Note for full-sky lensing reconstruction of gradient and curl modes from CMB

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1 Purpose

Several studies have investigated a method to reconstruct the deflection angle, which characterizes the effect of weak lensing on CMB maps (e.g., [1, 2, 3, 4, 5, 6, 7, 8, 9]). In general, the deflection angle in a direction $\hat{\boldsymbol{n}}$, $\boldsymbol{d}(\hat{\boldsymbol{n}})$, where $\hat{\boldsymbol{n}}$ is the unit vector defined on the unit sphere, is decomposed into gradient and curl part as [7, 10]

$$\boldsymbol{d}(\hat{\boldsymbol{n}}) = \boldsymbol{\nabla}\phi(\hat{\boldsymbol{n}}) + (\star\boldsymbol{\nabla})\varpi(\hat{\boldsymbol{n}})\,, \tag{1}$$

where the first term, $\boldsymbol{\nabla}\phi(\hat{\boldsymbol{n}})$, and second term, $(\star\boldsymbol{\nabla})\varpi(\hat{\boldsymbol{n}})$, represent gradient and curl mode of deflection angle, respectively, and, in the polar coordinate, the covariant derivative on the unit sphere is given by $\boldsymbol{\nabla} = \boldsymbol{e}_\theta(\partial/\partial\theta) + (\boldsymbol{e}_\varphi/\sin\theta)(\partial/\partial\varphi)$ with $\boldsymbol{e}_\theta$ and $\boldsymbol{e}_\varphi$ describing the basis vectors in the polar coordinate. The symbol, $\star$, denotes a operation which rotates the angle of two-dimensional vector counterclockwise by 90-degree; for a vector on the unit sphere expressed in terms of the basis vectors, $\boldsymbol{a} = a_\theta\boldsymbol{e}_\theta + a_\varphi\boldsymbol{e}_\varphi$, the operator, $\star$, act on $\boldsymbol{a}$ as $(\star\boldsymbol{a}) = a_\theta\boldsymbol{e}_\varphi - a_\varphi\boldsymbol{e}_\theta$ [7]. Hereafter, we call the potentials, ϕ and ϖ, "scalar lensing potential" and "pseudo-scalar lensing potential", respectively. In this work, we consider a reconstruction method for both gradient and curl modes from CMB maps without flat-sky approximation.

2 Weak lensing of the CMB

Here we briefly review the lensing effect on the CMB anisotropies in the flat- and full-sky cases, including both the gradient and curl modes of deflection angle. We first discuss the lensing effect on CMB in the flat-sky limit, and the similar discussion for full sky is given in the next. The detailed calculation of lensing effect is presented in, e.g., Ref. [11] in the absence of the curl mode, and in Ref. [12] including the curl mode.

The lensed CMB anisotropies, $\widetilde{X}(\hat{\boldsymbol{n}})$ $(=\Theta, Q\pm \mathrm{i}U)$, are expressed by the unlensed anisotropies in the deflected direction, $X(\hat{\boldsymbol{n}}+\boldsymbol{d})$. Then, the lensed CMB anisotropies is expanded in terms of the CMB deflection angle $\boldsymbol{d}$ [13];

$$\widetilde{X}(\hat{\boldsymbol{n}}) = X(\hat{\boldsymbol{n}}+\boldsymbol{d}) = X(\hat{\boldsymbol{n}}) + \boldsymbol{d}\cdot\boldsymbol{\nabla}X(\hat{\boldsymbol{n}}) + \mathcal{O}(|\boldsymbol{d}|^2)\,. \tag{2}$$

where the deflection angle of CMB can be expressed in Eq. (1). Usually, the deflection angle is a small perturbed quantity, $|\boldsymbol{d}| \ll 1$. Hereafter we neglect the contributions of $\mathcal{O}(|\boldsymbol{d}|^2)$ in Eq. (2).

2.1 Flat sky limit

Let us consider the lensing effect on the CMB temperature, $\Theta(\hat{\boldsymbol{n}})$, and polarization $Q(\hat{\boldsymbol{n}}) \pm \mathrm{i}U(\hat{\boldsymbol{n}})$ in the flat-sky limit, where the fluctuations are decomposed with the two-dimensional plane wave. In the two-dimensional Fourier space, instead of the spin-2 quantities $Q_{\boldsymbol{\ell}} \pm \mathrm{i}U_{\boldsymbol{\ell}}$, we use the rotational invariant combination, E- and B-mode. We define the Fourier coefficients, $\Theta_{\boldsymbol{\ell}}$, $E_{\boldsymbol{\ell}}$ and $B_{\boldsymbol{\ell}}$ as [11, 5]

$$\Theta_{\boldsymbol{\ell}} = \int \mathrm{d}^2\hat{\boldsymbol{n}}\; \mathrm{e}^{-\mathrm{i}\hat{\boldsymbol{n}}\cdot\boldsymbol{\ell}}\,\Theta(\hat{\boldsymbol{n}})\,, \tag{3}$$

$$[E \pm \mathrm{i}B]_{\boldsymbol{\ell}} = \int \mathrm{d}^2\hat{\boldsymbol{n}}\; \mathrm{e}^{-\mathrm{i}\hat{\boldsymbol{n}}\cdot\boldsymbol{\ell}}\,[Q \pm \mathrm{i}U](\hat{\boldsymbol{n}})\,\mathrm{e}^{\pm 2\mathrm{i}(\varphi_{\boldsymbol{\ell}} - \varphi_{\hat{n}})}\,. \tag{4}$$

The lensed fields are expressed in terms of the unlensed fields, $X_{\boldsymbol{\ell}}$ as

$$\widetilde{X}_{\boldsymbol{\ell}} = X_{\boldsymbol{\ell}} + \sum_{X'=\Theta,E,B} \hat{L}^X_{\boldsymbol{\ell}}(X')\,, \tag{5}$$

where we introduce the operators $\hat{L}^X_{\boldsymbol{\ell}}$:

$$\hat{L}^X_{\boldsymbol{\ell}}(X') = -\sum_x \int \frac{\mathrm{d}^2\boldsymbol{\ell}'}{(2\pi)^2} \{(\boldsymbol{\ell} - \boldsymbol{\ell}') \odot_x \boldsymbol{\ell}'\} R^{XX'}_{\boldsymbol{\ell},\boldsymbol{\ell}'}\, x_{\boldsymbol{\ell}-\boldsymbol{\ell}'} X'_{\boldsymbol{\ell}'}\,, \tag{6}$$

The symbols ($\odot_\phi$ and $\odot_\varpi$) and the matrix $R^{XX'}_{\boldsymbol{\ell},\boldsymbol{\ell}'}$ are defined as

$$\boldsymbol{a} \odot_\phi \boldsymbol{b} \equiv \boldsymbol{a} \cdot \boldsymbol{b} = a_1 b_1 + a_2 b_2\,, \tag{7}$$

$$\boldsymbol{a} \odot_\varpi \boldsymbol{b} \equiv (\star\boldsymbol{a}) \cdot \boldsymbol{b} = a_2 b_1 - a_1 b_2 = -\boldsymbol{b} \odot_\varpi \boldsymbol{a}\,, \tag{8}$$

$$R^{XX'}_{\boldsymbol{\ell},\boldsymbol{\ell}'} \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos 2\varphi_{\boldsymbol{\ell}',\boldsymbol{\ell}} & -\sin 2\varphi_{\boldsymbol{\ell}',\boldsymbol{\ell}} \\ 0 & \sin 2\varphi_{\boldsymbol{\ell}',\boldsymbol{\ell}} & \cos 2\varphi_{\boldsymbol{\ell}',\boldsymbol{\ell}} \end{pmatrix}\,. \tag{9}$$

with $\varphi_{\boldsymbol{\ell}',\boldsymbol{\ell}} = \varphi_{\boldsymbol{\ell}'} - \varphi_{\boldsymbol{\ell}}$. Note that $-R^{EB}_{\boldsymbol{\ell},\boldsymbol{\ell}'} = R^{BE}_{\boldsymbol{\ell},\boldsymbol{\ell}'} = \sin 2\varphi_{\boldsymbol{\ell}',\boldsymbol{\ell}}$ and $R^{XX'}_{\boldsymbol{\ell},-\boldsymbol{\ell}'} = -R^{XX'}_{\boldsymbol{\ell},\boldsymbol{\ell}'}$ for $XX' = EB$ or BE.

2.2 Full sky

2.2.1 Lensing effect on CMB anisotropies: temperature

Let us next discuss the lensing effect on the temperature anisotropies in full sky. The lensed temperature fluctuations $\widetilde{\Theta}(\hat{\boldsymbol{n}})$, are transformed into the harmonic space according to

$$\widetilde{\Theta}_{\ell m} = \int \mathrm{d}^2\hat{\boldsymbol{n}}\; Y^*_{\ell m}(\hat{\boldsymbol{n}})\widetilde{\Theta}(\hat{\boldsymbol{n}})\,, \tag{10}$$

with the quantities, $\widetilde{\Theta}_{\ell m}$ and $Y_{\ell m}(\hat{\boldsymbol{n}})$, describing the harmonic coefficients, and the spin-0 spherical harmonics, respectively. The harmonic coefficients of the lensed quantities are obtained by transforming Eq. (2) into the harmonic space, according to Eq. (10). Using the expression of deflection angle (1), the lensed temperature anisotropies in the harmonic space are given by [12]

$$\begin{aligned}\widetilde{\Theta}_{LM} &= \Theta_{LM} + \int \mathrm{d}^2\hat{\boldsymbol{n}}\ Y^*_{LM}(\hat{\boldsymbol{n}})\big[\boldsymbol{\nabla}\phi(\hat{\boldsymbol{n}}) + (\star\boldsymbol{\nabla})\varpi(\hat{\boldsymbol{n}})\big]\cdot\boldsymbol{\nabla}\Theta(\hat{\boldsymbol{n}}) \\ &= \Theta_{LM} + \sum_{\ell\ell' mm'} \Theta_{\ell' m'}(-1)^M \begin{pmatrix} L & \ell & \ell' \\ -M & m & m' \end{pmatrix} \sum_x \mathcal{S}^{x,0}_{L\ell\ell'} x_{\ell m}\,, \end{aligned} \tag{11}$$

where the summation $\sum_x$ is over $x = \phi$ and ϖ, and the quantities ${}_0\mathcal{S}^x_{L\ell\ell'}$ are defined as

$$\begin{pmatrix} L & \ell & \ell' \\ M & m & m' \end{pmatrix} \mathcal{S}^{x,0}_{L\ell\ell'} = \int \mathrm{d}\hat{\boldsymbol{n}}\ Y_{LM}(\boldsymbol{\nabla}Y_{\ell m}) \odot_x (\boldsymbol{\nabla}Y_{\ell' m'})\,. \tag{12}$$

The quantities $\mathcal{S}^{x,0}_{L\ell,\ell'}$ are expressed in terms of the Wigner-$3j$ symbols:

$$\mathcal{S}^{\phi,0}_{L\ell\ell'} = \sqrt{\frac{(2\ell+1)(2\ell'+1)(2L+1)}{16\pi}}[-L(L+1)+\ell(\ell+1)+\ell'(\ell'+1)]\begin{pmatrix} L & \ell & \ell' \\ 0 & 0 & 0 \end{pmatrix}\,, \tag{13}$$

$$\mathcal{S}^{\varpi,0}_{L\ell\ell'} = -\mathrm{i}\sqrt{\frac{(2\ell+1)(2\ell'+1)(2L+1)}{16\pi}}\sqrt{\ell(\ell+1)}\sqrt{\ell'(\ell'+1)}\left[\begin{pmatrix} L & \ell & \ell' \\ 0 & -1 & 1 \end{pmatrix} - \begin{pmatrix} L & \ell & \ell' \\ 0 & 1 & -1 \end{pmatrix}\right]\,. \tag{14}$$

We note that the quantities ${}_0\mathcal{S}^x_{L\ell\ell'}$ satisfy

$$\mathcal{S}^{\phi,0}_{L\ell\ell'} = (-1)^{L+\ell+\ell'}\mathcal{S}^{\phi,0}_{L\ell\ell'}\,, \qquad \mathcal{S}^{\varpi,0}_{L\ell\ell'} = -(-1)^{L+\ell+\ell'}\mathcal{S}^{\varpi,0}_{L\ell\ell'}\,. \tag{15}$$

The above equations come from the parity symmetry of Θ, ϕ and ϖ; the temperature anisotropies and the scalar lensing potential are even parity, while the pseudo-scalar lensing potential is odd parity. In fact, Eq. (15) is checked by changing the variable, $\hat{\boldsymbol{n}} \to -\hat{\boldsymbol{n}}$ in the r.h.s. of Eq. (12). Under this transformation, the spin-0 spherical harmonics are multiplied by a factor $(-1)^\ell$, and the derivatives become $\boldsymbol{\nabla} \to -\boldsymbol{\nabla}$ and $(\star\boldsymbol{\nabla}) \to (\star\boldsymbol{\nabla})$, respectively. As a result, the r.h.s. of Eq. (12) are multiplied by a factor of $(-1)^{L+\ell+\ell'}$ and $-(-1)^{L+\ell+\ell'}$, respectively. Eq. (15) is also checked with the formulas of the Wigner $3j$ symbols.

