Computing quadratic estimator, delensing in curvedsky

Toshiya Namikawa

Latest version

Abstract

Here, I describe an algorithm for computing the quadratic estimator and its normalization of the lensing, cosmic bi-refringence, patchy reionization, and so on.

Contents

1 Notations

In the followings, we use small letters for multipoles of the CMB anisotropies (e.g., ℓ), while large letters are used for multipoles of the distortion fields (lensing, rotation, etc).

1.1 CMB

Θ denotes the CMB temperature fluctuations, and Q and U denote the Stokes parameters of the CMB linear polarization. The following equation defines the harmonic coefficients of the temperature anisotropies (and, in general, any scalar quantities x):

$$x_{LM} = \int \mathrm{d}^2\hat{\boldsymbol{n}}\ Y^*_{LM}(\hat{\boldsymbol{n}})x(\hat{\boldsymbol{n}})\,. \tag{1}$$

where Y_{LM} is the spin-0 spherical harmonics. On the other hand, Q and U are changed by the rotation of the sphere, and are therefore usually transformed into the rotational invariant quantities, the E and B modes, as

$$[E \pm \mathrm{i}B]_{\ell m} = \int \mathrm{d}^2\hat{\boldsymbol{n}}\ [Y^{\pm 2}_{\ell m}(\hat{\boldsymbol{n}})]^*[Q \pm \mathrm{i}U](\hat{\boldsymbol{n}})\,. \tag{2}$$

Here, $Y^{\pm 2}_{\ell m}$ is the spin-2 spherical harmonics. For short notation, we also use

$$\begin{aligned} \Xi^{\pm} &= E \pm \mathrm{i}B\,, \\ P^{\pm} &= Q \pm \mathrm{i}U \end{aligned} \tag{3}$$

1.2 Gravitational weak lensing

The lensing effect on CMB anisotropies is described as remapping of the unlensed CMB anisotropies by the deflection angle [1, 2]

$$X(\hat{\boldsymbol{n}}) = X[\hat{\boldsymbol{n}} + \boldsymbol{d}(\hat{\boldsymbol{n}})]\,, \tag{4}$$

where X is Θ or $P^{\pm}$. The deflection angle of the CMB lensing is decomposed into the lensing potential, ϕ, and curl mode, ϖ, as [3]

$$\boldsymbol{d}(\hat{\boldsymbol{n}}) = \boldsymbol{\nabla}\phi(\hat{\boldsymbol{n}}) + (\star\boldsymbol{\nabla})\varpi(\hat{\boldsymbol{n}}) \,, \tag{5}$$

where the operator $\star\boldsymbol{\nabla}$ denotes the derivatives with 90° rotation counterclockwise on the plane perpendicular to the line-of-sight direction and then operation. The harmonic coefficients of ϕ and ϖ are given by Eq. (1). The remapping of the CMB anisotropies is then given by

$$X(\hat{\boldsymbol{n}}) = X(\hat{\boldsymbol{n}}) + [\boldsymbol{\nabla}\phi(\hat{\boldsymbol{n}}) + (\star\boldsymbol{\nabla})\varpi(\hat{\boldsymbol{n}})]\cdot\boldsymbol{\nabla}X + \mathcal{O}(\phi^2,\varpi^2) \,. \tag{6}$$

1.3 Polarization angle rotation

If the rotation angle is small, the modulation of polarization after a rotation by an angle α is given by (e.g. [4])

$$\delta P^{\pm}(\hat{\boldsymbol{n}}) = \pm 2\mathrm{i}\alpha(\hat{\boldsymbol{n}})P^{\pm}(\hat{\boldsymbol{n}}) \,. \tag{7}$$

The harmonic coefficients of α is given by Eq. (1).

1.4 Amplitude modulations

Survey window, gain fluctuations, and the inhomogeneities of the reionization, could vary the amplitudes of the CMB fluctuations across the sky. Denoting the modulations as $1+\tau(\hat{\boldsymbol{n}})$, this leads to the modulation in CMB temperature and polarization as (e.g. [5])

$$\begin{aligned}\delta\Theta(\hat{\boldsymbol{n}}) &= \tau(\hat{\boldsymbol{n}})\Theta(\hat{\boldsymbol{n}}) \,, \\ \delta P^{\pm}(\hat{\boldsymbol{n}}) &= \tau(\hat{\boldsymbol{n}})P^{\pm}(\hat{\boldsymbol{n}}) \,.\end{aligned} \tag{8}$$

The harmonic coefficients of τ is given by Eq. (1).

1.5 Spherical Harmonics and Wigner-3j

The spherical harmonics is related to the Wigner-3j symbols as [6]

$$\int \mathrm{d}^2\hat{\boldsymbol{n}}\; Y^{s_1}_{\ell_1 m_1} Y^{s_2}_{\ell_2 m_2} Y^{s_3}_{\ell_3 m_3} = \sqrt{\frac{(2\ell_1+1)(2\ell_2+1)(2\ell_3+1)}{4\pi}} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ -s_1 & -s_2 & -s_3 \end{pmatrix} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \,, \tag{9}$$

with $s_1+s_2+s_3=0$ and $m_1+m_2+m_3=0$.

1.6 Derivatives of Spherical Harmonics

In general, denoting $a^s_\ell = \sqrt{(\ell-s)(\ell+s)/2}$, the derivative of the spherical harmonics is given by [6]

$$\boldsymbol{\nabla}Y^s_{\ell m} = a^s_\ell Y^{s+1}_{\ell m}\boldsymbol{e}^* - a^{-s}_\ell Y^{s-1}_{\ell m}\boldsymbol{e} \,. \tag{10}$$

Here, we introduce the polarization vector $\boldsymbol{e}$ which are defined

$$\boldsymbol{e} = \frac{\boldsymbol{e}_1 + \mathrm{i}\boldsymbol{e}_2}{\sqrt{2}} \tag{11}$$

with $\boldsymbol{e}_i$ denoting the basis vectors orthogonal to the radial vector. The polarization vector satisfies $\boldsymbol{e}\cdot\boldsymbol{e} = 0$, $\boldsymbol{e}\cdot\boldsymbol{e}^* = 1$, $\star\boldsymbol{e} = -\mathrm{i}\boldsymbol{e}$. In particular, for $s=0$,

$$\boldsymbol{\nabla}Y_{\ell m} = a^0_\ell\big(Y^1_{\ell m}\boldsymbol{e}^* - Y^{-1}_{\ell m}\boldsymbol{e}\big) \,, \tag{12}$$

and, for $s=\pm 2$, denoting $a^{\pm} = a^{\pm 2}_\ell$,

$$\begin{aligned}\boldsymbol{\nabla}Y^2_{\ell m} &= a^+_\ell Y^3_{\ell m}\boldsymbol{e}^* - a^-_\ell Y^1_{\ell m}\boldsymbol{e} \,, \\ \boldsymbol{\nabla}Y^{-2}_{\ell m} &= a^-_\ell Y^{-1}_{\ell m}\boldsymbol{e}^* - a^+_\ell Y^{-3}_{\ell m}\boldsymbol{e} \,.\end{aligned} \tag{13}$$

1.7 Map derivatives

Derivative of scalar quantities such as the CMB temperature fluctuations and lensing potential is

$$\boldsymbol{\nabla} x = \sum_{LM} x_{LM} \boldsymbol{\nabla} Y_{LM} = \sum_{LM} x_{LM} a_L^0 \left(Y_{LM}^1 \boldsymbol{e}^* - Y_{LM}^{-1} \boldsymbol{e} \right) = x^+ \boldsymbol{e}^* - x^- \boldsymbol{e}\,. \tag{14}$$

where we define

$$x^\pm \equiv \sum_{LM} x_{LM} a_L^0 Y_{LM}^{\pm 1}\,, \tag{15}$$

and $(x^+)^* = -x^-$. The rotation of a pseudo-scalar quantity is given by

$$(\star\boldsymbol{\nabla})\varpi = \sum_{LM} \varpi_{LM} (\star\boldsymbol{\nabla}) Y_{LM} = \sum_{LM} \varpi_{LM} a_L^0 \mathrm{i} \left(Y_{LM}^1 \boldsymbol{e}^* + Y_{LM}^{-1} \boldsymbol{e} \right) = \mathrm{i}(\varpi^+ \boldsymbol{e}^* + \varpi^- \boldsymbol{e})\,, \tag{16}$$

and $(\varpi^+)^* = -\varpi^-$. Spin-2 fields such as the CMB linear polarization is given by

$$\boldsymbol{\nabla} P^+ = \sum_{\ell m} \Xi_{\ell m}^+ \boldsymbol{\nabla} Y_{\ell m}^2 = \sum_{\ell m} \Xi_{\ell m}^+ \left(a_\ell^+ Y_{\ell m}^3 \boldsymbol{e}^* - a_\ell^- Y_{\ell m}^1 \boldsymbol{e} \right) = \Xi^{+^+} \boldsymbol{e}^* - \Xi^{+^-} \boldsymbol{e}\,, \tag{17}$$

$$\boldsymbol{\nabla} P^- = (\boldsymbol{\nabla} P^+)^* = \sum_{\ell m} \Xi_{\ell m}^- \boldsymbol{\nabla} Y_{\ell m}^{-2} = \sum_{\ell m} \Xi_{\ell m}^- \left(a_\ell^- Y_{\ell m}^{-1} \boldsymbol{e}^* - a_\ell^+ Y_{\ell m}^{-3} \boldsymbol{e} \right) = \Xi^{-^+} \boldsymbol{e}^* - \Xi^{-^-} \boldsymbol{e}\,. \tag{18}$$

Note that $(\Xi^{+^+})^* = -\Xi^{-^-}$ and $(\Xi^{+^-})^* = -\Xi^{-^+}$.

2 Distortion of CMB anisotropies

In the following, we first define useful quantities to compute the distortion effect. A multipole factor is defined as

$$\gamma_{\ell_1\ell_2\ell_3} \equiv \sqrt{\frac{(2\ell_1+1)(2\ell_2+1)(2\ell_3+1)}{4\pi}} \,. \tag{19}$$

The convolution operator in full sky is defined as

$$\widetilde{\sum_{LM\ell'm'}}^{(\ell m)} \equiv \sum_{LM\ell'm'} (-1)^m \begin{pmatrix} \ell & L & \ell' \\ -m & M & m' \end{pmatrix} \,. \tag{20}$$

We introduce the following coefficients;

$$c_\phi = 1\,, \tag{21}$$
$$c_\varpi = -\mathrm{i}\,, \tag{22}$$
$$c_\alpha = 1\,, \tag{23}$$
$$c_\tau = 1\,, \tag{24}$$

and

$$\zeta^+ = 1\,, \tag{25}$$
$$\zeta^- = \mathrm{i}\,. \tag{26}$$

A parity symmetry indicator is given by

$$p^{x,\pm}_{\ell_1\ell_2\ell_3} \equiv c_x \frac{1 \pm c_x^2(-1)^{\ell_1+\ell_2+\ell_3}}{2} \,, \tag{27}$$

and simply

$$p^{\pm}_{\ell_1\ell_2\ell_3} \equiv \frac{1 \pm (-1)^{\ell_1+\ell_2+\ell_3}}{2} \,. \tag{28}$$

2.1 Lensing distortion

The lensing contributions in the position space become

$$\begin{aligned}
\delta^\phi\Theta &= \boldsymbol{\nabla}\phi\cdot\boldsymbol{\nabla}\Theta = -\phi^-\Theta^+ - \phi^+\Theta^- \,, \\
\delta^\varpi\Theta &= (\star\boldsymbol{\nabla})\varpi\cdot\boldsymbol{\nabla}\Theta = \mathrm{i}(\varpi^-\Theta^+ - \varpi^+\Theta^-) \,, \\
\delta^\phi P^\pm &= \boldsymbol{\nabla}\phi\cdot\boldsymbol{\nabla}P^\pm = -\phi^-\Xi^{\pm^+} - \phi^+\Xi^{\pm^-} \,, \\
\delta^\varpi P^\pm &= (\star\boldsymbol{\nabla})\varpi\cdot\boldsymbol{\nabla}P^\pm = \mathrm{i}(\varpi^-\Xi^{\pm^+} - \varpi^+\Xi^{\pm^-}) \,.
\end{aligned} \tag{29}$$

The harmonics transform of the lensing contributions in temperature is

$$\begin{aligned}
\delta\Theta_{\ell m} &= -c_x \int \mathrm{d}^2\hat{\boldsymbol{n}}\; Y^*_{\ell m}[x^-\Theta^+ + c_x^2 x^+\Theta^-] \\
&= -c_x \sum_{LM\ell'm'} x_{LM}\Theta_{\ell'm'} a^0_L a^0_{\ell'} \int \mathrm{d}^2\hat{\boldsymbol{n}}\; (-1)^m Y_{\ell,-m}[Y^{-1}_{LM}Y^1_{\ell'm'} + c_x^2 Y^1_{LM}Y^{-1}_{\ell'm'}] \\
&= -\sum_{LM\ell'm'} x_{LM}\Theta_{\ell'm'} 2a^0_L a^0_{\ell'} p^{x,+}_{\ell L\ell'} \gamma_{\ell L\ell'} (-1)^m \begin{pmatrix} \ell & L & \ell' \\ -m & M & m' \end{pmatrix} \begin{pmatrix} \ell & L & \ell' \\ 0 & 1 & -1 \end{pmatrix} \\
&= -\widetilde{\sum_{LM\ell'm'}}^{(\ell m)} x_{LM}\Theta_{\ell'm'} 2a^0_L a^0_{\ell'} p^{x,+}_{\ell L\ell'} \gamma_{\ell L\ell'} \begin{pmatrix} \ell & L & \ell' \\ 0 & 1 & -1 \end{pmatrix} \\
&= \widetilde{\sum_{LM\ell'm'}}^{(\ell m)} x_{LM}\Theta_{\ell'm'} W^{x,0}_{\ell L\ell'} \,.
\end{aligned} \tag{30}$$

Here we denote

$$W^{x,0}_{\ell_1\ell_2\ell_3} = -2a^0_{\ell_2}a^0_{\ell_3}p^{x,+}_{\ell_1\ell_2\ell_3}\gamma_{\ell_1\ell_2\ell_3}\begin{pmatrix}\ell_1 & \ell_2 & \ell_3 \\ 0 & 1 & -1\end{pmatrix} . \tag{31}$$

Here, $(W^{\phi,0}_{\ell_1\ell_2\ell_3})^* = W^{\phi,0}_{\ell_1\ell_2\ell_3}$. Note that the above quantity is consistent with Ref. [7] and also $(W^{\varpi,0}_{\ell_1\ell_2\ell_3})^* = (-1)^{\ell_1+\ell_2+\ell_3}W^{\varpi,0}_{\ell_1\ell_2\ell_3}$.

