[Note]
Likelihood for Lensed and Delensed B-mode

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Contents

1 Likelihood for single random variable with Edgeworth expansion

1.1 Edgeworth expansion

If the skewness is zero, the Edgeworth expansion up to second order is equivalent to the Gram-Charlier A series up to the kurtosis order. The probability distribution for a zero-mean random field x with variance C is given by

$$\mathcal{L}(x) = \left[1 + \frac{k_4}{24}\left(\frac{x^4}{C^2} - 6\frac{x^2}{C} + 3\right)\right] \frac{1}{\sqrt{2\pi C}} \exp\left[-\frac{x^2}{2C}\right] . \quad (1)$$

Here we use $k_4 \equiv \kappa_4/C^2$ in stead of the kurtosis κ_4. This probability function satisfies [1]

$$\int \mathrm{d}x \ \mathcal{L} = 1\,, \tag{2}$$

where we use

$$\int \mathrm{d}x \ \frac{x^{2n}}{C^n}\frac{1}{\sqrt{2\pi C}}\exp\left[-\frac{x^2}{2C}\right] = (2n-1)(2n-3)\cdots 3\,. \tag{3}$$

The chi-square is

$$\chi^2(x) \equiv -2\ln\mathcal{L} = -2\ln\left[1+\frac{k_4}{24}\left(\frac{x^4}{C^2}-6\frac{x^2}{C}+3\right)\right] + \ln C + \frac{x^2}{C}\,. \tag{4}$$

1.2 Likelihood for the amplitude parameter

We first discuss the likelihood function for the amplitude of the power spectrum A. Replacing C with AC, we obtain

$$\mathcal{L}(x|A) = \left[1+\frac{k_4}{24A^2}\left(\frac{x^4}{A^2C^2}-6\frac{x^2}{AC}+3\right)\right]\frac{1}{\sqrt{2\pi AC}}\exp\left[-\frac{x^2}{2AC}\right]\,. \tag{5}$$

We define an observed amplitude defined as

$$\widehat{A} = \frac{x^2}{C}\,. \tag{6}$$

Using $dx/d\widehat{A} = \sqrt{C/\widehat{A}}/2$, the Likelihood is then given by

$$\mathcal{L}(\widehat{A}|A) = \left[1+\frac{k_4}{24A^2}\left(\frac{\widehat{A}^2}{A^2}-6\frac{\widehat{A}}{A}+3\right)\right]\frac{1}{2\sqrt{2\pi A\widehat{A}}}\exp\left[-\frac{\widehat{A}}{2A}\right]\,. \tag{7}$$

1.2.1 Statistical properties

The ensemble average of the observed amplitude is

$$\langle\widehat{A}\rangle = \int \mathrm{d}\widehat{A}\ \widehat{A}\mathcal{L}(\widehat{A}|A) = 1\,. \tag{8}$$

The variance of this estimator is

$$\begin{aligned}\langle\widehat{A}^2\rangle - 1 &= \int \mathrm{d}x\ \frac{x^4}{C^2}\mathcal{L}(x) - 1 \\ &= 2 + \frac{k_4}{24}\left\langle\frac{x^8}{C^4}-6\frac{x^6}{C^3}+3\frac{x^4}{C^2}\right\rangle \\ &= 2+\frac{k_4}{24}(105-90+9) = 2+k_4\,. \end{aligned} \tag{9}$$

The above naive estimator is not biased due to the presence of kurtosis correction term, but the variance is increased.

1.2.2 Maximum likelihood point

The chi-square is

$$\chi^2(\widehat{A}|A) = -2\ln\left[1+\frac{k_4}{24}\left(\frac{\widehat{A}^2}{A^4}-\frac{6\widehat{A}}{A^3}+\frac{3}{A^2}\right)\right] + \ln A + \ln\widehat{A} + \frac{\widehat{A}}{A}\,. \tag{10}$$

[1] Note that, in general, probability distribution with Edgeworth expansion can be negative, and is not normalized to unity.

The derivative with respect to A is

$$0=\frac{\partial\chi^2(\widehat{A}|A)}{\partial A}=-\frac{k_4}{12[1+K(A)]}\left(\frac{-4\widehat{A}}{A^5}+\frac{18\widehat{A}}{A^4}-\frac{6}{A^3}\right)+\frac{1}{A}-\frac{\widehat{A}}{A^2}\,. \tag{11}$$

Here

$$K(A)=\frac{k_4}{24}\left(\frac{\widehat{A}^2}{A^4}-\frac{6\widehat{A}}{A^3}+\frac{3}{A^2}\right)\,. \tag{12}$$

Rewriting the above equation and ignoring $K(A)$, we obtain

$$0=-\frac{k_4}{6}\left(-2\widehat{A}^2+9A\widehat{A}-3A^2\right)+A^3(A-\widehat{A})\,. \tag{13}$$

In the limit of $k_4\to 0$, the solution is

$$A_{\rm MLE}|_{k_4\to 0}=\widehat{A}\,. \tag{14}$$

We assume that the solution with $k_4\neq 0$ is given by

$$A_{\rm MLE}=\widehat{A}+k_4A_1\,, \tag{15}$$

and substituting this into Eq. (13) leads to

$$A_1=\frac{2}{3\widehat{A}}\,. \tag{16}$$

1.2.3 Fisher estimator

In the BICEP experiment, the Fisher estiamtor is considered as an observeble. Here we discuss the statistical properties of the Fisher estimator. The Fisher estimator is given by

$$\widehat{A}_{\rm F}=1+\frac{1}{F}\frac{\partial\ln\mathcal{L}}{\partial A}\bigg|_{A=1}=1-\frac{1}{2F}\frac{\partial\chi^2}{\partial A}\bigg|_{A=1}\,, \tag{17}$$

where F is the Fisher matrix at $A=1$ (in this case, F is just a number)

$$F=\left\langle\left(\frac{\partial\ln\mathcal{L}}{\partial A}\right)^2\right\rangle\bigg|_{A=1}=\frac{1}{4}\left\langle\left(\frac{\partial\chi^2}{\partial A}\right)^2\right\rangle\bigg|_{A=1}\,. \tag{18}$$

Eq. (17) is written as

$$\begin{aligned}\widehat{A}_{\rm F}&=1+\frac{1}{F}\left[\frac{-k_4}{12}\left(2\widehat{A}^2-9\widehat{A}+3\right)-\frac{1}{2}(1-\widehat{A})\right]\\&=1+\frac{1}{2F}\left[\kappa_4A_1'-1+\widehat{A}\right]\\&\simeq 1+\frac{1}{2F}\left[\widehat{A}'-1\right]\,,\end{aligned} \tag{19}$$

where we define $\widehat{A}'=\widehat{A}+k_4A_1'$ and

$$A_1'=\frac{1}{6}\left(-2\widehat{A}^2+9\widehat{A}-3\right)\,. \tag{20}$$

The Fisher matrix is rewritten as

$$\begin{aligned}4F&=\langle(\widehat{A}'-1)^2\rangle\\&=\langle(\widehat{A}-1)^2\rangle+2k_4\langle(\widehat{A}-1)A_1'\rangle+\mathcal{O}(\kappa_4^2)\\&=2+k_4+2k_4\langle\widehat{A}A_1'\rangle+\mathcal{O}(\kappa_4^2)\\&=2-k_4+\mathcal{O}(\kappa_4^2)\,.\end{aligned} \tag{21}$$