From Eq. (15), $\mathcal{S}^{\phi,0}_{L\ell\ell'}$ becomes zero if $L+\ell+\ell'$ is an odd integer, and the coefficient $\mathcal{S}^{\varpi,0}_{L\ell\ell'}$ vanishes when $L+\ell+\ell'$ is an even integer. These properties are essential for a separate reconstruction of gradient and curl modes in subsequent analysis.

2.2.2 Lensing effect on CMB anisotropies: polarizations

Finally we consider the lensing effect on the CMB polarizations. We are especially concerned with the rotationally invariant combinations (E- and B-mode polarizations [12]):

$$[\widetilde{E} \pm \mathrm{i}\widetilde{B}]_{LM} = [E \pm \mathrm{i}B]_{LM} + \int \mathrm{d}^2\hat{\boldsymbol{n}}\ (Y^{\pm 2}_{LM})^*(\hat{\boldsymbol{n}})\big[\boldsymbol{\nabla}\phi(\hat{\boldsymbol{n}}) + (\star\boldsymbol{\nabla})\varpi(\hat{\boldsymbol{n}})\big]\boldsymbol{\nabla}(Q \pm \mathrm{i}U)(\hat{\boldsymbol{n}}) \tag{16}$$

$$= [E \pm \mathrm{i}B]_{LM} + \sum_{\ell\ell' mm'} [E \pm \mathrm{i}B]_{\ell' m'}(-1)^M \begin{pmatrix} L & \ell & \ell' \\ -M & m & m' \end{pmatrix} \sum_x \mathcal{S}^{x,\pm 2}_{L\ell\ell'} x_{\ell m}\,, \tag{17}$$

with the quantities, $\mathcal{S}^{x,\pm 2}_{L\ell\ell'}$ defined by

$$\begin{pmatrix} L & \ell & \ell' \\ M & m & m' \end{pmatrix} \mathcal{S}^{x,\pm 2}_{L\ell\ell'} = \int \mathrm{d}^2\hat{\boldsymbol{n}}\; Y^{\pm 2}_{LM}[\boldsymbol{\nabla} Y_{\ell m}] \odot_x [\boldsymbol{\nabla} Y^{\pm 2}_{\ell' m'}] \,. \tag{18}$$

The quantity $Y^{\pm 2}_{\ell' m'}(\hat{\boldsymbol{n}})$ denotes the spin-± 2 spherical harmonics. Similar to the case of temperature anisotropies, the quantities $\mathcal{S}^{x,\pm 2}_{L\ell\ell'}$ are written as [12]

$$\mathcal{S}^{\phi,\pm 2}_{L\ell\ell'} = \sqrt{\frac{(2\ell+1)(2\ell'+1)(2L+1)}{16\pi}} [\ell(\ell+1) + \ell'(\ell'+1) - L(L+1)] \begin{pmatrix} L & \ell & \ell' \\ \pm 2 & 0 & \mp 2 \end{pmatrix} , \tag{19}$$

$$\mathcal{S}^{\varpi,\pm 2}_{L\ell\ell'} = -\mathrm{i}\sqrt{\frac{(2\ell+1)(2\ell'+1)(2L+1)}{16\pi}} \sqrt{\ell(\ell+1)} \sqrt{(\ell' \pm 2)(\ell'+1\pm 2)} \times \left[\sqrt{\frac{\ell'+1\mp 2}{\ell'+1\pm 2}} \begin{pmatrix} L & \ell & \ell' \\ \pm 2 & -1 & 1\mp 2 \end{pmatrix} - \sqrt{\frac{\ell'\mp 2}{\ell'\pm 2}} \begin{pmatrix} L & \ell & \ell' \\ \pm 2 & 1 & -1\mp 2 \end{pmatrix} \right] . \tag{20}$$

Eq. (17) is rewritten in the separable form for E- and B-mode polarizations:

$$\widetilde{E}_{LM} = E_{LM} + \sum_{\ell\ell' mm'} (-1)^M \begin{pmatrix} L & \ell & \ell' \\ -M & m & m' \end{pmatrix} \sum_x x_{\ell m} \{ \mathcal{S}^{x,+}_{L\ell\ell'} E_{\ell' m'} - \mathcal{S}^{x,-}_{L\ell\ell'} B_{\ell' m'} \} , \tag{21}$$

$$\widetilde{B}_{LM} = B_{LM} + \sum_{\ell\ell' mm'} (-1)^M \begin{pmatrix} L & \ell & \ell' \\ -M & m & m' \end{pmatrix} \sum_x x_{\ell m} \{ \mathcal{S}^{x,-}_{L,\ell,\ell'} E_{\ell' m'} + \mathcal{S}^{x,+}_{L,\ell,\ell'} B_{\ell' m'} \} , \tag{22}$$

where we define

$$\mathcal{S}^{x,+}_{L\ell\ell'} = \frac{\mathcal{S}^{x,2}_{L\ell\ell'} + \mathcal{S}^{x,-2}_{L\ell\ell'}}{2} , \qquad \mathcal{S}^{x,-}_{L\ell\ell'} = \frac{\mathcal{S}^{x,2}_{L\ell\ell'} - \mathcal{S}^{x,-2}_{L\ell\ell'}}{2\mathrm{i}} . \tag{23}$$

Note again that, for an even integer of $L+\ell+\ell'$, the coefficients $\mathcal{S}^{\varpi,+}$ and $\mathcal{S}^{\phi,-}$ vanish. On the other hand, the quantities $\mathcal{S}^{\phi,+}$ and $\mathcal{S}^{\varpi,-}$ vanish when $L+\ell+\ell'$ is an odd integer. Similar to the case of temperature, these properties come from the fact that E-mode polarization and scalar lensing potential are even parity, while B-mode polarization and pseudo-scalar lensing potential are odd parity.

2.2.3 Expression with matrix

The lensing effect on the anisotropies are given by the unified form:

$$\widetilde{X}_{LM} = X_{LM} + \sum_x \sum_{\ell\ell' mm'} \sum_{X'=\Theta,E,B} x_{\ell m} (R^x_{\ell\ell' L})^{XX'} X'_{\ell' m'} (-1)^M \begin{pmatrix} \ell & \ell' & L \\ m & m' & -M \end{pmatrix} , \tag{24}$$

with the 3×3 matrix $(R^x_{\ell\ell' L})^{XX'}$

$$(R^x_{\ell\ell' L})^{XX'} = \begin{pmatrix} \mathcal{S}^{x,0}_{\ell\ell' L} & 0 & 0 \\ 0 & \mathcal{S}^{x,+}_{\ell\ell' L} & -\mathcal{S}^{x,-}_{\ell\ell' L} \\ 0 & \mathcal{S}^{x,-}_{\ell\ell' L} & \mathcal{S}^{x,+}_{\ell\ell' L} \end{pmatrix} \tag{25}$$

Table 1: Functional forms of $\overline{f}^{x,(XX')}_{\boldsymbol{\ell},\boldsymbol{L},\boldsymbol{L}'}$ in the flat-sky case.

XX'	$\overline{f}^{\phi,(XX')}_{\boldsymbol{\ell},\boldsymbol{L},\boldsymbol{L}'}$	$\overline{f}^{\varpi,(XX')}_{\boldsymbol{\ell},\boldsymbol{L},\boldsymbol{L}'}$
ΘΘ	$C^{\Theta\Theta}_L \boldsymbol{\ell}\cdot\boldsymbol{L} + C^{\Theta\Theta}_{L'}\boldsymbol{\ell}\cdot\boldsymbol{L}'$	$C^{\Theta\Theta}_L(\star\boldsymbol{\ell})\cdot\boldsymbol{L} + C^{\Theta\Theta}_{L'}(\star\boldsymbol{\ell})\cdot\boldsymbol{L}'$
ΘE	$C^{\Theta\mathrm{E}}_L \boldsymbol{\ell}\cdot\boldsymbol{L}\cos 2\varphi_{L,L'} + C^{\Theta\mathrm{E}}_{L'}\boldsymbol{\ell}\cdot\boldsymbol{L}'$	$C^{\Theta\mathrm{E}}_L(\star\boldsymbol{\ell})\cdot\boldsymbol{L}\cos 2\varphi_{L,L'} + C^{\Theta\mathrm{E}}_{L'}(\star\boldsymbol{\ell})\cdot\boldsymbol{L}'$
ΘB	$C^{\Theta\mathrm{E}}_L \boldsymbol{\ell}\cdot\boldsymbol{L}\sin 2\varphi_{L,L'}$	$C^{\Theta\mathrm{E}}_L(\star\boldsymbol{\ell})\cdot\boldsymbol{L}\sin 2\varphi_{L,L'}$
EE	$[\boldsymbol{\ell}\cdot\boldsymbol{L}C^{\mathrm{EE}}_L + \boldsymbol{\ell}\cdot\boldsymbol{L}'C^{\mathrm{EE}}_{L'}]\cos 2\varphi_{L,L'}$	$[(\star\boldsymbol{\ell})\cdot\boldsymbol{L}C^{\mathrm{EE}}_L + (\star\boldsymbol{\ell})\cdot\boldsymbol{L}'C^{\mathrm{EE}}_{L'}]\cos 2\varphi_{L,L'}$
EB	$[\boldsymbol{\ell}\cdot\boldsymbol{L}C^{\mathrm{EE}}_L - \boldsymbol{\ell}\cdot\boldsymbol{L}'C^{\mathrm{BB}}_{L'}]\sin 2\varphi_{L,L'}$	$[(\star\boldsymbol{\ell})\cdot\boldsymbol{L}C^{\mathrm{EE}}_L - (\star\boldsymbol{\ell})\cdot\boldsymbol{L}'C^{\mathrm{BB}}_{L'}]\sin 2\varphi_{L,L'}$
BB	$[\boldsymbol{\ell}\cdot\boldsymbol{L}C^{\mathrm{BB}}_L + \boldsymbol{\ell}\cdot\boldsymbol{L}'C^{\mathrm{BB}}_{L'}]\cos 2\varphi_{L,L'}$	$[(\star\boldsymbol{\ell})\cdot\boldsymbol{L}C^{\mathrm{BB}}_L + (\star\boldsymbol{\ell})\cdot\boldsymbol{L}'C^{\mathrm{BB}}_{L'}]\cos 2\varphi_{L,L'}$

3 Quadratic estimator in the presence of curl mode

3.1 Optimal quadratic estimator : flat sky

3.1.1 Ensemble average with a fixed realization of lensing fields

The statistical ensemble of primary CMB alone gives [5]

$$\begin{aligned}\langle \widetilde{X}_{\boldsymbol{\ell}_1}\widetilde{X}'_{\boldsymbol{\ell}_2}\rangle_{\mathrm{CMB}} &= \left\langle \left(X_{\boldsymbol{\ell}_1} + \sum_{Y=\Theta,E,B}\hat{L}^X_{\boldsymbol{\ell}_1}(Y)\right)\left(X'_{\boldsymbol{\ell}_2} + \sum_{Y'=\Theta,E,B}\hat{L}^{X'}_{\boldsymbol{\ell}_2}(Y')\right)\right\rangle_{\mathrm{CMB}} \\ &= \langle X_{\boldsymbol{\ell}_1}X'_{\boldsymbol{\ell}_2}\rangle_{\mathrm{CMB}} + \sum_{Y=\Theta,E,B}\left\{\langle X_{\boldsymbol{\ell}_1}\hat{L}^{X'}_{\boldsymbol{\ell}_2}(Y)\rangle_{\mathrm{CMB}} + \langle X'_{\boldsymbol{\ell}_2}\hat{L}^{X}_{\boldsymbol{\ell}_1}(Y)\rangle_{\mathrm{CMB}}\right\} . \end{aligned} \tag{26}$$

Note that the ensemble average $\langle\cdots\rangle_{\mathrm{CMB}}$ denotes the average over the primary CMB anisotropies. We note that

$$X_{\boldsymbol{\ell}'}\hat{L}^{X'}_{\boldsymbol{\ell}}(Y) = \sum_x \{(\boldsymbol{\ell}+\boldsymbol{\ell}')\odot_x \boldsymbol{\ell}'\} x_{\boldsymbol{\ell}+\boldsymbol{\ell}'} R^{X'Y}_{\boldsymbol{\ell},-\boldsymbol{\ell}'} C^{YX}_{\ell'} . \tag{27}$$

