Computing quadratic estimator, delensing in curvedsky

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Abstract

Here, I describe an algorithm for computing the quadratic estimator normalization of the lensing, cosmic bi-refringence, patchy reionization, and so on.

Contents

1 Notations

In the followings, we use small letters for multipoles of the CMB anisotropies (e.g., ℓ), while large letters are used for multipoles of the distortion fields (lensing, rotation, etc).

1.1 CMB

Θ denotes the CMB temperature fluctuations, and Q and U denote the Stokes parameters of the CMB linear polarization. The following equation defines the harmonic coefficients of the temperature anisotropies (and, in general, any scalar quantities x):

$$x_{LM} = \int \mathrm{d}^2\hat{\boldsymbol{n}}\; Y^*_{LM}(\hat{\boldsymbol{n}})x(\hat{\boldsymbol{n}})\,. \tag{1}$$

where Y_{LM} is the spin-0 spherical harmonics. On the other hand, Q and U are changed by the rotation of the sphere, and are therefore usually transformed into the rotational invariant quantities, the E and B modes, as

$$[E \pm \mathrm{i}B]_{\ell m} = \int \mathrm{d}^2\hat{\boldsymbol{n}}\; (Y^{\pm 2}_{\ell m})^*(\hat{\boldsymbol{n}})[Q \pm \mathrm{i}U](\hat{\boldsymbol{n}})\,. \tag{2}$$

Here, $Y^{\pm 2}_{\ell m}$ is the spin-2 spherical harmonics. For short notation, we also use

$$\begin{aligned} \Xi^{\pm} &= E \pm \mathrm{i}B\,, \\ P^{\pm} &= Q \pm \mathrm{i}U \end{aligned} \tag{3}$$

1.2 Lensing

The lensing effect on CMB anisotropies is described as remapping of the unlensed CMB anisotropies by the deflection angle [1, 2]

$$X(\hat{\boldsymbol{n}}) = X(\hat{\boldsymbol{n}} + \boldsymbol{d})\,, \tag{4}$$

where X is Θ or $P^{\pm}$. The deflection angle of the CMB lensing is decomposed into the lensing potential, ϕ, and curl mode, ϖ, as [3]

$$\boldsymbol{d} = \boldsymbol{\nabla}\phi + \boldsymbol{\Delta}\varpi\,, \tag{5}$$

where the operator $\boldsymbol{\Delta} = \star\boldsymbol{\nabla}$ denotes the derivatives with 90° rotation counterclockwise on the plane perpendicular to the line-of-sight direction and then operation. The harmonic coefficients of ϕ and ϖ are given by Eq. (1). The remapping of the CMB anisotropies is then given by

$$X(\hat{\boldsymbol{n}}) = X(\hat{\boldsymbol{n}}) + [\boldsymbol{\nabla}\phi + \boldsymbol{\Delta}\varpi]\cdot\boldsymbol{\nabla}X + \mathcal{O}(\phi^2, \varpi^2)\,. \tag{6}$$

1.3 Rotation

If the rotation angle is small, the modulation of polarization after a rotation by an angle α is given by (e.g. [4])

$$\delta P^{\pm} = \pm 2\alpha P^{\pm}\,. \tag{7}$$

The harmonic coefficients of α is given by Eq. (1).

1.4 Inhomogeneous Reionization

The inhomogeneities of the reionization could vary the optical depth τ across the CMB sky. If the spatial variation of τ is very small, this leads to the modulation in CMB temperature and polarization as (e.g. [5])

$$\Theta \to \Theta + \tau\Theta \,.P^{\pm} \to P^{\pm} + \tau P^{\pm} \,. \tag{8}$$

The harmonic coefficients of τ is given by Eq. (1).

1.5 Spherical Harmonics and Wigner-3j

The spherical harmonics is related to the Wigner-3j symbols as [6]

$$\int d^2\hat{\boldsymbol{n}}\; Y^{s_1}_{\ell_1 m_1} Y^{s_2}_{\ell_2 m_2} Y^{s_3}_{\ell_3 m_3} = \sqrt{\frac{(2\ell_1+1)(2\ell_2+1)(2\ell_3+1)}{4\pi}} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ -s_1 & -s_2 & -s_3 \end{pmatrix} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \,. \tag{9}$$

1.6 Derivatives of Spherical Harmonics

In general, denoting $a^s_\ell = \sqrt{(\ell-s)(\ell+s)/2}$, the derivative of the spherical harmonics is given by [6]

$$\boldsymbol{\nabla} Y^s_{\ell m} = a^s_\ell Y^{s+1}_{\ell m}\boldsymbol{e}^* - a^{-s}_\ell Y^{s-1}_{\ell m}\boldsymbol{e} \,. \tag{10}$$

Here, we introduce the polarization vector $\boldsymbol{e}$ which are defined

$$\boldsymbol{e} = \frac{\boldsymbol{e}_1 + i\boldsymbol{e}_2}{\sqrt{2}} \tag{11}$$

with $\boldsymbol{e}_i$ denoting the basis vectors orthogonal to the radial vector. The polarization vector satisfies $\boldsymbol{e}\cdot\boldsymbol{e} = 0$, $\boldsymbol{e}\cdot\boldsymbol{e}^* = 1$, $\star\boldsymbol{e} = -i\boldsymbol{e}$. In particular, for $s = 0$,

$$\boldsymbol{\nabla} Y_{\ell m} = a^0_\ell \big(Y^1_{\ell m}\boldsymbol{e}^* - Y^{-1}_{\ell m}\boldsymbol{e} \big) \,, \tag{12}$$

and, for $s = \pm 2$, denoting $a^{\pm} = a^{\pm 2}_\ell$,

$$\begin{aligned} \boldsymbol{\nabla} Y^2_{\ell m} &= a^+_\ell Y^3_{\ell m}\boldsymbol{e}^* - a^-_\ell Y^1_{\ell m}\boldsymbol{e} \,, \\ \boldsymbol{\nabla} Y^{-2}_{\ell m} &= a^-_\ell Y^{-1}_{\ell m}\boldsymbol{e}^* - a^+_\ell Y^{-3}_{\ell m}\boldsymbol{e} \,. \end{aligned} \tag{13}$$

1.7 Map derivatives

Derivative of scalar quantities such as the CMB temperature fluctuations and lensing potential is

$$\boldsymbol{\nabla} x = \sum_{LM} x_{LM}\boldsymbol{\nabla} Y_{LM} = \sum_{LM} x_{LM} a^0_L \left(Y^1_{LM}\boldsymbol{e}^* - Y^{-1}_{LM}\boldsymbol{e} \right) = x^+\boldsymbol{e}^* - x^-\boldsymbol{e} \,. \tag{14}$$

where we define

$$x^{\pm} \equiv \sum_{LM} x_{LM} a^0_L Y^{\pm 1}_{LM} \,, \tag{15}$$

and $(x^+)^* = -x^-$. The rotation of a pseudo-scalar quantity is given by

$$\boldsymbol{\Delta}\varpi = \sum_{LM} \varpi_{LM}\, \boldsymbol{\Delta}\, Y_{LM} = \sum_{LM} \varpi_{LM} a^0_L i \left(Y^1_{LM}\boldsymbol{e}^* + Y^{-1}_{LM}\boldsymbol{e} \right) = i(\varpi^+\boldsymbol{e}^* + \varpi^-\boldsymbol{e}) \,, \tag{16}$$

and $(\varpi^+)^* = -\varpi^-$. Spin-2 fields such as the CMB linear polarization is given by

$$\boldsymbol{\nabla} P^+ = \sum_{\ell m} \Xi^+_{\ell m}\boldsymbol{\nabla} Y^2_{\ell m} = \sum_{\ell m} \Xi^+_{\ell m} \left(a^+_\ell Y^3_{\ell m}\boldsymbol{e}^* - a^-_\ell Y^1_{\ell m}\boldsymbol{e} \right) = \Xi^{+^+}\boldsymbol{e}^* - \Xi^{+^-}\boldsymbol{e} \,, \tag{17}$$

$$\boldsymbol{\nabla} P^- = (\boldsymbol{\nabla} P^+)^* = \sum_{\ell m} \Xi^-_{\ell m}\boldsymbol{\nabla} Y^{-2}_{\ell m} = \sum_{\ell m} \Xi^-_{\ell m} \left(a^-_\ell Y^{-1}_{\ell m}\boldsymbol{e}^* - a^+_\ell Y^{-3}_{\ell m}\boldsymbol{e} \right) = \Xi^{-^+}\boldsymbol{e}^* - \Xi^{-^-}\boldsymbol{e} \,. \tag{18}$$

Note that $(\Xi^{+^+})^* = -\Xi^{-^-}$ and $(\Xi^{+^-})^* = -\Xi^{-^+}$.

2 Distortion of CMB anisotropies

In the following, we first define useful quantities to compute the distortion effect. The parity symmetry indicator is given by

$$p^{\pm}_{\ell_1\ell_2\ell_3} \equiv \frac{1 \pm (-1)^{\ell_1+\ell_2+\ell_3}}{2} \,, \tag{19}$$

$$\tag{20}$$

An even (odd) parity quantity contains p^+ (p^-). A multipole factor is defined as

$$\gamma_{\ell_1\ell_2\ell_3} \equiv \sqrt{\frac{(2\ell_1+1)(2\ell_2+1)(2\ell_3+1)}{4\pi}} \,. \tag{21}$$

The convolution operator in full sky is defined as

$$\widetilde{\sum_{LM\ell'm'}}^{(\ell m)} \equiv \sum_{LM\ell'm'} (-1)^m \begin{pmatrix} \ell & L & \ell' \\ -m & M & m' \end{pmatrix} \,. \tag{22}$$

2.1 Lensing distortion

The lensing contributions in the position space become

$$\begin{aligned} \delta^\phi\Theta &= \boldsymbol{\nabla}\phi\cdot\boldsymbol{\nabla}\Theta = -\phi^-\Theta^+ - \phi^+\Theta^- \,, \\ \delta^\varpi\Theta &= \boldsymbol{\Delta}\varpi\cdot\boldsymbol{\nabla}\Theta = \mathrm{i}(\varpi^-\Theta^+ - \varpi^+\Theta^-) \,, \\ \delta^\phi P^\pm &= \boldsymbol{\nabla}\phi\cdot\boldsymbol{\nabla}P^\pm = -\phi^-\Xi^{\pm^+} - \phi^+\Xi^{\pm^-} \,, \\ \delta^\varpi P^\pm &= \boldsymbol{\Delta}\varpi\cdot\boldsymbol{\nabla}P^\pm = \mathrm{i}(\varpi^-\Xi^{\pm^+} - \varpi^+\Xi^{\pm^-}) \,. \end{aligned} \tag{23}$$

2.1.1 Lens distortion in harmonic space: Temperature

The harmonics transform of the lensing contributions is

$$\begin{aligned} \delta^\phi\Theta_{\ell m} &= -\int \mathrm{d}^2\hat{\boldsymbol{n}}\; Y^*_{\ell m}[\phi^-\Theta^+ + \phi^+\Theta^-] \\ &= -\sum_{LM\ell'm'} \phi_{LM}\Theta_{\ell'm'} a^0_L a^0_{\ell'} \int \mathrm{d}^2\hat{\boldsymbol{n}}\; (-1)^m Y_{\ell,-m}[Y^{-1}_{LM}Y^1_{\ell'm'} + Y^1_{LM}Y^{-1}_{\ell'm'}] \\ &= -\sum_{LM\ell'm'} \phi_{LM}\Theta_{\ell'm'} 2a^0_L a^0_{\ell'} p^+_{\ell L\ell'}\gamma_{\ell L\ell'}(-1)^m \begin{pmatrix} \ell & L & \ell' \\ -m & M & m' \end{pmatrix} \begin{pmatrix} \ell & L & \ell' \\ 0 & 1 & -1 \end{pmatrix} \\ &= -\widetilde{\sum_{LM\ell'm'}}^{(\ell m)} \phi_{LM}\Theta_{\ell'm'} 2a^0_L a^0_{\ell'} p^+_{\ell L\ell'}\gamma_{\ell L\ell'} \begin{pmatrix} \ell & L & \ell' \\ 0 & 1 & -1 \end{pmatrix} \\ &= \widetilde{\sum_{LM\ell'm'}}^{(\ell m)} \phi_{LM}\Theta_{\ell'm'} W^{\phi,0}_{\ell L\ell'} \,. \end{aligned} \tag{24}$$

Here we introduce coefficients $c_\phi = 1$ and $c_\varpi = -\mathrm{i}$, and denote

$$W^{\phi,0}_{\ell_1\ell_2\ell_3} = -2c_\phi a^0_{\ell_2} a^0_{\ell_3} p^+_{\ell_1\ell_2\ell_3}\gamma_{\ell_1\ell_2\ell_3} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ 0 & 1 & -1 \end{pmatrix} \,. \tag{25}$$

Note that $(W^{\phi,0}_{\ell_1\ell_2\ell_3})^* = W^{\phi,0}_{\ell_1\ell_2\ell_3}$.

