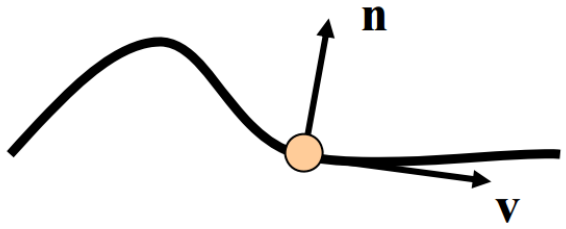
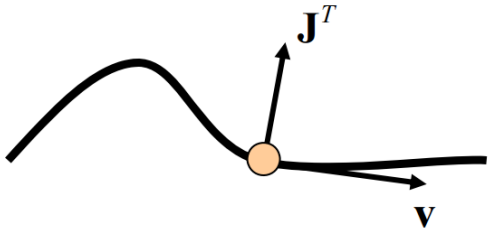
Lecture IX: Position-Based Dynamics

Previously: Constraints Calculus

- Constraint: $C(p, p', p'') = 0$.
- Velocity always tangent: $\mathrm{J}\vec{v} = 0$
- Constraint force always normal (virtual work):

$$\overrightarrow{F_c} = \sum_i \lambda_i \mathrm{J}_{i,.} = J^T \lambda$$





Previously: Sequential Impulses

- **while** (!done) {
  for all constraints c **do** solve c;
  }

- Computing the impulse:
  - For old velocity: $J_i v \neq 0$
  - For new velocity: $J_i(v + \Delta v) = 0$.
  - Apply impulse: $P = M\Delta v = {J_i}^T \lambda_i$, M is mass matrix.
- To compute λ_i

$$J_i\left(v + M^{-1}{J_i}^T\lambda_i\right) = 0$$

$$\lambda_i = \frac{-J_i v}{J_i M^{-1}{J_i}^T}$$

Previously: Semi-implicit Integration

- Also known as sympletic.
- Velocity integrated forward:

$$v(t + \Delta t) = v(t) + \frac{F}{m}\Delta t$$

- Position integrated backwards:

$$x(t + \Delta t) = x(t) + \Delta t v(t + \Delta t)$$

Position-Based Dynamics Principles

- Velocity always set to match position change at each iteration:

