[Note]
Geodesic equation for scalar, vector and tensor perturbations

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1 Curl mode induced by vector and tensor perturbation

1.1 Metric Perturbation and Affine Connection

The line element up to first order is given by

$$\mathrm{d}s^2 = a^2(\eta)[-(1+2A)\mathrm{d}\eta - 2B_i\mathrm{d}\eta dx^i + (\gamma_{ij}+2C_{ij})\mathrm{d}x^i\mathrm{d}x^j] \,. \tag{1}$$

Since the path of null geodesic is not affected by conformal transformation, we set the scale factor $a=1$.

The first order perturbation of the affine connection, $\delta\Gamma^\mu_{\nu\rho}$, is given by

$$\delta\Gamma^\mu_{\nu\rho} = \frac{1}{2}\bar{g}^{\mu\nu}[-2h_{\rho\sigma}\bar{\Gamma}^\sigma_{\nu\lambda} + h_{\rho\nu,\lambda} + h_{\rho\lambda,\nu} - h_{\nu\lambda,\rho}] \,. \tag{2}$$

Then, the affine connection $\Gamma^\mu_{\nu\rho}(=\bar{\Gamma}^\mu_{\nu\rho}+\delta\Gamma^\mu_{\nu\rho})$ up to first order becomes

$$\begin{aligned}
\Gamma^0_{00} &= A' \,, \qquad \Gamma^0_{0i} = \Gamma^0_{i0} = A_{,i} \,, \qquad \Gamma^0_{ij} = \frac{1}{2}(B_{i|j}+B_{j|i}) + (C_{ij})' \,, \\
\Gamma^i_{00} &= \gamma^{ij}A_{,j} - (\gamma^{ij}B_j)' \,, \qquad \Gamma^i_{0j} = \frac{1}{2}(B_j^{|i} - B^i_{|j}) + (C^i{}_j)' \,, \\
\overline{\Gamma}^\chi_{ij} &= -(\delta_{i\theta}\delta_{j\theta} + \delta_{i\phi}\delta_{j\phi}\sin^2\theta)\chi \,, \\
\overline{\Gamma}^\theta_{ij} &= \frac{1}{\chi}(\delta_{i\theta}\delta_{j\chi} + \delta_{i\chi}\delta_{j\theta} - \delta_{i\phi}\delta_{j\phi}\chi\sin\theta\cos\theta) \,, \\
\overline{\Gamma}^\phi_{ij} &= \frac{1}{\chi\sin\theta}[(\delta_{i\phi}\delta_{j\chi} + \delta_{i\chi}\delta_{j\phi})\sin\theta + (\delta_{i\phi}\delta_{j\theta} + \delta_{i\theta}\delta_{j\phi})\chi\cos\theta] \,, \\
\delta\Gamma^i_{jk} &= \frac{1}{2}\bar{g}^{ii}[-2C_{il}\bar{\Gamma}^l_{jk} + C_{ij,k} + C_{ik,j} - C_{jk,i}] \,,
\end{aligned} \tag{3}$$

where the prime denotes the derivative with respect to η. Note that components of the affine connection except for Γ^i_{jk} are first order quantity.

1.2 Geodesic Equation

The null geodesics parametrized by the affine parameter $\hat{\lambda}$ are solutions of the geodesic equation,

$$\frac{\mathrm{d}^2 x^\mu(\hat{\lambda})}{\mathrm{d}\hat{\lambda}^2} + \Gamma^\mu_{\nu\rho}(x(\hat{\lambda}))\frac{\mathrm{d}x^\nu(\hat{\lambda})}{\mathrm{d}\hat{\lambda}}\frac{\mathrm{d}x^\rho(\hat{\lambda})}{\mathrm{d}\hat{\lambda}} = 0\,, \tag{4}$$

with $\hat{g}_{\mu\nu}(dx^\mu/d\hat{\lambda})(dx^\nu/d\hat{\lambda}) = 0$ and $d\hat{s}^2 = 0$. The 0-component of the geodesic equation (4) gives