The estimator is written as

$$\widehat{A}_{\rm F}=1+\frac{2}{2-k_4}(\widehat{A}'-1)\,. \tag{22}$$

This estimator is no longer unbiased if $A\neq 1$. To see this, we take the ensamble average of the estimator, assuming that $\langle x^2\rangle = AC$:

$$\begin{aligned}
\langle\widehat{A}_{\rm F}\rangle - A &= 1-A+\frac{2}{2-k_4}(\langle\widehat{A}\rangle + k_4\langle A'_1\rangle -1)\\
&= 1-A+\frac{2}{2-k_4}\left[A+k_4\left[-(A^2-1)+\frac{3}{2}(A-1)\right]-1\right]\\
&= \frac{2}{2-k_4}\left[\frac{2-k_4}{2}(1-A)-(1-A)-k_4\left[(A^2-1)+\frac{3}{2}(1-A)\right]\right]\\
&= \frac{2}{2-k_4}(1-A)\left[\frac{2-k_4}{2}-1-k_4\left[-A+\frac{1}{2}\right]\right]\\
&= \frac{-2k_4}{2-k_4}(1-A)^2\,.
\end{aligned} \tag{23}$$

The variance of the estimator is given by $1/F$:

$$\frac{1}{F}=\frac{4}{2-k_4}\simeq 2+k_4\,. \tag{24}$$

2 Application of Edgeworth expansion to Lensed B-mode

2.1 Likelihood for lensed B-mode

The probability distribution of $\boldsymbol{a}=\{a_{\ell m}\}$ for a given covariance $\mathbf{C}=\langle \boldsymbol{a}\boldsymbol{a}^t\rangle$ is give by

$$\mathcal{L}(\boldsymbol{a}|\mathbf{C})\propto[1+k]\frac{1}{\sqrt{\det\mathbf{C}}}\,{\rm e}^{-\frac{1}{2}\boldsymbol{a}\mathbf{C}^{-1}\boldsymbol{a}}\,, \tag{25}$$

where k is [1] [2]

$$\begin{aligned}
k &= \frac{1}{24}\sum_{\ell,m}\langle a_{\ell_1m_1}a_{\ell_2m_2}a_{\ell_3m_3}a_{\ell_4m_4}\rangle_c\\
&\quad\times\left(\overline{a}_{\ell_1m_1}\overline{a}_{\ell_2m_2}\overline{a}_{\ell_3m_3}\overline{a}_{\ell_4m_4}-6\mathbf{C}^{-1}_{\ell_1m_1,\ell_2m_2}\overline{a}_{\ell_3m_3}\overline{a}_{\ell_4m_4}+3\mathbf{C}^{-1}_{\ell_1m_1,\ell_2m_2}\mathbf{C}^{-1}_{\ell_3m_3,\ell_4m_4}\right)\\
&= {\rm e}^{\frac{1}{2}\boldsymbol{a}\mathbf{C}^{-1}\boldsymbol{a}}\frac{1}{24}\sum_{\ell_im_i}\langle a_{\ell_1m_1}a_{\ell_2m_2}a_{\ell_3m_3}a_{\ell_4m_4}\rangle_c\frac{\partial}{\partial a_{\ell_1m_1}}\frac{\partial}{\partial a_{\ell_2m_2}}\frac{\partial}{\partial a_{\ell_3m_3}}\frac{\partial}{\partial a_{\ell_4m_4}}{\rm e}^{-\frac{1}{2}\boldsymbol{a}\mathbf{C}^{-1}\boldsymbol{a}}\,.
\end{aligned} \tag{26}$$

The covariance of the lensed B-mode anisotropies is diagonal:

$$\mathbf{C}_{\ell m,\ell' m'}=\delta_{\ell,\ell'}\delta_{m,-m'}(-1)^mC_\ell\,. \tag{27}$$

This leads to

$$\begin{aligned}
\mathcal{L}(\boldsymbol{a}|\mathbf{C}) &\propto [1+k]\prod_\ell\frac{1}{C_\ell^{(2\ell+1)/2}}\exp\left[-\frac{1}{2}\sum_{m=-\ell}^{\ell}\frac{|a_{\ell m}|^2}{C_\ell}\right]\\
&= [1+k]\prod_\ell\frac{1}{C_\ell^{(2\ell+1)/2}}\prod_{m=-\ell}^{\ell}\exp\left[-\frac{1}{2}\frac{|a_{\ell m}|^2}{C_\ell}\right]\\
&= [1+k]\mathcal{L}_{\rm g}(\boldsymbol{a}|\mathbf{C})\,.
\end{aligned} \tag{28}$$

[2] $4!=24$ is missing in Regan et al.

Here we introduce the Gaussian Likelihood:

$$\mathcal{L}_{\rm g}(\boldsymbol{a}|\mathbf{C}) \equiv \prod_\ell \frac{1}{C_\ell^{(2\ell+1)/2}} \prod_{m=-\ell}^{\ell} \exp\left[-\frac{1}{2}\frac{|a_{\ell m}|^2}{C_\ell}\right] . \tag{29}$$

Note that the likelihood described above is used to derive the optimal trispectrum estimator of the lensing potential power spectrum [2, 3]. In such case, we extract the off-diagonal elements of the two alm's. This implies that the estimator of the lensing potential power spectrum uses anisotropic information on the lensed B-mode multipoles which is different from that used in the amplitude parameter.

2.2 Likelihood for amplitude parameter

Replacing C_ℓ with AC_ℓ, we obtian

$$\mathcal{L}(\boldsymbol{a}|\mathbf{C}) = [1+k(A)] \prod_\ell \frac{1}{(AC_\ell)^{(2\ell+1)/2}} \prod_{m=-\ell}^{\ell} \exp\left[-\frac{1}{2}\frac{|a_{\ell m}|^2}{AC_\ell}\right] , \tag{30}$$

and

$$\begin{aligned} k = &\frac{1}{24}\sum_{\ell,m}\langle a_{\ell_1 m_1}a_{\ell_2 m_2}a_{\ell_3 m_3}a_{\ell_4 m_4}\rangle_c \\ &\times\left(\frac{1}{A^4}\overline{a}_{\ell_1 m_1}\overline{a}_{\ell_2 m_2}\overline{a}_{\ell_3 m_3}\overline{a}_{\ell_4 m_4} - \frac{6}{A^3 C_{\ell_1}}\delta_{\ell_1\ell_2}\delta_{m_1,-m_2}\overline{a}_{\ell_3 m_3}\overline{a}_{\ell_4 m_4} + \frac{3}{A^2 C_{\ell_1}C_{\ell_3}}\delta_{\ell_1\ell_2}\delta_{m_1,-m_2}\delta_{\ell_3\ell_4}\delta_{m_3,-m_4}\right) \end{aligned} \tag{31}$$

Using $\widehat{A}_\ell = \sum_{m=-\ell}^{\ell}|a_\ell^2|/(2\ell+1)/C_\ell$,

$$\mathcal{L}(\boldsymbol{a}|A) \propto [1+k]A^{-\sum_\ell (2\ell+1)/2}\prod_\ell C_\ell^{-(2\ell+1)/2}\exp\left[\frac{-1}{2A}\sum_\ell(2\ell+1)\widehat{A}_\ell\right] . \tag{32}$$