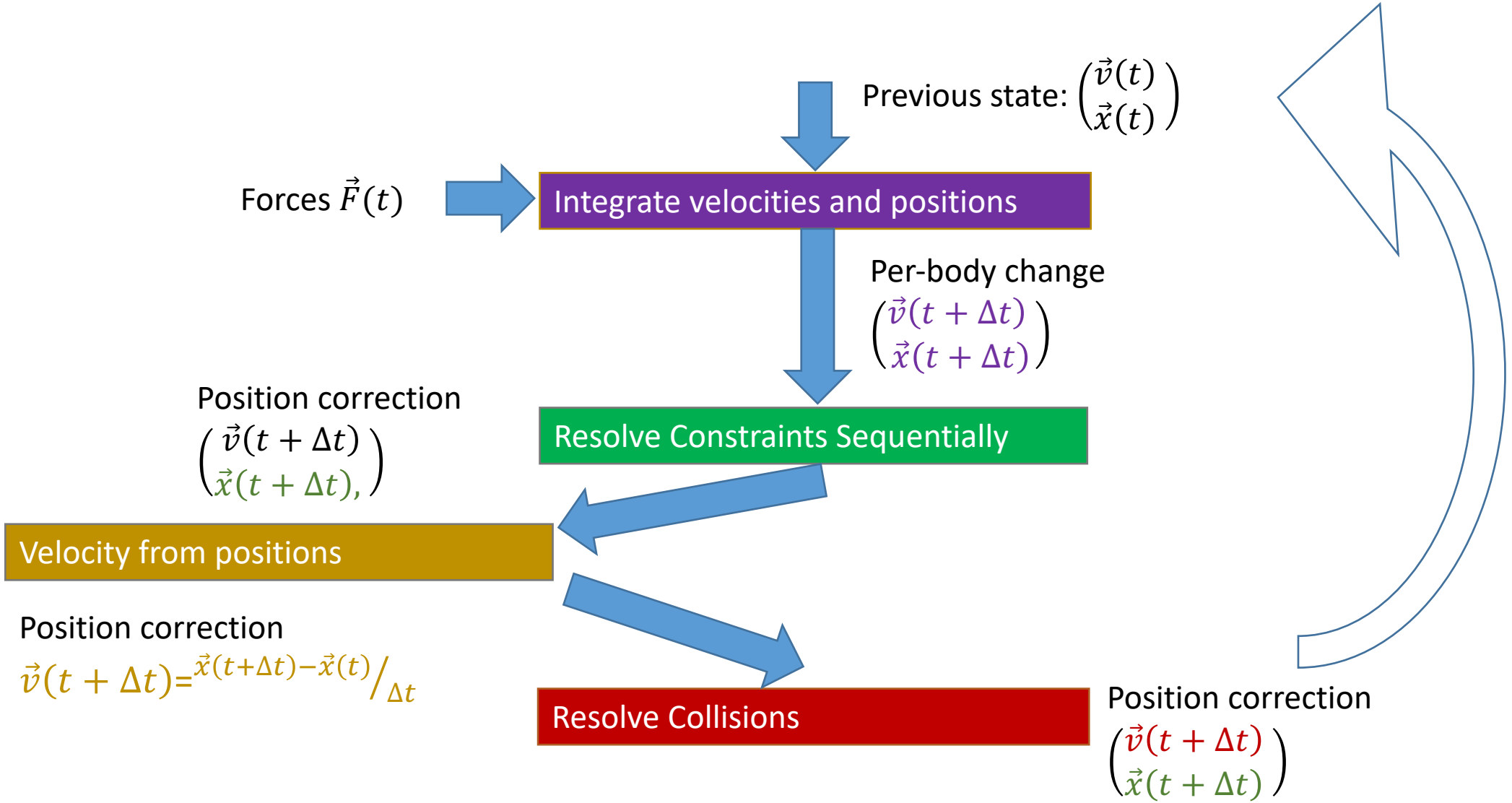
$$v = {\Delta p}/{\Delta t}$$

- Constraints expressed by positions alone

$$C(p, p', p'') = C(p) = 0$$

- “Classic” version: works on particles.
  - Connection (like rigidity) modeled with constraints.
  - Does not explicitly handle rotations

The Position-Based Game-Engine Loop



Position-Based Dynamics

- Advantages
  - Unconditional stability
  - Modularity and uniformity
- Disadvantages
  - Not physically accurate (but visually OK)
  - Resolves rotation by constraint solving
    - But then uniformly handles non-rigidity.

https://www.youtube.com/watch?v=j5igW5-h4ZM

Constraints Solving

- For body positions $\mathrm{X} = x_1, x_2, \cdots, x_m$.
- Set of equality constraints $C_i(X) = 0$.
  - lengths, connectivity.
- Set of inequality constraints $C_i(X) \geq 0$.
  - Collision, deformation limits.
- Sequential solution:
  - For each violating $C_i(X)$ in turn, compute ΔX s.t. $C_i(X + \Delta X)$ is valid.

Conservation

- Conserving linear momentum:

$$\sum_{i=1}^{m} m_i \Delta x_i = 0$$

- In matrix writing: $M\Delta X = 0$.
  - $M = diag(m_1, m_1, m_1, \cdots, m_n, m_n, m_n)$.
- Conserving angular momentum (r_i to a fixed origin)

$$\sum_{i=1}^{m} r_i \times m_i \Delta x_i = 0$$

$$\lambda_i = \frac{-J_i v}{J_i M^{-1} {J_i}^T}$$

Resolving Constraints

- Linear approximation:

$$C_i(p + \Delta p) \approx C_i(p) + \nabla C_i \cdot \Delta p$$

- Closest point: move in the gradient direction.

$$\Delta p = \lambda \nabla C_i$$

- To conserve momenta: Alternative weighting

$$\Delta p = \lambda_i M^{-1} \nabla C_i$$

  - Intuitive explanation: exchanging impulses and not velocities.