On the other hand, the lensed anisotropies for polarization are given by

$$\begin{aligned}\delta\Xi^{\pm}_{\ell m} &= -c_x\int \mathrm{d}^2\hat{\boldsymbol{n}}\ (Y^{\pm2}_{\ell m})^*[x^-\Xi^{\pm^+} + c_x^2 x^+\Xi^{\pm^-}] \\ &= -c_x\sum_{LM\ell'm'}\phi_{LM}\Xi^{\pm}_{\ell'm'}a^0_L\int \mathrm{d}^2\hat{\boldsymbol{n}}\ (Y^{\pm2}_{\ell m})^*[a^+_{\ell'}Y^{\mp1}_{LM}Y^{\pm3}_{\ell'm'} + c_x^2a^-_{\ell'}Y^{\pm1}_{LM}Y^{\pm1}_{\ell'm'}] \\ &= -c_x\sum_{LM\ell'm'}(-1)^m\begin{pmatrix}\ell & L & \ell' \\ -m & M & m'\end{pmatrix}\phi_{LM}\Xi^{\pm}_{\ell'm'}\gamma_{\ell L\ell'}a^0_L\left[a^+_{\ell'}\begin{pmatrix}\ell & L & \ell' \\ \mp2 & \mp1 & \pm3\end{pmatrix} + c_x^2a^-_{\ell'}\begin{pmatrix}\ell & L & \ell' \\ \mp2 & \pm1 & \pm1\end{pmatrix}\right] \\ &= \widetilde{\sum_{LM\ell'm'}}^{(\ell m)}\phi_{LM}\Xi^{\pm}_{\ell'm'}W^{\phi,\pm2}_{\ell L\ell'} ,\end{aligned} \tag{32}$$

with

$$\begin{aligned}W^{x,2}_{\ell_1\ell_2\ell_3} &= (-1)^{\ell_1+\ell_2+\ell_3}W^{x,-2}_{\ell_1\ell_2\ell_3} \\ &= -c_x\gamma_{\ell_1\ell_2\ell_3}a^0_{\ell_2}\left[a^+_{\ell_3}\begin{pmatrix}\ell_1 & \ell_2 & \ell_3 \\ -2 & -1 & 3\end{pmatrix} + c_x^2a^-_{\ell_3}\begin{pmatrix}\ell_1 & \ell_2 & \ell_3 \\ -2 & 1 & 1\end{pmatrix}\right] .\end{aligned} \tag{33}$$

2.2 Rotation distortion

The E and B modes after the rotation are given by

$$\begin{aligned}\delta\Xi^{\pm}_{\ell m} &= \pm2\mathrm{i}\int \mathrm{d}^2\hat{\boldsymbol{n}}\ (Y^{\pm2}_{\ell m})^*\alpha P^{\pm} \\ &= \pm2\mathrm{i}\sum_{LM\ell'm'}\alpha_{LM}\Xi^{\pm}_{\ell'm'}\int \mathrm{d}^2\hat{\boldsymbol{n}}\ (Y^{\pm2}_{\ell m})^*Y_{LM}Y^{\pm2}_{\ell'm'} \\ &= \pm2\mathrm{i}\sum_{LM\ell'm'}(-1)^m\begin{pmatrix}\ell & L & \ell' \\ -m & M & m'\end{pmatrix}\alpha_{LM}\Xi^{\pm}_{\ell'm'}\gamma_{\ell L\ell'}\begin{pmatrix}\ell & L & \ell' \\ \mp2 & 0 & \pm2\end{pmatrix} \\ &= \widetilde{\sum_{LM\ell'm'}}^{(\ell m)}\alpha_{LM}\Xi^{\pm}_{\ell'm'}W^{\alpha,\pm2}_{\ell L\ell'} ,\end{aligned} \tag{34}$$

with

$$W^{\alpha,\pm2}_{\ell_1\ell_2\ell_3} = \pm2\mathrm{i}\gamma_{\ell_1\ell_2\ell_3}\begin{pmatrix}\ell_1 & \ell_2 & \ell_3 \\ \mp2 & 0 & \pm2\end{pmatrix} . \tag{35}$$

2.3 Amplitude distortion

The harmonics transform of $\tau(\hat{\boldsymbol{n}})\Theta(\hat{\boldsymbol{n}})$ is

$$\begin{aligned}\delta\Theta_{\ell m} &= \int \mathrm{d}^2\hat{\boldsymbol{n}}\ Y^*_{\ell m}\tau(\hat{\boldsymbol{n}})\Theta(\hat{\boldsymbol{n}}) \\ &= \sum_{LM\ell'm'}\tau_{LM}\Theta_{\ell'm'}\int \mathrm{d}^2\hat{\boldsymbol{n}}\ Y^*_{\ell m}Y_{LM}Y_{\ell'm'} \\ &= \sum_{LM\ell'm'}\tau_{LM}\Theta_{\ell'm'}p^+_{\ell L\ell'}\gamma_{\ell L\ell'}(-1)^m\begin{pmatrix}\ell & L & \ell' \\ -m & M & m'\end{pmatrix}\begin{pmatrix}\ell & L & \ell' \\ 0 & 0 & 0\end{pmatrix} \\ &= \widetilde{\sum_{LM\ell'm'}}^{(\ell m)}\tau_{LM}\Theta_{\ell'm'}W^{\tau,0}_{\ell L\ell'} ,\end{aligned} \tag{36}$$

where

$$W^{\tau,0}_{\ell L\ell'} = p^{+}_{\ell L\ell'}\gamma_{\ell L\ell'}\begin{pmatrix}\ell & L & \ell' \\ 0 & 0 & 0\end{pmatrix} . \tag{37}$$

The polarization anisotropies with the amplitude distortion are given by

$$\begin{aligned}\delta\Xi^{\pm}_{\ell m} &= \int \mathrm{d}^2\hat{\boldsymbol{n}}\ (Y^{\pm 2}_{\ell m})^* \tau P^{\pm} \\ &= \widetilde{\sum_{LM\ell'm'}}^{(\ell m)} \tau_{LM}\Xi^{\pm}_{\ell' m'}W^{\tau,\pm 2}_{\ell L\ell'} ,\end{aligned} \tag{38}$$

with

$$W^{\tau,\pm 2}_{\ell_1\ell_2\ell_3} = \gamma_{\ell_1\ell_2\ell_3}\begin{pmatrix}\ell_1 & \ell_2 & \ell_3 \\ \mp 2 & 0 & \pm 2\end{pmatrix} . \tag{39}$$

2.4 Translate into E/B

Now we consider the distorted E/B modes separately. In general, if the distortion is given by

$$\delta\Xi^{\pm}_{\ell m} = \widetilde{\sum_{LM\ell'm'}}^{(\ell m)} x_{LM}\Xi^{\pm}_{\ell' m'}W^{x,\pm 2}_{\ell L\ell'} , \tag{40}$$

we obtain

$$\delta E_{\ell m} = \widetilde{\sum_{LM\ell'm'}}^{(\ell m)} x_{LM}\left(E_{\ell' m'}W^{x,+}_{\ell L\ell'} + B_{\ell' m'}W^{x,-}_{\ell L\ell'}\right) , \tag{41}$$

$$\delta B_{\ell m} = \widetilde{\sum_{LM\ell'm'}}^{(\ell m)} x_{LM}\left(-E_{\ell' m'}W^{x,-}_{\ell L\ell'} + B_{\ell' m'}W^{x,+}_{\ell L\ell'}\right) , \tag{42}$$

where we define

$$W^{x,\pm}_{\ell L\ell'} \equiv \zeta^{\pm}\frac{W^{x,+2}_{\ell L\ell'} \pm W^{x,-2}_{\ell L\ell'}}{2} . \tag{43}$$

For lensing, the functional form of W is given by

$$\begin{aligned}W^{x,\pm}_{\ell_1\ell_2\ell_3} &= \zeta^{\pm}\frac{1 \pm c_x^2(-1)^{\ell_1+\ell_2+\ell_3}}{2}W^{x,2}_{\ell_1\ell_2\ell_3} \\ &= -\zeta^{\pm}p^{x,\pm}_{\ell_1\ell_2\ell_3}\gamma_{\ell_1\ell_2\ell_3}a^0_{\ell_2}\left[a^+_{\ell_3}\begin{pmatrix}\ell_1 & \ell_2 & \ell_3 \\ -2 & -1 & 3\end{pmatrix} + c_x^2 a^-_{\ell_3}\begin{pmatrix}\ell_1 & \ell_2 & \ell_3 \\ -2 & 1 & 1\end{pmatrix}\right] ,\end{aligned} \tag{44}$$

For polarization rotation, we obtain

$$W^{\alpha,\pm}_{\ell_1\ell_2\ell_3} = \pm 2\zeta^{\mp}p^{\mp}_{\ell_1\ell_2\ell_3}\gamma_{\ell_1\ell_2\ell_3}\begin{pmatrix}\ell_1 & \ell_2 & \ell_3 \\ -2 & 0 & 2\end{pmatrix} . \tag{45}$$

This is consistent with [8] in the absence of B-mode.

For amplitude modulations, we find

$$W^{\tau,\pm}_{\ell_1\ell_2\ell_3} = \zeta^{\pm}p^{\pm}_{\ell_1\ell_2\ell_3}\gamma_{\ell_1\ell_2\ell_3}\begin{pmatrix}\ell_1 & \ell_2 & \ell_3 \\ -2 & 0 & 2\end{pmatrix} . \tag{46}$$

Note that

$$\begin{aligned}W^{\alpha,+} &= 2W^{\tau,-}_{\ell_1\ell_2\ell_3} , \\ W^{\alpha,-} &= -2W^{\tau,+}_{\ell_1\ell_2\ell_3} .\end{aligned} \tag{47}$$

2.5 Summary

The above all distortions are described in the following form:

$$\delta\Theta_{\ell m} = \widetilde{\sum_{LM\ell'm'}}^{(\ell m)} x_{LM}\Theta_{\ell'm'}W^{x,0}_{\ell L\ell'} \,, \tag{48}$$

$$\delta E_{\ell m} = \widetilde{\sum_{LM\ell'm'}}^{(\ell m)} x_{LM}\left(E_{\ell'm'}W^{x,+}_{\ell L\ell'} + B_{\ell'm'}W^{x,-}_{\ell L\ell'}\right) \,, \tag{49}$$

$$\delta B_{\ell m} = \widetilde{\sum_{LM\ell'm'}}^{(\ell m)} x_{LM}\left(-E_{\ell'm'}W^{x,-}_{\ell L\ell'} + B_{\ell'm'}W^{x,+}_{\ell L\ell'}\right) \tag{50}$$

where x is a distortion field.

The property of W is also important. If x is parity even, $W^{x,+}_{\ell L\ell'}$ and $W^{x,-}_{\ell L\ell'}$ are non-zero only when $\ell+L+\ell'$ is even and odd, respectively. If x is parity odd, $W^{x,-}_{\ell L\ell'}$ and $W^{x,+}_{\ell L\ell'}$ are non-zero only when $\ell+L+\ell'$ is even and odd, respectively. $W^{x,0}$ is the same as $W^{x,+}$.

3 Quadratic estimator

3.1 Distortion induced anisotropies

The distortion fields x described above induce the off-diagonal elements of the covariance ($\ell \neq \ell'$ or $m \neq m'$), [9, 10]

$$\langle \widetilde{X}_{\ell m} \widetilde{Y}_{\ell' m'} \rangle_{\rm CMB} = \sum_{LM} \begin{pmatrix} \ell & \ell' & L \\ m & m' & M \end{pmatrix} f^{x,({\rm XY})}_{\ell L \ell'} x^*_{LM} \,, \tag{51}$$

where $\langle \cdots \rangle_{\rm CMB}$ denotes the ensemble average over the primary CMB anisotropies with a fixed realization of the distortion fields. We ignore the higher-order terms of the distortion fields. The functional form of the weight functions f are discussed later.

With a quadratic combination of observed CMB anisotropies, $\widehat{X}$ and $\widehat{Y}$, the general quadratic estimators are formed as

$$[\widehat{x}^{\rm XY}_{LM}]^* = A^{x,({\rm XY})}_L \sum_{\ell \ell' m m'} \begin{pmatrix} \ell & \ell' & L \\ m & m' & M \end{pmatrix} g^{x,({\rm XY})}_{\ell L \ell'} \widehat{X}_{\ell m} \widehat{Y}_{\ell' m'} \,. \tag{52}$$

Here we define

$$g^{x,({\rm XY})}_{\ell L \ell'} = \frac{[f^{x,({\rm XY})}_{\ell L \ell'}]^*}{\Delta^{\rm XY} \widehat{C}^{\rm XX}_\ell \widehat{C}^{\rm YY}_{\ell'}} \tag{53}$$

$$A^{x,({\rm XY})}_L = \frac{1}{2L+1} \sum_{\ell \ell'} f^{x,({\rm XY})}_{\ell L \ell'} g^{x,({\rm XY})}_{\ell L \ell'} \,, \tag{54}$$

where $\Delta^{\rm XX} = 2$, $\Delta^{\rm EB} = \Delta^{\rm TB} = 1$, and $\widehat{C}^{\rm XX}_\ell$ ($\widehat{C}^{\rm YY}_\ell$) is the observed power spectrum.

3.2 Weight Function: Derivations

3.2.1 $\Theta\Theta$

Let us first consider the temperature case. There are two contributions to the temperature quadratic estimator, and the one is given as

$$\begin{aligned} \langle (\delta\Theta_{\ell m}) \Theta_{\ell' m'} \rangle &= \sum_{LM\ell'' m''} (-1)^{m'} \begin{pmatrix} \ell & L & \ell'' \\ -m & M & m'' \end{pmatrix} W^{x,0}_{\ell L \ell''} \langle \Theta_{\ell'' m''} \Theta_{\ell' m'} \rangle \\ &= \sum_{LM\ell'' m''} (-1)^{m'} \begin{pmatrix} \ell & L & \ell'' \\ -m & M & m'' \end{pmatrix} W^{x,0}_{\ell L \ell'} \delta_{\ell'' \ell'} \delta_{m'', -m'} (-1)^{m'} C^{\Theta\Theta}_{\ell'} \\ &= \sum_{LM} \begin{pmatrix} \ell & \ell' & L \\ m & m' & M \end{pmatrix} x^*_{LM} W^{x,0}_{\ell L \ell'} C^{\Theta\Theta}_{\ell'} \,. \end{aligned} \tag{55}$$

The other term is obtained by $(\ell'', m'') \leftrightarrow (\ell, m)$ and is given by

$$\langle \Theta_{\ell m} \delta\Theta_{\ell' m'} \rangle = \sum_{LM} (-1)^{\ell + \ell' + L} \begin{pmatrix} \ell & \ell' & L \\ m & m' & M \end{pmatrix} x^*_{LM} W^{x,0}_{\ell' L \ell} C^{\Theta\Theta}_{\ell} \,. \tag{56}$$

The sum of the above two equations yield

$$f^{x,(\Theta\Theta)}_{\ell L \ell'} = W^{x,0}_{\ell L \ell'} C^{\Theta\Theta}_{\ell'} + (-1)^{\ell + \ell' + L} W^{x,0}_{\ell' L \ell} C^{\Theta\Theta}_{\ell} \,. \tag{57}$$

The sign $(-1)^{\ell + \ell'' + L}$ depends on the parity of W; $(-1)^{\ell + \ell'' + L} = 1$ for the even parity fields (e.g. $x = \phi, \tau$) and -1 for the odd parity fields (e.g. $x = \varpi, \alpha$).

