# Note for TAM expansion and its applications

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# 1 Difficulties in deriving Cl's with conventional approach

## 1.1 Introduction

Let us consider the scalar/ pseudo-scalar lensing potentials expressed as

$$\boldsymbol{\nabla}^2\phi = \int_0^{\chi_{\rm s}} {\rm d}\chi \left[\kappa_0(\chi,\chi_{\rm s})\eth^-\eth^+\mathcal{P}_0 + \kappa_1(\chi,\chi_{\rm s})\frac{\eth^-\mathcal{P}_+ + \eth^+\mathcal{P}_-}{2}\right] \,, \tag{1}$$

$$\boldsymbol{\nabla}^2\varpi = \int_0^{\chi_{\rm s}} {\rm d}\chi\, \kappa_1(\chi,\chi_{\rm s})\frac{\eth^-\mathcal{P}_+ + \eth^+\mathcal{P}_-}{2} \,. \tag{2}$$

where

$$\kappa_0(\chi,\chi_{\rm s}) = \frac{1}{\chi} - \frac{1}{\chi_{\rm s}} \tag{3}$$

$$\kappa_1(\chi,\chi_{\rm s}) = \frac{1}{\chi} - \frac{\chi_{\rm s}-\chi}{\chi_{\rm s}}\delta_{\rm D}(\chi) \tag{4}$$

and

$$\mathcal{P}_0 = \Psi - \Phi + \sigma_i\hat{n}^i + \frac{1}{2}h_{ij}\hat{n}^i\hat{n}^j$$
$$\mathcal{P}_\pm = (\sigma_i e^i_a + h_{ij}e^i_a\hat{n}^j)e^a_\pm \,. \tag{5}$$

**[Spin-operated spherical harmonics]**

Spin operators acting on a spin $s$ function are given by

$$\eth^\pm_s = \bar{\mu}^{\pm s}\left[\bar{\mu}\partial_\mu \mp \frac{i}{\bar{\mu}}\partial_\varphi\right]\bar{\mu}^{\mp s} \,, \tag{6}$$

where we define $\bar{\mu} = \sqrt{1-\mu^2}$. With $\alpha_{\ell,m} = (-1)^{(-m+|m|)/2}[(2\ell+1)(\ell-|m|)!/4\pi(\ell+|m|)!]^{1/2}$, a spin-operated spherical harmonics is defined as

$$\begin{aligned}\eth^\pm_0 Y_{\ell,m}(\mu,\varphi) &= \alpha_{\ell,m}\left[\sqrt{1-\mu^2}\partial_\mu \mp \frac{i}{\sqrt{1-\mu^2}}\partial_\varphi\right]e^{im\varphi}P^m_\ell(\mu) \\ &= \alpha_{\ell,m}e^{im\varphi}\left[\sqrt{1-\mu^2}\partial_\mu \pm \frac{m}{\sqrt{1-\mu^2}}\right]P^m_\ell(\mu) \equiv \alpha_{\ell,m}e^{im\varphi}{}_{\pm1}P^m_\ell(\mu) \,. \end{aligned} \tag{7}$$

Similarly, we obtain

$$\eth_1^{\pm}\eth_0^{\pm}Y_{\ell,m}(\mu,\varphi) = \alpha_{\ell,m}\left[(1-\mu^2)\partial_\mu^2 \mp 2i\left(\partial_\mu + \frac{\mu}{1-\mu^2}\right)\partial_\varphi - \frac{1}{1-\mu^2}\partial_\varphi^2\right] e^{im\varphi}P_\ell^m(\mu)$$
$$= \alpha_{\ell,m}e^{im\varphi}\left[(1-\mu^2)\partial_\mu^2 \pm 2m\left(\partial_\mu + \frac{\mu}{1-\mu^2}\right) + \frac{m^2}{1-\mu^2}\right]P_\ell^m(\mu) \equiv \alpha_{\ell,m}e^{im\varphi}{}_{\pm2}P_\ell^m(\mu)\,. \tag{8}$$

