{L_1',L_1,\ell}C^{\phi\phi}_{\ell} \\ &= \sum_{\ell,m}\mathcal{W}^{\phi}_{\ell}\sum_{L',M'}\begin{pmatrix} L & L' & \ell \\ M & M' & m\end{pmatrix}\begin{pmatrix} L & L' & \ell \\ M & M' & m\end{pmatrix}\frac{(f^{(EB)}_{L',L,\ell})^*}{\widehat{C}^{EE,{\rm LB}}_{L'}}f^{(EB)}_{L',L,\ell}C^{\phi\phi}_{\ell} \\ &= \Xi_L[\mathcal{W}^{E,{\rm LB}}C^{\rm EE},\mathcal{W}^{\phi}C^{\phi\phi}]\end{aligned} \tag{27}$$

Then, combining Eq. (47), we obtain

$$\langle \widehat{B}^{\rm lens}_{L,M}(\widehat{B}^{\rm LB}_{L,M})^*\rangle = N^{(c,0)}_L + \Xi_L[\mathcal{W}^{E,{\rm LB}}C^{\rm EE},\mathcal{W}^{\phi}C^{\phi\phi}]\,. \tag{28}$$

2.1.2 Auto correlation

Next we consider the third term of Eq. (43):

$$\begin{aligned}\langle|\widehat{B}^{\rm lens}_{L,M}|^2\rangle &= \sum_{L_1,M_1}\sum_{\ell_1,m_1}\sum_{L_2,M_2}\sum_{\ell_2,m_2}\begin{pmatrix} L & L_1 & \ell_1 \\ M & M_1 & m_1\end{pmatrix}\mathcal{S}^{(-)}_{L,L_1,\ell_1}\begin{pmatrix} L & L_2 & \ell_2 \\ M & M_2 & m_2\end{pmatrix}\mathcal{S}^{(-)}_{L,L_2,\ell_2} \\ &\quad\times\mathcal{W}^{E,{\rm LB}}_{L_1}\mathcal{W}^{E,{\rm LB}}_{L_2}\mathcal{W}^{\phi}_{\ell_1}\mathcal{W}^{\phi}_{\ell_2}\langle(\widehat{E}^{\rm LB}_{L_1,M_1}\widehat{\phi}_{\ell_1,m_1})^*\widehat{E}^{\rm LB}_{L_2,M_2}\widehat{\phi}_{\ell_2,m_2}\rangle \\ &= \sum_{L_1,M_1}\sum_{\ell_1,m_1}\sum_{L_2,M_2}\sum_{\ell_2,m_2}\begin{pmatrix} L & L_1 & \ell_1 \\ M & M_1 & m_1\end{pmatrix}\mathcal{S}^{(-)}_{L,L_1,\ell_1}\begin{pmatrix} L & L_2 & \ell_2 \\ M & M_2 & m_2\end{pmatrix}\mathcal{S}^{(-)}_{L,L_2,\ell_2} \\ &\quad\times\mathcal{W}^{E,{\rm LB}}_{L_1}\mathcal{W}^{E,{\rm LB}}_{L_2}\mathcal{W}^{\phi}_{\ell_1}\mathcal{W}^{\phi}_{\ell_2}N^{(EB)}_{\ell_1}N^{(EB)}_{\ell_2}\sum_{L_1',M_1'}\sum_{L_1'',M_1''}\sum_{L_2',M_2'}\sum_{L_2'',M_2''} \\ &\quad\times\begin{pmatrix} L_1' & L_1'' & \ell_1 \\ M_1' & M_1'' & m_1\end{pmatrix}\begin{pmatrix} L_2' & L_2'' & \ell_2 \\ M_2' & M_2'' & m_2\end{pmatrix}(g^{(EB)}_{L_1'',L_1',\ell_1})^* g^{(EB)}_{L_2'',L_2',\ell_2} \\ &\quad\times\langle(\widehat{E}^{\rm LB}_{L_1,M_1})^*\widehat{B}^{\rm G}_{L_1',M_1'}\widehat{E}^{\rm G}_{L_1'',M_1''}\widehat{E}^{\rm LB}_{L_2,M_2}(\widehat{B}^{\rm G}_{L_2',M_2'}\widehat{E}^{\rm G}_{L_2'',M_2''})^*\rangle\,.\end{aligned} \tag{29}$$

Since the connected part of six-point correlation involved in the above equation is at least the 2nd order of $C^{\phi\phi}_{\ell}$, we only consider the disconnected part.

The most significant contributions in Eq. (51) would come from

$$\begin{aligned}\langle|\widehat{B}^{\rm lens}_{L,M}|^2\rangle &\simeq \sum_{L_1,M_1}\sum_{\ell_1,m_1}\sum_{L_2,M_2}\sum_{\ell_2,m_2}\begin{pmatrix} L & L_1 & \ell_1 \\ M & M_1 & m_1\end{pmatrix}\mathcal{S}^{(-)}_{L,L_1,\ell_1}\begin{pmatrix} L & L_2 & \ell_2 \\ M & M_2 & m_2\end{pmatrix}\mathcal{S}^{(-)}_{L,L_2,\ell_2} \\ &\quad\times\mathcal{W}^{E,{\rm LB}}_{L_1}\mathcal{W}^{E,{\rm LB}}_{L_2}\mathcal{W}^{\phi}_{\ell_1}\mathcal{W}^{\phi}_{\ell_2}\langle(\widehat{E}^{\rm LB}_{L_1,M_1}\widehat{E}^{\rm LB}_{L_2,M_2}\rangle\langle(\widehat{\phi}_{\ell_1,m_1})^*\widehat{\phi}_{\ell_2,m_2}\rangle \\ &\simeq \sum_{L_1,M_1}\sum_{\ell_1,m_1}\begin{pmatrix} L & L_1 & \ell_1 \\ M & M_1 & m_1\end{pmatrix}\begin{pmatrix} L & L_1 & \ell_1 \\ M & M_1 & m_1\end{pmatrix}(\mathcal{S}^{(-)}_{L,L_1,\ell_1})^2(\mathcal{W}^{E,{\rm LB}}_{L_1}\mathcal{W}^{\phi}_{\ell_1})^2\widehat{C}^{\rm EE,LB}_{L_1}(N^{(EB)}_{\ell_1}+C^{\phi\phi}_{\ell_1}) \\ &= \frac{1}{2L+1}\sum_{L_1,\ell_1}(\mathcal{S}^{(-)}_{L,L_1,\ell_1})^2\mathcal{W}^{E,{\rm LB}}_{L_1}\mathcal{W}^{\phi}_{\ell_1}C^{\rm EE}_{L_1}C^{\phi\phi}_{\ell_1} \\ &= \Xi_L[\mathcal{W}^{E,{\rm LB}}C^{\rm EE},\mathcal{W}^{\phi}C^{\phi\phi}]\,.\end{aligned} \tag{30}$$

where, from the first to the last equation, we ignore the higher order biases [4, 5, 6, 7].