$$\begin{aligned} 0 &= \frac{\mathrm{d}^2\eta}{\mathrm{d}\hat{\lambda}^2} + \Gamma^0_{\nu\rho}\frac{\mathrm{d}x^\nu}{\mathrm{d}\hat{\lambda}}\frac{\mathrm{d}x^\rho}{\mathrm{d}\hat{\lambda}} = \frac{\mathrm{d}^2\eta}{\mathrm{d}\hat{\lambda}^2} + \Gamma^0_{00}\left(\frac{\mathrm{d}\eta}{\mathrm{d}\hat{\lambda}}\right)^2 + 2\Gamma^0_{0i}\frac{\mathrm{d}\eta}{\mathrm{d}\hat{\lambda}}\frac{\mathrm{d}x^i}{\mathrm{d}\hat{\lambda}} + \Gamma^0_{ij}\frac{\mathrm{d}x^i}{\mathrm{d}\hat{\lambda}}\frac{\mathrm{d}x^j}{\mathrm{d}\hat{\lambda}} \\ &= \frac{\mathrm{d}^2\eta}{\mathrm{d}\hat{\lambda}^2} + \left(\frac{\mathrm{d}\eta}{\mathrm{d}\hat{\lambda}}\right)^2\left[\Gamma^0_{00} + 2\Gamma^0_{0i}\frac{\mathrm{d}x^i}{\mathrm{d}\eta} + \Gamma^0_{ij}\frac{\mathrm{d}x^i}{\mathrm{d}\eta}\frac{\mathrm{d}x^j}{\mathrm{d}\eta}\right] \end{aligned} \tag{5}$$

The i-component of the geodesic equation gives

$$\begin{aligned} 0 &= \frac{\mathrm{d}^2 x^i}{\mathrm{d}\hat{\lambda}^2} + \Gamma^i_{\nu\rho}\frac{\mathrm{d}x^\nu}{\mathrm{d}\hat{\lambda}}\frac{\mathrm{d}x^\rho}{\mathrm{d}\hat{\lambda}} = \frac{\mathrm{d}^2 x^i}{\mathrm{d}\hat{\lambda}^2} + \Gamma^i_{00}\left(\frac{\mathrm{d}\eta}{\mathrm{d}\hat{\lambda}}\right)^2 + 2\Gamma^i_{0j}\frac{\mathrm{d}\eta}{\mathrm{d}\hat{\lambda}}\frac{\mathrm{d}x^j}{\mathrm{d}\hat{\lambda}} + \Gamma^i_{jk}\frac{\mathrm{d}x^j}{\mathrm{d}\hat{\lambda}}\frac{\mathrm{d}x^k}{\mathrm{d}\hat{\lambda}} \\ &= \left[\left(\frac{\mathrm{d}\eta}{\mathrm{d}\hat{\lambda}}\right)^2\frac{\mathrm{d}^2 x^i}{\mathrm{d}\eta^2} + \frac{\mathrm{d}x^i}{\mathrm{d}\eta}\frac{\mathrm{d}^2\eta}{\mathrm{d}\hat{\lambda}^2}\right] + \left[\Gamma^i_{00} + 2\Gamma^i_{0j}\frac{\mathrm{d}x^j}{\mathrm{d}\eta} + \Gamma^i_{jk}\frac{\mathrm{d}x^j}{\mathrm{d}\eta}\frac{\mathrm{d}x^k}{\mathrm{d}\eta}\right]\left(\frac{\mathrm{d}\eta}{\mathrm{d}\hat{\lambda}}\right)^2 \\ &= \frac{\mathrm{d}x^i}{\mathrm{d}\eta}\frac{\mathrm{d}^2\eta}{\mathrm{d}\hat{\lambda}^2} + \left[\frac{\mathrm{d}^2 x^i}{\mathrm{d}\eta^2} + \Gamma^i_{00} + 2\Gamma^i_{0j}\frac{\mathrm{d}x^j}{\mathrm{d}\eta} + \Gamma^i_{jk}\frac{\mathrm{d}x^j}{\mathrm{d}\eta}\frac{\mathrm{d}x^k}{\mathrm{d}\eta}\right]\left(\frac{\mathrm{d}\eta}{\mathrm{d}\hat{\lambda}}\right)^2 . \end{aligned} \tag{6}$$