### 1.1.1 Formulas

$$\int_{-1}^{1}\frac{d\mu}{2}(1-\mu^2)^{m/2}P_{\ell,m}(\mu)e^{-ix\mu} = (-i)^{\ell+m}\frac{(\ell+m)!}{(\ell-m)!}\frac{j_\ell(x)}{x^m}\,. \tag{9}$$

## 1.2

### 1.2.1 Scalar/pseudo-scalar lensing potentials

To obtain harmonics coefficients, we need to calculate

$$\int d^2\hat{\boldsymbol{n}}_{\pm s}h(\eth^{\mp})^sY_{\ell,m} = \sum_{S=\pm2}\frac{1}{\sqrt{2}}\int\frac{d^3\boldsymbol{k}}{(2\pi)^3}\int d^2\hat{\boldsymbol{n}}(\eth^{\mp})^sY_{\ell,m}e^{-ix\mu}(1-\mu^2)^{1-s/2}(\mu\mp S/2)^sh^{(S)}e^{iS\varphi}\,. \tag{10}$$

Integrating in terms of $\varphi$, we obtain

$$\int d^2\hat{\boldsymbol{n}}_{\pm s}h_{\mp s}Y_{\ell,m} = \sum_{m=\pm2}\frac{2\pi}{\sqrt{2}}\int\frac{d^3\boldsymbol{k}}{(2\pi)^3}h^{(m)}\alpha_{\ell,m\mp s}A_{\ell,m}(\boldsymbol{k})\,. \tag{11}$$

where we define

$${}_{\mp s}A_{\ell,m}(\boldsymbol{k}) = \int_{-1}^{1}\frac{d\mu}{2}{}_{\mp s}P_{\ell,m}(\mu)e^{-ix\mu}(1-\mu^2)^{1-s/2}(\mu\mp m/2)^s\,. \tag{12}$$

Note that

- $s=0$:

$${}_0A_{\ell,\pm2}(\boldsymbol{k}) = \int_{-1}^{1}\frac{d\mu}{2}P_\ell^2(\mu)e^{-ix\mu}(1-\mu^2) = -(-i)^\ell\frac{(\ell+2)!}{(\ell-2)!}\frac{j_\ell(x)}{x^2}\,. \tag{13}$$

- $s=1$:

$$\begin{aligned}
{}_{+1}A_{\ell,\pm2}(\boldsymbol{k}) &= \int_{-1}^{1}\frac{d\mu}{2}\left[\partial_\mu P_\ell^2 \pm \frac{2P_\ell^2}{1-\mu^2}\right]e^{-ix\mu}(1-\mu^2)(\mu\mp1) \\
&= \int_{-1}^{1}\frac{d\mu}{2}\left[-\left(-ix(1-\mu^2)(\mu\mp1) - 2\mu(\mu\mp1) + (1-\mu^2)\right) \pm 2(\mu\mp1)\right]P_\ell^2e^{-ix\mu} \\
&= \int_{-1}^{1}\frac{d\mu}{2}(1-\mu^2)\left(ix(\mu\mp1)-3\right)P_\ell^2e^{-ix\mu} = -\left(x(\partial_x-1)+3\right)\int_{-1}^{1}\frac{d\mu}{2}(1-\mu^2)P_\ell^2e^{-ix\mu} \\
&= (-i)^{\ell+2}\frac{(\ell+2)!}{(\ell-2)!}\left(x(\partial_x\mp1)+3\right)\frac{j_\ell(x)}{x^2} \\
{}_{-1}A_{\ell,\pm2}(\boldsymbol{k}) &= (-i)^{\ell+2}\frac{(\ell+2)!}{(\ell-2)!}\left(x(\partial_x\pm1)+3\right)\frac{j_\ell(x)}{x^2}\,.
\end{aligned} \tag{14}$$