Substituting Eqs. (27) into Eq. (26), we obtain

$$\langle \widetilde{X}_{\boldsymbol{\ell}_1} \widetilde{X}'_{\boldsymbol{\ell}_2} \rangle_{\rm CMB} = C^{XX'}_{\ell_1} \delta_{\boldsymbol{\ell}_1,-\boldsymbol{\ell}_2} + \sum_x \overline{f}^{x,(XX')}_{\boldsymbol{L},\boldsymbol{\ell}_1,\boldsymbol{\ell}_2} x_{\boldsymbol{L}} \,, \tag{28}$$

where we define $\boldsymbol{L} = \boldsymbol{\ell}_1 + \boldsymbol{\ell}_2$ and denote

$$\overline{f}^{\phi,(XX')}_{\boldsymbol{L},\boldsymbol{\ell}_1,\boldsymbol{\ell}_2} = \sum_{Y=\Theta,E,B} \left\{ R^{X'Y}_{\boldsymbol{\ell}_2,-\boldsymbol{\ell}_1} \{\boldsymbol{L} \odot_\phi \boldsymbol{\ell}_1\} C^{YX}_{\ell_1} + R^{XY}_{\boldsymbol{\ell}_1,-\boldsymbol{\ell}_2} \{\boldsymbol{L} \odot_\phi \boldsymbol{\ell}_2\} C^{YX'}_{\ell_2} \right\} , \tag{29}$$

$$\overline{f}^{\varpi,(XX')}_{\boldsymbol{L},\boldsymbol{\ell}_1,\boldsymbol{\ell}_2} = \sum_{Y=\Theta,E,B} \left\{ R^{X'Y}_{\boldsymbol{\ell}_2,-\boldsymbol{\ell}_1} \{\boldsymbol{L} \odot_\varpi \boldsymbol{\ell}_1\} C^{YX}_{\ell_1} + R^{XY}_{\boldsymbol{\ell}_2,-\boldsymbol{\ell}_1} \{\boldsymbol{L} \odot_\varpi \boldsymbol{\ell}_2\} C^{YX'}_{\ell_2} \right\} , \tag{30}$$

The explicit form of $\overline{f}^{x,(\alpha)}$ is summarized in Table 1. Note that

$$\int \frac{\mathrm{d}^2\boldsymbol{\ell}'}{(2\pi)^2} \overline{f}^{\phi,(\alpha)}_{\boldsymbol{\ell}',\boldsymbol{\ell}-\boldsymbol{\ell}'} \overline{f}^{\varpi,(\alpha)}_{\boldsymbol{\ell}',\boldsymbol{\ell}-\boldsymbol{\ell}'} = 0 \,. \tag{31}$$

Hereafter, for arbitrary functions, $\mathcal{W}^{x,(\alpha)}_{\boldsymbol{\ell}_1,\boldsymbol{\ell}_2}$ and $\mathcal{Z}^{x,(\alpha)}_{\boldsymbol{\ell}_1,\boldsymbol{\ell}_2}$, we define

$$[\mathcal{W}^x, \mathcal{Z}^x]^{(\alpha)}_{\boldsymbol{\ell}} \equiv \int \frac{\mathrm{d}^2\boldsymbol{\ell}'}{(2\pi)^2} \mathcal{W}^{x,(\alpha)}_{\boldsymbol{\ell}',\boldsymbol{\ell}-\boldsymbol{\ell}'} \mathcal{Z}^{x,(\alpha)}_{\boldsymbol{\ell}',\boldsymbol{\ell}-\boldsymbol{\ell}'} \,. \tag{32}$$

3.1.2 Formalism

We define an optimal estimator for ϕ and ϖ as

$$\widehat{x}^{(\alpha)}_{\boldsymbol{\ell}} = \int \frac{\mathrm{d}^2\boldsymbol{\ell}'}{(2\pi)^2} \overline{F}^{x,(\alpha)}_{\boldsymbol{\ell}',\boldsymbol{\ell}-\boldsymbol{\ell}'} \widetilde{X}_{\boldsymbol{\ell}'} \widetilde{Y}_{\boldsymbol{\ell}-\boldsymbol{\ell}'} \,, \tag{33}$$

where x denotes ϕ or ϖ. The estimators should satisfy

$$\langle \widehat{x}^{(\alpha)}_{\boldsymbol{\ell}} \rangle_{\rm CMB} = x_{\boldsymbol{\ell}} \,. \tag{34}$$

Then, according to the optimal condition, we uniquely determine the functional form of $F^{x,(\alpha)}$.

When we substitute Eq. (33) into Eq. (34), we need to compute the ensemble average of each pair of two lensing fields, whose form is described in Eq. (26). By applying Eq. (34), neglecting higher order corrections and zero-mode ($C_0 = 0$), we obtain

$$\langle \text{Eq. (33)} \rangle_{\rm CMB} = \int \frac{\mathrm{d}^2\boldsymbol{\ell}'}{(2\pi)^2} \overline{F}^{x,(\alpha)}_{\boldsymbol{\ell}',\boldsymbol{\ell}-\boldsymbol{\ell}'} \langle \widetilde{X}_{\boldsymbol{\ell}'} \widetilde{Y}_{\boldsymbol{\ell}-\boldsymbol{\ell}'} \rangle_{\rm CMB} \tag{35}$$

$$= \int \frac{\mathrm{d}^2\boldsymbol{\ell}'}{(2\pi)^2} \overline{F}^{x,(\alpha)}_{\boldsymbol{\ell}',\boldsymbol{\ell}-\boldsymbol{\ell}'} \sum_{x'} f^{x',(\alpha)}_{\boldsymbol{\ell}',\boldsymbol{\ell}-\boldsymbol{\ell}'} x'_{\boldsymbol{\ell}} \tag{36}$$

$$= \sum_{x'} [\overline{F}^x, \overline{f}^{x'}]^{(\alpha)}_{\boldsymbol{\ell}} x'_{\boldsymbol{\ell}} \,. \tag{37}$$

From Eq. (34), we obtain

$$[\overline{F}^x,\overline{f}^{x'}]^{(\alpha)}_{\boldsymbol{\ell}} = \delta_{xx'} \,, \tag{38}$$

Next, we determine the function $\overline{F}^{x,(\alpha)}$ from the optimal condition by using the Lagrange-Multiplier method. The Gaussian variance of the optimal estimator can be written, including the Lagrangian multiplier, as

$$\begin{aligned}\langle|\widehat{x}^{(\alpha)}_{\boldsymbol{\ell}}|^2\rangle_G &= \int\frac{\mathrm{d}^2\boldsymbol{\ell}'}{(2\pi)^2}\int\frac{\mathrm{d}^2\boldsymbol{\ell}''}{(2\pi)^2}\,(\overline{F}^{x,(\alpha)}_{\boldsymbol{\ell}',\boldsymbol{\ell}-\boldsymbol{\ell}'})^*\overline{F}^{x,(\alpha)}_{\boldsymbol{\ell}'',\boldsymbol{\ell}-\boldsymbol{\ell}''}\langle\widetilde{X}^*_{\boldsymbol{\ell}'}\widetilde{Y}^*_{\boldsymbol{\ell}-\boldsymbol{\ell}'}\widetilde{X}_{\boldsymbol{\ell}''}\widetilde{Y}_{\boldsymbol{\ell}-\boldsymbol{\ell}''}\rangle_G \\ &\quad-\sum_{xx'}\lambda^x_{x'}([\overline{F}^x,\overline{f}^{x'}]^{(\alpha)}_{\boldsymbol{\ell}}-\delta_{xx'}) \\ &= \int\frac{\mathrm{d}^2\boldsymbol{\ell}'}{(2\pi)^2}\,(\overline{F}^{x,(\alpha)}_{\boldsymbol{\ell}',\boldsymbol{\ell}-\boldsymbol{\ell}'})^*\{\overline{F}^{x,(\alpha)}_{\boldsymbol{\ell}',\boldsymbol{\ell}-\boldsymbol{\ell}'}\mathcal{C}^{XX}_{\ell'}\mathcal{C}^{YY}_{|\boldsymbol{\ell}-\boldsymbol{\ell}'|}+\overline{F}^{x,(\alpha)}_{\boldsymbol{\ell}-\boldsymbol{\ell}',\boldsymbol{\ell}'}\mathcal{C}^{XY}_{\ell'}\mathcal{C}^{XY}_{|\boldsymbol{\ell}-\boldsymbol{\ell}'|}\} \\ &\quad-\sum_{xx'}\lambda^x_{x'}([\overline{F}^x,\overline{f}^{x'}]^{(\alpha)}_{\boldsymbol{\ell}}-\delta_{xx'})\,. \end{aligned}\tag{39}$$

A function $\overline{F}^{x,(\alpha)}$ which minimize the variance is obtained from

$$0 = \frac{\delta}{\delta\,\overline{F}^{x,(\alpha)}_{\boldsymbol{\ell}',\boldsymbol{\ell}-\boldsymbol{\ell}'}}\langle|\widehat{x}^{(\alpha)}_{\boldsymbol{\ell}}|^2\rangle_G \propto 2(\overline{F}^{x,(\alpha)}_{\boldsymbol{\ell}',\boldsymbol{\ell}-\boldsymbol{\ell}'}\mathcal{C}^{XX}_{\ell'}\mathcal{C}^{YY}_{|\boldsymbol{\ell}-\boldsymbol{\ell}'|}+\overline{F}^{x,(\alpha)}_{\boldsymbol{\ell}-\boldsymbol{\ell}',\boldsymbol{\ell}'}\mathcal{C}^{XY}_{\ell'}\mathcal{C}^{XY}_{|\boldsymbol{\ell}-\boldsymbol{\ell}'|})-\sum_{x'}\lambda^x_{x'}\overline{f}^{x',(\alpha)}_{\boldsymbol{\ell}',\boldsymbol{\ell}-\boldsymbol{\ell}'}\,. \tag{40}$$

$$0 = \frac{\delta}{\delta\,\overline{F}^{x,(\alpha)}_{\boldsymbol{\ell}-\boldsymbol{\ell}',\boldsymbol{\ell}'}}\langle|\widehat{x}^{(\alpha)}_{\boldsymbol{\ell}}|^2\rangle_G \propto 2(\overline{F}^{x,(\alpha)}_{\boldsymbol{\ell}-\boldsymbol{\ell}',\boldsymbol{\ell}'}\mathcal{C}^{YY}_{\ell'}\mathcal{C}^{XX}_{|\boldsymbol{\ell}-\boldsymbol{\ell}'|}+\overline{F}^{x,(\alpha)}_{\boldsymbol{\ell}',\boldsymbol{\ell}-\boldsymbol{\ell}'}\mathcal{C}^{XY}_{\ell'}\mathcal{C}^{XY}_{|\boldsymbol{\ell}-\boldsymbol{\ell}'|})-\sum_{x'}\lambda^x_{x'}\overline{f}^{x',(\alpha)}_{\boldsymbol{\ell}',\boldsymbol{\ell}-\boldsymbol{\ell}'} \tag{41}$$

Then, we obtain relations between $\overline{F}^{x,(\alpha)}$ and the Lagrange multiplier as

$$\overline{F}^{\alpha}_{\boldsymbol{\ell}',\boldsymbol{\ell}-\boldsymbol{\ell}'} = \sum_{x'}\lambda^x_{x'}\overline{g}^{x',(\alpha)}_{\boldsymbol{\ell}',\boldsymbol{\ell}-\boldsymbol{\ell}'}\,, \tag{42}$$

with the quantities being

$$\overline{g}^{x,(\alpha)}_{\boldsymbol{\ell}',\boldsymbol{\ell}-\boldsymbol{\ell}'} = \frac{\mathcal{C}^{XX}_{|\boldsymbol{\ell}-\boldsymbol{\ell}'|}\mathcal{C}^{YY}_{\ell}\overline{f}^{x,(\alpha)}_{\boldsymbol{\ell}',\boldsymbol{\ell}-\boldsymbol{\ell}'}-\mathcal{C}^{XY}_{|\boldsymbol{\ell}-\boldsymbol{\ell}'|}\mathcal{C}^{XY}_{\ell}\overline{f}^{x,(\alpha)}_{\boldsymbol{\ell}-\boldsymbol{\ell}',\boldsymbol{\ell}'}}{\mathcal{C}^{XX}_{\ell'}\mathcal{C}^{YY}_{|\boldsymbol{\ell}-\boldsymbol{\ell}'|}\mathcal{C}^{XX}_{|\boldsymbol{\ell}-\boldsymbol{\ell}'|}\mathcal{C}^{YY}_{\ell}-(\mathcal{C}^{XY}_{\ell'}\mathcal{C}^{XY}_{|\boldsymbol{\ell}-\boldsymbol{\ell}'|})^2}\,. \tag{43}$$

Substituting this into Eq. (38), λ become

$$\lambda^x_{x'} = \frac{\delta_{xx'}}{[\overline{f}^x,\overline{g}^x]^{(\alpha)}_{\boldsymbol{\ell}}}\,. \tag{44}$$

Substituting this into Eq. (42), we find

$$\overline{F}^{x,(\alpha)}_{\boldsymbol{\ell}',\boldsymbol{\ell}-\boldsymbol{\ell}'} = \frac{\overline{g}^{x,(\alpha)}_{\boldsymbol{\ell}',\boldsymbol{\ell}-\boldsymbol{\ell}'}}{[\overline{f}^x,\overline{g}^x]^{(\alpha)}_{\boldsymbol{\ell}}}\,. \tag{45}$$

Note that if we neglect the power spectrum of curl mode, $C^{\varpi\varpi}_{\ell}$, in the lensed power spectrum, we obtain the quadratic estimator derived in Ref. [5].