There are other terms which must be taken into account. As an example of these terms, we here consider the disconnected part at zeroth order of $C_\ell^{\phi\phi}$:

$$\begin{aligned}
&\langle(\widehat{E}^{\rm LB}_{L_1,M_1})^*\widehat{B}^{\rm G}_{L_1',M_1'}\widehat{E}^{\rm G}_{L_1'',M_1''}\widehat{E}^{\rm LB}_{L_2,M_2}(\widehat{B}^{\rm G}_{L_2',M_2'}\widehat{E}^{\rm G}_{L_2'',M_2''})^*\rangle_{\phi=0} \\
&= \delta_{L_1',L_2'}\delta_{M_1',M_2'}\widehat{C}^{\rm BB,G}_{L_1'}\bigg(\delta_{L_1,L_1''}\delta_{M_1,M_1''}\delta_{L_2,L_2''}\delta_{M_2,M_2''}\widetilde{C}^{\rm EE}_{L_1}\widetilde{C}^{\rm EE}_{L_2} \\
&\quad + \delta_{L_1,L_2}\delta_{M_1,M_2}\delta_{L_1'',L_2''}\delta_{M_1'',M_2''}\widehat{C}^{\rm EE,LB}_{L_1}\widehat{C}^{\rm EE,G}_{L_1''} \\
&\quad + \delta_{L_1,L_2''}\delta_{M_1,M_2''}\delta_{L_1'',L_2}\delta_{M_1'',M_2}\widetilde{C}^{\rm EE}_{L_1}\widetilde{C}^{\rm EE}_{L_1''}(-1)^{M_1+M_1''}\bigg) \,. \qquad (31)
\end{aligned}$$

Note that the second term is involved in Eq. (52). The first term in the above equation leads to the discontinuity in the residual B-mode power spectrum. To see this, substituting the first term into Eq. (51), we obtain,

$$\begin{aligned}
N_L^{(\rm a,0)} &\equiv \sum_{L_1,M_1}\sum_{\ell_1,m_1}\sum_{L_2,M_2}\sum_{\ell_2,m_2}\begin{pmatrix} L & L_1 & \ell_1 \\ M & M_1 & m_1\end{pmatrix}\mathcal{S}^{(-)}_{L,L_1,\ell_1}\begin{pmatrix} L & L_2 & \ell_2 \\ M & M_2 & m_2\end{pmatrix}\mathcal{S}^{(-)}_{L,L_2,\ell_2} \\
&\quad\times \mathcal{W}^{E,\rm LB}_{L_1}\mathcal{W}^{E,\rm LB}_{L_2}\mathcal{W}^{\phi}_{\ell_1}\mathcal{W}^{\phi}_{\ell_2}N^{(EB)}_{\ell_1}N^{(EB)}_{\ell_2}\sum_{L_1',M_1'}\sum_{L_1'',M_1''}\sum_{L_2',M_2'}\sum_{L_2'',M_2''} \\
&\quad\times \begin{pmatrix} L_1' & L_1'' & \ell_1 \\ M_1' & M_1'' & m_1\end{pmatrix}\begin{pmatrix} L_2' & L_2'' & \ell_2 \\ M_2' & M_2'' & m_2\end{pmatrix}(g^{(EB)}_{L_1'',L_1',\ell_1})^*g^{(EB)}_{L_2'',L_2',\ell_2} \\
&\quad\times \delta_{L_1',L_2'}\delta_{M_1',M_2'}\delta_{L_1,L_1''}\delta_{M_1,M_1''}\delta_{L_2,L_2''}\delta_{M_2,M_2''}\widehat{C}^{\rm BB,G}_{L_1'}\widetilde{C}^{\rm EE}_{L_1}\widetilde{C}^{\rm EE}_{L_2} \\
&= \sum_{L_1,M_1}\sum_{\ell_1,m_1}\sum_{L_2,M_2}\sum_{\ell_2,m_2}\sum_{L_1',M_1'}\begin{pmatrix} L & L_1 & \ell_1 \\ M & M_1 & m_1\end{pmatrix}\mathcal{S}^{(-)}_{L,L_1,\ell_1}\begin{pmatrix} L & L_2 & \ell_2 \\ M & M_2 & m_2\end{pmatrix}\mathcal{S}^{(-)}_{L,L_2,\ell_2} \\
&\quad\times \mathcal{W}^{E,\rm LB}_{L_1}\mathcal{W}^{E,\rm LB}_{L_2}\mathcal{W}^{\phi}_{\ell_1}\mathcal{W}^{\phi}_{\ell_2}N^{(EB)}_{\ell_1}N^{(EB)}_{\ell_2} \\
&\quad\times \begin{pmatrix} L_1' & L_1 & \ell_1 \\ M_1' & M_1 & m_1\end{pmatrix}\begin{pmatrix} L_1' & L_2 & \ell_2 \\ M_1' & M_2 & m_2\end{pmatrix}(g^{(EB)}_{L_1,L_1',\ell_1})^*g^{(EB)}_{L_2,L_1',\ell_2}\widehat{C}^{\rm BB,G}_{L_1'}\widetilde{C}^{\rm EE}_{L_1}\widetilde{C}^{\rm EE}_{L_2} \,. \qquad (32)
\end{aligned}$$

Using the orthogonality relation, we find

$$\begin{aligned}
N_L^{(\rm a,0)} &= \frac{\widehat{C}^{\rm BB,G}_L}{(2L+1)^2}\sum_{L_1,\ell_1}\sum_{L_2,\ell_2}\mathcal{S}^{(-)}_{L,L_1,\ell_1}\mathcal{S}^{(-)}_{L,L_2,\ell_2} \\
&\quad\times \mathcal{W}^{E,\rm LB}_{L_1}\mathcal{W}^{E,\rm LB}_{L_2}\mathcal{W}^{\phi}_{\ell_1}\mathcal{W}^{\phi}_{\ell_2}N^{(EB)}_{\ell_1}N^{(EB)}_{\ell_2}(g^{(EB)}_{L_1,L,\ell_1})^*g^{(EB)}_{L_2,L,\ell_2}\widetilde{C}^{\rm EE}_{L_1}\widetilde{C}^{\rm EE}_{L_2} \\
&= \widehat{C}^{\rm BB,G}_L\left|\frac{1}{2L+1}\sum_{L_1,\ell_1}\mathcal{S}^{(-)}_{L,L_1,\ell_1}\mathcal{W}^{E,\rm LB}_{L_1}\mathcal{W}^{\phi}_{\ell_1}N^{(EB)}_{\ell_1}g^{(EB)}_{L_1,L,\ell_1}\widetilde{C}^{\rm EE}_{L_1}\right|^2 \\
&= \frac{1}{\widehat{C}^{\rm BB,G}_L}\left|\Xi_L[\mathcal{W}^{E,\rm LB}\mathcal{W}^{E,\rm G}\widetilde{C}^{\rm EE},\mathcal{W}^{\phi}N^{(EB)}]\right|^2 \,. \qquad (33)
\end{aligned}$$

The above quantity vanishes if we do not use the B-mode polarization at $L < L_{\rm min}$ since, from the second to third equality, we omit $\delta_{L,L_1'}$.