For $s=0$, the spin operator is expressed as

$$\eth = \sqrt{1-\mu^2}\left(\frac{\partial}{\partial\mu} - \frac{i}{1-\mu^2}\frac{\partial}{\partial\varphi}\right) . \tag{15}$$

# 2 Total-Angular-Momentum (TAM) Basis

## 2.1 TAM expansion

The TAM basis is given by

$${}_s\mathcal{G}_\ell^m(\chi,\hat{\boldsymbol{n}},\boldsymbol{k}) = (-i)^\ell\sqrt{\frac{4\pi}{2\ell+1}}{}_s\mathcal{Y}_\ell^m(\hat{\boldsymbol{n}})e^{-i\boldsymbol{k}\cdot\chi\hat{\boldsymbol{n}}} . \tag{16}$$

Note that the plane wave is decomposed into

$$e^{-i\boldsymbol{k}\cdot\chi\hat{\boldsymbol{n}}} = \sum_{L=0}^{\infty}(-i)^L\sqrt{4\pi(2L+1)}j_L(k\chi){}_s\mathcal{Y}_L^m(\hat{\boldsymbol{n}}_{\boldsymbol{k}}) \tag{17}$$

where $\hat{\boldsymbol{n}}_{\boldsymbol{k}}$ is the unit vector obtained by rotating $\hat{\boldsymbol{n}}$ so that $\boldsymbol{k}\to\boldsymbol{e}_z$. This leads to

$$\begin{aligned}{}_s\mathcal{G}_\ell^m(\chi,\hat{\boldsymbol{n}},\boldsymbol{k}) &= {}_s\mathcal{G}_\ell^m(\chi,\hat{\boldsymbol{n}}_{\boldsymbol{k}},k\boldsymbol{e}_z) \\ &= (-i)^\ell\sqrt{\frac{4\pi}{2\ell+1}}{}_s\mathcal{Y}_\ell^m(\hat{\boldsymbol{n}}_{\boldsymbol{k}})e^{-ik\chi\boldsymbol{e}_z\cdot\hat{\boldsymbol{n}}_{\boldsymbol{k}}} \\ &= \sum_{L=0}^{\infty}(-i)^L\sqrt{4\pi(2L+1)}{}_s\mathcal{J}_L^{(\ell,m)}(k\chi){}_s\mathcal{Y}_L^m(\hat{\boldsymbol{n}}_{\boldsymbol{k}}) .\end{aligned} \tag{18}$$

where we define

$${}_s\mathcal{J}_L^{(\ell,m)}(x) = \frac{(-i)^{\ell-L}}{\sqrt{(2\ell+1)(2L+1)}}\int_{-1}^{1}d\mu\int_{-\pi}^{\pi}d\varphi({}_s\mathcal{Y}_L^m)^*{}_s\mathcal{Y}_\ell^m e^{-ik\chi\mu} . \tag{19}$$

TAM transform is defined as

$${}_sX(\chi,\hat{\boldsymbol{n}}) = \int \mathrm{d}^3\boldsymbol{k}\sum_{\ell=0}^{\infty}\sum_{m=-2}^{2}{}_sX_\ell^{(m)}(\chi,k){}_s\mathcal{G}_\ell^m(\chi,\hat{\boldsymbol{n}},\boldsymbol{k}) . \tag{20}$$

### 2.1.1 $(\ell,m)=(1,0)$

For ${}_1\mathcal{J}_L^{(1,0)}$, we need

$${}_1\mathcal{Y}_{1,0} = \sqrt{\frac{3}{8\pi}}\sqrt{1-\mu^2} , \tag{21}$$

$${}_1\mathcal{Y}_{L,0} = \frac{\sqrt{1-\mu^2}}{\sqrt{L(L+1)}}\sqrt{\frac{2L+1}{4\pi}}\frac{\mathrm{d}P_L(\mu)}{\mathrm{d}\mu} . \tag{22}$$