- For equality constraints:

$$C_i(p) + \lambda_i \nabla {C_i}^T M^{-1} \nabla C_i = 0 \rightarrow$$

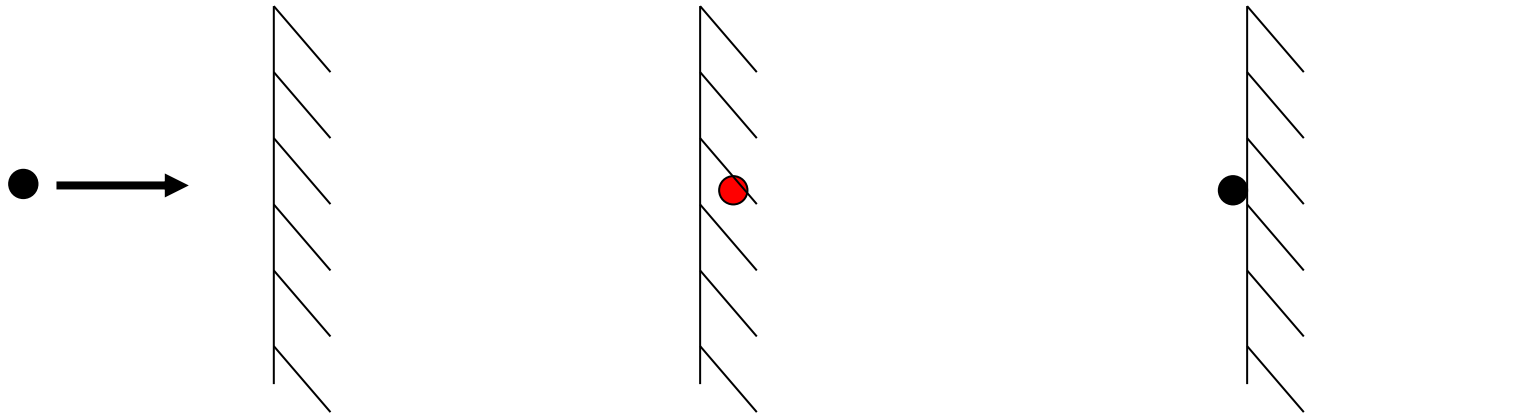
$$\lambda_i = \frac{-C_i(p)}{\nabla {C_i}^T M^{-1} \nabla C_i}$$

Inequality Constraints

- Essentially resolved the same

$$C_i(p + \Delta p) \approx C_i(p) + \nabla C_i \cdot \Delta p \geq 0$$

- Main difference: projection only performed if constraint is violated
  - Example: if collision happened.



Constraint Stiffness

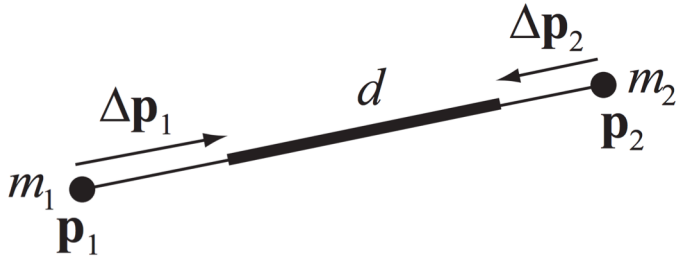
- Scaling each constraint by a scalar $k_i \in [0,1]$.
- Then we use: $\Delta p = \lambda_i k_i M^{-1} \nabla C_i$
- Makes the constraint “less” or “more” important.
  - The solution space is biased towards a solution that is closest to the components of the previous solution that satisfy the important constraints.

Stretch Constraint

- Between two points: $C(x_1, x_2) = |x_1 - x_2| - d_{12}$.
- Gradient components: $\nabla_1 C = \hat{n}, \nabla_2 C = -\hat{n}, \hat{n} = \frac{x_1 - x_2}{|x_1 - x_2|}$
- Resulting movement ($w = m^{-1}$) :

$$\Delta x_1 = \frac{-w_1}{w_1 + w_2} C(x_1, x_2)\hat{n}$$

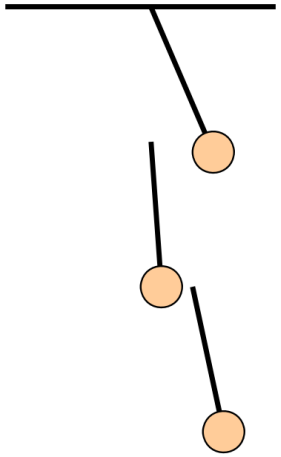
$$\Delta x_2 = \frac{w_2}{w_1 + w_2} C(x_1, x_2)\hat{n}$$