At first order of $C_\ell^{\phi\phi}$, there are also additional terms which have non-negligible contributions. To see this, we consider the following six-point correlation which is involved in Eq. (51):

$$\begin{aligned}
&\langle (\widehat{E}^{\rm LB}_{L_1,M_1})^*\widehat{E}^{\rm G}_{L_1'',M_1''}\rangle\langle\langle \widehat{B}^{\rm G}_{L_1',M_1'}\widehat{E}^{\rm LB}_{L_2,M_2}\rangle_{\rm CMB}\langle(\widehat{B}^{\rm G}_{L_2',M_2'}\widehat{E}^{\rm G}_{L_2'',M_2''})^*\rangle_{\rm CMB}\rangle \\
&= \delta_{L_1,L_1''}\delta_{M_1,M_1''}\widetilde{C}^{\rm EE}_{L_1}\sum_{\ell',m'}\begin{pmatrix} L_1' & L_2 & \ell' \\ M_1' & M_2 & m'\end{pmatrix}\begin{pmatrix} L_2' & L_2'' & \ell' \\ M_2' & M_2'' & m'\end{pmatrix}(f^{(EB)}_{L_2,L_1',\ell'})^* f^{(EB)}_{L_2'',L_2',\ell'}C^{\phi\phi}_{\ell'} \,. \qquad (34)
\end{aligned}$$

Substituting the above equation into Eq. (51), we obtain

$$\begin{aligned}
N^{(\rm a,1)}_L &= 2\sum_{L_1,M_1}\sum_{\ell_1,m_1}\sum_{L_2,M_2}\sum_{\ell_2,m_2}\begin{pmatrix} L & L_1 & \ell_1 \\ M & M_1 & m_1\end{pmatrix}\mathcal{S}^{(-)}_{L,L_1,\ell_1}\begin{pmatrix} L & L_2 & \ell_2 \\ M & M_2 & m_2\end{pmatrix}\mathcal{S}^{(-)}_{L,L_2,\ell_2} \\
&\quad\times \mathcal{W}^{E,\rm LB}_{L_1}\mathcal{W}^{E,\rm LB}_{L_2}\mathcal{W}^{\phi}_{\ell_1}\mathcal{W}^{\phi}_{\ell_2}N^{(EB)}_{\ell_1}N^{(EB)}_{\ell_2}\sum_{L_1',M_1'}\sum_{L_1'',M_1''}\sum_{L_2',M_2'}\sum_{L_2'',M_2''} \\
&\quad\times \begin{pmatrix} L_1' & L_1'' & \ell_1 \\ M_1' & M_1'' & m_1\end{pmatrix}\begin{pmatrix} L_2' & L_2'' & \ell_2 \\ M_2' & M_2'' & m_2\end{pmatrix}(g^{(EB)}_{L_1'',L_1',\ell_1})^* g^{(EB)}_{L_2'',L_2',\ell_2} \\
&\quad\times \delta_{L_1,L_1''}\delta_{M_1,M_1''}\widetilde{C}^{\rm EE}_{L_1}\sum_{\ell',m'}\begin{pmatrix} L_1' & L_2 & \ell' \\ M_1' & M_2 & m'\end{pmatrix}\begin{pmatrix} L_2' & L_2'' & \ell' \\ M_2' & M_2'' & m'\end{pmatrix}(f^{(EB)}_{L_2,L_1',\ell'})^* f^{(EB)}_{L_2'',L_2',\ell'}C^{\phi\phi}_{\ell'} \\
&= 2\sum_{L_1,M_1}\sum_{\ell_1,m_1}\sum_{L_2,M_2}\sum_{\ell_2,m_2}\sum_{L_1',M_1'}\sum_{L_2',M_2'}\sum_{L_2'',M_2''}\sum_{\ell',m'}\begin{pmatrix} L & L_1 & \ell_1 \\ M & M_1 & m_1\end{pmatrix}\begin{pmatrix} L_1' & L_1 & \ell_1 \\ M_1' & M_1 & m_1\end{pmatrix} \\
&\quad\times \begin{pmatrix} L & L_2 & \ell_2 \\ M & M_2 & m_2\end{pmatrix}\begin{pmatrix} L_1' & L_2 & \ell' \\ M_1' & M_2 & m'\end{pmatrix}\begin{pmatrix} L_2' & L_2'' & \ell_2 \\ M_2' & M_2'' & m_2\end{pmatrix}\begin{pmatrix} L_2' & L_2'' & \ell' \\ M_2' & M_2'' & m'\end{pmatrix}\mathcal{S}^{(-)}_{L,L_1,\ell_1}\mathcal{S}^{(-)}_{L,L_2,\ell_2} \\
&\quad\times \mathcal{W}^{E,\rm LB}_{L_1}\mathcal{W}^{E,\rm LB}_{L_2}\mathcal{W}^{\phi}_{\ell_1}\mathcal{W}^{\phi}_{\ell_2}N^{(EB)}_{\ell_1}N^{(EB)}_{\ell_2}\widetilde{C}^{\rm EE}_{L_1}C^{\phi\phi}_{\ell'}(f^{(EB)}_{L_2,L_1',\ell'}g^{(EB)}_{L_1,L_1',\ell_1})^* f^{(EB)}_{L_2'',L_2',\ell'}g^{(EB)}_{L_2'',L_2',\ell_2} \,. \qquad (35)
\end{aligned}$$

In the above equation, we multiply the factor $1/2$. This is because the same equation is obtained from counterpart of Eq. (58) (i.e., the term obtained by exchanging $L_1\leftrightarrow L_2$ and $L_1''\leftrightarrow L_2''$ in Eq. (58)). The total contribution from Eq. (58) and its counterpart therefore becomes $N^{(\rm a,1)}_L$. Using the orthogonality relation of the Wigner 3-j symbols, we find

$$\begin{aligned}
N^{(\rm a,1)}_L &= \frac{2}{(2L+1)^2}\sum_{L_1,\ell_1}\sum_{L_2,\ell_2}\sum_{L_2',L_2''}\mathcal{S}^{(-)}_{L,L_1,\ell_1}\mathcal{S}^{(-)}_{L,L_2,\ell_2}\mathcal{W}^{E,\rm LB}_{L_1}\mathcal{W}^{E,\rm LB}_{L_2}\mathcal{W}^{\phi}_{\ell_1}\mathcal{W}^{\phi}_{\ell_2}N^{(EB)}_{\ell_1}N^{(EB)}_{\ell_2}\widetilde{C}^{\rm EE}_{L_1}C^{\phi\phi}_{\ell_2} \\
&\quad\times\frac{1}{2\ell_2+1}(f^{(EB)}_{L_2,L,\ell_2}g^{(EB)}_{L_1,L,\ell_1})^* f^{(EB)}_{L_2'',L_2',\ell_2}g^{(EB)}_{L_2'',L_2',\ell_2} \\
&= \frac{2}{(2L+1)^2\widehat{C}^{\rm BB,G}_L}\sum_{L_1,\ell_1}\sum_{L_2,\ell_2}[\mathcal{S}^{(-)}_{L,L_1,\ell_1}\mathcal{S}^{(-)}_{L,L_2,\ell_2}]^2\mathcal{W}^{E,\rm LB}_{L_1}\mathcal{W}^{E,\rm LB}_{L_2}\mathcal{W}^{\phi}_{\ell_1}\mathcal{W}^{\phi}_{\ell_2}N^{(EB)}_{\ell_1}C^{\phi\phi}_{\ell_2}C^{\rm EE}_{L_2}\mathcal{W}^{E,\rm G}_{L_1}\widetilde{C}^{\rm EE}_{L_1} \\
&= \frac{2}{\widehat{C}^{\rm BB,G}_L}\Xi_L[\mathcal{W}^{E,\rm LB}\mathcal{W}^{E,\rm G}\widetilde{C}^{\rm EE},\mathcal{W}^{\phi}N^{(EB)}]\Xi_L[\mathcal{W}^{E,\rm LB}C^{\rm EE},\mathcal{W}^{\phi}C^{\phi\phi}] \,. \qquad (36)
\end{aligned}$$

Similar to $N^{(\rm a,0)}_L$, the above quantity vanishes if $L<L_{\rm min}$.