Then

$$\int d\varphi \int d\mu e^{-ix\mu} {}_1\mathcal{Y}_{1,0} {}_1\mathcal{Y}^*_{L,0} = 2\pi \int d\mu e^{-ix\mu} \sqrt{\frac{3(2L+1)}{8L(L+1)}} \sqrt{1-\mu^2}(-P_{L,1})$$
$$= -\sqrt{\frac{3(2L+1)}{8L(L+1)}} \int d\mu e^{-ix\mu} \sqrt{1-\mu^2} P_{L,1}$$
$$= -\sqrt{\frac{3(2L+1)L(L+1)}{2}} (-i)^{L+1} \frac{j_L(x)}{x} \,. \tag{23}$$

Thus

$${}_1\epsilon_L^{(1,0)} = -\frac{(-i)^{1-L}}{\sqrt{3(2L+1)}} \sqrt{\frac{3(2L+1)L(L+1)}{2}} (-i)^{L+1} \frac{j_L(x)}{x} = \sqrt{\frac{L(L+1)}{2}} \frac{j_L(x)}{x} \,. \tag{24}$$

### 2.1.2 $(\ell, m) = (1, \pm 1)$

For ${}_1\mathcal{J}_L^{(1,\pm1)}$, we need

$${}_1\mathcal{Y}_{1,\pm1} = \sqrt{\frac{3}{16\pi}} (\mu \mp 1) e^{\pm i\varphi} \,, \tag{25}$$
$${}_1\mathcal{Y}_{L,\pm1} = \frac{\sqrt{1-\mu^2}}{L(L+1)} \sqrt{\frac{2L+1}{4\pi}} \left( \frac{\partial}{\partial\mu} - \frac{i}{1-\mu^2} \frac{\partial}{\partial\varphi} \right) P_{L,1} e^{\pm i\varphi}$$
$$= \frac{\sqrt{1-\mu^2}}{L(L+1)} \sqrt{\frac{2L+1}{4\pi}} \left( \frac{\mathrm{d}P_{L,1}}{\mathrm{d}\mu} \pm \frac{P_{L,1}}{1-\mu^2} \right) e^{\pm i\varphi} \,. \tag{26}$$

This leads to

$$\int d\varphi \int d\mu e^{-ix\mu} {}_1\mathcal{Y}_{1,\pm1} {}_1\mathcal{Y}^*_{L,\pm1} = \frac{\sqrt{3(2L+1)}}{4L(L+1)} \int d\mu e^{-ix\mu} (\mu \mp 1) \sqrt{1-\mu^2} \left( \frac{\mathrm{d}P_{L,1}}{\mathrm{d}\mu} \pm \frac{P_{L,1}}{1-\mu^2} \right)$$
$$= \frac{\sqrt{3(2L+1)}}{4L(L+1)} \int d\mu e^{-ix\mu} P_{L,1}$$
$$\times \left[ ix(\mu \mp 1)\sqrt{1-\mu^2} - \sqrt{1-\mu^2} + \frac{\mu(\mu \mp 1)}{\sqrt{1-\mu^2}} + \left( \pm \frac{(\mu \mp 1)}{\sqrt{1-\mu^2}} \right) \right] \,. \tag{27}$$

Note that

$$\left[ ix(\mu \mp 1)\sqrt{1-\mu^2} - \sqrt{1-\mu^2} + \frac{\mu(\mu \mp 1)}{\sqrt{1-\mu^2}} + \left( \pm \frac{(\mu \mp 1)}{\sqrt{1-\mu^2}} \right) \right] = \left[ ix(\mu \mp 1)\sqrt{1-\mu^2} - \sqrt{1-\mu^2} + \frac{(\mu \pm 1 \ldots}{\sqrt{1 \ldots}} \right.$$
$$= \left[ ix(1 \mp \mu)\sqrt{1-\mu^2} - \sqrt{1-\mu^2} - \sqrt{1-\mu \ldots} \right.$$
$$= \left[ ix(\mu \mp 1) - 2 \right] \sqrt{1-\mu^2} \tag{28}$$