Using Eq.(5), we obtain

$$0 = -F\dot{x}^i + \ddot{x}^i + \Gamma^i_{00} + 2\Gamma^i_{0j}\dot{x}^j + \Gamma^i_{jk}\dot{x}^j\dot{x}^k\,, \tag{7}$$

where the overdot represents the derivative with respect to η and we define

$$F \equiv \Gamma^0_{00} + 2\Gamma^0_{0i}\dot{x}^i + \Gamma^0_{ij}\dot{x}^i\dot{x}^j\,. \tag{8}$$

1.3 Solution of Geodesic Equations

In the absence of the metric perturbation, the photons emitted at the last scattering surface would go straight. This means that, if we observe the photons in a direction (θ,φ), the unperturbed photons path is given by $\bar{x}^i = (\eta_0 - \eta, \theta, \varphi)$, where the quantity η_0 is the conformal time at the observer. To solve Eq. (7), we introduce the perturbed components of the null geodesic, ξ^i, such that $x^i = \bar{x}^i + \xi^i$. Then, at first order, Eq. (7) becomes

$$\begin{aligned} 0 &= F\delta^i_\chi + \ddot{\xi}^i + \Gamma^i_{00} - 2\Gamma^i_{0\chi} - 2\bar{\Gamma}^i_{\chi k}\dot{\xi}^k + 2\delta\Gamma^i_{\chi\chi} \\ &= \ddot{\xi}^i - 2\bar{\Gamma}^i_{\chi k}\dot{\xi}^k + Fn^i + \Gamma^i_{00} - 2\Gamma^i_{0\chi} + 2\delta\Gamma^i_{\chi\chi} \\ &= \ddot{\xi}^i - 2\bar{\Gamma}^i_{\chi k}\dot{\xi}^k + G^i\,, \end{aligned} \tag{9}$$

where we define

$$G^i \equiv F\delta^i_\chi + \Gamma^i_{00} - 2\Gamma^i_{0\chi} + 2\delta\Gamma^i_{\chi\chi}\,. \tag{10}$$

Using Eq. (3), we obtain

$$0 = \ddot{\xi}^\chi + G^\chi \,, \tag{11}$$

$$0 = \ddot{\xi}^a - \frac{2}{\chi}\dot{\xi}^a + G^a \,. \tag{12}$$

where $a = 2$ and 3. Under the condition $\xi^i(\eta_0) = \bar{x}^i$, the solutions of Eq. (12) are

$$\xi^\chi(\eta) = -\int_{\eta_0}^{\eta} \mathrm{d}\eta_1 \int_{\eta_0}^{\eta_1} G^\chi \mathrm{d}\eta_2 \,, \tag{13}$$

$$\xi^a(\eta) = \bar{x}^a + \int_{\eta_0}^{\eta} \mathrm{d}\eta_1 \left[\exp \int_{\eta_0}^{\eta_1} \frac{2\mathrm{d}\eta_2}{\chi}\right] \left[C^a - \int_{\eta_0}^{\eta_1} \left(G^a \mathrm{d}\eta_2 \exp \int_{\eta_0}^{\eta_2} \frac{-2\mathrm{d}\eta_3}{\chi}\right)\right] \,, \tag{14}$$

with the quantities C^θ and C^ϕ being constant. If we use the Born approximation, the expression for ξ^a becomes

$$\xi^a(\eta) - \bar{x}^a = \lim_{\epsilon\to 0} \int_{\eta_0-\epsilon}^{\eta} \frac{\epsilon^2 \mathrm{d}\eta_1}{(\eta_0-\eta_1)^2} \left[C^a - \int_{\eta_0-\epsilon}^{\eta_1} \left(\frac{G^a(\eta_2, \chi(\eta_2)\hat{\boldsymbol{n}})\mathrm{d}\eta_2(\eta_0-\eta_2)^2}{\epsilon^2}\right)\right] \tag{15}$$