2.1.3 Total

By combining Eqs. (50), (52), (55), and (58), we obtain

$$
\begin{aligned}
C_L^{\mathrm{BB,res}} &= \widehat{C}_L^{\mathrm{BB,LB}} - 2N_L^{(\mathrm{c},0)} + N_L^{(\mathrm{a},0)} + N_L^{(\mathrm{a},1)} - \frac{1}{2L+1}\sum_{\ell,L'}(\mathcal{S}^{(-)}_{L,L',\ell})^2\mathcal{W}^{E,LB}_{L'}\mathcal{W}^{\phi,G}_{\ell}C^{EE}_{L'}C^{\phi\phi}_{\ell} \\
&\simeq C_L^{\mathrm{BB}} + N_L^{\mathrm{BB,LB}} - 2N_L^{(\mathrm{c},0)} + N_L^{(\mathrm{a},0)} + N_L^{(\mathrm{a},1)} \\
&\quad + \frac{1}{2L+1}\sum_{\ell,L'}(\mathcal{S}^{(-)}_{L,L',\ell})^2(1-\mathcal{W}^{E,LB}_{L'}\mathcal{W}^{\phi,G}_{\ell})C^{EE}_{L'}C^{\phi\phi}_{\ell}\,,
\end{aligned} \tag{37}
$$

where, from the first to second line, we ignore $\mathcal{O}[(C^{\phi\phi}_\ell)^2]$ in the lensing B-mode power spectrum $\widetilde{C}_L^{\mathrm{BB}}$ involved in $\widehat{C}_L^{\mathrm{BB,LB}}$. If we ignore bias terms, $N_L^{(\mathrm{c},0)}$, $N_L^{(\mathrm{a},0)}$ and $N_L^{(\mathrm{a},1)}$, we obtain the expression for the residual B-mode power spectrum obtained in Ref. [8].

2.2 EE quadratic estimator

Next we consider the case with EE-quadratic estimator for lensing reconstruction. The corresponding equation of Eq. (45) is

$$
\begin{aligned}
\langle \widehat{B}^{\mathrm{lens}}_{L,M}(\widehat{B}^{\mathrm{LB}}_{L,M})^*\rangle &= -\mathrm{i}\sum_{L',M'}\sum_{\ell,m}\begin{pmatrix} L & L' & \ell \\ M & M' & m\end{pmatrix}\mathcal{S}^{(-)}_{L,L',\ell}\mathcal{W}^{E}_{L'}\mathcal{W}^{\phi}_{\ell} \\
&\quad \times N^{(EE)}_{\ell}\sum_{L_1,L_2}\sum_{M_1,M_2}\begin{pmatrix} L_1 & L_2 & \ell \\ M_1 & M_2 & m\end{pmatrix} g^{(EE)}_{L_1,L_2,\ell}\langle(\widehat{E}^{\mathrm{LB}}_{L',M'})^*\widehat{E}^{\mathrm{G}}_{L_1,M_1}\widehat{E}^{\mathrm{G}}_{L_2,M_2}(\widehat{B}^{\mathrm{LB}}_{L,M})^*\rangle\,.
\end{aligned} \tag{38}
$$

Contrary to the EB-estimator, the disconnected part of the four-point correlation vanishes and the connected part is given by

$$
\langle \widehat{E}^{\mathrm{G}}_{L_1,M_1}\widehat{E}^{\mathrm{G}}_{L_2,M_2}(\widehat{E}^{\mathrm{LB}}_{L',M'}\widehat{B}^{\mathrm{obs}}_{L,M})^*\rangle_{\mathrm{c}} \simeq \langle\langle \widehat{E}^{\mathrm{G}}_{L_1,M_1}\widehat{E}^{\mathrm{G}}_{L_2,M_2}\rangle_{\mathrm{CMB}}\langle(\widehat{E}^{\mathrm{LB}}_{L',M'}\widehat{B}^{\mathrm{LB}}_{L,M})^*\rangle_{\mathrm{CMB}}\rangle\,. \tag{39}
$$

If we also assume that the third term of Eq. (43) is given by

$$
\langle\widehat{\phi}_{\ell_1,m_1}\widehat{E}^{\mathrm{LB}}_{L'_1,M'_1}(\widehat{\phi}_{\ell_2,m_2}\widehat{E}^{\mathrm{LB}}_{L'_2,M'_2})^*\rangle \simeq \delta_{\ell_1,\ell_2}\delta_{m_1,m_2}\delta_{L'_1,L'_2}\delta_{M'_1,M'_2}(N^{EE}_{\ell_1}+C^{\phi\phi}_{\ell_1})\widehat{C}^{\mathrm{EE,LB}}_{L'_1}\,, \tag{40}
$$

we obtain the same expression given in [8]:

$$
C_L^{\mathrm{BB,res}} = C_L^{\mathrm{BB}} + N_L^{\mathrm{BB,LB}} + \frac{1}{2L+1}\sum_{\ell,L'}(\mathcal{S}^{(-)}_{L,L',\ell})^2(1-\mathcal{W}^{E,LB}_{L'}\mathcal{W}^{\phi,G}_{\ell})C^{EE}_{L'}C^{\phi\phi}_{\ell}\,, \tag{41}
$$

3 Delensing with single experiment

The observed B-mode is described as

$$
\langle|\widehat{B}_{L,M}|^2\rangle = C_L^{\mathrm{BB}} + \widetilde{C}_L^{\mathrm{BB}} + N_L^{BB} = C_L^{\mathrm{BB,tot}} + N_L^{BB} = \widehat{C}_L^{\mathrm{BB}}\,. \tag{42}
$$

The residual B-mode power spectrum given in Eq. (15) is decomposed into the following three terms:

$$
\begin{aligned}
\langle|\widehat{B}^{\mathrm{res}}_{L,M}|^2\rangle &= \langle(\widehat{B}_{L,M}-\widehat{B}^{\mathrm{lens}}_{L,M})(\widehat{B}_{L,M}-\widehat{B}^{\mathrm{lens}}_{L,M})^*\rangle \\
&= \langle|\widehat{B}_{L,M}|^2\rangle - 2\Re\langle\widehat{B}^{\mathrm{lens}}_{L,M}(\widehat{B}_{L,M})^*\rangle + \langle|\widehat{B}^{\mathrm{lens}}_{L,M}|^2\rangle\,.
\end{aligned} \tag{43}
$$

3.1 Delensing bias

3.1.1 Cross correlation

With Eq. (14), the ensemble average involved in the second term of Eq. (43) becomes

$$\langle \widehat{B}^{\rm lens}_{L,M}(\widehat{B}_{L,M})^*\rangle = \sum_{\ell,L'} \mathcal{W}^\phi_\ell \frac{(f^{(EB)}_{L',L,\ell})^*}{\widehat{C}^{\rm EE}_{L'}} \sum_{m,M'} \begin{pmatrix} L & L' & \ell \\ M & M' & m \end{pmatrix} \langle (\widehat{B}_{L,M}\widehat{E}_{L',M'}\widehat{\phi}_{\ell,m})^*\rangle \,. \tag{44}$$

Using the expression of the EB-estimator (13), the above equation becomes

$$\langle \widehat{B}^{\rm lens}_{L,M}(\widehat{B}_{L,M})^*\rangle = \sum_{\ell,L'} \mathcal{W}^\phi_\ell N_\ell \frac{(f_{L',L,\ell})^*}{\widehat{C}^{\rm EE}_{L'}} \sum_{m,M'} \begin{pmatrix} L & L' & \ell \\ M & M' & m \end{pmatrix}$$
$$\times \sum_{L_1,L'_1} \sum_{M_1,M'_1} \begin{pmatrix} L_1 & L'_1 & \ell \\ M_1 & M'_1 & m \end{pmatrix} (g_{L'_1,L_1,\ell})^* \langle (\widehat{B}_{L,M}\widehat{E}_{L',M'})^* \widehat{B}_{L_1,M_1}\widehat{E}_{L'_1,M'_1}\rangle \,. \tag{45}$$

The above four-point correlation is decomposed into the disconnected and connected parts.