Thus

$$\int d\varphi \int d\mu e^{-ix\mu}{}_1\mathcal{Y}_{1,\pm 1 1}\mathcal{Y}^*_{L,\pm 1} = \frac{\sqrt{3(2L+1)}}{4L(L+1)} \int d\mu e^{-ix\mu} P_{L,1}\left[ix(\mu \mp 1) - 2\right]\sqrt{1-\mu^2}$$
$$= \frac{\sqrt{3(2L+1)}}{2L(L+1)}\left[ix(i\partial_x \mp 1) - 2\right](-i)^{L+1}L(L+1)\frac{j_L(x)}{x}\,. \tag{29}$$

Then

$$ {}_1\mathcal{J}_L^{(1,\pm 1)} = -\frac{1}{2}\left[-x\partial_x \mp ix - 2\right]\frac{j_L(x)}{x} = \frac{1}{2}\left[\frac{j_L(x)}{x} + j_L' \pm ix\frac{j_L(x)}{x}\right]\,. \tag{30}$$

### 2.1.3 $(\ell, m) = (2, 0)$

For ${}_1\mathcal{J}_L^{(2,0)}$, we need

$$ {}_1\mathcal{Y}_{2,0} = \sqrt{\frac{15}{8\pi}}\mu\sqrt{1-\mu^2}\,, \tag{31}$$

$$ {}_1\mathcal{Y}_{L,0} = \frac{\sqrt{1-\mu^2}}{\sqrt{L(L+1)}}\sqrt{\frac{2L+1}{4\pi}}\frac{\mathrm{d}P_L(\mu)}{\mathrm{d}\mu}\,. \tag{32}$$

This leads to

$$\int d\varphi \int d\mu e^{-ix\mu}{}_1\mathcal{Y}_{2,0 1}\mathcal{Y}^*_{L,0} = \sqrt{\frac{15(2L+1)}{8L(L+1)}}\int d\mu e^{-ix\mu}\mu(1-\mu^2)\frac{\mathrm{d}P_L(\mu)}{\mathrm{d}\mu}$$
$$= -\sqrt{\frac{15(2L+1)}{8L(L+1)}}\int d\mu e^{-ix\mu}\mu\sqrt{(1-\mu^2)}P_{L,1}(\mu)$$
$$= -\sqrt{\frac{15(2L+1)}{8L(L+1)}}(i\partial_x)\int d\mu e^{-ix\mu}\sqrt{(1-\mu^2)}P_{L,1}(\mu)$$
$$= -\sqrt{\frac{15(2L+1)}{8L(L+1)}}(i\partial_x)2(-i)^{L+1}L(L+1)\frac{j_L(x)}{x}$$
$$= (-i)^{L+2}L(L+1)\sqrt{\frac{15(2L+1)}{2L(L+1)}}\left(\frac{j_L(x)}{x}\right)'\,. \tag{33}$$

Therefore

$$ {}_1\mathcal{J}_L^{(2,0)} = \frac{(-i)^{2-L}}{\sqrt{5(2L+1)}}(-i)^{L+2}L(L+1)\sqrt{\frac{15(2L+1)}{2L(L+1)}}\left(\frac{j_L(x)}{x}\right)' = \sqrt{\frac{3L(L+1)}{2}}\left(\frac{j_L(x)}{x}\right)'\,. \tag{34}$$

### 2.1.4 $(\ell, m) = (2, \pm 1)$

For ${}_1\mathcal{J}_L^{(2,\pm1)}$, we need

$$
{}_1\mathcal{Y}_{2,\pm1} = -\sqrt{\frac{5}{16\pi}}(1 \mp \mu)(1 \pm 2\mu)e^{\pm i\varphi}\,, \tag{35}
$$

$$
{}_1\mathcal{Y}_{L,\pm1} = \frac{\sqrt{1-\mu^2}}{L(L+1)}\sqrt{\frac{2L+1}{4\pi}}\left(\frac{\mathrm{d}P_{L,1}}{\mathrm{d}\mu} \pm \frac{P_{L,1}}{1-\mu^2}\right)e^{\pm i\varphi}\,. \tag{36}
$$