On the other hand, for curl mode,

$$\begin{aligned}\delta^{\varpi}\Theta_{\ell m} &= \mathrm{i}\int \mathrm{d}^2\hat{\boldsymbol{n}}\; Y^*_{\ell m}[\varpi^-\Theta^+ - \varpi^+\Theta^-] \\ &= \sum_{LM\ell'm'} \varpi_{LM}\Theta_{\ell'm'} 2\mathrm{i}a^0_L a^0_{\ell'} p^-_{\ell L\ell'}\gamma_{\ell L\ell'}(-1)^m \begin{pmatrix} \ell & L & \ell' \\ -m & M & m' \end{pmatrix}\begin{pmatrix} \ell & L & \ell' \\ 0 & 1 & -1 \end{pmatrix} \\ &= \widetilde{\sum_{LM\ell'm'}}^{(\ell m)} \varpi_{LM}\Theta_{\ell'm'} W^{\varpi,0}_{\ell L\ell'}\,, \end{aligned} \tag{26}$$

with

$$W^{\varpi,0}_{\ell_1\ell_2\ell_3} = -2c_{\varpi}a^0_{\ell_2}a^0_{\ell_3}p^-_{\ell_1\ell_2\ell_3}\gamma_{\ell_1\ell_2\ell_3}\begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ 0 & 1 & -1 \end{pmatrix}\,. \tag{27}$$

Note that the above quantity is consistent with Ref. [7] and also $(W^{\varpi,0}_{\ell_1\ell_2\ell_3})^* = (-1)^{\ell_1+\ell_2+\ell_3}W^{\varpi,0}_{\ell_1\ell_2\ell_3}$ which is consistent with (27).

2.1.2 Lens distortion in harmonic space: Polarization

The lensed anisotropies for polarizations are given by

$$\begin{aligned}\delta^{\phi}\Xi^{\pm}_{\ell m} &= -\int \mathrm{d}^2\hat{\boldsymbol{n}}\;(Y^{\pm 2}_{\ell m})^*[\phi^-\Xi^{\pm^+} + \phi^+\Xi^{\pm^-}] \\ &= -\sum_{LM\ell'm'}\phi_{LM}\Xi^{\pm}_{\ell'm'}a^0_L\int \mathrm{d}^2\hat{\boldsymbol{n}}\;(Y^{\pm 2}_{\ell m})^*[a^+_{\ell'}Y^{\mp 1}_{LM}Y^{\pm 3}_{\ell'm'} + a^-_{\ell'}Y^{\pm 1}_{LM}Y^{\pm 1}_{\ell'm'}] \\ &= -\sum_{LM\ell'm'}(-1)^m\begin{pmatrix} \ell & L & \ell' \\ -m & M & m' \end{pmatrix}\phi_{LM}\Xi^{\pm}_{\ell'm'}\gamma_{\ell L\ell'}a^0_L\left[a^+_{\ell'}\begin{pmatrix} \ell & L & \ell' \\ \mp 2 & \mp 1 & \pm 3 \end{pmatrix} + a^-_{\ell'}\begin{pmatrix} \ell & L & \ell' \\ \mp 2 & \pm 1 & \pm 1 \end{pmatrix}\right] \\ &= \widetilde{\sum_{LM\ell'm'}}^{(\ell m)}\phi_{LM}\Xi^{\pm}_{\ell'm'}W^{\phi,\pm 2}_{\ell L\ell'}\,, \end{aligned} \tag{28}$$

with

$$S^{\phi,2}_{\ell_1\ell_2\ell_3} = (-1)^{\ell_1+\ell_2+\ell_3}S^{\phi,-2}_{\ell_1\ell_2\ell_3} = -c_{\phi}\gamma_{\ell_1\ell_2\ell_3}a^0_{\ell_2}\left[a^+_{\ell_3}\begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ -2 & -1 & 3 \end{pmatrix} + a^-_{\ell_3}\begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ -2 & 1 & 1 \end{pmatrix}\right]\,. \tag{29}$$

For curl mode,

$$\begin{aligned}\delta^{\varpi}\Xi^{\pm}_{\ell m} &= \mathrm{i}\int \mathrm{d}^2\hat{\boldsymbol{n}}\;(Y^{\pm 2}_{\ell m})^*[\varpi^-\Xi^{\pm^+} - \varpi^+\Xi^{\pm^-}] \\ &= \pm\mathrm{i}\sum_{LM\ell'm'}(-1)^m\begin{pmatrix} \ell & L & \ell' \\ -m & M & m' \end{pmatrix}\varpi_{LM}\Xi^{\pm}_{\ell'm'}a^0_L\gamma_{\ell L\ell'}\left[a^+_{\ell'}\begin{pmatrix} \ell & L & \ell' \\ \mp 2 & \mp 1 & \pm 3 \end{pmatrix} - a^-_{\ell'}\begin{pmatrix} \ell & L & \ell' \\ \mp 2 & \pm 1 & \pm 1 \end{pmatrix}\right] \\ &= \widetilde{\sum_{LM\ell'm'}}^{(\ell m)}\varpi_{LM}\Xi^{\pm}_{\ell'm'}W^{\varpi,\pm 2}_{\ell L\ell'}\,, \end{aligned} \tag{30}$$

with

$$W^{\varpi,2}_{\ell_1\ell_2\ell_3} = -(-1)^{\ell_1+\ell_2+\ell_3}W^{\varpi,-2}_{\ell_1\ell_2\ell_3} = -c_{\varpi}\gamma_{\ell_1\ell_2\ell_3}a^0_{\ell_2}\left[a^+_{\ell_3}\begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ -2 & -1 & 3 \end{pmatrix} - a^-_{\ell_3}\begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ -2 & 1 & 1 \end{pmatrix}\right]\,. \tag{31}$$

Now we consider the lensed E/B modes separately. In general, for $X^{\pm} = A \pm \mathrm{i}B = (a \pm \mathrm{i}b)c^{(\pm)}$,

$$A = \frac{X^+ + X^-}{2} = \left(a\frac{c^{(+)} + c^{(-)}}{2} + \mathrm{i}b\frac{c^{(+)} - c^{(-)}}{2}\right)\,, \tag{32}$$

$$B = \frac{X^+ - X^-}{2\mathrm{i}} = \left(-a\mathrm{i}\frac{c^{(+)} - c^{(-)}}{2} + b\frac{c^{(+)} + c^{(-)}}{2}\right) \tag{33}$$

The lensing correction terms for E/B modes are then given by

$$\delta^x E_{\ell m} = \widetilde{\sum_{LM\ell'm'}}^{(\ell m)} \phi_{LM} \left[W^{x,+}_{\ell L\ell'} E_{\ell'm'} + W^{x,-}_{\ell L\ell'} B_{\ell'm'}\right] \,, \tag{34}$$

$$\delta^x B_{\ell m} = \widetilde{\sum_{LM\ell'm'}}^{(\ell m)} \phi_{LM} \left[-W^{x,-}_{\ell L\ell'} E_{\ell'm'} + W^{x,+}_{\ell L\ell'} B_{\ell'm'}\right] \,. \tag{35}$$

Here we define

$$\begin{aligned}
W^{x,+}_{\ell_1\ell_2\ell_3} &\equiv \frac{W^{x,2}_{\ell_1\ell_2\ell_3} + W^{x,-2}_{\ell_1\ell_2\ell_3}}{2} = \frac{1+c_x^2(-1)^{\ell_1+\ell_2+\ell_3}}{2} W^{x,2}_{\ell_1\ell_2\ell_3} \\
&= -\wp^{x,+}_{\ell_1\ell_2\ell_3}\gamma_{\ell_1\ell_2\ell_3} a^0_{\ell_2}\left[a^+_{\ell_3}\begin{pmatrix}\ell_1 & \ell_2 & \ell_3 \\ -2 & -1 & 3\end{pmatrix} + c_x^2 a^-_{\ell_3}\begin{pmatrix}\ell_1 & \ell_2 & \ell_3 \\ -2 & 1 & 1\end{pmatrix}\right] \,, \\
W^{x,-}_{\ell_1\ell_2\ell_3} &\equiv \mathrm{i}\,\frac{W^{x,2}_{\ell_1\ell_2\ell_3} - W^{x,-2}_{\ell_1\ell_2\ell_3}}{2} = \mathrm{i}\,\frac{1-c_x^2(-1)^{\ell_1+\ell_2+\ell_3}}{2} W^{x,2}_{\ell_1\ell_2\ell_3} \\
&= -\wp^{x,-}_{\ell_1\ell_2\ell_3}\gamma_{\ell_1\ell_2\ell_3} a^0_{\ell_2}\left[a^+_{\ell_3}\begin{pmatrix}\ell_1 & \ell_2 & \ell_3 \\ -2 & -1 & 3\end{pmatrix} + c_x^2 a^-_{\ell_3}\begin{pmatrix}\ell_1 & \ell_2 & \ell_3 \\ -2 & 1 & 1\end{pmatrix}\right] \,,
\end{aligned} \tag{36}$$

where we reintroduce a parity indicator as

$$\begin{aligned}
\wp^{x,+}_{\ell_1\ell_2\ell_3} &= c_x \frac{1+c_x^2(-1)^{\ell_1+\ell_2+\ell_3}}{2} \,, \\
\wp^{x,-}_{\ell_1\ell_2\ell_3} &= \mathrm{i}c_x \frac{1-c_x^2(-1)^{\ell_1+\ell_2+\ell_3}}{2} \,.
\end{aligned} \tag{37}$$

2.2 Rotation distortion

The E and B modes after the rotation are given by

$$\begin{aligned}
\delta\Xi^\pm_{\ell m} &= \pm 2\int \mathrm{d}^2\hat{\boldsymbol{n}}\; (Y^{\pm 2}_{\ell m})^* \alpha P^\pm \\
&= \pm 2 \sum_{LM\ell'm'} \alpha_{LM}\Xi^\pm_{\ell'm'} \int \mathrm{d}^2\hat{\boldsymbol{n}}\; (Y^{\pm 2}_{\ell m})^* Y_{LM} Y^{\pm 2}_{\ell'm'} \\
&= \pm 2 \sum_{LM\ell'm'} (-1)^m \begin{pmatrix}\ell & L & \ell' \\ -m & M & m'\end{pmatrix} \alpha_{LM}\Xi^\pm_{\ell'm'}\gamma_{\ell L\ell'} \begin{pmatrix}\ell & L & \ell' \\ \mp 2 & 0 & \pm 2\end{pmatrix} \\
&= \widetilde{\sum_{LM\ell'm'}}^{(\ell m)} \alpha_{LM}\Xi^\pm_{\ell'm'} W^{\alpha,\pm 2}_{\ell L\ell'} \,,
\end{aligned} \tag{38}$$

with

$$W^{\alpha,\pm 2}_{\ell_1\ell_2\ell_3} = \pm 2\gamma_{\ell_1\ell_2\ell_3}\begin{pmatrix}\ell_1 & \ell_2 & \ell_3 \\ \mp 2 & 0 & \pm 2\end{pmatrix} \,. \tag{39}$$

The distorted E and B modes are then described as

$$\delta E_{\ell m} = \widetilde{\sum_{LM\ell'm'}}^{(\ell m)} \alpha_{LM}\left(E_{\ell'm'}W^{\alpha,+}_{\ell L\ell'} + B_{\ell'm'}W^{\alpha,-}_{\ell L\ell'}\right) \,, \tag{40}$$

$$\delta B_{\ell m} = \widetilde{\sum_{LM\ell'm'}}^{(\ell m)} \alpha_{LM}\left(-E_{\ell'm'}W^{\alpha,-}_{\ell L\ell'} + B_{\ell'm'}W^{\alpha,+}_{\ell L\ell'}\right) \tag{41}$$

where we define $c_\alpha = 1$ and

$$W^{\alpha,\pm}_{\ell_1\ell_2\ell_3} = 2\wp^{\alpha,\mp}_{\ell_1\ell_2\ell_3}\gamma_{\ell_1\ell_2\ell_3}\begin{pmatrix}\ell_1 & \ell_2 & \ell_3 \\ -2 & 0 & 2\end{pmatrix} \tag{42}$$

2.3 Patchy-tau distortion

The harmonics transform of $\tau(\hat{\boldsymbol{n}})\Theta(\hat{\boldsymbol{n}})$ is

$$\begin{aligned}
\delta\Theta_{\ell m} &= \int \mathrm{d}^2\hat{\boldsymbol{n}}\; Y^*_{\ell m}\tau(\hat{\boldsymbol{n}})\Theta(\hat{\boldsymbol{n}}) \\
&= \sum_{LM\ell' m'} \tau_{LM}\Theta_{\ell' m'} \int \mathrm{d}^2\hat{\boldsymbol{n}}\; Y^*_{\ell m} Y_{LM} Y_{\ell' m'} \\
&= \sum_{LM\ell' m'} \tau_{LM}\Theta_{\ell' m'} p^+_{\ell L\ell'}\gamma_{\ell L\ell'}(-1)^m \begin{pmatrix} \ell & L & \ell' \\ -m & M & m' \end{pmatrix} \begin{pmatrix} \ell & L & \ell' \\ 0 & 0 & 0 \end{pmatrix} \\
&= \widetilde{\sum_{LM\ell' m'}}^{(\ell m)} \tau_{LM}\Theta_{\ell' m'} W^{\tau,0}_{\ell L\ell'} \,,
\end{aligned} \tag{43}$$

where

$$W^{\tau,0}_{\ell L\ell'} = p^+_{\ell L\ell'}\gamma_{\ell L\ell'} \begin{pmatrix} \ell & L & \ell' \\ 0 & 0 & 0 \end{pmatrix} \,. \tag{44}$$

2.4 Summary

The above all distortions are described in the following form:

$$\delta\Theta_{\ell m} = \widetilde{\sum_{LM\ell' m'}}^{(\ell m)} x_{LM}\Theta_{\ell' m'} W^{x,0}_{\ell L\ell'} \,, \tag{45}$$

$$\delta E_{\ell m} = \widetilde{\sum_{LM\ell' m'}}^{(\ell m)} x_{LM}\left(E_{\ell' m'} W^{x,+}_{\ell L\ell'} + B_{\ell' m'} W^{x,-}_{\ell L\ell'}\right) \,, \tag{46}$$

$$\delta B_{\ell m} = \widetilde{\sum_{LM\ell' m'}}^{(\ell m)} x_{LM}\left(-E_{\ell' m'} W^{x,-}_{\ell L\ell'} + B_{\ell' m'} W^{x,+}_{\ell L\ell'}\right) \tag{47}$$

where x is a distortion field. The functional form of W is given above.

