Computing quadratic estimator, delensing in curvedsky

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Latest version

Abstract

Here, I describe an algorithm for computing the quadratic estimator and its normalization of the lensing, cosmic bi-refringence, patchy reionization, and so on.

Contents

1 Notations

In the followings, we use small letters for multipoles of the CMB anisotropies (e.g., ℓ), while large letters are used for multipoles of the distortion fields (lensing, rotation, etc).

1.1 CMB

Θ denotes the CMB temperature fluctuations, and Q and U denote the Stokes parameters of the CMB linear polarization. The following equation defines the harmonic coefficients of the temperature anisotropies (and, in general, any scalar quantities x):

$$x_{LM} = \int \mathrm{d}^2\hat{\boldsymbol{n}}\; Y^*_{LM}(\hat{\boldsymbol{n}})x(\hat{\boldsymbol{n}})\,. \tag{1}$$

where Y_{LM} is the spin-0 spherical harmonics. On the other hand, Q and U are changed by the rotation of the sphere, and are therefore usually transformed into the rotational invariant quantities, the E and B modes, as [1]

$$[E \pm \mathrm{i}B]_{\ell m} = \int \mathrm{d}^2\hat{\boldsymbol{n}}\; [Y^{\pm 2}_{\ell m}(\hat{\boldsymbol{n}})]^*[Q \pm \mathrm{i}U](\hat{\boldsymbol{n}})\,. \tag{2}$$

Here, $Y^{\pm 2}_{\ell m}$ is the spin-2 spherical harmonics. For short notation, we also use

$$\begin{aligned} \Xi^{\pm} &= E \pm \mathrm{i}B\,, \\ P^{\pm} &= Q \pm \mathrm{i}U \end{aligned} \tag{3}$$

1.2 Gravitational weak lensing

The lensing effect on CMB anisotropies is described as remapping of the unlensed CMB anisotropies by the deflection angle [1, 2]

$$X(\hat{\boldsymbol{n}}) = X[\hat{\boldsymbol{n}} + \boldsymbol{d}(\hat{\boldsymbol{n}})]\,, \tag{4}$$

where X is Θ or $P^{\pm}$. The deflection angle of the CMB lensing is decomposed into the lensing potential, ϕ, and curl mode, ϖ, as [3]

$$\boldsymbol{d}(\hat{\boldsymbol{n}}) = \boldsymbol{\nabla}\phi(\hat{\boldsymbol{n}}) + (\star\boldsymbol{\nabla})\varpi(\hat{\boldsymbol{n}})\,, \tag{5}$$

where the operator $\star\boldsymbol{\nabla}$ denotes the derivatives with 90° rotation counterclockwise on the plane perpendicular to the line-of-sight direction and then operation. The harmonic coefficients of ϕ and ϖ are given by Eq. (1). The remapping of the CMB anisotropies is then given by

$$X(\hat{\boldsymbol{n}}) = X(\hat{\boldsymbol{n}}) + [\boldsymbol{\nabla}\phi(\hat{\boldsymbol{n}}) + (\star\boldsymbol{\nabla})\varpi(\hat{\boldsymbol{n}})] \cdot \boldsymbol{\nabla}X + \mathcal{O}(\phi^2, \varpi^2)\,. \tag{6}$$

1.3 Polarization angle rotation

If the rotation angle is small, the modulation of polarization after a rotation by an angle α is given by (e.g. [4])

$$\delta P^{\pm}(\hat{\boldsymbol{n}}) = \pm 2\mathrm{i}\alpha(\hat{\boldsymbol{n}})P^{\pm}(\hat{\boldsymbol{n}})\,. \tag{7}$$

The harmonic coefficients of α is given by Eq. (1).

1.4 Amplitude modulations

Survey window, gain fluctuations, and the inhomogeneities of the reionization, could vary the amplitudes of the CMB fluctuations across the sky. Denoting the modulations as $1 + \tau(\hat{\boldsymbol{n}})$, this leads to the modulation in CMB temperature and polarization as (e.g. [5])

$$\begin{aligned} \delta\Theta(\hat{\boldsymbol{n}}) &= \tau(\hat{\boldsymbol{n}})\Theta(\hat{\boldsymbol{n}})\,, \\ \delta P^{\pm}(\hat{\boldsymbol{n}}) &= \tau(\hat{\boldsymbol{n}})P^{\pm}(\hat{\boldsymbol{n}})\,. \end{aligned} \tag{8}$$

The harmonic coefficients of τ is given by Eq. (1).

[1] This definition is different from the `Healpix` by its sign.

1.5 Spherical Harmonics and Wigner-3j

The spherical harmonics is related to the Wigner-3j symbols as [6]

$$\int \mathrm{d}^2\hat{\boldsymbol{n}}\; Y^{s_1}_{\ell_1 m_1} Y^{s_2}_{\ell_2 m_2} Y^{s_3}_{\ell_3 m_3} = \sqrt{\frac{(2\ell_1+1)(2\ell_2+1)(2\ell_3+1)}{4\pi}} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ -s_1 & -s_2 & -s_3 \end{pmatrix} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix} , \tag{9}$$

with $s_1+s_2+s_3=0$ and $m_1+m_2+m_3=0$.

1.6 Derivatives of Spherical Harmonics

In general, denoting $a^s_\ell = -\sqrt{(\ell-s)(\ell+s+1)/2}$, the derivative of the spherical harmonics is given by [6]

$$\boldsymbol{\nabla} Y^s_{\ell m} = a^s_\ell Y^{s+1}_{\ell m}\boldsymbol{e}^* - a^{-s}_\ell Y^{s-1}_{\ell m}\boldsymbol{e} \,. \tag{10}$$

Here, we introduce the polarization vector $\boldsymbol{e}$ which are defined

$$\boldsymbol{e} = \frac{\boldsymbol{e}_1 + \mathrm{i}\boldsymbol{e}_2}{\sqrt{2}} \tag{11}$$

with $\boldsymbol{e}_i$ denoting the basis vectors orthogonal to the radial vector. The polarization vector satisfies $\boldsymbol{e}\cdot\boldsymbol{e}=0$, $\boldsymbol{e}\cdot\boldsymbol{e}^*=1$, $\star\boldsymbol{e}=-\mathrm{i}\boldsymbol{e}$. In particular, for $s=0$,

$$\boldsymbol{\nabla} Y_{\ell m} = a^0_\ell \big(Y^1_{\ell m}\boldsymbol{e}^* - Y^{-1}_{\ell m}\boldsymbol{e}\big) \,, \tag{12}$$

and, for $s=\pm 2$, denoting $a^\pm = a^{\pm 2}_\ell$,

$$\begin{aligned} \boldsymbol{\nabla} Y^2_{\ell m} &= a^+_\ell Y^3_{\ell m}\boldsymbol{e}^* - a^-_\ell Y^1_{\ell m}\boldsymbol{e} \,, \\ \boldsymbol{\nabla} Y^{-2}_{\ell m} &= a^-_\ell Y^{-1}_{\ell m}\boldsymbol{e}^* - a^+_\ell Y^{-3}_{\ell m}\boldsymbol{e} \,. \end{aligned} \tag{13}$$

1.7 Map derivatives

Derivative of scalar quantities such as the CMB temperature fluctuations and lensing potential is

$$\boldsymbol{\nabla} x = \sum_{LM} x_{LM}\boldsymbol{\nabla} Y_{LM} = \sum_{LM} x_{LM} a^0_L \left(Y^1_{LM}\boldsymbol{e}^* - Y^{-1}_{LM}\boldsymbol{e}\right) = x^+\boldsymbol{e}^* - x^-\boldsymbol{e} \,. \tag{14}$$

where we define

$$x^\pm \equiv \sum_{LM} x_{LM} a^0_L Y^{\pm 1}_{LM} \,, \tag{15}$$

and $(x^+)^* = -x^-$. The rotation of a pseudo-scalar quantity is given by

$$(\star\boldsymbol{\nabla})\varpi = \sum_{LM} \varpi_{LM}(\star\boldsymbol{\nabla})Y_{LM} = \sum_{LM} \varpi_{LM} a^0_L \mathrm{i} \left(Y^1_{LM}\boldsymbol{e}^* + Y^{-1}_{LM}\boldsymbol{e}\right) = \mathrm{i}(\varpi^+\boldsymbol{e}^* + \varpi^-\boldsymbol{e}) \,, \tag{16}$$

and $(\varpi^+)^* = -\varpi^-$. Spin-2 fields such as the CMB linear polarization is given by

$$\boldsymbol{\nabla} P^+ = \sum_{\ell m} \Xi^+_{\ell m}\boldsymbol{\nabla} Y^2_{\ell m} = \sum_{\ell m} \Xi^+_{\ell m}\left(a^+_\ell Y^3_{\ell m}\boldsymbol{e}^* - a^-_\ell Y^1_{\ell m}\boldsymbol{e}\right) = \Xi^{+^+}\boldsymbol{e}^* - \Xi^{+^-}\boldsymbol{e} \,, \tag{17}$$

$$\boldsymbol{\nabla} P^- = (\boldsymbol{\nabla} P^+)^* = \sum_{\ell m} \Xi^-_{\ell m}\boldsymbol{\nabla} Y^{-2}_{\ell m} = \sum_{\ell m} \Xi^-_{\ell m}\left(a^-_\ell Y^{-1}_{\ell m}\boldsymbol{e}^* - a^+_\ell Y^{-3}_{\ell m}\boldsymbol{e}\right) = \Xi^{-^+}\boldsymbol{e}^* - \Xi^{-^-}\boldsymbol{e} \,. \tag{18}$$

Note that $(\Xi^{+^+})^* = -\Xi^{-^-}$ and $(\Xi^{+^-})^* = -\Xi^{-^+}$.

2 Distortion of CMB anisotropies

In the following, we first define useful quantities to compute the distortion effect. A multipole factor is defined as

$$\gamma_{\ell_1\ell_2\ell_3} \equiv \sqrt{\frac{(2\ell_1+1)(2\ell_2+1)(2\ell_3+1)}{4\pi}} \,. \tag{19}$$

The convolution operator in full sky is defined as

$$\widetilde{\sum_{LM\ell'm'}}^{(\ell m)} \equiv \sum_{LM\ell'm'} (-1)^m \begin{pmatrix} \ell & L & \ell' \\ -m & M & m' \end{pmatrix} \,. \tag{20}$$

We introduce the following coefficients;

$$c_\phi = 1 \,, \tag{21}$$
$$c_\varpi = -\mathrm{i} \,, \tag{22}$$
$$c_\alpha = 1 \,, \tag{23}$$
$$c_\tau = 1 \,, \tag{24}$$

and

$$\zeta^+ = 1 \,, \tag{25}$$
$$\zeta^- = \mathrm{i} \,. \tag{26}$$

A parity symmetry indicator is given by

$$p^{x,\pm}_{\ell_1\ell_2\ell_3} \equiv c_x \frac{1 \pm c_x^2(-1)^{\ell_1+\ell_2+\ell_3}}{2} \,, \tag{27}$$

and simply

$$p^{\pm}_{\ell_1\ell_2\ell_3} \equiv \frac{1 \pm (-1)^{\ell_1+\ell_2+\ell_3}}{2} \,. \tag{28}$$

2.1 Lensing distortion

The lensing contributions in the position space become

$$\begin{aligned}
\delta^\phi\Theta &= \boldsymbol{\nabla}\phi\cdot\boldsymbol{\nabla}\Theta = -\phi^-\Theta^+ - \phi^+\Theta^- \,, \\
\delta^\varpi\Theta &= (\star\boldsymbol{\nabla})\varpi\cdot\boldsymbol{\nabla}\Theta = \mathrm{i}(\varpi^-\Theta^+ - \varpi^+\Theta^-) \,, \\
\delta^\phi P^\pm &= \boldsymbol{\nabla}\phi\cdot\boldsymbol{\nabla}P^\pm = -\phi^-\Xi^{\pm^+} - \phi^+\Xi^{\pm^-} \,, \\
\delta^\varpi P^\pm &= (\star\boldsymbol{\nabla})\varpi\cdot\boldsymbol{\nabla}P^\pm = \mathrm{i}(\varpi^-\Xi^{\pm^+} - \varpi^+\Xi^{\pm^-}) \,.
\end{aligned} \tag{29}$$

The harmonics transform of the lensing contributions in temperature is

$$\begin{aligned}
\delta\Theta_{\ell m} &= -c_x \int \mathrm{d}^2\hat{\boldsymbol{n}}\; Y^*_{\ell m}[x^-\Theta^+ + c_x^2 x^+\Theta^-] \\
&= -c_x \sum_{LM\ell'm'} x_{LM}\Theta_{\ell'm'} a^0_L a^0_{\ell'} \int \mathrm{d}^2\hat{\boldsymbol{n}}\; (-1)^m Y_{\ell,-m}[Y^{-1}_{LM}Y^1_{\ell'm'} + c_x^2 Y^1_{LM}Y^{-1}_{\ell'm'}] \\
&= -\sum_{LM\ell'm'} x_{LM}\Theta_{\ell'm'} 2a^0_L a^0_{\ell'} p^{x,+}_{\ell L\ell'}\gamma_{\ell L\ell'}(-1)^m \begin{pmatrix} \ell & L & \ell' \\ -m & M & m' \end{pmatrix} \begin{pmatrix} \ell & L & \ell' \\ 0 & 1 & -1 \end{pmatrix} \\
&= -\widetilde{\sum_{LM\ell'm'}}^{(\ell m)} x_{LM}\Theta_{\ell'm'} 2a^0_L a^0_{\ell'} p^{x,+}_{\ell L\ell'}\gamma_{\ell L\ell'} \begin{pmatrix} \ell & L & \ell' \\ 0 & 1 & -1 \end{pmatrix} \\
&= \widetilde{\sum_{LM\ell'm'}}^{(\ell m)} x_{LM}\Theta_{\ell'm'} W^{x,0}_{\ell L\ell'} \,.
\end{aligned} \tag{30}$$

Here we denote

$$W^{x,0}_{\ell_1\ell_2\ell_3} = -2a^0_{\ell_2}a^0_{\ell_3}p^{x,+}_{\ell_1\ell_2\ell_3}\gamma_{\ell_1\ell_2\ell_3}\begin{pmatrix}\ell_1 & \ell_2 & \ell_3\\ 0 & 1 & -1\end{pmatrix} . \tag{31}$$

Here, $(W^{\phi,0}_{\ell_1\ell_2\ell_3})^* = W^{\phi,0}_{\ell_1\ell_2\ell_3}$. Note that the above quantity is consistent with Ref. [7] and also $(W^{\varpi,0}_{\ell_1\ell_2\ell_3})^* = (-1)^{\ell_1+\ell_2+\ell_3}W^{\varpi,0}_{\ell_1\ell_2\ell_3}$.

On the other hand, the lensed anisotropies for polarization are given by

$$\begin{aligned}
\delta\Xi^{\pm}_{\ell m} &= -c_x\int \mathrm{d}^2\hat{\boldsymbol{n}}\ (Y^{\pm2}_{\ell m})^*[x^-\Xi^{\pm^+} + c_x^2x^+\Xi^{\pm^-}] \\
&= -c_x\sum_{LM\ell'm'}\phi_{LM}\Xi^{\pm}_{\ell'm'}a^0_L\int \mathrm{d}^2\hat{\boldsymbol{n}}\ (Y^{\pm2}_{\ell m})^*[a^+_{\ell'}Y^{\mp1}_{LM}Y^{\pm3}_{\ell'm'} + c_x^2a^-_{\ell'}Y^{\pm1}_{LM}Y^{\pm1}_{\ell'm'}] \\
&= -c_x\sum_{LM\ell'm'}(-1)^m\begin{pmatrix}\ell & L & \ell'\\ -m & M & m'\end{pmatrix}\phi_{LM}\Xi^{\pm}_{\ell'm'}\gamma_{\ell L\ell'}a^0_L\left[a^+_{\ell'}\begin{pmatrix}\ell & L & \ell'\\ \mp2 & \mp1 & \pm3\end{pmatrix} + c_x^2a^-_{\ell'}\begin{pmatrix}\ell & L & \ell'\\ \mp2 & \pm1 & \pm1\end{pmatrix}\right] \\
&= \widetilde{\sum_{LM\ell'm'}}^{(\ell m)}\phi_{LM}\Xi^{\pm}_{\ell'm'}W^{\phi,\pm2}_{\ell L\ell'} ,
\end{aligned} \tag{32}$$

with

$$\begin{aligned}
W^{x,2}_{\ell_1\ell_2\ell_3} &= (-1)^{\ell_1+\ell_2+\ell_3}W^{x,-2}_{\ell_1\ell_2\ell_3} \\
&= -c_x\gamma_{\ell_1\ell_2\ell_3}a^0_{\ell_2}\left[a^+_{\ell_3}\begin{pmatrix}\ell_1 & \ell_2 & \ell_3\\ -2 & -1 & 3\end{pmatrix} + c_x^2a^-_{\ell_3}\begin{pmatrix}\ell_1 & \ell_2 & \ell_3\\ -2 & 1 & 1\end{pmatrix}\right] .
\end{aligned} \tag{33}$$

2.2 Rotation distortion

The E and B modes after the rotation are given by

$$\begin{aligned}
\delta\Xi^{\pm}_{\ell m} &= \pm2\mathrm{i}\int \mathrm{d}^2\hat{\boldsymbol{n}}\ (Y^{\pm2}_{\ell m})^*\alpha P^{\pm} \\
&= \pm2\mathrm{i}\sum_{LM\ell'm'}\alpha_{LM}\Xi^{\pm}_{\ell'm'}\int \mathrm{d}^2\hat{\boldsymbol{n}}\ (Y^{\pm2}_{\ell m})^*Y_{LM}Y^{\pm2}_{\ell'm'} \\
&= \pm2\mathrm{i}\sum_{LM\ell'm'}(-1)^m\begin{pmatrix}\ell & L & \ell'\\ -m & M & m'\end{pmatrix}\alpha_{LM}\Xi^{\pm}_{\ell'm'}\gamma_{\ell L\ell'}\begin{pmatrix}\ell & L & \ell'\\ \mp2 & 0 & \pm2\end{pmatrix} \\
&= \widetilde{\sum_{LM\ell'm'}}^{(\ell m)}\alpha_{LM}\Xi^{\pm}_{\ell'm'}W^{\alpha,\pm2}_{\ell L\ell'} ,
\end{aligned} \tag{34}$$

with

$$W^{\alpha,\pm2}_{\ell_1\ell_2\ell_3} = \pm2\mathrm{i}\gamma_{\ell_1\ell_2\ell_3}\begin{pmatrix}\ell_1 & \ell_2 & \ell_3\\ \mp2 & 0 & \pm2\end{pmatrix} . \tag{35}$$

2.3 Amplitude distortion

The harmonics transform of $\tau(\hat{\boldsymbol{n}})\Theta(\hat{\boldsymbol{n}})$ is

$$\begin{aligned}
\delta\Theta_{\ell m} &= \int \mathrm{d}^2\hat{\boldsymbol{n}}\ Y^*_{\ell m}\tau(\hat{\boldsymbol{n}})\Theta(\hat{\boldsymbol{n}}) \\
&= \sum_{LM\ell'm'}\tau_{LM}\Theta_{\ell'm'}\int \mathrm{d}^2\hat{\boldsymbol{n}}\ Y^*_{\ell m}Y_{LM}Y_{\ell'm'} \\
&= \sum_{LM\ell'm'}\tau_{LM}\Theta_{\ell'm'}p^+_{\ell L\ell'}\gamma_{\ell L\ell'}(-1)^m\begin{pmatrix}\ell & L & \ell'\\ -m & M & m'\end{pmatrix}\begin{pmatrix}\ell & L & \ell'\\ 0 & 0 & 0\end{pmatrix} \\
&= \widetilde{\sum_{LM\ell'm'}}^{(\ell m)}\tau_{LM}\Theta_{\ell'm'}W^{\tau,0}_{\ell L\ell'} ,
\end{aligned} \tag{36}$$

where

$$W^{\tau,0}_{\ell L\ell'} = p^{+}_{\ell L\ell'}\gamma_{\ell L\ell'}\begin{pmatrix}\ell & L & \ell' \\ 0 & 0 & 0\end{pmatrix} = \gamma_{\ell L\ell'}\begin{pmatrix}\ell & L & \ell' \\ 0 & 0 & 0\end{pmatrix} . \tag{37}$$