3.2.2 EB

In the EB estimator, the two contributions are given as

$$\begin{aligned}
\langle E_{\ell m}\delta B_{\ell' m'}\rangle &= \sum_{LM\ell''m''}(-1)^{m'}\begin{pmatrix}\ell' & L & \ell'' \\ -m' & M & m''\end{pmatrix} x_{LM}[-\langle E_{\ell m}E_{\ell''m''}\rangle W^{x,-}_{\ell' L\ell''} + \langle E_{\ell m}B_{\ell''m''}\rangle W^{x,+}_{\ell' L\ell''}] \\
&= \sum_{LM}(-1)^{m+m'}\begin{pmatrix}\ell' & L & \ell \\ -m' & M & -m\end{pmatrix} x_{LM}[-C^{\rm EE}_\ell W^{x,-}_{\ell' L\ell} + C^{\rm EB}_\ell W^{x,+}_{\ell' L\ell}] \\
&= \sum_{LM}(-1)^{\ell+L+\ell'+1}\begin{pmatrix}\ell' & L & \ell \\ m' & M & m\end{pmatrix} x^*_{LM}[C^{\rm EE}_\ell W^{x,-}_{\ell' L\ell} - C^{\rm EB}_\ell W^{x,+}_{\ell' L\ell}] \\
&= \sum_{LM}(-1)^{\ell+L+\ell'+1}\begin{pmatrix}\ell & \ell' & L \\ m & m' & M\end{pmatrix} x^*_{LM}[C^{\rm EE}_\ell W^{x,-}_{\ell' L\ell} - C^{\rm EB}_\ell W^{x,+}_{\ell' L\ell}]\,,\,,
\end{aligned} \tag{58}$$

and

$$\begin{aligned}
\langle(\delta E_{\ell m})B_{\ell' m'}\rangle &= \sum_{LM\ell''m''}(-1)^{m}\begin{pmatrix}\ell & L & \ell'' \\ -m & M & m''\end{pmatrix} x_{LM}[\langle B_{\ell' m'}E_{\ell''m''}\rangle W^{x,+}_{\ell L\ell''} + \langle B_{\ell' m'}B_{\ell''m''}\rangle W^{x,-}_{\ell L\ell''}] \\
&= \sum_{LM}(-1)^{m+m'}\begin{pmatrix}\ell & L & \ell' \\ -m & M & -m'\end{pmatrix} x_{LM}[C^{\rm EB}_{\ell'} W^{x,+}_{\ell L\ell'} + C^{\rm BB}_{\ell'} W^{x,-}_{\ell L\ell'}] \\
&= \sum_{LM}\begin{pmatrix}\ell & \ell' & L \\ m & m' & M\end{pmatrix} x^*_{LM}[C^{\rm EB}_{\ell'} W^{x,+}_{\ell L\ell'} + C^{\rm BB}_{\ell'} W^{x,-}_{\ell L\ell'}]\,.
\end{aligned} \tag{59}$$

Combining the above two terms, we find

$$f^{x,(EB)}_{\ell L\ell'} = C^{\rm EB}_{\ell'} W^{x,+}_{\ell L\ell'} + C^{\rm BB}_{\ell'} W^{x,-}_{\ell L\ell'} + (-1)^{\ell+L+\ell'+1}[C^{\rm EE}_\ell W^{x,-}_{\ell' L\ell} - C^{\rm EB}_\ell W^{x,+}_{\ell' L\ell}]\,, \tag{60}$$

If we decompose the terms into the following two parts,

$$\begin{aligned}
f^{x,(EB),+}_{\ell L\ell'} &= C^{\rm BB}_{\ell'} W^{x,-}_{\ell L\ell'} - (-1)^{\ell+L+\ell'} C^{\rm EE}_\ell W^{x,-}_{\ell' L\ell} \\
f^{x,(EB),-}_{\ell L\ell'} &= C^{\rm EB}_{\ell'} W^{x,+}_{\ell L\ell'} + (-1)^{\ell+L+\ell'} C^{\rm EB}_\ell W^{x,+}_{\ell' L\ell}\,,
\end{aligned} \tag{61}$$

the above two parts are orthogonal each other. This indicates that, if $C^{\rm EB}_\ell$ is non-zero due to the global rotation, even parity fields (lensing, window) leak into the odd parity estimator (rotation, curl mode) and introduce a mean-field;

$$\langle\widehat{\alpha}_{LM}\rangle = \alpha_{LM} + A^{\alpha,EB}_L \sum_{x=\phi,\tau,\dots} x_{LM}\frac{1}{2L+1}\sum_{\ell\ell'} g^{\alpha,EB}_{\ell L\ell'} f^{x,EB,{\rm even}}_{\ell L\ell'}\,. \tag{62}$$

3.3 Summary

The weight functions are, in general, given as

$$f^{x,(\Theta\Theta)}_{\ell L\ell'} = W^{x,0}_{\ell L\ell'} C^{\Theta\Theta}_{\ell'} + p_x W^{x,0}_{\ell' L\ell} C^{\Theta\Theta}_\ell\,, \tag{63}$$

$$f^{x,(\Theta E)}_{\ell L\ell'} = W^{x,0}_{\ell L\ell'} C^{\Theta {\rm E}}_{\ell'} + p_x W^{x,+}_{\ell' L\ell} C^{\Theta {\rm E}}_\ell\,, \tag{64}$$

$$f^{x,(\Theta B)}_{\ell L\ell'} = p_x W^{x,-}_{\ell' L\ell} C^{\Theta {\rm E}}_\ell\,, \tag{65}$$

$$f^{x,(EE)}_{\ell L\ell'} = W^{x,+}_{\ell L\ell'} C^{\rm EE}_{\ell'} + p_x W^{x,+}_{\ell' L\ell} C^{\rm EE}_\ell\,, \tag{66}$$

$$f^{x,(EB)}_{\ell L\ell'} = W^{x,-}_{\ell L\ell'} C^{\rm BB}_{\ell'} + p_x W^{x,-}_{\ell' L\ell} C^{\rm EE}_\ell\,, \tag{67}$$

$$f^{x,(BB)}_{\ell L\ell'} = W^{x,+}_{\ell L\ell'} C^{\rm BB}_{\ell'} + p_x W^{x,+}_{\ell' L\ell} C^{\rm BB}_\ell\,. \tag{68}$$

Here, the parity index is $p_\phi = p_\epsilon = 1$ and $p_\varpi = p_\alpha = -1$. Strictly speaking, p_x should be $(-1)^{\ell'+L+\ell}$. However, W is only non-zero when $\ell'+L+\ell$ is even, and vice versa. The parity even quantities are $x=\phi$ and

ϵ. The odd parity quantities are $x = \varpi$ and α. Note that the above weight functions are consistent with Ref. [7] ($W^{x,-}_{\ell L\ell'} = -{}_{\ominus}S^{x}_{\ell L\ell'}$) for the lensing case.

In addition, the weight functions due to the presence of ΘB and EB are given by

$$f^{x,(\Theta E)}_{\ell L\ell'} = , \tag{69}$$

$$f^{x,(\Theta B)}_{\ell L\ell'} = , \tag{70}$$

$$f^{x,(EE)}_{\ell L\ell'} = , \tag{71}$$

$$f^{x,(EB)}_{\ell L\ell'} = W^{x,+}_{\ell L\ell'} C^{\rm EB}_{\ell'} + p_x W^{x,+}_{\ell' L\ell} C^{\rm EB}_{\ell} , \tag{72}$$

$$f^{x,(BB)}_{\ell L\ell'} = \ . \tag{73}$$

4 Computing quadratic estimator

4.1 Spherical Harmonics

The polarization vectors satisfy, $\boldsymbol{e}\cdot\boldsymbol{e}^*=1$, and, $\boldsymbol{e}\cdot\boldsymbol{e}=\boldsymbol{e}^*\cdot\boldsymbol{e}^*=0$. We obtain

$$\boldsymbol{\nabla} Y^s_{\ell m} = \sqrt{\frac{(\ell-s)(\ell+s+1)}{2}}Y^{s+1}_{\ell m}\boldsymbol{e}^* - \sqrt{\frac{(\ell+s)(\ell-s+1)}{2}}Y^{s-1}_{\ell m}\boldsymbol{e}\,. \tag{74}$$

The complex conjugate is $(Y^s_{\ell m})^*=(-1)^{s+m}Y^{-s}_{\ell,-m}$. In particular, for $s=0$,

$$\boldsymbol{\nabla} Y^*_{\ell m} = \sqrt{\frac{\ell(\ell+1)}{2}}\left((Y^1_{\ell m})^*\boldsymbol{e} - (Y^{-1}_{\ell m})^*\boldsymbol{e}^*\right), \tag{75}$$

and, for $s=-2$,

$$\begin{aligned}\boldsymbol{\nabla} Y^{-2}_{\ell m} &= \sqrt{\frac{(\ell+2)(\ell-1)}{2}}Y^{-1}_{\ell m}\boldsymbol{e}^* - \sqrt{\frac{(\ell-2)(\ell+3)}{2}}Y^{-3}_{\ell m}\boldsymbol{e}\,,\\ \boldsymbol{\nabla}(Y^{-2}_{\ell m})^* &= \sqrt{\frac{(\ell+2)(\ell-1)}{2}}(Y^{-1}_{\ell m})^*\boldsymbol{e} - \sqrt{\frac{(\ell-2)(\ell+3)}{2}}(Y^{-3}_{\ell m})^*\boldsymbol{e}^*\,.\end{aligned} \tag{76}$$

4.2 Healpix

`Healpix` is a useful public package for fullsky analysis [11]. Here, we consider the `Healpix` spin-s harmonic transform of a map $S(\hat{\boldsymbol{n}})=S^+(\hat{\boldsymbol{n}})+\mathrm{i}S^-(\hat{\boldsymbol{n}})$ where $S^\pm$ is real and $s\geq 0$. The harmonic coefficient is given by

$$S^+ + \mathrm{i}S^- = \sum_{\ell m} a^s_{\ell m}Y^s_{\ell m}\,. \tag{77}$$

Note that $a^{-s}_{\ell m}$ is defined as

$$S^+ - \mathrm{i}S^- = \sum_{\ell m} a^{-s}_{\ell m}Y^{-s}_{\ell m}\,. \tag{78}$$

Then we obtain $(a^s_{\ell m})^*=(-1)^{m+s}a^{-s}_{\ell,-m}$. The subroutine `map2alm_spin` transform $S^\pm$ to $a^{s,\pm}_{\ell m}$ where

$$a^{s,+}_{\ell m} = -\frac{a^s_{\ell m}+(-1)^s a^{-s}_{\ell m}}{2} \tag{79}$$

$$a^{s,-}_{\ell m} = -\frac{a^s_{\ell m}-(-1)^s a^{-s}_{\ell m}}{2\mathrm{i}}\,, \tag{80}$$

are the rotational invariant coefficients with parity even and odd, respectively. Since $(a^s_{\ell m})^*=(-1)^{m+s}a^{-s}_{\ell,-m}$, the above coefficients satisfy

$$(a^{s,\pm}_{\ell m})^* = (-1)^m a^{s,\pm}_{\ell,-m}\,. \tag{81}$$

On the other hand, `alm2map_spin` transform $a^{s,\pm}_{\ell m}$ to $S^\pm$, but $a^{s,\pm}_{\ell m}$ should satisfy the above condition. Note that, with $S\equiv S^+ + \mathrm{i}S^-$, we find

$$a^{s,+}_{\ell m} = -\frac{1}{2}\int \mathrm{d}\hat{\boldsymbol{n}}\ [(Y^s_{\ell m})^*S + (-1)^s(Y^{-s}_{\ell m})^*S^*]\,, \tag{82}$$

$$a^{s,-}_{\ell m} = -\frac{1}{2\mathrm{i}}\int \mathrm{d}\hat{\boldsymbol{n}}\ [(Y^s_{\ell m})^*S - (-1)^s(Y^{-s}_{\ell m})^*S^*]\,. \tag{83}$$

Let us consider the case we want to transform $a_{\ell m}$ with a spin-s spherical harmonics using `alm2map_spin`. The outputs, $S^\pm$, are given by:

$$S^+ + \mathrm{i}S^- = \sum_{\ell m} a_{\ell m}Y^s_{\ell m}\,. \tag{84}$$

The complex conjugate of the above quantity becomes

$$S^+ - \mathrm{i}S^- = (-1)^s \sum_{\ell m} a_{\ell m} Y_{\ell m}^{-s} \,. \tag{85}$$

The inputs of `alm2map_spin` become

$$a_{\ell m}^{s,+} = -a_{\ell m} \,, \tag{86}$$
$$a_{\ell m}^{s,-} = 0 \,. \tag{87}$$

4.3 Lensing

In fullsky, the quadratic estimator of the gradient and curl modes are given by [10, 7]:

$$\widehat{\phi}_{\ell m}^{(\alpha)} = A_\ell^{\phi,(\alpha)} \int \mathrm{d}^2\hat{\boldsymbol{n}} \; [\boldsymbol{\nabla} Y_{\ell m}^*(\hat{\boldsymbol{n}})] \cdot \boldsymbol{v}^{(\alpha)}(\hat{\boldsymbol{n}}) \,, \tag{88}$$

$$\widehat{\varpi}_{\ell m}^{(\alpha)} = A_\ell^{\varpi,(\alpha)} \int \mathrm{d}^2\hat{\boldsymbol{n}} \; [(\star\boldsymbol{\nabla}) Y_{\ell m}^*(\hat{\boldsymbol{n}})] \cdot \boldsymbol{v}^{(\alpha)}(\hat{\boldsymbol{n}}) \,, \tag{89}$$

where $N_\ell^{x,(\alpha)}$ is the normalization of the quadratic estimator and we define

$$\boldsymbol{v}^{\Theta\Theta}(\hat{\boldsymbol{n}}) = A_\Theta^0 \boldsymbol{\nabla} A_{\Theta\Theta}^0 \,, \tag{90}$$
$$\boldsymbol{v}^{\Theta E}(\hat{\boldsymbol{n}}) = \Re(A_E^2 \boldsymbol{\nabla} A_{\Theta E}^{-2}) + A_\Theta \boldsymbol{\nabla} A_{E\Theta} \,, \tag{91}$$
$$\boldsymbol{v}^{\Theta B}(\hat{\boldsymbol{n}}) = \Re(A_{iB}^2 \boldsymbol{\nabla} A_{\Theta E}^{-2}) \,, \tag{92}$$
$$\boldsymbol{v}^{EE}(\hat{\boldsymbol{n}}) = \Re(A_E^2 \boldsymbol{\nabla} A_{EE}^{-2}) \,, \tag{93}$$
$$\boldsymbol{v}^{EB}(\hat{\boldsymbol{n}}) = \Re(A_{iB}^2 \boldsymbol{\nabla} A_{EE}^{-2}) + \Re(A_E^2 \boldsymbol{\nabla} A_{iBiB}^{-2}) \,, \tag{94}$$
$$\boldsymbol{v}^{BB}(\hat{\boldsymbol{n}}) = \Re(A_{iB}^2 \boldsymbol{\nabla} A_{iBiB}^{-2}) \,. \tag{95}$$

Here we define

$$A_X^s(\hat{\boldsymbol{n}}) = \sum_{\ell m} \overline{X}_{\ell m} Y_{\ell m}^s(\hat{\boldsymbol{n}}) \,, \tag{96}$$

$$A_{XY}^s(\hat{\boldsymbol{n}}) = \sum_{\ell m} C_\ell^{XY} \overline{X}_{\ell m} Y_{\ell m}^s(\hat{\boldsymbol{n}}) \,, \tag{97}$$

with $\overline{X}_{\ell m} = \widehat{X}_{\ell m}/\widehat{C}_\ell^{XY}$ being the inverse-variance filtered multipoles by the lensed angular power spectra including instrumental noise. The quantity $\boldsymbol{v}^{\Theta E}$ gives the nearly optimal estimator [10].