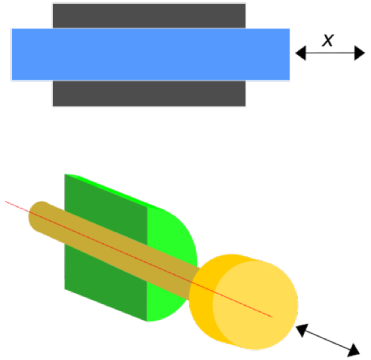


Connectors (Attachment Constraint)

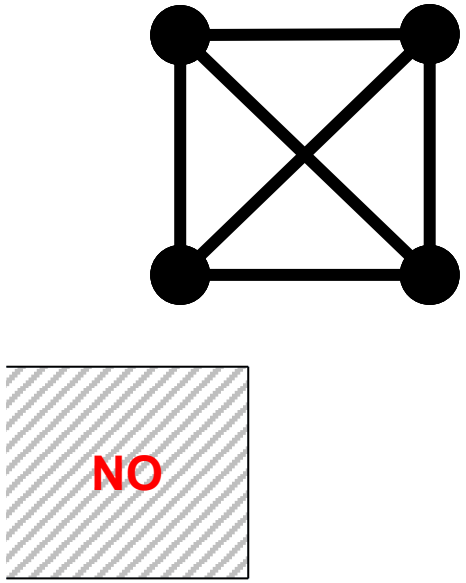
- If two (rigid) objects need to connect at two points x_1, x_2.
- Add connector constraint $C_i(x_1, x_2)$=$x_1 - x_2$.
  - Usually with the highest stiffness $k_i = 1$.
- Extensions: 1D translational constraint along $\hat{n}$:

$$C_i(x_1, x_2) = (x_1 - x_2)\hat{n} = 0$$

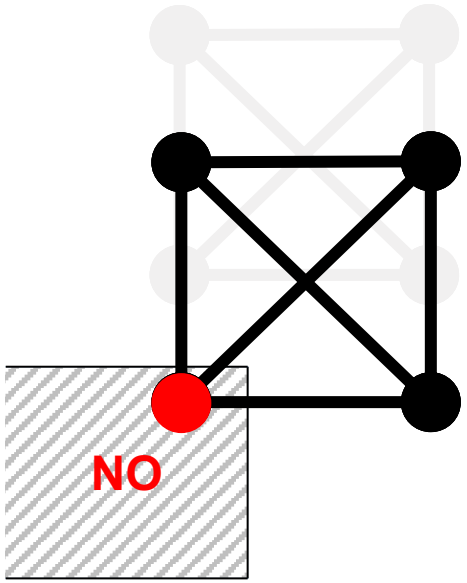




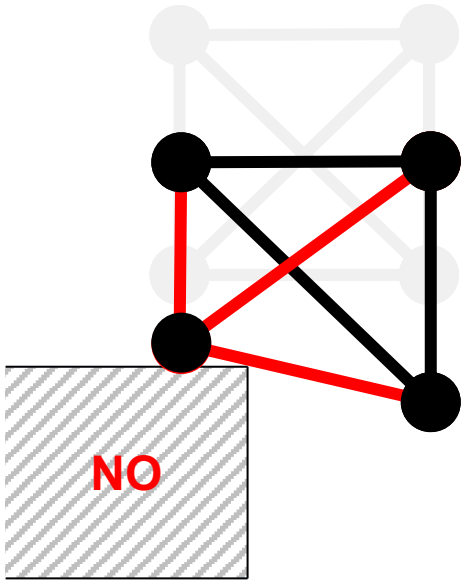
Example (rigid collision)