$$= -\int_{\eta_0}^{\eta} \mathrm{d}\eta_1 \int_{\eta_0}^{\eta_1} \mathrm{d}\eta_2 \frac{(\eta_0-\eta_2)^2}{(\eta_0-\eta_1)^2} G^a(\eta_2, \chi(\eta_2)\hat{\boldsymbol{n}}) \tag{16}$$

$$= -\int_{\eta_0}^{\eta} \mathrm{d}\eta_2 \int_{\eta_2}^{\eta} \mathrm{d}\eta_1 \frac{(\eta_0-\eta_2)^2}{(\eta_0-\eta_1)^2} G^a(\eta_2, \chi(\eta_2)\hat{\boldsymbol{n}}) \tag{17}$$

$$= -\int_{\eta_0}^{\eta} \mathrm{d}\eta_2 \frac{(\eta_0-\eta_2)(\eta-\eta_2)}{\eta_0-\eta} G^a(\eta_2, \chi(\eta_2)\hat{\boldsymbol{n}}) \tag{18}$$

$$= -\int_{0}^{\chi} \mathrm{d}\chi' \frac{\chi'(\chi-\chi')}{\chi} G^a(\eta_0-\chi', \chi'\hat{\boldsymbol{n}}) \,. \tag{19}$$

Using the expression for the affine connection, we can rewrite Eq. (10) as

$$G^a = \Gamma^a_{00} - 2\Gamma^a_{0\chi} + 2\delta\Gamma^a_{\chi\chi} \tag{20}$$

$$= \gamma^{aa} A_{,a} - (\gamma^{aa} B_a)' - \gamma^{aa}(B_{\chi|a} - B_{a|\chi}) - 2(\gamma^{aa} C_{a\chi})' + \bar{g}^{aa}(2C_{a\chi,\chi} - C_{\chi\chi,a}) \tag{21}$$

$$= \gamma^{aa} \left\{A_{,a} - (B_a + 2C_{a\chi})' - B_{\chi,a} + B_{a,\chi} + 2C_{a\chi,\chi} - C_{\chi\chi,a}\right\} \tag{22}$$

$$= \gamma^{aa} \left\{\frac{\partial}{\partial a}(A - B_\chi - C_{\chi\chi}) - (B_a + 2C_{a\chi})' + B_{a,\chi} + 2C_{a\chi,\chi}\right\} \tag{23}$$

$$= \gamma^{aa} \left\{\frac{\partial}{\partial a}(A - B_\chi - C_{\chi\chi}) - \frac{\mathrm{d}}{\mathrm{d}\chi}(B_a + 2C_{a\chi})\right\} \,, \tag{24}$$

where we use

$$\frac{\mathrm{d}}{\mathrm{d}\chi} = -\frac{\partial}{\partial\eta} + \frac{\mathrm{d}x^i}{\mathrm{d}\eta}\frac{\partial}{\partial x^i} \,. \tag{25}$$

Hereafter, we use

$$\Psi = A - B_\chi - C_{\chi\chi} \,, \tag{26}$$

$$\Omega_a = -B_a - 2C_{a\chi} \,. \tag{27}$$

1.4 Deflection Angle

Using the basis in the polar coordinate

$$\begin{aligned}
\boldsymbol{e}_r &= \boldsymbol{e}_x \sin\theta\cos\varphi + \boldsymbol{e}_y \sin\theta\sin\varphi + \boldsymbol{e}_z\cos\theta && (28)\\
&= \boldsymbol{e}_x\sqrt{1-\mu^2}\cos\varphi + \boldsymbol{e}_y\sqrt{1-\mu^2}\sin\varphi + \boldsymbol{e}_z\mu\,, && (29)\\
\boldsymbol{e}_\theta &= \boldsymbol{e}_x\cos\theta\cos\varphi + \boldsymbol{e}_y\cos\theta\sin\varphi - \boldsymbol{e}_z\sin\theta && (30)\\
&= \boldsymbol{e}_x\mu\cos\varphi + \boldsymbol{e}_y\mu\sin\varphi - \boldsymbol{e}_z\sqrt{1-\mu^2}\,, && (31)\\
\boldsymbol{e}_\varphi &= -\boldsymbol{e}_x\sin\varphi + \boldsymbol{e}_y\cos\varphi\,, && (32)
\end{aligned}$$

