1 EE method

$$d_i^{(j)} = \frac{Y(\mathbf{X}_{\sim\mathbf{i}}^{(\mathbf{j})}, X_i^{(j)} + \Delta^{(i,j)})}{\Delta^{(i,j)}}, \tag{1}$$

$$\mu_i = \frac{1}{r}\sum_{j=1}^{r} d_i^{(j)}. \tag{2}$$

$$\sigma_i = \frac{1}{r}\sum_{j=1}^{r} (d_i^{(j)} - \mu_i) \tag{3}$$

$$\mu_i^* = \frac{1}{r}\sum_{j=1}^{r} \left|d_i^{(j)}\right|. \tag{4}$$

$$\mu_{i,\sigma}^* = \mu_i^* \frac{\sigma_{X_i}}{\sigma_Y}. \tag{5}$$

Scaling provides link from "deterministic effect" to impact on variation of Y. Parameters may have large effects on Y but not on its variation: The impact of X may be strong but always the same if $\sigma_x = \epsilon$. Especially tricky, if Y does also not vary a lot.

Example: linear function with two parameters: high coeff - low SD and low coeff, high SD.

2 Sampling schemes

One EE per parameter per subsample.

$$\underset{(k+1)\times k}{\mathbf{R}} = \begin{pmatrix} a_1 & a_2 & ... & a_k \\ \mathbf{b_1} & a_2 & ... & a_k \\ a_1 & \mathbf{b_2} & ... & a_k \\ \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & ... & \mathbf{b_k} \end{pmatrix} \tag{6}$$

Contains choice of (quasi-random) sequence. Implies share of very high steps.

$$\underset{(k+1)\times k}{\mathbf{T}} = \begin{pmatrix} a_1 & a_2 & ... & a_k \\ \mathbf{b_1} & a_2 & ... & a_k \\ \mathbf{b_1} & \mathbf{b_2} & ... & a_k \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{b_1} & \mathbf{b_2} & ... & \mathbf{b_k} \end{pmatrix} \tag{7}$$

Contains choice of two numerical parameters. Lower bound of step is 0.5 given step function.

3 EE method for correlated inputs

1. $\mathbf{z} = \boldsymbol{\Phi^{-1}(u)}$
2. $\mathbf{z_c} = \mathbf{Q^T z^T}$
3. $\mathbf{x} = \boldsymbol{\mu} + \mathbf{z_c}(\mathbf{i})\boldsymbol{\sigma(i)}$

Transform samples. $N(3k+1)$ and $3Nk$ function evals.
Correlations include step.

$$\mathcal{T}_1(\mathbf{T_{i+1,*}}; i-1) = \begin{pmatrix} a_k & a_1 & ... & ... & a_{k-1} \\ \mathbf{b_1} & a_2 & ... & ... & a_k \\ \mathbf{b_2} & a_3 & ... & ... & \mathbf{b_1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{b_k} & \mathbf{b_1} & ... & ... & \mathbf{b_{k-1}} \end{pmatrix} \tag{8}$$

$$\mathcal{T}_1(\mathbf{T_{i,*}}; i-1) = \begin{pmatrix} a_1 & a_2 & ... & ... & a_k \\ a_2 & ... & ... & a_k & \mathbf{b_1} \\ a_3 & ... & ... & \mathbf{b_1} & \mathbf{b_2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{b_1} & \mathbf{b_2} & ... & ... & \mathbf{b_k} \end{pmatrix} \tag{9}$$

Correlations exclude step.

$$\mathcal{T}_1(\mathbf{T_{i,*}}; i-1) = \begin{pmatrix} a_2 & a_3 & ... & ... & a_1 \\ a_3 & ... & a_k & \mathbf{b_1} & a_2 \\ a_4 & ... & \mathbf{b_1} & \mathbf{b_2} & a_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{b_2} & \mathbf{b_3} & ... & ... & \mathbf{b_1} \end{pmatrix} \tag{10}$$

Trajectory design only.