3 Quadratic estimator

3.1 Distortion induced anisotropies

The distortion fields x described above induce the off-diagonal elements of the covariance ($\ell \neq \ell'$ or $m \neq m'$), [8, 9]

$$\langle \widetilde{X}_{\ell m}\widetilde{Y}_{\ell' m'}\rangle_{\rm CMB} = \sum_{LM} \begin{pmatrix} \ell & \ell' & L \\ m & m' & M \end{pmatrix} f^{x,{\rm XY}}_{\ell L\ell'} x^*_{LM} \,, \tag{48}$$

where $\langle\cdots\rangle_{\rm CMB}$ denotes the ensemble average over the primary CMB anisotropies with a fixed realization of the distortion fields. We ignore the higher-order terms of the distortion fields. The functional form of the weight functions f are discussed later.

With a quadratic combination of observed CMB anisotropies, $\widehat{X}$ and $\widehat{Y}$, the general quadratic estimators are formed as

$$[\widehat{x}^{\rm XY}_{LM}]^* = A^{x,{\rm XY}}_L \sum_{\ell\ell' mm'} \begin{pmatrix} \ell & \ell' & L \\ m & m' & M \end{pmatrix} g^{x,{\rm XY}}_{\ell L\ell'} \widehat{X}_{\ell m}\widehat{Y}_{\ell' m'} \,. \tag{49}$$

Here we define

$$g^{x,{\rm XY}}_{\ell L\ell'} = \frac{[f^{x,{\rm XY}}_{\ell L\ell'}]^*}{\Delta^{\rm XY}\widehat{C}^{\rm XX}_\ell \widehat{C}^{\rm YY}_{\ell'}} \tag{50}$$

$$A^{x,{\rm XY}}_L = \frac{1}{2L+1}\sum_{\ell\ell'} f^{x,{\rm XY}}_{\ell L\ell'} g^{x,{\rm XY}}_{\ell L\ell'} \,, \tag{51}$$

where $\Delta^{\rm XX} = 2$, $\Delta^{\rm EB} = \Delta^{\rm TB} = 1$, and $\widehat{C}^{\rm XX}_\ell$ ($\widehat{C}^{\rm YY}_\ell$) is the observed power spectrum.

3.2 Weight Function

The weight functions are, in general, given as

$$f^{x,(\Theta\Theta)}_{\ell L\ell'} = W^{x,0}_{\ell L\ell'} C^{\Theta\Theta}_{\ell'} + p_x W^{x,0}_{\ell' L\ell} C^{\Theta\Theta}_{\ell} \,, \tag{52}$$

$$f^{x,(\Theta E)}_{\ell L\ell'} = W^{x,0}_{\ell L\ell'} C^{\Theta{\rm E}}_{\ell'} + p_x W^{x,+}_{\ell' L\ell} C^{\Theta{\rm E}}_{\ell} \,, \tag{53}$$

$$f^{x,(\Theta B)}_{\ell L\ell'} = p_x W^{x,-}_{\ell' L\ell} C^{\Theta{\rm E}}_{\ell} \,, \tag{54}$$

$$f^{x,(EE)}_{\ell L\ell'} = W^{x,+}_{\ell L\ell'} C^{\rm EE}_{\ell'} + p_x W^{x,+}_{\ell' L\ell} C^{\rm EE}_{\ell} \,, \tag{55}$$

$$f^{x,(EB)}_{\ell L\ell'} = W^{x,-}_{\ell L\ell'} C^{\rm BB}_{\ell'} + p_x W^{x,-}_{\ell' L\ell} C^{\rm EE}_{\ell} \,, \tag{56}$$

$$f^{x,(BB)}_{\ell L\ell'} = W^{x,+}_{\ell L\ell'} C^{\rm BB}_{\ell'} + p_x W^{x,+}_{\ell' L\ell} C^{\rm BB}_{\ell} \,. \tag{57}$$

Here, the parity index is $p_\phi = p_\epsilon = 1$ and $p_\varpi = p_\alpha = -1$. Strictly speaking, p_x should be $(-1)^{\ell'+L+\ell}$. However, W is only non-zero when $\ell'+L+\ell$ is even, and vice versa. The parity even quantities are $x=\phi$ and ϵ. The odd parity quantities are $x=\varpi$ and α. Note that the above weight functions are consistent with Ref. [7] ($W^{x,-}_{\ell L\ell'} = -{}_{\ominus}S^x_{\ell L\ell'}$) for the lensing case.

3.3 Weight Function: Derivations

Let us first consider the temperature case. There are two contributions to the temperature quadratic estimator, and the one is given as

$$\begin{aligned}\langle\Theta_{\ell''m''}\delta\Theta_{\ell m}\rangle &= \widetilde{\sum_{LM\ell'm'}}^{(\ell m)} x_{LM}\langle\Theta_{\ell''m''}\Theta_{\ell'm'}\rangle W^{x,0}_{\ell L\ell'} \\ &= \widetilde{\sum_{LM\ell'm'}}^{(\ell m)} x_{LM}\delta_{\ell''\ell'}\delta_{m'',-m'}(-1)^{m'}C^{\Theta\Theta}_{\ell'}W^{x,0}_{\ell L\ell'} \\ &= \sum_{LM}\begin{pmatrix}\ell & \ell'' & L\\ m & m'' & M\end{pmatrix} x^*_{LM}C^{\Theta\Theta}_{\ell''}W^{x,0}_{\ell L\ell''}\,. \end{aligned} \tag{58}$$

Here, we use

$$\begin{aligned}\widetilde{\sum_{LM\ell'm'}}^{(\ell m)}\delta_{\ell''\ell'}\delta_{m'',-m'}(-1)^{m'}x_{LM} &= \sum_{LM}(-1)^{m+m'}\begin{pmatrix}\ell & L & \ell''\\ -m & M & -m''\end{pmatrix} x_{LM} \\ &= \sum_{LM}(-1)^{-M}\begin{pmatrix}\ell & \ell'' & L\\ m & m'' & -M\end{pmatrix} x_{LM} \\ &= \sum_{LM}\begin{pmatrix}\ell & \ell'' & L\\ m & m'' & M\end{pmatrix} x^*_{LM}\end{aligned} \tag{59}$$

The other term is obtained by $(\ell'',m'')\leftrightarrow(\ell,m)$ and is given by

$$\langle\Theta_{\ell m}\delta\Theta_{\ell''m''}\rangle = \sum_{LM}(-1)^{\ell+\ell''+L}\begin{pmatrix}\ell & \ell'' & L\\ m & m'' & M\end{pmatrix} x^*_{LM}C^{\Theta\Theta}_{\ell}W^{x,0}_{\ell''L\ell}\,. \tag{60}$$

The sign $(-1)^{\ell+\ell''+L}$ depends on the parity of W.

In the EB estimator, the two contributions are given as

$$\begin{aligned}E_{\ell''m''}\delta B_{\ell m} &= -\widetilde{\sum_{LM\ell'm'}}^{(\ell m)} x_{LM}\langle E_{\ell''m''}E_{\ell'm'}\rangle W^{x,-}_{\ell L\ell'} \\ &= -\widetilde{\sum_{LM\ell'm'}}^{(\ell m)} x_{LM}(-1)^{m''}\delta_{\ell''\ell'}\delta_{m'',-m'}C^{\rm EE}_{\ell''}W^{x,-}_{\ell L\ell'} \\ &= -\sum_{LM}\begin{pmatrix}\ell & \ell'' & L\\ m & m'' & M\end{pmatrix} x^*_{LM}C^{\rm EE}_{\ell''}W^{x,-}_{\ell L\ell'}\,,\end{aligned} \tag{61}$$

and

$$\begin{aligned}B_{\ell''m''}\delta E_{\ell m} &= \widetilde{\sum_{LM\ell'm'}}^{(\ell m)} x_{LM}\langle E_{\ell''m''}E_{\ell'm'}\rangle W^{x,-}_{\ell L\ell'} \\ &= \sum_{LM}(-1)^{\ell+\ell''+L}\begin{pmatrix}\ell & \ell'' & L\\ m & m'' & M\end{pmatrix} x^*_{LM}C^{\rm EE}_{\ell}W^{x,-}_{\ell''L\ell}\,.\end{aligned} \tag{62}$$

4 Computing quadratic estimator

4.1 Spherical Harmonics

The polarization vectors satisfy, $\boldsymbol{e}\cdot\boldsymbol{e}^*=1$, and, $\boldsymbol{e}\cdot\boldsymbol{e}=\boldsymbol{e}^*\cdot\boldsymbol{e}^*=0$. We obtain

$$\boldsymbol{\nabla} Y^s_{\ell m} = \sqrt{\frac{(\ell-s)(\ell+s+1)}{2}}Y^{s+1}_{\ell m}\boldsymbol{e}^* - \sqrt{\frac{(\ell+s)(\ell-s+1)}{2}}Y^{s-1}_{\ell m}\boldsymbol{e}\,. \tag{63}$$

The complex conjugate is $(Y^s_{\ell m})^*=(-1)^{s+m}Y^{-s}_{\ell,-m}$. In particular, for $s=0$,

$$\boldsymbol{\nabla} Y^*_{\ell m} = \sqrt{\frac{\ell(\ell+1)}{2}}\left((Y^1_{\ell m})^*\boldsymbol{e} - (Y^{-1}_{\ell m})^*\boldsymbol{e}^*\right), \tag{64}$$

and, for $s=-2$,

$$\begin{aligned}
\boldsymbol{\nabla} Y^{-2}_{\ell m} &= \sqrt{\frac{(\ell+2)(\ell-1)}{2}}Y^{-1}_{\ell m}\boldsymbol{e}^* - \sqrt{\frac{(\ell-2)(\ell+3)}{2}}Y^{-3}_{\ell m}\boldsymbol{e}\,, \\
\boldsymbol{\nabla} (Y^{-2}_{\ell m})^* &= \sqrt{\frac{(\ell+2)(\ell-1)}{2}}(Y^{-1}_{\ell m})^*\boldsymbol{e} - \sqrt{\frac{(\ell-2)(\ell+3)}{2}}(Y^{-3}_{\ell m})^*\boldsymbol{e}^*\,.
\end{aligned} \tag{65}$$

4.2 Healpix

`Healpix` is a useful public package for fullsky analysis [10]. Here, we consider the `Healpix` spin-s harmonic transform of a map $S(\hat{\boldsymbol{n}})=S^+(\hat{\boldsymbol{n}})+\mathrm{i}S^-(\hat{\boldsymbol{n}})$ where $S^\pm$ is real and $s\geq 0$. The harmonic coefficient is given by

$$S^+ + \mathrm{i}S^- = \sum_{\ell m} a^s_{\ell m}Y^s_{\ell m}\,. \tag{66}$$

Note that $a^{-s}_{\ell m}$ is defined as

$$S^+ - \mathrm{i}S^- = \sum_{\ell m} a^{-s}_{\ell m}Y^{-s}_{\ell m}\,. \tag{67}$$

Then we obtain $(a^s_{\ell m})^*=(-1)^{m+s}a^{-s}_{\ell,-m}$. The subroutine `map2alm_spin` transform $S^\pm$ to $a^{s,\pm}_{\ell m}$ where

$$a^{s,+}_{\ell m} = -\frac{a^s_{\ell m}+(-1)^s a^{-s}_{\ell m}}{2} \tag{68}$$

$$a^{s,-}_{\ell m} = -\frac{a^s_{\ell m}-(-1)^s a^{-s}_{\ell m}}{2\mathrm{i}}\,, \tag{69}$$

are the rotational invariant coefficients with parity even and odd, respectively. Since $(a^s_{\ell m})^* = (-1)^{m+s}a^{-s}_{\ell,-m}$, the above coefficients satisfy

$$(a^{s,\pm}_{\ell m})^* = (-1)^m a^{s,\pm}_{\ell,-m}\,. \tag{70}$$

On the other hand, `alm2map_spin` transform $a^{s,\pm}_{\ell m}$ to $S^\pm$, but $a^{s,\pm}_{\ell m}$ should satisfy the above condition. Note that, with $S\equiv S^+ + \mathrm{i}S^-$, we find

$$a^{s,+}_{\ell m} = -\frac{1}{2}\int \mathrm{d}\hat{\boldsymbol{n}}\ [(Y^s_{\ell m})^*S + (-1)^s(Y^{-s}_{\ell m})^*S^*]\,, \tag{71}$$

$$a^{s,-}_{\ell m} = -\frac{1}{2\mathrm{i}}\int \mathrm{d}\hat{\boldsymbol{n}}\ [(Y^s_{\ell m})^*S - (-1)^s(Y^{-s}_{\ell m})^*S^*]\,. \tag{72}$$