This leads to

$$
\begin{aligned}
&\int d\varphi \int d\mu e^{-ix\mu}{}_1\mathcal{Y}_{2,\pm1 1}\mathcal{Y}^*_{L,\pm1} \\
&= -\frac{1}{L(L+1)}\sqrt{\frac{5(2L+1)}{16}}\int d\mu e^{-ix\mu}\sqrt{1-\mu^2}(1\mp\mu)(1\pm2\mu)\left(\frac{\mathrm{d}P_{L,1}}{\mathrm{d}\mu} \pm \frac{P_{L,1}}{1-\mu^2}\right) \\
&= -\frac{1}{L(L+1)}\sqrt{\frac{5(2L+1)}{16}}\int d\mu e^{-ix\mu} \\
&\quad \times \Bigg[-\Bigg(-ix\sqrt{1-\mu^2}(1\mp\mu)(1\pm2\mu) - \frac{\mu(1\mp\mu)(1\pm2\mu)}{\sqrt{1-\mu^2}} \mp \sqrt{1-\mu^2}(1\pm2\mu) \pm 2(1\mp\mu)\sqrt{1-\mu^2}\Bigg)P \\
&\quad \pm (1\mp\mu)(1\pm2\mu)\frac{P_{L,1}}{\sqrt{1-\mu^2}}\Bigg] \\
&= -\frac{1}{L(L+1)}\sqrt{\frac{5(2L+1)}{16}}\int d\mu e^{-ix\mu} \\
&\quad \times \left[ix(1\mp\mu)(1\pm2\mu) + \frac{\mu(1\mp\mu)(1\pm2\mu)}{1-\mu^2} + 4\mu \mp 1 \pm \frac{(1\mp\mu)(1\pm2\mu)}{1-\mu^2}\right]\sqrt{1-\mu^2}P_{L,1} \\
&= -\frac{1}{L(L+1)}\sqrt{\frac{5(2L+1)}{16}}\int d\mu e^{-ix\mu}\Big[ix(1\mp\mu)(1\pm2\mu) + 6\mu\Big]\sqrt{1-\mu^2}P_{L,1} \\
&= -\frac{1}{L(L+1)}\sqrt{\frac{5(2L+1)}{16}}\Big[ix(1+2\partial_x^2 \pm i\partial_x) + 6i\partial_x\Big]\int d\mu e^{-ix\mu}\sqrt{1-\mu^2}P_{L,1} \\
&= -\sqrt{\frac{5(2L+1)}{4}}\Big[ix(1+2\partial_x^2 \pm i\partial_x) + 6i\partial_x\Big](-i)^{L+1}\frac{j_L}{x} \\
&= (-i)^{L+2}\sqrt{\frac{5(2L+1)}{4}}\left[j_L + 2j_L'' - 4\frac{j_L'}{x} + 4\frac{j_L}{x^2} \pm ij_L' \mp i\frac{j_L}{x} + 6\frac{j_L'}{x} - 6\frac{j_L}{x^2}\right] \\
&= (-i)^{L+2}\sqrt{\frac{5(2L+1)}{4}}\left[j_L + 2j_L'' + 2\frac{j_L'}{x} - 2\frac{j_L}{x^2} \pm ij_L' \mp i\frac{j_L}{x}\right].
\end{aligned} \tag{37}
$$

Therefore

$$
{}_1\mathcal{J}_L^{(2,\pm1)} = \frac{1}{2}\left[j_L + 2j_L'' + 2\frac{j_L'}{x} - 2\frac{j_L}{x^2} \pm ij_L' \mp i\frac{j_L}{x}\right]. \tag{38}
$$