In general, we can decompose the 2D vector, $\boldsymbol{v}^\alpha$, into

$$\boldsymbol{v}^\alpha = \frac{v_-^\alpha \boldsymbol{e} + v_+^\alpha \boldsymbol{e}^*}{\sqrt{2}} \,. \tag{98}$$

Since $\boldsymbol{v}^\alpha$ is real, we find $(v_-^\alpha)^* = v_+^\alpha \equiv v^\alpha$. Then we obtain

$$\begin{aligned}\widehat{\phi}_{\ell m}^{(\alpha)} &= \frac{\sqrt{\ell(\ell+1)} A_\ell^{\phi,(\alpha)}}{2} \int \mathrm{d}^2\hat{\boldsymbol{n}} \; [(Y_{\ell m}^1)^* v^\alpha - (Y_{\ell m}^{-1})^* (v^\alpha)^*] \\ &= -\sqrt{\ell(\ell+1)} A_\ell^{\phi,(\alpha)} v_{\ell m}^{1,+} \,,\end{aligned} \tag{99}$$

$$\begin{aligned}\widehat{\varpi}_{\ell m}^{(\alpha)} &= -\mathrm{i}\frac{\sqrt{\ell(\ell+1)} A_\ell^{\varpi,(\alpha)}}{2} \int \mathrm{d}^2\hat{\boldsymbol{n}} \; [(Y_{\ell m}^1)^* v^\alpha + (Y_{\ell m}^{-1})^* (v^\alpha)^*] \\ &= -\sqrt{\ell(\ell+1)} A_\ell^{\varpi,(\alpha)} v_{\ell m}^{1,-} \,,\end{aligned} \tag{100}$$

where $v_{\ell m}^{1,\pm}$ are the outputs of `map2alm_spin` by inputting, $S = v^\alpha$, with $s = 1$. In the following subsections, we show v^α for each quadratic estimator.

4.3.1 $\Theta\Theta$

The estimator for $\Theta\Theta$ contains

$$\begin{aligned}
\boldsymbol{v}^{\Theta\Theta} &= \sum_{\ell m} \overline{\Theta}_{\ell m} Y_{\ell m} \sum_{\ell m} C_\ell^{\Theta\Theta} \overline{\Theta}_{\ell m} \boldsymbol{\nabla} Y_{\ell m} \\
&= \sum_{\ell m} \overline{\Theta}_{\ell m} Y_{\ell m} \sum_{\ell m} C_\ell^{\Theta\Theta} \overline{\Theta}_{\ell m} \sqrt{\frac{\ell(\ell+1)}{2}} \left(Y^1_{\ell m} \boldsymbol{e}^* - Y^{-1}_{\ell m} \boldsymbol{e} \right) \\
&= \frac{1}{\sqrt{2}} \Theta^0 \left[(\Theta^+ + \mathrm{i}\Theta^-) \boldsymbol{e}^* + (\Theta^+ - \mathrm{i}\Theta^-) \boldsymbol{e} \right] , \qquad (101)
\end{aligned}$$

where we define

$$\Theta^0 = \sum_{\ell m} \overline{\Theta}_{\ell m} Y_{\ell m} \,, \qquad (102)$$

$$\Theta^+ + \mathrm{i}\Theta^- = \sum_{\ell m} C_\ell^{\Theta\Theta} \overline{\Theta}_{\ell m} \sqrt{\ell(\ell+1)} Y^1_{\ell m} \,. \qquad (103)$$

We obtain

$$v^{\Theta\Theta} = \Theta^0 (\Theta^+ + \mathrm{i}\Theta^-) \,. \qquad (104)$$

4.3.2 ΘE

The ΘE estimator contains;

$$\begin{aligned}
\boldsymbol{v}^{\Theta E} &= \Re \left[\sum_{\ell m} \overline{E}_{\ell m} Y^{+2}_{\ell m} \sum_{\ell m} C_\ell^{\Theta E} \overline{\Theta}_{\ell m} \left(\sqrt{\frac{(\ell+2)(\ell-1)}{2}} Y^{-1}_{\ell m} \boldsymbol{e}^* - \sqrt{\frac{(\ell-2)(\ell+3)}{2}} Y^{-3}_{\ell m} \boldsymbol{e} \right) \right] \\
&\quad + \sum_{\ell m} \overline{\Theta}_{\ell m} Y_{\ell m} \sum_{\ell m} C_\ell^{\Theta E} \overline{E}_{\ell m} \sqrt{\frac{\ell(\ell+1)}{2}} \left(Y^1_{\ell m} \boldsymbol{e}^* - Y^{-1}_{\ell m} \boldsymbol{e} \right) \\
&= \frac{1}{2\sqrt{2}} \left[(Q^E + \mathrm{i}U^E) [-(\Theta_1^+ - \mathrm{i}\Theta_1^-) \boldsymbol{e}^* + (\Theta_3^+ - \mathrm{i}\Theta_3^-) \boldsymbol{e}] + \text{c.c.} \right] \\
&\quad + \frac{1}{\sqrt{2}} \overline{\Theta} [(E_1^+ + \mathrm{i}E_1^-) \boldsymbol{e}^* + (E_1^+ - \mathrm{i}E_1^-) \boldsymbol{e}] \,. \qquad (105)
\end{aligned}$$

where we define

$$\begin{aligned}
Q^E + \mathrm{i}U^E &\equiv \sum_{\ell m} Y^2_{\ell m} \overline{E}_{\ell m} = A_E^2 \,, \\
\Theta_1^+ + \mathrm{i}\Theta_1^- &\equiv \sum_{\ell m} Y^1_{\ell m} \overline{\Theta}_{\ell m} C_\ell^{\Theta E} \sqrt{(\ell+2)(\ell-1)} \,, \\
\Theta_3^+ + \mathrm{i}\Theta_3^- &\equiv \sum_{\ell m} Y^3_{\ell m} \overline{\Theta}_{\ell m} C_\ell^{\Theta E} \sqrt{(\ell-2)(\ell+3)} \,, \\
E_1^+ + \mathrm{i}E_1^- &\equiv \sum_{\ell m} Y^1_{\ell m} \overline{E}_{\ell m} C_\ell^{\Theta E} \sqrt{\ell(\ell+1)} \,. \qquad (106)
\end{aligned}$$

The above quantities are obtained by `map2alm_spin`. We find that

$$\begin{aligned}
v^{\Theta E} &= \frac{1}{2} [(Q^E + \mathrm{i}U^E)(-\Theta_1^+ + \mathrm{i}\Theta_1^-) + (Q^E - \mathrm{i}U^E)(\Theta_3^+ + \mathrm{i}\Theta_3^-)] + \overline{\Theta}(E_1^+ + \mathrm{i}E_1^-) \\
&= \frac{1}{2} [Q^E(\Theta_3^+ - \Theta_1^+) + U^E(\Theta_3^- - \Theta_1^-) + \mathrm{i}[Q^E(\Theta_3^- + \Theta_1^-) - U^E(\Theta_3^+ + \Theta_1^+)]] + \overline{\Theta}(E_1^+ + \mathrm{i}E_1^-) \,. \qquad (107)
\end{aligned}$$

4.3.3 ΘB

The ΘB estimator is obtained by replacing E to $\mathrm{i}B$ in the ΘE estimator and ignore the second term;

$$v^{\Theta B} = \frac{1}{2}[Q^B(\Theta_3^+ - \Theta_1^+) + U^B(\Theta_3^- - \Theta_1^-) + \mathrm{i}[Q^B(\Theta_3^- + \Theta_1^-) - U^B(\Theta_3^+ + \Theta_1^+)]] \,, \tag{108}$$

where we define the Q/U map from B-mode alone;

$$Q^B + \mathrm{i}U^B \equiv \sum_{\ell m} Y^2_{\ell m}\mathrm{i}\overline{B}_{\ell m} \,. \tag{109}$$

4.3.4 EE

The EE estimator contains;

$$\begin{aligned} \boldsymbol{v}^{EE} &= \frac{1}{2}(Q^E + \mathrm{i}U^E)\sum_{\ell m} C_\ell^{EE}\overline{E}_{\ell m}\Big(\sqrt{\frac{(\ell+2)(\ell-1)}{2}}Y^{-1}_{\ell m}\boldsymbol{e}^* - \sqrt{\frac{(\ell-2)(\ell+3)}{2}}Y^{-3}_{\ell m}\boldsymbol{e}\Big) + \mathrm{c.c} \\ &= \frac{1}{2\sqrt{2}}(Q^E + \mathrm{i}U^E)[-(\mathcal{E}_1^+ - \mathrm{i}\mathcal{E}_1^-)\boldsymbol{e}^* + (\mathcal{E}_3^+ - \mathrm{i}\mathcal{E}_3^-)\boldsymbol{e}] + \mathrm{c.c} \,, \end{aligned} \tag{110}$$

where we define

$$\begin{aligned} \mathcal{E}_1^+ + \mathrm{i}\mathcal{E}_1^- &\equiv \sum_{\ell m} Y^1_{\ell m}\overline{E}_{\ell m}C_\ell^{EE}\sqrt{(\ell+2)(\ell-1)} \,, \\ \mathcal{E}_3^+ + \mathrm{i}\mathcal{E}_3^- &\equiv \sum_{\ell m} Y^3_{\ell m}\overline{E}_{\ell m}C_\ell^{EE}\sqrt{(\ell-2)(\ell+3)} \,. \end{aligned} \tag{111}$$

Then we obtain

$$\begin{aligned} v^{EE} &= \frac{1}{2}(Q^E + \mathrm{i}U^E)[-\mathcal{E}_1^+ + \mathrm{i}\mathcal{E}_1^-] + \frac{1}{2}(Q^E - \mathrm{i}U^E)[\mathcal{E}_3^+ + \mathrm{i}\mathcal{E}_3^-] \\ &= \frac{1}{2}[Q^E(\mathcal{E}_3^+ - \mathcal{E}_1^+) + U^E(\mathcal{E}_3^- - \mathcal{E}_1^-)] + \frac{\mathrm{i}}{2}[Q^E(\mathcal{E}_3^- + \mathcal{E}_1^-) - U^E(\mathcal{E}_3^+ + \mathcal{E}_1^+)] \,, \end{aligned} \tag{112}$$

4.3.5 BB

The BB estimator is the same as EE estimator but using B mode, and the result is;

$$v^{BB} = \frac{1}{2}[Q^B(\mathcal{B}_3^+ - \mathcal{B}_1^+) + U^B(\mathcal{B}_3^- - \mathcal{B}_1^-)] + \frac{\mathrm{i}}{2}[Q^B(\mathcal{B}_3^- + \mathcal{B}_1^-) - U^B(\mathcal{B}_3^+ + \mathcal{B}_1^+)] \,, \tag{113}$$

where we define

$$\begin{aligned} \mathcal{B}_1^+ + \mathrm{i}\mathcal{B}_1^- &\equiv \sum_{\ell m} Y^1_{\ell m}\mathrm{i}\overline{B}_{\ell m}C_\ell^{BB}\sqrt{(\ell+2)(\ell-1)} \,, \\ \mathcal{B}_3^+ + \mathrm{i}\mathcal{B}_3^- &\equiv \sum_{\ell m} Y^3_{\ell m}\mathrm{i}\overline{B}_{\ell m}C_\ell^{BB}\sqrt{(\ell-2)(\ell+3)} \,. \end{aligned} \tag{114}$$

4.3.6 EB

The first term of the EB estimator is obtained by replacing E to $\mathrm{i}B$ in the first half of the EE estimator. Similarly, the second term of the BB estimator is given by replacing $\mathrm{i}B$ to E in the first half of the BB estimator. The result is;

$$\begin{aligned} v^{EB} &= \frac{1}{2}[Q^B(\mathcal{E}_3^+ - \mathcal{E}_1^+) + U^B(\mathcal{E}_3^- - \mathcal{E}_1^-)] + \frac{\mathrm{i}}{2}[Q^B(\mathcal{E}_3^- + \mathcal{E}_1^-) - U^B(\mathcal{E}_3^+ + \mathcal{E}_1^+)] \\ &+ \frac{1}{2}[Q^E(\mathcal{B}_3^+ - \mathcal{B}_1^+) + U^E(\mathcal{B}_3^- - \mathcal{B}_1^-)] + \frac{\mathrm{i}}{2}[Q^E(\mathcal{B}_3^- + \mathcal{B}_1^-) - U^E(\mathcal{B}_3^+ + \mathcal{B}_1^+)] \,. \end{aligned} \tag{115}$$

4.4 Polarization rotation angle

4.4.1 EB

The EB quadratic estimator for the polarization rotation is given by

$$[\widehat{\alpha}^{\rm EB}_{LM}]^* = A^{\alpha,{\rm EB}}_L \sum_{\ell\ell' mm'} \begin{pmatrix} \ell & \ell' & L \\ m & m' & M \end{pmatrix} \left[W^{\alpha,-}_{\ell L\ell'} C^{\rm BB}_{\ell'} - W^{\alpha,-}_{\ell' L\ell} C^{\rm EE}_{\ell}\right] \overline{E}_{\ell m}\overline{B}_{\ell' m'} \,. \tag{116}$$