Let us consider the case we want to transform $a_{\ell m}$ with a spin-s spherical harmonics using `alm2map_spin`. The outputs, $S^\pm$, are given by:

$$S^+ + \mathrm{i}S^- = \sum_{\ell m} a_{\ell m}Y^s_{\ell m}\,. \tag{73}$$

The complex conjugate of the above quantity becomes

$$S^+ - \mathrm{i}S^- = (-1)^s \sum_{\ell m} a_{\ell m} Y_{\ell m}^{-s} \,. \tag{74}$$

The inputs of `alm2map_spin` become

$$a_{\ell m}^{s,+} = -\frac{1+(-1)^s}{2} a_{\ell m} \,, \tag{75}$$

$$a_{\ell m}^{s,-} = -\frac{1-(-1)^s}{2\mathrm{i}} a_{\ell m} \,. \tag{76}$$

4.3 Lensing quadratic estimator

In fullsky, the quadratic estimator of the gradient and curl modes are given by [9, 7]:

$$\widehat{\phi}_{\ell m}^{(\alpha)} = A_\ell^{\phi,(\alpha)} \int \mathrm{d}^2\hat{\boldsymbol{n}}\ [\boldsymbol{\nabla} Y_{\ell m}^*(\hat{\boldsymbol{n}})] \cdot \boldsymbol{v}^{(\alpha)}(\hat{\boldsymbol{n}}) \,, \tag{77}$$

$$\widehat{\varpi}_{\ell m}^{(\alpha)} = A_\ell^{\varpi,(\alpha)} \int \mathrm{d}^2\hat{\boldsymbol{n}}\ [(\star\boldsymbol{\nabla}) Y_{\ell m}^*(\hat{\boldsymbol{n}})] \cdot \boldsymbol{v}^{(\alpha)}(\hat{\boldsymbol{n}}) \,, \tag{78}$$

where $N_\ell^{x,(\alpha)}$ is the normalization of the quadratic estimator and we define

$$\boldsymbol{v}^{\Theta\Theta}(\hat{\boldsymbol{n}}) = A_\Theta^0 \boldsymbol{\nabla} A_{\Theta\Theta}^0 \,, \tag{79}$$

$$\boldsymbol{v}^{\Theta E}(\hat{\boldsymbol{n}}) = \Re(A_E^2 \boldsymbol{\nabla} A_{\Theta E}^{-2}) + A_\Theta \boldsymbol{\nabla} A_{E\Theta} \,, \tag{80}$$

$$\boldsymbol{v}^{\Theta B}(\hat{\boldsymbol{n}}) = \Re(A_{iB}^2 \boldsymbol{\nabla} A_{\Theta E}^{-2}) \,, \tag{81}$$

$$\boldsymbol{v}^{EE}(\hat{\boldsymbol{n}}) = \Re(A_E^2 \boldsymbol{\nabla} A_{EE}^{-2}) \,, \tag{82}$$

$$\boldsymbol{v}^{EB}(\hat{\boldsymbol{n}}) = \Re(A_{iB}^2 \boldsymbol{\nabla} A_{EE}^{-2}) + \Re(A_E^2 \boldsymbol{\nabla} A_{iBiB}^{-2}) \,, \tag{83}$$

$$\boldsymbol{v}^{BB}(\hat{\boldsymbol{n}}) = \Re(A_{iB}^2 \boldsymbol{\nabla} A_{iBiB}^{-2}) \,. \tag{84}$$

Here we define

$$A_X^s(\hat{\boldsymbol{n}}) = \sum_{\ell m} \overline{X}_{\ell m} Y_{\ell m}^s(\hat{\boldsymbol{n}}) \,, \tag{85}$$

$$A_{XY}^s(\hat{\boldsymbol{n}}) = \sum_{\ell m} C_\ell^{XY} \overline{X}_{\ell m} Y_{\ell m}^s(\hat{\boldsymbol{n}}) \,, \tag{86}$$

with $\overline{X}_{\ell m} = \widehat{X}_{\ell m}/\widehat{C}_\ell^{XY}$ being the inverse-variance filtered multipoles by the lensed angular power spectra including instrumental noise. The quantity $\boldsymbol{v}^{\Theta E}$ gives the nearly optimal estimator [9].

In general, we can decompose the 2D vector, $\boldsymbol{v}^{(\alpha)}$, into

$$\boldsymbol{v}^{(\alpha)} = \frac{1}{\sqrt{2}}[] v_-^{(\alpha)} \boldsymbol{e} + v_+^{(\alpha)} \boldsymbol{e}^*] \,. \tag{87}$$

Since $\boldsymbol{v}^{(\alpha)}$ is real, we find $(v_-^{(\alpha)})^* = v_+^{(\alpha)} \equiv v^{(\alpha)}$. Then we obtain

$$\begin{aligned} \widehat{\phi}_{\ell m}^{(\alpha)} &= \frac{\sqrt{\ell(\ell+1)} A_\ell^{\phi,(\alpha)}}{2} \int \mathrm{d}^2\hat{\boldsymbol{n}}\ [(Y_{\ell m}^1)^* v^{(\alpha)} - (Y_{\ell m}^{-1})^* (v^{(\alpha)})^*] \\ &= -\sqrt{\ell(\ell+1)} A_\ell^{\phi,(\alpha)} v_{\ell m}^{1,+} \,, \end{aligned} \tag{88}$$

$$\begin{aligned} \widehat{\varpi}_{\ell m}^{(\alpha)} &= -\mathrm{i}\frac{\sqrt{\ell(\ell+1)} A_\ell^{\varpi,(\alpha)}}{2} \int \mathrm{d}^2\hat{\boldsymbol{n}}\ [(Y_{\ell m}^1)^* v^{(\alpha)} + (Y_{\ell m}^{-1})^* (v^{(\alpha)})^*] \\ &= -\sqrt{\ell(\ell+1)} A_\ell^{\varpi,(\alpha)} v_{\ell m}^{1,-} \,, \end{aligned} \tag{89}$$

where $v_{\ell m}^{1,\pm}$ are the outputs of `map2alm_spin` by inputting, $S = v^{(\alpha)}$, with $s = 1$.

4.3.1 TE

$$
\begin{aligned}
\boldsymbol{v}^{\Theta E} = \Re\Bigg[&\sum_{\ell m}\overline{E}_{\ell m}Y^{+2}_{\ell m}\sum_{\ell m}C_\ell^{\Theta E}\overline{\Theta}_{\ell m}\left(\sqrt{\frac{(\ell+2)(\ell-1)}{2}}Y^{-1}_{\ell m}\boldsymbol{e}^* - \sqrt{\frac{(\ell-2)(\ell+3)}{2}}Y^{-3}_{\ell m}\boldsymbol{e}\right)\Bigg] \\
&+ \sum_{\ell m}\overline{\Theta}_{\ell m}Y_{\ell m}\sum_{\ell m}C_\ell^{\Theta E}\overline{E}_{\ell m}\sqrt{\frac{\ell(\ell+1)}{2}}\left(Y^1_{\ell m}\boldsymbol{e}^* - Y^{-1}_{\ell m}\boldsymbol{e}\right) \\
= &\frac{1}{2\sqrt{2}}\left[(Q^E+\mathrm{i}U^E)[-(\Theta_1^+-\mathrm{i}\Theta_1^-)\boldsymbol{e}^* + (\Theta_3^+-\mathrm{i}\Theta_3^-)\boldsymbol{e}] + \text{c.c.}\right] \\
&+ \frac{1}{\sqrt{2}}\overline{\Theta}[(E_1^+ + \mathrm{i}E_1^-)\boldsymbol{e}^* + (E_1^+ - \mathrm{i}E_1^-)\boldsymbol{e}] \,.
\end{aligned}
\tag{90}
$$

where we define

$$
\begin{aligned}
Q^E + \mathrm{i}U^E &\equiv \sum_{\ell m} Y^2_{\ell m}\overline{E}_{\ell m} = A_E^2 \,, \\
\Theta_1^+ + \mathrm{i}\Theta_1^- &\equiv \sum_{\ell m} Y^1_{\ell m}\overline{\Theta}_{\ell m}C_\ell^{\Theta E}\sqrt{(\ell+2)(\ell-1)} \,, \\
\Theta_3^+ + \mathrm{i}\Theta_3^- &\equiv \sum_{\ell m} Y^3_{\ell m}\overline{\Theta}_{\ell m}C_\ell^{\Theta E}\sqrt{(\ell-2)(\ell+3)} \,, \\
E_1^+ + \mathrm{i}E_1^- &\equiv \sum_{\ell m} Y^1_{\ell m}\overline{E}_{\ell m}C_\ell^{\Theta E}\sqrt{\ell(\ell+1)} \,.
\end{aligned}
\tag{91}
$$

The above quantities are obtained by `map2alm_spin`. We find that

$$
\begin{aligned}
v^{\Theta E} &= \frac{1}{2}[(Q^E+\mathrm{i}U^E)(-\Theta_1^+ + \mathrm{i}\Theta_1^-) + (Q^E-\mathrm{i}U^E)(\Theta_3^+ + \mathrm{i}\Theta_3^-)] + \overline{\Theta}(E_1^+ + \mathrm{i}E_1^-) \\
&= \frac{1}{2}[Q^E(\Theta_3^+ - \Theta_1^+) + U^E(\Theta_3^- - \Theta_1^-) + \mathrm{i}[Q^E(\Theta_3^- + \Theta_1^-) - U^E(\Theta_3^+ + \Theta_1^+)]] + \overline{\Theta}(E_1^+ + \mathrm{i}E_1^-) \,.
\end{aligned}
\tag{92}
$$

4.3.2 EE

$$
\begin{aligned}
\boldsymbol{v}^{(EE)} &= \frac{1}{2}(Q^E+\mathrm{i}U^E)\sum_{\ell m}C_\ell^{EE}\overline{E}_{\ell m}\Big(\sqrt{\frac{(\ell+2)(\ell-1)}{2}}Y^{-1}_{\ell m}\boldsymbol{e}^* - \sqrt{\frac{(\ell-2)(\ell+3)}{2}}Y^{-3}_{\ell m}\boldsymbol{e}\Big) + \text{c.c} \\
&= \frac{1}{2\sqrt{2}}(Q^E+\mathrm{i}U^E)[-(\mathcal{E}_1^+ - \mathrm{i}\mathcal{E}_1^-)\boldsymbol{e}^* + (\mathcal{E}_3^+ - \mathrm{i}\mathcal{E}_3^-)\boldsymbol{e}] + \text{c.c} \,,
\end{aligned}
\tag{93}
$$

where we define

$$
\begin{aligned}
\mathcal{E}_1^+ + \mathrm{i}\mathcal{E}_1^- &\equiv \sum_{\ell m} Y^1_{\ell m}\overline{E}_{\ell m}C_\ell^{EE}\sqrt{(\ell+2)(\ell-1)} \,, \\
\mathcal{E}_3^+ + \mathrm{i}\mathcal{E}_3^- &\equiv \sum_{\ell m} Y^3_{\ell m}\overline{E}_{\ell m}C_\ell^{EE}\sqrt{(\ell-2)(\ell+3)} \,.
\end{aligned}
\tag{94}
$$

Then we obtain

$$
\begin{aligned}
v^{(EE)} &= \frac{1}{2}(Q^E+\mathrm{i}U^E)[-\mathcal{E}_1^+ + \mathrm{i}\mathcal{E}_1^-] + \frac{1}{2}(Q^E-\mathrm{i}U^E)[\mathcal{E}_3^+ + \mathrm{i}\mathcal{E}_3^-] \\
&= \frac{1}{2}[Q^E(\mathcal{E}_3^+ - \mathcal{E}_1^+) + U^E(\mathcal{E}_3^- - \mathcal{E}_1^-)] + \frac{\mathrm{i}}{2}[Q^E(\mathcal{E}_3^- + \mathcal{E}_1^-) - U^E(\mathcal{E}_3^+ + \mathcal{E}_1^+)] \,,
\end{aligned}
\tag{95}
$$

4.4 Polarization rotation quadratic estimator

4.4.1 EB

The EB quadratic estimator for the polarization rotation is given by

$$
[\widehat{\alpha}^{\rm EB}_{LM}]^* = A_L^{\alpha,{\rm EB}}\sum_{\ell\ell' mm'}\begin{pmatrix}\ell & \ell' & L \\ m & m' & M\end{pmatrix}\left[W^{\alpha,-}_{\ell L\ell'}C^{\rm BB}_{\ell'} - W^{\alpha,-}_{\ell' L\ell}C^{\rm EE}_{\ell}\right]\overline{E}_{\ell m}\overline{B}_{\ell' m'} \,. \tag{96}
$$