The disconnected part is given by

$$\langle (\widehat{B}_{L,M}\widehat{E}_{L',M'})^* \widehat{B}_{L_1,M_1}\widehat{E}_{L'_1,M'_1}\rangle_{\rm d} = \delta_{L,L_1}\delta_{M,M_1}\delta_{L',L'_1}\delta_{M',M'_1}\widehat{C}^{\rm BB}_L \widehat{C}^{\rm EE}_{L'} \,. \tag{46}$$

Substituting the above equation into Eq. (45), we obtain

$$\begin{aligned}
\langle \widehat{B}^{\rm lens}_{L,M}(\widehat{B}_{L,M})^*\rangle_{\rm d} &= \sum_{\ell,L'} \mathcal{W}^\phi_\ell N_\ell \sum_{m,M'} \begin{pmatrix} L & L' & \ell \\ M & M' & m \end{pmatrix} \begin{pmatrix} L & L' & \ell \\ M & M' & m \end{pmatrix} (f_{L',L,\ell}g_{L',L,\ell})^* \widehat{C}^{\rm BB}_L \\
&= \frac{1}{2L+1} \sum_{\ell,L'} \mathcal{W}^\phi_\ell N_\ell (f_{L',L,\ell}g_{L',L,\ell})^* \widehat{C}^{\rm BB}_L \\
&= \frac{1}{2L+1} \sum_{\ell,L'} (\mathcal{S}^{(-)}_{L,L',\ell})^2 [\mathcal{W}^{\rm E}_{L'} C^{\rm EE}_{L'}][\mathcal{W}^\phi_\ell N_\ell] \\
&= \left(\frac{1}{\widehat{C}^{\rm BB}_L} \Xi_L[\mathcal{W}^{\rm E}C^{\rm EE}, \mathcal{W}^\phi N] \right) \widehat{C}^{\rm BB}_L \\
&\equiv A_L \widehat{C}^{\rm BB}_L \,.
\end{aligned} \tag{47}$$

On the other hand, the connected part becomes (e.g., [3])

$$\langle (\widehat{B}_{L,M}\widehat{E}_{L',M'})^* \widehat{B}_{L_1,M_1}\widehat{E}_{L'_1,M'_1}\rangle_{\rm c} \simeq \sum_{\ell',m'} \begin{pmatrix} L & L' & \ell' \\ M & M' & m' \end{pmatrix} \begin{pmatrix} L_1 & L'_1 & \ell' \\ M_1 & M'_1 & m' \end{pmatrix} (f_{L',L,\ell'})^* f_{L'_1,L_1,\ell'} C^{\phi\phi}_{\ell'} \,. \tag{48}$$

This leads to

$$\begin{aligned}
C_L^{\mathrm{BB,W}} &\equiv \langle \widehat{B}^{\mathrm{lens}}_{L,M}(\widehat{B}_{L,M})^*\rangle \\
&= \sum_{\ell,m}\mathcal{W}^\phi_\ell N_\ell \sum_{L',M'}\begin{pmatrix} L & L' & \ell \\ M & M' & m\end{pmatrix}\frac{(f_{L',L,\ell})^*}{\widehat{C}^{\mathrm{EE}}_{L'}} \\
&\quad\times\frac{1}{2\ell+1}\sum_{L_1,L_1'}(g_{L_1',L_1,\ell})^*\begin{pmatrix} L & L' & \ell \\ M & M' & m\end{pmatrix}(f_{L',L,\ell})^* f_{L_1',L_1,\ell}C^{\phi\phi}_\ell \\
&= \sum_{\ell,m}\mathcal{W}^\phi_\ell \sum_{L',M'}\begin{pmatrix} L & L' & \ell \\ M & M' & m\end{pmatrix}\begin{pmatrix} L & L' & \ell \\ M & M' & m\end{pmatrix}\frac{(f_{L',L,\ell})^*}{\widehat{C}^{\mathrm{EE}}_{L'}}f_{L',L,\ell}C^{\phi\phi}_\ell \\
&= \Xi_L[\mathcal{W}^E C^{\mathrm{EE}},\mathcal{W}^\phi C^{\phi\phi}]\,.
\end{aligned}\tag{49}$$

Then, combining Eq. (47), we obtain

$$\langle \widehat{B}^{\mathrm{lens}}_{L,M}(\widehat{B}^{\mathrm{LB}}_{L,M})^*\rangle = A_L\widehat{C}^{\mathrm{BB}}_L + C_L^{\mathrm{BB,W}}\,. \tag{50}$$

3.1.2 Auto correlation

Next we consider the third term of Eq. (43):

$$\begin{aligned}
\langle|\widehat{B}^{\mathrm{lens}}_{L,M}|^2\rangle &= \sum_{L_1,M_1}\sum_{\ell_1,m_1}\sum_{L_2,M_2}\sum_{\ell_2,m_2}\begin{pmatrix} L & L_1 & \ell_1 \\ M & M_1 & m_1\end{pmatrix}\mathcal{S}^{(-)}_{L,L_1,\ell_1}\begin{pmatrix} L & L_2 & \ell_2 \\ M & M_2 & m_2\end{pmatrix}\mathcal{S}^{(-)}_{L,L_2,\ell_2} \\
&\quad\times\mathcal{W}^E_{L_1}\mathcal{W}^E_{L_2}\mathcal{W}^\phi_{\ell_1}\mathcal{W}^\phi_{\ell_2}\langle(\widehat{E}_{L_1,M_1}\widehat{\phi}_{\ell_1,m_1})^*\widehat{E}_{L_2,M_2}\widehat{\phi}_{\ell_2,m_2}\rangle \\
&= \sum_{L_1,M_1}\sum_{\ell_1,m_1}\sum_{L_2,M_2}\sum_{\ell_2,m_2}\begin{pmatrix} L & L_1 & \ell_1 \\ M & M_1 & m_1\end{pmatrix}\mathcal{S}^{(-)}_{L,L_1,\ell_1}\begin{pmatrix} L & L_2 & \ell_2 \\ M & M_2 & m_2\end{pmatrix}\mathcal{S}^{(-)}_{L,L_2,\ell_2} \\
&\quad\times\mathcal{W}^E_{L_1}\mathcal{W}^E_{L_2}\mathcal{W}^\phi_{\ell_1}\mathcal{W}^\phi_{\ell_2}N_{\ell_1}N_{\ell_2}\sum_{L_1',M_1'}\sum_{L_1'',M_1''}\sum_{L_2',M_2'}\sum_{L_2'',M_2''} \\
&\quad\times\begin{pmatrix} L_1' & L_1'' & \ell_1 \\ M_1' & M_1'' & m_1\end{pmatrix}\begin{pmatrix} L_2' & L_2'' & \ell_2 \\ M_2' & M_2'' & m_2\end{pmatrix}(g_{L_1'',L_1',\ell_1})^* g_{L_2'',L_2',\ell_2} \\
&\quad\times\langle(\widehat{E}_{L_1,M_1})^*\widehat{B}_{L_1',M_1'}\widehat{E}_{L_1'',M_1''}\widehat{E}_{L_2,M_2}(\widehat{B}_{L_2',M_2'}\widehat{E}_{L_2'',M_2''})^*\rangle\,.
\end{aligned}\tag{51}$$

Since the connected part of six-point correlation involved in the above equation is at least the 2nd order of $C^{\phi\phi}_\ell$, we only consider the disconnected part.