2.3 Statistical properties of amplitude estimator

In the case of Gaussian field, the amplitude estimator is given by [4]

$$\widehat{A} = \frac{\sum_\ell(2\ell+1)\widehat{A}_\ell}{\sum_\ell(2\ell+1)} . \tag{33}$$

Even if $k_4 \neq 0$, the mean of the estimator satisfies

$$\langle\widehat{A}\rangle = 1 . \tag{34}$$

On the other hand, the variance of the estimator is

$$\begin{aligned} \langle\widehat{A}^2\rangle - 1 &= -1 + \frac{1}{[\sum_\ell(2\ell+1)]^2}\sum_{\ell\ell'}(2\ell+1)(2\ell'+1)\langle\widehat{A}_\ell\widehat{A}_{\ell'}\rangle \\ &= -1 + \frac{1}{[\sum_\ell(2\ell+1)]^2}\sum_{\ell\ell'}(2\ell+1)(2\ell'+1)\frac{\langle\widehat{C}_\ell\widehat{C}_{\ell'}\rangle}{C_\ell C_{\ell'}} . \end{aligned} \tag{35}$$

Now we evaluate the covariance of the power spectrum:

$$\begin{aligned} \langle\widehat{C}_\ell\widehat{C}_{\ell'}\rangle &= \frac{1}{(2\ell+1)(2\ell'+1)}\sum_{mm'}\int \mathrm{d}\boldsymbol{a}\ \mathcal{L}(\boldsymbol{a})a_{\ell m}a^*_{\ell m}a_{\ell' m'}a^*_{\ell' m'} \\ &= C_\ell C_{\ell'}\left(1+\frac{2\delta_{\ell\ell'}}{2\ell+1}\right) + \frac{1}{(2\ell+1)(2\ell'+1)}\sum_{mm'}\int\mathrm{d}\boldsymbol{a}\ a_{\ell m}a^*_{\ell m}a_{\ell' m'}a^*_{\ell' m'}k(\boldsymbol{a})\mathcal{L}_{\rm g} . \end{aligned} \tag{36}$$

Using Eq. (26), and introducing an operator c_i as $c_i a_{\ell_i m_i} = a^*_{\ell_i m_i}$,

$$
\begin{aligned}
&\int \mathrm{d}\boldsymbol{a}\; a_{\ell m} a^*_{\ell m} a_{\ell' m'} a^*_{\ell' m'} \frac{\partial}{\partial a_{\ell_1 m_1}} \frac{\partial}{\partial a_{\ell_2 m_2}} \frac{\partial}{\partial a_{\ell_3 m_3}} \frac{\partial}{\partial a_{\ell_4 m_4}} \mathcal{L}_{\rm g} \\
&= -\int \mathrm{d}\boldsymbol{a}\; [\delta_{\ell\ell_1}\delta_{mm_1}(a^*_{\ell m} + c_1 a_{\ell m}) a_{\ell' m'} a^*_{\ell' m'} + (\ell m \leftrightarrow \ell' m')] \\
&\quad\times \frac{\partial}{\partial a_{\ell_2 m_2}} \frac{\partial}{\partial a_{\ell_3 m_3}} \frac{\partial}{\partial a_{\ell_4 m_4}} \mathcal{L}_{\rm g} \\
&= \int \mathrm{d}\boldsymbol{a}\; [\delta_{\ell\ell_1}\delta_{mm_1}\delta_{\ell\ell_2}\delta_{mm_2}(c_2+c_1) a_{\ell' m'} a^*_{\ell' m'} + \delta_{\ell\ell_1}\delta_{mm_1}(a^*_{\ell m} + c_1 a_{\ell m})\delta_{\ell'\ell_2}\delta_{m'm_2}(a^*_{\ell' m'} + c_2 a_{\ell' m'}) + (\ell m \leftrightarrow \ell' m')] \\
&\quad\times \frac{\partial}{\partial a_{\ell_3 m_3}} \frac{\partial}{\partial a_{\ell_4 m_4}} \mathcal{L}_{\rm g} \\
&= -\int \mathrm{d}\boldsymbol{a}\; [\delta_{\ell\ell_1}\delta_{mm_1}\delta_{\ell\ell_2}\delta_{mm_2}(c_2+c_1)\delta_{\ell'\ell_3}\delta_{m'm_3}(a^*_{\ell' m'} + c_3 a_{\ell' m'}) + \delta_{\ell\ell_1}\delta_{mm_1}\delta_{\ell\ell_3}\delta_{mm_3}(c_3+c_1)\delta_{\ell'\ell_2}\delta_{m'm_2}(a^*_{\ell' m'} + c_2 a_{\ell' m'}) \\
&\quad + \delta_{\ell\ell_1}\delta_{mm_1}(a^*_{\ell m} + c_1 a_{\ell m})\delta_{\ell'\ell_2}\delta_{m'm_2}\delta_{\ell'\ell_3}\delta_{m'm_3}(c_3+c_2)] + (\ell m \leftrightarrow \ell' m')] \\
&\quad\times \frac{\partial}{\partial a_{\ell_4 m_4}} \mathcal{L}_{\rm g} \\
&= \int \mathrm{d}\boldsymbol{a}\; \frac{1}{4} \sum_{h,i,j,k=(1,2,3,4)} \delta_{\ell\ell_h}\delta_{mm_h}\delta_{\ell\ell_i}\delta_{mm_i}\delta_{\ell'\ell_j}\delta_{mm_j}\delta_{\ell'\ell_k}\delta_{m'm_k}(c_h+c_i)(c_j+c_k)\mathcal{L}_{\rm g}\,. \qquad (37)
\end{aligned}
$$

This leads to

$$
\begin{aligned}
\frac{1}{(2\ell+1)(2\ell'+1)} \sum_{mm'} \int \mathrm{d}\boldsymbol{a}\; a_{\ell m} a^*_{\ell m} a_{\ell' m'} a^*_{\ell' m'} k(\boldsymbol{a}) \mathcal{L}_{\rm g} &= \frac{1}{(2\ell+1)(2\ell'+1)} \sum_{mm'} \int \mathrm{d}\boldsymbol{a}\; \langle C_\ell C_{\ell'} \rangle_{\rm c} \mathcal{L}_{\rm g} \\
&= \langle C_\ell C_{\ell'} \rangle_{\rm c}\,. \qquad (38)
\end{aligned}
$$

We finally obtain the expression for the variance:

$$
\begin{aligned}
\langle \widehat{A}^2 \rangle &= \frac{2}{[\sum_\ell (2\ell+1)]^2} \sum_{\ell\ell'} \left[(2\ell+1)\delta_{\ell\ell'} + (2\ell+1)(2\ell'+1) \frac{\langle C_\ell C_{\ell'} \rangle_{\rm c}}{C_\ell C_{\ell'}} \right] \\
&= \frac{2}{[\sum_\ell (2\ell+1)]^2} \sum_{\ell\ell'} \sqrt{(2\ell+1)(2\ell'+1)} R_{\ell\ell'}\,. \qquad (39)
\end{aligned}
$$