Activates distortions by numerical parameters and general differences in drawing the elements between schemes.

$$d_i^{full,T} = \frac{f\Big(\mathcal{T}(\mathbf{T_{i+1,*}}; i-1)\Big) - f\Big(\mathcal{T}(\mathbf{T_{i,*}}; i)\Big)}{\Delta}. \tag{11}$$

$$d_i^{ind,T} = \frac{f\Big(\mathcal{T}(\mathbf{T_{i+1,*}}; i)\Big) - f\Big(\mathcal{T}(\mathbf{T_{i,*}}; i)\Big)}{\Delta}. \tag{12}$$

Improvements. Denominator can be written much simpler as $\mathcal{T}(b_i) - \mathcal{T}(a_i)$.

$$d_i^{c,T} = \frac{f\Big(\mathcal{T}(\mathbf{T_{i+1,*}}; i-1)\Big) - f\Big(\mathcal{T}(\mathbf{T_{i-1,*}}; i)\Big)}{F^{-1}\Big(\Phi^u(b_i)\Big) - F^{-1}\Big(\Phi^u(a_i)\Big)} \tag{13}$$

$$d_i^{u,T} = \frac{f\Big(\mathcal{T}(\mathbf{T_{i+1,*}}; i)\Big) - f\Big(\mathcal{T}(\mathbf{T_{i,*}}; i)\Big)}{F^{-1}\Big({Q^T}_{k,*k-1}(j)T^T_{i+1,*k-1}(j) + {Q^T}_{k,k}\Phi^u(b_i)\Big) - F^{-1}\Big({Q^T}_{k,*k-1}(j)T^T_{i,*k-1}(j) + {Q^T}_{k,k}\Phi^u(a_i)\Big)} \tag{14}$$

4 Replication and Validation

Linear function with three parameters and Correlations equal to 0.9, 0.4, 0.01. Numerical parameter lead to re-lineariation of measures in GM'17.

Let $f(X_1, ..., X_k) = \sum_{i=1}^{k} c_i X_i$ be an arbitrary linear function. Let $\rho_{i,j}$ be the linear correlation between X_i and X_j. Then, for all $i \in 1, ..., k$, I expect:[1]

$$d_i^{u,*} = c_i \tag{15}$$

$$d_i^{c,*} = \sum_{j=1}^{k} \rho_{i,j} c_j.. \tag{16}$$

Both equations state that, conceptually, the result does not depend on the sampling scheme.

[1] These results correspond to the intuition provided by the example in Saltelli et al. (2008), p. 123.

Table 1. Replication and validation - trajectory design

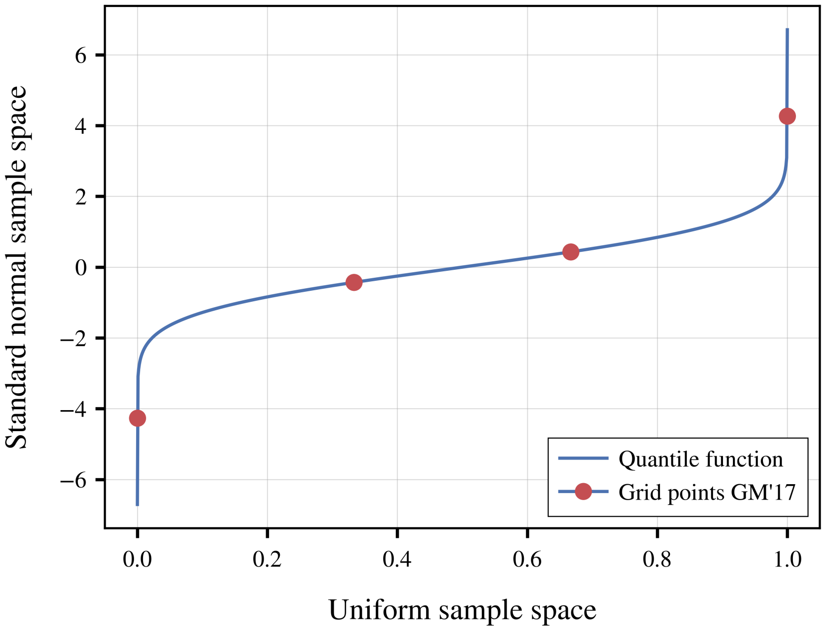
Measure	GM'17	Repl. $\mu^{*\dagger}$	Repl. $\sigma^{\ddagger}$	S'20
	1.20	1.36	0.83	1.00
$\mu^{*,ind}$	1.30	1.48	0.91	1.00
	3.20	3.11	1.94	1.00
	0.55	0.00	0.56	0.00
σ^{ind}	0.60	0.00	0.62	0.00
	1.30	0.00	1.32	0.00
	14.90	16.20	9.97	2.30
$\mu^{*,full}$	12.50	13.45	8.31	1.91
	10.00	9.93	6.18	1.41
	6.50	0.00	6.74	0.00
σ^{full}	5.50	0.00	5.63	0.00
	4.00	0.00	4.20	0.00

† $0^{num} = 0.00001$ and $l = 4$.
‡ $0^{num} = 0.00000001$ and $l = 24$.