The most significant contributions in Eq. (51) would come from

$$\begin{aligned}
\langle|\widehat{B}^{\mathrm{lens}}_{L,M}|^2\rangle &\simeq \sum_{L_1,M_1}\sum_{\ell_1,m_1}\sum_{L_2,M_2}\sum_{\ell_2,m_2}\begin{pmatrix} L & L_1 & \ell_1 \\ M & M_1 & m_1\end{pmatrix}\mathcal{S}^{(-)}_{L,L_1,\ell_1}\begin{pmatrix} L & L_2 & \ell_2 \\ M & M_2 & m_2\end{pmatrix}\mathcal{S}^{(-)}_{L,L_2,\ell_2} \\
&\quad\times\mathcal{W}^E_{L_1}\mathcal{W}^E_{L_2}\mathcal{W}^\phi_{\ell_1}\mathcal{W}^\phi_{\ell_2}\langle(\widehat{E}_{L_1,M_1}\widehat{E}_{L_2,M_2}\rangle\langle(\widehat{\phi}_{\ell_1,m_1})^*\widehat{\phi}_{\ell_2,m_2}\rangle \\
&\simeq \sum_{L_1,M_1}\sum_{\ell_1,m_1}\begin{pmatrix} L & L_1 & \ell_1 \\ M & M_1 & m_1\end{pmatrix}\begin{pmatrix} L & L_1 & \ell_1 \\ M & M_1 & m_1\end{pmatrix}(\mathcal{S}^{(-)}_{L,L_1,\ell_1})^2(\mathcal{W}^E_{L_1}\mathcal{W}^\phi_{\ell_1})^2\widehat{C}^{\mathrm{EE}}_{L_1}(N_{\ell_1}+C^{\phi\phi}_{\ell_1}) \\
&= \frac{1}{2L+1}\sum_{L_1,\ell_1}(\mathcal{S}^{(-)}_{L,L_1,\ell_1})^2\mathcal{W}^E_{L_1}\mathcal{W}^\phi_{\ell_1}C^{\mathrm{EE}}_{L_1}C^{\phi\phi}_{\ell_1} \\
&= \Xi_L[\mathcal{W}^E C^{\mathrm{EE}},\mathcal{W}^\phi C^{\phi\phi}] \\
&= C_L^{\mathrm{BB,W}}\,.
\end{aligned}\tag{52}$$

where, from the first to the last equation, we ignore the higher order biases [4, 5, 6, 7].

There are other terms which must be taken into account. As an example of these terms, we here consider the disconnected part at zeroth order of $C_\ell^{\phi\phi}$:

$$\begin{aligned}
&\langle(\widehat{E}_{L_1,M_1})^*\widehat{B}_{L_1',M_1'}\widehat{E}_{L_1'',M_1''}\widehat{E}_{L_2,M_2}(\widehat{B}_{L_2',M_2'}\widehat{E}_{L_2'',M_2''})^*\rangle_{\phi=0} \\
&=\delta_{L_1',L_2'}\delta_{M_1',M_2'}\widehat{C}^{\rm BB}_{L_1'}\bigg(\delta_{L_1,L_1''}\delta_{M_1,M_1''}\delta_{L_2,L_2''}\delta_{M_2,M_2''}\widehat{C}^{\rm EE}_{L_1}\widehat{C}^{\rm EE}_{L_2} \\
&\qquad+\delta_{L_1,L_2}\delta_{M_1,M_2}\delta_{L_1'',L_2''}\delta_{M_1'',M_2''}\widehat{C}^{\rm EE}_{L_1}\widehat{C}^{\rm EE}_{L_1''} \\
&\qquad+\delta_{L_1,L_2''}\delta_{M_1,M_2''}\delta_{L_1'',L_2}\delta_{M_1'',M_2}\widehat{C}^{\rm EE}_{L_1}\widehat{C}^{\rm EE}_{L_1''}(-1)^{M_1+M_1''}\bigg)\,.
\end{aligned} \tag{53}$$

Note that the second term is involved in Eq. (52). The first term in the above equation leads to the discontinuity in the residual B-mode power spectrum. To see this, substituting the first term into Eq. (51), we obtain,

$$\begin{aligned}
N_L^{\rm (a,0)}&\equiv\sum_{L_1,M_1}\sum_{\ell_1,m_1}\sum_{L_2,M_2}\sum_{\ell_2,m_2}\begin{pmatrix}L&L_1&\ell_1\\M&M_1&m_1\end{pmatrix}\mathcal{S}^{(-)}_{L,L_1,\ell_1}\begin{pmatrix}L&L_2&\ell_2\\M&M_2&m_2\end{pmatrix}\mathcal{S}^{(-)}_{L,L_2,\ell_2} \\
&\quad\times\mathcal{W}^E_{L_1}\mathcal{W}^E_{L_2}\mathcal{W}^\phi_{\ell_1}\mathcal{W}^\phi_{\ell_2}N_{\ell_1}N_{\ell_2}\sum_{L_1',M_1'}\sum_{L_1'',M_1''}\sum_{L_2',M_2'}\sum_{L_2'',M_2''} \\
&\quad\times\begin{pmatrix}L_1'&L_1''&\ell_1\\M_1'&M_1''&m_1\end{pmatrix}\begin{pmatrix}L_2'&L_2''&\ell_2\\M_2'&M_2''&m_2\end{pmatrix}(g_{L_1'',L_1',\ell_1})^*g_{L_2'',L_2',\ell_2} \\
&\quad\times\delta_{L_1',L_2'}\delta_{M_1',M_2'}\delta_{L_1,L_1''}\delta_{M_1,M_1''}\delta_{L_2,L_2''}\delta_{M_2,M_2''}\widehat{C}^{\rm BB}_{L_1'}\widehat{C}^{\rm EE}_{L_1}\widehat{C}^{\rm EE}_{L_2} \\
&=\sum_{L_1,M_1}\sum_{\ell_1,m_1}\sum_{L_2,M_2}\sum_{\ell_2,m_2}\sum_{L_1',M_1'}\begin{pmatrix}L&L_1&\ell_1\\M&M_1&m_1\end{pmatrix}\mathcal{S}^{(-)}_{L,L_1,\ell_1}\begin{pmatrix}L&L_2&\ell_2\\M&M_2&m_2\end{pmatrix}\mathcal{S}^{(-)}_{L,L_2,\ell_2} \\
&\quad\times\mathcal{W}^E_{L_1}\mathcal{W}^E_{L_2}\mathcal{W}^\phi_{\ell_1}\mathcal{W}^\phi_{\ell_2}N_{\ell_1}N_{\ell_2} \\
&\quad\times\begin{pmatrix}L_1'&L_1&\ell_1\\M_1'&M_1&m_1\end{pmatrix}\begin{pmatrix}L_1'&L_2&\ell_2\\M_1'&M_2&m_2\end{pmatrix}(g_{L_1,L_1',\ell_1})^*g_{L_2,L_1',\ell_2}\widehat{C}^{\rm BB}_{L_1'}\widehat{C}^{\rm EE}_{L_1}\widehat{C}^{\rm EE}_{L_2}\,.
\end{aligned} \tag{54}$$