STEP 0

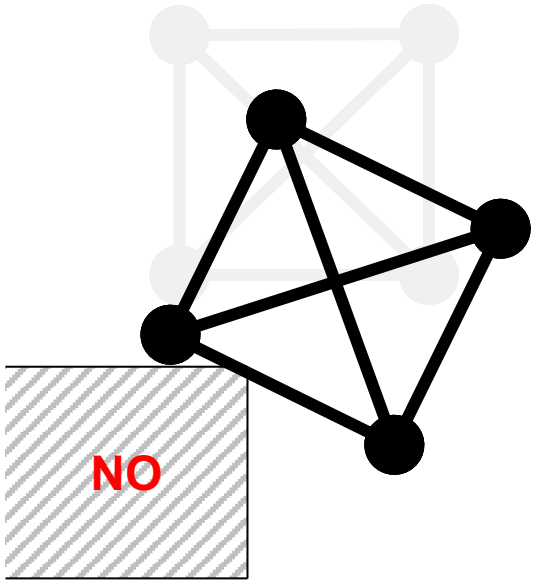


STEP 1
before constraints

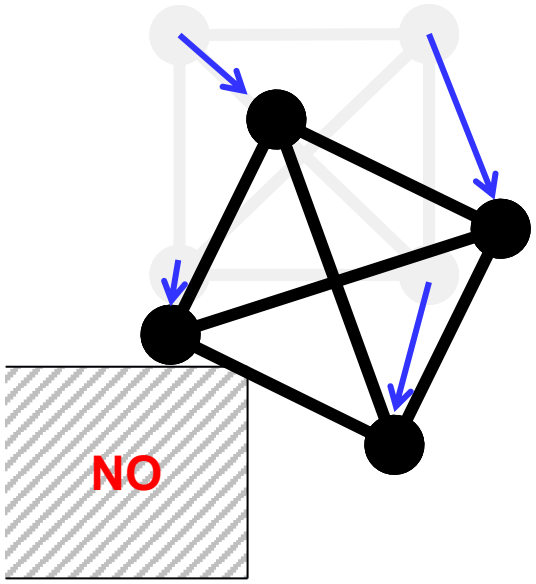


STEP 1
after 1st constraint

Example



STEP 1
after all constraints
multiple times



STEP 1
(implicit) velocities

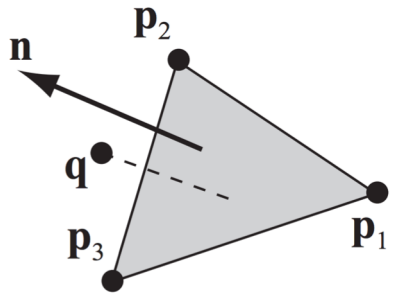
Rotation is induced by rigidity!

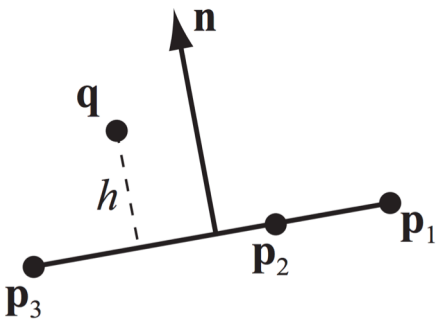
Collision Constraint

- In previous iteration: x_i is out of the object.
- After position integration: x_{i+1} is penetrating.
- Penetration normal: $\hat{n}$ (pointing outwards)
- Must make sure: the projected point x_{i+1} is non-penetrating.
- Constraint: $C(x_{i+1}) = (x_{i+1} - x_i)\hat{n} \geq 0$.
  - with the highest stiffness $k_i = 1$
- Disadvantage: not re-colliding objects.
  - Artefacts usually negligible.

Specific Collision Constraint: Point through Triangle

- A point orientation vs. triangle normal:
- $C_{ptt}(q, x_1, x_2, x_3) = (q - x_1) \cdot \frac{(x_2 - x_1) \times (x_3 - x_1)}{|(x_2 - x_1) \times (x_3 - x_1)|} - h$
- h - thickness of triangle
  - Used for cloth simulation.





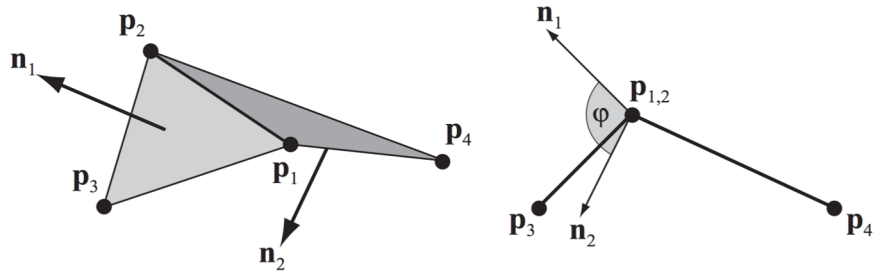
Bending Constraint

- Trying to preserve dihedral angle ϑ_{fg} between triangle f, g as much as possible.