Table 2: The functional forms of $f^{x,(XX')}_{\ell LL'}$. The label "even" and "odd" indicate that the function are non-zero only when $\ell+L+L'$ is even or odd, respectively.

XX'	$f^{\phi,(XX')}_{\ell LL'}$	$f^{\varpi,(XX')}_{\ell LL'}$
$\Theta\Theta$	$\mathcal{S}^{\phi,0}_{L\ell L'}C^{\Theta\Theta}_{L'}+\mathcal{S}^{\phi,0}_{L'\ell L}C^{\Theta\Theta}_{L}$ (even)	$\mathcal{S}^{\varpi,0}_{L\ell L'}C^{\Theta\Theta}_{L'}-\mathcal{S}^{\varpi,0}_{L'\ell L}C^{\Theta\Theta}_{L}$ (odd)
ΘE	$\mathcal{S}^{\phi,0}_{L\ell L'}C^{\Theta\mathrm{E}}_{L'}+\mathcal{S}^{\phi,+}_{L'\ell L}C^{\Theta\mathrm{E}}_{L}$ (even)	$\mathcal{S}^{\varpi,0}_{L\ell L'}C^{\Theta\mathrm{E}}_{L'}-\mathcal{S}^{\varpi,+}_{L'\ell L}C^{\Theta\mathrm{E}}_{L}$ (odd)
ΘB	$-\mathcal{S}^{\phi,-}_{L'\ell L}C^{\Theta\mathrm{E}}_{L}$ (odd)	$\mathcal{S}^{\varpi,-}_{L'\ell L}C^{\Theta\mathrm{E}}_{L}$ (even)
EE	$\mathcal{S}^{\phi,+}_{L\ell L'}C^{\mathrm{EE}}_{L'}+\mathcal{S}^{\phi,+}_{L'\ell L}C^{\mathrm{EE}}_{L}$ (even)	$\mathcal{S}^{\varpi,+}_{L\ell L'}C^{\mathrm{EE}}_{L'}-\mathcal{S}^{\varpi,+}_{L'\ell L}C^{\mathrm{EE}}_{L}$ (odd)
EB	$-\mathcal{S}^{\phi,-}_{L\ell L'}C^{\mathrm{BB}}_{L}-\mathcal{S}^{\phi,-}_{L'\ell L}C^{\mathrm{EE}}_{L}$ (odd)	$-\mathcal{S}^{\varpi,-}_{L\ell L'}C^{\mathrm{BB}}_{L}+\mathcal{S}^{\varpi,-}_{L'\ell L}C^{\mathrm{EE}}_{L}$ (even)
BB	$\mathcal{S}^{\phi,+}_{L\ell L'}C^{\mathrm{BB}}_{L'}+\mathcal{S}^{\phi,+}_{L'\ell L}C^{\mathrm{BB}}_{L}$ (even)	$\mathcal{S}^{\varpi,+}_{L\ell L'}C^{\mathrm{BB}}_{L'}-\mathcal{S}^{\varpi,+}_{L'\ell L}C^{\mathrm{BB}}_{L}$ (odd)

3.2 Full sky formalism

In this section, we present a full-sky reconstruction method of $\phi_{\ell m}$ and $\varpi_{\ell m}$.

3.2.1 Lensing fields as a quadratic statistics

The off-diagonal elements of two lensed anisotropies in full sky are given by

$$\langle\widetilde{X}_{LM}\widetilde{X}'_{L'M'}\rangle_{\mathrm{CMB}}=\sum_{\ell m}(-1)^m\begin{pmatrix}\ell & L & L'\\ -m & M & M'\end{pmatrix}\sum_x f^{x,(XX')}_{\ell LL'}x_{\ell m}\,. \tag{46}$$

The functional forms of $f^{x,(XX')}_{\ell LL'}$ are summarized in Table 2, and the labels "even" and "odd" indicate that the function are non-zero only when $\ell+L+L'$ is even or odd, respectively. Note that "even" and "odd" come from the parity of ϕ, ϖ, Θ, E and B. To find expressions for $\phi_{\ell m}$ and $\varpi_{\ell m}$ from Eq. (46), we multiply

$$(-1)^{m'}\begin{pmatrix}\ell' & L & L'\\ -m' & M & M'\end{pmatrix}f^{\phi,(XX')}_{\ell' LL'}\,, \tag{47}$$

in both sides of Eq. (46), and sum over M and M'. As a result, we find that, for all combination of ℓ, L and L', Eq. (46) becomes

$$\phi_{\ell m} = \frac{2\ell+1}{f^{\phi,(XX')}_{\ell LL'}} \sum_{MM'} (-1)^m \begin{pmatrix} \ell & L & L' \\ -m & M & M' \end{pmatrix} \langle \widetilde{X}_{LM} \widetilde{X}'_{L'M'} \rangle_{\rm CMB} \,. \tag{48}$$

Notice that the term involving ϖ in Eq. (46) vanishes. This is because, from Table 2, the symmetric property of parity leads the following equations :

$$f^{\phi,(XX')}_{\ell LL'} f^{\varpi,(XX')}_{\ell LL'} = 0 \qquad (\text{for all } \ell,\, L \text{ and } L') \,. \tag{49}$$

The similar expression for $\varpi_{\ell m}$ is obtained by replacing $f^{\phi,(XX')}_{\ell LL'}$ in Eq. (47) with $f^{\varpi,(XX')}_{\ell LL'}$, and the result is

$$\varpi_{\ell m} = \frac{2\ell+1}{f^{\varpi,(XX')}_{\ell LL'}} \sum_{MM'} (-1)^m \begin{pmatrix} \ell & L & L' \\ -m & M & M' \end{pmatrix} \langle \widetilde{X}_{LM} \widetilde{X}'_{L'M'} \rangle_{\rm CMB} \,. \tag{50}$$

Eqs. (48) and (50) imply that the scalar and pseudo-scalar lensing potential can be reconstructed independently, using the quadratic combination of lensed CMB anisotropies.

3.2.2 Estimator

Based on Eqs. (48) and (50), we first naively define the estimator for the lensing potential x ($= \phi$ or ϖ) as follows:

$$\frac{2\ell+1}{f^{x,(\alpha)}_{\ell LL'}} \sum_{MM'} (-1)^m \begin{pmatrix} \ell & L & L' \\ -m & M & M' \end{pmatrix} \widetilde{X}_{LM} \widetilde{X}'_{L'M'} \,, \tag{51}$$

where the subscript, α, means a pair of two CMB maps, e.g., $\alpha = \Theta\Theta$ or EB. The multipoles, L and L', are chosen so that $f^{x,(\alpha)}_{\ell LL'} \neq 0$. With Eqs. (48) and (50), the estimator is rewritten as

$$\text{Eq. (51)} = x_{\ell m} + n^{x,(\alpha)}_{\ell m LL'} \,, \tag{52}$$

where the quantity, $n^{x,(\alpha)}_{\ell m LL'}$, is given by

$$n^{x,(\alpha)}_{\ell m LL'} = (-1)^m \frac{2\ell+1}{f^{x,(\alpha)}_{\ell LL'}} \sum_{MM'} \begin{pmatrix} \ell & L & L' \\ -m & M & M' \end{pmatrix} \left(\widetilde{X}_{LM} \widetilde{X}'_{L'M'} - \langle \widetilde{X}_{LM} \widetilde{X}'_{L'M'} \rangle_{\rm CMB} \right) \tag{53}$$

Note that the above equation can be expressed without the quantity, $\langle \cdots \rangle_{\rm CMB}$. The above estimator (51) is not optimal, and we redefine the estimator for ϕ and ϖ by introducing a weight function, $F^{x,(\alpha)}_{\ell LL'}$, in order to reduce the contribution from $n^{x,(\alpha)}_{\ell m LL'}$. Summing up all combination of L and L', we write the estimator of the lensing potentials as

$$\widehat{x}^{(\alpha)}_{\ell m} = \sum_{LL'} F^{x,(\alpha)}_{\ell LL'} \sum_{MM'} (-1)^m \begin{pmatrix} \ell & L & L' \\ -m & M & M' \end{pmatrix} \widetilde{X}_{LM} \widetilde{X}'_{L'M'} \tag{54}$$

$$= \sum_{LL'} F^{x,(\alpha)}_{\ell LL'} \sum_{x'} \frac{f^{x',(\alpha)}_{\ell LL'}}{2\ell+1} x'_{\ell m} + n^{x,(\alpha)}_{\ell m} = \sum_{x'} [F^x, f^{x'}]^{(\alpha)}_\ell x'_{\ell m} + n^{x,(\alpha)}_{\ell m} \,. \tag{55}$$

where the inner product $[a^x, b^{x'}]_\ell^{(\alpha)}$ for arbitrary two quantities, $a^{x,(\alpha)}_{\ell LL'}$ and $b^{x',(\alpha)}_{\ell LL'}$, is defined by

$$[a^x, b^{x'}]_\ell^{(\alpha)} \equiv \frac{1}{2\ell+1}\sum_{LL'} a^{x,(\alpha)}_{\ell LL'} b^{x',(\alpha)}_{\ell LL'} \,. \tag{56}$$

The quantity, $n^{x,(\alpha)}_{\ell m}$, is defined as

$$n^{x,(\alpha)}_{\ell m} \equiv \sum_{LL'} F^{x,(\alpha)}_{\ell LL'} \frac{f^{x,(\alpha)}_{\ell LL'}}{2\ell+1} n^{x,(\alpha)}_{\ell m LL'} \,. \tag{57}$$

The functional form of the weight function is determined so that the noise contribution is minimized as follows. Eq. (55) implies that the estimator would be an unbiased estimator if we impose the following condition:

$$[F^x, f^{x'}]_\ell^{(\alpha)} = \delta_{x,x'} \,. \tag{58}$$

Mathematically, this is equivalent to $\langle \widehat{x}^{(\alpha)}_{\ell m}\rangle_{\rm CMB} = x_{\ell m}$. Also, we wish to suppress the noise contributions, $n^{x,(\alpha)}_{\ell m}$, imposing the following condition:

$$\frac{\delta}{\delta\, F^{x,(\alpha)}_{\ell LL'}} \langle |n^{x,(\alpha)}_{\ell m}|^2\rangle = 0 \,. \tag{59}$$

Let us determine the functional form of the weight function under the conditions, (58) and (59), with the Lagrange-multiplier method. The variance of $n^{(\alpha)}_{\ell m}$ is given by

$$\langle |n^{(\alpha)}_{\ell m}|^2\rangle = \frac{1}{2\ell+1}\sum_{LL'} (F^{x,(\alpha)}_{\ell LL'})^* \left(F^{x,(\alpha)}_{\ell LL'} \widetilde{\mathcal{C}}^{XX'}_L \widetilde{\mathcal{C}}^{YY'}_{L'} + F^{x,(\alpha)}_{\ell L'L} (-1)^{\ell+L+L'} \widetilde{\mathcal{C}}^{XY'}_L \widetilde{\mathcal{C}}^{X'Y}_{L'} \right) \,, \tag{60}$$

where the quantity, $\widetilde{\mathcal{C}}^{XY}_L$, is the lensed angular power spectrum including the contributions from instrumental noise. Then, Eq. (59) under the constraint (58) is equivalent to

$$\begin{aligned} \frac{\delta}{\delta\, F^{x,(\alpha)}_{\ell LL'}} \Bigg\{ &\frac{1}{2\ell+1}\sum_{LL'} (F^{x,(\alpha)}_{\ell LL'})^* \left(F^{x,(\alpha)}_{\ell LL'} \widetilde{\mathcal{C}}^{XX}_L \widetilde{\mathcal{C}}^{YY}_{L'} + (-1)^{\ell+L+L'} F^{x,(\alpha)}_{\ell L'L} \widetilde{\mathcal{C}}^{XY}_L \widetilde{\mathcal{C}}^{XY}_{L'} \right) \\ &+ \sum_{x'} \lambda^x_{x'} \left([F^x, f^{x'}]_\ell^{(\alpha)} - \delta_{xx'} \right) \Bigg\} = 0 \,. \end{aligned} \tag{61}$$