Here we introduce the correlation coefficients of the power spectrum covariance

$$
R_{\ell\ell'} = \frac{\sqrt{(2\ell+1)(2\ell'+1)}}{2} \frac{\langle C_\ell C_{\ell'} \rangle_{\rm c}}{C_\ell C_{\ell'}}\,. \qquad (40)
$$

3 Correlation Coefficients of ClBB

3.1 Lensing B-mode

Lensed B-mode power spectrum is given by

$$
C_\ell = \frac{1}{2\ell+1} \sum_{\ell' L} (\mathcal{S}^{(-)}_{\ell\ell' L})^2 C^{\rm EE}_{\ell'} C^{\phi\phi}_L \equiv \Xi_\ell[C^{\rm EE}, C^{\phi\phi}]\,, \qquad (41)
$$

where we define a convolution operator:

$$
\Xi_\ell[A,B] = \frac{1}{2\ell+1} \sum_{\ell_1\ell_2} (\mathcal{S}^{(-)}_{\ell\ell_1\ell_2})^2 A_{\ell_1} B_{\ell_2}\,. \qquad (42)
$$

The fluctuation of the E-mode and lensing potential power spectra leads to

$$\delta C_\ell = \sum_{\ell'} \frac{\partial C_\ell}{\partial C^{\rm EE}_{\ell'}} \delta C^{\rm EE}_{\ell'} + \sum_{\ell'} \frac{\partial C_\ell}{\partial C^{\phi\phi}_{\ell'}} \delta C^{\phi\phi}_{\ell'} \,. \tag{43}$$

The covariance of the power spectrum from this contributions is therefore given by

$$\langle C_\ell C_{\ell'}\rangle_{\rm c} = \sum_L \frac{\partial C_\ell}{\partial C^{\rm EE}_L} \frac{2(C^{\rm EE}_L)^2}{2L+1} \frac{\partial C_{\ell'}}{\partial C^{\rm EE}_L} + \sum_L \frac{\partial C_\ell}{\partial C^{\phi\phi}_L} \frac{2(C^{\phi\phi}_L)^2}{2L+1} \frac{\partial C_{\ell'}}{\partial C^{\phi\phi}_L} \,, \tag{44}$$

where we use

$$\langle \delta C^X_\ell \delta C^X_{\ell'}\rangle = \frac{2C^X_\ell C^X_{\ell'}}{2\ell+1} \delta_{\ell\ell'} \,. \tag{45}$$

We omit the fully connected term which has negligible contribution [5], and describe the covariance matrix as [6]

$$\begin{aligned}
{\rm Cov}^{\rm BB}_{\ell\ell'} \equiv \langle C_\ell C_{\ell'}\rangle - \langle C_\ell\rangle\langle C_{\ell'}\rangle &= \frac{2}{2\ell+1} C^2_\ell \delta_{\ell\ell'} + \sum_L \frac{\partial C_\ell}{\partial C^{\rm EE}_L} \frac{2(C^{\rm EE}_L)^2}{2L+1} \frac{\partial C_{\ell'}}{\partial C^{\rm EE}_L} + \sum_L \frac{\partial C_\ell}{\partial C^{\phi\phi}_L} \frac{2(C^{\phi\phi}_L)^2}{2L+1} \frac{\partial C_{\ell'}}{\partial C^{\phi\phi}_L} \\
&\equiv \frac{2}{2\ell+1} C^2_\ell \delta_{\ell\ell'} + {\rm Cov}^{\rm E}_{\ell\ell'} + {\rm Cov}^{\phi}_{\ell\ell'} \,,
\end{aligned} \tag{46}$$

where we denote the second and third terms as ${\rm Cov}^{\rm E}$ and ${\rm Cov}^{\phi}$, respectively.

3.1.1 Efficient computation

Ref. [6] evaluates the derivatives using the finite difference between the angular power spectra obtained from `CAMB`. This method naturally takes into account the effect of the higher-order contributions of the remapping. On the other hand, in our case, since the large scale B-mode power spectrum is well evaluated by the convolution of the E and lensing potential power spectra [7], we compute the covariance based on an analysitc approach described below.

To evaluate the connected part of the covariance, ${\rm Cov}^{\rm E}$ and ${\rm Cov}^{\rm grad}$, we rewrite the derivative as

$$\frac{\partial C_\ell}{\partial C^{\rm EE}_L} = \Xi^{\phi}_{\ell L}[C^{\phi\phi}] \,, \tag{47}$$

$$\frac{\partial C_\ell}{\partial C^{\phi\phi}_L} = \Xi^{\rm E}_{\ell L}[C^{\rm EE}] \,. \tag{48}$$

Here we define

$$\Xi^{\phi}_{\ell L}[A] \equiv \frac{1}{2\ell+1} \sum_{L'} (\mathcal{S}^{(-)}_{\ell L L'})^2 A_{L'} \,, \tag{49}$$

$$\Xi^{\rm E}_{\ell L}[A] \equiv \frac{1}{2\ell+1} \sum_{L'} (\mathcal{S}^{(-)}_{\ell L' L})^2 A_{L'} \,. \tag{50}$$

Note that

$$\Xi_\ell[A,B] = \sum_L A_L \Xi^{\phi}_{\ell L}[B] = \sum_L B_L \Xi^{\rm E}_{\ell L}[A] \,. \tag{51}$$

The covariance is then given by

$$\begin{aligned}
{\rm Cov}^{\rm E}_{\ell\ell'} &= \sum_L \frac{\partial C_\ell}{\partial C^{\rm EE}_L} \frac{2(C^{\rm EE}_L)^2}{2L+1} \frac{\partial C_{\ell'}}{\partial C^{\rm EE}_L} \\
&= \sum_L \Xi^{\phi}_{\ell L}[C^{\phi\phi}] \frac{2(C^{\rm EE}_L)^2}{2L+1} \Xi^{\phi}_{\ell' L}[C^{\phi\phi}] \,.
\end{aligned} \tag{52}$$

Denoting

$$F_L^{\ell'} = \frac{2(C_L^{\rm EE})^2}{2L+1}\Xi^\phi_{\ell' L}[C^{\phi\phi}]\,, \tag{53}$$

we obtain

$$\mathrm{Cov}^{\rm E}_{\ell\ell'} = \Xi_\ell[F^{\ell'}, C^{\phi\phi}]\,. \tag{54}$$

Similarly, we find

$$\mathrm{Cov}^{\phi}_{\ell\ell'} = \Xi_\ell[C^{\rm EE}, G^{\ell'}]\,, \tag{55}$$

where we defin

$$G_L^{\ell'} = \frac{2(C_L^{\phi\phi})^2}{2L+1}\Xi^{\rm E}_{\ell' L}[C^{\rm EE}]\,. \tag{56}$$

The summations Ξ_ℓ, $\Xi^{\rm E}_{\ell L}$ and $\Xi^\phi_{\ell L}$ are efficiently evaluated using the reduced wigner d functions as described in Ref. [7].