$$C_{bend}(x_1, x_2, x_3, x_4) = acos\left(\hat{n}_f \cdot \hat{n}_g\right) - \vartheta_{fg}$$

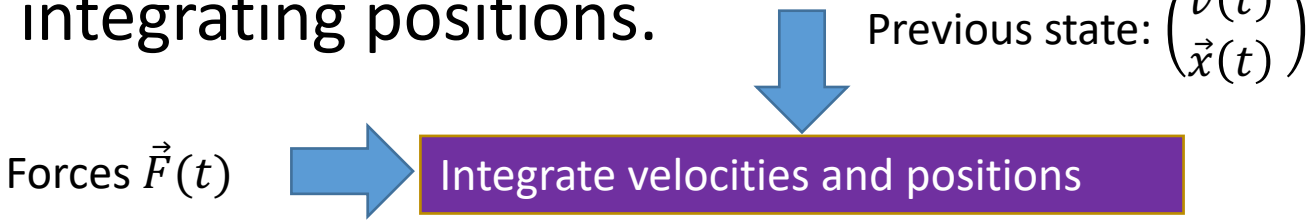
- $\hat{n}_f$: normal to triangle f (resp. g)

$$\hat{n}_f = \frac{(x_2 - x_1) \times (x_3 - x_1)}{|(x_2 - x_1) \times (x_3 - x_1)|}$$



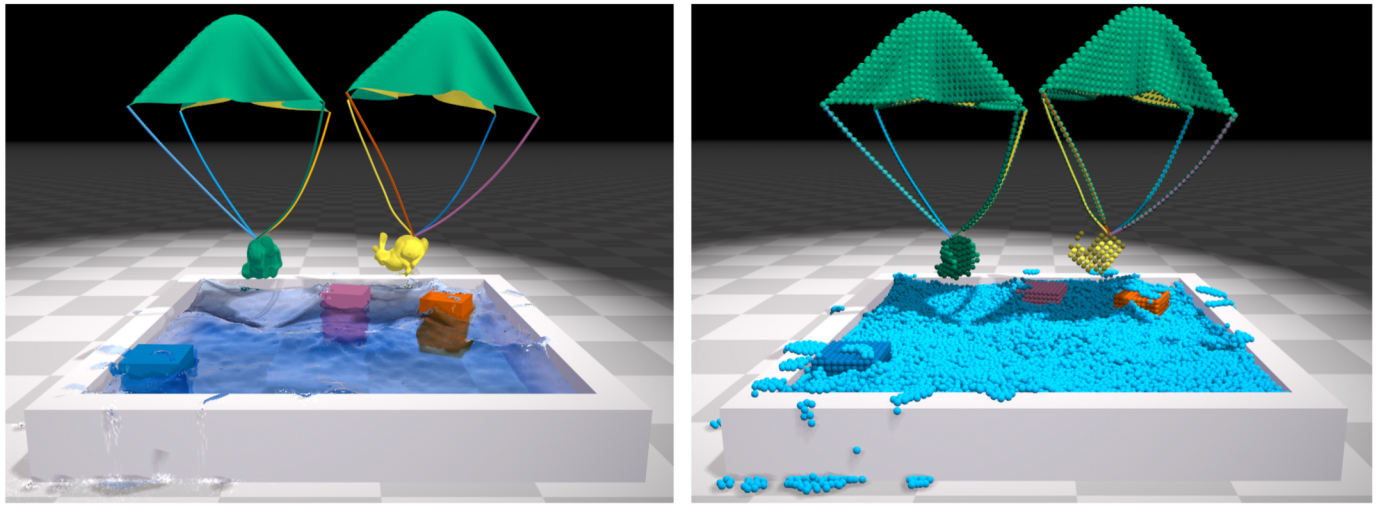
Velocity Damping

- For integrating velocities, before integrating positions.
- $v_i \leftarrow v_i + k_{damp}\Delta v_i$
- Many schemes exist.