the deflection angle, $\boldsymbol{d}$, is defined by [1]

$$\boldsymbol{d}(\theta,\varphi) \equiv \boldsymbol{e}_r(\theta+\xi^\theta,\varphi+\xi^\varphi) - \boldsymbol{e}_r(\theta,\varphi) \simeq \boldsymbol{e}_\theta\xi^\theta + \boldsymbol{e}_\varphi\xi^\varphi\sin\theta + \mathcal{O}(\xi^2)\,. \qquad (33)$$

The “convergence” and “rotation” ($D^\oplus$ and $D^\otimes$ [2]) are given by

$$\begin{aligned}
2D^\oplus \equiv -\boldsymbol{\nabla}\cdot\boldsymbol{d} &= -\int_0^{\chi_s}\mathrm{d}\chi\frac{(\chi_s-\chi)}{\chi_s\chi}\boldsymbol{\nabla}\cdot\left[\boldsymbol{e}_\theta\left(\frac{\partial\Psi}{\partial\theta}-\frac{\mathrm{d}\Omega_\theta}{\mathrm{d}\chi}\right)+\frac{\boldsymbol{e}_\varphi}{\sin\theta}\left(\frac{\partial\Psi}{\partial\varphi}-\frac{\mathrm{d}\Omega_\varphi}{\mathrm{d}\chi}\right)\right] && (34)\\
&= -\int_0^{\chi_s}\mathrm{d}\chi\frac{(\chi_s-\chi)}{\chi_s\chi}\boldsymbol{\nabla}\cdot\left[\boldsymbol{\nabla}\Psi-\left(\boldsymbol{e}_\theta\frac{\mathrm{d}\Omega_\theta}{\mathrm{d}\chi}+\frac{\boldsymbol{e}_\varphi}{\sin\theta}\frac{\mathrm{d}\Omega_\varphi}{\mathrm{d}\chi}\right)\right] && (35)\\
&= -\int_0^{\chi_s}\mathrm{d}\chi\frac{(\chi_s-\chi)}{\chi_s\chi}\left[\nabla^2\Psi-\boldsymbol{\nabla}\cdot\left(\boldsymbol{e}_\theta\frac{\mathrm{d}\Omega_\theta}{\mathrm{d}\chi}+\frac{\boldsymbol{e}_\varphi}{\sin\theta}\frac{\mathrm{d}\Omega_\varphi}{\mathrm{d}\chi}\right)\right]\,, && (36)\\
2D^\otimes \equiv (\star\boldsymbol{\nabla})\cdot\boldsymbol{d} &= -\int_0^{\chi_s}\mathrm{d}\chi\frac{(\chi_s-\chi)}{\chi_s\chi}(\star\boldsymbol{\nabla})\cdot\left[\boldsymbol{\nabla}\Psi-\left(\boldsymbol{e}_\theta\frac{\mathrm{d}\Omega_\theta}{\mathrm{d}\chi}+\frac{\boldsymbol{e}_\varphi}{\sin\theta}\frac{\mathrm{d}\Omega_\varphi}{\mathrm{d}\chi}\right)\right] && (37)\\
&= \int_0^{\chi_s}\mathrm{d}\chi\frac{(\chi_s-\chi)}{\chi_s\chi}\left(\boldsymbol{e}_\varphi\frac{\partial}{\partial\theta}-\frac{\boldsymbol{e}_\theta}{\sin\theta}\frac{\partial}{\partial\varphi}\right)\cdot\left(\boldsymbol{e}_\theta\frac{\mathrm{d}\Omega_\theta}{\mathrm{d}\chi}+\frac{\boldsymbol{e}_\varphi}{\sin\theta}\frac{\mathrm{d}\Omega_\varphi}{\mathrm{d}\chi}\right) && (38)\\
&= \int_0^{\chi_s}\mathrm{d}\chi\frac{(\chi_s-\chi)}{\chi_s\chi}\frac{1}{\sin\theta}\left(\frac{\partial}{\partial\varphi}\frac{\mathrm{d}\Omega_\theta}{\mathrm{d}\chi}-\frac{\partial}{\partial\theta}\frac{\mathrm{d}\Omega_\varphi}{\mathrm{d}\chi}\right)\,. && (39)
\end{aligned}$$