Table 2. Replication and validation - radial design

Measure	GM'17	Replication	S'20
	0.60	0.57	1.00
$\mu^{*,ind}$	0.75	0.85	1.00
	1.50	1.31	1.00
	0.20	0.10	0.00
σ^{ind}	0.30	0.41	0.00
	0.85	0.22	0.00
	7.50	6.84	2.30
$\mu^{*,full}$	6.80	7.77	1.91
	4.75	4.19	1.41
	2.90	1.15	0.00
σ^{full}	2.65	3.68	0.00
	2.50	0.70	0.00

Figure 1. Grid points in standard normal sample space for trajectory design with $l = 4$



5 Results: Uncertainty Analysis

Figure 2. Probability distribution of quantity of interest q

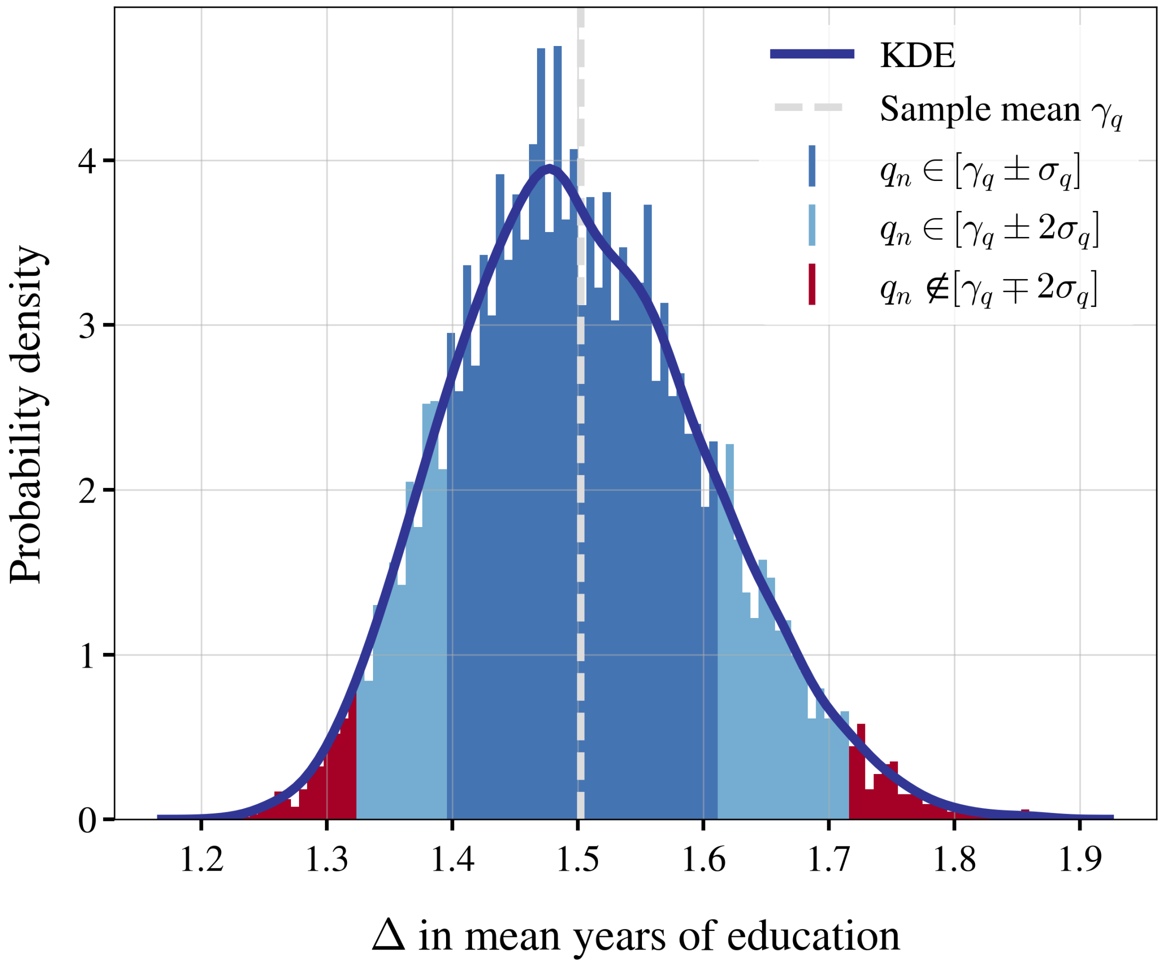
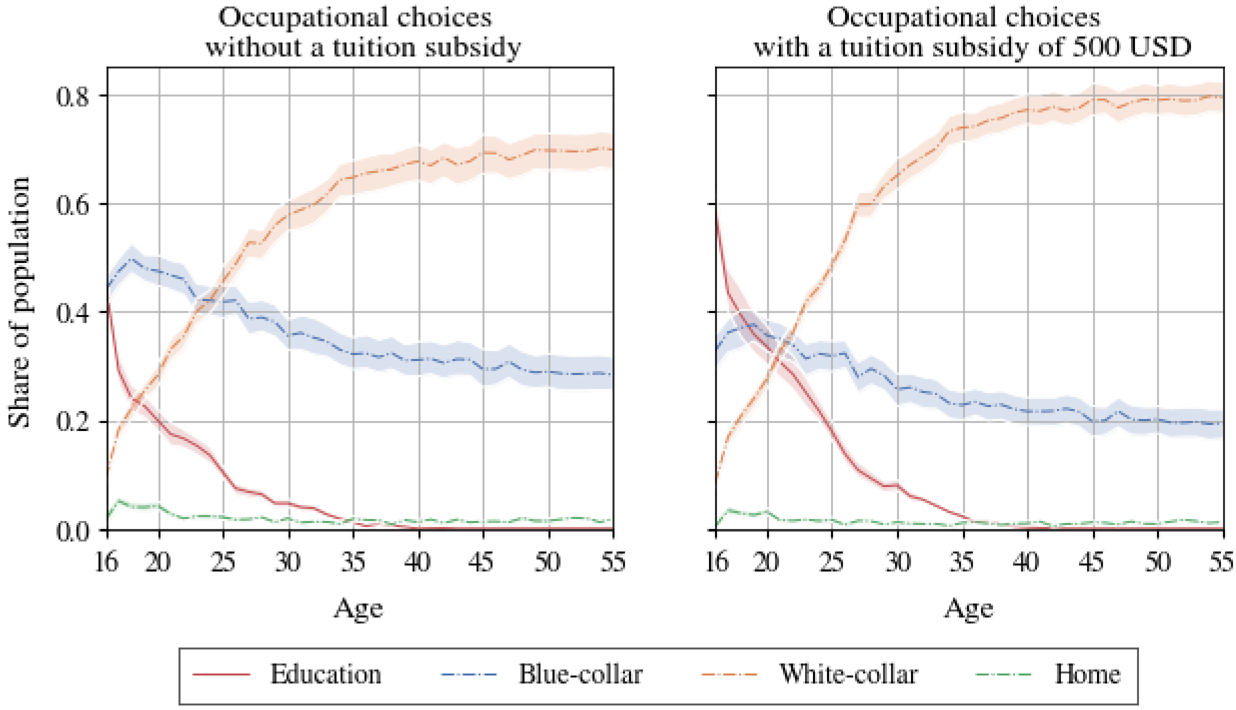