The polarization anisotropies with the amplitude distortion are given by

$$\begin{aligned}\delta\Xi^{\pm}_{\ell m} &= \int \mathrm{d}^2\hat{\boldsymbol{n}}\ (Y^{\pm 2}_{\ell m})^* \tau P^{\pm} \\ &= \widetilde{\sum_{LM\ell'm'}}^{(\ell m)} \tau_{LM}\Xi^{\pm}_{\ell'm'}W^{\tau,\pm 2}_{\ell L\ell'} ,\end{aligned} \tag{38}$$

with

$$W^{\tau,\pm 2}_{\ell_1\ell_2\ell_3} = \gamma_{\ell_1\ell_2\ell_3}\begin{pmatrix}\ell_1 & \ell_2 & \ell_3 \\ \mp 2 & 0 & \pm 2\end{pmatrix} . \tag{39}$$

2.4 Translate into E/B

Now we consider the distorted E/B modes separately. In general, if the distortion is given by

$$\delta\Xi^{\pm}_{\ell m} = \widetilde{\sum_{LM\ell'm'}}^{(\ell m)} x_{LM}\Xi^{\pm}_{\ell'm'}W^{x,\pm 2}_{\ell L\ell'} , \tag{40}$$

we obtain

$$\delta E_{\ell m} = \widetilde{\sum_{LM\ell'm'}}^{(\ell m)} x_{LM}\left(E_{\ell'm'}W^{x,+}_{\ell L\ell'} + B_{\ell'm'}W^{x,-}_{\ell L\ell'}\right) , \tag{41}$$

$$\delta B_{\ell m} = \widetilde{\sum_{LM\ell'm'}}^{(\ell m)} x_{LM}\left(-E_{\ell'm'}W^{x,-}_{\ell L\ell'} + B_{\ell'm'}W^{x,+}_{\ell L\ell'}\right) , \tag{42}$$

where we define

$$W^{x,\pm}_{\ell L\ell'} \equiv \zeta^{\pm}\frac{W^{x,+2}_{\ell L\ell'} \pm W^{x,-2}_{\ell L\ell'}}{2} . \tag{43}$$

For lensing, the functional form of W is given by

$$\begin{aligned}W^{x,\pm}_{\ell_1\ell_2\ell_3} &= \zeta^{\pm}\frac{1 \pm c_x^2(-1)^{\ell_1+\ell_2+\ell_3}}{2}W^{x,2}_{\ell_1\ell_2\ell_3} \\ &= -\zeta^{\pm}p^{x,\pm}_{\ell_1\ell_2\ell_3}\gamma_{\ell_1\ell_2\ell_3}a^0_{\ell_2}\left[a^+_{\ell_3}\begin{pmatrix}\ell_1 & \ell_2 & \ell_3 \\ -2 & -1 & 3\end{pmatrix} + c_x^2 a^-_{\ell_3}\begin{pmatrix}\ell_1 & \ell_2 & \ell_3 \\ -2 & 1 & 1\end{pmatrix}\right] ,\end{aligned} \tag{44}$$

For polarization rotation, we obtain

$$W^{\alpha,\pm}_{\ell_1\ell_2\ell_3} = \pm 2\zeta^{\mp}p^{\mp}_{\ell_1\ell_2\ell_3}\gamma_{\ell_1\ell_2\ell_3}\begin{pmatrix}\ell_1 & \ell_2 & \ell_3 \\ -2 & 0 & 2\end{pmatrix} . \tag{45}$$

This is consistent with [8] in the absence of B-modes. For amplitude modulations, we find

$$W^{\tau,\pm}_{\ell_1\ell_2\ell_3} = \zeta^{\pm}p^{\pm}_{\ell_1\ell_2\ell_3}\gamma_{\ell_1\ell_2\ell_3}\begin{pmatrix}\ell_1 & \ell_2 & \ell_3 \\ -2 & 0 & 2\end{pmatrix} . \tag{46}$$

2.5 Summary

The above all distortions are described in the following form:

$$\delta\Theta_{\ell m} = \widetilde{\sum_{LM\ell' m'}}^{(\ell m)} x_{LM}\Theta_{\ell' m'} W^{x,0}_{\ell L \ell'} \,, \tag{47}$$

$$\delta E_{\ell m} = \widetilde{\sum_{LM\ell' m'}}^{(\ell m)} x_{LM}\left(E_{\ell' m'} W^{x,+}_{\ell L \ell'} + B_{\ell' m'} W^{x,-}_{\ell L \ell'}\right) \,, \tag{48}$$

$$\delta B_{\ell m} = \widetilde{\sum_{LM\ell' m'}}^{(\ell m)} x_{LM}\left(-E_{\ell' m'} W^{x,-}_{\ell L \ell'} + B_{\ell' m'} W^{x,+}_{\ell L \ell'}\right) \tag{49}$$

where x is a distortion field.

The property of W is also important. If x is parity even, $W^{x,+}_{\ell L \ell'}$ and $W^{x,-}_{\ell L \ell'}$ are non-zero only when $\ell + L + \ell'$ is even and odd, respectively. If x is parity odd, $W^{x,-}_{\ell L \ell'}$ and $W^{x,+}_{\ell L \ell'}$ are non-zero only when $\ell + L + \ell'$ is even and odd, respectively. $W^{x,0}$ is the same as $W^{x,+}$. Note that

$$\begin{aligned} W^{\alpha,+}_{\ell_1\ell_2\ell_3} &= 2W^{\tau,-}_{\ell_1\ell_2\ell_3} \,, \\ W^{\alpha,-}_{\ell_1\ell_2\ell_3} &= -2W^{\tau,+}_{\ell_1\ell_2\ell_3} \,, \end{aligned} \tag{50}$$

and

$$W^{\alpha,\pm}_{\ell_3\ell_2\ell_1} = W^{\alpha,\pm}_{\ell_1\ell_2\ell_3} \,, \tag{51}$$

$$W^{\tau,s}_{\ell_3\ell_2\ell_1} = W^{\tau,s}_{\ell_1\ell_2\ell_3} \,, \tag{52}$$

where $s = 0, \pm$.

3 Quadratic estimator

3.1 Distortion induced anisotropies

The distortion fields x described above induce the off-diagonal elements of the covariance ($\ell \neq \ell'$ or $m \neq m'$), [9, 10]

$$\langle \widetilde{X}_{\ell m} \widetilde{Y}_{\ell' m'} \rangle_{\rm CMB} = \sum_{LM} \begin{pmatrix} \ell & \ell' & L \\ m & m' & M \end{pmatrix} f^{x,({\rm XY})}_{\ell L \ell'} x^*_{LM} \,, \tag{53}$$

where $\langle \cdots \rangle_{\rm CMB}$ denotes the ensemble average over the primary CMB anisotropies with a fixed realization of the distortion fields. We ignore the higher-order terms of the distortion fields.

The functional form of the weight functions f are discussed in Sec. 3.3.

3.2 Quadratic estimator

With a quadratic combination of observed CMB anisotropies, $\widehat{X}$ and $\widehat{Y}$, the general quadratic estimators are formed as

$$[\widehat{x}^{\rm XY}_{LM}]^* = A^{x,({\rm XY})}_L \sum_{\ell \ell' m m'} \begin{pmatrix} \ell & \ell' & L \\ m & m' & M \end{pmatrix} g^{x,({\rm XY})}_{\ell L \ell'} \widehat{X}_{\ell m} \widehat{Y}_{\ell' m'} \,. \tag{54}$$

Here we define

$$g^{x,({\rm XY})}_{\ell L \ell'} = \frac{[f^{x,({\rm XY})}_{\ell L \ell'}]^*}{\Delta^{\rm XY} \widehat{C}^{\rm XX}_{\ell} \widehat{C}^{\rm YY}_{\ell'}} \tag{55}$$

$$A^{x,({\rm XY})}_L = \frac{1}{2L+1} \sum_{\ell \ell'} f^{x,({\rm XY})}_{\ell L \ell'} g^{x,({\rm XY})}_{\ell L \ell'} \,, \tag{56}$$

where $\Delta^{\rm XX} = 2$, $\Delta^{\rm EB} = \Delta^{\rm TB} = 1$, and $\widehat{C}^{\rm XX}_\ell$ ($\widehat{C}^{\rm YY}_\ell$) is the observed power spectrum.

3.3 Weight Function: Derivations

3.3.1 $\Theta\Theta$

Let us first consider the temperature case. There are two contributions to the temperature quadratic estimator, and the one is given as

$$\begin{aligned} \langle (\delta\Theta_{\ell m}) \Theta_{\ell' m'} \rangle &= \sum_{LM\ell'' m''} (-1)^m \begin{pmatrix} \ell & L & \ell'' \\ -m & M & m'' \end{pmatrix} x_{LM} W^{x,0}_{\ell L \ell''} \langle \Theta_{\ell'' m''} \Theta_{\ell' m'} \rangle \\ &= \sum_{LM\ell'' m''} (-1)^m \begin{pmatrix} \ell & L & \ell'' \\ -m & M & m'' \end{pmatrix} x_{LM} W^{x,0}_{\ell L \ell'} \delta_{\ell'' \ell'} \delta_{m'', -m'} (-1)^{m'} C^{\Theta\Theta}_{\ell'} \\ &= \sum_{LM} (-1)^{m+m'} \begin{pmatrix} \ell & L & \ell' \\ -m & M & -m' \end{pmatrix} x_{LM} W^{x,0}_{\ell L \ell'} C^{\Theta\Theta}_{\ell'} \\ &= \sum_{LM} \begin{pmatrix} \ell & \ell' & L \\ m & m' & M \end{pmatrix} x^*_{LM} W^{x,0}_{\ell L \ell'} C^{\Theta\Theta}_{\ell'} \,. \end{aligned} \tag{57}$$

In the above, from the third to the last equation, we use $m + m' = -M$, change the sign of $m, m'M$, $M \to -M$, and further change the order of column in the Wigner 3j. The other term is obtained by $(\ell'', m'') \leftrightarrow (\ell, m)$ and is given by

$$\langle \Theta_{\ell m} \delta\Theta_{\ell' m'} \rangle = \sum_{LM} (-1)^{\ell+\ell'+L} \begin{pmatrix} \ell & \ell' & L \\ m & m' & M \end{pmatrix} x^*_{LM} W^{x,0}_{\ell' L \ell} C^{\Theta\Theta}_{\ell} \,. \tag{58}$$

The sum of the above two equations yield

$$f^{x,(\Theta\Theta)}_{\ell L\ell'} = W^{x,0}_{\ell L\ell'} C^{\Theta\Theta}_{\ell'} + (-1)^{\ell+\ell'+L} W^{x,0}_{\ell' L\ell} C^{\Theta\Theta}_{\ell} \,. \tag{59}$$

The sign $(-1)^{\ell+\ell''+L}$ depends on the parity of W; $(-1)^{\ell+\ell''+L} = 1$ for the even parity fields (e.g. $x = \phi, \tau$) and -1 for the odd parity fields (e.g. $x = \varpi, \alpha$).

3.3.2 EB

In the EB estimator, the two contributions are given as

$$\begin{aligned}
\langle E_{\ell m}\delta B_{\ell' m'}\rangle &= \sum_{LM\ell'' m''} (-1)^{m'} \begin{pmatrix} \ell' & L & \ell'' \\ -m' & M & m'' \end{pmatrix} x_{LM}[-\langle E_{\ell m}E_{\ell'' m''}\rangle W^{x,-}_{\ell' L\ell''} + \langle E_{\ell m}B_{\ell'' m''}\rangle W^{x,+}_{\ell' L\ell''}] \\
&= \sum_{LM} (-1)^{m+m'} \begin{pmatrix} \ell' & L & \ell \\ -m' & M & -m \end{pmatrix} x_{LM}[-C^{\rm EE}_{\ell} W^{x,-}_{\ell' L\ell} + C^{\rm EB}_{\ell} W^{x,+}_{\ell' L\ell}] \\
&= \sum_{LM} (-1)^{\ell+L+\ell'+1} \begin{pmatrix} \ell' & L & \ell \\ m' & M & m \end{pmatrix} x^*_{LM}[C^{\rm EE}_{\ell} W^{x,-}_{\ell' L\ell} - C^{\rm EB}_{\ell} W^{x,+}_{\ell' L\ell}] \\
&= \sum_{LM} (-1)^{\ell+L+\ell'+1} \begin{pmatrix} \ell & \ell' & L \\ m & m' & M \end{pmatrix} x^*_{LM}[C^{\rm EE}_{\ell} W^{x,-}_{\ell' L\ell} - C^{\rm EB}_{\ell} W^{x,+}_{\ell' L\ell}] \,, \,,
\end{aligned} \tag{60}$$

and

$$\begin{aligned}
\langle (\delta E_{\ell m}) B_{\ell' m'}\rangle &= \sum_{LM\ell'' m''} (-1)^{m} \begin{pmatrix} \ell & L & \ell'' \\ -m & M & m'' \end{pmatrix} x_{LM}[\langle B_{\ell' m'}E_{\ell'' m''}\rangle W^{x,+}_{\ell L\ell''} + \langle B_{\ell' m'}B_{\ell'' m''}\rangle W^{x,-}_{\ell L\ell''}] \\
&= \sum_{LM} (-1)^{m+m'} \begin{pmatrix} \ell & L & \ell' \\ -m & M & -m' \end{pmatrix} x_{LM}[C^{\rm EB}_{\ell'} W^{x,+}_{\ell L\ell'} + C^{\rm BB}_{\ell'} W^{x,-}_{\ell L\ell'}] \\
&= \sum_{LM} \begin{pmatrix} \ell & \ell' & L \\ m & m' & M \end{pmatrix} x^*_{LM}[C^{\rm EB}_{\ell'} W^{x,+}_{\ell L\ell'} + C^{\rm BB}_{\ell'} W^{x,-}_{\ell L\ell'}] \,.
\end{aligned} \tag{61}$$

Combining the above two terms, we find

$$f^{x,(EB)}_{\ell L\ell'} = C^{\rm EB}_{\ell'} W^{x,+}_{\ell L\ell'} + C^{\rm BB}_{\ell'} W^{x,-}_{\ell L\ell'} + (-1)^{\ell+L+\ell'+1}[C^{\rm EE}_{\ell} W^{x,-}_{\ell' L\ell} - C^{\rm EB}_{\ell} W^{x,+}_{\ell' L\ell}] \,, \tag{62}$$

If we decompose the terms into the following two parts,

$$\begin{aligned}
f^{x,(EB),+}_{\ell L\ell'} &= C^{\rm BB}_{\ell'} W^{x,-}_{\ell L\ell'} - (-1)^{\ell+L+\ell'} C^{\rm EE}_{\ell} W^{x,-}_{\ell' L\ell} \\
f^{x,(EB),-}_{\ell L\ell'} &= C^{\rm EB}_{\ell'} W^{x,+}_{\ell L\ell'} + (-1)^{\ell+L+\ell'} C^{\rm EB}_{\ell} W^{x,+}_{\ell' L\ell} \,,
\end{aligned} \tag{63}$$

the above two parts are orthogonal each other. This indicates that, if $C^{\rm EB}_{\ell}$ is non-zero due to the global rotation, even parity fields (lensing, window) leak into the odd parity estimator (rotation, curl mode) and introduce a mean-field;

$$\langle \widehat{\alpha}_{LM}\rangle = \alpha_{LM} + A^{\alpha,EB}_L \sum_{x=\phi,\tau,\dots} x_{LM} \frac{1}{2L+1} \sum_{\ell\ell'} g^{\alpha,EB}_{\ell L\ell'} f^{x,EB,{\rm even}}_{\ell L\ell'} \,. \tag{64}$$

3.4 Additive distortions

Point-source (or inhomogeneous noise) can also produce mode couplings. Assuming that the fields are uncorrelated between pixels, the additive anisotropies in the temperature quadratic estimator are given by

$$\begin{aligned}\langle n_{\ell m}n_{\ell' m'}\rangle &= \int \mathrm{d}^2\hat{\boldsymbol{n}} \int \mathrm{d}^2\hat{\boldsymbol{n}}'\ Y^*_{\ell m}(\hat{\boldsymbol{n}})Y^*_{\ell' m'}(\hat{\boldsymbol{n}})\langle n(\hat{\boldsymbol{n}})n(\hat{\boldsymbol{n}}')\rangle \\ &= \int \mathrm{d}^2\hat{\boldsymbol{n}}\ Y^*_{\ell m}(\hat{\boldsymbol{n}})Y^*_{\ell' m'}(\hat{\boldsymbol{n}})\langle s(\hat{\boldsymbol{n}})\rangle \\ &= \int \mathrm{d}^2\hat{\boldsymbol{n}}\ Y^*_{\ell m}(\hat{\boldsymbol{n}})Y^*_{\ell' m'}(\hat{\boldsymbol{n}})\sum_{LM} s_{LM}Y_{LM}(\hat{\boldsymbol{n}}) \\ &= \sum_{LM} s_{LM}\int \mathrm{d}^2\hat{\boldsymbol{n}}\ (-1)^{m+m'}Y_{\ell,-m}(\hat{\boldsymbol{n}})Y_{\ell',-m'}(\hat{\boldsymbol{n}})Y_{LM}(\hat{\boldsymbol{n}}) \\ &= \sum_{LM}\begin{pmatrix}\ell & \ell' & L \\ m & m' & M\end{pmatrix} f^{s,(\Theta\Theta)}_{\ell L\ell'}s^*_{LM}\,, \end{aligned} \tag{65}$$

where

$$f^{s,(\Theta\Theta)}_{\ell L\ell'} = \gamma_{\ell L\ell'}\begin{pmatrix}\ell & L & \ell' \\ 0 & 0 & 0\end{pmatrix} = W^{\tau,0}_{\ell L\ell'}\,. \tag{66}$$

Thus, we obtain the following relation:

$$f^{s,(\Theta\Theta)}_{\ell L\ell'} = f^{\tau,(\Theta\Theta)}_{\ell L\ell'}|_{C^{\Theta\Theta}_\ell=1/2}\,. \tag{67}$$

3.5 Summary

The weight functions are given as

$$f^{x,(\Theta\Theta)}_{\ell L\ell'} = W^{x,0}_{\ell L\ell'}C^{\Theta\Theta}_{\ell'} + p_x W^{x,0}_{\ell' L\ell}C^{\Theta\Theta}_{\ell}\,, \tag{68}$$
$$f^{x,(\Theta E)}_{\ell L\ell'} = W^{x,0}_{\ell L\ell'}C^{\Theta \mathrm{E}}_{\ell'} + p_x W^{x,+}_{\ell' L\ell}C^{\Theta \mathrm{E}}_{\ell}\,, \tag{69}$$
$$f^{x,(\Theta B)}_{\ell L\ell'} = p_x W^{x,-}_{\ell' L\ell}C^{\Theta \mathrm{E}}_{\ell}\,, \tag{70}$$
$$f^{x,(EE)}_{\ell L\ell'} = W^{x,+}_{\ell L\ell'}C^{\mathrm{EE}}_{\ell'} + p_x W^{x,+}_{\ell' L\ell}C^{\mathrm{EE}}_{\ell}\,, \tag{71}$$
$$f^{x,(EB)}_{\ell L\ell'} = W^{x,-}_{\ell L\ell'}C^{\mathrm{BB}}_{\ell'} + p_x W^{x,-}_{\ell' L\ell}C^{\mathrm{EE}}_{\ell}\,, \tag{72}$$
$$f^{x,(BB)}_{\ell L\ell'} = W^{x,+}_{\ell L\ell'}C^{\mathrm{BB}}_{\ell'} + p_x W^{x,+}_{\ell' L\ell}C^{\mathrm{BB}}_{\ell}\,. \tag{73}$$

Here, the parity index is $p_\phi = p_\epsilon = 1$ and $p_\varpi = p_\alpha = -1$. Strictly speaking, p_x should be $(-1)^{\ell'+L+\ell}$. However, W is only non-zero when $\ell'+L+\ell$ is even, and vice versa. The parity even quantities are $x=\phi$ and ϵ. The odd parity quantities are $x=\varpi$ and α. Note that the above weight functions are consistent with Ref. [7] ($W^{x,-}_{\ell L\ell'} = -_\ominus S^x_{\ell L\ell'}$) for the lensing case.