Note that, if we only consider the scalar perturbations in the metric, the rotation vanishes and the expression for convergence is consistent with the results obtained in Ref. [3].

1.5 Angular Power Spectrum

Let us consider the vector mode in the metric perturbation, $\boldsymbol{\Omega}$;

$$\boldsymbol{\Omega}(\eta_0-\chi,\chi,\theta,\varphi) = \int\frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3}\,\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot(\chi\boldsymbol{e}_r)}\{\Omega_{\boldsymbol{k},+}\boldsymbol{e}_+ + \Omega_{\boldsymbol{k},-}\boldsymbol{e}_-\}\,. \qquad (40)$$

The perturbation $\boldsymbol{\Omega}$ is divergence-less ($\boldsymbol{\nabla}\cdot\boldsymbol{\Omega}=0$). Note that the quantities $\Omega_{\boldsymbol{k},+}$ and $\Omega_{\boldsymbol{k},-}$ are the function of χ. If we choose

$$\boldsymbol{k} = k\boldsymbol{e}_z, \qquad \boldsymbol{e}_\pm = \boldsymbol{e}_x \pm \mathrm{i}\boldsymbol{e}_y, \qquad (41)$$

we obtain

$$\boldsymbol{k}\cdot\boldsymbol{e}_r = k\mu\,, \qquad \boldsymbol{e}_\pm\cdot\boldsymbol{e}_\theta = \mu\,\mathrm{e}^{\pm\mathrm{i}\varphi}\,, \qquad \boldsymbol{e}_\pm\cdot\boldsymbol{e}_\varphi = -\sin\varphi\pm\mathrm{i}\cos\varphi = \pm\mathrm{i}\,\mathrm{e}^{\pm\mathrm{i}\varphi}\,. \qquad (42)$$

Since

$$\Omega_i \mathrm{d}x^i = \boldsymbol{\Omega}\cdot\mathrm{d}\boldsymbol{x} \tag{43}$$
$$= (\tilde{\Omega}_r\boldsymbol{e}_r + \tilde{\Omega}_\theta\boldsymbol{e}_\theta + \tilde{\Omega}_\varphi\boldsymbol{e}_\varphi)\cdot(\boldsymbol{e}_r\mathrm{d}\chi + \chi\boldsymbol{e}_\theta\mathrm{d}\theta + \chi\boldsymbol{e}_\varphi\sin\theta\mathrm{d}\varphi) \tag{44}$$
$$= \tilde{\Omega}_r\mathrm{d}\chi + \tilde{\Omega}_\theta\chi\mathrm{d}\theta + \tilde{\Omega}_\varphi\chi\sin\theta\mathrm{d}\varphi \tag{45}$$

we obtain

$$\Omega_\theta = \chi(\boldsymbol{\Omega}\cdot\boldsymbol{e}_\theta) = \chi\mu\int\frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3}\,\mathrm{e}^{-\mathrm{i}k\chi\mu}\{\Omega_{\boldsymbol{k},+}\mathrm{e}^{\mathrm{i}\varphi} + \Omega_{\boldsymbol{k},-}\,\mathrm{e}^{-\mathrm{i}\varphi}\}\,, \tag{46}$$