Using the property of the Wigner 3j, we obtain

$$\begin{aligned}\begin{pmatrix}\ell & \ell' & L \\ m & m' & M\end{pmatrix} W^{\alpha,-}_{\ell L\ell'} &= 2\wp^{\alpha,+}_{\ell L\ell'}\gamma_{\ell L\ell'}\begin{pmatrix}\ell & L & \ell' \\ -2 & 0 & 2\end{pmatrix}\begin{pmatrix}\ell & \ell' & L \\ m & m' & M\end{pmatrix} \\ &= [1+(-1)^{\ell+L+\ell'}]\gamma_{\ell L\ell'}\begin{pmatrix}\ell & \ell' & L \\ -2 & 2 & 0\end{pmatrix}\begin{pmatrix}\ell & \ell' & L \\ m & m' & M\end{pmatrix} \\ &= \gamma_{\ell L\ell'}\left[\begin{pmatrix}\ell & \ell' & L \\ -2 & 2 & 0\end{pmatrix}+\begin{pmatrix}\ell & \ell' & L \\ 2 & -2 & 0\end{pmatrix}\right]\begin{pmatrix}\ell & \ell' & L \\ m & m' & M\end{pmatrix} \\ &= \int \mathrm{d}\hat{\boldsymbol{n}}\; Y_{LM}[Y^{-2}_{\ell m}Y^{2}_{\ell'm'}+Y^{2}_{\ell m}Y^{-2}_{\ell'm'}]\,. \end{aligned} \tag{97}$$

The second term is also the same as the first term;

$$\begin{pmatrix}\ell & \ell' & L \\ m & m' & M\end{pmatrix} W^{\alpha,-}_{\ell' L\ell} = \int \mathrm{d}\hat{\boldsymbol{n}}\; Y_{LM}[Y^{-2}_{\ell m}Y^{2}_{\ell'm'}+Y^{2}_{\ell m}Y^{-2}_{\ell'm'}]\,. \tag{98}$$

We then obtain

$$\begin{aligned}\widehat{\alpha}^{\mathrm{EB}}_{LM} &= A^{\alpha,\mathrm{EB}}_L\int \mathrm{d}\hat{\boldsymbol{n}}\; Y^*_{LM}\big[\sum_{\ell m}Y^{-2}_{\ell m}\overline{E}_{\ell m}\sum_{\ell'm'}Y^{2}_{\ell'm'}C^{\mathrm{BB}}_{\ell'}\overline{B}_{\ell'm'}+\sum_{\ell m}Y^{2}_{\ell m}\overline{E}_{\ell m}\sum_{\ell'm'}Y^{-2}_{\ell'm'}C^{\mathrm{BB}}_{\ell'}\overline{B}_{\ell'm'} \\ &\quad -\sum_{\ell m}Y^{2}_{\ell m}C^{\mathrm{EE}}_{\ell}\overline{E}_{\ell m}\sum_{\ell'm'}Y^{-2}_{\ell'm'}\overline{B}_{\ell'm'}-\sum_{\ell m}Y^{-2}_{\ell m}C^{\mathrm{EE}}_{\ell}\overline{E}_{\ell m}\sum_{\ell'm'}Y^{2}_{\ell'm'}\overline{B}_{\ell'm'}\big] \\ &= -\mathrm{i}A^{\alpha,\mathrm{EB}}_L\int \mathrm{d}\hat{\boldsymbol{n}}\; Y^*_{LM}\left[(Q^E-\mathrm{i}U^E)(\mathcal{Q}^B+\mathrm{i}\mathcal{U}^B)+(\mathcal{Q}^E+\mathrm{i}\mathcal{U}^E)(Q^B-\mathrm{i}U^B)-\mathrm{c.c.}\right] \\ &= 2A^{\alpha,\mathrm{EB}}_L\int \mathrm{d}\hat{\boldsymbol{n}}\; Y^*_{LM}\left[Q^E\mathcal{U}^B-U^E\mathcal{Q}^B+\mathcal{U}^EQ^B-\mathcal{Q}^EU^B\right]\,. \end{aligned} \tag{99}$$

where we define

$$\begin{aligned} Q^E+\mathrm{i}U^E &= \sum_{\ell m}Y^{2}_{\ell m}\overline{E}_{\ell m} = (\sum_{\ell m}Y^{-2}_{\ell m}\overline{E}_{\ell m})^* \\ \mathcal{Q}^B+\mathrm{i}\mathcal{U}^B &= \sum_{\ell m}Y^{2}_{\ell m}\mathrm{i}C^{\mathrm{BB}}_{\ell}\overline{B}_{\ell m} \\ \mathcal{Q}^E+\mathrm{i}\mathcal{U}^E &= \sum_{\ell m}Y^{2}_{\ell m}C^{\mathrm{EE}}_{\ell}\overline{E}_{\ell m} \\ Q^B+\mathrm{i}U^B &= \sum_{\ell m}Y^{2}_{\ell m}\mathrm{i}\overline{B}_{\ell m} = -(\sum_{\ell m}Y^{-2}_{\ell m}\mathrm{i}\overline{B}_{\ell m})^*\,. \end{aligned} \tag{100}$$

5 Computing Quadratic Estimator Normalization

Here, I generalize the algorithm of [11] to the case including the cosmic bi-refringence, patchy reionization, and so on.

5.1 Normalization and Kernel function

The normalization of the ΘΘ quadratic estimator is

$$\frac{1}{A_L^{x,(\Theta\Theta)}} = \frac{1}{2L+1}\sum_{\ell\ell'}\frac{\left[W^{x,0}_{\ell L\ell'}C^{\Theta\Theta}_{\ell'} + p_x W^{x,0}_{\ell' L\ell}C^{\Theta\Theta}_{\ell}\right]^2}{2\widehat{C}^{\Theta\Theta}_\ell\widehat{C}^{\Theta\Theta}_{\ell'}}$$
$$= \frac{1}{2}\Sigma^{(0),x}_L\left[\frac{1}{\widehat{C}^{\Theta\Theta}},\frac{(C^{\Theta\Theta})^2}{\widehat{C}^{\Theta\Theta}}\right] + p_x\Gamma^{(0),x}_L\left[\frac{C^{\Theta\Theta}}{\widehat{C}^{\Theta\Theta}},\frac{C^{\Theta\Theta}}{\widehat{C}^{\Theta\Theta}}\right] + \frac{1}{2}\Sigma^{(0),x}_L\left[\frac{(C^{\Theta\Theta})^2}{\widehat{C}^{\Theta\Theta}},\frac{1}{\widehat{C}^{\Theta\Theta}}\right] , \qquad (101)$$

where we define kernel functions as

$$\Sigma^{(0),x}_L[A,B] = \frac{1}{2L+1}\sum_{\ell\ell'}(W^{x,0}_{\ell L\ell'})^2 A_\ell B_{\ell'} , \qquad (102)$$
$$\Gamma^{(0),x}_L[A,B] = \frac{1}{2L+1}\sum_{\ell\ell'}W^{x,0}_{\ell L\ell'}W^{x,0}_{\ell' L\ell} A_\ell B_{\ell'} . \qquad (103)$$

For ΘE,

$$\frac{1}{A_L^{x,(\Theta E)}} = \frac{1}{2L+1}\sum_{\ell\ell'}\frac{|W^{x,0}_{\ell L\ell'}C^{\Theta \mathrm{E}}_{\ell'} + p_x W^{x,+}_{\ell' L\ell}C^{\Theta \mathrm{E}}_{\ell}|^2}{\widehat{C}^{\Theta\Theta}_\ell\widehat{C}^{\mathrm{EE}}_{\ell'}}$$
$$= \frac{1}{2L+1}\sum_{\ell\ell'}\left[(W^{x,0}_{\ell L\ell'})^2\frac{(C^{\Theta\mathrm{E}}_{\ell'})^2}{\widehat{C}^{\Theta\Theta}_\ell\widehat{C}^{\mathrm{EE}}_{\ell'}} + 2p_x W^{x,0}_{\ell L\ell'}W^{x,+}_{\ell' L\ell}\frac{C^{\Theta\mathrm{E}}_{\ell'}C^{\Theta\mathrm{E}}_{\ell}}{\widehat{C}^{\Theta\Theta}_\ell\widehat{C}^{\mathrm{EE}}_{\ell'}} + (W^{x,+}_{\ell' L\ell})^2\frac{(C^{\Theta\mathrm{E}}_{\ell})^2}{\widehat{C}^{\Theta\Theta}_\ell\widehat{C}^{\mathrm{EE}}_{\ell'}}\right]$$
$$= \Sigma^{(0),x}_L\left[\frac{1}{\widehat{C}^{\Theta\Theta}},\frac{(C^{\Theta\mathrm{E}})^2}{\widehat{C}^{\mathrm{EE}}}\right] + 2p_x\Gamma^{(\times),x}_L\left[\frac{C^{\Theta\mathrm{E}}}{\widehat{C}^{\Theta\Theta}},\frac{C^{\Theta\mathrm{E}}}{\widehat{C}^{\mathrm{EE}}}\right] + \Sigma^{(+),x}_L\left[\frac{1}{\widehat{C}^{\mathrm{EE}}},\frac{(C^{\Theta\mathrm{E}})^2}{\widehat{C}^{\Theta\Theta}}\right] , \qquad (104)$$

where kernel functions are defined as

$$\Gamma^{(\times),x}_L[A,B] = \frac{1}{2L+1}\sum_{\ell\ell'}W^{x,0}_{\ell L\ell'}W^{x,+}_{\ell' L\ell} A_\ell B_{\ell'} ,$$
$$\Sigma^{(+),x}_L[A,B] = \frac{1}{2L+1}\sum_{\ell\ell'}(W^{x,+}_{\ell L\ell'})^2 A_\ell B_{\ell'} . \qquad (105)$$

For ΘB,

$$\frac{1}{A_L^{x,(\Theta B)}} = \frac{1}{2L+1}\sum_{\ell\ell'}\frac{|W^{x,-}_{\ell' L\ell}C^{\Theta\mathrm{E}}_{\ell}|^2}{\widehat{C}^{\Theta\Theta}_\ell\widehat{C}^{\mathrm{BB}}_{\ell'}}$$
$$= \Sigma^{(-),x}_L\left[\frac{1}{\widehat{C}^{\mathrm{BB}}},\frac{(C^{\Theta\mathrm{E}})^2}{\widehat{C}^{\Theta\Theta}}\right] , \qquad (106)$$

where

$$\Sigma^{(-),x}_L[A,B] = \frac{1}{2L+1}\sum_{\ell\ell'}|W^{x,-}_{\ell L\ell'}|^2 A_\ell B_{\ell'} . \qquad (107)$$

For EE (and for BB by replacing EE → BB),

$$\frac{1}{A_L^{x}} = \frac{1}{2L+1}\sum_{\ell\ell'}\frac{|W^{x,+}_{\ell L\ell'}C^{\mathrm{EE}}_{\ell'} + p_x W^{x,+}_{\ell' L\ell}C^{\mathrm{EE}}_{\ell}|^2}{2\widehat{C}^{\mathrm{EE}}_\ell\widehat{C}^{\mathrm{EE}}_{\ell'}}$$
$$= \Sigma^{(+),x}_L\left[\frac{1}{\widehat{C}^{\mathrm{EE}}},\frac{(C^{\mathrm{EE}})^2}{\widehat{C}^{\mathrm{EE}}}\right] + p_x\Gamma^{(+),x}_L\left[\frac{C^{\mathrm{EE}}}{\widehat{C}^{\mathrm{EE}}},\frac{C^{\mathrm{EE}}}{\widehat{C}^{\mathrm{EE}}}\right] , \qquad (108)$$

where

$$\Gamma^{(+),x}_L[A,B] = \frac{1}{2L+1}\sum_{\ell\ell'} W^{x,+}_{\ell' L\ell} W^{x,+}_{\ell L\ell'} A_\ell B_{\ell'} = \Gamma^{(+),x}_L[B,A]\,. \tag{109}$$

For EB,

$$\begin{aligned}\frac{1}{A^{x,(EB)}_L} &= \frac{1}{2L+1}\sum_{\ell\ell'} \frac{|W^{x,-}_{\ell L\ell'} C^{\rm BB}_{\ell'} + p_x W^{x,-}_{\ell' L\ell} C^{\rm EE}_\ell|^2}{\widehat{C}^{\rm EE}_\ell \widehat{C}^{\rm BB}_{\ell'}} \\ &= \Sigma^{(-),x}_L\left[\frac{1}{\widehat{C}^{\rm EE}}, \frac{(C^{\rm BB})^2}{\widehat{C}^{\rm BB}}\right] + 2p_x\Gamma^{(-),x}_L\left[\frac{C^{\rm EE}}{\widehat{C}^{\rm EE}}, \frac{C^{\rm BB}}{\widehat{C}^{\rm BB}}\right] + \Sigma^{(-),x}_L\left[\frac{1}{\widehat{C}^{\rm BB}}, \frac{(C^{\rm EE})^2}{\widehat{C}^{\rm EE}}\right]\,,\end{aligned} \tag{110}$$

where

$$\Gamma^{(-),x}_L[A,B] = \frac{1}{2L+1}\sum_{\ell\ell'} [W^{x,-}_{\ell L\ell'}]^* W^{x,-}_{\ell' L\ell} A_\ell B_{\ell'} = \Gamma^{(-),x}_L[B,A]\,. \tag{111}$$