In addition, the weight functions due to the presence of ΘB and EB are given by

$$f^{x,(\Theta E)}_{\ell L\ell'} = \,, \tag{74}$$
$$f^{x,(\Theta B)}_{\ell L\ell'} = \,, \tag{75}$$
$$f^{x,(EE)}_{\ell L\ell'} = \,, \tag{76}$$
$$f^{x,(EB)}_{\ell L\ell'} = W^{x,+}_{\ell L\ell'}C^{\mathrm{EB}}_{\ell'} + p_x W^{x,+}_{\ell' L\ell}C^{\mathrm{EB}}_{\ell}\,, \tag{77}$$
$$f^{x,(BB)}_{\ell L\ell'} = \ . \tag{78}$$

4 Computing quadratic estimator

4.1 Spherical Harmonics

The polarization vectors satisfy, $\boldsymbol{e}\cdot\boldsymbol{e}^*=1$, and, $\boldsymbol{e}\cdot\boldsymbol{e}=\boldsymbol{e}^*\cdot\boldsymbol{e}^*=0$. We obtain

$$-\boldsymbol{\nabla}Y^s_{\ell m}=\sqrt{\frac{(\ell-s)(\ell+s+1)}{2}}Y^{s+1}_{\ell m}\boldsymbol{e}^*-\sqrt{\frac{(\ell+s)(\ell-s+1)}{2}}Y^{s-1}_{\ell m}\boldsymbol{e}\,. \tag{79}$$

The complex conjugate is $(Y^s_{\ell m})^*=(-1)^{s+m}Y^{-s}_{\ell,-m}$. In particular, for $s=0$,

$$-\boldsymbol{\nabla}Y^*_{\ell m}=\sqrt{\frac{\ell(\ell+1)}{2}}\left((Y^1_{\ell m})^*\boldsymbol{e}-(Y^{-1}_{\ell m})^*\boldsymbol{e}^*\right), \tag{80}$$

and, for $s=-2$,

$$\begin{aligned}-\boldsymbol{\nabla}Y^{-2}_{\ell m}&=\sqrt{\frac{(\ell+2)(\ell-1)}{2}}Y^{-1}_{\ell m}\boldsymbol{e}^*-\sqrt{\frac{(\ell-2)(\ell+3)}{2}}Y^{-3}_{\ell m}\boldsymbol{e}\,,\\ -\boldsymbol{\nabla}(Y^{-2}_{\ell m})^*&=\sqrt{\frac{(\ell+2)(\ell-1)}{2}}(Y^{-1}_{\ell m})^*\boldsymbol{e}-\sqrt{\frac{(\ell-2)(\ell+3)}{2}}(Y^{-3}_{\ell m})^*\boldsymbol{e}^*\,.\end{aligned} \tag{81}$$

4.2 Healpix

`Healpix` is a useful public package for fullsky analysis [11]. Here, we consider the `Healpix` spin-s harmonic transform of a map $S(\hat{\boldsymbol{n}})=S^+(\hat{\boldsymbol{n}})+\mathrm{i}S^-(\hat{\boldsymbol{n}})$ where $S^\pm$ is real and $s\geq 0$. The harmonic coefficient is given by

$$S^++\mathrm{i}S^-=\sum_{\ell m}a^s_{\ell m}Y^s_{\ell m}\,. \tag{82}$$

Note that $a^{-s}_{\ell m}$ is defined as

$$S^+-\mathrm{i}S^-=\sum_{\ell m}a^{-s}_{\ell m}Y^{-s}_{\ell m}\,. \tag{83}$$

Then we obtain $(a^s_{\ell m})^*=(-1)^{m+s}a^{-s}_{\ell,-m}$. The subroutine `map2alm_spin` transform $S^\pm$ to $a^{s,\pm}_{\ell m}$ where

$$a^{s,+}_{\ell m}=-\frac{a^s_{\ell m}+(-1)^sa^{-s}_{\ell m}}{2} \tag{84}$$

$$a^{s,-}_{\ell m}=-\frac{a^s_{\ell m}-(-1)^sa^{-s}_{\ell m}}{2\mathrm{i}}\,, \tag{85}$$

are the rotational invariant coefficients with parity even and odd, respectively. Since $(a^s_{\ell m})^*=(-1)^{m+s}a^{-s}_{\ell,-m}$, the above coefficients satisfy

$$(a^{s,\pm}_{\ell m})^*=(-1)^ma^{s,\pm}_{\ell,-m}\,. \tag{86}$$

On the other hand, `alm2map_spin` transform $a^{s,\pm}_{\ell m}$ to $S^\pm$, but $a^{s,\pm}_{\ell m}$ should satisfy the above condition. Note that, with $S\equiv S^++\mathrm{i}S^-$, we find

$$a^{s,+}_{\ell m}=-\frac{1}{2}\int\mathrm{d}\hat{\boldsymbol{n}}\,\left[(Y^s_{\ell m})^*S+(-1)^s(Y^{-s}_{\ell m})^*S^*\right], \tag{87}$$

$$a^{s,-}_{\ell m}=-\frac{1}{2\mathrm{i}}\int\mathrm{d}\hat{\boldsymbol{n}}\,\left[(Y^s_{\ell m})^*S-(-1)^s(Y^{-s}_{\ell m})^*S^*\right]. \tag{88}$$

Let us consider the case we want to transform $a_{\ell m}$ with a spin-s spherical harmonics using `alm2map_spin`. The outputs, $S^\pm$, are given by:

$$S^++\mathrm{i}S^-=\sum_{\ell m}a_{\ell m}Y^s_{\ell m}\,. \tag{89}$$

The complex conjugate of the above quantity becomes

$$S^+ - \mathrm{i}S^- = (-1)^s \sum_{\ell m} a_{\ell m} Y^{-s}_{\ell m} \,. \tag{90}$$

The inputs of `alm2map_spin` become

$$a^{s,+}_{\ell m} = -a_{\ell m} \,, \tag{91}$$

$$a^{s,-}_{\ell m} = 0 \,. \tag{92}$$

4.3 Lensing

In fullsky, the quadratic estimator of the gradient and curl modes are given by [10, 7]:

$$\widehat{\phi}^{(\alpha)}_{\ell m} = A^{\phi,(\alpha)}_\ell \int \mathrm{d}^2\hat{\boldsymbol{n}}\ [\boldsymbol{\nabla} Y^*_{\ell m}(\hat{\boldsymbol{n}})] \cdot \boldsymbol{v}^{(\alpha)}(\hat{\boldsymbol{n}}) \,, \tag{93}$$

$$\widehat{\varpi}^{(\alpha)}_{\ell m} = A^{\varpi,(\alpha)}_\ell \int \mathrm{d}^2\hat{\boldsymbol{n}}\ [(\star\boldsymbol{\nabla}) Y^*_{\ell m}(\hat{\boldsymbol{n}})] \cdot \boldsymbol{v}^{(\alpha)}(\hat{\boldsymbol{n}}) \,, \tag{94}$$

where $N^{x,(\alpha)}_\ell$ is the normalization of the quadratic estimator and we define

$$\boldsymbol{v}^{\Theta\Theta}(\hat{\boldsymbol{n}}) = A^0_\Theta \boldsymbol{\nabla} A^0_{\Theta\Theta} \,, \tag{95}$$

$$\boldsymbol{v}^{\Theta E}(\hat{\boldsymbol{n}}) = \Re(A^2_E \boldsymbol{\nabla} A^{-2}_{\Theta E}) + A_\Theta \boldsymbol{\nabla} A_{E\Theta} \,, \tag{96}$$

$$\boldsymbol{v}^{\Theta B}(\hat{\boldsymbol{n}}) = \Re(A^2_{iB} \boldsymbol{\nabla} A^{-2}_{\Theta E}) \,, \tag{97}$$

$$\boldsymbol{v}^{EE}(\hat{\boldsymbol{n}}) = \Re(A^2_E \boldsymbol{\nabla} A^{-2}_{EE}) \,, \tag{98}$$

$$\boldsymbol{v}^{EB}(\hat{\boldsymbol{n}}) = \Re(A^2_{iB} \boldsymbol{\nabla} A^{-2}_{EE}) + \Re(A^2_E \boldsymbol{\nabla} A^{-2}_{iBiB}) \,, \tag{99}$$

$$\boldsymbol{v}^{BB}(\hat{\boldsymbol{n}}) = \Re(A^2_{iB} \boldsymbol{\nabla} A^{-2}_{iBiB}) \,. \tag{100}$$

Here we define

$$A^s_X(\hat{\boldsymbol{n}}) = \sum_{\ell m} \overline{X}_{\ell m} Y^s_{\ell m}(\hat{\boldsymbol{n}}) \,, \tag{101}$$

$$A^s_{XY}(\hat{\boldsymbol{n}}) = \sum_{\ell m} C^{XY}_\ell \overline{X}_{\ell m} Y^s_{\ell m}(\hat{\boldsymbol{n}}) \,, \tag{102}$$

with $\overline{X}_{\ell m} = \widehat{X}_{\ell m}/\widehat{C}^{XY}_\ell$ being the inverse-variance filtered multipoles by the lensed angular power spectra including instrumental noise. The quantity $\boldsymbol{v}^{\Theta E}$ gives the nearly optimal estimator [10].

In general, we can decompose the 2D vector, $\boldsymbol{v}^\alpha$, into

$$\boldsymbol{v}^\alpha = \frac{v^\alpha_- \boldsymbol{e} + v^\alpha_+ \boldsymbol{e}^*}{\sqrt{2}} \,. \tag{103}$$

Since $\boldsymbol{v}^\alpha$ is real, we find $(v^\alpha_-)^* = v^\alpha_+ \equiv v^\alpha$. Then we obtain

$$\begin{aligned}\widehat{\phi}^{(\alpha)}_{\ell m} &= -\frac{\sqrt{\ell(\ell+1)}A^{\phi,(\alpha)}_\ell}{2} \int \mathrm{d}^2\hat{\boldsymbol{n}}\ [(Y^1_{\ell m})^* v^\alpha - (Y^{-1}_{\ell m})^* (v^\alpha)^*] \\ &= \sqrt{\ell(\ell+1)}A^{\phi,(\alpha)}_\ell v^{1,+}_{\ell m} \,,\end{aligned} \tag{104}$$

$$\begin{aligned}\widehat{\varpi}^{(\alpha)}_{\ell m} &= \mathrm{i}\frac{\sqrt{\ell(\ell+1)}A^{\varpi,(\alpha)}_\ell}{2} \int \mathrm{d}^2\hat{\boldsymbol{n}}\ [(Y^1_{\ell m})^* v^\alpha + (Y^{-1}_{\ell m})^* (v^\alpha)^*] \\ &= \sqrt{\ell(\ell+1)}A^{\varpi,(\alpha)}_\ell v^{1,-}_{\ell m} \,,\end{aligned} \tag{105}$$

where $v^{1,\pm}_{\ell m}$ are the outputs of `map2alm_spin` by inputting, $S = v^\alpha$, with $s = 1$. In the following subsections, we show v^α for each quadratic estimator.

4.3.1 $\Theta\Theta$

The estimator for $\Theta\Theta$ contains

$$\begin{aligned}
\boldsymbol{v}^{\Theta\Theta} &= \sum_{\ell m} \overline{\Theta}_{\ell m} Y_{\ell m} \sum_{\ell m} C_\ell^{\Theta\Theta} \overline{\Theta}_{\ell m} \boldsymbol{\nabla} Y_{\ell m} \\
&= \sum_{\ell m} \overline{\Theta}_{\ell m} Y_{\ell m} \sum_{\ell m} C_\ell^{\Theta\Theta} \overline{\Theta}_{\ell m} \sqrt{\frac{\ell(\ell+1)}{2}} \big(- Y^1_{\ell m} \boldsymbol{e}^* + Y^{-1}_{\ell m} \boldsymbol{e} \big) \\
&= \frac{1}{\sqrt{2}} \Theta^0 \big[(\Theta^+ + \mathrm{i}\Theta^-) \boldsymbol{e}^* + (\Theta^+ - \mathrm{i}\Theta^-) \boldsymbol{e} \big] \,, \qquad (106)
\end{aligned}$$

where we define

$$\Theta^0 = \sum_{\ell m} \overline{\Theta}_{\ell m} Y_{\ell m} \,, \qquad (107)$$

$$\Theta^+ + \mathrm{i}\Theta^- = - \sum_{\ell m} \overline{\Theta}_{\ell m} C_\ell^{\Theta\Theta} \sqrt{\ell(\ell+1)} Y^1_{\ell m} \,. \qquad (108)$$

We obtain

$$v^{\Theta\Theta} = \Theta^0 (\Theta^+ + \mathrm{i}\Theta^-) \,. \qquad (109)$$

4.3.2 ΘE

The ΘE estimator contains;

$$\begin{aligned}
\boldsymbol{v}^{\Theta E} &= \Re \left[\sum_{\ell m} \overline{E}_{\ell m} Y^{+2}_{\ell m} \sum_{\ell m} C_\ell^{\Theta E} \overline{\Theta}_{\ell m} \left(- \sqrt{\frac{(\ell+2)(\ell-1)}{2}} Y^{-1}_{\ell m} \boldsymbol{e}^* + \sqrt{\frac{(\ell-2)(\ell+3)}{2}} Y^{-3}_{\ell m} \boldsymbol{e} \right) \right] \\
&\quad + \sum_{\ell m} \overline{\Theta}_{\ell m} Y_{\ell m} \sum_{\ell m} C_\ell^{\Theta E} \overline{E}_{\ell m} \sqrt{\frac{\ell(\ell+1)}{2}} \left(- Y^1_{\ell m} \boldsymbol{e}^* + Y^{-1}_{\ell m} \boldsymbol{e} \right) \\
&= \frac{1}{2\sqrt{2}} \left[(Q^E + \mathrm{i}U^E)[-(\Theta_1^+ - \mathrm{i}\Theta_1^-) \boldsymbol{e}^* + (\Theta_3^+ - \mathrm{i}\Theta_3^-) \boldsymbol{e}] + \text{c.c.} \right] \\
&\quad + \frac{1}{\sqrt{2}} \overline{\Theta} [(E_1^+ + \mathrm{i}E_1^-) \boldsymbol{e}^* + (E_1^+ - \mathrm{i}E_1^-) \boldsymbol{e}] \,. \qquad (110)
\end{aligned}$$

where we define

$$\begin{aligned}
Q^E + \mathrm{i}U^E &\equiv \sum_{\ell m} Y^2_{\ell m} \overline{E}_{\ell m} = A^2_E \,, \\
\Theta_1^+ + \mathrm{i}\Theta_1^- &\equiv - \sum_{\ell m} Y^1_{\ell m} \overline{\Theta}_{\ell m} C_\ell^{\Theta E} \sqrt{(\ell+2)(\ell-1)} \,, \\
\Theta_3^+ + \mathrm{i}\Theta_3^- &\equiv - \sum_{\ell m} Y^3_{\ell m} \overline{\Theta}_{\ell m} C_\ell^{\Theta E} \sqrt{(\ell-2)(\ell+3)} \,, \\
E_1^+ + \mathrm{i}E_1^- &\equiv - \sum_{\ell m} Y^1_{\ell m} \overline{E}_{\ell m} C_\ell^{\Theta E} \sqrt{\ell(\ell+1)} \,. \qquad (111)
\end{aligned}$$

The above quantities are obtained by `map2alm_spin`. We find that

$$\begin{aligned}
v^{\Theta E} &= \frac{1}{2} [(Q^E + \mathrm{i}U^E)(-\Theta_1^+ + \mathrm{i}\Theta_1^-) + (Q^E - \mathrm{i}U^E)(\Theta_3^+ + \mathrm{i}\Theta_3^-)] + \overline{\Theta}(E_1^+ + \mathrm{i}E_1^-) \\
&= \frac{1}{2} [Q^E(\Theta_3^+ - \Theta_1^+) + U^E(\Theta_3^- - \Theta_1^-) + \mathrm{i}[Q^E(\Theta_3^- + \Theta_1^-) - U^E(\Theta_3^+ + \Theta_1^+)]] + \overline{\Theta}(E_1^+ + \mathrm{i}E_1^-) \,. \qquad (112)
\end{aligned}$$

4.3.3 ΘB

The ΘB estimator is obtained by replacing E to $\mathrm{i}B$ in the ΘE estimator and ignore the second term;

$$v^{\Theta B}=\frac{1}{2}[Q^B(\Theta_3^+-\Theta_1^+)+U^B(\Theta_3^--\Theta_1^-)+\mathrm{i}[Q^B(\Theta_3^-+\Theta_1^-)-U^B(\Theta_3^++\Theta_1^+)]]\,, \tag{113}$$

where we define the Q/U map from B-mode alone;

$$Q^B+\mathrm{i}U^B\equiv\sum_{\ell m}Y_{\ell m}^2\mathrm{i}\overline{B}_{\ell m}\,. \tag{114}$$

4.3.4 EE

The EE estimator contains;

$$\begin{aligned}\boldsymbol{v}^{EE}&=\frac{1}{2}(Q^E+\mathrm{i}U^E)\sum_{\ell m}C_\ell^{EE}\overline{E}_{\ell m}\Big(-\sqrt{\frac{(\ell+2)(\ell-1)}{2}}Y_{\ell m}^{-1}\boldsymbol{e}^*+\sqrt{\frac{(\ell-2)(\ell+3)}{2}}Y_{\ell m}^{-3}\boldsymbol{e}\Big)+\text{c.c}\\&=\frac{1}{2\sqrt{2}}(Q^E+\mathrm{i}U^E)[-(\mathcal{E}_1^+-\mathrm{i}\mathcal{E}_1^-)\boldsymbol{e}^*+(\mathcal{E}_3^+-\mathrm{i}\mathcal{E}_3^-)\boldsymbol{e}]+\text{c.c}\,,\end{aligned} \tag{115}$$

where we define

$$\begin{aligned}\mathcal{E}_1^++\mathrm{i}\mathcal{E}_1^-&\equiv-\sum_{\ell m}Y_{\ell m}^1\overline{E}_{\ell m}C_\ell^{EE}\sqrt{(\ell+2)(\ell-1)}\,,\\\mathcal{E}_3^++\mathrm{i}\mathcal{E}_3^-&\equiv-\sum_{\ell m}Y_{\ell m}^3\overline{E}_{\ell m}C_\ell^{EE}\sqrt{(\ell-2)(\ell+3)}\,.\end{aligned} \tag{116}$$

Then we obtain

$$\begin{aligned}v^{EE}&=\frac{1}{2}(Q^E+\mathrm{i}U^E)[-\mathcal{E}_1^++\mathrm{i}\mathcal{E}_1^-]+\frac{1}{2}(Q^E-\mathrm{i}U^E)[\mathcal{E}_3^++\mathrm{i}\mathcal{E}_3^-]\\&=\frac{1}{2}[Q^E(\mathcal{E}_3^+-\mathcal{E}_1^+)+U^E(\mathcal{E}_3^--\mathcal{E}_1^-)]+\frac{\mathrm{i}}{2}[Q^E(\mathcal{E}_3^-+\mathcal{E}_1^-)-U^E(\mathcal{E}_3^++\mathcal{E}_1^+)]\,,\end{aligned} \tag{117}$$