### 2.1.5 $(\ell, m) = (2, \pm 2)$

For ${}_1\mathcal{J}_L^{(2,\pm2)}$, we need

$$
\begin{aligned}
{}_1\mathcal{Y}_{2,\pm2} &= -\sqrt{\frac{5}{16\pi}}\sqrt{1-\mu^2}(\mu \mp 1)e^{\pm 2i\varphi}\,, && (39)\\
{}_1\mathcal{Y}_{L,\pm2} &= \sqrt{\frac{1-\mu^2}{L(L+1)}\frac{(L-2)!}{(L+2)!}}\sqrt{\frac{2L+1}{4\pi}}\left(\frac{\partial}{\partial\mu} - \frac{i}{1-\mu^2}\frac{\partial}{\partial\varphi}\right)P_{L,2}e^{\pm 2i\varphi}\\
&= \sqrt{\frac{1-\mu^2}{L(L+1)}\frac{(L-2)!}{(L+2)!}}\sqrt{\frac{2L+1}{4\pi}}\left(\frac{\mathrm{d}P_{L,2}}{\mathrm{d}\mu} \pm 2\frac{P_{L,2}}{1-\mu^2}\right)e^{\pm 2i\varphi}\,. && (40)
\end{aligned}
$$

This leads to

$$
\begin{aligned}
&\int d\varphi \int d\mu e^{-ix\mu}{}_1\mathcal{Y}_{2,\pm2}{}_1\mathcal{Y}^*_{L,\pm2}\\
&= -\frac{1}{\sqrt{L(L+1)}}\sqrt{\frac{(L-2)!}{(L+2)!}}\sqrt{\frac{5(2L+1)}{16}}\int d\mu e^{-ix\mu}(1-\mu^2)(\mu\mp1)\left(\frac{\mathrm{d}P_{L,2}}{\mathrm{d}\mu} \pm 2\frac{P_{L,1}}{1-\mu^2}\right)\\
&= -\frac{1}{\sqrt{L(L+1)}}\sqrt{\frac{(L-2)!}{(L+2)!}}\sqrt{\frac{5(2L+1)}{16}}\int d\mu e^{-ix\mu}\left[(\mu\mp1)(1-\mu^2)\frac{\mathrm{d}P_{L,2}}{\mathrm{d}\mu} \pm 2(\mu\mp1)P_{L,1}\right]\\
&= -\frac{1}{\sqrt{L(L+1)}}\sqrt{\frac{(L-2)!}{(L+2)!}}\sqrt{\frac{5(2L+1)}{16}}\int d\mu e^{-ix\mu}\left[ix(\mu\mp1)(1-\mu^2) - (1-\mu^2) + 2\mu(\mu\mp1) \pm 2(\mu\mp1)\right]\\
&= -\frac{1}{\sqrt{L(L+1)}}\sqrt{\frac{(L-2)!}{(L+2)!}}\sqrt{\frac{5(2L+1)}{16}}\int d\mu e^{-ix\mu}\left[ix(\mu\mp1) - 3\right](1-\mu^2)P_{L,2}\\
&= -\frac{1}{\sqrt{L(L+1)}}\sqrt{\frac{(L-2)!}{(L+2)!}}\sqrt{\frac{5(2L+1)}{4}}\left[-x\partial_x \mp ix - 3\right](-i)^{L+2}\frac{(L+2)!}{(L-2)!}\frac{j_L}{x^2}\\
&= -(-i)^{L+2}\sqrt{\frac{5(2L+1)(L+2)(L-1)}{4}}\left[-\frac{j_L'}{x} \mp i\frac{j_L}{x} - \frac{j_L}{x^2}\right]\,. \qquad (41)
\end{aligned}
$$

Therefore

$$
{}_1\mathcal{J}_L^{(2,\pm2)} = \frac{\sqrt{(L+2)(L-1)}}{2}\left[\frac{j_L'}{x} \pm i\frac{j_L}{x} + \frac{j_L}{x^2}\right]\,. \qquad (42)
$$