Using the property of the Wigner 3j, we obtain

$$\begin{aligned} \begin{pmatrix} \ell & \ell' & L \\ m & m' & M \end{pmatrix} W^{\alpha,-}_{\ell L\ell'} &= 2\wp^{\alpha,+}_{\ell L\ell'}\gamma_{\ell L\ell'} \begin{pmatrix} \ell & L & \ell' \\ -2 & 0 & 2 \end{pmatrix}\begin{pmatrix} \ell & \ell' & L \\ m & m' & M \end{pmatrix} \\ &= [1+(-1)^{\ell+L+\ell'}]\gamma_{\ell L\ell'} \begin{pmatrix} \ell & \ell' & L \\ -2 & 2 & 0 \end{pmatrix}\begin{pmatrix} \ell & \ell' & L \\ m & m' & M \end{pmatrix} \\ &= \gamma_{\ell L\ell'} \left[\begin{pmatrix} \ell & \ell' & L \\ -2 & 2 & 0 \end{pmatrix} + \begin{pmatrix} \ell & \ell' & L \\ 2 & -2 & 0 \end{pmatrix}\right]\begin{pmatrix} \ell & \ell' & L \\ m & m' & M \end{pmatrix} \\ &= \int {\rm d}\hat{\boldsymbol{n}}\, Y_{LM}[Y^{-2}_{\ell m}Y^{2}_{\ell' m'} + Y^{2}_{\ell m}Y^{-2}_{\ell' m'}] \,. \end{aligned} \tag{117}$$

The second term is also the same as the first term;

$$\begin{pmatrix} \ell & \ell' & L \\ m & m' & M \end{pmatrix} W^{\alpha,-}_{\ell' L\ell} = \int {\rm d}\hat{\boldsymbol{n}}\, Y_{LM}[Y^{-2}_{\ell m}Y^{2}_{\ell' m'} + Y^{2}_{\ell m}Y^{-2}_{\ell' m'}] \,. \tag{118}$$

We then obtain

$$\begin{aligned} \widehat{\alpha}^{\rm EB}_{LM} &= A^{\alpha,{\rm EB}}_L \int {\rm d}\hat{\boldsymbol{n}}\, Y^*_{LM}\Big[\sum_{\ell m} Y^{-2}_{\ell m}\overline{E}_{\ell m}\sum_{\ell' m'} Y^{2}_{\ell' m'}C^{\rm BB}_{\ell'}\overline{B}_{\ell' m'} + \sum_{\ell m} Y^{2}_{\ell m}\overline{E}_{\ell m}\sum_{\ell' m'} Y^{-2}_{\ell' m'}C^{\rm BB}_{\ell'}\overline{B}_{\ell' m'} \\ &\qquad - \sum_{\ell m} Y^{2}_{\ell m}C^{\rm EE}_{\ell}\overline{E}_{\ell m}\sum_{\ell' m'} Y^{-2}_{\ell' m'}\overline{B}_{\ell' m'} - \sum_{\ell m} Y^{-2}_{\ell m}C^{\rm EE}_{\ell}\overline{E}_{\ell m}\sum_{\ell' m'} Y^{2}_{\ell' m'}\overline{B}_{\ell' m'}\Big] \\ &= -{\rm i}A^{\alpha,{\rm EB}}_L \int {\rm d}\hat{\boldsymbol{n}}\, Y^*_{LM}\left[(Q^E-{\rm i}U^E)(\mathcal{Q}^B+{\rm i}\mathcal{U}^B) + (\mathcal{Q}^E+{\rm i}\mathcal{U}^E)(Q^B-{\rm i}U^B) - {\rm c.c.}\right] \\ &= 2A^{\alpha,{\rm EB}}_L \int {\rm d}\hat{\boldsymbol{n}}\, Y^*_{LM}\left[Q^E\mathcal{U}^B - U^E\mathcal{Q}^B + \mathcal{U}^E Q^B - \mathcal{Q}^E U^B\right] \,. \end{aligned} \tag{119}$$

where we define

$$\begin{aligned} Q^E+{\rm i}U^E &= \sum_{\ell m} Y^2_{\ell m}\overline{E}_{\ell m} = (\sum_{\ell m} Y^{-2}_{\ell m}\overline{E}_{\ell m})^* \\ \mathcal{Q}^B+{\rm i}\mathcal{U}^B &= \sum_{\ell m} Y^2_{\ell m}{\rm i}C^{\rm BB}_{\ell}\overline{B}_{\ell m} \\ \mathcal{Q}^E+{\rm i}\mathcal{U}^E &= \sum_{\ell m} Y^2_{\ell m}C^{\rm EE}_{\ell}\overline{E}_{\ell m} \\ Q^B+{\rm i}U^B &= \sum_{\ell m} Y^2_{\ell m}{\rm i}\overline{B}_{\ell m} = -(\sum_{\ell m} Y^{-2}_{\ell m}{\rm i}\overline{B}_{\ell m})^* \,. \end{aligned} \tag{120}$$

4.5 Amplitude modulation

4.5.1 EB

The EB quadratic estimator for the amplitude modulation is given by

$$[\widehat{\epsilon}^{\rm EB,-}_{LM}]^* = A^{\tau,{\rm EB},-}_L \sum_{\ell\ell' mm'} \begin{pmatrix} \ell & \ell' & L \\ m & m' & M \end{pmatrix} \left[C^{\rm EB}_{\ell'} W^{\tau,+}_{\ell L\ell'} + C^{\rm EB}_{\ell} W^{\tau,+}_{\ell' L\ell}\right] \overline{E}_{\ell m}\overline{B}_{\ell' m'} \,. \tag{121}$$

Note that $W^{\alpha,-}_{\ell' L\ell} = -2W^{\tau,+}_{\ell' L\ell}$. We then obtain the estimator by replacing $C^{\rm BB}_{\ell'}$ and $C^{\rm EE}_{\ell}$ with $-C^{\rm EB}_{\ell'}$ and $C^{\rm EB}_{\ell}$, respectively, in the polarization rotation estimator, and multiplying $1/2$, yielding

$$\widehat{\epsilon}^{\rm EB,-}_{LM} = A^{\tau,\rm EB}_L \int \mathrm{d}\hat{\boldsymbol{n}}\; Y^*_{LM} \left[Q^E\mathcal{U}^B - U^E\mathcal{Q}^B + Q^B\mathcal{U}^E - U^B\mathcal{Q}^E \right] . \tag{122}$$

where we define

$$\begin{aligned}
Q^E + \mathrm{i}U^E &= \sum_{\ell m} Y^2_{\ell m}\overline{E}_{\ell m} \\
\mathcal{Q}^B + \mathrm{i}\mathcal{U}^B &= -\sum_{\ell m} Y^2_{\ell m}\mathrm{i}C^{\rm EB}_{\ell}\overline{B}_{\ell m} \\
\mathcal{Q}^E + \mathrm{i}\mathcal{U}^E &= \sum_{\ell m} Y^2_{\ell m}C^{\rm EB}_{\ell}\overline{E}_{\ell m} \\
Q^B + \mathrm{i}U^B &= \sum_{\ell m} Y^2_{\ell m}\mathrm{i}\overline{B}_{\ell m} \, .
\end{aligned} \tag{123}$$

5 Computing Quadratic Estimator Normalization

Here, I generalize the algorithm of [12] to the case including the cosmic bi-refringence, patchy reionization, and so on.

Using $s = 0, \pm$, we define the following kernel functions;

$$\Sigma_L^{(s),x}[A,B] = \frac{1}{2L+1}\sum_{\ell\ell'} |W^{x,s}_{\ell L\ell'}|^2 A_\ell B_{\ell'} \,, \tag{124}$$

$$\Sigma_L^{(\times),x}[A,B] = \frac{1}{2L+1}\sum_{\ell\ell'} W^{x,0}_{\ell L\ell'} W^{x,+}_{\ell L\ell'} A_\ell B_{\ell'} \,, \tag{125}$$

$$\Gamma_L^{(s),x}[A,B] = \frac{1}{2L+1}\sum_{\ell\ell'} [W^{x,s}_{\ell L\ell'}]^* W^{x,s}_{\ell' L\ell} A_\ell B_{\ell'} \,, \tag{126}$$

$$\Gamma_L^{(\times),x}[A,B] = \frac{1}{2L+1}\sum_{\ell\ell'} W^{x,0}_{\ell L\ell'} W^{x,+}_{\ell' L\ell} A_\ell B_{\ell'} \,. \tag{127}$$

Note that

$$\Gamma_L^{(\pm),x}[A,B] = \Gamma_L^{(\pm),x}[B,A] \,. \tag{128}$$

5.1 Normalization

5.1.1 $\Theta\Theta$

The normalization of the $\Theta\Theta$ quadratic estimator is given by

$$\begin{aligned}
[A_L^{x,(\Theta\Theta)}]^{-1} &= \frac{1}{2L+1}\sum_{\ell\ell'} \frac{\left[W^{x,0}_{\ell L\ell'} C^{\Theta\Theta}_{\ell'} + p_x W^{x,0}_{\ell' L\ell} C^{\Theta\Theta}_{\ell}\right]^2}{2\widehat{C}^{\Theta\Theta}_\ell \widehat{C}^{\Theta\Theta}_{\ell'}} \\
&= \frac{1}{2}\Sigma_L^{(0),x}\left[\frac{1}{\widehat{C}^{\Theta\Theta}}, \frac{(C^{\Theta\Theta})^2}{\widehat{C}^{\Theta\Theta}}\right] + p_x\Gamma_L^{(0),x}\left[\frac{C^{\Theta\Theta}}{\widehat{C}^{\Theta\Theta}}, \frac{C^{\Theta\Theta}}{\widehat{C}^{\Theta\Theta}}\right] + \frac{1}{2}\Sigma_L^{(0),x}\left[\frac{(C^{\Theta\Theta})^2}{\widehat{C}^{\Theta\Theta}}, \frac{1}{\widehat{C}^{\Theta\Theta}}\right] \,.
\end{aligned} \tag{129}$$

5.1.2 ΘE

The normalization of the quadratic ΘE estimator is given by

$$\begin{aligned}
[A_L^{x,(\Theta E)}]^{-1} &= \frac{1}{2L+1}\sum_{\ell\ell'} \frac{|W^{x,0}_{\ell L\ell'} C^{\Theta\mathrm{E}}_{\ell'} + p_x W^{x,+}_{\ell' L\ell} C^{\Theta\mathrm{E}}_{\ell}|^2}{\widehat{C}^{\Theta\Theta}_\ell \widehat{C}^{\mathrm{EE}}_{\ell'}} \\
&= \frac{1}{2L+1}\sum_{\ell\ell'}\left[(W^{x,0}_{\ell L\ell'})^2 \frac{(C^{\Theta\mathrm{E}}_{\ell'})^2}{\widehat{C}^{\Theta\Theta}_\ell \widehat{C}^{\mathrm{EE}}_{\ell'}} + 2p_x W^{x,0}_{\ell L\ell'} W^{x,+}_{\ell' L\ell}\frac{C^{\Theta\mathrm{E}}_{\ell'} C^{\Theta\mathrm{E}}_{\ell}}{\widehat{C}^{\Theta\Theta}_\ell \widehat{C}^{\mathrm{EE}}_{\ell'}} + (W^{x,+}_{\ell' L\ell})^2 \frac{(C^{\Theta\mathrm{E}}_{\ell})^2}{\widehat{C}^{\Theta\Theta}_\ell \widehat{C}^{\mathrm{EE}}_{\ell'}}\right] \\
&= \Sigma_L^{(0),x}\left[\frac{1}{\widehat{C}^{\Theta\Theta}}, \frac{(C^{\Theta\mathrm{E}})^2}{\widehat{C}^{\mathrm{EE}}}\right] + 2p_x\Gamma_L^{(\times),x}\left[\frac{C^{\Theta\mathrm{E}}}{\widehat{C}^{\Theta\Theta}}, \frac{C^{\Theta\mathrm{E}}}{\widehat{C}^{\mathrm{EE}}}\right] + \Sigma_L^{(+),x}\left[\frac{1}{\widehat{C}^{\mathrm{EE}}}, \frac{(C^{\Theta\mathrm{E}})^2}{\widehat{C}^{\Theta\Theta}}\right] \,,
\end{aligned} \tag{130}$$

5.1.3 ΘE

The normalization of the quadratic ΘB estimator is given by

$$\begin{aligned}
[A_L^{x,(\Theta B)}]^{-1} &= \frac{1}{2L+1}\sum_{\ell\ell'} \frac{|W^{x,-}_{\ell' L\ell} C^{\Theta\mathrm{E}}_{\ell}|^2}{\widehat{C}^{\Theta\Theta}_\ell \widehat{C}^{\mathrm{BB}}_{\ell'}} \\
&= \Sigma_L^{(-),x}\left[\frac{1}{\widehat{C}^{\mathrm{BB}}}, \frac{(C^{\Theta\mathrm{E}})^2}{\widehat{C}^{\Theta\Theta}}\right] \,,
\end{aligned} \tag{131}$$

5.1.4 EE and BB

The normalization of the quadratic EE estimator (and for the BB estimator by replacing the $EE \to BB$ spectrum) is given by

$$
\begin{aligned}
[A_L^{x,(EE)}]^{-1} &= \frac{1}{2L+1}\sum_{\ell\ell'}\frac{|W^{x,+}_{\ell L\ell'}C^{\rm EE}_{\ell'}+p_xW^{x,+}_{\ell' L\ell}C^{\rm EE}_{\ell}|^2}{2\widehat{C}^{\rm EE}_{\ell}\widehat{C}^{\rm EE}_{\ell'}} \\
&= \Sigma^{(+),x}_L\left[\frac{1}{\widehat{C}^{\rm EE}},\frac{(C^{\rm EE})^2}{\widehat{C}^{\rm EE}}\right]+p_x\Gamma^{(+),x}_L\left[\frac{C^{\rm EE}}{\widehat{C}^{\rm EE}},\frac{C^{\rm EE}}{\widehat{C}^{\rm EE}}\right] , \qquad (132)
\end{aligned}
$$

5.1.5 EB

The normalization of the quadratic EB estimator is given by

$$
\begin{aligned}
[A_L^{x,(EB)}]^{-1} &= \frac{1}{2L+1}\sum_{\ell\ell'}\frac{|W^{x,-}_{\ell L\ell'}C^{\rm BB}_{\ell'}+p_xW^{x,-}_{\ell' L\ell}C^{\rm EE}_{\ell}|^2}{\widehat{C}^{\rm EE}_{\ell}\widehat{C}^{\rm BB}_{\ell'}} \\
&= \Sigma^{(-),x}_L\left[\frac{1}{\widehat{C}^{\rm EE}},\frac{(C^{\rm BB})^2}{\widehat{C}^{\rm BB}}\right]+2p_x\Gamma^{(-),x}_L\left[\frac{C^{\rm EE}}{\widehat{C}^{\rm EE}},\frac{C^{\rm BB}}{\widehat{C}^{\rm BB}}\right]+\Sigma^{(-),x}_L\left[\frac{1}{\widehat{C}^{\rm BB}},\frac{(C^{\rm EE})^2}{\widehat{C}^{\rm EE}}\right] , \qquad (133)
\end{aligned}
$$