Figure 3. Comparison of shares of occupation decision over time between scenarios with cone plots



5.1 Results: Qualitative Sensitivity Analysis

Be cautious!

Table 3. Mean absolute correlated and uncorrelated elementary effects (based on 150 subsamples in trajectory and radial design)

Parameter	$\mu_T^{*,c}$	$\mu_R^{*,c}$	$\mu_T^{*,u}$	$\mu_R^{*,u}$
General				
δ	17	23	476	415
Blue-collar				
β^b	1	3	43	88
β^b_e	11	14	406	443
β^b_b	25	51	688	1169
β^b_{bb}	871	934	15 540	17 860
β^b_w	29	48	73	143
β^b_{ww}	389	460	869	1183
White-collar				
β^w	1	3	50	117
β^w_e	26	28	943	852
β^w_w	24	47	718	1521
β^w_{ww}	933	997	12 257	18 069
β^w_b	131	127	309	356
β^w_{bb}	120	1352	2088	2477
Education				
β^e	0.0008	0.0002	0.001	0.003
β^e_{he}	0.0001	0.0002	0.001	0.001
β^e_{re}	0.0003	0.0002	0.0003	0.0006
Home				
β^h	0.0003	0.0003	0.000 02	0.000 02
Lower Triangular Cholesky Matrix				
c_1	8	16	18	37
c_2	8	11	22	24
c_3	0.0004	0.0004	0.0004	0.0007
c_4	0.0004	0.000 08	0.0002	0.0003
$c_{1,2}$	4	4	10	10
$c_{1,3}$	0.0005	0.0006	0.0006	0.0005
$c_{2,3}$	0.0003	0.0005	0.0006	0.001
$c_{1,4}$	0.000 04	0.000 05	0.0004	0.0005
$c_{2,4}$	0.0001	0.0002	0.0001	0.0002
$c_{3,4}$	0.0001	0.0001	0.000 08	0.0001