$$\Omega_\varphi = \chi\sqrt{1-\mu^2}(\boldsymbol{\Omega}\cdot\boldsymbol{e}_\varphi) = \mathrm{i}\chi\sqrt{1-\mu^2}\int\frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3}\,\mathrm{e}^{-\mathrm{i}k\chi\mu}\{\Omega_{\boldsymbol{k},+}\mathrm{e}^{\mathrm{i}\varphi} - \Omega_{\boldsymbol{k},-}\,\mathrm{e}^{-\mathrm{i}\varphi}\}\,, \tag{47}$$

$$\Omega_{\theta,\varphi} = \mathrm{i}\chi\mu\int\frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3}\,\mathrm{e}^{-\mathrm{i}k\chi\mu}\{\Omega_{\boldsymbol{k},+}\mathrm{e}^{\mathrm{i}\varphi} - \Omega_{\boldsymbol{k},-}\,\mathrm{e}^{-\mathrm{i}\varphi}\}\,, \tag{48}$$

$$\Omega_{\varphi,\theta} = \int\frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3}\,[\mathrm{i}\chi\mu - k\chi^2(1-\mu^2)]\,\mathrm{e}^{-\mathrm{i}k\chi\mu}\{\Omega_{\boldsymbol{k},+}\mathrm{e}^{\mathrm{i}\varphi} - \Omega_{\boldsymbol{k},-}\,\mathrm{e}^{-\mathrm{i}\varphi}\} \tag{49}$$

Substituting this into Eq. (39), the rotation becomes

$$D^{\otimes} = \frac{1}{2}\int_0^{\chi_s}\mathrm{d}\chi\frac{(\chi_s-\chi)}{\chi_s\chi}\frac{\mathrm{d}}{\mathrm{d}\chi}\int\frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3}\,\mathrm{e}^{-\mathrm{i}k\chi\mu}k\chi^2\sqrt{1-\mu^2}\{\Omega_{\boldsymbol{k},+}\mathrm{e}^{\mathrm{i}\varphi} - \Omega_{\boldsymbol{k},-}\,\mathrm{e}^{-\mathrm{i}\varphi}\} \tag{50}$$
$$= \frac{1}{2}\int_0^{\chi_s}\mathrm{d}\chi\int\frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3}\,\mathrm{e}^{-\mathrm{i}k\chi\mu}k\sqrt{1-\mu^2}\{\Omega_{\boldsymbol{k},+}\mathrm{e}^{\mathrm{i}\varphi} - \Omega_{\boldsymbol{k},-}\,\mathrm{e}^{-\mathrm{i}\varphi}\}\,. \tag{51}$$

The multipole coefficients of the rotation then become

$$D^{\otimes}_{\ell,m} = \frac{1}{2}\int_0^{\chi_s}\mathrm{d}\chi\int\frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3}\int_{-1}^{1}\mathrm{d}\mu\int_0^{2\pi}\mathrm{d}\varphi Y_{\ell,m}(\hat{\boldsymbol{n}})\,\mathrm{e}^{-\mathrm{i}k\chi\mu}k(1-\mu^2)^{1/2}\{\Omega_{\boldsymbol{k},+}\mathrm{e}^{\mathrm{i}\varphi} - \Omega_{\boldsymbol{k},-}\,\mathrm{e}^{-\mathrm{i}\varphi}\} \tag{52}$$

$$= \frac{1}{2}\sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}\int_0^{\chi_s}\mathrm{d}\chi\int\frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3}$$
$$\times\int_{-1}^{1}\mathrm{d}\mu\,\mathrm{e}^{-\mathrm{i}k\chi\mu}k(1-\mu^2)^{1/2}P_{\ell,m}(\mu)\int_0^{2\pi}\mathrm{d}\varphi\,\mathrm{e}^{\mathrm{i}m\varphi}\{\Omega_{\boldsymbol{k},+}\mathrm{e}^{\mathrm{i}\varphi} - \Omega_{\boldsymbol{k},-}e^{-\mathrm{i}\varphi}\} \tag{53}$$