The quantities, λ^x_ϕ and λ^x_ϖ, are the Lagrange multiplier whose functional form is specified below. Eq. (61) leads to the following two equations:

$$(F^{x,(\alpha)}_{\ell LL'})^* \widetilde{\mathcal{C}}^{XX}_L \widetilde{\mathcal{C}}^{YY}_{L'} + (F^{x,(\alpha)}_{\ell L'L})^* (-1)^{\ell+L+L'} \widetilde{\mathcal{C}}^{XY}_L \widetilde{\mathcal{C}}^{XY}_{L'} + \sum_{x'} \lambda^x_{x'} f^{x',(\alpha)}_{\ell LL'} = 0 \,, \tag{62}$$

$$(F^{x,(\alpha)}_{\ell L'L})^* \widetilde{\mathcal{C}}^{XX}_{L'} \widetilde{\mathcal{C}}^{YY}_{L} + (F^{x,(\alpha)}_{\ell LL'})^* (-1)^{\ell+L+L'} \widetilde{\mathcal{C}}^{XY}_L \widetilde{\mathcal{C}}^{XY}_{L'} + \sum_{x'} \lambda^x_{x'} f^{x',(\alpha)}_{\ell L'L} = 0 \,. \tag{63}$$

Multiplying the factors $\widetilde{\mathcal{C}}^{XX}_{L'}\widetilde{\mathcal{C}}^{YY}_{L}$ and $-(-1)^{\ell+L+L'}\widetilde{\mathcal{C}}^{XY}_{L}\widetilde{\mathcal{C}}^{XY}_{L'}$ with Eqs. (62) and (63), respectively, the sum of Eqs. (62) and (63) gives

$$F^{x,(\alpha)}_{\ell LL'} + \sum_{x'}(\lambda^{x}_{x'})^{*} g^{x',(\alpha)}_{\ell LL'} = 0\,, \tag{64}$$

where we define

$$g^{x,(\alpha)}_{\ell LL'} = \frac{(f^{x,(\alpha)}_{\ell LL'})^{*}\widetilde{\mathcal{C}}^{XX}_{L'}\widetilde{\mathcal{C}}^{YY}_{L} - (-1)^{\ell+L+L'}\widetilde{\mathcal{C}}^{XY}_{L}\widetilde{\mathcal{C}}^{XY}_{L'}(f^{x,(\alpha)}_{\ell L'L})^{*}}{\widetilde{\mathcal{C}}^{XX}_{L}\widetilde{\mathcal{C}}^{YY}_{L'}\widetilde{\mathcal{C}}^{XX}_{L'}\widetilde{\mathcal{C}}^{YY}_{L} - (\widetilde{\mathcal{C}}^{XY}_{L}\widetilde{\mathcal{C}}^{XY}_{L'})^{2}}\,. \tag{65}$$

Substituting Eq. (64) into Eq. (58), we obtain

$$-\sum_{x''}(\lambda^{x}_{x''})^{*}[g^{x''}, f^{x'}]^{(\alpha)}_{\ell} = \delta_{xx'}\,. \tag{66}$$

From Eq. (49), we find

$$[g^{x''}, f^{x'}]^{(\alpha)}_{\ell} = \delta_{x''x'}[g^{x'}, f^{x'}]^{(\alpha)}_{\ell}\,. \tag{67}$$

Combining the above equation with Eq. (66), we obtain the explicit form of the Lagrange multiplier

$$(\lambda^{x}_{x'})^{*} = -\frac{\delta_{xx'}}{[f^{x}, g^{x}]^{(\alpha)}_{\ell}}\,. \tag{68}$$

Then, from Eq. (64), we finally obtain the expression for the weight function:

$$F^{x,(\alpha)}_{\ell LL'} = \frac{g^{x,(\alpha)}_{\ell LL'}}{[f^{x}, g^{x}]^{(\alpha)}_{\ell}}\,. \tag{69}$$

Note that, with the explicit expression (69), the noise variance, $N^{x,(\alpha)}_{\ell}$, given in Eq. (60) becomes

$$\begin{aligned} N^{x,(\alpha)}_{\ell} &\equiv \langle |n^{x,(\alpha)}_{\ell m}|^{2}\rangle \\ &= \frac{1}{2\ell+1}\frac{1}{[f^{x}, g^{x}]^{(\alpha)}_{\ell}}\sum_{LL'}(g^{x,(\alpha)}_{\ell LL'})^{*}\left(F^{x,(\alpha)}_{\ell LL'}\widetilde{\mathcal{C}}^{XX}_{L}\widetilde{\mathcal{C}}^{YY}_{L'} + F^{x,(\alpha)}_{\ell L'L}(-1)^{\ell+L+L'}\widetilde{\mathcal{C}}^{XY}_{L}\widetilde{\mathcal{C}}^{XY}_{L'}\right) \\ &= \frac{1}{[f^{x}, g^{x}]^{(\alpha)}_{\ell}}\,, \end{aligned} \tag{70}$$

where we use the relations given in Eqs. (62) and (68). Thus, the weight function can be recast as

$$F^{x,(\alpha)}_{\ell LL'} = N^{x,(\alpha)}_{\ell} g^{x,(\alpha)}_{\ell LL'}\,. \tag{71}$$

With the weight function given above, the estimators defined by Eq. (54) become optimal, i.e., the noise contribution is minimized. Eq. (69) or Eq. (71) is one of the main results. Note that, if the curl mode is absent, $\varpi = 0$, the resultant form of the weight function for ϕ exactly coincides with the one obtained in Ref. [6]. The difference appears when the angular power spectrum of the pseudo-scalar lensing potential, $C^{\varpi\varpi}_{\ell}$, included in the lensed angular power spectrum becomes non-vanishing. Note again that, in practical case, to use the estimator, the angular power spectrum of primary CMB anisotropies should be a priori known (i.e., $f^{x,(\alpha)}_{\ell LL'}$ is given).

3.2.3 Optimal combination

As discussed in Ref. [6], the noise contribution can be suppressed by combining the estimators. The minimum variance estimators of the lensing potential and lensing cross-potential are given by

$$\widehat{x}^{(\mathrm{mv})}_{\ell m} = \sum_{\alpha} W^{x,(\alpha)}\widehat{x}^{(\alpha)}_{\ell m} \,. \tag{72}$$

The weight functions, $W^{x,(\alpha)}$, are determined so that the estimator satisfy $\langle\widehat{x}^{(\mathrm{mv})}_{\ell m}\rangle_{\mathrm{CMB}} = x_{\ell m}$, and the variance of the estimator is minimum. The minimum variance estimator is then determined as

$$\widehat{x}^{(\mathrm{mv})}_{\ell m} = N^{x,(\mathrm{mv})}_{\ell} \sum_{\alpha,\beta}\{(\boldsymbol{N}^{x}_{\ell})^{-1}\}^{\alpha,\beta}\widehat{x}^{(\alpha)}_{\ell m} \,, \tag{73}$$

where the minimum variance, $N^{x,(\mathrm{mv})}_{\ell}$, are defined by

$$\frac{1}{N^{x,(\mathrm{mv})}_{\ell}} = \sum_{\beta,\beta'}\{(\boldsymbol{N}^{x}_{\ell})^{-1}\}_{\beta\beta'} \,, \tag{74}$$

and the components of the matrix, $\{\boldsymbol{N}^{x}_{\ell}\}_{\alpha,\beta}$, is given by $N^{x,(\alpha,\beta)}_{\ell}$.

4 Numerical Computation Tips

4.1 Non-Perturbative Formalism of Lensed Angular Power Spectrum

We basically follow Ref. [14], but including the curl mode. Let us consider the correlation of lensed fields in two directions, $\hat{\boldsymbol{n}}_1 = (\theta,\phi)$ and $\hat{\boldsymbol{n}}_2 = (\theta+\beta,\phi)$. The two point correlation of lensed field $\widehat{\xi}(\beta)$ is defined by

$$\begin{aligned}\widehat{\xi} &= \langle\Theta(\hat{\boldsymbol{n}}_1+\boldsymbol{d}_1)\Theta(\hat{\boldsymbol{n}}_2+\boldsymbol{d}_2)\rangle = \sum_{\ell m} C^{\Theta\Theta}_{\ell}\langle Y_{\ell m}(\hat{\boldsymbol{n}}_1+\boldsymbol{d}_1)Y_{\ell m}(\hat{\boldsymbol{n}}_2+\boldsymbol{d}_2)\rangle \\ &= \sum_{\ell m m'} C^{\Theta\Theta}_{\ell} d^{\ell}_{mm'}(\beta)\langle Y_{\ell m}(\alpha_1,\psi_1)Y^{*}_{\ell m'}(\alpha_2,\psi_2)\rangle \,,\end{aligned} \tag{75}$$

where we use

$$Y_{\ell m}(\hat{\boldsymbol{n}}_2) = \sum_{m'} d^{\ell}_{mm'}(\beta)Y_{\ell m'}(\hat{\boldsymbol{n}}_1) \,, \qquad \boldsymbol{d}_i = \alpha_i(\cos\psi_i,\sin\psi_i) \,. \tag{76}$$

To evaluate the ensemble average, we first compute the probability distribution function of $(\alpha_1,\alpha_2,\psi_1,\psi_2)$. The deflection angle is decomposed into two fields, i.e. the gradient (ϕ) and curl (ϖ) modes. Using

$$\boldsymbol{\nabla} Y_{\ell m} = -\sqrt{\frac{\ell(\ell+1)}{2}}(Y^{1}_{\ell m}\bar{\boldsymbol{e}} - Y^{-1}_{\ell m}\boldsymbol{e}) \,, \tag{77}$$

we obtain

$$\begin{aligned} \boldsymbol{d}(\hat{\boldsymbol{n}}) = \boldsymbol{\nabla}\phi + (\star\boldsymbol{\nabla})\varpi &= -\sum_{\ell m}\sqrt{\frac{\ell(\ell+1)}{2}}\left[\phi_{\ell m}(Y^1_{\ell m}\bar{\boldsymbol{e}} + Y^{-1}_{\ell m}\boldsymbol{e}) + \mathrm{i}\varpi_{\ell m}(Y^1_{\ell m}\bar{\boldsymbol{e}} - Y^{-1}_{\ell m}\boldsymbol{e})\right] \\ &= -\sum_{\ell m}\sqrt{\frac{\ell(\ell+1)}{2}}\left[(\phi_{\ell m} + \mathrm{i}\varpi_{\ell m})Y^1_{\ell m}\bar{\boldsymbol{e}} + (\phi_{\ell m} - i\varpi_{\ell m})Y^{-1}_{\ell m}\boldsymbol{e}\right]\,. \end{aligned} \tag{78}$$

Since the deflection angle has both spin 1 and -1 components, we introduce a spin 1 quantity

$$\alpha(\hat{\boldsymbol{n}}) = \boldsymbol{d}\cdot(-\sqrt{2}\boldsymbol{e}) = \sum_{\ell m}(\phi_{\ell m} + \mathrm{i}\varpi_{\ell m})Y^1_{\ell m}\,. \tag{79}$$

Denoting $\alpha(\hat{\boldsymbol{n}}_i) = \alpha_i e^{i\psi_i}$, the covariance of the spin -1 deflection field becomes

$$\begin{aligned} \langle\alpha(\hat{\boldsymbol{n}}_1)\alpha(\hat{\boldsymbol{n}}_2)\rangle &= \sum_{\ell_1,m_1}\ell_1(\ell_1+1)(-1)^{m_1}(C^{\phi\phi}_{\ell_1} - C^{\varpi\varpi}_{\ell_1})Y^1_{\ell_1 m_1}(\hat{\boldsymbol{n}}_1)Y^1_{\ell_1 m_1}(\hat{\boldsymbol{n}}_2) \\ &= \sum_{\ell_1}\ell_1(\ell_1+1)(C^{\phi\phi}_{\ell_1} - C^{\varpi\varpi}_{\ell_1})\sum_{m_1}(Y^1_{\ell_1 m_1})^*(\hat{\boldsymbol{n}}_1)Y^1_{\ell_1 m_1}(\hat{\boldsymbol{n}}_2) \\ &= -\sum_{\ell_1}\ell_1(\ell_1+1)(C^{\phi\phi}_{\ell_1} - C^{\varpi\varpi}_{\ell_1})\frac{2\ell+1}{4\pi}d^{\ell}_{-11}(\beta) \equiv -C_{\mathrm{gl},2}(\beta) \end{aligned} \tag{80}$$

$$\langle\alpha^*(\hat{\boldsymbol{n}}_1)\alpha(\hat{\boldsymbol{n}}_2)\rangle = \sum_{\ell_1}\ell_1(\ell_1+1)(C^{\phi\phi}_{\ell_1} + C^{\varpi\varpi}_{\ell_1})\frac{2\ell+1}{4\pi}d^{\ell}_{11}(\beta) \equiv C_{\mathrm{gl}}(\beta)\,. \tag{81}$$