4.3.5 BB

The BB estimator is the same as EE estimator but using B modes, and the result is;

$$v^{BB}=\frac{1}{2}[Q^B(\mathcal{B}_3^+-\mathcal{B}_1^+)+U^B(\mathcal{B}_3^--\mathcal{B}_1^-)]+\frac{\mathrm{i}}{2}[Q^B(\mathcal{B}_3^-+\mathcal{B}_1^-)-U^B(\mathcal{B}_3^++\mathcal{B}_1^+)]\,, \tag{118}$$

where we define

$$\begin{aligned}\mathcal{B}_1^++\mathrm{i}\mathcal{B}_1^-&\equiv-\sum_{\ell m}Y_{\ell m}^1\mathrm{i}\overline{B}_{\ell m}C_\ell^{BB}\sqrt{(\ell+2)(\ell-1)}\,,\\\mathcal{B}_3^++\mathrm{i}\mathcal{B}_3^-&\equiv-\sum_{\ell m}Y_{\ell m}^3\mathrm{i}\overline{B}_{\ell m}C_\ell^{BB}\sqrt{(\ell-2)(\ell+3)}\,.\end{aligned} \tag{119}$$

4.3.6 EB

The first term of the EB estimator is obtained by replacing E to $\mathrm{i}B$ in the first half of the EE estimator. Similarly, the second term of the BB estimator is given by replacing $\mathrm{i}B$ to E in the first half of the BB estimator. The result is;

$$\begin{aligned}v^{EB}&=\frac{1}{2}[Q^B(\mathcal{E}_3^+-\mathcal{E}_1^+)+U^B(\mathcal{E}_3^--\mathcal{E}_1^-)]+\frac{\mathrm{i}}{2}[Q^B(\mathcal{E}_3^-+\mathcal{E}_1^-)-U^B(\mathcal{E}_3^++\mathcal{E}_1^+)]\\&+\frac{1}{2}[Q^E(\mathcal{B}_3^+-\mathcal{B}_1^+)+U^E(\mathcal{B}_3^--\mathcal{B}_1^-)]+\frac{\mathrm{i}}{2}[Q^E(\mathcal{B}_3^-+\mathcal{B}_1^-)-U^E(\mathcal{B}_3^++\mathcal{B}_1^+)]\,.\end{aligned} \tag{120}$$

4.4 Polarization rotation angle

4.4.1 EB

The EB quadratic estimator for the polarization rotation is given by

$$[\widehat{\alpha}^{\rm EB}_{LM}]^* = A^{\alpha,{\rm EB}}_L \sum_{\ell\ell' mm'} \begin{pmatrix} \ell & \ell' & L \\ m & m' & M \end{pmatrix} \left[W^{\alpha,-}_{\ell L\ell'} C^{\rm BB}_{\ell'} - W^{\alpha,-}_{\ell' L\ell} C^{\rm EE}_{\ell}\right] \overline{E}_{\ell m}\overline{B}_{\ell' m'} \,. \tag{121}$$

Using the property of the Wigner 3j, we obtain

$$\begin{aligned}\begin{pmatrix} \ell & \ell' & L \\ m & m' & M \end{pmatrix} W^{\alpha,-}_{\ell L\ell'} &= -2p^+_{\ell L\ell'}\gamma_{\ell L\ell'} \begin{pmatrix} \ell & L & \ell' \\ -2 & 0 & 2 \end{pmatrix}\begin{pmatrix} \ell & \ell' & L \\ m & m' & M \end{pmatrix} \\ &= -[1+(-1)^{\ell+L+\ell'}]\gamma_{\ell L\ell'} \begin{pmatrix} \ell & \ell' & L \\ -2 & 2 & 0 \end{pmatrix}\begin{pmatrix} \ell & \ell' & L \\ m & m' & M \end{pmatrix} \\ &= -\gamma_{\ell L\ell'} \left[\begin{pmatrix} \ell & \ell' & L \\ -2 & 2 & 0 \end{pmatrix} + \begin{pmatrix} \ell & \ell' & L \\ 2 & -2 & 0 \end{pmatrix}\right]\begin{pmatrix} \ell & \ell' & L \\ m & m' & M \end{pmatrix} \\ &= -\int {\rm d}\hat{\boldsymbol{n}}\; Y_{LM}[Y^{-2}_{\ell m}Y^2_{\ell' m'} + Y^2_{\ell m}Y^{-2}_{\ell' m'}] \,. \end{aligned} \tag{122}$$

The second term is also the same as the first term;

$$\begin{pmatrix} \ell & \ell' & L \\ m & m' & M \end{pmatrix} W^{\alpha,-}_{\ell' L\ell} = -\int {\rm d}\hat{\boldsymbol{n}}\; Y_{LM}[Y^{-2}_{\ell m}Y^2_{\ell' m'} + Y^2_{\ell m}Y^{-2}_{\ell' m'}] \,. \tag{123}$$

We then obtain

$$\begin{aligned}\widehat{\alpha}^{\rm EB}_{LM} &= -A^{\alpha,{\rm EB}}_L \int {\rm d}\hat{\boldsymbol{n}}\; Y^*_{LM}\Big[\sum_{\ell m} Y^{-2}_{\ell m}\overline{E}_{\ell m}\sum_{\ell' m'} Y^2_{\ell' m'}C^{\rm BB}_{\ell'}\overline{B}_{\ell' m'} + \sum_{\ell m} Y^2_{\ell m}\overline{E}_{\ell m}\sum_{\ell' m'} Y^{-2}_{\ell' m'}C^{\rm BB}_{\ell'}\overline{B}_{\ell' m'} \\ &\qquad - \sum_{\ell m} Y^2_{\ell m}C^{\rm EE}_\ell\overline{E}_{\ell m}\sum_{\ell' m'} Y^{-2}_{\ell' m'}\overline{B}_{\ell' m'} - \sum_{\ell m} Y^{-2}_{\ell m}C^{\rm EE}_\ell\overline{E}_{\ell m}\sum_{\ell' m'} Y^{2}_{\ell' m'}\overline{B}_{\ell' m'}\Big] \\ &= {\rm i}A^{\alpha,{\rm EB}}_L \int {\rm d}\hat{\boldsymbol{n}}\; Y^*_{LM}\left[(Q^E - {\rm i}U^E)(\mathcal{Q}^B + {\rm i}\mathcal{U}^B) + (\mathcal{Q}^E + {\rm i}\mathcal{U}^E)(Q^B - {\rm i}U^B) - {\rm c.c.}\right] \\ &= -2A^{\alpha,{\rm EB}}_L \int {\rm d}\hat{\boldsymbol{n}}\; Y^*_{LM}\left[Q^E\mathcal{U}^B - U^E\mathcal{Q}^B + \mathcal{U}^E Q^B - \mathcal{Q}^E U^B\right] \,. \end{aligned} \tag{124}$$

where we define

$$\begin{aligned} Q^E + {\rm i}U^E &= \sum_{\ell m} Y^2_{\ell m}\overline{E}_{\ell m} = (\sum_{\ell m} Y^{-2}_{\ell m}\overline{E}_{\ell m})^* \\ \mathcal{Q}^B + {\rm i}\mathcal{U}^B &= \sum_{\ell m} Y^2_{\ell m}{\rm i}C^{\rm BB}_\ell\overline{B}_{\ell m} \\ \mathcal{Q}^E + {\rm i}\mathcal{U}^E &= \sum_{\ell m} Y^2_{\ell m}C^{\rm EE}_\ell\overline{E}_{\ell m} \\ Q^B + {\rm i}U^B &= \sum_{\ell m} Y^2_{\ell m}{\rm i}\overline{B}_{\ell m} = -(\sum_{\ell m} Y^{-2}_{\ell m}{\rm i}\overline{B}_{\ell m})^* \,. \end{aligned} \tag{125}$$

4.5 Amplitude modulation

4.5.1 ΘΘ

The unnormalized estimator for ΘΘ is given by

$$[\overline{\epsilon}^{\Theta\Theta}_{LM}]^* = \sum_{\ell\ell' mm'} \begin{pmatrix} \ell & \ell' & L \\ m & m' & M \end{pmatrix} \frac{[W^{\epsilon,0}_{\ell L\ell'}C^{\Theta\Theta}_{\ell'} + W^{\epsilon,0}_{\ell' L\ell}C^{\Theta\Theta}_{\ell}]}{2\widehat{C}^{\Theta\Theta}_\ell\widehat{C}^{\Theta\Theta}_{\ell'}}\widehat{\Theta}_{\ell m}\widehat{\Theta}_{\ell' m'} \,, \tag{126}$$

and the sum is non-zero only when $\ell + L + \ell'$ is even. Note that

$$W^{\epsilon,0}_{\ell L\ell'} = \gamma_{\ell L\ell'} \begin{pmatrix} \ell & L & \ell' \\ 0 & 0 & 0 \end{pmatrix} = W^{\epsilon,0}_{\ell' L\ell} \,. \tag{127}$$

The estimator contains

$$\gamma_{\ell L\ell'} \begin{pmatrix} \ell & \ell' & L \\ m & m' & M \end{pmatrix} W^{\epsilon,0}_{\ell L\ell'} = \int \mathrm{d}\hat{\boldsymbol{n}}\; Y_{\ell m} Y_{LM} Y_{\ell' m'} \,, \tag{128}$$

Substituting the above equation to Eq. (126), we obtain

$$\begin{aligned} \bar{\epsilon}^{\Theta\Theta}_{LM} &= \sum_{\ell\ell' mm'} \int \mathrm{d}\hat{\boldsymbol{n}}\; Y^*_{\ell m} Y^*_{LM} Y^*_{\ell' m'} \frac{[C^{\Theta\Theta}_{\ell'} + C^{\Theta\Theta}_{\ell}]}{2\widehat{C}^{\Theta\Theta}_{\ell}\widehat{C}^{\Theta\Theta}_{\ell'}} \widehat{\Theta}^*_{\ell m}\widehat{\Theta}^*_{\ell' m'} \\ &= \int \mathrm{d}\hat{\boldsymbol{n}}\; Y^*_{LM} \left[\sum_{\ell m} \frac{C^{\Theta\Theta}_{\ell}\widehat{\Theta}_{\ell m} Y_{\ell m}}{\widehat{C}^{\Theta\Theta}_{\ell}}\right] \left[\sum_{\ell' m'} \frac{C^{\Theta\Theta}_{\ell'}\widehat{\Theta}_{\ell' m'} Y_{\ell' m'}}{\widehat{C}^{\Theta\Theta}_{\ell'}}\right] . \end{aligned} \tag{129}$$

4.5.2 EB

The EB quadratic estimator for the amplitude modulation is given by

$$[\widehat{\epsilon}^{\rm EB}_{LM}]^* = A^{\tau,\rm EB}_L \sum_{\ell\ell' mm'} \begin{pmatrix} \ell & \ell' & L \\ m & m' & M \end{pmatrix} [W^{\tau,-}_{\ell L\ell'} C^{\rm BB}_{\ell'} + W^{\tau,-}_{\ell' L\ell} C^{\rm EE}_{\ell}]\, \overline{E}_{\ell m}\overline{B}_{\ell' m'} \,. \tag{130}$$

Using the property of the Wigner 3j, we obtain

$$\begin{aligned} \begin{pmatrix} \ell & \ell' & L \\ m & m' & M \end{pmatrix} W^{\tau,-}_{\ell L\ell'} &= \mathrm{i}\frac{[1-(-1)^{\ell+L+\ell'}]}{2}\gamma_{\ell L\ell'} \begin{pmatrix} \ell & L & \ell' \\ -2 & 0 & 2 \end{pmatrix} \begin{pmatrix} \ell & \ell' & L \\ m & m' & M \end{pmatrix} \\ &= \frac{\mathrm{i}}{2}\gamma_{\ell L\ell'} \left[\begin{pmatrix} \ell & \ell' & L \\ 2 & -2 & 0 \end{pmatrix} - \begin{pmatrix} \ell & \ell' & L \\ -2 & 2 & 0 \end{pmatrix} \right] \begin{pmatrix} \ell & \ell' & L \\ m & m' & M \end{pmatrix} \\ &= \frac{\mathrm{i}}{2}\int \mathrm{d}\hat{\boldsymbol{n}}\; Y_{LM}[Y^2_{\ell m} Y^{-2}_{\ell' m'} - Y^{-2}_{\ell m} Y^2_{\ell' m'}] \,. \end{aligned} \tag{131}$$

The second term is also the same as the first term;

$$\begin{pmatrix} \ell & \ell' & L \\ m & m' & M \end{pmatrix} W^{\tau,-}_{\ell' L\ell} = \frac{\mathrm{i}}{2}\int \mathrm{d}\hat{\boldsymbol{n}}\; Y_{LM}[Y^{-2}_{\ell m} Y^2_{\ell' m'} - Y^2_{\ell m} Y^{-2}_{\ell' m'}] \,. \tag{132}$$

We then obtain

$$\begin{aligned} \widehat{\epsilon}^{\rm EB}_{LM} &= \frac{\mathrm{i}}{2}A^{\tau,\rm EB}_L \int \mathrm{d}\hat{\boldsymbol{n}}\; Y^*_{LM} [\sum_{\ell m} Y^2_{\ell m}\overline{E}_{\ell m} \sum_{\ell' m'} Y^{-2}_{\ell' m'} C^{\rm BB}_{\ell'}\overline{B}_{\ell' m'} - \sum_{\ell m} Y^{-2}_{\ell m}\overline{E}_{\ell m} \sum_{\ell' m'} Y^2_{\ell' m'} C^{\rm BB}_{\ell'}\overline{B}_{\ell' m'} \\ &\quad + \sum_{\ell m} Y^{-2}_{\ell m} C^{\rm EE}_{\ell}\overline{E}_{\ell m} \sum_{\ell' m'} Y^2_{\ell' m'}\overline{B}_{\ell' m'} - \sum_{\ell m} Y^2_{\ell m} C^{\rm EE}_{\ell}\overline{E}_{\ell m} \sum_{\ell' m'} Y^{-2}_{\ell' m'}\overline{B}_{\ell' m'}] \\ &= \frac{1}{2}A^{\tau,\rm EB}_L \int \mathrm{d}\hat{\boldsymbol{n}}\; Y^*_{LM} \left[-(Q^E + \mathrm{i}U^E)(\mathcal{Q}^B - \mathrm{i}\mathcal{U}^B) + (\mathcal{Q}^E - \mathrm{i}\mathcal{U}^E)(Q^B + \mathrm{i}U^B) + \text{c.c.} \right] \\ &= A^{\tau,\rm EB}_L \int \mathrm{d}\hat{\boldsymbol{n}}\; Y^*_{LM} \left[-Q^E\mathcal{Q}^B - U^E\mathcal{U}^B + \mathcal{Q}^E Q^B + \mathcal{U}^E U^B \right] \,. \end{aligned} \tag{133}$$

4.5.3 EB (odd)

The EB quadratic estimator for the amplitude modulation is given by

$$[\widehat{\epsilon}^{\rm EB,-}_{LM}]^* = A^{\tau,\rm EB,-}_L \sum_{\ell\ell' mm'} \begin{pmatrix} \ell & \ell' & L \\ m & m' & M \end{pmatrix} [C^{\rm EB}_{\ell'} W^{\tau,+}_{\ell L\ell'} + C^{\rm EB}_{\ell} W^{\tau,+}_{\ell' L\ell}]\, \overline{E}_{\ell m}\overline{B}_{\ell' m'} \,. \tag{134}$$

Note that $W^{\alpha,-}_{\ell' L\ell} = -2W^{\tau,+}_{\ell' L\ell}$. We then obtain the estimator by replacing $C^{\rm BB}_{\ell'}$ and $C^{\rm EE}_{\ell}$ with $-C^{\rm EB}_{\ell'}$ and $C^{\rm EB}_{\ell}$, respectively, in the polarization rotation estimator, and multiplying $1/2$, yielding

$$\widehat{\epsilon}^{\rm EB,-}_{LM} = -A^{\tau,\rm EB}_L \int {\rm d}\hat{\boldsymbol{n}}\; Y^*_{LM} \left[Q^E\mathcal{U}^B - U^E\mathcal{Q}^B + Q^B\mathcal{U}^E - U^B\mathcal{Q}^E\right] . \tag{135}$$

where we define

$$\begin{aligned} Q^E + {\rm i}U^E &= \sum_{\ell m} Y^2_{\ell m}\overline{E}_{\ell m} \\ \mathcal{Q}^B + {\rm i}\mathcal{U}^B &= -\sum_{\ell m} Y^2_{\ell m}{\rm i}C^{\rm EB}_{\ell}\overline{B}_{\ell m} \\ \mathcal{Q}^E + {\rm i}\mathcal{U}^E &= \sum_{\ell m} Y^2_{\ell m}C^{\rm EB}_{\ell}\overline{E}_{\ell m} \\ Q^B + {\rm i}U^B &= \sum_{\ell m} Y^2_{\ell m}{\rm i}\overline{B}_{\ell m} \,. \end{aligned} \tag{136}$$

5 Computing Quadratic Estimator Normalization

Here, we generalize the algorithm of [12] to the case including the cosmic bi-refringence, patchy reionization, and so on.

Using $s = 0, \pm$, we define the following kernel functions;

$$\Sigma_L^{(s),x}[A,B] = \frac{1}{2L+1}\sum_{\ell\ell'} |W^{x,s}_{\ell L\ell'}|^2 A_\ell B_{\ell'} \,, \tag{137}$$

$$\Sigma_L^{(\times),x}[A,B] = \frac{1}{2L+1}\sum_{\ell\ell'} W^{x,0}_{\ell L\ell'} W^{x,+}_{\ell L\ell'} A_\ell B_{\ell'} \,, \tag{138}$$

$$\Gamma_L^{(s),x}[A,B] = \frac{1}{2L+1}\sum_{\ell\ell'} [W^{x,s}_{\ell L\ell'}]^* W^{x,s}_{\ell' L\ell} A_\ell B_{\ell'} \,, \tag{139}$$

$$\Gamma_L^{(\times),x}[A,B] = \frac{1}{2L+1}\sum_{\ell\ell'} W^{x,0}_{\ell L\ell'} W^{x,+}_{\ell' L\ell} A_\ell B_{\ell'} \,. \tag{140}$$

Note that

$$\Gamma_L^{(\pm),x}[A,B] = \Gamma_L^{(\pm),x}[B,A] \,. \tag{141}$$

5.1 Normalization

5.1.1 $\Theta\Theta$

The normalization of the $\Theta\Theta$ quadratic estimator is given by

$$\begin{aligned} [A_L^{x,(\Theta\Theta)}]^{-1} &= \frac{1}{2L+1}\sum_{\ell\ell'} \frac{\left[W^{x,0}_{\ell L\ell'} C^{\Theta\Theta}_{\ell'} + p_x W^{x,0}_{\ell' L\ell} C^{\Theta\Theta}_{\ell}\right]^2}{2\widehat{C}^{\Theta\Theta}_\ell \widehat{C}^{\Theta\Theta}_{\ell'}} \\ &= \Sigma_L^{(0),x}\left[\frac{1}{\widehat{C}^{\Theta\Theta}}, \frac{(C^{\Theta\Theta})^2}{\widehat{C}^{\Theta\Theta}}\right] + p_x \Gamma_L^{(0),x}\left[\frac{C^{\Theta\Theta}}{\widehat{C}^{\Theta\Theta}}, \frac{C^{\Theta\Theta}}{\widehat{C}^{\Theta\Theta}}\right] \,. \end{aligned} \tag{142}$$

Note that, for point sources, the normalization is obtained by substituting $C^{\Theta\Theta}_\ell = 1/2$ in the numerator for $x = \tau$.