3.2 Delensed B-mode

Next we discuss the covariance of the delensed B-mode power spectrum. To account for this effect, next we assume that the lensing potential is decomposed into the lensing potential and noise terms:

$$\widehat{\phi}_{\ell m} = \phi_{\ell m} + n_{\ell m}\,. \tag{57}$$

The delensed B-mode is given as a sum of two components

$$B' = [E\star(\phi-\phi^W)] + [E\star n^W]\,, \tag{58}$$

where ϕ^W and n^W are the Wiener filtered multipoles, and we assume that the E-mode Wiener filter is unity. The delensed B-mode power spectrum is then given by [7, 8]

$$C'_\ell = \Xi_\ell[\widehat{C}^{\rm EE}, \widehat{C}^{\phi\phi}(1-W^\phi)^2 + \widehat{N}^{\phi\phi}(W^\phi)^2]\,. \tag{59}$$

where the function W is the Wiener fileter:

$$W^{\rm E}_\ell = \frac{C_\ell^{\rm EE}}{C_\ell^{\rm EE}+N_\ell^{\rm P}}\,, \tag{60}$$

$$W^{\phi}_\ell = \frac{C_\ell^{\phi\phi}}{C_\ell^{\phi\phi}+N_\ell^{\phi}}\,. \tag{61}$$

3.2.1 naive approximation

Now we use the same procedure as in the case of the lensed B-mode power spectrum, i.e.,

$$\begin{aligned}\langle C'_\ell C'_\ell\rangle_{\rm c} &= \sum_L \frac{\partial C'_\ell}{\partial \widehat{C}_L^{\rm EE}}\frac{2(C_L^{\rm EE})^2}{2L+1}\frac{\partial C'_{\ell'}}{\partial \widehat{C}_L^{\rm EE}} + \sum_L \frac{\partial C'_\ell}{\partial \widehat{C}_L^{\phi\phi}}\frac{2(C_L^{\phi\phi})^2}{2L+1}\frac{\partial C'_{\ell'}}{\partial \widehat{C}_L^{\phi\phi}} + \sum_L \frac{\partial C'_\ell}{\partial \widehat{N}_L^{\phi\phi}}\frac{2(N_L^{\phi\phi})^2}{2L+1}\frac{\partial C'_{\ell'}}{\partial \widehat{N}_L^{\phi\phi}} \\ &= \sum_L \frac{\partial C'_\ell}{\partial \widehat{C}_L^{\rm EE}}\frac{2(C_L^{\rm EE})^2}{2L+1}\frac{\partial C'_{\ell'}}{\partial \widehat{C}_L^{\rm EE}} + \sum_L \frac{\partial C^{\rm BB}_\ell}{\partial C_L^{\phi\phi}}\frac{2(\widehat{C}_L^{\phi\phi})^2}{2L+1}\frac{\partial C^{\rm BB}_{\ell'}}{\partial C_L^{\phi\phi}}\left[(1-W_L^\phi)^4 + \left(\frac{\widehat{N}_L^{\phi\phi}}{\widehat{C}_L^{\phi\phi}}\right)^2 (W_L^\phi)^4\right]\,.\end{aligned} \tag{62}$$

Here we assume that the power spectra $\widehat{C}^{\rm EE}$, $\widehat{C}^{\phi\phi}$ and $\widehat{N}^{\phi\phi}$ has Gaussain covariance. In the second line of the last equation, we use the derivative of the lensed B-mode in terms of the lensing potential. Note that

$$\left[(1-W_L^\phi)^4 + \left(\frac{\widehat{N}_L^{\phi\phi}}{\widehat{C}_L^{\phi\phi}}\right)^2 (W_L^\phi)^4\right] = (1-W_L^\phi)^2\frac{(C_L^{\phi\phi})^2+(N_L^{\phi\phi})^2}{(C_L^{\phi\phi}+N^{\phi\phi})^2} \equiv (1-W_L^\phi)^2\beta(r_L)\,, \tag{63}$$

where we define

$$r_L \equiv \frac{C^{\phi\phi}}{N^{\phi\phi}}\,, \qquad \beta(x) \equiv \frac{x^2+1}{(x+1)^2} \tag{64}$$

This implies that, if $1-W^{\phi}_L = \alpha$ is nearly constant, the covariance is given by

$$\frac{1}{\alpha^2}\langle C'_\ell C'_\ell\rangle_{\rm c} = \sum_L \frac{\partial C^{\rm BB}_\ell}{\partial C^{\rm EE}_L}\frac{2(C^{\rm EE}_L)^2}{2L+1}\frac{\partial C^{\rm BB}_{\ell'}}{\partial C^{\rm EE}_L} + \sum_L \frac{\partial C^{\rm BB}_\ell}{\partial C^{\phi\phi}_L}\frac{2(\widehat{C}^{\phi\phi}_L)^2}{2L+1}\frac{\partial C^{\rm BB}_{\ell'}}{\partial C^{\phi\phi}_L}\beta(r_L)\,. \tag{65}$$

The only deference from the lensed B-mode covariance is the presence of $\beta(r_L)$. This function is minimized when $r_L = 1$, i.e., the signal and noise are comparable in the map to be used for the lensing reconstruction. On the other hand, the noise or signal dominant case ($r_L = 0$ or $r = \infty$), the function becomes unity, and the correlation coefficient of the covariance is the same as that of the lensed B-mode.

3.2.2 Additional correction

The above formula does not take into account the terms coming from e.g. a connected correlation between 4 E-modes, 2 lensing potential and noise: $\langle E^1E^2\rangle\langle E^3E^4\rangle\langle\phi^1\phi^3\rangle\langle n^2n^4\rangle$. This correction is simply included by performing the following approximation

$$\mathrm{Cov}^{\rm BB,del}_{\ell\ell'} = \frac{2}{2\ell+1}(C'_\ell)^2\delta_{\ell\ell'} + \sum_L \frac{\partial C'_\ell}{\partial C^{\rm EE}_L}\frac{2(C^{\rm EE}_L)^2}{2L+1}\frac{\partial C'_{\ell'}}{\partial C^{\rm EE}_L} + \sum_L \frac{\partial C'_\ell}{\partial C^{\phi\phi}_L}\frac{2(C^{\phi\phi}_L)^2}{2L+1}\frac{\partial C'_{\ell'}}{\partial C^{\phi\phi}_L}\,, \tag{66}$$

where C' is defined as

$$C'_\ell = \Xi_\ell[\widehat{C}^{\rm EE}, \widehat{C}^{\phi\phi}(1-W^\phi)^2 + \widehat{N}^{\phi\phi}(W^\phi)^2] \to \Xi_\ell[\widehat{C}^{\rm EE}, C^{\phi\phi}(1-W^\phi)]\,. \tag{67}$$

The correlation coefficients are given by

$$R_{\ell\ell'} = \delta_{\ell\ell'} + \sqrt{(2\ell+1)(2\ell'+1)}\sum_L\left[\frac{\partial\ln C'_\ell}{\partial C^{\rm EE}_L}\frac{(C^{\rm EE}_L)^2}{2L+1}\frac{\partial\ln C'_{\ell'}}{\partial C^{\rm EE}_L} + \frac{\partial\ln C'_\ell}{\partial C^{\phi\phi}_L}\frac{(C^{\phi\phi}_L)^2}{2L+1}\frac{\partial\ln C'_{\ell'}}{\partial C^{\phi\phi}_L}\right]\,. \tag{68}$$

The simulated result with $\sigma_{\rm P} = 6\mu$K-arcmin and $\theta = 4$ arcmin shows that the variance of the amplitude parameter is $\sigma(A) = 0.005388$. Using the naive formula, the estimated variance of the amplitude parameter becomes $\sigma(A) = 0.005149$ (4.6% smaller). If we add the correction terms, we obtain $\sigma(A) = 0.005264$ (2.4% smaller). Note that, in the Gaussian case, $\sigma(A) = 0.004698$.