This leads to the following relations;

$$\begin{aligned} \langle\Re\alpha(\hat{\boldsymbol{n}}_1)\Re\alpha(\hat{\boldsymbol{n}}_2)\rangle &= \frac{C_{\mathrm{gl}}(\beta) - C_{\mathrm{gl},2}(\beta)}{2}\,, \\ \langle\Im\alpha(\hat{\boldsymbol{n}}_1)\Im\alpha(\hat{\boldsymbol{n}}_2)\rangle &= \frac{C_{\mathrm{gl}}(\beta) + C_{\mathrm{gl},2}(\beta)}{2}\,, \\ \langle\Re\alpha(\hat{\boldsymbol{n}}_1)\Im\alpha(\hat{\boldsymbol{n}}_2)\rangle &= \langle\Im\alpha(\hat{\boldsymbol{n}}_1)\Re\alpha(\hat{\boldsymbol{n}}_2)\rangle = 0\,. \end{aligned} \tag{82}$$

If we choose $\beta = 0$, we obtain

$$\langle\Re\alpha^2(\hat{\boldsymbol{n}}_1)\rangle + \langle\Im\alpha^2(\hat{\boldsymbol{n}}_1)\rangle = C_{\mathrm{gl}}(0)\,. \tag{83}$$

Assuming $\langle\Re\alpha^2(\hat{\boldsymbol{n}}_1)\rangle = \langle\Im\alpha^2(\hat{\boldsymbol{n}}_1)\rangle$ and $\langle\Re\alpha^2(\hat{\boldsymbol{n}}_2)\rangle = \langle\Im\alpha^2(\hat{\boldsymbol{n}}_2)\rangle$, we obtain

$$\begin{aligned} \langle(\Re\alpha(\hat{\boldsymbol{n}}_1) + \Re\alpha(\hat{\boldsymbol{n}}_2))^2\rangle &= C_{\mathrm{gl}}(0) + C_{\mathrm{gl}}(\beta) - C_{\mathrm{gl},2}(\beta)\,, \\ \langle(\Im\alpha(\hat{\boldsymbol{n}}_1) + \Im\alpha(\hat{\boldsymbol{n}}_2))^2\rangle &= C_{\mathrm{gl}}(0) + C_{\mathrm{gl}}(\beta) + C_{\mathrm{gl},2}(\beta)\,, \\ \langle(\Re\alpha(\hat{\boldsymbol{n}}_1) - \Re\alpha(\hat{\boldsymbol{n}}_2))^2\rangle &= C_{\mathrm{gl}}(0) - C_{\mathrm{gl}}(\beta) + C_{\mathrm{gl},2}(\beta)\,, \\ \langle(\Im\alpha(\hat{\boldsymbol{n}}_1) - \Im\alpha(\hat{\boldsymbol{n}}_2))^2\rangle &= C_{\mathrm{gl}}(0) - C_{\mathrm{gl}}(\beta) - C_{\mathrm{gl},2}(\beta)\,. \end{aligned} \tag{84}$$

Assuming the lensing fields are Gaussian and using $\sigma^2 \equiv C_{\rm gl}(0) - C_{\rm gl}(\beta)$, and transforming variables to $(\alpha_1, \alpha_2, \psi_1, \psi_2)$, the probability distribution function is given by [14]

$$\begin{aligned} P(\alpha_1, \alpha_2, \psi_1, \psi_2) = 4\alpha_1\alpha_2 &\mathcal{G}(\Re\alpha(\hat{\boldsymbol{n}}_1) + \Re\alpha(\hat{\boldsymbol{n}}_2), \sigma^2 + 2C_{\rm gl} - C_{\rm gl,2}) \\ &\times \mathcal{G}(\Re\alpha(\hat{\boldsymbol{n}}_1) - \Re\alpha(\hat{\boldsymbol{n}}_2), \sigma^2 + 2C_{\rm gl} + C_{\rm gl,2}) \\ &\times \mathcal{G}(\Im\alpha(\hat{\boldsymbol{n}}_1) + \Im\alpha(\hat{\boldsymbol{n}}_2), \sigma^2 + C_{\rm gl,2}) \\ &\times \mathcal{G}(\Im\alpha(\hat{\boldsymbol{n}}_1) - \Im\alpha(\hat{\boldsymbol{n}}_2), \sigma^2 - C_{\rm gl,2}) \,, \end{aligned} \tag{85}$$

where the function $\mathcal{G}(x, \sigma^2)$ is the normal distribution function;

$$\mathcal{G}(x, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \mathrm{e}^{-x^2/\sigma^2} \,. \tag{86}$$

Next, we evaluate the ensemble average of $\langle Y_{\ell m}(\alpha_1, \psi_1) Y^*_{\ell m'}(\alpha_2, \psi_2)\rangle$ by using the probability distribution function. We assume $\sigma^2 \gg C_{\rm gl}$, $C_{\rm gl,2}$. This assumption leads to

$$\mathcal{G}(X, \sigma^2 + C) \simeq \left(1 - \frac{C}{2\sigma^2}\right) \mathcal{G}\left[X\left(1 - \frac{C}{2\sigma^2}\right), \sigma^2\right] \,, \tag{87}$$

and Eq.(85) becomes

$$\begin{aligned} P(\alpha_1, \alpha_2, \psi_1, \psi_2) &\simeq \frac{4\alpha_1\alpha_2}{(2\pi)^2}\left(1 - 2\frac{C_{\rm gl}}{\sigma^2}\right) \mathrm{e}^{-(\alpha_1^2+\alpha_2^2)/\sigma^2} \mathrm{e}^{2(\alpha_1\alpha_2/\sigma^4)[-C_{\rm gl,2}\cos(\psi_1+\psi_2) + C_{\rm gl}\cos(\psi_1-\psi_2)]} \\ &\simeq \frac{4\alpha_1\alpha_2}{(2\pi)^2} \mathrm{e}^{-(\alpha_1^2+\alpha_2^2)/\sigma^2} \\ &\quad \times \left\{1 - 2\frac{C_{\rm gl}}{\sigma^2} + \frac{2\alpha_1\alpha_2}{\sigma^4}[-C_{\rm gl,2}\cos(\psi_1+\psi_2) + C_{\rm gl}\cos(\psi_1 - \psi_2)]\right\} . \end{aligned} \tag{88}$$

Thus, the integral over ψ_1 and ψ_2 in the ensemble average $\langle Y_{\ell m}(\alpha_1, \psi_1) Y^*_{\ell m'}(\alpha_2, \psi_2)\rangle$ becomes

$$\begin{aligned} \langle Y_{\ell m}(\alpha_1, \psi_1) Y^*_{\ell m'}(\alpha_2, \psi_2)\rangle = &\int_0^\infty \mathrm{d}\alpha_1 \int_0^\infty \mathrm{d}\alpha_2 \left\{4\alpha_1\alpha_2\left(1 - 2\frac{C_{\rm gl}}{\sigma^2}\right)\delta_{m,0}\delta_{m',0}\right. \\ &+ \frac{8(\alpha_1\alpha_2)^2}{\sigma^4}(-C_{\rm gl,2}(\delta_{m,-1}\delta_{m',-1} + \delta_{m,1}\delta_{m',1}) \\ &\left. + C_{\rm gl}(\delta_{m,-1}\delta_{m',1} + \delta_{m,1}\delta_{m',-1}))\right\} \mathrm{e}^{-(\alpha_1^2+\alpha_2^2)/\sigma^2} Y_{\ell m}(\alpha_1, 0) Y^*_{\ell m'}(\alpha_2, 0) \,. \end{aligned} \tag{89}$$

Denoting

$$X_{imn} = \int_0^\infty \mathrm{d}\alpha \, \frac{2\alpha}{\sigma^2}\left(\frac{\alpha}{\sigma^2}\right)^i \mathrm{e}^{-\alpha^2/\sigma^2} d^\ell_{mn}(\alpha) \,, \tag{90}$$

and neglecting the term including $C_{\rm gl}$, we finally obtain [14]

$$\widehat{\xi} \simeq \sum_\ell \frac{2\ell+1}{4\pi} C_\ell^{\Theta\Theta}\left\{X_{000}^2 d^\ell_{00} + \frac{8}{\ell(\ell+1)} C_{\rm gl,2}(X'_{000})^2 d^\ell_{1,-1} + C^2_{\rm gl,2}(X'_{000})^2 d^\ell_{00} + X^2_{220} d^\ell_{2,-2}\right\} . \tag{91}$$

where primes denote differentiation in terms of σ^2. Similar results can be obtained for polarization;

$$\widehat{\xi}_+ \simeq \sum_\ell \frac{2\ell+1}{4\pi}(C_\ell^{EE}+C_\ell^{BB})\{X_{022}^2 d_{22}^\ell + 2C_{\rm gl,2}X_{132}X_{121}d_{31}^\ell + C_{\rm gl,2}^2[X_{022}^2 d_{22}^\ell + X_{242}X_{220}d_{40}^\ell] \tag{92}$$

$$\widehat{\xi}_- \simeq \sum_\ell \frac{2\ell+1}{4\pi}(C_\ell^{EE}-C_\ell^{BB})\Big\{X_{022}^2 d_{2,-2}^\ell + C_{\rm gl,2}[X_{121}^2 d_{1,-1}^\ell + X_{132}^2 d_{3,-3}^\ell] + \frac{1}{2}C_{\rm gl,2}^2[2X_{022}^2 d_{2,-2}^\ell + X_{220}^2 d_{00}^\ell + X_{242}^2 d_{4,-4}^\ell]\Big\} \tag{93}$$

$$\widehat{\xi}_{\rm c} \simeq \sum_\ell \frac{2\ell+1}{4\pi}C_\ell^{\Theta E}\Big\{X_{022}X_{000}d_{02}^\ell + C_{\rm gl,2}\frac{2X_{000}}{\sqrt{\ell(\ell+1)}}[X_{112}d_{11}^\ell + X_{132}d_{3,-1}^\ell] + \frac{1}{2}C_{\rm gl,2}^2[(2X_{022}X_{000}+X_{220}^2)d_{20}^\ell + X_{220}X_{242}d_{-2,4}^\ell]\Big\}. \tag{94}$$

4.2 Computation of Wigner $3j$ symbols

We compute the Wigner $3j$ symbols recursively. Given ℓ, there are two loops, i.e., L_1 and L_2. In the loop of L_1, the loop of L_2 is start from $L_2 = \ell + L_1$ and down to $L_2 = \min(L_1, \ell_{\rm min})$. Then, we need to the initial values of Wigner $3j$ symbols for $L_2 = \ell + L_1$. In general,

$$\begin{aligned}\begin{pmatrix} L_1 & \ell & \ell+L_1 \\ -M_1 & -M_2 & M_1+M_2 \end{pmatrix} &= \frac{(-1)^{\ell+L_1+M_1+M_2}}{\sqrt{2\ell+2L_1+1}} \\ &\quad\times\left[\frac{(2\ell)!(2L_1)!(\ell+L_1+M_1+M_2)!(\ell+L_1-M_1-M_2)!}{(2\ell+2L_1)!(L_1+M_1)!(L_1-M_1)!(\ell+M_2)!(\ell-M_2)!}\right]^{1/2}, \\ \begin{pmatrix} L_1 & \ell & L_1+\ell-1 \\ -M_1 & -M_2 & M_1+M_2 \end{pmatrix} &= (\ell M_2 - L_1 M_1) \\ &\quad\times\left[\frac{2\ell+2L_1+1}{\ell L_1[(\ell+L_1)^2-(M_1+M_2)^2]}\right]^{1/2}\begin{pmatrix} L_1 & \ell & \ell+L_1 \\ -M_1 & -M_2 & M_1+M_2 \end{pmatrix}. \end{aligned} \tag{95}$$

In our calculation, five cases are needed, i.e., $(M_1 = 0, M_2 = 0)$, $(M_1 = 1, M_2 = -1)$, $(M_1 = 2, M_2 = 0)$, $(M_1 = 1, M_2 = 1)$, and $(M_1 = 2, M_2 = 1)$. These can be obtained by the recursion relation of Wigner $3j$ symbols. Initial conditions for $(M_1 = 1, M_2 = -1)$, $(M_1 = 2, M_2 = 0)$, $(M_1 = 1, M_2 = 1)$, and $(M_1 = 2, M_2 = 1)$, are related to that for $(M_1 = 0, M_2 = 0)$:

$$\begin{pmatrix} L_1 & \ell & \ell+L_1 \\ -M_1 & -M_2 & M_1+M_2 \end{pmatrix} = (-1)^{M_1+M_2}\frac{(L_1)!(\ell)!}{(\ell+L_1)!}\left[\frac{(\ell+L_1+M_1+M_2)!(\ell+L_1-M_1-M_2)!}{(L_1+M_1)!(L_1-M_1)!(\ell+M_2)!(\ell-M_2)!}\right]^{1/2} \times\begin{pmatrix} L_1 & \ell & \ell+L_1 \\ 0 & 0 & 0 \end{pmatrix}. \tag{96}$$