Also,

$$
[A_L^{x,(EB),-}]^{-1} = \Sigma^{(+),x}_L\left[\frac{1}{\widehat{C}^{\rm EE}},\frac{(C^{\rm EB})^2}{\widehat{C}^{\rm BB}}\right]+2p_x\Gamma^{(+),x}_L\left[\frac{C^{\rm EB}}{\widehat{C}^{\rm EE}},\frac{C^{\rm EB}}{\widehat{C}^{\rm BB}}\right]+\Sigma^{(+),x}_L\left[\frac{1}{\widehat{C}^{\rm BB}},\frac{(C^{\rm EB})^2}{\widehat{C}^{\rm EE}}\right] . \qquad (134)
$$

5.2 Noise covariance and kernel function

5.2.1 $\Theta\Theta\Theta E$

The noise covariance between the $\Theta\Theta$ and ΘE estimators is given by

$$
\begin{aligned}
\frac{A_L^{x,(\Theta\Theta)}A_L^{x,(\Theta E)}}{N_L^{x,(\Theta\Theta\Theta E)}} &= \frac{1}{2L+1}\sum_{\ell\ell'}\left[\frac{W^{x,0}_{\ell L\ell'}C^{\Theta\Theta}_{\ell'}}{2\widehat{C}^{\Theta\Theta}_{\ell}\widehat{C}^{\Theta\Theta}_{\ell'}}+p_x(\ell\leftrightarrow\ell')\right]\left[\frac{(W^{x,0}_{\ell L\ell'}C^{\Theta\rm E}_{\ell'}+p_xW^{x,+}_{\ell' L\ell}C^{\Theta\rm E}_{\ell})\widehat{C}^{\Theta\rm E}_{\ell'}}{\widehat{C}^{\rm EE}_{\ell'}}+p_x(\ell\leftrightarrow\ell')\right] \\
&= \frac{1}{2L+1}\sum_{\ell\ell'}\left[\frac{W^{x,0}_{\ell L\ell'}C^{\Theta\Theta}_{\ell'}}{\widehat{C}^{\Theta\Theta}_{\ell}\widehat{C}^{\Theta\Theta}_{\ell'}}\frac{(W^{x,0}_{\ell L\ell'}C^{\Theta\rm E}_{\ell'}+p_xW^{x,+}_{\ell' L\ell}C^{\Theta\rm E}_{\ell})\widehat{C}^{\Theta\rm E}_{\ell'}}{\widehat{C}^{\rm EE}_{\ell'}}\right. \\
&\qquad \left. +p_x\frac{W^{x,0}_{\ell L\ell'}C^{\Theta\Theta}_{\ell'}}{\widehat{C}^{\Theta\Theta}_{\ell}\widehat{C}^{\Theta\Theta}_{\ell'}}\frac{(W^{x,0}_{\ell' L\ell}C^{\Theta\rm E}_{\ell}+p_xW^{x,+}_{\ell L\ell'}C^{\Theta\rm E}_{\ell'})\widehat{C}^{\Theta\rm E}_{\ell}}{\widehat{C}^{\rm EE}_{\ell}}\right] \\
&= \Sigma^{(0),x}_L\left[\frac{1}{\widehat{C}^{\Theta\Theta}},\frac{C^{\Theta\Theta}C^{\Theta\rm E}\widehat{C}^{\Theta\rm E}}{\widehat{C}^{\Theta\Theta}\widehat{C}^{\rm EE}}\right]+p_x\Gamma^{(\times),x}_L\left[\frac{C^{\Theta\rm E}}{\widehat{C}^{\Theta\Theta}},\frac{C^{\Theta\Theta}\widehat{C}^{\Theta\rm E}}{\widehat{C}^{\Theta\Theta}\widehat{C}^{\rm EE}}\right] \\
&\qquad +p_x\Gamma^{(0),x}_L\left[\frac{C^{\Theta\rm E}\widehat{C}^{\Theta\rm E}}{\widehat{C}^{\Theta\Theta}\widehat{C}^{\rm EE}},\frac{C^{\Theta\Theta}}{\widehat{C}^{\Theta\Theta}}\right]+\Sigma^{(\times),x}_L\left[\frac{\widehat{C}^{\Theta\rm E}}{\widehat{C}^{\Theta\Theta}\widehat{C}^{\rm EE}},\frac{C^{\Theta\rm E}C^{\Theta\Theta}}{\widehat{C}^{\Theta\Theta}}\right] , \qquad (135)
\end{aligned}
$$

5.2.2 $\Theta\Theta EE$

The noise covariance between the $\Theta\Theta$ and EE estimators is given by

$$\begin{aligned}
\frac{A_L^{x,(\Theta\Theta)}A_L^{x,(EE)}}{N_L^{x,(\Theta\Theta EE)}} &= \frac{1}{2L+1}\sum_{\ell\ell'}\left[\frac{W^{x,0}_{\ell L\ell'}C^{\Theta\Theta}_{\ell'}}{2\widehat{C}^{\Theta\Theta}_\ell\widehat{C}^{\Theta\Theta}_{\ell'}}+p_x(\ell\leftrightarrow\ell')\right]\left[\frac{(W^{x,+}_{\ell L\ell'}C^{\rm EE}_{\ell'}+p_xW^{x,+}_{\ell' L\ell}C^{\rm EE}_{\ell})\widehat{C}^{\Theta\rm E}_\ell\widehat{C}^{\Theta\rm E}_{\ell'}}{2\widehat{C}^{\rm EE}_\ell\widehat{C}^{\rm EE}_{\ell'}}+p_x(\ell\leftrightarrow\ell')\right] \\
&= \frac{1}{2L+1}\sum_{\ell\ell'}\frac{W^{x,0}_{\ell L\ell'}C^{\Theta\Theta}_{\ell'}}{\widehat{C}^{\Theta\Theta}_\ell\widehat{C}^{\Theta\Theta}_{\ell'}}\left[\frac{(W^{x,+}_{\ell L\ell'}C^{\rm EE}_{\ell'}+p_xW^{x,+}_{\ell' L\ell}C^{\rm EE}_{\ell})\widehat{C}^{\Theta\rm E}_\ell\widehat{C}^{\Theta\rm E}_{\ell'}}{2\widehat{C}^{\rm EE}_\ell\widehat{C}^{\rm EE}_{\ell'}}+p_x(\ell\leftrightarrow\ell')\right] \\
&= \frac{1}{2L+1}\sum_{\ell\ell'}\frac{W^{x,0}_{\ell L\ell'}C^{\Theta\Theta}_{\ell'}}{\widehat{C}^{\Theta\Theta}_\ell\widehat{C}^{\Theta\Theta}_{\ell'}}\frac{(W^{x,+}_{\ell L\ell'}C^{\rm EE}_{\ell'}+p_xW^{x,+}_{\ell' L\ell}C^{\rm EE}_{\ell})\widehat{C}^{\Theta\rm E}_\ell\widehat{C}^{\Theta\rm E}_{\ell'}}{\widehat{C}^{\rm EE}_\ell\widehat{C}^{\rm EE}_{\ell'}} \\
&= \Sigma^{(0),x}_L\left[\frac{\widehat{C}^{\Theta\rm E}}{\widehat{C}^{\Theta\Theta}\widehat{C}^{\rm EE}},\frac{C^{\Theta\Theta}C^{\rm EE}\widehat{C}^{\Theta\rm E}}{\widehat{C}^{\Theta\Theta}\widehat{C}^{\rm EE}}\right]+p_x\Gamma^{(\times),x}_L\left[\frac{\widehat{C}^{\Theta\rm E}C^{\rm EE}}{\widehat{C}^{\Theta\Theta}\widehat{C}^{\rm EE}},\frac{C^{\Theta\Theta}\widehat{C}^{\Theta\rm E}}{\widehat{C}^{\Theta\Theta}\widehat{C}^{\rm EE}}\right] .
\end{aligned} \tag{136}$$

5.2.3 ΘEEE

The noise covariance between the ΘE and EE estimators is given by

$$\begin{aligned}
\frac{A_L^{x,(\Theta E)}A_L^{x,(EE)}}{N_L^{x,(\Theta EEE)}} &= \frac{1}{2L+1}\sum_{\ell\ell'}\left[\frac{W^{x,+}_{\ell L\ell'}C^{\rm EE}_{\ell'}}{2\widehat{C}^{\rm EE}_\ell\widehat{C}^{\rm EE}_{\ell'}}+p_x(\ell\leftrightarrow\ell')\right]\left[\frac{(W^{x,0}_{\ell L\ell'}C^{\Theta\rm E}_{\ell'}+p_xW^{x,+}_{\ell' L\ell}C^{\Theta\rm E}_{\ell})\widehat{C}^{\Theta\rm E}_\ell}{\widehat{C}^{\Theta\Theta}_\ell}+p_x(\ell\leftrightarrow\ell')\right] \\
&= \frac{1}{2L+1}\sum_{\ell\ell'}\left[\frac{W^{x,+}_{\ell L\ell'}C^{\rm EE}_{\ell'}}{\widehat{C}^{\rm EE}_\ell\widehat{C}^{\rm EE}_{\ell'}}+p_x\frac{W^{x,+}_{\ell' L\ell}C^{\rm EE}_{\ell}}{\widehat{C}^{\rm EE}_\ell\widehat{C}^{\rm EE}_{\ell'}}\right]\left[\frac{(W^{x,0}_{\ell L\ell'}C^{\Theta\rm E}_{\ell'}+p_xW^{x,+}_{\ell' L\ell}C^{\Theta\rm E}_{\ell})\widehat{C}^{\Theta\rm E}_\ell}{\widehat{C}^{\Theta\Theta}_\ell}\right] \\
&= \Sigma^{(\times),x}_L\left[\frac{\widehat{C}^{\Theta\rm E}}{\widehat{C}^{\Theta\Theta}\widehat{C}^{\rm EE}},\frac{C^{\Theta\rm E}C^{\rm EE}}{\widehat{C}^{\rm EE}}\right]+p_x\Gamma^{(+),x}_L\left[\frac{C^{\Theta\rm E}\widehat{C}^{\Theta\rm E}}{\widehat{C}^{\Theta\Theta}\widehat{C}^{\rm EE}},\frac{C^{\rm EE}}{\widehat{C}^{\rm EE}}\right] \\
&\quad +p_x\Gamma^{(\times),x}_L\left[\frac{\widehat{C}^{\Theta\rm E}C^{\rm EE}}{\widehat{C}^{\Theta\Theta}\widehat{C}^{\rm EE}},\frac{C^{\Theta\rm E}}{\widehat{C}^{\rm EE}}\right]+\Sigma^{(+),x}_L\left[\frac{C^{\Theta\rm E}\widehat{C}^{\Theta\rm E}C^{\rm EE}}{\widehat{C}^{\Theta\Theta}\widehat{C}^{\rm EE}},\frac{1}{\widehat{C}^{\rm EE}}\right] .
\end{aligned} \tag{137}$$

5.2.4 ΘBEB

The noise covariance between the ΘB and EB estimators is given by

$$\begin{aligned}
\frac{A_L^{x,(\Theta B)}A_L^{x,(EB)}}{N_L^{x,(\Theta BEB)}} &= \frac{1}{2L+1}\sum_{\ell\ell'}\left[\frac{(W^{x,-}_{\ell L\ell'})^*C^{\rm BB}_{\ell'}-p_x(W^{x,-}_{\ell' L\ell})^*C^{\rm EE}_{\ell}}{\widehat{C}^{\rm EE}_\ell\widehat{C}^{\rm BB}_{\ell'}}\right]\left[\frac{-p_xW^{x,-}_{\ell' L\ell}C^{\Theta\rm E}_\ell\widehat{C}^{\Theta\rm E}_\ell}{\widehat{C}^{\Theta\Theta}_\ell}\right] \\
&= -p_x\Gamma^{(-),x}_L\left[\frac{C^{\Theta\rm E}\widehat{C}^{\Theta\rm E}}{\widehat{C}^{\Theta\Theta}\widehat{C}^{\rm EE}},\frac{C^{\rm BB}}{\widehat{C}^{\rm BB}}\right]+\Sigma^{(-),x}_L\left[\frac{C^{\Theta\rm E}\widehat{C}^{\Theta\rm E}C^{\rm EE}}{\widehat{C}^{\Theta\Theta}\widehat{C}^{\rm EE}},\frac{1}{\widehat{C}^{\rm BB}}\right] .
\end{aligned} \tag{138}$$

6 Explicit Kernel Functions

Here we consider expression for the Kernel functions in terms of the Wigner d-functions. In the following calculations, we frequently use

$$\int_{-1}^{1}\mathrm{d}\mu\; d^{\ell_1}_{s_1,s'_1}(\beta)d^{\ell_2}_{s_2,s'_2}(\beta)d^{\ell_3}_{s_3,s'_3}(\beta) = 2\begin{pmatrix}\ell_1 & \ell_2 & \ell_3\\ s_1 & s_2 & s_3\end{pmatrix}\begin{pmatrix}\ell_1 & \ell_2 & \ell_3\\ s'_1 & s'_2 & s'_3\end{pmatrix}\,, \tag{139}$$

with $s_1+s_2+s_3=s'_1+s'_2+s'_3=0$ and $\mu=\cos\beta$, and the symmetric property:

$$d^{\ell}_{mm'}(\beta) = (-1)^{m-m'}d^{\ell}_{-m,-m'}(\beta) = (-1)^{m-m'}d^{\ell}_{m'm}(\beta) \tag{140}$$

$$d^{\ell}_{mm'}(\beta) = (-1)^{\ell+m}d^{\ell}_{m,-m'}(\pi-\beta)\,. \tag{141}$$