The above formula, however, can not explain the increase of the correlation coefficients in the case of the quadratic delensing in the CV-limit.

A Correlation Coefficients of ClBB in the flat sky limit

In this section, we derive the flat sky counterpart of the $C^{\rm BB}_\ell$covariance matrix and correlation coefficients.

A.1 B-mode in the flat sky

The lensing B-mode in the flat sky is given by

$$\begin{aligned} B_{\boldsymbol{\ell}} &= \int\frac{\mathrm{d}^2\boldsymbol{L}}{(2\pi)^2}\,\boldsymbol{L}\cdot(\boldsymbol{\ell}-\boldsymbol{L})\phi_{\boldsymbol{L}}E_{\boldsymbol{\ell}-\boldsymbol{L}}\sin 2(\varphi_{\boldsymbol{\ell}-\boldsymbol{L}}-\varphi_{\boldsymbol{\ell}}) \\ &\equiv \xi^{\boldsymbol{L}}_{\boldsymbol{\ell}}\phi_{\boldsymbol{L}}E_{\boldsymbol{\ell}-\boldsymbol{L}}\,, \end{aligned} \tag{69}$$

where we introduce a convolution operation:

$$\xi^{\boldsymbol{L}}_{\boldsymbol{\ell}} \equiv \int\frac{\mathrm{d}^2\boldsymbol{L}}{(2\pi)^2}\,\boldsymbol{L}\cdot(\boldsymbol{\ell}-\boldsymbol{L})\sin 2(\varphi_{\boldsymbol{\ell}-\boldsymbol{L}}-\varphi_{\boldsymbol{\ell}})\,. \tag{70}$$

Note that $\xi^{\boldsymbol{L}}_{\boldsymbol{\ell}} = \xi^{-\boldsymbol{L}}_{-\boldsymbol{\ell}}$. The delensed B-mode is expressed as

$$B^{\rm del}_{\boldsymbol{\ell}} = \xi^{\boldsymbol{L}}_{\boldsymbol{\ell}}(E_{\boldsymbol{\ell}-\boldsymbol{L}}\phi_{\boldsymbol{L}} - W^{\rm E}_{\boldsymbol{\ell}-\boldsymbol{L}}E_{\boldsymbol{\ell}-\boldsymbol{L}}W^{\phi}_{\boldsymbol{L}}\widehat{\phi}_{\boldsymbol{L}})\,. \tag{71}$$

Assuming that $W^{\rm E}=1$ and $\widehat{\phi}=\phi+n$, we rewrite the above equation as

$$B^{\rm del}_{\boldsymbol{\ell}} = \xi^{\boldsymbol{L}}_{\boldsymbol{\ell}}[E_{\boldsymbol{\ell}-\boldsymbol{L}}(1-W^{\phi}_{\boldsymbol{L}})\phi_{\boldsymbol{L}} - E_{\boldsymbol{\ell}-\boldsymbol{L}}W^{\phi}_{\boldsymbol{L}}n_{\boldsymbol{L}}] \equiv \xi^{\boldsymbol{L}}_{\boldsymbol{\ell}}[E_{\boldsymbol{\ell}-\boldsymbol{L}}\phi^w_{\boldsymbol{L}} - E_{\boldsymbol{\ell}-\boldsymbol{L}}n^w_{\boldsymbol{L}}]\,. \tag{72}$$

Here we define $\phi^w_{\boldsymbol{\ell}} = (1-W^{\phi}_{\boldsymbol{\ell}})\phi_{\boldsymbol{\ell}}$ and $n^w_{\boldsymbol{\ell}} = W^{\phi}_{\boldsymbol{\ell}}n_{\boldsymbol{\ell}}$.

A.2 B-mode power spectrum

The angular power spectrum of the lensing B-mode is given by

$$\begin{aligned}\delta^{\rm D}_{\boldsymbol{0}}C^{\rm BB}_{\ell} &= \langle|B_{\boldsymbol{\ell}}|^2\rangle = \xi^{\boldsymbol{L}}_{\boldsymbol{\ell}}\xi^{\boldsymbol{L}'}_{\boldsymbol{\ell}}\langle E_{\boldsymbol{\ell}-\boldsymbol{L}}\phi_{\boldsymbol{L}}E^*_{\boldsymbol{\ell}-\boldsymbol{L}'}\phi^*_{\boldsymbol{L}'}\rangle \\ &= \xi^{\boldsymbol{L}}_{\boldsymbol{\ell}}\xi^{\boldsymbol{L}'}_{\boldsymbol{\ell}}\delta^{\rm D}_{\boldsymbol{L}-\boldsymbol{L}'}\delta^{\rm D}_{\boldsymbol{0}}C^{\rm EE}_{|\boldsymbol{\ell}-\boldsymbol{L}|}C^{\phi\phi}_{L} \equiv \delta^{\rm D}_{\boldsymbol{0}}\widetilde{\xi}^{\boldsymbol{L}}_{\boldsymbol{\ell\ell}}C^{\rm EE}_{|\boldsymbol{\ell}-\boldsymbol{L}|}C^{\phi\phi}_{L}\,,\end{aligned} \tag{73}$$

where $\delta^{\rm D}_{\boldsymbol{\ell}}$ is the Dirac delta function in two dimension. We define

$$\widetilde{\xi}^{\boldsymbol{L}}_{\boldsymbol{\ell\ell}'} \equiv \xi^{\boldsymbol{L}}_{\boldsymbol{\ell}}\xi^{\boldsymbol{L}}_{\boldsymbol{\ell}'}\delta^{\rm D}_{\boldsymbol{L}-\boldsymbol{L}'} = \int\frac{{\rm d}^2\boldsymbol{L}}{(2\pi)^2}\,\boldsymbol{L}\cdot(\boldsymbol{\ell}-\boldsymbol{L})\boldsymbol{L}\cdot(\boldsymbol{\ell}'-\boldsymbol{L})\sin 2(\varphi_{\boldsymbol{\ell}'-\boldsymbol{L}}-\varphi_{\boldsymbol{\ell}'})\sin 2(\varphi_{\boldsymbol{\ell}'-\boldsymbol{L}}-\varphi_{\boldsymbol{\ell}'})\,. \tag{74}$$

The delensed B-mode power spectrum is obtained in the similar manner, and the result is

$$\begin{aligned}\delta^{\rm D}_{\boldsymbol{0}}C^{\rm BB,del}_{\ell} &= \widetilde{\xi}^{\boldsymbol{L}}_{\boldsymbol{\ell\ell}}C^{\rm EE}_{|\boldsymbol{\ell}-\boldsymbol{L}|}[(1-W^{\phi}_L)^2C^{\phi\phi}_L + (W^{\phi}_L)^2N^{\phi\phi}_L] \\ &= \widetilde{\xi}^{\boldsymbol{L}}_{\boldsymbol{\ell\ell}}C^{\rm EE}_{|\boldsymbol{\ell}-\boldsymbol{L}|}C^{\phi\phi}_L(1-W^{\phi}_L)\,.\end{aligned} \tag{75}$$