According to the above initial conditions, next we compute the cases with $L_1 + \ell - 2, \cdots, \min(L_1, \ell_{\rm min})$. From Eq.(25) in Sec.8.6 of Ref.[15],

$$a\begin{pmatrix} L_1 & \ell & L_2 \\ M_1 & M_2 & M_3 \end{pmatrix} = b\begin{pmatrix} L_1 & \ell & L_2+1 \\ M_1 & M_2 & M_3 \end{pmatrix} + c\begin{pmatrix} L_1 & \ell & L_2+2 \\ M_1 & M_2 & M_3 \end{pmatrix}. \tag{97}$$

where the coefficients in general case are described by

$$
\begin{aligned}
a &= \left\{(L_2+1)^2 - M_3^2\right\}^{1/2}\alpha\,,\\
b &= -M_2\beta_1 + M_3\beta_2\,,\\
c &= -\left\{(L_2+2) - M_3^2\right\}^{1/2}\gamma\,.
\end{aligned}
\tag{98}
$$

The quantities α, β_i, γ are

$$
\begin{aligned}
\alpha &= (L_2+2)\sqrt{(-L_2+\ell+L_1)(L_2-\ell+L_1+1)(L_2+\ell-L_1+1)(L_2+\ell+L_1+2)}\,,\\
\beta_1 &= 2(L_2+1)(L_2+2)(2L_2+3)\,,\\
\beta_2 &= (2L_2+3)[(L_2+1)(L_2+2)+\ell(\ell+1)-L_1(L_1-1)]\,,\\
\gamma &= (L_2+1)\sqrt{(-L_2+\ell+L_1-1)(L_2-\ell+L_1+2)(L_2+\ell-L_1+2)(L_2+\ell+L_1+3)}\,.
\end{aligned}
\tag{99}
$$

A Flat-Sky Approximation

A.1 Fourier Transformation

Here, we introduce the Fourier transformation of the function, $x(\hat{\boldsymbol{n}})$, defined on the unit sphere:

$$
x_{\boldsymbol{\ell}} = \int \mathrm{d}^2\hat{\boldsymbol{n}}\; x(\hat{\boldsymbol{n}})\,\mathrm{e}^{-i\hat{\boldsymbol{n}}\cdot\boldsymbol{\ell}}\,,
\tag{100}
$$

where $\boldsymbol{\ell} = \ell(\cos\varphi_{\boldsymbol{\ell}}, \sin\varphi_{\boldsymbol{\ell}}, 0)$, and the integration is

$$
\int \mathrm{d}^2\hat{\boldsymbol{n}} \;\equiv \int_{-\pi}^{\pi} d\varphi \int_{-1}^{1} d\mu\,.
\tag{101}
$$

A.1.1 Formula

Here, we summarize the formula related to the Fourier transformation. Let us first consider the Fourier coefficient of $\boldsymbol{\nabla}X(\hat{\boldsymbol{n}})$. the derivative of X in Fourier space becomes

$$
\begin{aligned}
\widehat{\xi}_s[\partial^i X](\boldsymbol{y}) &= \int d\boldsymbol{x} a_s(\boldsymbol{x},\boldsymbol{y})[\partial^i X(\boldsymbol{x})]\\
&= \int d\boldsymbol{x} a_s(\boldsymbol{x},\boldsymbol{y})\left(\int d\boldsymbol{y}'\widehat{\xi}_{s'}[X](\boldsymbol{y}')[\partial^i a^*_{s'}(\boldsymbol{y}',\boldsymbol{x})]\right)\\
&= \int d\boldsymbol{x} a_s(\boldsymbol{x},\boldsymbol{y})\int d\boldsymbol{y}'\widehat{\xi}_{s'}[X](\boldsymbol{y}')i\left[\boldsymbol{y}' + s'\frac{\boldsymbol{e}_{\varphi_{\boldsymbol{x}}}}{x}\right]a^*_{s'}(\boldsymbol{y}',\boldsymbol{x})\\
&= \int d\boldsymbol{y}'\widehat{\xi}_{s'}[X](\boldsymbol{y}')\,\mathrm{e}^{-i(s\varphi_{\boldsymbol{y}}+s'\varphi_{\boldsymbol{y}'})}\int d\boldsymbol{x}\left[\mathrm{i}\boldsymbol{y}' + is'\frac{\boldsymbol{e}_{\varphi_{\boldsymbol{x}}}}{x}\right]\mathrm{e}^{\mathrm{i}\boldsymbol{x}\cdot(\boldsymbol{y}-\boldsymbol{y}')}\,\mathrm{e}^{\mathrm{i}(s-s')\varphi_{\boldsymbol{x}}}\\
&= \int d\boldsymbol{y}'\hat{\xi}_{s'}[X](\boldsymbol{y}')\,\mathrm{e}^{\mathrm{i}(s_X-s)\varphi_{\boldsymbol{y}}}i\boldsymbol{y}'\delta_D(\boldsymbol{y}-\boldsymbol{y}')\delta_D(s-s')\\
&= \mathrm{i}\boldsymbol{y}\delta_D(s-s')\widehat{\xi}_{s'}[X](\boldsymbol{y})\,.
\end{aligned}
\tag{102}
$$

The product of two functions, X and Y, becomes

$$\begin{aligned}\widehat{\xi}_s[XY](\boldsymbol{y}) &= \int d\boldsymbol{x} a_s(\boldsymbol{x},\boldsymbol{y})X(\boldsymbol{x})Y(\boldsymbol{x}) \\ &= \int d\boldsymbol{x} a_s(\boldsymbol{x},\boldsymbol{y}) \int d\boldsymbol{y}'\widehat{\xi}_{s'}[X](\boldsymbol{y}')a^*_{s'}(\boldsymbol{x},\boldsymbol{y}') \int d\boldsymbol{y}''\widehat{\xi}_{s''}[Y](\boldsymbol{y}'')a^*_{s''}(\boldsymbol{x},\boldsymbol{y}'') \\ &= \int d\boldsymbol{y}'\widehat{\xi}_{s'}[X](\boldsymbol{y}') \int d\boldsymbol{y}''\widehat{\xi}_{s''}[Y](\boldsymbol{y}'') \int d\boldsymbol{x}\, \mathrm{e}^{\mathrm{i}\boldsymbol{x}\cdot(\boldsymbol{y}-\boldsymbol{y}'-\boldsymbol{y}'')}\, \mathrm{e}^{\mathrm{i}(s-s'-s'')\varphi_{\boldsymbol{x}}}\, \mathrm{e}^{\mathrm{i}(s\varphi_{\boldsymbol{y}}-s'\varphi_{\boldsymbol{y}'}-s''\varphi_{\boldsymbol{y}''})} \\ &= \delta_D(s-s'-s'') \int d\boldsymbol{y}'\widehat{\xi}_{s'}[X](\boldsymbol{y}')\, \mathrm{e}^{\mathrm{i}(s-s'')\varphi_{\boldsymbol{y}}-(s'-s'')\varphi_{\boldsymbol{y}'}}\widehat{\xi}_{s''}[Y](\boldsymbol{y}-\boldsymbol{y}') \end{aligned} \tag{103}$$

From these relations, we obtain

$$\begin{aligned}\widehat{\xi}_s[\partial^i X\partial_i Y](\boldsymbol{y}) &= \delta_D(s-s'-s'') \int d\boldsymbol{y}'\widehat{\xi}_{s'}[\partial^i X](\boldsymbol{y}')\widehat{\xi}_{s''}[\partial_i Y](\boldsymbol{y}-\boldsymbol{y}') \\ &= -\delta_D(s-s'-s'') \int d\boldsymbol{y}'\boldsymbol{y}'\cdot(\boldsymbol{y}-\boldsymbol{y}')\, \mathrm{e}^{i(s-s'')\varphi_{\boldsymbol{y}}-(s'-s'')\varphi_{\boldsymbol{y}'}}\widehat{\xi}_{s'}[X](\boldsymbol{y}')\widehat{\xi}_{s''}[Y](\boldsymbol{y}-\boldsymbol{y}')\,, \end{aligned} \tag{104}$$

and

$$\begin{aligned}\widehat{\xi}_s[(\star\boldsymbol{\nabla}X)\cdot\boldsymbol{\nabla}Y](\boldsymbol{y}) &= \epsilon^{ij}\widehat{\xi}_s[\partial_j X\partial_i Y](\boldsymbol{y}) \\ &= -\delta(s-s'-s'') \int d\boldsymbol{y}'(\star\boldsymbol{y}')\cdot(\boldsymbol{y}-\boldsymbol{y}')\, \mathrm{e}^{\mathrm{i}(s-s'')\varphi_{\boldsymbol{y}}-(s'-s'')\varphi_{\boldsymbol{y}'}}\widehat{\xi}_{s'}[X](\boldsymbol{y}')\widehat{\xi}_{s''}[Y](\boldsymbol{y}-\boldsymbol{y}')\,. \end{aligned} \tag{105}$$

A.1.2 Examples

The lensing effect distort the temperature and polarization as

$$\widetilde{X}(\hat{\boldsymbol{n}}) = X(\hat{\boldsymbol{n}}) + \boldsymbol{\nabla}\phi\cdot\boldsymbol{\nabla}X(\hat{\boldsymbol{n}}) + (\star\boldsymbol{\nabla})\varpi\cdot\boldsymbol{\nabla}X(\hat{\boldsymbol{n}})\,. \tag{106}$$

Operating $\widehat{\xi}$ to both side of the equation, we obtain

$$\begin{aligned}\widehat{\xi}_s[\widetilde{X}](\boldsymbol{\ell}) &= \widehat{\xi}_s[X](\boldsymbol{\ell}) + \widehat{\xi}_s[\boldsymbol{\nabla}\phi\cdot\boldsymbol{\nabla}X](\boldsymbol{\ell}) + \xi_s[(\star\boldsymbol{\nabla})\varpi\cdot\boldsymbol{\nabla}X](\boldsymbol{\ell}) \\ &= \widehat{\xi}_s[X](\boldsymbol{\ell}) - \int d\boldsymbol{y}'\boldsymbol{y}'\cdot(\boldsymbol{y}-\boldsymbol{y}')\, \mathrm{e}^{\mathrm{i}s(\varphi_{\boldsymbol{y}}-\varphi_{\boldsymbol{y}'})}\widehat{\xi}_s[X](\boldsymbol{y}')\widehat{\xi}_0[\phi](\boldsymbol{y}-\boldsymbol{y}') \\ &\quad - \int d\boldsymbol{y}'(\star\boldsymbol{y}')\cdot(\boldsymbol{y}-\boldsymbol{y}')\, \mathrm{e}^{\mathrm{i}s(\varphi_{\boldsymbol{y}}-\varphi_{\boldsymbol{y}'})}\widehat{\xi}_s[X](\boldsymbol{y}')\widehat{\xi}_0[\varpi](\boldsymbol{y}-\boldsymbol{y}')\,. \end{aligned} \tag{107}$$

B Lensed CMB power spectrum

The lensed fields are expressed in terms of the unlensed fields, $X_{\boldsymbol{\ell}}$, up to second order of lensing potentials:

$$\widetilde{X}_{\boldsymbol{\ell}} = X_{\boldsymbol{\ell}} + \sum_Y \{\hat{L}^{X,(1)}_{\boldsymbol{\ell}}(Y) + \hat{L}^{X,(2)}_{\boldsymbol{\ell}}(Y)\}\,, \tag{108}$$

where we introduce the operators $\hat{A}^{XY}_{\boldsymbol{\ell},\langle\boldsymbol{\ell}'\rangle}$ and $\hat{B}^{XY}_{\boldsymbol{\ell},\langle\boldsymbol{\ell}'\rangle}$:

$$\hat{L}^{X,(1)}_{\boldsymbol{\ell}} = -\sum_x \int \frac{\mathrm{d}^2\boldsymbol{\ell}'}{(2\pi)^2} \{\boldsymbol{\ell}' \odot_x (\boldsymbol{\ell} - \boldsymbol{\ell}')\} R^{XY}_{\boldsymbol{\ell}-\boldsymbol{\ell}'} x_{\boldsymbol{\ell}-\boldsymbol{\ell}'} \,, \tag{109}$$

$$\hat{L}^{X,(2)}_{\boldsymbol{\ell}} = \frac{1}{2}\sum_x \int \frac{\mathrm{d}^2\boldsymbol{\ell}'}{(2\pi)^2} \int \frac{\mathrm{d}^2\boldsymbol{\ell}''}{(2\pi)^2} \{\boldsymbol{\ell}' \odot_x \boldsymbol{\ell}''\}\{\boldsymbol{\ell}' \odot_x (\boldsymbol{\ell} - \boldsymbol{\ell}' - \boldsymbol{\ell}'')\} R^{XY}_{\boldsymbol{\ell}-\boldsymbol{\ell}'} x_{\boldsymbol{\ell}''} x^*_{-\boldsymbol{\ell}+\boldsymbol{\ell}'+\boldsymbol{\ell}''} \,. \tag{110}$$