5.2 Noise covariance and kernel function

For $\Theta\Theta\Theta E$,

$$\begin{aligned}\frac{A^{x,(\Theta\Theta)}_L A^{x,(\Theta E)}_L}{N^{x,(\Theta\Theta\Theta E)}_L} &= \frac{1}{2L+1}\sum_{\ell\ell'}\left[\frac{W^{x,0}_{\ell L\ell'} C^{\Theta\Theta}_{\ell'}}{2\widehat{C}^{\Theta\Theta}_\ell \widehat{C}^{\Theta\Theta}_{\ell'}} + p_x(\ell\leftrightarrow\ell')\right]\left[\frac{(W^{x,0}_{\ell L\ell'} C^{\Theta\rm E}_{\ell'} + p_x W^{x,+}_{\ell' L\ell} C^{\Theta\rm E}_\ell)\widehat{C}^{\Theta\rm E}_{\ell'}}{\widehat{C}^{\rm EE}_{\ell'}} + p_x(\ell\leftrightarrow\ell')\right] \\ &= \frac{1}{2L+1}\sum_{\ell\ell'}\left[\frac{W^{x,0}_{\ell L\ell'} C^{\Theta\Theta}_{\ell'}}{\widehat{C}^{\Theta\Theta}_\ell \widehat{C}^{\Theta\Theta}_{\ell'}}\frac{(W^{x,0}_{\ell L\ell'} C^{\Theta\rm E}_{\ell'} + p_x W^{x,+}_{\ell' L\ell} C^{\Theta\rm E}_\ell)\widehat{C}^{\Theta\rm E}_{\ell'}}{\widehat{C}^{\rm EE}_{\ell'}}\right. \\ &\quad\left. + p_x \frac{W^{x,0}_{\ell L\ell'} C^{\Theta\Theta}_{\ell'}}{\widehat{C}^{\Theta\Theta}_\ell \widehat{C}^{\Theta\Theta}_{\ell'}}\frac{(W^{x,0}_{\ell' L\ell} C^{\Theta\rm E}_{\ell} + p_x W^{x,+}_{\ell L\ell'} C^{\Theta\rm E}_{\ell'})\widehat{C}^{\Theta\rm E}_{\ell}}{\widehat{C}^{\rm EE}_{\ell}}\right] \\ &= \Sigma^{(0),x}_L\left[\frac{1}{\widehat{C}^{\Theta\Theta}}, \frac{C^{\Theta\Theta}C^{\Theta\rm E}\widehat{C}^{\Theta\rm E}}{\widehat{C}^{\Theta\Theta}\widehat{C}^{\rm EE}}\right] + p_x\Gamma^{(\times),x}_L\left[\frac{C^{\Theta\rm E}}{\widehat{C}^{\Theta\Theta}}, \frac{C^{\Theta\Theta}\widehat{C}^{\Theta\rm E}}{\widehat{C}^{\Theta\Theta}\widehat{C}^{\rm EE}}\right] \\ &\quad + p_x\Gamma^{(0),x}_L\left[\frac{C^{\Theta\rm E}\widehat{C}^{\Theta\rm E}}{\widehat{C}^{\Theta\Theta}\widehat{C}^{\rm EE}}, \frac{C^{\Theta\Theta}}{\widehat{C}^{\Theta\Theta}}\right] + \Sigma^{(\times),x}_L\left[\frac{\widehat{C}^{\Theta\rm E}}{\widehat{C}^{\Theta\Theta}\widehat{C}^{\rm EE}}, \frac{C^{\Theta\rm E}C^{\Theta\Theta}}{\widehat{C}^{\Theta\Theta}}\right]\,,\end{aligned} \tag{112}$$

where

$$\Sigma^{(\times),x}_L[A,B] = \frac{1}{2L+1}\sum_{\ell\ell'} W^{x,0}_{\ell L\ell'} W^{x,+}_{\ell L\ell'} A_\ell B_{\ell'}\,. \tag{113}$$

For $\Theta\Theta EE$,

$$\begin{aligned}\frac{A^{x,(\Theta\Theta)}_L A^{x,(EE)}_L}{N^{x,(\Theta\Theta EE)}_L} &= \frac{1}{2L+1}\sum_{\ell\ell'}\left[\frac{W^{x,0}_{\ell L\ell'} C^{\Theta\Theta}_{\ell'}}{2\widehat{C}^{\Theta\Theta}_\ell \widehat{C}^{\Theta\Theta}_{\ell'}} + p_x(\ell\leftrightarrow\ell')\right]\left[\frac{(W^{x,+}_{\ell L\ell'} C^{\rm EE}_{\ell'} + p_x W^{x,+}_{\ell' L\ell} C^{\rm EE}_\ell)\widehat{C}^{\Theta\rm E}_\ell\widehat{C}^{\Theta\rm E}_{\ell'}}{2\widehat{C}^{\rm EE}_\ell\widehat{C}^{\rm EE}_{\ell'}} + p_x(\ell\leftrightarrow\ell')\right] \\ &= \frac{1}{2L+1}\sum_{\ell\ell'}\frac{W^{x,0}_{\ell L\ell'} C^{\Theta\Theta}_{\ell'}}{\widehat{C}^{\Theta\Theta}_\ell \widehat{C}^{\Theta\Theta}_{\ell'}}\left[\frac{(W^{x,+}_{\ell L\ell'} C^{\rm EE}_{\ell'} + p_x W^{x,+}_{\ell' L\ell} C^{\rm EE}_\ell)\widehat{C}^{\Theta\rm E}_\ell\widehat{C}^{\Theta\rm E}_{\ell'}}{2\widehat{C}^{\rm EE}_\ell\widehat{C}^{\rm EE}_{\ell'}} + p_x(\ell\leftrightarrow\ell')\right] \\ &= \frac{1}{2L+1}\sum_{\ell\ell'}\frac{W^{x,0}_{\ell L\ell'} C^{\Theta\Theta}_{\ell'}}{\widehat{C}^{\Theta\Theta}_\ell \widehat{C}^{\Theta\Theta}_{\ell'}}\frac{(W^{x,+}_{\ell L\ell'} C^{\rm EE}_{\ell'} + p_x W^{x,+}_{\ell' L\ell} C^{\rm EE}_\ell)\widehat{C}^{\Theta\rm E}_\ell\widehat{C}^{\Theta\rm E}_{\ell'}}{\widehat{C}^{\rm EE}_\ell\widehat{C}^{\rm EE}_{\ell'}} \\ &= \Sigma^{(0),x}_L\left[\frac{\widehat{C}^{\Theta\rm E}}{\widehat{C}^{\Theta\Theta}\widehat{C}^{\rm EE}}, \frac{C^{\Theta\Theta}C^{\rm EE}\widehat{C}^{\Theta\rm E}}{\widehat{C}^{\Theta\Theta}\widehat{C}^{\rm EE}}\right] + p_x\Gamma^{(\times),x}_L\left[\frac{\widehat{C}^{\Theta\rm E}C^{\rm EE}}{\widehat{C}^{\Theta\Theta}\widehat{C}^{\rm EE}}, \frac{C^{\Theta\Theta}\widehat{C}^{\Theta\rm E}}{\widehat{C}^{\Theta\Theta}\widehat{C}^{\rm EE}}\right]\,.\end{aligned} \tag{114}$$

For ΘEEE,

$$
\begin{aligned}
\frac{A_L^{x,(\Theta E)}A_L^{x,(EE)}}{N_L^{x,(\Theta EEE)}} &= \frac{1}{2L+1}\sum_{\ell\ell'}\left[\frac{W^{x,+}_{\ell L\ell'}C^{\mathrm{EE}}_{\ell'}}{2\widehat{C}^{\mathrm{EE}}_\ell\widehat{C}^{\mathrm{EE}}_{\ell'}}+p_x(\ell\leftrightarrow\ell')\right]\left[\frac{(W^{x,0}_{\ell L\ell'}C^{\Theta\mathrm{E}}_{\ell'}+p_xW^{x,+}_{\ell' L\ell}C^{\Theta\mathrm{E}}_{\ell})\widehat{C}^{\Theta\mathrm{E}}_\ell}{\widehat{C}^{\Theta\Theta}_\ell}+p_x(\ell\leftrightarrow\ell')\right] \\
&= \frac{1}{2L+1}\sum_{\ell\ell'}\left[\frac{W^{x,+}_{\ell L\ell'}C^{\mathrm{EE}}_{\ell'}}{\widehat{C}^{\mathrm{EE}}_\ell\widehat{C}^{\mathrm{EE}}_{\ell'}}+p_x\frac{W^{x,+}_{\ell' L\ell}C^{\mathrm{EE}}_{\ell}}{\widehat{C}^{\mathrm{EE}}_\ell\widehat{C}^{\mathrm{EE}}_{\ell'}}\right]\left[\frac{(W^{x,0}_{\ell L\ell'}C^{\Theta\mathrm{E}}_{\ell'}+p_xW^{x,+}_{\ell' L\ell}C^{\Theta\mathrm{E}}_{\ell})\widehat{C}^{\Theta\mathrm{E}}_\ell}{\widehat{C}^{\Theta\Theta}_\ell}\right] \\
&= \Sigma^{(\times),x}_L\left[\frac{\widehat{C}^{\Theta\mathrm{E}}}{\widehat{C}^{\Theta\Theta}\widehat{C}^{\mathrm{EE}}},\frac{C^{\Theta\mathrm{E}}C^{\mathrm{EE}}}{\widehat{C}^{\mathrm{EE}}}\right]+p_x\Gamma^{(+),x}_L\left[\frac{C^{\Theta\mathrm{E}}\widehat{C}^{\Theta\mathrm{E}}}{\widehat{C}^{\Theta\Theta}\widehat{C}^{\mathrm{EE}}},\frac{C^{\mathrm{EE}}}{\widehat{C}^{\mathrm{EE}}}\right] \\
&\quad +p_x\Gamma^{(\times),x}_L\left[\frac{\widehat{C}^{\Theta\mathrm{E}}C^{\mathrm{EE}}}{\widehat{C}^{\Theta\Theta}\widehat{C}^{\mathrm{EE}}},\frac{C^{\Theta\mathrm{E}}}{\widehat{C}^{\mathrm{EE}}}\right]+\Sigma^{(+),x}_L\left[\frac{C^{\Theta\mathrm{E}}\widehat{C}^{\Theta\mathrm{E}}C^{\mathrm{EE}}}{\widehat{C}^{\Theta\Theta}\widehat{C}^{\mathrm{EE}}},\frac{1}{\widehat{C}^{\mathrm{EE}}}\right]\,.
\end{aligned}
\tag{115}
$$

For ΘBEB,

$$
\begin{aligned}
\frac{A_L^{x,(\Theta B)}A_L^{x,(EB)}}{N_L^{x,(\Theta BEB)}} &= \frac{1}{2L+1}\sum_{\ell\ell'}\left[\frac{(W^{x,-}_{\ell L\ell'})^*C^{\mathrm{BB}}_{\ell'}-p_x(W^{x,-}_{\ell' L\ell})^*C^{\mathrm{EE}}_{\ell}}{\widehat{C}^{\mathrm{EE}}_\ell\widehat{C}^{\mathrm{BB}}_{\ell'}}\right]\left[\frac{-p_xW^{x,-}_{\ell' L\ell}C^{\Theta\mathrm{E}}_\ell\widehat{C}^{\Theta\mathrm{E}}_\ell}{\widehat{C}^{\Theta\Theta}_\ell}\right] \\
&= -p_x\Gamma^{(-),x}_L\left[\frac{C^{\Theta\mathrm{E}}\widehat{C}^{\Theta\mathrm{E}}}{\widehat{C}^{\Theta\Theta}\widehat{C}^{\mathrm{EE}}},\frac{C^{\mathrm{BB}}}{\widehat{C}^{\mathrm{BB}}}\right]+\Sigma^{(-),x}_L\left[\frac{C^{\Theta\mathrm{E}}\widehat{C}^{\Theta\mathrm{E}}C^{\mathrm{EE}}}{\widehat{C}^{\Theta\Theta}\widehat{C}^{\mathrm{EE}}},\frac{1}{\widehat{C}^{\mathrm{BB}}}\right]\,.
\end{aligned}
\tag{116}
$$

6 Explicit Kernel Functions

Here we consider expression for the Kernel functions in terms of the Wigner d-functions. In the following calculations, we frequently use

$$\int_{-1}^{1}\mathrm{d}\mu\; d^{\ell_1}_{s_1,s_1'}(\beta)d^{\ell_2}_{s_2,s_2'}(\beta)d^{\ell_3}_{s_3,s_3'}(\beta) = 2\begin{pmatrix}\ell_1 & \ell_2 & \ell_3\\ s_1 & s_2 & s_3\end{pmatrix}\begin{pmatrix}\ell_1 & \ell_2 & \ell_3\\ s_1' & s_2' & s_3'\end{pmatrix}\,, \tag{117}$$

with $s_1+s_2+s_3=s_1'+s_2'+s_3'=0$ and $\mu=\cos\beta$, and the symmetric property:

$$d^{\ell}_{mm'}(\beta) = (-1)^{m-m'}d^{\ell}_{-m,-m'}(\beta) = (-1)^{m-m'}d^{\ell}_{m'm}(\beta) \tag{118}$$

$$d^{\ell}_{mm'}(\beta) = (-1)^{\ell+m}d^{\ell}_{m,-m'}(\pi-\beta)\,. \tag{119}$$