We also define

$$X^{p\dots q} = a^p_\ell\dots a^q_\ell X_\ell\,. \tag{142}$$

6.1 Kernel Functions: Lensing

We obtain

$$\begin{aligned}
\Sigma^{(0),x}_L[A,B] &= \frac{1}{2L+1}\sum_{\ell\ell'}|W^{x,0}_{\ell L\ell'}|^2A_\ell B_{\ell'}\\
&= \sum_{\ell\ell'}4\pi L(L+1)\frac{2\ell+1}{4\pi}A_\ell\frac{2\ell'+1}{4\pi}B_{\ell'}\ell'(\ell'+1)\frac{1+c_x^2(-1)^{\ell+L+\ell'}}{2}\begin{pmatrix}\ell & L & \ell'\\ 0 & 1 & -1\end{pmatrix}^2\\
&= \pi L(L+1)\sum_{\ell\ell'}\frac{2\ell+1}{4\pi}A_\ell\frac{2\ell'+1}{4\pi}B_{\ell'}2\ell'(\ell'+1)\left[\begin{pmatrix}\ell & L & \ell'\\ 0 & 1 & -1\end{pmatrix}^2 + c_x^2\begin{pmatrix}\ell & L & \ell'\\ 0 & -1 & 1\end{pmatrix}\begin{pmatrix}\ell & L & \ell'\\ 0 & 1 & -1\end{pmatrix}\right]\\
&= \int_{-1}^{1}\mathrm{d}\mu\;\pi L(L+1)\sum_{\ell\ell'}\frac{2\ell+1}{4\pi}A_\ell\frac{2\ell'+1}{4\pi}B_{\ell'}\ell'(\ell'+1)[d^\ell_{00}d^L_{11}d^{\ell'}_{11}+c_x^2d^\ell_{00}d^L_{1,-1}d^{\ell'}_{1,-1}]\\
&= \int_{-1}^{1}\mathrm{d}\mu\;\pi L(L+1)\{\xi_{00}[A]\xi_{11}[B^{00}]d^L_{11}+c_x^2\xi_{00}[A]\xi_{1,-1}[B^{00}]d^L_{1,-1}\}\,,
\end{aligned} \tag{143}$$

where

$$\xi_{mm'}[A] = \sum_\ell\frac{2\ell+1}{4\pi}A_\ell d^\ell_{mm'}\,. \tag{144}$$

The cross-term is

$$\begin{aligned}
\Gamma^{(0),x}_L[A,B] &= \frac{1}{2L+1}\sum_{\ell\ell'}(W^{x,0}_{\ell L\ell'})^*W^{x,0}_{\ell' L\ell}A_\ell B_{\ell'}\\
&= \sum_{\ell\ell'}4\pi L(L+1)\frac{2\ell+1}{4\pi}A_\ell\frac{2\ell'+1}{4\pi}B_{\ell'}a^0_\ell a^0_{\ell'}\frac{1+c_x^2(-1)^{\ell+L+\ell'}}{2}\begin{pmatrix}\ell & L & \ell'\\ 0 & 1 & -1\end{pmatrix}\begin{pmatrix}\ell' & L & \ell\\ 0 & 1 & -1\end{pmatrix}\\
&= \pi L(L+1)\sum_{\ell\ell'}\frac{2\ell+1}{4\pi}A^0_\ell\frac{2\ell'+1}{4\pi}B^0_{\ell'}2\left[\begin{pmatrix}\ell & L & \ell'\\ 0 & 1 & -1\end{pmatrix}\begin{pmatrix}\ell & L & \ell'\\ 1 & -1 & 0\end{pmatrix} + c_x^2\begin{pmatrix}\ell & L & \ell'\\ 0 & 1 & -1\end{pmatrix}\begin{pmatrix}\ell & L & \ell'\\ -1 & 1 & 0\end{pmatrix}\right]\\
&= \int_{-1}^{1}\mathrm{d}\mu\;\pi L(L+1)\sum_{\ell\ell'}\frac{2\ell+1}{4\pi}A^0_\ell\frac{2\ell'+1}{4\pi}B^0_{\ell'}[d^\ell_{01}d^L_{1,-1}d^{\ell'}_{-1,0}+c_x^2d^\ell_{0,-1}d^L_{11}d^{\ell'}_{-1,0}]\\
&= -\int_{-1}^{1}\mathrm{d}\mu\;\pi L(L+1)\{\xi_{01}[A^0]\xi_{0,-1}[B^0]d^L_{1,-1}+c_x^2\xi_{01}[A^0]\xi_{01}[B^0]d^L_{11}\}\,.
\end{aligned} \tag{145}$$

Denoting $p=\pm$ and $x=\phi,\varpi$, we rewrite the kernel for polarization as

$$\begin{aligned}
&\Sigma_L^{(p),x}[A,B] = \frac{1}{2L+1}\sum_{\ell\ell'}|W^{x,p}_{\ell L\ell'}|^2 A_\ell B_{\ell'} \\
&= \pi L(L+1)\sum_{\ell\ell'}\frac{2\ell+1}{4\pi}A_\ell\frac{2\ell'+1}{4\pi}B_{\ell'}2[1+pc_x^2(-1)^{\ell+L+\ell'}]\left[a^+_{\ell'}\begin{pmatrix}\ell & L & \ell' \\ -2 & -1 & 3\end{pmatrix}+c_x^2a^-_{\ell'}\begin{pmatrix}\ell & L & \ell' \\ -2 & 1 & 1\end{pmatrix}\right]^2 \\
&= \pi L(L+1)\sum_{\ell\ell'}\frac{2\ell+1}{4\pi}A_\ell\frac{2\ell'+1}{4\pi}B_{\ell'}[1+pc_x^2(-1)^{\ell+L+\ell'}] \\
&\quad\times 2\left[(a^+_{\ell'})^2\begin{pmatrix}\ell & L & \ell' \\ -2 & -1 & 3\end{pmatrix}^2+(a^-_{\ell'})^2\begin{pmatrix}\ell & L & \ell' \\ -2 & 1 & 1\end{pmatrix}^2+2c_x^2a^+_{\ell'}a^-_{\ell'}\begin{pmatrix}\ell & L & \ell' \\ -2 & 1 & 1\end{pmatrix}\begin{pmatrix}\ell & L & \ell' \\ -2 & -1 & 3\end{pmatrix}\right] \\
&= \pi L(L+1)\sum_{\ell\ell'}\frac{2\ell+1}{4\pi}A_\ell\frac{2\ell'+1}{4\pi}B_{\ell'} \\
&\quad\times 2\left[(a^+_{\ell'})^2\begin{pmatrix}\ell & L & \ell' \\ -2 & -1 & 3\end{pmatrix}^2+(a^-_{\ell'})^2\begin{pmatrix}\ell & L & \ell' \\ -2 & 1 & 1\end{pmatrix}^2+2c_x^2a^+_{\ell'}a^-_{\ell'}\begin{pmatrix}\ell & L & \ell' \\ -2 & 1 & 1\end{pmatrix}\begin{pmatrix}\ell & L & \ell' \\ -2 & -1 & 3\end{pmatrix}\right. \\
&\quad\left.+pc_x^2(a^+_{\ell'})^2\begin{pmatrix}\ell & L & \ell' \\ -2 & -1 & 3\end{pmatrix}\begin{pmatrix}\ell & L & \ell' \\ 2 & 1 & -3\end{pmatrix}+pc_x^2(a^-_{\ell'})^2\begin{pmatrix}\ell & L & \ell' \\ -2 & 1 & 1\end{pmatrix}\begin{pmatrix}\ell & L & \ell' \\ 2 & -1 & -1\end{pmatrix}+2pa^+_{\ell'}a^-_{\ell'}\begin{pmatrix}\ell & L & \ell' \\ -2 & 1 & 1\end{pmatrix}\begin{pmatrix}\ell & L & \ell' \\ 2 & 1 & -3\end{pmatrix}\right] \\
&= \int_{-1}^{1}\mathrm{d}\mu\ \pi L(L+1)\sum_{\ell\ell'}\frac{2\ell+1}{4\pi}A_\ell\frac{2\ell'+1}{4\pi}B_{\ell'}[(a^+_{\ell'})^2d^\ell_{22}d^L_{11}d^{\ell'}_{33}+(a^-_{\ell'})^2d^\ell_{22}d^L_{11}d^{\ell'}_{11} \\
&\quad+2c_x^2a^+_{\ell'}a^-_{\ell'}d^\ell_{22}d^L_{1,-1}d^{\ell'}_{13}+pc_x^2(a^+_{\ell'})^2d^\ell_{-2,2}d^L_{-1,1}d^{\ell'}_{3,-3}+pc_x^2(a^-_{\ell'})^2d^\ell_{-2,2}d^L_{1,-1}d^{\ell'}_{1,-1}+2pa^+_{\ell'}a^-_{\ell'}d^\ell_{-2,2}d^L_{11}d^{\ell'}_{1,-3}] \\
&= \int_{-1}^{1}\mathrm{d}\mu\ \pi L(L+1)[(\xi_{22}[A]\xi_{33}[B^{++}]+\xi_{22}[A]\xi_{11}[B^{--}]+2p\xi_{2,-2}[A]\xi_{3,-1}[B^{+-}])d^L_{11} \\
&\quad+c_x^2(p\xi_{2,-2}[A]\xi_{3,-3}[B^{++}]+p\xi_{2,-2}[A]\xi_{1,-1}[B^{--}]+2\xi_{22}[A]\xi_{31}[B^{+-}])d^L_{1,-1}]\,,
\end{aligned}\tag{146}$$

and

$$
\begin{aligned}
\Gamma_L^{(p),x}[A,B] &= \frac{1}{2L+1}\sum_{\ell\ell'}(W^{x,p}_{\ell L\ell'})^* W^{x,p}_{\ell' L\ell}A_\ell B_{\ell'} \\
&= \pi L(L+1)\sum_{\ell\ell'}\frac{2\ell+1}{4\pi}A_\ell\frac{2\ell'+1}{4\pi}B_{\ell'}2[1+pc_x^2(-1)^{\ell+L+\ell'}] \\
&\quad\times\left[a^+_{\ell'}\begin{pmatrix}\ell & L & \ell' \\ -2 & -1 & 3\end{pmatrix}+c_x^2a^-_{\ell'}\begin{pmatrix}\ell & L & \ell' \\ -2 & 1 & 1\end{pmatrix}\right]\left[a^+_{\ell}\begin{pmatrix}\ell' & L & \ell \\ -2 & -1 & 3\end{pmatrix}+c_x^2a^-_{\ell}\begin{pmatrix}\ell' & L & \ell \\ -2 & 1 & 1\end{pmatrix}\right] \\
&= \pi L(L+1)\sum_{\ell\ell'}\frac{2\ell+1}{4\pi}A_\ell\frac{2\ell'+1}{4\pi}B_{\ell'}2[(-1)^{\ell+L+\ell'}+pc_x^2] \\
&\quad\times\left[a^+_{\ell'}\begin{pmatrix}\ell & L & \ell' \\ -2 & -1 & 3\end{pmatrix}+c_x^2a^-_{\ell'}\begin{pmatrix}\ell & L & \ell' \\ -2 & 1 & 1\end{pmatrix}\right]\left[a^+_{\ell}\begin{pmatrix}\ell & L & \ell' \\ 3 & -1 & -2\end{pmatrix}+c_x^2a^-_{\ell}\begin{pmatrix}\ell & L & \ell' \\ 1 & 1 & -2\end{pmatrix}\right] \\
&= \pi L(L+1)\sum_{\ell\ell'}\frac{2\ell+1}{4\pi}A_\ell\frac{2\ell'+1}{4\pi}B_{\ell'} \\
&\quad\times 2\Bigg\{\left[a^+_{\ell'}\begin{pmatrix}\ell & L & \ell' \\ 2 & 1 & -3\end{pmatrix}+c_x^2a^-_{\ell'}\begin{pmatrix}\ell & L & \ell' \\ 2 & -1 & -1\end{pmatrix}\right]\left[a^+_{\ell}\begin{pmatrix}\ell & L & \ell' \\ 3 & -1 & -2\end{pmatrix}+c_x^2a^-_{\ell}\begin{pmatrix}\ell & L & \ell' \\ 1 & 1 & -2\end{pmatrix}\right] \\
&\quad+p\left[c_x^2a^+_{\ell'}\begin{pmatrix}\ell & L & \ell' \\ -2 & -1 & 3\end{pmatrix}+a^-_{\ell'}\begin{pmatrix}\ell & L & \ell' \\ -2 & 1 & 1\end{pmatrix}\right]\left[a^+_{\ell}\begin{pmatrix}\ell & L & \ell' \\ 3 & -1 & -2\end{pmatrix}+c_x^2a^-_{\ell}\begin{pmatrix}\ell & L & \ell' \\ 1 & 1 & -2\end{pmatrix}\right]\Bigg\} \\
&= \int_{-1}^{1}\mathrm{d}\mu\ \pi L(L+1)\sum_{\ell\ell'}\frac{2\ell+1}{4\pi}A_\ell\frac{2\ell'+1}{4\pi}B_{\ell'} \\
&\quad\times[a^+_{\ell'}a^+_\ell d^\ell_{23}d^L_{1,-1}d^{\ell'}_{-3,-2}+c_x^2a^+_{\ell'}a^-_\ell d^\ell_{21}d^L_{11}d^{\ell'}_{-3,-2}+c_x^2a^-_{\ell'}a^+_\ell d^\ell_{23}d^L_{11}d^{\ell'}_{-1,-2}+a^-_{\ell'}a^-_\ell d^\ell_{21}d^L_{1,-1}d^{\ell'}_{-1,-2} \\
&\quad+p(c_x^2a^+_{\ell'}a^+_\ell d^\ell_{-2,3}d^L_{11}d^{\ell'}_{3,-2}+a^+_{\ell'}a^-_\ell d^\ell_{-2,1}d^L_{1,-1}d^{\ell'}_{3,-2}+a^-_{\ell'}a^+_\ell d^\ell_{-2,3}d^L_{1,-1}d^{\ell'}_{1,-2}+c_x^2a^-_{\ell'}a^-_\ell d^\ell_{-2,1}d^L_{11}d^{\ell'}_{1,-2})] \\
&= \int_{-1}^{1}\mathrm{d}\mu\ \pi L(L+1)[-c_x^2(\xi_{21}[A^-]\xi_{32}[B^+]+\xi_{32}[A^+]\xi_{21}[B^-]+p\xi_{3,-2}[A^+]\xi_{3,-2}[B^+]+p\xi_{2,-1}[A^-]\xi_{2,-1}[B^-])d^L_{11} \\
&\quad+(\xi_{32}[A^+]\xi_{32}[B^+]+\xi_{21}[A^-]\xi_{21}[B^-]-p\xi_{2,-1}[A^-]\xi_{3,-2}[B^+]-p\xi_{3,-2}[A^+]\xi_{2,-1}[B^-])d^L_{1,-1}]\,.
\end{aligned}
\tag{147}
$$