Figure 4. Sigma-normalized mean absolute Elementary Effects for trajectory design

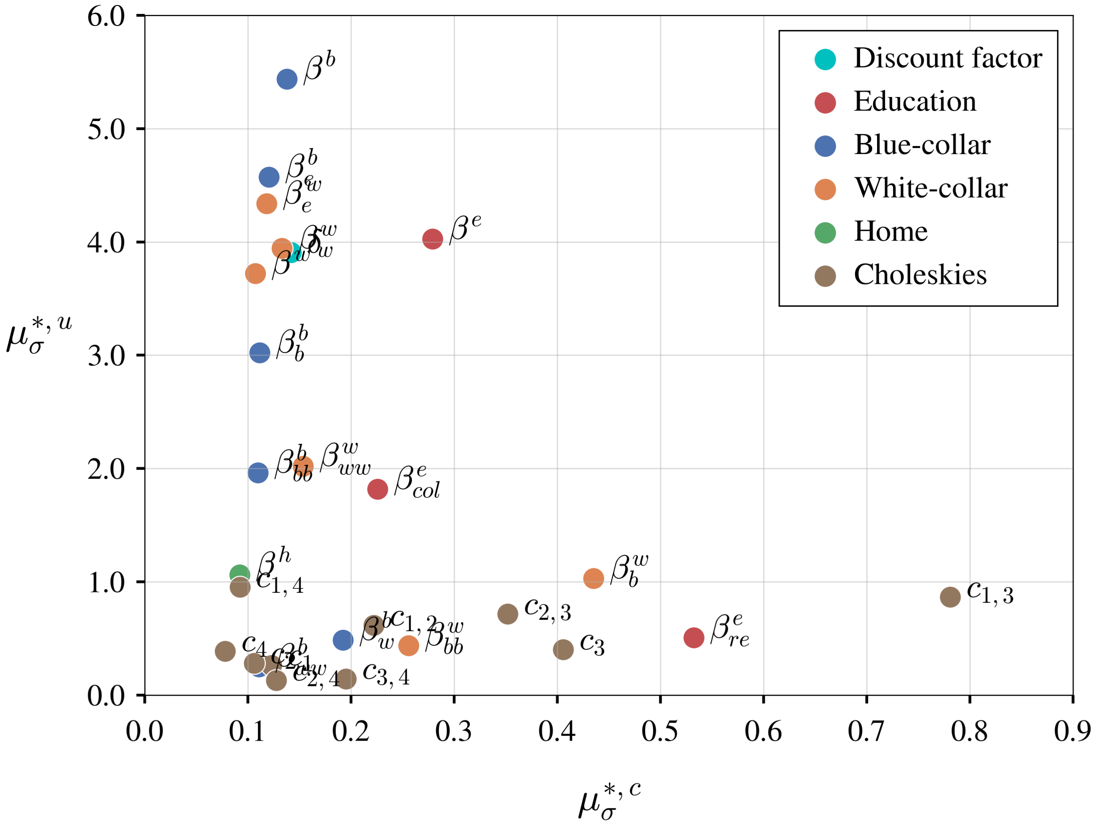
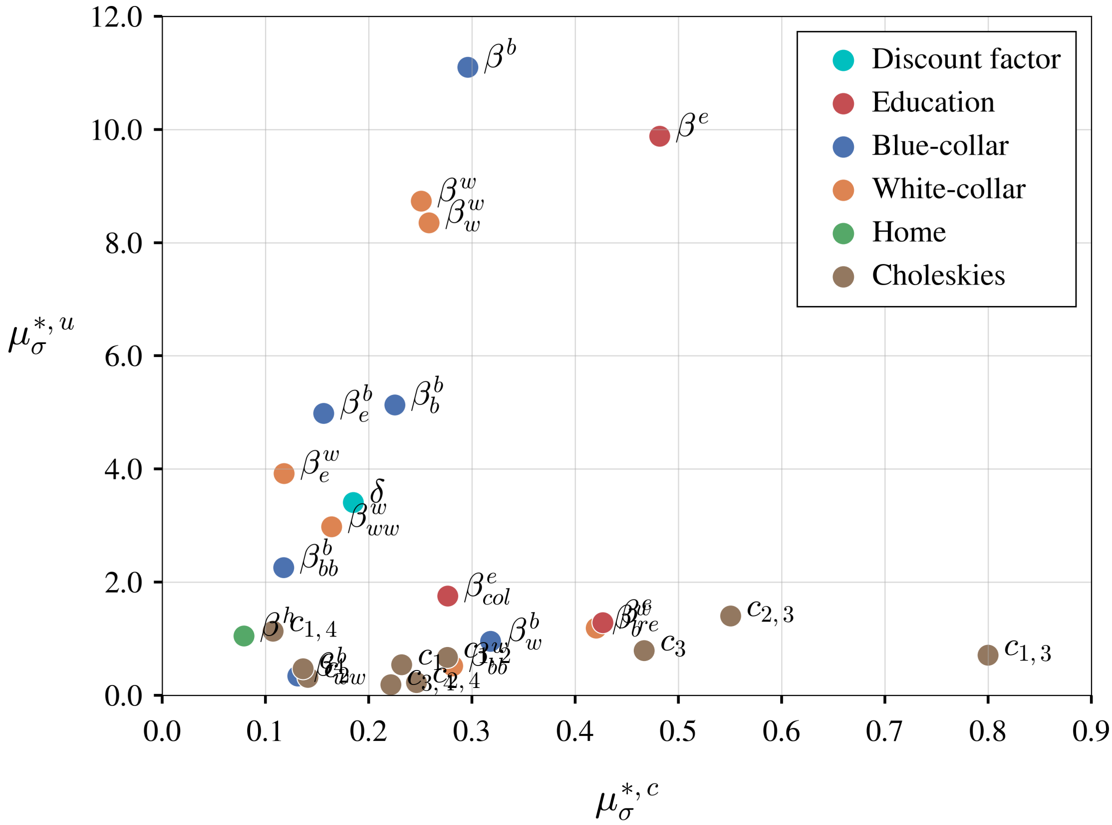