We also define

$$X^{p\dots q} = a^p_\ell\dots a^q_\ell X_\ell\,. \tag{120}$$

6.1 Kernel Functions: Lensing

We obtain

$$\begin{aligned}
\Sigma^{(0),x}_L[A,B] &= \frac{1}{2L+1}\sum_{\ell\ell'}|W^{x,0}_{\ell L\ell'}|^2A_\ell B_{\ell'} \\
&= \sum_{\ell\ell'}4\pi L(L+1)\frac{2\ell+1}{4\pi}A_\ell\frac{2\ell'+1}{4\pi}B_{\ell'}\ell'(\ell'+1)\frac{1+c_x^2(-1)^{\ell+L+\ell'}}{2}\begin{pmatrix}\ell & L & \ell'\\ 0 & 1 & -1\end{pmatrix}^2 \\
&= \pi L(L+1)\sum_{\ell\ell'}\frac{2\ell+1}{4\pi}A_\ell\frac{2\ell'+1}{4\pi}B_{\ell'}2\ell'(\ell'+1)\left[\begin{pmatrix}\ell & L & \ell'\\ 0 & 1 & -1\end{pmatrix}^2 + c_x^2\begin{pmatrix}\ell & L & \ell'\\ 0 & -1 & 1\end{pmatrix}\begin{pmatrix}\ell & L & \ell'\\ 0 & 1 & -1\end{pmatrix}\right] \\
&= \int_{-1}^{1}\mathrm{d}\mu\;\pi L(L+1)\sum_{\ell\ell'}\frac{2\ell+1}{4\pi}A_\ell\frac{2\ell'+1}{4\pi}B_{\ell'}\ell'(\ell'+1)[d^\ell_{00}d^L_{11}d^{\ell'}_{11}+c_x^2d^\ell_{00}d^L_{1,-1}d^{\ell'}_{1,-1}] \\
&= \int_{-1}^{1}\mathrm{d}\mu\;\pi L(L+1)\{\xi_{00}[A]\xi_{11}[B^{00}]d^L_{11}+c_x^2\xi_{00}[A]\xi_{1,-1}[B^{00}]d^L_{1,-1}\}\,,
\end{aligned} \tag{121}$$

where

$$\xi_{mm'}[A] = \sum_\ell\frac{2\ell+1}{4\pi}A_\ell d^\ell_{mm'}\,. \tag{122}$$

The cross-term is

$$\begin{aligned}
\Gamma^{(0),x}_L[A,B] &= \frac{1}{2L+1}\sum_{\ell\ell'}(W^{x,0}_{\ell L\ell'})^*W^{x,0}_{\ell' L\ell}A_\ell B_{\ell'} \\
&= \sum_{\ell\ell'}4\pi L(L+1)\frac{2\ell+1}{4\pi}A_\ell\frac{2\ell'+1}{4\pi}B_{\ell'}a^0_\ell a^0_{\ell'}\frac{1+c_x^2(-1)^{\ell+L+\ell'}}{2}\begin{pmatrix}\ell & L & \ell'\\ 0 & 1 & -1\end{pmatrix}\begin{pmatrix}\ell' & L & \ell\\ 0 & 1 & -1\end{pmatrix} \\
&= \pi L(L+1)\sum_{\ell\ell'}\frac{2\ell+1}{4\pi}A^0_\ell\frac{2\ell'+1}{4\pi}B^0_{\ell'}2\left[\begin{pmatrix}\ell & L & \ell'\\ 0 & 1 & -1\end{pmatrix}\begin{pmatrix}\ell & L & \ell'\\ 1 & -1 & 0\end{pmatrix} + c_x^2\begin{pmatrix}\ell & L & \ell'\\ 0 & 1 & -1\end{pmatrix}\begin{pmatrix}\ell & L & \ell'\\ -1 & 1 & 0\end{pmatrix}\right] \\
&= \int_{-1}^{1}\mathrm{d}\mu\;\pi L(L+1)\sum_{\ell\ell'}\frac{2\ell+1}{4\pi}A^0_\ell\frac{2\ell'+1}{4\pi}B^0_{\ell'}[d^\ell_{01}d^L_{1,-1}d^{\ell'}_{-1,0}+c_x^2d^\ell_{0,-1}d^L_{11}d^{\ell'}_{-1,0}] \\
&= -\int_{-1}^{1}\mathrm{d}\mu\;\pi L(L+1)\{\xi_{01}[A^0]\xi_{0,-1}[B^0]d^L_{1,-1}+c_x^2\xi_{01}[A^0]\xi_{01}[B^0]d^L_{11}\}\,.
\end{aligned} \tag{123}$$

Denoting $p = \pm$ and $x = \phi, \varpi$, we rewrite the kernel for polarization as

$$\Sigma_L^{(p),x}[A,B] = \frac{1}{2L+1}\sum_{\ell\ell'}|W^{x,p}_{\ell L\ell'}|^2 A_\ell B_{\ell'}$$
$$= \pi L(L+1)\sum_{\ell\ell'}\frac{2\ell+1}{4\pi}A_\ell\frac{2\ell'+1}{4\pi}B_{\ell'}2[1+pc_x^2(-1)^{\ell+L+\ell'}]\left[a^+_{\ell'}\begin{pmatrix}\ell & L & \ell'\\ -2 & -1 & 3\end{pmatrix} + c_x^2 a^-_{\ell'}\begin{pmatrix}\ell & L & \ell'\\ -2 & 1 & 1\end{pmatrix}\right]^2$$
$$= \pi L(L+1)\sum_{\ell\ell'}\frac{2\ell+1}{4\pi}A_\ell\frac{2\ell'+1}{4\pi}B_{\ell'}[1+pc_x^2(-1)^{\ell+L+\ell'}]$$
$$\times 2\left[(a^+_{\ell'})^2\begin{pmatrix}\ell & L & \ell'\\ -2 & -1 & 3\end{pmatrix}^2 + (a^-_{\ell'})^2\begin{pmatrix}\ell & L & \ell'\\ -2 & 1 & 1\end{pmatrix}^2 + 2c_x^2a^+_{\ell'}a^-_{\ell'}\begin{pmatrix}\ell & L & \ell'\\ -2 & 1 & 1\end{pmatrix}\begin{pmatrix}\ell & L & \ell'\\ -2 & -1 & 3\end{pmatrix}\right]$$
$$= \pi L(L+1)\sum_{\ell\ell'}\frac{2\ell+1}{4\pi}A_\ell\frac{2\ell'+1}{4\pi}B_{\ell'}$$
$$\times 2\left[(a^+_{\ell'})^2\begin{pmatrix}\ell & L & \ell'\\ -2 & -1 & 3\end{pmatrix}^2 + (a^-_{\ell'})^2\begin{pmatrix}\ell & L & \ell'\\ -2 & 1 & 1\end{pmatrix}^2 + 2c_x^2a^+_{\ell'}a^-_{\ell'}\begin{pmatrix}\ell & L & \ell'\\ -2 & 1 & 1\end{pmatrix}\begin{pmatrix}\ell & L & \ell'\\ -2 & -1 & 3\end{pmatrix}\right.$$
$$\left.+ pc_x^2(a^+_{\ell'})^2\begin{pmatrix}\ell & L & \ell'\\ -2 & -1 & 3\end{pmatrix}\begin{pmatrix}\ell & L & \ell'\\ 2 & 1 & -3\end{pmatrix} + pc_x^2(a^-_{\ell'})^2\begin{pmatrix}\ell & L & \ell'\\ -2 & 1 & 1\end{pmatrix}\begin{pmatrix}\ell & L & \ell'\\ 2 & -1 & -1\end{pmatrix} + 2pa^+_{\ell'}a^-_{\ell'}\begin{pmatrix}\ell & L & \ell'\\ -2 & 1 & 1\end{pmatrix}\begin{pmatrix}\ell & L & \ell'\\ 2 & 1 & -3\end{pmatrix}\right]$$
$$= \int_{-1}^{1}\mathrm{d}\mu\ \pi L(L+1)\sum_{\ell\ell'}\frac{2\ell+1}{4\pi}A_\ell\frac{2\ell'+1}{4\pi}B_{\ell'}[(a^+_{\ell'})^2d^\ell_{22}d^L_{11}d^{\ell'}_{33} + (a^-_{\ell'})^2d^\ell_{22}d^L_{11}d^{\ell'}_{11}$$
$$+ 2c_x^2a^+_{\ell'}a^-_{\ell'}d^\ell_{22}d^L_{1,-1}d^{\ell'}_{13} + pc_x^2(a^+_{\ell'})^2d^\ell_{-2,2}d^L_{-1,1}d^{\ell'}_{3,-3} + pc_x^2(a^-_{\ell'})^2d^\ell_{-2,2}d^L_{1,-1}d^{\ell'}_{1,-1} + 2pa^+_{\ell'}a^-_{\ell'}d^\ell_{-2,2}d^L_{11}d^{\ell'}_{1,-3}]$$
$$= \int_{-1}^{1}\mathrm{d}\mu\ \pi L(L+1)[(\xi_{22}[A]\xi_{33}[B^{++}] + \xi_{22}[A]\xi_{11}[B^{--}] + 2p\xi_{2,-2}[A]\xi_{3,-1}[B^{+-}])d^L_{11}$$
$$+ c_x^2(p\xi_{2,-2}[A]\xi_{3,-3}[B^{++}] + p\xi_{2,-2}[A]\xi_{1,-1}[B^{--}] + 2\xi_{22}[A]\xi_{31}[B^{+-}])d^L_{1,-1}]\,, \quad (124)$$

and

$$
\begin{aligned}
\Gamma_L^{(p),x}[A,B] &= \frac{1}{2L+1}\sum_{\ell\ell'}(W^{x,p}_{\ell L\ell'})^* W^{x,p}_{\ell' L\ell} A_\ell B_{\ell'} \\
&= \pi L(L+1)\sum_{\ell\ell'}\frac{2\ell+1}{4\pi}A_\ell\frac{2\ell'+1}{4\pi}B_{\ell'}2[1+pc_x^2(-1)^{\ell+L+\ell'}] \\
&\quad\times\left[a^+_{\ell'}\begin{pmatrix}\ell & L & \ell' \\ -2 & -1 & 3\end{pmatrix}+c_x^2a^-_{\ell'}\begin{pmatrix}\ell & L & \ell' \\ -2 & 1 & 1\end{pmatrix}\right]\left[a^+_{\ell}\begin{pmatrix}\ell' & L & \ell \\ -2 & -1 & 3\end{pmatrix}+c_x^2a^-_{\ell}\begin{pmatrix}\ell' & L & \ell \\ -2 & 1 & 1\end{pmatrix}\right] \\
&= \pi L(L+1)\sum_{\ell\ell'}\frac{2\ell+1}{4\pi}A_\ell\frac{2\ell'+1}{4\pi}B_{\ell'}2[(-1)^{\ell+L+\ell'}+pc_x^2] \\
&\quad\times\left[a^+_{\ell'}\begin{pmatrix}\ell & L & \ell' \\ -2 & -1 & 3\end{pmatrix}+c_x^2a^-_{\ell'}\begin{pmatrix}\ell & L & \ell' \\ -2 & 1 & 1\end{pmatrix}\right]\left[a^+_{\ell}\begin{pmatrix}\ell & L & \ell' \\ 3 & -1 & -2\end{pmatrix}+c_x^2a^-_{\ell}\begin{pmatrix}\ell & L & \ell' \\ 1 & 1 & -2\end{pmatrix}\right] \\
&= \pi L(L+1)\sum_{\ell\ell'}\frac{2\ell+1}{4\pi}A_\ell\frac{2\ell'+1}{4\pi}B_{\ell'} \\
&\quad\times 2\Bigg\{\left[a^+_{\ell'}\begin{pmatrix}\ell & L & \ell' \\ 2 & 1 & -3\end{pmatrix}+c_x^2a^-_{\ell'}\begin{pmatrix}\ell & L & \ell' \\ 2 & -1 & -1\end{pmatrix}\right]\left[a^+_{\ell}\begin{pmatrix}\ell & L & \ell' \\ 3 & -1 & -2\end{pmatrix}+c_x^2a^-_{\ell}\begin{pmatrix}\ell & L & \ell' \\ 1 & 1 & -2\end{pmatrix}\right] \\
&\quad+p\left[c_x^2a^+_{\ell'}\begin{pmatrix}\ell & L & \ell' \\ -2 & -1 & 3\end{pmatrix}+a^-_{\ell'}\begin{pmatrix}\ell & L & \ell' \\ -2 & 1 & 1\end{pmatrix}\right]\left[a^+_{\ell}\begin{pmatrix}\ell & L & \ell' \\ 3 & -1 & -2\end{pmatrix}+c_x^2a^-_{\ell}\begin{pmatrix}\ell & L & \ell' \\ 1 & 1 & -2\end{pmatrix}\right]\Bigg\} \\
&= \int_{-1}^1 \mathrm{d}\mu\ \pi L(L+1)\sum_{\ell\ell'}\frac{2\ell+1}{4\pi}A_\ell\frac{2\ell'+1}{4\pi}B_{\ell'} \\
&\quad\times[a^+_{\ell'}a^+_\ell d^\ell_{23}d^L_{1,-1}d^{\ell'}_{-3,-2}+c_x^2a^+_{\ell'}a^-_\ell d^\ell_{21}d^L_{11}d^{\ell'}_{-3,-2}+c_x^2a^-_{\ell'}a^+_\ell d^\ell_{23}d^L_{11}d^{\ell'}_{-1,-2}+a^-_{\ell'}a^-_\ell d^\ell_{21}d^L_{1,-1}d^{\ell'}_{-1,-2} \\
&\quad+p(c_x^2a^+_{\ell'}a^+_\ell d^\ell_{-2,3}d^L_{11}d^{\ell'}_{3,-2}+a^+_{\ell'}a^-_\ell d^\ell_{-2,1}d^L_{1,-1}d^{\ell'}_{3,-2}+a^-_{\ell'}a^+_\ell d^\ell_{-2,3}d^L_{1,-1}d^{\ell'}_{1,-2}+c_x^2a^-_{\ell'}a^-_\ell d^\ell_{-2,1}d^L_{11}d^{\ell'}_{1,-2})] \\
&= \int_{-1}^1 \mathrm{d}\mu\ \pi L(L+1)[-c_x^2(\xi_{21}[A^-]\xi_{32}[B^+]+\xi_{32}[A^+]\xi_{21}[B^-]+p\xi_{3,-2}[A^+]\xi_{3,-2}[B^+]+p\xi_{2,-1}[A^-]\xi_{2,-1}[B^-])d^L_{11} \\
&\qquad+(\xi_{32}[A^+]\xi_{32}[B^+]+\xi_{21}[A^-]\xi_{21}[B^-]-p\xi_{2,-1}[A^-]\xi_{3,-2}[B^+]-p\xi_{3,-2}[A^+]\xi_{2,-1}[B^-])d^L_{1,-1}\,.
\end{aligned}
\tag{125}
$$