5.1.2 ΘE

The normalization of the quadratic ΘE estimator is given by

$$\begin{aligned} [A_L^{x,(\Theta E)}]^{-1} &= \frac{1}{2L+1}\sum_{\ell\ell'} \frac{|W^{x,0}_{\ell L\ell'} C^{\Theta \mathrm{E}}_{\ell'} + p_x W^{x,+}_{\ell' L\ell} C^{\Theta\mathrm{E}}_{\ell}|^2}{\widehat{C}^{\Theta\Theta}_\ell \widehat{C}^{\mathrm{EE}}_{\ell'}} \\ &= \frac{1}{2L+1}\sum_{\ell\ell'} \left[(W^{x,0}_{\ell L\ell'})^2 \frac{(C^{\Theta\mathrm{E}}_{\ell'})^2}{\widehat{C}^{\Theta\Theta}_\ell \widehat{C}^{\mathrm{EE}}_{\ell'}} + 2p_x W^{x,0}_{\ell L\ell'} W^{x,+}_{\ell' L\ell} \frac{C^{\Theta\mathrm{E}}_{\ell'} C^{\Theta\mathrm{E}}_{\ell}}{\widehat{C}^{\Theta\Theta}_\ell \widehat{C}^{\mathrm{EE}}_{\ell'}} + (W^{x,+}_{\ell' L\ell})^2 \frac{(C^{\Theta\mathrm{E}}_{\ell})^2}{\widehat{C}^{\Theta\Theta}_\ell \widehat{C}^{\mathrm{EE}}_{\ell'}}\right] \\ &= \Sigma_L^{(0),x}\left[\frac{1}{\widehat{C}^{\Theta\Theta}}, \frac{(C^{\Theta\mathrm{E}})^2}{\widehat{C}^{\mathrm{EE}}}\right] + 2p_x \Gamma_L^{(\times),x}\left[\frac{C^{\Theta\mathrm{E}}}{\widehat{C}^{\Theta\Theta}}, \frac{C^{\Theta\mathrm{E}}}{\widehat{C}^{\mathrm{EE}}}\right] + \Sigma_L^{(+),x}\left[\frac{1}{\widehat{C}^{\mathrm{EE}}}, \frac{(C^{\Theta\mathrm{E}})^2}{\widehat{C}^{\Theta\Theta}}\right] \,, \end{aligned} \tag{143}$$

5.1.3 ΘE

The normalization of the quadratic ΘB estimator is given by

$$\begin{aligned} [A_L^{x,(\Theta B)}]^{-1} &= \frac{1}{2L+1}\sum_{\ell\ell'} \frac{|W^{x,-}_{\ell' L\ell} C^{\Theta\mathrm{E}}_{\ell}|^2}{\widehat{C}^{\Theta\Theta}_\ell \widehat{C}^{\mathrm{BB}}_{\ell'}} \\ &= \Sigma_L^{(-),x}\left[\frac{1}{\widehat{C}^{\mathrm{BB}}}, \frac{(C^{\Theta\mathrm{E}})^2}{\widehat{C}^{\Theta\Theta}}\right] \,, \end{aligned} \tag{144}$$

5.1.4 EE and BB

The normalization of the quadratic EE estimator (and for the BB estimator by replacing the $EE \to BB$ spectrum) is given by

$$\begin{aligned}[A_L^{x,(EE)}]^{-1} &= \frac{1}{2L+1}\sum_{\ell\ell'}\frac{|W^{x,+}_{\ell L\ell'}C^{\rm EE}_{\ell'} + p_x W^{x,+}_{\ell' L\ell}C^{\rm EE}_{\ell}|^2}{2\widehat{C}^{\rm EE}_{\ell}\widehat{C}^{\rm EE}_{\ell'}} \\ &= \Sigma^{(+),x}_L\left[\frac{1}{\widehat{C}^{\rm EE}},\frac{(C^{\rm EE})^2}{\widehat{C}^{\rm EE}}\right] + p_x\Gamma^{(+),x}_L\left[\frac{C^{\rm EE}}{\widehat{C}^{\rm EE}},\frac{C^{\rm EE}}{\widehat{C}^{\rm EE}}\right] ,\end{aligned} \tag{145}$$

5.1.5 EB

The normalization of the quadratic EB estimator is given by

$$\begin{aligned}[A_L^{x,(EB)}]^{-1} &= \frac{1}{2L+1}\sum_{\ell\ell'}\frac{|W^{x,-}_{\ell L\ell'}C^{\rm BB}_{\ell'} + p_x W^{x,-}_{\ell' L\ell}C^{\rm EE}_{\ell}|^2}{\widehat{C}^{\rm EE}_{\ell}\widehat{C}^{\rm BB}_{\ell'}} \\ &= \Sigma^{(-),x}_L\left[\frac{1}{\widehat{C}^{\rm EE}},\frac{(C^{\rm BB})^2}{\widehat{C}^{\rm BB}}\right] + 2p_x\Gamma^{(-),x}_L\left[\frac{C^{\rm EE}}{\widehat{C}^{\rm EE}},\frac{C^{\rm BB}}{\widehat{C}^{\rm BB}}\right] + \Sigma^{(-),x}_L\left[\frac{1}{\widehat{C}^{\rm BB}},\frac{(C^{\rm EE})^2}{\widehat{C}^{\rm EE}}\right] ,\end{aligned} \tag{146}$$

Also,

$$[A_L^{x,(EB),-}]^{-1} = \Sigma^{(+),x}_L\left[\frac{1}{\widehat{C}^{\rm EE}},\frac{(C^{\rm EB})^2}{\widehat{C}^{\rm BB}}\right] + 2p_x\Gamma^{(+),x}_L\left[\frac{C^{\rm EB}}{\widehat{C}^{\rm EE}},\frac{C^{\rm EB}}{\widehat{C}^{\rm BB}}\right] + \Sigma^{(+),x}_L\left[\frac{1}{\widehat{C}^{\rm BB}},\frac{(C^{\rm EB})^2}{\widehat{C}^{\rm EE}}\right] . \tag{147}$$

5.2 Cross normalization

5.2.1 $\Theta\Theta$

The cross normalization of the $\Theta\Theta$ quadratic estimator is given by

$$\begin{aligned}[A_L^{xy,(\Theta\Theta)}]^{-1} &= \frac{1}{2L+1}\sum_{\ell\ell'}\frac{\left[W^{x,0}_{\ell L\ell'}C^{\Theta\Theta}_{\ell'} + p_x W^{x,0}_{\ell' L\ell}C^{\Theta\Theta}_{\ell}\right]\left[W^{y,0}_{\ell L\ell'}C^{\Theta\Theta}_{\ell'} + p_y W^{y,0}_{\ell' L\ell}C^{\Theta\Theta}_{\ell}\right]}{2\widehat{C}^{\Theta\Theta}_{\ell}\widehat{C}^{\Theta\Theta}_{\ell'}} \\ &= \frac{1+p_xp_y}{2}\Sigma^{(0),xy}_L\left[\frac{1}{\widehat{C}^{\Theta\Theta}},\frac{(C^{\Theta\Theta})^2}{\widehat{C}^{\Theta\Theta}}\right] + \frac{p_x+p_y}{2}\Gamma^{(0),xy}_L\left[\frac{C^{\Theta\Theta}}{\widehat{C}^{\Theta\Theta}},\frac{C^{\Theta\Theta}}{\widehat{C}^{\Theta\Theta}}\right] .\end{aligned} \tag{148}$$

If either x or y are point sources, the cross normalization is given by substituting $x=\tau$ (or $y=\tau$) and $C^{\Theta\Theta}_\ell = 1/2$:

$$\begin{aligned}[A_L^{xs,(\Theta\Theta)}]^{-1} &= \frac{1}{2L+1}\sum_{\ell\ell'}\frac{\left[W^{x,0}_{\ell L\ell'}C^{\Theta\Theta}_{\ell'} + p_x W^{x,0}_{\ell' L\ell}C^{\Theta\Theta}_{\ell}\right]\left[W^{\tau,0}_{\ell L\ell'} + W^{\tau,0}_{\ell' L\ell}\right]}{4\widehat{C}^{\Theta\Theta}_{\ell}\widehat{C}^{\Theta\Theta}_{\ell'}} \\ &= \frac{1+p_x}{4}\Sigma^{(0),x\tau}_L\left[\frac{1}{\widehat{C}^{\Theta\Theta}},\frac{C^{\Theta\Theta}}{\widehat{C}^{\Theta\Theta}}\right] + \frac{1+p_x}{4}\Gamma^{(0),x\tau}_L\left[\frac{1}{\widehat{C}^{\Theta\Theta}},\frac{C^{\Theta\Theta}}{\widehat{C}^{\Theta\Theta}}\right] .\end{aligned} \tag{149}$$

5.2.2 EB

The cross normalization of the even quadratic EB estimators is given by

$$\begin{aligned}[A_L^{xy,(EB)}]^{-1} &= \frac{1}{2L+1}\sum_{\ell\ell'}\frac{(W^{x,-}_{\ell L\ell'}C^{\rm BB}_{\ell'} - W^{x,-}_{\ell' L\ell}C^{\rm EE}_{\ell})(W^{y,-}_{\ell L\ell'}C^{\rm BB}_{\ell'} - W^{y,-}_{\ell' L\ell}C^{\rm EE}_{\ell})}{\widehat{C}^{\rm EE}_{\ell}\widehat{C}^{\rm BB}_{\ell'}} \\ &= \Sigma^{(-),xy}_L\left[\frac{1}{\widehat{C}^{\rm EE}},\frac{(C^{\rm BB})^2}{\widehat{C}^{\rm BB}}\right] - \Gamma^{(-),xy}_L\left[\frac{C^{\rm EE}}{\widehat{C}^{\rm EE}},\frac{C^{\rm BB}}{\widehat{C}^{\rm BB}}\right] - \Gamma^{(-),xy}_L\left[\frac{C^{\rm BB}}{\widehat{C}^{\rm BB}},\frac{C^{\rm EE}}{\widehat{C}^{\rm EE}}\right] + \Sigma^{(-),xy}_L\left[\frac{1}{\widehat{C}^{\rm BB}},\frac{(C^{\rm EE})^2}{\widehat{C}^{\rm EE}}\right] ,\end{aligned} \tag{150}$$

5.3 Noise covariance

5.3.1 $\Theta\Theta\Theta E$

The noise covariance between the $\Theta\Theta$ and ΘE estimators is given by

$$\begin{aligned}
\frac{A_L^{x,(\Theta\Theta)}A_L^{x,(\Theta E)}}{N_L^{x,(\Theta\Theta\Theta E)}} &= \frac{1}{2L+1}\sum_{\ell\ell'}\left[\frac{W^{x,0}_{\ell L\ell'}C^{\Theta\Theta}_{\ell'}}{2\widehat{C}^{\Theta\Theta}_{\ell}\widehat{C}^{\Theta\Theta}_{\ell'}}+p_x(\ell\leftrightarrow\ell')\right]\left[\frac{(W^{x,0}_{\ell L\ell'}C^{\Theta\mathrm{E}}_{\ell'}+p_xW^{x,+}_{\ell' L\ell}C^{\Theta\mathrm{E}}_{\ell})\widehat{C}^{\Theta\mathrm{E}}_{\ell'}}{\widehat{C}^{\mathrm{EE}}_{\ell'}}+p_x(\ell\leftrightarrow\ell')\right] \\
&= \frac{1}{2L+1}\sum_{\ell\ell'}\left[\frac{W^{x,0}_{\ell L\ell'}C^{\Theta\Theta}_{\ell'}}{\widehat{C}^{\Theta\Theta}_{\ell}\widehat{C}^{\Theta\Theta}_{\ell'}}\frac{(W^{x,0}_{\ell L\ell'}C^{\Theta\mathrm{E}}_{\ell'}+p_xW^{x,+}_{\ell' L\ell}C^{\Theta\mathrm{E}}_{\ell})\widehat{C}^{\Theta\mathrm{E}}_{\ell'}}{\widehat{C}^{\mathrm{EE}}_{\ell'}}\right. \\
&\quad \left.+\,p_x\frac{W^{x,0}_{\ell L\ell'}C^{\Theta\Theta}_{\ell'}}{\widehat{C}^{\Theta\Theta}_{\ell}\widehat{C}^{\Theta\Theta}_{\ell'}}\frac{(W^{x,0}_{\ell' L\ell}C^{\Theta\mathrm{E}}_{\ell}+p_xW^{x,+}_{\ell L\ell'}C^{\Theta\mathrm{E}}_{\ell'})\widehat{C}^{\Theta\mathrm{E}}_{\ell}}{\widehat{C}^{\mathrm{EE}}_{\ell}}\right] \\
&= \Sigma^{(0),x}_L\left[\frac{1}{\widehat{C}^{\Theta\Theta}},\frac{C^{\Theta\Theta}C^{\Theta\mathrm{E}}\widehat{C}^{\Theta\mathrm{E}}}{\widehat{C}^{\Theta\Theta}\widehat{C}^{\mathrm{EE}}}\right]+p_x\Gamma^{(\times),x}_L\left[\frac{C^{\Theta\mathrm{E}}}{\widehat{C}^{\Theta\Theta}},\frac{C^{\Theta\Theta}\widehat{C}^{\Theta\mathrm{E}}}{\widehat{C}^{\Theta\Theta}\widehat{C}^{\mathrm{EE}}}\right] \\
&\quad +p_x\Gamma^{(0),x}_L\left[\frac{C^{\Theta\mathrm{E}}\widehat{C}^{\Theta\mathrm{E}}}{\widehat{C}^{\Theta\Theta}\widehat{C}^{\mathrm{EE}}},\frac{C^{\Theta\Theta}}{\widehat{C}^{\Theta\Theta}}\right]+\Sigma^{(\times),x}_L\left[\frac{\widehat{C}^{\Theta\mathrm{E}}}{\widehat{C}^{\Theta\Theta}\widehat{C}^{\mathrm{EE}}},\frac{C^{\Theta\mathrm{E}}C^{\Theta\Theta}}{\widehat{C}^{\Theta\Theta}}\right], \qquad (151)
\end{aligned}$$

5.3.2 $\Theta\Theta EE$

The noise covariance between the $\Theta\Theta$ and EE estimators is given by

$$\begin{aligned}
\frac{A_L^{x,(\Theta\Theta)}A_L^{x,(EE)}}{N_L^{x,(\Theta\Theta EE)}} &= \frac{1}{2L+1}\sum_{\ell\ell'}\left[\frac{W^{x,0}_{\ell L\ell'}C^{\Theta\Theta}_{\ell'}}{2\widehat{C}^{\Theta\Theta}_{\ell}\widehat{C}^{\Theta\Theta}_{\ell'}}+p_x(\ell\leftrightarrow\ell')\right]\left[\frac{(W^{x,+}_{\ell L\ell'}C^{\mathrm{EE}}_{\ell'}+p_xW^{x,+}_{\ell' L\ell}C^{\mathrm{EE}}_{\ell})\widehat{C}^{\Theta\mathrm{E}}_{\ell}\widehat{C}^{\Theta\mathrm{E}}_{\ell'}}{2\widehat{C}^{\mathrm{EE}}_{\ell}\widehat{C}^{\mathrm{EE}}_{\ell'}}+p_x(\ell\leftrightarrow\ell')\right] \\
&= \frac{1}{2L+1}\sum_{\ell\ell'}\frac{W^{x,0}_{\ell L\ell'}C^{\Theta\Theta}_{\ell'}}{\widehat{C}^{\Theta\Theta}_{\ell}\widehat{C}^{\Theta\Theta}_{\ell'}}\left[\frac{(W^{x,+}_{\ell L\ell'}C^{\mathrm{EE}}_{\ell'}+p_xW^{x,+}_{\ell' L\ell}C^{\mathrm{EE}}_{\ell})\widehat{C}^{\Theta\mathrm{E}}_{\ell}\widehat{C}^{\Theta\mathrm{E}}_{\ell'}}{2\widehat{C}^{\mathrm{EE}}_{\ell}\widehat{C}^{\mathrm{EE}}_{\ell'}}+p_x(\ell\leftrightarrow\ell')\right] \\
&= \frac{1}{2L+1}\sum_{\ell\ell'}\frac{W^{x,0}_{\ell L\ell'}C^{\Theta\Theta}_{\ell'}}{\widehat{C}^{\Theta\Theta}_{\ell}\widehat{C}^{\Theta\Theta}_{\ell'}}\frac{(W^{x,+}_{\ell L\ell'}C^{\mathrm{EE}}_{\ell'}+p_xW^{x,+}_{\ell' L\ell}C^{\mathrm{EE}}_{\ell})\widehat{C}^{\Theta\mathrm{E}}_{\ell}\widehat{C}^{\Theta\mathrm{E}}_{\ell'}}{\widehat{C}^{\mathrm{EE}}_{\ell}\widehat{C}^{\mathrm{EE}}_{\ell'}} \\
&= \Sigma^{(0),x}_L\left[\frac{\widehat{C}^{\Theta\mathrm{E}}}{\widehat{C}^{\Theta\Theta}\widehat{C}^{\mathrm{EE}}},\frac{C^{\Theta\Theta}C^{\mathrm{EE}}\widehat{C}^{\Theta\mathrm{E}}}{\widehat{C}^{\Theta\Theta}\widehat{C}^{\mathrm{EE}}}\right]+p_x\Gamma^{(\times),x}_L\left[\frac{\widehat{C}^{\Theta\mathrm{E}}C^{\mathrm{EE}}}{\widehat{C}^{\Theta\Theta}\widehat{C}^{\mathrm{EE}}},\frac{C^{\Theta\Theta}\widehat{C}^{\Theta\mathrm{E}}}{\widehat{C}^{\Theta\Theta}\widehat{C}^{\mathrm{EE}}}\right]. \qquad (152)
\end{aligned}$$

5.3.3 ΘEEE

The noise covariance between the ΘE and EE estimators is given by

$$\begin{aligned}
\frac{A_L^{x,(\Theta E)}A_L^{x,(EE)}}{N_L^{x,(\Theta EEE)}} &= \frac{1}{2L+1}\sum_{\ell\ell'}\left[\frac{W^{x,+}_{\ell L\ell'}C^{\mathrm{EE}}_{\ell'}}{2\widehat{C}^{\mathrm{EE}}_{\ell}\widehat{C}^{\mathrm{EE}}_{\ell'}}+p_x(\ell\leftrightarrow\ell')\right]\left[\frac{(W^{x,0}_{\ell L\ell'}C^{\Theta\mathrm{E}}_{\ell'}+p_xW^{x,+}_{\ell' L\ell}C^{\Theta\mathrm{E}}_{\ell})\widehat{C}^{\Theta\mathrm{E}}_{\ell}}{\widehat{C}^{\Theta\Theta}_{\ell}}+p_x(\ell\leftrightarrow\ell')\right] \\
&= \frac{1}{2L+1}\sum_{\ell\ell'}\left[\frac{W^{x,+}_{\ell L\ell'}C^{\mathrm{EE}}_{\ell'}}{\widehat{C}^{\mathrm{EE}}_{\ell}\widehat{C}^{\mathrm{EE}}_{\ell'}}+p_x\frac{W^{x,+}_{\ell' L\ell}C^{\mathrm{EE}}_{\ell}}{\widehat{C}^{\mathrm{EE}}_{\ell}\widehat{C}^{\mathrm{EE}}_{\ell'}}\right]\left[\frac{(W^{x,0}_{\ell L\ell'}C^{\Theta\mathrm{E}}_{\ell'}+p_xW^{x,+}_{\ell' L\ell}C^{\Theta\mathrm{E}}_{\ell})\widehat{C}^{\Theta\mathrm{E}}_{\ell}}{\widehat{C}^{\Theta\Theta}_{\ell}}\right] \\
&= \Sigma^{(\times),x}_L\left[\frac{\widehat{C}^{\Theta\mathrm{E}}}{\widehat{C}^{\Theta\Theta}\widehat{C}^{\mathrm{EE}}},\frac{C^{\Theta\mathrm{E}}C^{\mathrm{EE}}}{\widehat{C}^{\mathrm{EE}}}\right]+p_x\Gamma^{(+),x}_L\left[\frac{C^{\Theta\mathrm{E}}\widehat{C}^{\Theta\mathrm{E}}}{\widehat{C}^{\Theta\Theta}\widehat{C}^{\mathrm{EE}}},\frac{C^{\mathrm{EE}}}{\widehat{C}^{\mathrm{EE}}}\right] \\
&\quad +p_x\Gamma^{(\times),x}_L\left[\frac{\widehat{C}^{\Theta\mathrm{E}}C^{\mathrm{EE}}}{\widehat{C}^{\Theta\Theta}\widehat{C}^{\mathrm{EE}}},\frac{C^{\Theta\mathrm{E}}}{\widehat{C}^{\mathrm{EE}}}\right]+\Sigma^{(+),x}_L\left[\frac{C^{\Theta\mathrm{E}}\widehat{C}^{\Theta\mathrm{E}}C^{\mathrm{EE}}}{\widehat{C}^{\Theta\Theta}\widehat{C}^{\mathrm{EE}}},\frac{1}{\widehat{C}^{\mathrm{EE}}}\right]. \qquad (153)
\end{aligned}$$