Eq.(108) gives the following relation

$$\begin{aligned}
\langle \widetilde{X}_{\boldsymbol{\ell}_1} \widetilde{X}'_{-\boldsymbol{\ell}_2} \rangle_{\mathrm{CMB}} &= \left\langle \left(X_{\boldsymbol{\ell}_1} + \hat{L}^{X,(1)}_{\boldsymbol{\ell}_1}(Y) + \hat{L}^{X,(2)}_{\boldsymbol{\ell}_1}(Y) \right) \left(X'_{-\boldsymbol{\ell}_2} + \hat{L}^{X',(1)}_{-\boldsymbol{\ell}_2}(Y') + \hat{L}^{X',(2)}_{-\boldsymbol{\ell}_2}(Y') \right)^* \right\rangle_{\mathrm{CMB}} \\
&= \langle X_{\boldsymbol{\ell}_1} X'_{-\boldsymbol{\ell}_2} \rangle_{\mathrm{CMB}} + \left\{ \langle \hat{L}^{X,(1)}_{\boldsymbol{\ell}_1}(Y) X'_{-\boldsymbol{\ell}_2} \rangle_{\mathrm{CMB}} + \langle \hat{L}^{X',(1)}_{-\boldsymbol{\ell}_2}(Y') X_{\boldsymbol{\ell}_1} \rangle_{\mathrm{CMB}} \right\} \\
&\quad + \langle \hat{L}^{X,(1)}_{\boldsymbol{\ell}_1}(Y) \hat{L}^{X',(1)}_{-\boldsymbol{\ell}_2}(Y') \rangle_{\mathrm{CMB}} \\
&\quad + \left\{ \langle \hat{L}^{X,(2)}_{\boldsymbol{\ell}_1}(Y) X'_{-\boldsymbol{\ell}_2} \rangle_{\mathrm{CMB}} + \langle \hat{L}^{X',(2)}_{-\boldsymbol{\ell}_2}(Y') X_{\boldsymbol{\ell}_1} \rangle_{\mathrm{CMB}} \right\} + \mathcal{O}(\phi^3, \varpi^3) \,.
\end{aligned} \tag{111}$$

Note that

$$\langle \hat{L}^{X,(1)}_{\boldsymbol{\ell}}(Y) X'_{\boldsymbol{L}} \rangle = -\sum_x (R_{\boldsymbol{\ell}-\boldsymbol{L}})^{XY} C^{YX'}_{\ell} \{\boldsymbol{L} \odot_x (\boldsymbol{\ell} - \boldsymbol{L})]\} x_{\boldsymbol{\ell}-\boldsymbol{L}} \,, \tag{112}$$

$$\begin{aligned}
\langle \hat{L}^{X,(1)}_{\boldsymbol{\ell}_1}(Y) \hat{L}^{X',(1)}_{-\boldsymbol{\ell}_2}(Y') \rangle &= \sum_x \int \frac{\mathrm{d}^2\boldsymbol{\ell}'_1}{(2\pi)^2} \{\boldsymbol{\ell}'_1 \odot_x (\boldsymbol{\ell}_1 - \boldsymbol{\ell}'_1)\}\{\boldsymbol{\ell}'_1 \odot_x (\boldsymbol{\ell}_2 - \boldsymbol{\ell}'_1)\} \\
&\quad \times (R_{\boldsymbol{\ell}_1-\boldsymbol{\ell}'_1})^{XY} C^{YY'}_{\ell} (R_{-\boldsymbol{\ell}_2+\boldsymbol{\ell}'_1})^{X'Y'} x_{\boldsymbol{\ell}_1-\boldsymbol{\ell}'_1} x_{-\boldsymbol{\ell}_2+\boldsymbol{\ell}'_1} \,,
\end{aligned} \tag{113}$$

$$\langle \hat{L}^{X,(2)}_{\boldsymbol{\ell}}(Y) X'_{\boldsymbol{L}} \rangle = \frac{1}{2}\sum_x \int \frac{\mathrm{d}^2\boldsymbol{\ell}''}{(2\pi)^2} (R_{\boldsymbol{\ell}-\boldsymbol{L}})^{XY} \{\boldsymbol{L} \odot_x \boldsymbol{\ell}''\}\{\boldsymbol{L} \odot_x (\boldsymbol{\ell} - \boldsymbol{L} - \boldsymbol{\ell}'')\} x_{\boldsymbol{\ell}''} x^*_{-\boldsymbol{\ell}+\boldsymbol{L}+\boldsymbol{\ell}''} \,. \tag{114}$$

Substituting Eqs. (112), (113) and (114) into Eq.(111), we obtain

$$\langle \widetilde{X}_{\boldsymbol{\ell}_1} \widetilde{X}'_{-\boldsymbol{\ell}_2} \rangle_{\mathrm{CMB}} = C^{XX'}_{\ell_1} \delta_{\boldsymbol{\ell}_1,\boldsymbol{\ell}_2} + \overline{f}^{XX',\phi}_{\boldsymbol{\ell}_1,\boldsymbol{\ell}_2} \phi_{\boldsymbol{\ell}_1-\boldsymbol{\ell}_2} + \overline{f}^{XX',\varpi}_{\boldsymbol{\ell}_1,\boldsymbol{\ell}_2} \varpi_{\boldsymbol{\ell}_1-\boldsymbol{\ell}_2} + \mathcal{O}(\phi^2, \varpi^2) \,, \tag{115}$$

where we define

$$\begin{aligned}
\overline{f}^{XX'}_{\boldsymbol{\ell}_1,\boldsymbol{\ell}_2} &= -\sum_{Y=\Theta,E,B} \left\{ R^{XY}_{\boldsymbol{\ell}_1-\boldsymbol{\ell}_2} \{\boldsymbol{\ell}_2 \odot_\phi (\boldsymbol{\ell}_1 - \boldsymbol{\ell}_2)\} C^{YX'}_{\ell_2} + R^{X'Y}_{\boldsymbol{\ell}_2-\boldsymbol{\ell}_1} \{\boldsymbol{\ell}_1 \odot_\phi (\boldsymbol{\ell}_2 - \boldsymbol{\ell}_1)\} C^{YX}_{\ell_1} \right\} \,, \\
\overline{f}^{XX'}_{\boldsymbol{\ell}_1\boldsymbol{\ell}_2} &= -\sum_{Y=\Theta,E,B} \left\{ R^{XY}_{\boldsymbol{\ell}_1-\boldsymbol{\ell}_2} \{\boldsymbol{\ell}_2 \odot_\varpi (\boldsymbol{\ell}_1 - \boldsymbol{\ell}_2)\} C^{YX'}_{\ell_2} + R^{X'Y}_{\boldsymbol{\ell}_2-\boldsymbol{\ell}_1} \{\boldsymbol{\ell}_1 \odot_\varpi (\boldsymbol{\ell}_2 - \boldsymbol{\ell}_1)\} C^{YX}_{\ell_1} \right\} \,.
\end{aligned} \tag{116}$$

Note that the second order is

$$\begin{aligned}
(\text{second order}) = \sum_x \Bigg\{ &\int \frac{\mathrm{d}^2\boldsymbol{\ell}'}{(2\pi)^2} \{\boldsymbol{\ell}' \odot_x (\boldsymbol{\ell}_1 - \boldsymbol{\ell}')\}\{\boldsymbol{\ell}' \odot_x (\boldsymbol{\ell}_2 - \boldsymbol{\ell}')\} R^{XY}_{\boldsymbol{\ell}_1-\boldsymbol{\ell}'} R^{X'Y'}_{\boldsymbol{\ell}_2-\boldsymbol{\ell}'} C^{YY'}_{\ell'} x_{\boldsymbol{\ell}_1-\boldsymbol{\ell}'} x_{-\boldsymbol{\ell}_2+\boldsymbol{\ell}'} \\
&+ \frac{1}{2} C^{YX'}_{\ell_2} \int \frac{\mathrm{d}^2\boldsymbol{\ell}'}{(2\pi)^2} (R_{\boldsymbol{\ell}_1-\boldsymbol{\ell}_2})^{XY} \{\boldsymbol{\ell}_2 \odot_x \boldsymbol{\ell}'\}\{\boldsymbol{\ell}_2 \odot_x (\boldsymbol{\ell}_1 - \boldsymbol{\ell}_2 - \boldsymbol{\ell}')\} x_{\boldsymbol{\ell}'} x_{\boldsymbol{\ell}_1-\boldsymbol{\ell}_2+\boldsymbol{\ell}'} \\
&+ \frac{1}{2} C^{Y'X}_{\ell_1} \int \frac{\mathrm{d}^2\boldsymbol{\ell}'}{(2\pi)^2} (R_{\boldsymbol{\ell}_2-\boldsymbol{\ell}_1})^{X'Y'} \{\boldsymbol{\ell}_1 \odot_x \boldsymbol{\ell}'\}\{\boldsymbol{\ell}_1 \odot_x (\boldsymbol{\ell}_2 - \boldsymbol{\ell}_1 - \boldsymbol{\ell}')\} x_{\boldsymbol{\ell}'} x_{\boldsymbol{\ell}_2-\boldsymbol{\ell}_1-\boldsymbol{\ell}'} \Bigg\} \,.
\end{aligned} \tag{117}$$

We now compute the angular power spectra of lensed CMB anisotropies at lowest order of $C_\ell^{\phi\phi}$ and $C_\ell^{\varpi\varpi}$. We can neglect the correlation $\langle X\phi\rangle$ and $\langle X\varpi\rangle$ since the small scale temperature and polarization at last scattering are almost uncorrelated with the late-time potentials, and correlation only has a tiny effect on the result. Then, the angular power spectra can be derived by computing up to second order of ϕ and ϖ, and the result is [11, 13]

$$\widetilde{C}_\ell^{\Theta\Theta} = (1-R_{\boldsymbol{\ell}})C_\ell^{\Theta\Theta} + \sum_x \int \frac{\mathrm{d}^2\boldsymbol{\ell}'}{2\pi}\,[\boldsymbol{\ell}'\odot_x(\boldsymbol{\ell}-\boldsymbol{\ell}')]^2 C^{xx}_{|\boldsymbol{\ell}-\boldsymbol{\ell}'|}C^{\Theta\Theta}_{\ell'}\,, \tag{118}$$

$$\widetilde{C}_\ell^{\Theta\mathrm{E}} = (1-R_{\boldsymbol{\ell}})C_\ell^{\Theta\mathrm{E}} + \sum_x \int \frac{\mathrm{d}^2\boldsymbol{\ell}'}{2\pi}\,[\boldsymbol{\ell}'\odot_x(\boldsymbol{\ell}-\boldsymbol{\ell}')]^2 C^{xx}_{|\boldsymbol{\ell}-\boldsymbol{\ell}'|}C^{\Theta\mathrm{E}}_{\ell'}\cos 2\varphi_{\boldsymbol{\ell}'}\,, \tag{119}$$

$$\widetilde{C}_\ell^{\mathrm{EE}} = (1-R_{\boldsymbol{\ell}})C_\ell^{EE} + \frac{1}{2}\sum_x \int \frac{\mathrm{d}^2\boldsymbol{\ell}'}{2\pi}\,[\boldsymbol{\ell}'\odot_x(\boldsymbol{\ell}-\boldsymbol{\ell}')]^2 C^{xx}_{|\boldsymbol{\ell}-\boldsymbol{\ell}'|}[(C^{EE}_{\ell'}+C^{BB}_{\ell'}) + (C^{EE}_{\ell'}-C^{BB}_{\ell'})\cos 4\varphi_{\boldsymbol{\ell}'}]\,, \tag{120}$$

$$\widetilde{C}_\ell^{\mathrm{BB}} = (1-R_{\boldsymbol{\ell}})C_\ell^{BB} + \frac{1}{2}\sum_x \int \frac{\mathrm{d}^2\boldsymbol{\ell}'}{2\pi}\,[\boldsymbol{\ell}'\odot_x(\boldsymbol{\ell}-\boldsymbol{\ell}')]^2 C^{xx}_{|\boldsymbol{\ell}-\boldsymbol{\ell}'|}[(C^{EE}_{\ell'}+C^{BB}_{\ell'}) - (C^{EE}_{\ell'}-C^{BB}_{\ell'})\cos 4\varphi_{\boldsymbol{\ell}'}]\,, \tag{121}$$

with the quantity $R_{\boldsymbol{\ell}}$ being

$$R_{\boldsymbol{\ell}} \equiv \sum_x \int \frac{\mathrm{d}^2\boldsymbol{\ell}'}{(2\pi)^2}\,(\boldsymbol{\ell}\odot_x\boldsymbol{\ell}')^2 C^{xx}_{\ell'}\,. \tag{122}$$

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