$$= \frac{1}{2}\sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}\int_0^{\chi_s}\mathrm{d}\chi\int\frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3}$$
$$\times\int_{-1}^{1}\mathrm{d}\mu\,\mathrm{e}^{-\mathrm{i}k\chi\mu}k(1-\mu^2)^{1/2}2\pi\{\Omega_{\boldsymbol{k},+}P_{\ell,-1}(\mu)\delta_{m,-1} - \Omega_{\boldsymbol{k},-}P_{\ell,1}(\mu)\delta_{m,1}\} \tag{54}$$

$$= \pi\sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-1)!}{(\ell+1)!}}\int_0^{\chi_s}\mathrm{d}\chi\int\frac{d^3\boldsymbol{k}}{(2\pi)^3}$$
$$\times\, k\,\{\Omega_{\boldsymbol{k},+}(-1)^m\delta_{m,-1} - \Omega_{\boldsymbol{k},-}\delta_{m,1}\}\int_{-1}^{1}\mathrm{d}\mu\,\mathrm{e}^{-\mathrm{i}k\chi\mu}(1-\mu^2)^{1/2}P_{\ell,1}(\mu)\,. \tag{55}$$

Using the following relation

$$\int_{-1}^{1} \mathrm{d}\mu\, \mathrm{e}^{\pm \mathrm{i}k\chi\mu}(1-\mu^2)^{1/2}P_{\ell,1}(\mu) = (\pm\mathrm{i})^{\ell+1}2\ell(\ell+1)\frac{j_\ell(x)}{x}\bigg|_{x=k\chi}, \tag{56}$$

Eq. (55) is rewritten as

$$D^{\otimes}_{\ell,m} = -2\pi(-\mathrm{i})^{\ell+1}\ell(\ell+1)\sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-1)!}{(\ell+1)!}}\int_0^{\chi_s}\mathrm{d}\chi\int\frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3}\,\frac{j_\ell(x)}{x}\bigg|_{x=k\chi}k\left\{\Omega_{\boldsymbol{k},+}(-1)^m\delta_{m,-1}-\Omega_{\boldsymbol{k},-}\delta_{m,1}\right\}\ . \tag{57}$$

Finally, the angular power spectrum of rotation becomes

$$C^{\otimes}_\ell = \langle\frac{1}{2\ell+1}\sum_{m=-\ell}^{\ell}|D^{\otimes}_{\ell,m}|^2\rangle = \frac{1}{2\ell+1}\Big(\langle|D^{\otimes}_{\ell,-1}|^2+|D^{\otimes}_{\ell,1}|^2\rangle\Big) \tag{58}$$

$$= \frac{4\pi^2\ell^2(\ell+1)^2}{2\ell+1}\frac{2\ell+1}{4\pi}\frac{(\ell-1)!}{(\ell+1)!}\int_0^{\chi_s}\mathrm{d}\chi\int_0^{\chi_s}\mathrm{d}\chi'\int\frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3}\int\frac{\mathrm{d}^3\boldsymbol{k}'}{(2\pi)^3} \tag{59}$$

$$\times\, k\frac{j_\ell(x)}{x}\bigg|_{x=k\chi}k'\frac{j_\ell(x')}{x'}\bigg|_{x'=k'\chi'}\Big(\langle\Omega^*_{\boldsymbol{k},+}\Omega_{\boldsymbol{k},+}\rangle+\langle\Omega^*_{\boldsymbol{k},-}\Omega_{\boldsymbol{k},-}\rangle\Big) \tag{60}$$

$$= \frac{\ell(\ell+1)}{2\pi}\int_0^{\chi_s}\frac{\mathrm{d}\chi}{\chi}\int_0^{\chi_s}\frac{\mathrm{d}\chi'}{\chi'}\int\mathrm{d}k\ k^2 j_\ell(k\chi)j_\ell(k\chi')P_\omega(k,\chi,\chi')\,. \tag{61}$$

References

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