5.3.4 ΘBEB

The noise covariance between the ΘB and EB estimators is given by

$$\frac{A_L^{x,(\Theta B)}A_L^{x,(EB)}}{N_L^{x,(\Theta BEB)}} = \frac{1}{2L+1}\sum_{\ell\ell'}\left[\frac{(W^{x,-}_{\ell L\ell'})^* C^{\rm BB}_{\ell'} - p_x(W^{x,-}_{\ell' L\ell})^* C^{\rm EE}_{\ell}}{\widehat{C}^{\rm EE}_{\ell}\widehat{C}^{\rm BB}_{\ell'}}\right]\left[\frac{-p_x W^{x,-}_{\ell' L\ell} C^{\Theta\rm E}_{\ell}\widehat{C}^{\Theta\rm E}_{\ell}}{\widehat{C}^{\Theta\Theta}_{\ell}}\right]$$
$$= -p_x\Gamma^{(-),x}_L\left[\frac{C^{\Theta\rm E}\widehat{C}^{\Theta\rm E}}{\widehat{C}^{\Theta\Theta}\widehat{C}^{\rm EE}},\frac{C^{\rm BB}}{\widehat{C}^{\rm BB}}\right] + \Sigma^{(-),x}_L\left[\frac{C^{\Theta\rm E}\widehat{C}^{\Theta\rm E}C^{\rm EE}}{\widehat{C}^{\Theta\Theta}\widehat{C}^{\rm EE}},\frac{1}{\widehat{C}^{\rm BB}}\right]. \tag{154}$$

6 Explicit Kernel Functions

Here we consider expression for the Kernel functions in terms of the Wigner d-functions. In the following calculations, we frequently use

$$\int_{-1}^{1}\mathrm{d}\mu\ d^{\ell_1}_{s_1,s_1'}(\beta)d^{\ell_2}_{s_2,s_2'}(\beta)d^{\ell_3}_{s_3,s_3'}(\beta)=2\begin{pmatrix}\ell_1 & \ell_2 & \ell_3\\ s_1 & s_2 & s_3\end{pmatrix}\begin{pmatrix}\ell_1 & \ell_2 & \ell_3\\ s_1' & s_2' & s_3'\end{pmatrix}, \tag{155}$$

with $s_1+s_2+s_3=s_1'+s_2'+s_3'=0$ and $\mu=\cos\beta$, and the symmetric property:

$$d^{\ell}_{mm'}(\beta)=(-1)^{m-m'}d^{\ell}_{-m,-m'}(\beta)=(-1)^{m-m'}d^{\ell}_{m'm}(\beta) \tag{156}$$

$$d^{\ell}_{mm'}(\beta)=(-1)^{\ell+m}d^{\ell}_{m,-m'}(\pi-\beta)\,. \tag{157}$$

Note that

$$(-1)^{\ell_1+\ell_2+\ell_3}\int_{-1}^{1}\mathrm{d}\mu\ d^{\ell_1}_{s_1,s_1'}d^{\ell_2}_{s_2,s_2'}d^{\ell_3}_{s_3,s_3'}=\int_{-1}^{1}\mathrm{d}\mu\ d^{\ell_1}_{s_1,-s_1'}d^{\ell_2}_{s_2,-s_2'}d^{\ell_3}_{s_3,-s_3'}\,. \tag{158}$$

We also define

$$X^{p\dots q}=(\sqrt{2}a^p_\ell)\dots(\sqrt{2}a^q_\ell)X_\ell\,. \tag{159}$$

and

$$\xi^A_{mm'}=\sum_\ell\frac{2\ell+1}{4\pi}A_\ell d^\ell_{mm'}\,. \tag{160}$$

6.1 Kernel Functions: Lensing

For lensing fields, $x=\phi$ or ϖ, we obtain

$$\begin{aligned}\Sigma^{(0),x}_L[A,B]&=\frac{1}{2L+1}\sum_{\ell\ell'}|W^{x,0}_{\ell L\ell'}|^2A_\ell B_{\ell'}\\
&=\sum_{\ell\ell'}4\pi\frac{2\ell+1}{4\pi}A_\ell\frac{2\ell'+1}{4\pi}B_{\ell'}\frac{L(L+1)}{2}\frac{\ell'(\ell'+1)}{2}[1+c_x^2(-1)^{\ell+L+\ell'}]2\begin{pmatrix}\ell & L & \ell'\\ 0 & 1 & -1\end{pmatrix}^2\\
&=\int_{-1}^{1}\mathrm{d}\mu\ \pi L(L+1)\sum_{\ell\ell'}\frac{2\ell+1}{4\pi}A_\ell\frac{2\ell'+1}{4\pi}B_{\ell'}\ell'(\ell'+1)[d^\ell_{00}d^L_{11}d^{\ell'}_{11}+c_x^2d^\ell_{00}d^L_{1,-1}d^{\ell'}_{1,-1}]\\
&=\int_{-1}^{1}\mathrm{d}\mu\ \pi L(L+1)\{\xi^A_{00}\xi^{B^{00}}_{11}d^L_{11}+c_x^2\xi^A_{00}\xi^{B^{00}}_{1,-1}d^L_{1,-1}\}\,.\end{aligned} \tag{161}$$

and

$$\begin{aligned}\Gamma^{(0),x}_L[A,B]&=\frac{1}{2L+1}\sum_{\ell\ell'}(W^{x,0}_{\ell L\ell'})^*W^{x,0}_{\ell' L\ell}A_\ell B_{\ell'}\\
&=\sum_{\ell\ell'}2\pi L(L+1)\frac{2\ell+1}{4\pi}A_\ell\frac{2\ell'+1}{4\pi}B_{\ell'}a^0_\ell a^0_{\ell'}[1+c_x^2(-1)^{\ell+L+\ell'}]2\begin{pmatrix}\ell & L & \ell'\\ 0 & 1 & -1\end{pmatrix}\begin{pmatrix}\ell' & L & \ell\\ 0 & 1 & -1\end{pmatrix}\\
&=\sum_{\ell\ell'}\pi L(L+1)\frac{2\ell+1}{4\pi}A^0_\ell\frac{2\ell'+1}{4\pi}B^0_{\ell'}[1+c_x^2(-1)^{\ell+L+\ell'}]2\begin{pmatrix}\ell & L & \ell'\\ 0 & 1 & -1\end{pmatrix}\begin{pmatrix}\ell & L & \ell'\\ 1 & -1 & 0\end{pmatrix}\\
&=\int_{-1}^{1}\mathrm{d}\mu\ \pi L(L+1)\sum_{\ell\ell'}\frac{2\ell+1}{4\pi}A^0_\ell\frac{2\ell'+1}{4\pi}B^0_{\ell'}[d^\ell_{01}d^L_{1,-1}d^{\ell'}_{-1,0}+c_x^2d^\ell_{0,-1}d^L_{11}d^{\ell'}_{-1,0}]\\
&=-\int_{-1}^{1}\mathrm{d}\mu\ \pi L(L+1)\{\xi^{A^0}_{01}\xi^{B^0}_{0,-1}d^L_{1,-1}+c_x^2\xi^{A^0}_{01}\xi^{B^0}_{01}d^L_{11}\}\,.\end{aligned} \tag{162}$$

Denoting $p = \pm$ and $x = \phi, \varpi$, we rewrite the kernel for polarization as

$$\Sigma_L^{(p),x}[A,B] = \frac{1}{2L+1}\sum_{\ell\ell'}|W^{x,p}_{\ell L\ell'}|^2 A_\ell B_{\ell'}$$
$$= \frac{\pi}{2}L(L+1)\sum_{\ell\ell'}\frac{2\ell+1}{4\pi}A_\ell\frac{2\ell'+1}{4\pi}B_{\ell'}2[1+pc_x^2(-1)^{\ell+L+\ell'}]\left[a^+_{\ell'}\begin{pmatrix}\ell & L & \ell'\\ -2 & -1 & 3\end{pmatrix} + c_x^2 a^-_{\ell'}\begin{pmatrix}\ell & L & \ell'\\ -2 & 1 & 1\end{pmatrix}\right]^2$$
$$= \frac{\pi}{2}L(L+1)\sum_{\ell\ell'}\frac{2\ell+1}{4\pi}A_\ell\frac{2\ell'+1}{4\pi}B_{\ell'}[1+pc_x^2(-1)^{\ell+L+\ell'}]$$
$$\times 2\left[(a^+_{\ell'})^2\begin{pmatrix}\ell & L & \ell'\\ -2 & -1 & 3\end{pmatrix}^2 + (a^-_{\ell'})^2\begin{pmatrix}\ell & L & \ell'\\ -2 & 1 & 1\end{pmatrix}^2 + 2c_x^2 a^+_{\ell'}a^-_{\ell'}\begin{pmatrix}\ell & L & \ell'\\ -2 & 1 & 1\end{pmatrix}\begin{pmatrix}\ell & L & \ell'\\ -2 & -1 & 3\end{pmatrix}\right]$$
$$= \frac{\pi}{2}\int_{-1}^{1}\mathrm{d}\mu\; L(L+1)\sum_{\ell\ell'}\frac{2\ell+1}{4\pi}A_\ell\frac{2\ell'+1}{4\pi}B_{\ell'}[(a^+_{\ell'})^2 d^\ell_{22}d^L_{11}d^{\ell'}_{33} + (a^-_{\ell'})^2 d^\ell_{22}d^L_{11}d^{\ell'}_{11}$$
$$+ 2c_x^2 a^+_{\ell'}a^-_{\ell'}d^\ell_{22}d^L_{1,-1}d^{\ell'}_{13} + pc_x^2(a^+_{\ell'})^2 d^\ell_{2,-2}d^L_{1,-1}d^{\ell'}_{3,-3} + pc_x^2(a^-_{\ell'})^2 d^\ell_{2,-2}d^L_{1,-1}d^{\ell'}_{1,-1} + 2pa^+_{\ell'}a^-_{\ell'}d^\ell_{2,-2}d^L_{11}d^{\ell'}_{1,-3}]$$
$$= \int_{-1}^{1}\mathrm{d}\mu\;\frac{\pi}{4}L(L+1)[(\xi^A_{22}\xi^{B^{++}}_{33} + \xi^A_{22}\xi_{11}[B^{--}] + 2p\xi^A_{2,-2}\xi^{B^{+-}}_{3,-1})d^L_{11}$$
$$+ c_x^2(p\xi^A_{2,-2}\xi^{B^{++}}_{3,-3} + p\xi^A_{2,-2}\xi^{B^{--}}_{1,-1} + 2\xi^A_{22}\xi^{B^{+-}}_{31}]d^L_{1,-1}]\,, \tag{163}$$

and

$$\Gamma_L^{(p),x}[A,B] = \frac{1}{2L+1}\sum_{\ell\ell'}(W^{x,p}_{\ell L\ell'})^* W^{x,p}_{\ell' L\ell}A_\ell B_{\ell'}$$
$$= \frac{\pi}{2}L(L+1)\sum_{\ell\ell'}\frac{2\ell+1}{4\pi}A_\ell\frac{2\ell'+1}{4\pi}B_{\ell'}2[1+pc_x^2(-1)^{\ell+L+\ell'}]$$
$$\times\left[a^+_{\ell'}\begin{pmatrix}\ell & L & \ell'\\ -2 & -1 & 3\end{pmatrix} + c_x^2 a^-_{\ell'}\begin{pmatrix}\ell & L & \ell'\\ -2 & 1 & 1\end{pmatrix}\right]\left[a^+_\ell\begin{pmatrix}\ell' & L & \ell\\ -2 & -1 & 3\end{pmatrix} + c_x^2 a^-_\ell\begin{pmatrix}\ell' & L & \ell\\ -2 & 1 & 1\end{pmatrix}\right]$$
$$= \frac{\pi}{2}L(L+1)\sum_{\ell\ell'}\frac{2\ell+1}{4\pi}A_\ell\frac{2\ell'+1}{4\pi}B_{\ell'}2[(-1)^{\ell+L+\ell'} + pc_x^2]$$
$$\times\left[a^+_{\ell'}\begin{pmatrix}\ell & L & \ell'\\ -2 & -1 & 3\end{pmatrix} + c_x^2 a^-_{\ell'}\begin{pmatrix}\ell & L & \ell'\\ -2 & 1 & 1\end{pmatrix}\right]\left[a^+_\ell\begin{pmatrix}\ell & L & \ell'\\ 3 & -1 & -2\end{pmatrix} + c_x^2 a^-_\ell\begin{pmatrix}\ell & L & \ell'\\ 1 & 1 & -2\end{pmatrix}\right]$$
$$= \int_{-1}^{1}\mathrm{d}\mu\;\frac{\pi}{2}L(L+1)\sum_{\ell\ell'}\frac{2\ell+1}{4\pi}A_\ell\frac{2\ell'+1}{4\pi}B_{\ell'}$$
$$\times[a^+_{\ell'}a^+_\ell d^\ell_{23}d^L_{1,-1}d^{\ell'}_{-3,-2} + c_x^2 a^+_{\ell'}a^-_\ell d^\ell_{21}d^L_{11}d^{\ell'}_{-3,-2} + c_x^2 a^-_{\ell'}a^+_\ell d^\ell_{23}d^L_{11}d^{\ell'}_{-1,-2} + a^-_{\ell'}a^-_\ell d^\ell_{21}d^L_{1,-1}d^{\ell'}_{-1,-2}$$
$$+ p(c_x^2 a^+_{\ell'}a^+_\ell d^\ell_{2,-3}d^L_{11}d^{\ell'}_{-3,2} + a^+_{\ell'}a^-_\ell d^\ell_{2,-1}d^L_{1,-1}d^{\ell'}_{-3,2} + a^-_{\ell'}a^+_\ell d^\ell_{2,-3}d^L_{1,-1}d^{\ell'}_{-1,2} + c_x^2 a^-_{\ell'}a^-_\ell d^\ell_{2,-1}d^L_{11}d^{\ell'}_{-1,2})]$$
$$= \int_{-1}^{1}\mathrm{d}\mu\;\frac{\pi}{4}L(L+1)[-(\xi^{A^-}_{21}\xi^{B^+}_{32} + \xi^{A^+}_{32}\xi^{B^-}_{21} + p\xi^{A^+}_{3,-2}\xi^{B^+}_{3,-2} + p\xi^{A^-}_{2,-1}\xi^{B^-}_{2,-1})c_x^2 d^L_{11}$$
$$+ (\xi^{A^+}_{32}\xi^{B^+}_{32} + \xi^{A^-}_{21}\xi^{B^-}_{21} - p\xi^{A^-}_{2,-1}\xi^{B^+}_{3,-2} - p\xi^{A^+}_{3,-2}\xi^{B^-}_{2,-1})d^L_{1,-1}]\,. \tag{164}$$

The other kernels are given by

$$
\begin{aligned}
\Sigma_L^{(\times),x}[A,B] &= \frac{1}{2L+1}\sum_{\ell\ell'}(W^{x,0}_{\ell L\ell'})^* W^{x,+}_{\ell L\ell'} A_\ell B_{\ell'} \\
&= \pi L(L+1)\sum_{\ell\ell'}\frac{2\ell+1}{4\pi}A_\ell\frac{2\ell'+1}{4\pi}B_{\ell'}a^0_{\ell'}2[1+c_x^2(-1)^{\ell+L+\ell'}] \\
&\quad\times\begin{pmatrix}\ell & L & \ell' \\ 0 & 1 & -1\end{pmatrix}\left[a^+_{\ell'}\begin{pmatrix}\ell & L & \ell' \\ -2 & -1 & 3\end{pmatrix}+c_x^2a^-_{\ell'}\begin{pmatrix}\ell & L & \ell' \\ -2 & 1 & 1\end{pmatrix}\right] \\
&= \int_{-1}^1 \mathrm{d}\mu\ \pi L(L+1)\sum_{\ell\ell'}\frac{2\ell+1}{4\pi}A_\ell\frac{2\ell'+1}{4\pi}B_{\ell'}a^0_{\ell'} \\
&\quad\times\left[a^+_{\ell'}d^\ell_{0,-2}d^L_{1,-1}d^{\ell'}_{-1,3}+c_x^2a^-_{\ell'}d^\ell_{0,-2}d^L_{11}d^{\ell'}_{-1,1}+c_x^2a^+_{\ell'}d^\ell_{02}d^L_{11}d^{\ell'}_{-1,-3}+a^-_{\ell'}d^\ell_{02}d^L_{1,-1}d^{\ell'}_{-1,-1}\right] \\
&= \int_{-1}^1 \mathrm{d}\mu\ \frac{\pi}{2}L(L+1)\{c_x^2(\xi^A_{20}\xi^{B^{0-}}_{1,-1}+\xi^A_{20}\xi^{B^{0+}}_{31})d^L_{11} \\
&\quad+(\xi^A_{20}\xi^{B^{0+}}_{3,-1}+\xi^A_{20}\xi^{B^{0-}}_{11})d^L_{1,-1}\}\,,
\end{aligned} \tag{165}
$$

and

$$
\begin{aligned}
\Gamma_L^{(\times),x}[A,B] &= \frac{1}{2L+1}\sum_{\ell\ell'}(W^{x,0}_{\ell L\ell'})^* W^{x,+}_{\ell' L\ell} A_\ell B_{\ell'} \\
&= \pi L(L+1)\sum_{\ell\ell'}\frac{2\ell+1}{4\pi}A_\ell\frac{2\ell'+1}{4\pi}B_{\ell'}a^0_{\ell'}2[1+c_x^2(-1)^{\ell+L+\ell'}] \\
&\quad\times\begin{pmatrix}\ell & L & \ell' \\ 0 & 1 & -1\end{pmatrix}\left[a^+_{\ell}\begin{pmatrix}\ell' & L & \ell \\ -2 & -1 & 3\end{pmatrix}+c_x^2a^-_{\ell}\begin{pmatrix}\ell' & L & \ell \\ -2 & 1 & 1\end{pmatrix}\right] \\
&= \int_{-1}^1 \mathrm{d}\mu\ \pi L(L+1)\sum_{\ell\ell'}\frac{2\ell+1}{4\pi}A_\ell\frac{2\ell'+1}{4\pi}B_{\ell'}a^0_{\ell'} \\
&\quad\times\left[a^+_{\ell}d^\ell_{0,-3}d^L_{11}d^{\ell'}_{-1,2}+c_x^2a^-_{\ell}d^\ell_{0,-1}d^L_{1,-1}d^{\ell'}_{-1,2}+c_x^2a^+_{\ell}d^\ell_{03}d^L_{1,-1}d^{\ell'}_{-1,-2}+a^-_{\ell}d^\ell_{01}d^L_{11}d^{\ell'}_{-1,-2}\right] \\
&= \int_{-1}^1 \mathrm{d}\mu\ \frac{\pi}{2}L(L+1)\{-(\xi^{A^+}_{30}\xi^{B^0}_{2,-1}+\xi^{A^-}_{10}\xi^{B^0}_{12})d^L_{11} \\
&\quad-c_x^2(\xi^{A^-}_{10}\xi^{B^0}_{2,-1}+\xi^{A^0}_{30}\xi^{B^-}_{21})d^L_{1,-1}\}\,.
\end{aligned} \tag{166}
$$