Figure 5. Sigma-normalized mean absolute Elementary Effects for radial design



5.2 Review: Estimation Results

Table 4. Estimates for the distribution of input parameters

Parameter	Mean	Standard error (SE)	SE in KW94
General			
δ	0.95	0.000 84	-
Blue-collar			
β^b	9.21	0.013	0.014
β^b_e	0.038	0.0011	0.0015
β^b_b	0.033	0.000 44	0.000 79
β^b_{bb}	−0.0005	0.000 013	0.000 019
β^b_w	0.0	0.000 67	0.0024
β^b_{ww}	0.0	0.000 029	0.000 096
White-collar			
β^w	8.48	0.0076	0.0123
β^w_e	0.07	0.000 47	0.000 96
β^w_w	0.067	0.000 55	0.000 90
β^w_{ww}	−0.001	0.000 017	0.000 070
β^w_b	0.022	0.000 33	0.0010
β^w_{bb}	−0.0005	0.000 021	0.000 030
Education			
β^e	0.0	330	459
β^e_{he}	0.0	155	410
β^e_{re}	−4000	202	660
Home			
β^h	17 750	390	1442
Lower Triangular Cholesky Matrix			
c_1	0.2	0.0015	0.0056
c_2	0.25	0.0013	0.0046
c_3	1500	108	350
c_4	1500	173	786
$c_{1,2}$	0.0	0.0064	0.023
$c_{1,3}$	0.0	143	0.412
$c_{2,3}$	0.0	116	0.379
$c_{1,4}$	0.0	232	0.911
$c_{2,4}$	0.0	130	0.624
$c_{3,4}$	0.0	177	0.870

5.3 Improvement: Sampling scheme tailored to Sobol' indices

Similar to trajectory design to have interactions still included. Base row is expectation. Shuffle row. Add random value between $[0, 0.5]$. Take square root of squared difference and divide by step.

References

Saltelli, A., M. Ratto, T. Andres, F. Campolongo, J. Cariboni, D. Gatelli, M. Saisana, and S. Tarantola (2008). *Global Sensitivity Analysis: The Primer.* John Wiley & Sons.