Using the orthogonality relation, we find

$$\begin{aligned}
N_L^{\rm (a,0)}&=\frac{\widehat{C}^{\rm BB}_L}{(2L+1)^2}\sum_{L_1,\ell_1}\sum_{L_2,\ell_2}\mathcal{S}^{(-)}_{L,L_1,\ell_1}\mathcal{S}^{(-)}_{L,L_2,\ell_2} \\
&\qquad\times\mathcal{W}^E_{L_1}\mathcal{W}^E_{L_2}\mathcal{W}^\phi_{\ell_1}\mathcal{W}^\phi_{\ell_2}N_{\ell_1}N_{\ell_2}(g_{L_1,L,\ell_1})^*g_{L_2,L,\ell_2}\widehat{C}^{\rm EE}_{L_1}\widehat{C}^{\rm EE}_{L_2} \\
&=\widehat{C}^{\rm BB}_L\left|\frac{1}{2L+1}\sum_{L_1,\ell_1}\mathcal{S}^{(-)}_{L,L_1,\ell_1}\mathcal{W}^E_{L_1}\mathcal{W}^\phi_{\ell_1}N_{\ell_1}g_{L_1,L,\ell_1}\widehat{C}^{\rm EE}_{L_1}\right|^2 \\
&=\left|\frac{1}{\widehat{C}^{\rm BB}_L}\Xi_L[\mathcal{W}^E\mathcal{W}^E\widehat{C}^{\rm EE},\mathcal{W}^\phi N]\right|^2\widehat{C}^{\rm BB}_L \qquad =A_L^2\widehat{C}^{\rm BB}_L\,.
\end{aligned} \tag{55}$$

The above quantity vanishes if we do not use the B-mode polarization at $L<L_{\rm min}$ since, from the second to third equality, we omit $\delta_{L,L_1'}$.

At first order of $C_\ell^{\phi\phi}$, there are also additional terms which have non-negligible contributions. To see this, we consider the following six-point correlation which is involved in Eq. (51):

$$\langle(\widehat{E}_{L_1,M_1})^*\widehat{E}_{L_1'',M_1''}\rangle\langle\langle\widehat{B}_{L_1',M_1'}\widehat{E}_{L_2,M_2}\rangle_{\rm CMB}\langle(\widehat{B}_{L_2',M_2'}\widehat{E}_{L_2'',M_2''})^*\rangle_{\rm CMB}\rangle$$
$$=\delta_{L_1,L_1''}\delta_{M_1,M_1''}\widehat{C}^{\rm EE}_{L_1}\sum_{\ell',m'}\begin{pmatrix}L_1' & L_2 & \ell'\\ M_1' & M_2 & m'\end{pmatrix}\begin{pmatrix}L_2' & L_2'' & \ell'\\ M_2' & M_2'' & m'\end{pmatrix}(f_{L_2,L_1',\ell'})^*f_{L_2'',L_2',\ell'}C^{\phi\phi}_{\ell'}\,. \tag{56}$$

Substituting the above equation into Eq. (51), we obtain

$$\begin{aligned}
N^{(\rm a,1)}_L &= 2\sum_{L_1,M_1}\sum_{\ell_1,m_1}\sum_{L_2,M_2}\sum_{\ell_2,m_2}\begin{pmatrix}L & L_1 & \ell_1\\ M & M_1 & m_1\end{pmatrix}\mathcal{S}^{(-)}_{L,L_1,\ell_1}\begin{pmatrix}L & L_2 & \ell_2\\ M & M_2 & m_2\end{pmatrix}\mathcal{S}^{(-)}_{L,L_2,\ell_2}\\
&\quad\times\mathcal{W}^E_{L_1}\mathcal{W}^E_{L_2}\mathcal{W}^\phi_{\ell_1}\mathcal{W}^\phi_{\ell_2}N_{\ell_1}N_{\ell_2}\sum_{L_1',M_1'}\sum_{L_1'',M_1''}\sum_{L_2',M_2'}\sum_{L_2'',M_2''}\\
&\quad\times\begin{pmatrix}L_1' & L_1'' & \ell_1\\ M_1' & M_1'' & m_1\end{pmatrix}\begin{pmatrix}L_2' & L_2'' & \ell_2\\ M_2' & M_2'' & m_2\end{pmatrix}(g_{L_1'',L_1',\ell_1})^*g_{L_2'',L_2',\ell_2}\\
&\quad\times\delta_{L_1,L_1''}\delta_{M_1,M_1''}\widehat{C}^{\rm EE}_{L_1}\sum_{\ell',m'}\begin{pmatrix}L_1' & L_2 & \ell'\\ M_1' & M_2 & m'\end{pmatrix}\begin{pmatrix}L_2' & L_2'' & \ell'\\ M_2' & M_2'' & m'\end{pmatrix}(f_{L_2,L_1',\ell'})^*f_{L_2'',L_2',\ell'}C^{\phi\phi}_{\ell'}\\
&= 2\sum_{L_1,M_1}\sum_{\ell_1,m_1}\sum_{L_2,M_2}\sum_{\ell_2,m_2}\sum_{L_1',M_1'}\sum_{L_2',M_2'}\sum_{L_2'',M_2''}\sum_{\ell',m'}\begin{pmatrix}L & L_1 & \ell_1\\ M & M_1 & m_1\end{pmatrix}\begin{pmatrix}L_1' & L_1 & \ell_1\\ M_1' & M_1 & m_1\end{pmatrix}\\
&\quad\times\begin{pmatrix}L & L_2 & \ell_2\\ M & M_2 & m_2\end{pmatrix}\begin{pmatrix}L_1' & L_2 & \ell'\\ M_1' & M_2 & m'\end{pmatrix}\begin{pmatrix}L_2' & L_2'' & \ell_2\\ M_2' & M_2'' & m_2\end{pmatrix}\begin{pmatrix}L_2' & L_2'' & \ell'\\ M_2' & M_2'' & m'\end{pmatrix}\mathcal{S}^{(-)}_{L,L_1,\ell_1}\mathcal{S}^{(-)}_{L,L_2,\ell_2}\\
&\quad\times\mathcal{W}^E_{L_1}\mathcal{W}^E_{L_2}\mathcal{W}^\phi_{\ell_1}\mathcal{W}^\phi_{\ell_2}N_{\ell_1}N_{\ell_2}\widetilde{C}^{\rm EE}_{L_1}C^{\phi\phi}_{\ell'}(f_{L_2,L_1',\ell'}g_{L_1,L_1',\ell_1})^*f_{L_2'',L_2',\ell'}g_{L_2'',L_2',\ell_2}\,.
\end{aligned}\tag{57}$$