6.2 Kernel Functions: Rotation

Next we consider the kernel functions for $x=\alpha$. If $p=-$ and $x=\alpha$,

$$
\begin{aligned}
\Sigma_L^{(-),\alpha}[A,B] &= \pi\sum_{\ell\ell'}\frac{2\ell+1}{4\pi}A_\ell\frac{2\ell'+1}{4\pi}B_{\ell'}8[1+(-1)^{\ell+L+\ell'}]\begin{pmatrix}\ell & L & \ell' \\ -2 & 0 & 2\end{pmatrix}^2 \\
&= \int_{-1}^1 \mathrm{d}\mu\ 4\pi\sum_{\ell\ell'}\frac{2\ell+1}{4\pi}A_\ell\frac{2\ell'+1}{4\pi}B_{\ell'}(d^\ell_{-2,-2}d^L_{00}d^{\ell'}_{22}+d^\ell_{-2,2}d^L_{00}d^{\ell'}_{2,-2}) \\
&= \int_{-1}^1 \mathrm{d}\mu\ 4\pi(\xi^A_{-2,-2}\xi^B_{22}+\xi^A_{-2,2}\xi^B_{2,-2})d^L_{00}\,,
\end{aligned} \tag{167}
$$

and, using the property of the weight function, we find

$$
\Gamma_L^{(-),\alpha}[A,B] = \Sigma_L^{(-),\alpha}[A,B]\,. \tag{168}
$$

Similarly, we obtain

$$\Sigma_L^{(+),\alpha}[A,B] = \pi \sum_{\ell\ell'} \frac{2\ell+1}{4\pi} A_\ell \frac{2\ell'+1}{4\pi} B_{\ell'} 8[1-(-1)^{\ell+L+\ell'}] \begin{pmatrix} \ell & L & \ell' \\ -2 & 0 & 2 \end{pmatrix}^2$$
$$= \int_{-1}^{1} \mathrm{d}\mu \; 4\pi(\xi^A_{-2,-2}\xi^B_{22} - \xi^A_{-2,2}\xi^B_{2,-2}) d^L_{00} \,, \tag{169}$$

and

$$\Gamma_L^{(+),\alpha}[A,B] = \Sigma_L^{(+),\alpha}[A,B] \,. \tag{170}$$

6.3 Kernel Functions: Amplitude

Here we consider $x = \tau$. For $s = 0$, we obtain

$$\Sigma_L^{0,\tau}[A,B] = \frac{1}{2L+1} \sum_{\ell\ell'} A_\ell B_{\ell'} p^+_{\ell L \ell'} (\gamma_{\ell L \ell'})^2 \begin{pmatrix} \ell & L & \ell' \\ 0 & 0 & 0 \end{pmatrix}^2$$
$$= \int_{-1}^{1} \mathrm{d}\mu \; \pi \sum_{\ell\ell'} \frac{2\ell+1}{4\pi} A_\ell \frac{2\ell'+1}{4\pi} B_{\ell'} 2 d^\ell_{00} d^L_{00} d^{\ell'}_{00}$$
$$= \int_{-1}^{1} \mathrm{d}\mu \; 2\pi \xi^A_{00} \xi^B_{00} d^L_{00} \,. \tag{171}$$

For $s = \pm$, the weight function is given by that of the rotation angle rotation, and we find

$$\Sigma_L^{(\pm),\tau}[A,B] = \frac{1}{4} \Sigma_L^{(\mp),\alpha}[A,B] \,, \tag{172}$$

and, using the property of the weight function, we find

$$\Gamma_L^{(\pm),\tau}[A,B] = \Sigma_L^{(\pm),\tau}[A,B] \,. \tag{173}$$

6.4 Response function

6.4.1 ϕ and τ

The lensing potential and amplitude modulation are both even. We then need to compute

$$W^{\phi,0}_{\ell L \ell'} W^{\tau,0}_{\ell L \ell'} = -2(p^+_{\ell L \ell'})^2 (\gamma_{\ell L \ell'})^2 a^0_L a^0_{\ell'} \begin{pmatrix} \ell & L & \ell' \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} \ell & L & \ell' \\ 0 & 0 & 0 \end{pmatrix}$$
$$= -\int_{-1}^{1} \mathrm{d}\mu \; (\gamma_{\ell L \ell'})^2 a^0_L a^0_{\ell'} d^\ell_{00} d^L_{10} d^{\ell'}_{-1,0} \,. \tag{174}$$

Then we obtain

$$\Sigma_L^{0,\phi\tau}[A,B] = \int_{-1}^{1} \mathrm{d}\mu \; 2\pi \sqrt{L(L+1)} d^L_{10} \xi^A_{00} \xi^{B^0}_{10} \,, \tag{175}$$

and

$$\Gamma_L^{0,\phi\tau}[A,B] = \Sigma_L^{0,\phi\tau}[A,B] \,. \tag{176}$$

For polarization,

$$W^{\phi,\pm}_{\ell L \ell'} W^{\tau,\pm}_{\ell L \ell'} = -(\zeta^\pm)^2 (p^\pm_{\ell L \ell'})^2 (\gamma_{\ell L \ell'})^2 a^0_L \left[a^+_{\ell'} \begin{pmatrix} \ell & L & \ell' \\ -2 & -1 & 3 \end{pmatrix} + a^-_{\ell'} \begin{pmatrix} \ell & L & \ell' \\ -2 & 1 & 1 \end{pmatrix} \right] \begin{pmatrix} \ell & L & \ell' \\ -2 & 0 & 2 \end{pmatrix} \,. \tag{177}$$

We obtain

$$W^{\phi,\pm}_{\ell L\ell'}W^{\tau,\pm}_{\ell L\ell'} = \mp p^{\pm}_{\ell L\ell'}(\gamma_{\ell L\ell'})^2 a^0_L \left[a^+_{\ell'}\begin{pmatrix}\ell & L & \ell'\\ -2 & -1 & 3\end{pmatrix}\begin{pmatrix}\ell & L & \ell'\\ -2 & 0 & 2\end{pmatrix} + a^-_{\ell'}\begin{pmatrix}\ell & L & \ell'\\ -2 & 1 & 1\end{pmatrix}\begin{pmatrix}\ell & L & \ell'\\ -2 & 0 & 2\end{pmatrix}\right]$$
$$= \int_{-1}^{1}\mathrm{d}\mu\ \frac{\mp 1}{4}(\gamma_{\ell L\ell'})^2 a^0_L \left[a^+_{\ell'}d^\ell_{22}d^L_{-1,0}d^{\ell'}_{32} + a^-_{\ell'}d^\ell_{22}d^L_{10}d^{\ell'}_{12} \pm a^+_{\ell'}d^\ell_{2,-2}d^L_{-1,0}d^{\ell'}_{3,-2} \pm a^-_{\ell'}d^\ell_{2,-2}d^L_{10}d^{\ell'}_{1,-2}\right] , \tag{178}$$

The kernel function is given by

$$\Sigma^{(\pm),\phi\tau}_L[A,B] = \mp\int_{-1}^{1}\mathrm{d}\mu\ \frac{\pi}{2}\sqrt{L(L+1)}d^L_{10}\left[\xi^A_{22}\xi^{B^+}_{32} - \xi^A_{22}\xi^{B^-}_{21} \pm \xi^A_{2,-2}\xi^{B^+}_{3,-2} \pm \xi^A_{2,-2}\xi^{B^-}_{2,-1}\right] , \tag{179}$$

and, using the property of the weight function, we find

$$\Gamma^{(\pm),\phi\tau}_L[A,B] = \Sigma^{(\pm),\phi\tau}_L[A,B] . \tag{180}$$

Then we obtain

$$\Sigma^{0,\phi\tau}_L[A,B] = \int_{-1}^{1}\mathrm{d}\mu\ 2\pi\sqrt{L(L+1)}d^L_{10}\xi^A_{00}\xi^{B^0}_{10} , \tag{181}$$

and

$$\Gamma^{0,\phi\tau}_L[A,B] = \Sigma^{0,\phi\tau}_L[A,B] . \tag{182}$$

For polarization,

$$W^{\phi,\pm}_{\ell L\ell'}W^{\tau,\pm}_{\ell L\ell'} = -(\zeta^\pm)^2(p^{\pm}_{\ell L\ell'})^2(\gamma_{\ell L\ell'})^2 a^0_L \left[a^+_{\ell'}\begin{pmatrix}\ell & L & \ell'\\ -2 & -1 & 3\end{pmatrix} + a^-_{\ell'}\begin{pmatrix}\ell & L & \ell'\\ -2 & 1 & 1\end{pmatrix}\right]\begin{pmatrix}\ell & L & \ell'\\ -2 & 0 & 2\end{pmatrix} . \tag{183}$$

We obtain

$$W^{\phi,\pm}_{\ell L\ell'}W^{\tau,\pm}_{\ell L\ell'} = \mp p^{\pm}_{\ell L\ell'}(\gamma_{\ell L\ell'})^2 a^0_L \left[a^+_{\ell'}\begin{pmatrix}\ell & L & \ell'\\ -2 & -1 & 3\end{pmatrix}\begin{pmatrix}\ell & L & \ell'\\ -2 & 0 & 2\end{pmatrix} + a^-_{\ell'}\begin{pmatrix}\ell & L & \ell'\\ -2 & 1 & 1\end{pmatrix}\begin{pmatrix}\ell & L & \ell'\\ -2 & 0 & 2\end{pmatrix}\right]$$
$$= \int_{-1}^{1}\mathrm{d}\mu\ \frac{\mp 1}{4}(\gamma_{\ell L\ell'})^2 a^0_L \left[a^+_{\ell'}d^\ell_{22}d^L_{-1,0}d^{\ell'}_{32} + a^-_{\ell'}d^\ell_{22}d^L_{10}d^{\ell'}_{12} \pm a^+_{\ell'}d^\ell_{2,-2}d^L_{-1,0}d^{\ell'}_{3,-2} \pm a^-_{\ell'}d^\ell_{2,-2}d^L_{10}d^{\ell'}_{1,-2}\right] , \tag{184}$$

The kernel function is given by

$$\Sigma^{(\pm),\phi\tau}_L[A,B] = \mp\int_{-1}^{1}\mathrm{d}\mu\ \frac{\pi}{2}\sqrt{L(L+1)}d^L_{10}\left[\xi^A_{22}\xi^{B^+}_{32} - \xi^A_{22}\xi^{B^-}_{21} \pm \xi^A_{2,-2}\xi^{B^+}_{3,-2} \pm \xi^A_{2,-2}\xi^{B^-}_{2,-1}\right] , \tag{185}$$

and, using the property of the weight function, we find

$$\Gamma^{(\pm),\phi\tau}_L[A,B] = \Sigma^{(\pm),\phi\tau}_L[A,B] . \tag{186}$$

6.4.2 ϕ and s

The lensing potential and sources are both even. For s, the weight is obtained by replacing $C^{\Theta\Theta}$ in the numerator with $1/2$.

6.4.3 α and τ

The response of the quadratic EB estimator is given by

$$
\begin{aligned}
[A_L^{\alpha\tau,(EB)}]^{-1} &= \frac{1}{2L+1}\sum_{\ell\ell'}\frac{(W^{\alpha,-}_{\ell L\ell'}C^{\rm BB}_{\ell'} - W^{\alpha,-}_{\ell' L\ell}C^{\rm EE}_{\ell})(W^{\tau,+}_{\ell L\ell'}C^{\rm EB}_{\ell'} + W^{\tau,+}_{\ell' L\ell}C^{\rm EB}_{\ell})}{\widehat{C}^{\rm EE}_{\ell}\widehat{C}^{\rm BB}_{\ell'}} \\
&= \frac{-1}{2(2L+1)}\sum_{\ell\ell'}\frac{(W^{\alpha,-}_{\ell L\ell'}C^{\rm BB}_{\ell'} - W^{\alpha,-}_{\ell' L\ell}C^{\rm EE}_{\ell})(W^{\alpha,-}_{\ell L\ell'}C^{\rm EB}_{\ell'} + W^{\alpha,-}_{\ell' L\ell}C^{\rm EB}_{\ell})}{\widehat{C}^{\rm EE}_{\ell}\widehat{C}^{\rm BB}_{\ell'}} \\
&= -\frac{1}{2}\Sigma^{(-),\alpha}_L\left[\frac{1}{\widehat{C}^{\rm EE}},\frac{C^{\rm EB}C^{\rm BB}}{\widehat{C}^{\rm BB}}\right] + \frac{1}{2}\Gamma^{(-),\alpha}_L\left[\frac{C^{\rm EE}}{\widehat{C}^{\rm EE}},\frac{(C^{\rm EB}-C^{\rm BB})}{\widehat{C}^{\rm BB}}\right] + \frac{1}{2}\Sigma^{(-),\alpha}_L\left[\frac{1}{\widehat{C}^{\rm BB}},\frac{C^{\rm EB}C^{\rm EE}}{\widehat{C}^{\rm EE}}\right],
\end{aligned}
\tag{187}
$$

7 Bias-hardened quadratic estimator

7.1 Definition

Assuming that an estimator has several mean-fields, the expectation of the estimator becomes:

$$\langle \widehat{x}_{LM} \rangle_{\rm CMB} = \sum_y R^{xy}_L y_{LM} \equiv \mathbf{R}_L \boldsymbol{y}_{LM} \,, \tag{188}$$

where $R^{xx}_L = 1$ by definition and R^{xy} is the response function. We can construct a bias-hardened estimators as [13]:

$$\widehat{x}^{\rm BH}_{LM} \equiv \sum_y [\mathbf{R}^{-1}]^{xy}_L \widehat{y}_{LM} \,, \tag{189}$$

which is insensitive to the source of mean-field bias:

$$\langle \widehat{x}^{\rm BH}_{LM} \rangle = x_{LM} \,. \tag{190}$$

7.2 Noise

The idealistic reconstruction noise is the diagonal elements of the following matrix:

$$\begin{aligned} \langle \boldsymbol{x}^{\rm BH} (\boldsymbol{x}^{\rm BH})^t \rangle = \mathbf{R}^{-1} \langle \boldsymbol{y}\boldsymbol{y}^t \rangle (\mathbf{R}^{-1})^T = \mathbf{R}^{-1}\mathbf{R} \begin{pmatrix} A^{y_1y_1} & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & A^{y_ny_n} \end{pmatrix} (\mathbf{R}^{-1})^T \\ = \begin{pmatrix} A^{y_1y_1} & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & A^{y_ny_n} \end{pmatrix} (\mathbf{R}^{-1})^T \end{aligned} \tag{191}$$

Thus, we obtain

$$A^{xx,({\rm BH})} = A^{xx} \{\mathbf{R}^{-1}\}_{xx} \tag{192}$$

For two estimator case, the above equation becomes

$$A^{xx,({\rm BH})} = \frac{A^{xx}}{1 - R^{xy}R^{yx}} = \frac{A^{xx}}{1 - A^{xx}A^{yy}(\bar{R}^{xy})^2} \,. \tag{193}$$

Here, $\bar{R}^{xy} = \bar{R}^{yx}$ is the unnormalized response function.

8 Computing delensed CMB anisotropies

8.1 Linear template of lensing B mode

The gradient of lensing potential $\boldsymbol{\nabla}\phi$ is transformed as

$$\begin{aligned}\boldsymbol{\nabla}\phi &= \sum_{\ell m}\boldsymbol{\nabla}Y_{\ell m}\phi_{\ell m} = -\sum_{\ell m}\sqrt{\frac{\ell(\ell+1)}{2}}\phi_{\ell m}(Y^1_{\ell m}\boldsymbol{e}^* - Y^{-1}_{\ell m}\boldsymbol{e}) \\ &= (\phi^+ + \mathrm{i}\phi^-)\boldsymbol{e}^* + (\phi^+ - \mathrm{i}\phi^-)\boldsymbol{e}\,, \end{aligned} \tag{194}$$

where $\phi^\pm$ is obtained by spin-1 inverse harmonic transform of $\phi_{\ell m}\sqrt{\ell(\ell+1)/2}$. Similarly the gradient of polarization $\boldsymbol{\nabla}P^\pm = \boldsymbol{\nabla}(Q \pm \mathrm{i}U)$ is

$$\begin{aligned}\boldsymbol{\nabla}P^+ &= -\sum_{\ell m}E_{\ell m}\boldsymbol{\nabla}Y^2_{\ell m} \\ &= \sum_{\ell m}E_{\ell m}\left(\sqrt{\frac{(\ell-2)(\ell+3)}{2}}Y^3_{\ell m}\boldsymbol{e}^* - \sqrt{\frac{(\ell+2)(\ell-1)}{2}}Y^1_{\ell m}\boldsymbol{e}\right) \\ &= -(E^+_3 + \mathrm{i}E^-_3)\boldsymbol{e}^* + (E^+_1 + \mathrm{i}E^-_1)\boldsymbol{e}\,, \end{aligned} \tag{195}$$

$$\boldsymbol{\nabla}P^- = (\boldsymbol{\nabla}P^+)^* = (E^+_1 - \mathrm{i}E^-_1)\boldsymbol{e}^* - (E^+_3 - \mathrm{i}E^-_3)\boldsymbol{e}\,. \tag{196}$$

This leads to

$$\boldsymbol{\nabla}\phi\cdot\boldsymbol{\nabla}P^+ = -(E^+_3 + \mathrm{i}E^-_3)(\phi^+ - \mathrm{i}\phi^-) + (E^+_1 + \mathrm{i}E^-_1)(\phi^+ + \mathrm{i}\phi^-) \tag{197}$$

$$\boldsymbol{\nabla}\phi\cdot\boldsymbol{\nabla}P^- = (\boldsymbol{\nabla}\phi\cdot\boldsymbol{\nabla}P^+)^*\,. \tag{198}$$

The harmonic transform of the above quantity becomes the leading-order lensing contributions to E/B.

9 Optimal filtering

9.1 Background

The inverse variance Wiener filtering is defined as

$$\left[\mathbf{C}^{-1}+\sum_k \mathbf{A}_k^\dagger \mathbf{N}_k^{-1}\mathbf{A}_k\right]\boldsymbol{x}=\sum_k \mathbf{A}_k^\dagger \mathbf{N}^{-1}\boldsymbol{d}_k\,, \tag{199}$$

where k is the index of frequency channels and different maps (e.g. LAT and SAT for SO), $\mathbf{C}$ is the signal covariance matrix, $\mathbf{N}_k$ is the noise covariance matrix in pixel space, $\mathbf{A}_k$ is a matrix that transforms the harmonic coefficients to a map in pixel space including beam and pixel convolution. From the data, $\boldsymbol{d}_k$, we solve $\boldsymbol{x}$ which is an array of the harmonic coefficients. The above equation is rewritten by the following numerically convenient form:

$$\left[1+\mathbf{C}^{1/2}\left(\sum_k \mathbf{Y}_k^\dagger \mathbf{N}_k^{-1}\mathbf{Y}_k\right)\mathbf{C}^{1/2}\right](\mathbf{C}^{-1/2}\boldsymbol{x})=\mathbf{C}^{1/2}\sum_k \mathbf{Y}_k^\dagger \mathbf{N}_k^{-1}\boldsymbol{d}_k\,, \tag{200}$$

where $(\mathbf{C}^{1/2})^2=\mathbf{C}$. Using the spherical harmonics, $Y_{\ell m}$, we define

$$\mathbf{Y}_k\boldsymbol{x}=\sum_\ell\sum_{m=-\ell}^{\ell} b_\ell^k x_{\ell m}Y_{\ell m}(\hat{\boldsymbol{n}})\,. \tag{201}$$

Here, b_ℓ^k is the one dimensional beam and pixel function, and $\hat{\boldsymbol{n}}_i$ denotes pixel position. Similarly,

$$\mathbf{Y}_k^\dagger\boldsymbol{x}=b_\ell^k\int \mathrm{d}^2\hat{\boldsymbol{n}}\; x(\hat{\boldsymbol{n}})Y_{\ell m}^*(\hat{\boldsymbol{n}})\,. \tag{202}$$

The operation involving the noise covariance is then becomes

$$\{\mathbf{Y}_k^\dagger\mathbf{N}_k^{-1}\mathbf{Y}_k\boldsymbol{x}\}_{\ell' m'}=\int \mathrm{d}^2\hat{\boldsymbol{n}}_j\; b_{\ell'}^k Y_{\ell' m'}^*(\hat{\boldsymbol{n}}_j)\int \mathrm{d}^2\hat{\boldsymbol{n}}_i\;\mathbf{N}^{-1}(\hat{\boldsymbol{n}}_i,\hat{\boldsymbol{n}}_j)\sum_{\ell m} b_\ell^k Y_{\ell m}(\hat{\boldsymbol{n}}_i)x_{\ell m}\,. \tag{203}$$

If the noise covariance is diagonal in pixel space and the signal matrix is diagonal in harmonic space, the matrix multiplication to an array of the harmonic coefficients becomes very simple. The conjugate gradient decent in the code solves $\boldsymbol{v}$ which satisfies

$$\mathbf{A}\boldsymbol{v}=\boldsymbol{b}\,, \tag{204}$$

where

$$\mathbf{A}=\left[1+\mathbf{C}^{1/2}\left(\sum_k \mathbf{Y}_k^\dagger \mathbf{N}_k^{-1}\mathbf{Y}_k\right)\mathbf{C}^{1/2}\right]\,, \tag{205}$$

$$\boldsymbol{b}=\mathbf{C}^{1/2}\sum_k \mathbf{Y}_k^\dagger \mathbf{N}_k^{-1}\boldsymbol{d}_k\,. \tag{206}$$

The solution, $\boldsymbol{v}$, is then transformed to $\boldsymbol{x}$.