The temperature-polarization kernel is

$$
\begin{aligned}
\Sigma_L^{(\times),x}[A,B] &= \frac{1}{2L+1}\sum_{\ell\ell'}(W^{x,0}_{\ell L\ell'})^* W^{x,+}_{\ell L\ell'} A_\ell B_{\ell'} \\
&= \pi L(L+1)\sum_{\ell\ell'}\frac{2\ell+1}{4\pi}A_\ell\frac{2\ell'+1}{4\pi}B_{\ell'}a^0_{\ell'}2[1+c_x^2(-1)^{\ell+L+\ell'}] \\
&\quad\times\begin{pmatrix}\ell & L & \ell' \\ 0 & 1 & -1\end{pmatrix}\left[a^+_{\ell'}\begin{pmatrix}\ell & L & \ell' \\ -2 & -1 & 3\end{pmatrix}+c_x^2a^-_{\ell'}\begin{pmatrix}\ell & L & \ell' \\ -2 & 1 & 1\end{pmatrix}\right] \\
&= \pi L(L+1)\sum_{\ell\ell'}\frac{2\ell+1}{4\pi}A_\ell\frac{2\ell'+1}{4\pi}B_{\ell'}a^0_{\ell'} \\
&\quad\times 2\left[\begin{pmatrix}\ell & L & \ell' \\ 0 & 1 & -1\end{pmatrix}+c_x^2\begin{pmatrix}\ell & L & \ell' \\ 0 & -1 & 1\end{pmatrix}\right]\left[a^+_{\ell'}\begin{pmatrix}\ell & L & \ell' \\ -2 & -1 & 3\end{pmatrix}+c_x^2a^-_{\ell'}\begin{pmatrix}\ell & L & \ell' \\ -2 & 1 & 1\end{pmatrix}\right] \\
&= \int_{-1}^1 \mathrm{d}\mu\ \pi L(L+1)\sum_{\ell\ell'}\frac{2\ell+1}{4\pi}A_\ell\frac{2\ell'+1}{4\pi}B_{\ell'}a^0_{\ell'} \\
&\quad\times\left[a^+_{\ell'}d^\ell_{0,-2}d^L_{1,-1}d^{\ell'}_{-1,3}+c_x^2a^-_{\ell'}d^\ell_{0,-2}d^L_{11}d^{\ell'}_{-1,1}+c_x^2a^+_{\ell'}d^\ell_{0,-2}d^L_{11}d^{\ell'}_{13}+a^-_{\ell'}d^\ell_{0,-2}d^L_{-1,1}d^{\ell'}_{11}\right] \\
&= \int_{-1}^1 \mathrm{d}\mu\ \pi L(L+1)\{c_x^2(\xi_{20}[A]\xi_{1,-1}[B^{0-}]+\xi_{20}[A]\xi_{31}[B^{0+}])d^L_{11} \\
&\qquad+(\xi_{20}[A]\xi_{3,-1}[B^{0+}]+\xi_{20}[A]\xi_{11}[B^{0-}])d^L_{1,-1}\}\,,
\end{aligned}
\tag{126}
$$

and

$$
\begin{aligned}
\Gamma_L^{(\times),x}[A,B] &= \frac{1}{2L+1}\sum_{\ell\ell'}(W^{x,0}_{\ell L\ell'})^* W^{x,+}_{\ell' L\ell}A_\ell B_{\ell'} \\
&= \pi L(L+1)\sum_{\ell\ell'}\frac{2\ell+1}{4\pi}A_\ell\frac{2\ell'+1}{4\pi}B_{\ell'}a^0_{\ell'}2[1+c_x^2(-1)^{\ell+L+\ell'}] \\
&\quad\times\begin{pmatrix}\ell & L & \ell' \\ 0 & 1 & -1\end{pmatrix}\left[a^+_\ell\begin{pmatrix}\ell' & L & \ell \\ -2 & -1 & 3\end{pmatrix}+c_x^2a^-_\ell\begin{pmatrix}\ell' & L & \ell \\ -2 & 1 & 1\end{pmatrix}\right] \\
&= \pi L(L+1)\sum_{\ell\ell'}\frac{2\ell+1}{4\pi}A_\ell\frac{2\ell'+1}{4\pi}B_{\ell'}a^0_{\ell'} \\
&\quad\times 2\left[\begin{pmatrix}\ell & L & \ell' \\ 0 & 1 & -1\end{pmatrix}+c_x^2\begin{pmatrix}\ell & L & \ell' \\ 0 & -1 & 1\end{pmatrix}\right]\left[a^+_\ell\begin{pmatrix}\ell & L & \ell' \\ -3 & 1 & 2\end{pmatrix}+c_x^2a^-_\ell\begin{pmatrix}\ell & L & \ell' \\ -1 & -1 & 2\end{pmatrix}\right] \\
&= \int_{-1}^1\mathrm{d}\mu\;\pi L(L+1)\sum_{\ell\ell'}\frac{2\ell+1}{4\pi}A_\ell\frac{2\ell'+1}{4\pi}B_{\ell'}a^0_{\ell'} \\
&\quad\times\left[a^+_\ell d^\ell_{0,-3}d^L_{11}d^{\ell'}_{-1,2}+c_x^2a^-_\ell d^\ell_{0,-1}d^L_{1,-1}d^{\ell'}_{-1,2}+c_x^2a^+_\ell d^\ell_{0,-3}d^L_{1,-1}d^{\ell'}_{12}+a^-_\ell d^\ell_{0,-1}d^L_{11}d^{\ell'}_{12}\right] \\
&= \int_{-1}^1\mathrm{d}\mu\;\pi L(L+1)\{-(\xi_{30}[A^+]\xi_{2,-1}[B^0]+\xi_{10}[A^-]\xi_{12}[B^0])d^L_{11} \\
&\quad-c_x^2(\xi_{10}[A^-]\xi_{2,-1}[B^0]+\xi_{30}[A^0]\xi_{21}[B^-])d^L_{1,-1}\}\,.
\end{aligned}
\tag{127}
$$

6.2 Kernel Functions: Rotation

Next we consider the kernel functions for $x=\alpha$. If $p=-$ and $x=\alpha$,

$$
\begin{aligned}
\Sigma_L^{(-),\alpha}[A,B] &= \pi\sum_{\ell\ell'}\frac{2\ell+1}{4\pi}A_\ell\frac{2\ell'+1}{4\pi}B_{\ell'}8[1+(-1)^{\ell+L+\ell'}]\begin{pmatrix}\ell & L & \ell' \\ -2 & 0 & 2\end{pmatrix}^2 \\
&= \pi L(L+1)\sum_{\ell\ell'}\frac{2\ell+1}{4\pi}A_\ell\frac{2\ell'+1}{4\pi}B_{\ell'}8\left[\begin{pmatrix}\ell & L & \ell' \\ -2 & 0 & 2\end{pmatrix}^2+\begin{pmatrix}\ell & L & \ell' \\ -2 & 0 & 2\end{pmatrix}\begin{pmatrix}\ell & L & \ell' \\ 2 & 0 & -2\end{pmatrix}\right] \\
&= \int_{-1}^1\mathrm{d}\mu\;\pi L(L+1)\sum_{\ell\ell'}\frac{2\ell+1}{4\pi}A_\ell\frac{2\ell'+1}{4\pi}B_{\ell'}4(d^\ell_{-2,-2}d^L_{00}d^{\ell'}_{22}+d^\ell_{-2,2}d^L_{00}d^{\ell'}_{2,-2}) \\
&= \int_{-1}^1\mathrm{d}\mu\;4\pi(\xi_{-2,-2}[A]\xi_{22}[B]+\xi_{-2,22}[A]\xi_{22}[B])d^L_{00}\,,
\end{aligned}
\tag{128}
$$

and

$$
\begin{aligned}
\Gamma_L^{(-),\alpha}[A,B] &= \pi\sum_{\ell\ell'}\frac{2\ell+1}{4\pi}A_\ell\frac{2\ell'+1}{4\pi}B_{\ell'}8[1+(-1)^{\ell+L+\ell'}]\begin{pmatrix}\ell & L & \ell' \\ -2 & 0 & 2\end{pmatrix}\begin{pmatrix}\ell' & L & \ell \\ -2 & 0 & 2\end{pmatrix} \\
&= \pi\sum_{\ell\ell'}\frac{2\ell+1}{4\pi}A_\ell\frac{2\ell'+1}{4\pi}B_{\ell'} \\
&\quad\times 8\begin{pmatrix}\ell & L & \ell' \\ -2 & 0 & 2\end{pmatrix}\left[\begin{pmatrix}\ell & L & \ell' \\ -2 & 0 & 2\end{pmatrix}+\begin{pmatrix}\ell & L & \ell' \\ 2 & 0 & -2\end{pmatrix}\right] \\
&= \int_{-1}^1\mathrm{d}\mu\;\pi\sum_{\ell\ell'}\frac{2\ell+1}{4\pi}A_\ell\frac{2\ell'+1}{4\pi}B_{\ell'}4[d^\ell_{-2,-2}d^L_{00}d^{\ell'}_{22}+d^\ell_{-2,2}d^L_{00}d^{\ell'}_{2,-2}] \\
&= \int_{-1}^1\mathrm{d}\mu\;4\pi(\xi_{-2,-2}[A]\xi_{22}[B]+\xi_{-2,2}[A]\xi_{2,-2}[B])d^L_{00}\,.
\end{aligned}
\tag{129}
$$

7 Computing delensed CMB anisotropies

7.1 Linear template of lensing B mode

The gradient of lensing potential $\boldsymbol{\nabla}\phi$ is transformed as

$$\begin{aligned}\boldsymbol{\nabla}\phi &= \sum_{\ell m}\boldsymbol{\nabla}Y_{\ell m}\phi_{\ell m} = \sum_{\ell m}\sqrt{\frac{\ell(\ell+1)}{2}}\phi_{\ell m}(Y^1_{\ell m}\boldsymbol{e}^* - Y^{-1}_{\ell m}\boldsymbol{e}) \\ &= (\phi^+ + \mathrm{i}\phi^-)\boldsymbol{e}^* + (\phi^+ - \mathrm{i}\phi^-)\boldsymbol{e}\,, \end{aligned} \tag{130}$$

where $\phi^\pm$ is obtained by spin-1 inverse harmonic transform of $\phi_{\ell m}\sqrt{\ell(\ell+1)/2}$. Similarly the gradient of polarization $\boldsymbol{\nabla}P^\pm = \boldsymbol{\nabla}(Q \pm \mathrm{i}U)$ is

$$\begin{aligned}\boldsymbol{\nabla}P^+ &= -\sum_{\ell m}E_{\ell m}\boldsymbol{\nabla}Y^2_{\ell m} \\ &= -\sum_{\ell m}E_{\ell m}\left(\sqrt{\frac{(\ell-2)(\ell+3)}{2}}Y^3_{\ell m}\boldsymbol{e}^* - \sqrt{\frac{(\ell+2)(\ell-1)}{2}}Y^1_{\ell m}\boldsymbol{e}\right) \\ &= -(E^+_3 + \mathrm{i}E^-_3)\boldsymbol{e}^* + (E^+_1 + \mathrm{i}E^-_1)\boldsymbol{e}\,, \end{aligned} \tag{131}$$

$$\boldsymbol{\nabla}P^- = (\boldsymbol{\nabla}P^+)^* = (E^+_1 - \mathrm{i}E^-_1)\boldsymbol{e}^* - (E^+_3 - \mathrm{i}E^-_3)\boldsymbol{e}\,. \tag{132}$$

This leads to

$$\boldsymbol{\nabla}\phi\cdot\boldsymbol{\nabla}P^+ = -(E^+_3 + \mathrm{i}E^-_3)(\phi^+ - \mathrm{i}\phi^-) + (E^+_1 + \mathrm{i}E^-_1)(\phi^+ + \mathrm{i}\phi^-) \tag{133}$$

$$\boldsymbol{\nabla}\phi\cdot\boldsymbol{\nabla}P^- = (\boldsymbol{\nabla}\phi\cdot\boldsymbol{\nabla}P^+)^*\,. \tag{134}$$

The harmonic transform of the above quantity becomes the E/B combinations.

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