In the above equation, we multiply the factor $1/2$. This is because the same equation is obtained from counterpart of Eq. (58) (i.e., the term obtained by exchanging $L_1\leftrightarrow L_2$ and $L_1''\leftrightarrow L_2''$ in Eq. (58)). The total contribution from Eq. (58) and its counterpart therefore becomes $N^{(\rm a,1)}_L$. Using the orthogonality relation of the Wigner 3-j symbols, we find

$$\begin{aligned}
N^{(\rm a,1)}_L &= \frac{2}{(2L+1)^2}\sum_{L_1,\ell_1}\sum_{L_2,\ell_2}\sum_{L_2',L_2''}\mathcal{S}^{(-)}_{L,L_1,\ell_1}\mathcal{S}^{(-)}_{L,L_2,\ell_2}\mathcal{W}^E_{L_1}\mathcal{W}^E_{L_2}\mathcal{W}^\phi_{\ell_1}\mathcal{W}^\phi_{\ell_2}N_{\ell_1}N_{\ell_2}\widehat{C}^{\rm EE}_{L_1}C^{\phi\phi}_{\ell_2}\\
&\quad\times\frac{1}{2\ell_2+1}(f_{L_2,L,\ell_2}g_{L_1,L,\ell_1})^*f_{L_2'',L_2',\ell_2}g_{L_2'',L_2',\ell_2}\\
&= \frac{2}{(2L+1)^2\widehat{C}^{\rm BB}_L}\sum_{L_1,\ell_1}\sum_{L_2,\ell_2}[\mathcal{S}^{(-)}_{L,L_1,\ell_1}\mathcal{S}^{(-)}_{L,L_2,\ell_2}]^2\mathcal{W}^E_{L_1}\mathcal{W}^E_{L_2}\mathcal{W}^\phi_{\ell_1}\mathcal{W}^\phi_{\ell_2}N_{\ell_1}C^{\phi\phi}_{\ell_2}C^{\rm EE}_{L_2}\mathcal{W}^E_{L_1}\widehat{C}^{\rm EE}_{L_1}\\
&= \frac{2}{\widehat{C}^{\rm BB}_L}\Xi_L[\mathcal{W}^E\mathcal{W}^E\widehat{C}^{\rm EE},\mathcal{W}^\phi N]\Xi_L[\mathcal{W}^EC^{\rm EE},\mathcal{W}^\phi C^{\phi\phi}]\\
&= 2A_LC^{\rm BB,W}_L\,.
\end{aligned}\tag{58}$$

Similar to $N^{(\rm a,0)}_L$, the above quantity vanishes if $L<L_{\rm min}$.

3.1.3 Total

By combining Eqs. (50), (52), (55), and (58), we obtain

$$\begin{aligned}
C_L^{\rm BB,res} &= \widehat{C}_L^{\rm BB,LB} - 2N_L^{\rm (c,0)} + N_L^{\rm (a,0)} + N_L^{\rm (a,1)} - C_L^{\rm BB,W} \\
&\simeq C_L^{\rm BB} + N_L^{\rm BB,LB} - 2N_L^{\rm (c,0)} + N_L^{\rm (a,0)} + N_L^{\rm (a,1)} \\
&\quad + \frac{1}{2L+1}\sum_{\ell,L'}(\mathcal{S}^{(-)}_{L,L',\ell})^2(1-\mathcal{W}^{E,LB}_{L'}\mathcal{W}^{\phi,G}_\ell)C^{EE}_{L'}C^{\phi\phi}_\ell \\
&= C_L^{\rm BB} + N_L^{\rm BB,LB} - 2A_L\widehat{C}_L^{\rm BB} + A_L^2\widehat{C}_L^{\rm BB} + A_LC_L^{\rm BB,W} \\
&\quad + \frac{1}{2L+1}\sum_{\ell,L'}(\mathcal{S}^{(-)}_{L,L',\ell})^2(1-\mathcal{W}^{E,LB}_{L'}\mathcal{W}^{\phi,G}_\ell)C^{EE}_{L'}C^{\phi\phi}_\ell\,,
\end{aligned} \tag{59}$$

where, from the first to second line, we ignore $\mathcal{O}[(C_\ell^{\phi\phi})^2]$ in the lensing B-mode power spectrum $\widetilde{C}_L^{\rm BB}$ involved in $\widehat{C}_L^{\rm BB,LB}$. If we ignore bias terms, $N_L^{\rm (c,0)}$, $N_L^{\rm (a,0)}$ and $N_L^{\rm (a,1)}$, we obtain the expression for the residual B-mode power spectrum obtained in Ref. [8].

3.2 EE quadratic estimator

Next we consider the case with EE-quadratic estimator for lensing reconstruction. The corresponding equation of Eq. (45) is

$$\begin{aligned}
\langle \widehat{B}^{\rm lens}_{L,M}(\widehat{B}^{\rm LB}_{L,M})^*\rangle &= -{\rm i}\sum_{L',M'}\sum_{\ell,m}\begin{pmatrix} L & L' & \ell \\ M & M' & m\end{pmatrix}\mathcal{S}^{(-)}_{L,L',\ell}\mathcal{W}^E_{L'}\mathcal{W}^\phi_\ell \\
&\quad\times N_\ell^{(EE)}\sum_{L_1,L_2}\sum_{M_1,M_2}\begin{pmatrix} L_1 & L_2 & \ell \\ M_1 & M_2 & m\end{pmatrix} g^{(EE)}_{L_1,L_2,\ell}\langle(\widehat{E}^{\rm LB}_{L',M'})^*\widehat{E}^{\rm G}_{L_1,M_1}\widehat{E}^{\rm G}_{L_2,M_2}(\widehat{B}^{\rm LB}_{L,M})^*\rangle\,.
\end{aligned} \tag{60}$$

Contrary to the EB-estimator, the disconnected part of the four-point correlation vanishes and the connected part is given by

$$\langle \widehat{E}^{\rm G}_{L_1,M_1}\widehat{E}^{\rm G}_{L_2,M_2}(\widehat{E}^{\rm LB}_{L',M'}\widehat{B}^{\rm obs}_{L,M})^*\rangle_{\rm c} \simeq \langle\langle \widehat{E}^{\rm G}_{L_1,M_1}\widehat{E}^{\rm G}_{L_2,M_2}\rangle_{\rm CMB}\langle(\widehat{E}^{\rm LB}_{L',M'}\widehat{B}^{\rm LB}_{L,M})^*\rangle_{\rm CMB}\rangle\,. \tag{61}$$

If we also assume that the third term of Eq. (43) is given by

$$\langle\widehat{\phi}_{\ell_1,m_1}\widehat{E}^{\rm LB}_{L_1',M_1'}(\widehat{\phi}_{\ell_2,m_2}\widehat{E}^{\rm LB}_{L_2',M_2'})^*\rangle \simeq \delta_{\ell_1,\ell_2}\delta_{m_1,m_2}\delta_{L_1',L_2'}\delta_{M_1',M_2'}(N^{EE}_{\ell_1}+C^{\phi\phi}_{\ell_1})\widehat{C}^{\rm EE,LB}_{L_1'}\,, \tag{62}$$

we obtain the same expression given in [8]:

$$C_L^{\rm BB,res} = C_L^{\rm BB} + N_L^{\rm BB,LB} + \frac{1}{2L+1}\sum_{\ell,L'}(\mathcal{S}^{(-)}_{L,L',\ell})^2(1-\mathcal{W}^{E,LB}_{L'}\mathcal{W}^{\phi,G}_\ell)C^{EE}_{L'}C^{\phi\phi}_\ell\,, \tag{63}$$

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