9.2 Inverse noise covariance

If the noise covariance in pixel space is diagonal,

$$\{\mathbf{N}\}_{ij}\equiv\langle n(\hat{\boldsymbol{n}}_i)n(\hat{\boldsymbol{n}}_j)\rangle=\delta(\hat{\boldsymbol{n}}_i-\hat{\boldsymbol{n}}_j)\sigma^2(\hat{\boldsymbol{n}}_i)\,, \tag{207}$$

we obtain

$$\{\mathbf{Y}^\dagger\mathbf{N}^{-1}\mathbf{Y}\boldsymbol{x}\}_{\ell'm'} = \int \mathrm{d}^2\hat{\boldsymbol{n}}_j\ Y^*_{\ell'm'}(\hat{\boldsymbol{n}}_j) \int \mathrm{d}^2\hat{\boldsymbol{n}}_i\ \sigma^2(\hat{\boldsymbol{n}}_i)\delta(\hat{\boldsymbol{n}}_i - \hat{\boldsymbol{n}}_j)\sum_{\ell m} Y_{\ell m}(\hat{\boldsymbol{n}}_i)x_{\ell m}$$
$$= \int \mathrm{d}^2\hat{\boldsymbol{n}}_j\ Y^*_{\ell'm'}(\hat{\boldsymbol{n}}_j)\sigma^2(\hat{\boldsymbol{n}}_j)\sum_{\ell m} Y_{\ell m}(\hat{\boldsymbol{n}}_j)x_{\ell m}\,, \tag{208}$$

where we ignore signal and beam. This operation is very efficient.

For a white uniform noise with σ [μK'], the noise covariance in pixel space becomes

$$\{\mathbf{N}\}_{ij} = \delta(\hat{\boldsymbol{n}}_i - \hat{\boldsymbol{n}}_j)\left(\frac{\sigma}{T_{\rm CMB}}\frac{\pi}{10800}\right)^2 \equiv \delta(\hat{\boldsymbol{n}}_i - \hat{\boldsymbol{n}}_j)N^{\rm white}\,. \tag{209}$$

Then, the above filtering is equivalent to the usual diagonal filtering:

$$\{\mathbf{A}\}_{\ell m,\ell'm'} = \delta_{\ell\ell'}\delta_{mm'}\left[1 + \frac{C_\ell}{N_\ell}\right]\,, \tag{210}$$

$$\{\boldsymbol{b}\}_{\ell m} = \frac{C_\ell^{1/2}}{N_\ell}(s_{\ell m} + n^b_{\ell m})\,, \tag{211}$$

where C_ℓ is the beam-deconvolved signal spectrum, $N_\ell = N^{\rm white}/b_\ell^2$, $s_{\ell m}$ is the signal and $n^b_{\ell m} = n_{\ell m}/b_\ell$ is the noise divided by beam. Substituting the above equations into Eq. (204), we obtain

$$x_{\ell m} = C_\ell^{1/2}v_{\ell m} = \frac{C_\ell}{C_\ell + N_\ell}(s_{\ell m} + n^b_{\ell m})\,. \tag{212}$$

The noise variance from some simulated noise is given by

$$\{\mathbf{N}\}_{ij} = W^{-1/2}(\hat{\boldsymbol{n}}_i)W^{-1/2}(\hat{\boldsymbol{n}}_j)\sum_{\ell m\ell'm'} Y_{\ell m}(\hat{\boldsymbol{n}}_i)Y_{\ell'm'}(\hat{\boldsymbol{n}}_j)\langle n_{\ell m}n_{\ell'm'}\rangle\,, \tag{213}$$

where W^{-1} represents inhomogeneities of scan. For a uniform noise with $\langle n_{\ell m}n_{\ell'm'}\rangle = \sigma_0^2\delta_{\ell\ell'}\delta_{mm'}$, the covariance is diagonal and we obtain

$$\{\mathbf{N}\}_{ii} = \frac{\sigma_0^2}{4\pi}\sum_{\ell=\ell_{\rm min}}^{\ell_{\rm max}}(2\ell+1) = \sigma_0^2\frac{(\ell_{\rm max}-\ell_{\rm min})(\ell_{\rm max}+\ell_{\rm min}+2)}{4\pi}\,. \tag{214}$$

Therefore, it is possible to construct an approximate noise covariance from simulation if $\langle n_{\ell m}n_{\ell'm'}\rangle \sim N_\ell\delta_{\ell\ell'}\delta_{mm'}$ and $N_\ell \sim \sigma_0^2$:

$$\sigma^2(\hat{\boldsymbol{n}}) \equiv \frac{4\pi\{\mathbf{N}\}_{ii}}{(\ell_{\rm max}-\ell_{\rm min})(\ell_{\rm max}+\ell_{\rm min}+2)}\,. \tag{215}$$

9.3 Preconditioner for the Conjugate Gradient Decent Algorithm

To solve the above equation, we use the preconditioned conjugate gradient decent algorithm. An appropriate preconditioner is essential to solve the equation efficiently. A simple way is to choose the following diagonal preconditioner:

$$\{\mathbf{M}\}_{(\ell m),(\ell m)} = 1 + \sum_k \frac{(b^k_\ell)^2C_\ell}{2\ell+1}\sum_m\int \mathrm{d}\hat{\boldsymbol{n}}_i\ Y^*_{\ell m}(\hat{\boldsymbol{n}}_i)N_k^{-1}(\hat{\boldsymbol{n}}_i)Y_{\ell m}(\hat{\boldsymbol{n}}_i)$$
$$= 1 + \sum_k \frac{(b^k_\ell)^2C_\ell}{4\pi}\int \mathrm{d}\hat{\boldsymbol{n}}_i\ N_k^{-1}(\hat{\boldsymbol{n}}_i)\,. \tag{216}$$

and zero otherwise. For a given σ_k in unit of μK' and a binary mask, $W_k(\hat{\boldsymbol{n}}_i)$, we obtain

$$N_k^{-1}(\hat{\boldsymbol{n}}_i) = W_k(\hat{\boldsymbol{n}}_i)\left[\frac{\sigma_k}{T_{CMB}} \times \frac{\pi}{10800}\right]^{-2} \tag{217}$$

Then, we find

$$\{\mathbf{M}\}_{(\ell m),(\ell m)} = 1 + \sum_k \frac{(b_\ell^k)^2 C_\ell}{4\pi}\left[\frac{\sigma_k}{T_{CMB}} \times \frac{\pi}{10800}\right]^{-2} \sum_i \frac{4\pi}{N_{\rm pix}} W_k(\hat{\boldsymbol{n}}_i)\,. \tag{218}$$

Another way is to split the preconditioner at some scale, $\ell = \ell_{\rm sp}$ and use different preconditioner to these scales. This is motivated by the fact that, for a low resolution map, or if enough computational memory is available, the dense inverse matrix up to $\ell_{\rm sp}$ can be saved. In this case, for the lower multipole, $\ell \leq \ell_{\rm sp}$, the dense inverse matrix is used for the preconditioner while the above approximate diagonal matrix is used for the preconditioner.

This approach is further extended to the multigrid preconditioner. In the multigrid method, we compute the preconditioner at $\ell \leq \ell_{\rm sp}$ from a lower resolution map, while the preconditioner at $\ell > \ell_{\rm sp}$ is given by the above diagonal matrix. At the lower resolution map, the preconditioner is obtained in the same way. By repeating this procedure, at the coarsest map, the preconditioner at $\ell \leq \ell_{\rm sp}$ is obtained by inverting the exact dense matrix.

The dense preconditioning matrix is obtained by substituting $a_{\ell m} = \delta_{\ell\ell_0}\delta_{mm_0}$ for $0 \leq \ell_0 \leq \ell_{\rm sp}$ and $0 \leq m_0 \leq \ell_0$ to the function:

$$a'_{\ell m} = \sum_{\ell' m'} \mathbf{A}_{\ell m, \ell' m'} a_{\ell' m'}\,. \tag{219}$$

Note that the spherical harmonic transform code allows $m \geq 0$ and the above operation gives:

$$a'_{\ell m} = a_{\ell_0 m_0} + \int \mathrm{d}^2\hat{\boldsymbol{n}}\; Y^*_{\ell m}(Y_{\ell_0 m_0} + Y^*_{\ell_0 m_0})N^{-1}\,. \tag{220}$$

We also substitute $a_{\ell m} = \mathrm{i}\delta_{\ell\ell_0}\delta_{mm_0}$ to obtain

$$a''_{\ell m} = a_{\ell_0 m_0} + \mathrm{i}\int \mathrm{d}^2\hat{\boldsymbol{n}}\; Y^*_{\ell m}(Y_{\ell_0 m_0} - Y^*_{\ell_0 m_0})N^{-1}\,. \tag{221}$$

Then, we obtain the matrix element as

$$\mathbf{A}_{\ell m, \ell_0 m_0} = \frac{a'_{\ell m} - \mathrm{i}a''_{\ell m}}{2}\,. \tag{222}$$

10 Skew-spectrum

10.1 Definition

The skewness relevant to the Minkowski functionals is given by

$$\begin{aligned}
S^0(\hat{\boldsymbol{n}}) &\equiv \langle \kappa^3(\hat{\boldsymbol{n}}) \rangle \,, \\
S^1(\hat{\boldsymbol{n}}) &\equiv -3 \langle \kappa^2(\hat{\boldsymbol{n}}) \boldsymbol{\nabla}^2 \kappa(\hat{\boldsymbol{n}}) \rangle \,, \\
S^2(\hat{\boldsymbol{n}}) &\equiv -6 \langle |\boldsymbol{\nabla}\kappa(\hat{\boldsymbol{n}})|^2 \boldsymbol{\nabla}^2 \kappa(\hat{\boldsymbol{n}}) \rangle \,.
\end{aligned} \tag{223}$$

From the above quantities, we obtain

$$\begin{aligned}
\overline{S}^0 &= \int d^2\hat{\boldsymbol{n}}\; S^0(\hat{\boldsymbol{n}}) = \int d^2\hat{\boldsymbol{n}} \sum_{\ell_i m_i} Y_{\ell_1 m_1} Y_{\ell_2 m_2} Y_{\ell_3 m_3} \langle \kappa_{\ell_1 m_1} \kappa_{\ell_2 m_2} \kappa_{\ell_3 m_3} \rangle \\
&= \int d^2\hat{\boldsymbol{n}} \sum_{\ell_i m_i} Y_{\ell_1 m_1} Y_{\ell_2 m_2} Y_{\ell_3 m_3} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix} h_{\ell_1 \ell_2 \ell_3} b_{\ell_1 \ell_2 \ell_3} \\
&= \sum_{\ell_i m_i} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix}^2 h^2_{\ell_1 \ell_2 \ell_3} b_{\ell_1 \ell_2 \ell_3} \\
&= \sum_{\ell_i} h^2_{\ell_1 \ell_2 \ell_3} b_{\ell_1 \ell_2 \ell_3} \,,
\end{aligned} \tag{224}$$

$$\begin{aligned}
\overline{S}^1 &= \int d^2\hat{\boldsymbol{n}}\; S^1(\hat{\boldsymbol{n}}) = 3 \int d^2\hat{\boldsymbol{n}} \sum_{\ell_i m_i} Y_{\ell_1 m_1} Y_{\ell_2 m_2} Y_{\ell_3 m_3} \ell_3(\ell_3+1) \langle \kappa_{\ell_1 m_1} \kappa_{\ell_2 m_2} \kappa_{\ell_3 m_3} \rangle \\
&= 3 \sum_{\ell_i m_i} \ell_3(\ell_3+1) \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix}^2 h^2_{\ell_1 \ell_2 \ell_3} b_{\ell_1 \ell_2 \ell_3} \\
&= \sum_{\ell_i} [\ell_1(\ell_1+1) + \ell_2(\ell_2+1) + \ell_3(\ell_3+1)] h^2_{\ell_1 \ell_2 \ell_3} b_{\ell_1 \ell_2 \ell_3} \,,
\end{aligned} \tag{225}$$

$$\begin{aligned}
\overline{S}^2 &= \int d^2\hat{\boldsymbol{n}}\; S^2(\hat{\boldsymbol{n}}) = 6 \int d^2\hat{\boldsymbol{n}} \sum_{\ell_i m_i} \boldsymbol{\nabla} Y_{\ell_1 m_1} \boldsymbol{\nabla} Y_{\ell_2 m_2} \boldsymbol{\nabla}^2 Y_{\ell_3 m_3} \ell_3(\ell_3+1) \langle \kappa_{\ell_1 m_1} \kappa_{\ell_2 m_2} \kappa_{\ell_3 m_3} \rangle \\
&= 3 \sum_{\ell_i m_i} \ell_3(\ell_3+1) [\ell_1(\ell_1+1) + \ell_2(\ell_2+1) - \ell_3(\ell_3+1)] \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix}^2 h^2_{\ell_1 \ell_2 \ell_3} b_{\ell_1 \ell_2 \ell_3} \\
&= \sum_{\ell_i} \{ \ell_3(\ell_3+1) [\ell_1(\ell_1+1) + \ell_2(\ell_2+1) - \ell_3(\ell_3+1)] + \text{cyc. perm.} \} h^2_{\ell_1 \ell_2 \ell_3} b_{\ell_1 \ell_2 \ell_3} \,.
\end{aligned} \tag{226}$$

Here, we use

$$\begin{aligned}
I &\equiv \int d^2\hat{\boldsymbol{n}}\; \boldsymbol{\nabla} Y_{\ell_1 m_1} \boldsymbol{\nabla} Y_{\ell_2 m_2} Y_{\ell_3 m_3} \\
&= \ell_2(\ell_2+1) \int d^2\hat{\boldsymbol{n}}\; Y_{\ell_1 m_1} Y_{\ell_2 m_2} Y_{\ell_3 m_3} - \int d^2\hat{\boldsymbol{n}}\; Y_{\ell_1 m_1} \boldsymbol{\nabla} Y_{\ell_2 m_2} \boldsymbol{\nabla} Y_{\ell_3 m_3} \\
&= [\ell_2(\ell_2+1) - \ell_3(\ell_3+1)] \int d^2\hat{\boldsymbol{n}}\; Y_{\ell_1 m_1} Y_{\ell_2 m_2} Y_{\ell_3 m_3} + \int d^2\hat{\boldsymbol{n}}\; \boldsymbol{\nabla} Y_{\ell_1 m_1} Y_{\ell_2 m_2} \boldsymbol{\nabla} Y_{\ell_3 m_3} \\
&= [\ell_2(\ell_2+1) - \ell_3(\ell_3+1) + \ell_1(\ell_1+1)] \int d^2\hat{\boldsymbol{n}}\; Y_{\ell_1 m_1} Y_{\ell_2 m_2} Y_{\ell_3 m_3} - I \,.
\end{aligned} \tag{227}$$

10.2 Spectrum

The skew spectra are defined as

$$S_\ell^{(0)} = \frac{1}{2\ell+1}\sum_m \kappa_{\ell m}(\kappa^2)^*_{\ell m} \tag{228}$$

$$S_\ell^{(1)} = \frac{-3}{2\ell+1}\sum_m (\boldsymbol{\nabla}^2\kappa)_{\ell m}(\kappa^2)^*_{\ell m} \tag{229}$$

$$S_\ell^{(2)} = \frac{-6}{2\ell+1}\sum_m (\boldsymbol{\nabla}\kappa\cdot\boldsymbol{\nabla}\kappa)_{\ell m}(\boldsymbol{\nabla}^2\kappa)^*_{\ell m}\,. \tag{230}$$

The expectation values become

$$\langle S_\ell^{(0)}\rangle = \frac{1}{2\ell+1}\sum_m \int \mathrm{d}^2\hat{\boldsymbol{n}} \sum_{\ell_1 m_1 \ell_2 m_2} Y_{\ell m}(\hat{\boldsymbol{n}})Y_{\ell_1 m_1}(\hat{\boldsymbol{n}})Y_{\ell_2 m_2}(\hat{\boldsymbol{n}})\langle \kappa_{\ell m}\kappa_{\ell_1 m_1}\kappa_{\ell_2 m_2}\rangle \tag{231}$$

$$= \frac{1}{2\ell+1}\sum_{\ell_1\ell_2} h^2_{\ell\ell_1\ell_2} b_{\ell\ell_1\ell_2}\,, \tag{232}$$

$$\langle S_\ell^{(1)}\rangle = \frac{3[\ell(\ell+1)]}{2\ell+1}\sum_m \int \mathrm{d}^2\hat{\boldsymbol{n}} \sum_{\ell_1 m_1 \ell_2 m_2} Y_{\ell m}(\hat{\boldsymbol{n}})Y_{\ell_1 m_1}(\hat{\boldsymbol{n}})Y_{\ell_2 m_2}(\hat{\boldsymbol{n}})\langle \kappa_{\ell m}\kappa_{\ell_1 m_1}\kappa_{\ell_2 m_2}\rangle \tag{233}$$

$$= \frac{3[\ell(\ell+1)]}{2\ell+1}\sum_{\ell_1\ell_2} h^2_{\ell\ell_1\ell_2} b_{\ell\ell_1\ell_2}\,, \tag{234}$$

$$\langle S_\ell^{(2)}\rangle = \frac{6[\ell(\ell+1)]}{2\ell+1}\sum_m \int \mathrm{d}^2\hat{\boldsymbol{n}} \sum_{\ell_1 m_1 \ell_2 m_2} Y_{\ell m}(\hat{\boldsymbol{n}})\boldsymbol{\nabla}Y_{\ell_1 m_1}(\hat{\boldsymbol{n}})\boldsymbol{\nabla}Y_{\ell_2 m_2}(\hat{\boldsymbol{n}})\langle \kappa_{\ell m}\kappa_{\ell_1 m_1}\kappa_{\ell_2 m_2}\rangle \tag{235}$$

$$= \frac{3[\ell(\ell+1)]}{2\ell+1}\sum_{\ell_1\ell_2}[-\ell(\ell+1)+\ell_1(\ell_1+1)+\ell_2(\ell_2+1)]h^2_{\ell\ell_1\ell_2} b_{\ell\ell_1\ell_2}\,. \tag{236}$$

The skew spectra, S_ℓ^i, satisfy

$$\overline{S}^i \equiv \sum_\ell (2\ell+1)S_\ell^i\,. \tag{237}$$

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