The temperature-polarization kernel is

$$
\begin{aligned}
\Sigma_L^{(\times),x}[A,B] &= \frac{1}{2L+1}\sum_{\ell\ell'}(W^{x,0}_{\ell L\ell'})^* W^{x,+}_{\ell L\ell'}A_\ell B_{\ell'} \\
&= \pi L(L+1)\sum_{\ell\ell'}\frac{2\ell+1}{4\pi}A_\ell\frac{2\ell'+1}{4\pi}B_{\ell'}a^0_{\ell'}2[1+c_x^2(-1)^{\ell+L+\ell'}] \\
&\quad\times\begin{pmatrix}\ell & L & \ell' \\ 0 & 1 & -1\end{pmatrix}\left[a^+_{\ell'}\begin{pmatrix}\ell & L & \ell' \\ -2 & -1 & 3\end{pmatrix}+c_x^2a^-_{\ell'}\begin{pmatrix}\ell & L & \ell' \\ -2 & 1 & 1\end{pmatrix}\right] \\
&= \pi L(L+1)\sum_{\ell\ell'}\frac{2\ell+1}{4\pi}A_\ell\frac{2\ell'+1}{4\pi}B_{\ell'}a^0_{\ell'} \\
&\quad\times 2\left[\begin{pmatrix}\ell & L & \ell' \\ 0 & 1 & -1\end{pmatrix}+c_x^2\begin{pmatrix}\ell & L & \ell' \\ 0 & -1 & 1\end{pmatrix}\right]\left[a^+_{\ell'}\begin{pmatrix}\ell & L & \ell' \\ -2 & -1 & 3\end{pmatrix}+c_x^2a^-_{\ell'}\begin{pmatrix}\ell & L & \ell' \\ -2 & 1 & 1\end{pmatrix}\right] \\
&= \int_{-1}^{1}\mathrm{d}\mu\ \pi L(L+1)\sum_{\ell\ell'}\frac{2\ell+1}{4\pi}A_\ell\frac{2\ell'+1}{4\pi}B_{\ell'}a^0_{\ell'} \\
&\quad\times\left[a^+_{\ell'}d^\ell_{0,-2}d^L_{1,-1}d^{\ell'}_{-1,3}+c_x^2a^-_{\ell'}d^\ell_{0,-2}d^L_{11}d^{\ell'}_{-1,1}+c_x^2a^+_{\ell'}d^\ell_{0,-2}d^L_{11}d^{\ell'}_{13}+a^-_{\ell'}d^\ell_{0,-2}d^L_{-1,1}d^{\ell'}_{11}\right] \\
&= \int_{-1}^{1}\mathrm{d}\mu\ \pi L(L+1)\{c_x^2(\xi_{20}[A]\xi_{1,-1}[B^{0-}]+\xi_{20}[A]\xi_{31}[B^{0+}])d^L_{11} \\
&\quad+(\xi_{20}[A]\xi_{3,-1}[B^{0+}]+\xi_{20}[A]\xi_{11}[B^{0-}])d^L_{1,-1}\}\,,
\end{aligned}
\tag{148}
$$

and

$$
\begin{aligned}
\Gamma_L^{(\times),x}[A,B] &= \frac{1}{2L+1}\sum_{\ell\ell'}(W^{x,0}_{\ell L\ell'})^* W^{x,+}_{\ell' L\ell} A_\ell B_{\ell'} \\
&= \pi L(L+1)\sum_{\ell\ell'}\frac{2\ell+1}{4\pi}A_\ell\frac{2\ell'+1}{4\pi}B_{\ell'}a^0_{\ell'}2[1+c_x^2(-1)^{\ell+L+\ell'}] \\
&\quad\times\begin{pmatrix}\ell & L & \ell'\\ 0 & 1 & -1\end{pmatrix}\left[a^+_\ell\begin{pmatrix}\ell' & L & \ell\\ -2 & -1 & 3\end{pmatrix}+c_x^2a^-_\ell\begin{pmatrix}\ell' & L & \ell\\ -2 & 1 & 1\end{pmatrix}\right] \\
&= \pi L(L+1)\sum_{\ell\ell'}\frac{2\ell+1}{4\pi}A_\ell\frac{2\ell'+1}{4\pi}B_{\ell'}a^0_{\ell'} \\
&\quad\times 2\left[\begin{pmatrix}\ell & L & \ell'\\ 0 & 1 & -1\end{pmatrix}+c_x^2\begin{pmatrix}\ell & L & \ell'\\ 0 & -1 & 1\end{pmatrix}\right]\left[a^+_\ell\begin{pmatrix}\ell & L & \ell'\\ -3 & 1 & 2\end{pmatrix}+c_x^2a^-_\ell\begin{pmatrix}\ell & L & \ell'\\ -1 & -1 & 2\end{pmatrix}\right] \\
&= \int_{-1}^{1}\mathrm{d}\mu\ \pi L(L+1)\sum_{\ell\ell'}\frac{2\ell+1}{4\pi}A_\ell\frac{2\ell'+1}{4\pi}B_{\ell'}a^0_{\ell'} \\
&\quad\times\left[a^+_\ell d^\ell_{0,-3}d^L_{11}d^{\ell'}_{-1,2}+c_x^2a^-_\ell d^\ell_{0,-1}d^L_{1,-1}d^{\ell'}_{-1,2}+c_x^2a^+_\ell d^\ell_{0,-3}d^L_{1,-1}d^{\ell'}_{12}+a^-_\ell d^\ell_{0,-1}d^L_{11}d^{\ell'}_{12}\right] \\
&= \int_{-1}^{1}\mathrm{d}\mu\ \pi L(L+1)\{-(\xi_{30}[A^+]\xi_{2,-1}[B^0]+\xi_{10}[A^-]\xi_{12}[B^0])d^L_{11} \\
&\quad - c_x^2(\xi_{10}[A^-]\xi_{2,-1}[B^0]+\xi_{30}[A^0]\xi_{21}[B^-])d^L_{1,-1}\}\,.
\end{aligned}
\tag{149}
$$

6.2 Kernel Functions: Rotation

Next we consider the kernel functions for $x=\alpha$. If $p=-$ and $x=\alpha$,

$$
\begin{aligned}
\Sigma_L^{(-),\alpha}[A,B] &= \pi\sum_{\ell\ell'}\frac{2\ell+1}{4\pi}A_\ell\frac{2\ell'+1}{4\pi}B_{\ell'}8[1+(-1)^{\ell+L+\ell'}]\begin{pmatrix}\ell & L & \ell'\\ -2 & 0 & 2\end{pmatrix}^2 \\
&= \pi L(L+1)\sum_{\ell\ell'}\frac{2\ell+1}{4\pi}A_\ell\frac{2\ell'+1}{4\pi}B_{\ell'}8\left[\begin{pmatrix}\ell & L & \ell'\\ -2 & 0 & 2\end{pmatrix}^2+\begin{pmatrix}\ell & L & \ell'\\ -2 & 0 & 2\end{pmatrix}\begin{pmatrix}\ell & L & \ell'\\ 2 & 0 & -2\end{pmatrix}\right] \\
&= \int_{-1}^{1}\mathrm{d}\mu\ \pi L(L+1)\sum_{\ell\ell'}\frac{2\ell+1}{4\pi}A_\ell\frac{2\ell'+1}{4\pi}B_{\ell'}4(d^\ell_{-2,-2}d^L_{00}d^{\ell'}_{22}+d^\ell_{-2,2}d^L_{00}d^{\ell'}_{2,-2}) \\
&= \int_{-1}^{1}\mathrm{d}\mu\ 4\pi(\xi_{-2,-2}[A]\xi_{22}[B]+\xi_{-2,2}[A]\xi_{2,-2}[B])d^L_{00}\,,
\end{aligned}
\tag{150}
$$

and

$$
\begin{aligned}
\Gamma_L^{(-),\alpha}[A,B] &= \pi\sum_{\ell\ell'}\frac{2\ell+1}{4\pi}A_\ell\frac{2\ell'+1}{4\pi}B_{\ell'}8[1+(-1)^{\ell+L+\ell'}]\begin{pmatrix}\ell & L & \ell'\\ -2 & 0 & 2\end{pmatrix}\begin{pmatrix}\ell' & L & \ell\\ -2 & 0 & 2\end{pmatrix} \\
&= \pi\sum_{\ell\ell'}\frac{2\ell+1}{4\pi}A_\ell\frac{2\ell'+1}{4\pi}B_{\ell'}8[1+(-1)^{\ell+L+\ell'}]\begin{pmatrix}\ell & L & \ell'\\ -2 & 0 & 2\end{pmatrix}^2 \\
&= \Sigma_L^{(-),\alpha}[A,B]\,.
\end{aligned}
\tag{151}
$$

6.3 Kernel Functions: Amplitude

Here we consider $x=\tau$.

$$
\Sigma_L^{(+),\tau}[A,B]=\frac{1}{4}\Sigma_L^{(-),\alpha}[A,B]\,,
\tag{152}
$$

and

$$
\Gamma_L^{(+),\tau}[A,B]=\frac{1}{4}\Gamma_L^{(-),\alpha}[A,B]\,.
\tag{153}
$$

6.4 Response function

6.4.1 α and τ

The response of the quadratic EB estimator is given by

$$
\begin{aligned}
[A_L^{\alpha\tau,(EB)}]^{-1} &= \frac{1}{2L+1}\sum_{\ell\ell'}\frac{(W^{\alpha,-}_{\ell L\ell'}C^{\rm BB}_{\ell'} - W^{\alpha,-}_{\ell' L\ell}C^{\rm EE}_{\ell})(W^{\tau,+}_{\ell L\ell'}C^{\rm EB}_{\ell'} + W^{\tau,+}_{\ell' L\ell}C^{\rm EB}_{\ell})}{\widehat{C}^{\rm EE}_{\ell}\widehat{C}^{\rm BB}_{\ell'}} \\
&= \frac{-1}{2(2L+1)}\sum_{\ell\ell'}\frac{(W^{\alpha,-}_{\ell L\ell'}C^{\rm BB}_{\ell'} - W^{\alpha,-}_{\ell' L\ell}C^{\rm EE}_{\ell})(W^{\alpha,-}_{\ell L\ell'}C^{\rm EB}_{\ell'} + W^{\alpha,-}_{\ell' L\ell}C^{\rm EB}_{\ell})}{\widehat{C}^{\rm EE}_{\ell}\widehat{C}^{\rm BB}_{\ell'}} \\
&= -\frac{1}{2}\Sigma_L^{(-),\alpha}\left[\frac{1}{\widehat{C}^{\rm EE}},\frac{C^{\rm EB}C^{\rm BB}}{\widehat{C}^{\rm BB}}\right] + \frac{1}{2}\Gamma_L^{(-),\alpha}\left[\frac{C^{\rm EE}}{\widehat{C}^{\rm EE}},\frac{(C^{\rm EB}-C^{\rm BB})}{\widehat{C}^{\rm BB}}\right] + \frac{1}{2}\Sigma_L^{(-),\alpha}\left[\frac{1}{\widehat{C}^{\rm BB}},\frac{C^{\rm EB}C^{\rm EE}}{\widehat{C}^{\rm EE}}\right] ,
\end{aligned}
\tag{154}
$$

7 Computing delensed CMB anisotropies

7.1 Linear template of lensing B mode

The gradient of lensing potential $\boldsymbol{\nabla}\phi$ is transformed as

$$\begin{aligned}\boldsymbol{\nabla}\phi &= \sum_{\ell m}\boldsymbol{\nabla}Y_{\ell m}\phi_{\ell m} = \sum_{\ell m}\sqrt{\frac{\ell(\ell+1)}{2}}\phi_{\ell m}(Y^1_{\ell m}\boldsymbol{e}^* - Y^{-1}_{\ell m}\boldsymbol{e}) \\ &= (\phi^+ + \mathrm{i}\phi^-)\boldsymbol{e}^* + (\phi^+ - \mathrm{i}\phi^-)\boldsymbol{e}\,, \end{aligned} \tag{155}$$

where $\phi^\pm$ is obtained by spin-1 inverse harmonic transform of $\phi_{\ell m}\sqrt{\ell(\ell+1)/2}$. Similarly the gradient of polarization $\boldsymbol{\nabla}P^\pm = \boldsymbol{\nabla}(Q \pm \mathrm{i}U)$ is

$$\begin{aligned}\boldsymbol{\nabla}P^+ &= -\sum_{\ell m}E_{\ell m}\boldsymbol{\nabla}Y^2_{\ell m} \\ &= -\sum_{\ell m}E_{\ell m}\left(\sqrt{\frac{(\ell-2)(\ell+3)}{2}}Y^3_{\ell m}\boldsymbol{e}^* - \sqrt{\frac{(\ell+2)(\ell-1)}{2}}Y^1_{\ell m}\boldsymbol{e}\right) \\ &= -(E^+_3 + \mathrm{i}E^-_3)\boldsymbol{e}^* + (E^+_1 + \mathrm{i}E^-_1)\boldsymbol{e}\,, \end{aligned} \tag{156}$$

$$\boldsymbol{\nabla}P^- = (\boldsymbol{\nabla}P^+)^* = (E^+_1 - \mathrm{i}E^-_1)\boldsymbol{e}^* - (E^+_3 - \mathrm{i}E^-_3)\boldsymbol{e}\,. \tag{157}$$

This leads to

$$\boldsymbol{\nabla}\phi\cdot\boldsymbol{\nabla}P^+ = -(E^+_3 + \mathrm{i}E^-_3)(\phi^+ - \mathrm{i}\phi^-) + (E^+_1 + \mathrm{i}E^-_1)(\phi^+ + \mathrm{i}\phi^-) \tag{158}$$

$$\boldsymbol{\nabla}\phi\cdot\boldsymbol{\nabla}P^- = (\boldsymbol{\nabla}\phi\cdot\boldsymbol{\nabla}P^+)^*\,. \tag{159}$$

The harmonic transform of the above quantity becomes the leading-order lensing contributions to E/B.

References

[1] A. Lewis and A. Challinor, *"Weak gravitational lensing of the CMB"*, *Phys. Rep.* **429** (2006) 1–65, [astro-ph/0601594].

[2] D. Hanson, A. Challinor, and A. Lewis, *"Weak lensing of the CMB"*, *Gen. Rel. Grav.* **42** (2010) 2197–2218, [arXiv:0911.0612].

[3] C. M. Hirata and U. Seljak, *"Reconstruction of lensing from the cosmic microwave background polarization"*, *Phys. Rev. D* **68** (2003) 083002, [astro-ph/0306354].

[4] V. Gluscevic, M. Kamionkowski, and A. Cooray, *"Derotation of the cosmic microwave background polarization: Full-sky formalism"*, *Phys. Rev. D* **80** (2009) 023510, [arXiv:0905.1687].

[5] C. Dvorkin and K. M. Smith, *"Reconstructing Patchy Reionization from the Cosmic Microwave Background"*, *Phys. Rev. D* **79** (2009) 043003, [arXiv:0812.1566].

[6] D. Varshalovich, A. Moskalev, and V. Kersonskii, *Quantum Theory of Angular Momentum*. World Scientific, Singapore, 1989.

[7] T. Namikawa, D. Yamauchi, and A. Taruya, *"Full-sky lensing reconstruction of gradient and curl modes from CMB maps"*, *J. Cosmol. Astropart. Phys.* **1201** (2012) 007, [arXiv:1110.1718].

[8] M. Kamionkowski, *"How to De-Rotate the Cosmic Microwave Background Polarization"*, *Phys. Rev. Lett.* **102** (2009) 111302, [arXiv:0810.1286].

[9] W. Hu and T. Okamoto, *"Mass Reconstruction with CMB Polarization"*, *Astrophys. J.* **574** (2002) 566–574, [astro-ph/0111606].

[10] T. Okamoto and W. Hu, *"CMB Lensing Reconstruction on the Full Sky"*, *Phys. Rev. D* **67** (2003) 083002, [astro-ph/0301031].

[11] K. Gorski *et al.*, *"HEALPix - A Framework for high resolution discretization, and fast analysis of data distributed on the sphere"*, *Astrophys. J.* **622** (2005) 759–771, [astro-ph/0409513].

[12] K. M. Smith, D. Hanson, M. LoVerde, C. M. Hirata, and O. Zahn, *"Delensing CMB Polarization with External Datasets"*, *J. Cosmol. Astropart. Phys.* **06** (2012) 014, [arXiv:1010.0048].