A.3 Lensed B-mode power spectrum covariance

Four point correlation of the lensed B-mode is given by

$$\begin{aligned}\langle|B_{\boldsymbol{\ell}}|^2|B_{\boldsymbol{\ell}'}|^2\rangle &= \xi^{\boldsymbol{L}_1}_{\boldsymbol{\ell}}\xi^{\boldsymbol{L}_2}_{\boldsymbol{\ell}}\xi^{\boldsymbol{L}_3}_{\boldsymbol{\ell}'}\xi^{\boldsymbol{L}_4}_{\boldsymbol{\ell}'}\langle E_{\boldsymbol{\ell}-\boldsymbol{L}_1}\phi_{\boldsymbol{L}_1}E^*_{\boldsymbol{\ell}-\boldsymbol{L}_2}\phi^*_{\boldsymbol{L}_2}E_{\boldsymbol{\ell}'-\boldsymbol{L}_3}\phi_{\boldsymbol{L}_3}E^*_{\boldsymbol{\ell}'-\boldsymbol{L}_4}\phi^*_{\boldsymbol{L}_4}\rangle \\ &= \xi^{\boldsymbol{L}_1}_{\boldsymbol{\ell}}\xi^{\boldsymbol{L}_2}_{\boldsymbol{\ell}}\xi^{\boldsymbol{L}_3}_{\boldsymbol{\ell}'}\xi^{\boldsymbol{L}_4}_{\boldsymbol{\ell}'}\langle E_{\boldsymbol{\ell}-\boldsymbol{L}_1}E^*_{\boldsymbol{\ell}-\boldsymbol{L}_2}E_{\boldsymbol{\ell}'-\boldsymbol{L}_3}E^*_{\boldsymbol{\ell}'-\boldsymbol{L}_4}\rangle\langle\phi_{\boldsymbol{L}_1}\phi^*_{\boldsymbol{L}_2}\phi_{\boldsymbol{L}_3}\phi^*_{\boldsymbol{L}_4}\rangle \\ &= \widetilde{\xi}^{\boldsymbol{L}_1}_{\boldsymbol{\ell\ell}}\widetilde{\xi}^{\boldsymbol{L}_3}_{\boldsymbol{\ell}'\boldsymbol{\ell}'}\langle E_{\boldsymbol{\ell}-\boldsymbol{L}_1}E^*_{\boldsymbol{\ell}-\boldsymbol{L}_1}E_{\boldsymbol{\ell}'-\boldsymbol{L}_3}E^*_{\boldsymbol{\ell}'-\boldsymbol{L}_3}\rangle C^{\phi\phi}_{L_1}C^{\phi\phi}_{L_3} \\ &\quad + 2\widetilde{\xi}^{\boldsymbol{L}_1}_{\boldsymbol{\ell},-\boldsymbol{\ell}'}\widetilde{\xi}^{\boldsymbol{L}_2}_{\boldsymbol{\ell},-\boldsymbol{\ell}'}\langle E_{\boldsymbol{\ell}-\boldsymbol{L}_1}E_{-\boldsymbol{\ell}+\boldsymbol{L}_2}E^*_{-\boldsymbol{\ell}'-\boldsymbol{L}_1}E^*_{\boldsymbol{\ell}'+\boldsymbol{L}_2}\rangle C^{\phi\phi}_{L_1}C^{\phi\phi}_{L_2}\,.\end{aligned} \tag{76}$$

The first term is rewritten as

$$\begin{aligned}\text{1st term} &= \int\frac{{\rm d}^2\boldsymbol{L}_1}{(2\pi)^2}\int\frac{{\rm d}^2\boldsymbol{L}_3}{(2\pi)^2}\frac{\partial C^{\rm BB}_{\ell}}{\partial C^{\rm EE}_{|\boldsymbol{\ell}-\boldsymbol{L}_1|}}\frac{\partial C^{\rm BB}_{\ell'}}{\partial C^{\rm EE}_{|\boldsymbol{\ell}-\boldsymbol{L}_3|}}\langle E_{\boldsymbol{\ell}-\boldsymbol{L}_1}E^*_{\boldsymbol{\ell}-\boldsymbol{L}_1}E_{\boldsymbol{\ell}'-\boldsymbol{L}_3}E^*_{\boldsymbol{\ell}'-\boldsymbol{L}_3}\rangle \\ &= \langle|B_{\boldsymbol{\ell}}|^2|B_{\boldsymbol{\ell}'}|^2\rangle + \int\frac{{\rm d}^2\boldsymbol{L}_1}{(2\pi)^2}\frac{\partial C^{\rm BB}_{\ell}}{\partial C^{\rm EE}_{|\boldsymbol{\ell}-\boldsymbol{L}_1|}}\frac{\partial C^{\rm BB}_{\ell'}}{\partial C^{\rm EE}_{|\boldsymbol{\ell}-\boldsymbol{L}_1|}}2(C^{\rm EE}_{|\boldsymbol{\ell}-\boldsymbol{L}_1|})^2\delta^{\rm D}_{\boldsymbol{0}}\,.\end{aligned} \tag{77}$$

A.4 Delensed B-mode power spectrum covariance

Assuming that ϕ and n are the random Gaussian field, four point correlation of the delensed B-mode is decomposed into three terms:

$$\langle|B_{\boldsymbol{\ell}}|^2|B_{\boldsymbol{\ell}'}|^2\rangle = T^{\phi^4}_{\boldsymbol{\ell\ell}'} + T^{\phi^2n^2}_{\boldsymbol{\ell\ell}'} + T^{n^4}_{\boldsymbol{\ell\ell}'}\,, \tag{78}$$

where

$$\begin{aligned}T^{\phi^4}_{\boldsymbol{\ell\ell}'} &= \xi^{\boldsymbol{L}_1}_{\boldsymbol{\ell}}\xi^{\boldsymbol{L}_2}_{\boldsymbol{\ell}}\xi^{\boldsymbol{L}_3}_{\boldsymbol{\ell}'}\xi^{\boldsymbol{L}_4}_{\boldsymbol{\ell}'}\langle E_{\boldsymbol{\ell}-\boldsymbol{L}_1}\phi^w_{\boldsymbol{L}_1}E_{\boldsymbol{\ell}-\boldsymbol{L}_2}\phi^w_{\boldsymbol{L}_2}E_{\boldsymbol{\ell}'-\boldsymbol{L}_3}\phi^w_{\boldsymbol{L}_3}E_{\boldsymbol{\ell}'-\boldsymbol{L}_4}\phi^w_{\boldsymbol{L}_4}\rangle \\ T^{n^4}_{\boldsymbol{\ell\ell}'} &= \xi^{\boldsymbol{L}_1}_{\boldsymbol{\ell}}\xi^{\boldsymbol{L}_2}_{\boldsymbol{\ell}}\xi^{\boldsymbol{L}_3}_{\boldsymbol{\ell}'}\xi^{\boldsymbol{L}_4}_{\boldsymbol{\ell}'}\langle E_{\boldsymbol{\ell}-\boldsymbol{L}_1}n^w_{\boldsymbol{L}_1}E_{\boldsymbol{\ell}-\boldsymbol{L}_2}n^w_{\boldsymbol{L}_2}E_{\boldsymbol{\ell}'-\boldsymbol{L}_3}n^w_{\boldsymbol{L}_3}E_{\boldsymbol{\ell}'-\boldsymbol{L}_4}n^w_{\boldsymbol{L}_4}\rangle \\ T^{\phi^2n^2}_{\boldsymbol{\ell\ell}'} &= \xi^{\boldsymbol{L}_1}_{\boldsymbol{\ell}}\xi^{\boldsymbol{L}_2}_{\boldsymbol{\ell}}\xi^{\boldsymbol{L}_3}_{\boldsymbol{\ell}'}\xi^{\boldsymbol{L}_4}_{\boldsymbol{\ell}'}[\langle E_{\boldsymbol{\ell}-\boldsymbol{L}_1}\phi^w_{\boldsymbol{L}_1}E_{\boldsymbol{\ell}-\boldsymbol{L}_2}\phi^w_{\boldsymbol{L}_2}E_{\boldsymbol{\ell}'-\boldsymbol{L}_3}n^w_{\boldsymbol{L}_3}E_{\boldsymbol{\ell}'-\boldsymbol{L}_4}n^w_{\boldsymbol{L}_4}\rangle + \text{(5 perms.)}]\,.\end{aligned} \tag{79}$$

We connect the lines under the restriction, i.e., only same type of lines can be connected. There are $3\times 3 = 9$ cases. Connecting the all two E and ϕ lines at 1 with those at 2, 3 or 4, leads to the disconnected part $(1-2{,}3-4)$ or diagonal part $(1-3{,}2-4)$ $(1-4{,}2-3)$ of the correlation.

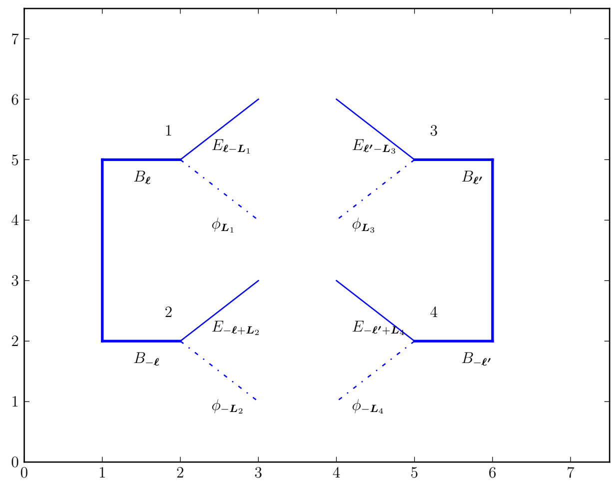

Figure 1: Diagram of the four point correlation.

B Application of Edgeworth expansion to Lensed B-mode

B.1 approximation of the kurtosis

The chi square is

$$\chi^2 \equiv -2\ln\mathcal{L} = -2\ln[1+k] + \mathrm{Tr}(\ln\mathbf{C}) + \boldsymbol{a}\mathbf{C}^{-1}\boldsymbol{a}\,, \tag{80}$$

If we assum

$$\overline{a}_{\ell m}\overline{a}_{\ell' m'} \simeq \overline{a}_{\ell m}\overline{a}_{\ell,-m}\delta_{\ell,\ell'}\delta_{m,-m'} \tag{81}$$

Then

$$\begin{aligned}
k &\simeq \frac{1}{24}\sum_{\ell_1 m_1 \ell_3 m_3}\langle a_{\ell_1 m_1}a^*_{\ell_1,m_1}a_{\ell_3 m_3}a^*_{\ell_3,m_3}\rangle_c\left(3\overline{a}_{\ell_1 m_1}\overline{a}^*_{\ell_1 m_1}\overline{a}_{\ell_3 m_3}\overline{a}^*_{\ell_3,m_3} - 6\frac{\overline{a}_{\ell_3 m_3}\overline{a}^*_{\ell_3,m_3}}{C_{\ell_1}} + 3\frac{1}{C_{\ell_1}C_{\ell_3}}\right) && (82)\\
&\simeq \frac{1}{24}\sum_{\ell_1 m_1 \ell_3 m_3}\frac{\langle C_{\ell_1}C_{\ell_3}\rangle_c}{C_{\ell_1}C_{\ell_3}}\left(3\frac{\widehat{C}_{\ell_1}}{C_{\ell_1}}\frac{\widehat{C}_{\ell_3}}{C_{\ell_3}} - 6\frac{\widehat{C}\ell_3}{C_{\ell_3}} + 3\right) && (83)\\
&= \frac{1}{8}\sum_{\ell_1\ell_3}(2\ell_1+1)(2\ell_3+1)R^c_{\ell_1\ell_3}\left(\widehat{A}_{\ell_1}\widehat{A}_{\ell_3} - 2\widehat{A}_{\ell_3} + 1\right)\,. && (84)
\end{aligned}$$

$$k = \frac{1}{8}\sum_{\ell_1\ell_3}(2\ell_1+1)(2\ell_3+1)R^c_{\ell_1\ell_3}\frac{1}{A^2}\left(\frac{\widehat{A}_{\ell_1}}{A}\frac{\widehat{A}_{\ell_3}}{A} - 2\frac{\widehat{A}_{\ell_3}}{A} + 1\right)\,. \tag{85}$$

B.2 Maximum likelihood estimator

$$\frac{\partial k}{\partial A} = \frac{1}{8}\sum_{\ell\ell'}(2\ell+1)(2\ell'+1)R^c_{\ell\ell'}\left(-\frac{4}{A^5}\widehat{A}_\ell\widehat{A}_{\ell'} + \frac{6}{A^4}\widehat{A}_{\ell'} - \frac{2}{A^3}\right)\,. \tag{86}$$

The maximum likelihood point is

$$0 = \frac{-2}{1+\kappa}\frac{\partial\kappa}{\partial A} + \frac{1}{A}\sum_{\ell}(2\ell+1) - \frac{1}{A^2}\sum_{\ell}(2\ell+1)\widehat{A}_{\ell}\,, \tag{87}$$

We find

$$0 \simeq \frac{1}{2}\sum_{\ell\ell'}(2\ell+1)(2\ell'+1)R^c_{\ell\ell'}\left(2\widehat{A}_\ell\widehat{A}_{\ell'} - 3A\widehat{A}_{\ell'} + A^2\right) + A^3[A\sum_{\ell}(2\ell+1) - \sum_{\ell}(2\ell+1)\widehat{A}_\ell]\,. \tag{88}$$

The solution is

$$A = A_0 + A_1 \tag{89}$$

where

$$A_0 = \frac{\sum_\ell(2\ell+1)\widehat{A}_\ell}{\sum_\ell(2\ell+1)} \tag{90}$$

and

$$A_1 = \frac{-1}{2A_0^3\sum_\ell(2\ell+1)}\sum_{\ell\ell'}(2\ell+1)(2\ell'+1)R^c_{\ell\ell'}\left(2\widehat{A}_\ell\widehat{A}_{\ell'} - 3A_0\widehat{A}_{\ell'} + A_0^2\right) \tag{91}$$

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