Position-Based Particles

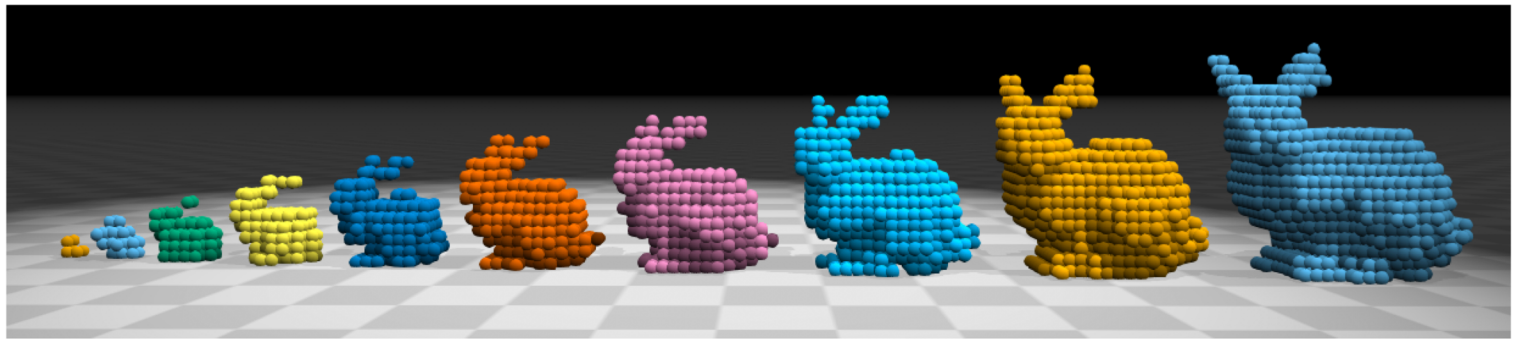
- Idea: represent entire mesh as set of particles and their inter-relations
- Advantage: much easier and more uniform constraints
- Disadvantage: hard to truly approximate surfaces



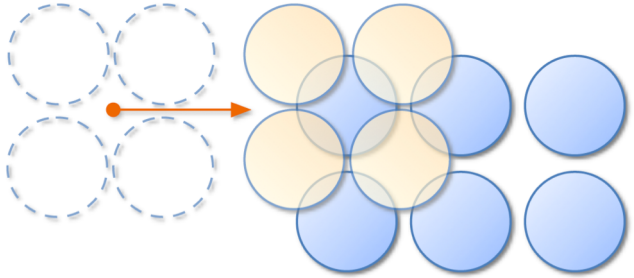
Unified Particle Physics for Real-Time Applications [Macklin *et al.* 2014]

Solid Representation

- Algorithm: the usual position-based approach.
  - Position $\vec{x}$
  - velocity $\vec{v}$
  - inverse mass w
  - radius r
  - affiliation (which mesh does particle belong to).
- Approximation: usually vertex-based.



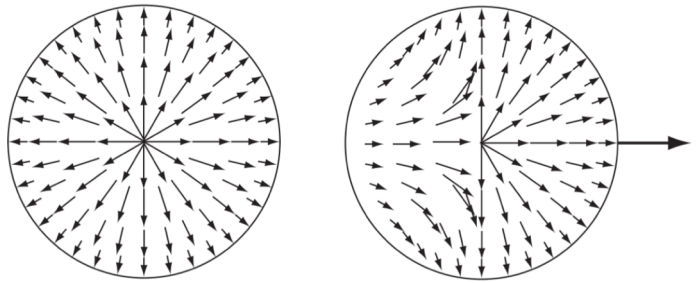
Collision Detection



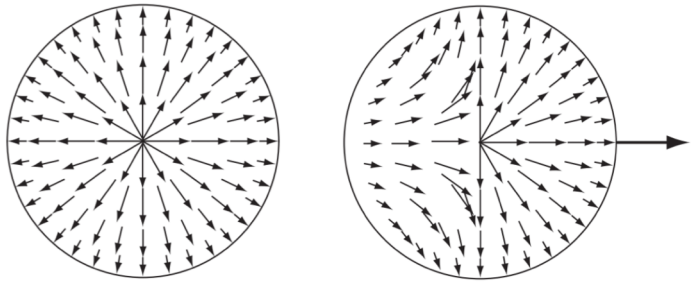
- Just sphere to sphere.
  - Or sphere to plane (floor)
- Naïve collision constraint:

$$c(\vec{x}_i, \vec{x}_j) = |\vec{x}_i - \vec{x}_j| - (r_i + r_j) > 0$$

- Problem: interlocking.
- Solution: Directional rejection.



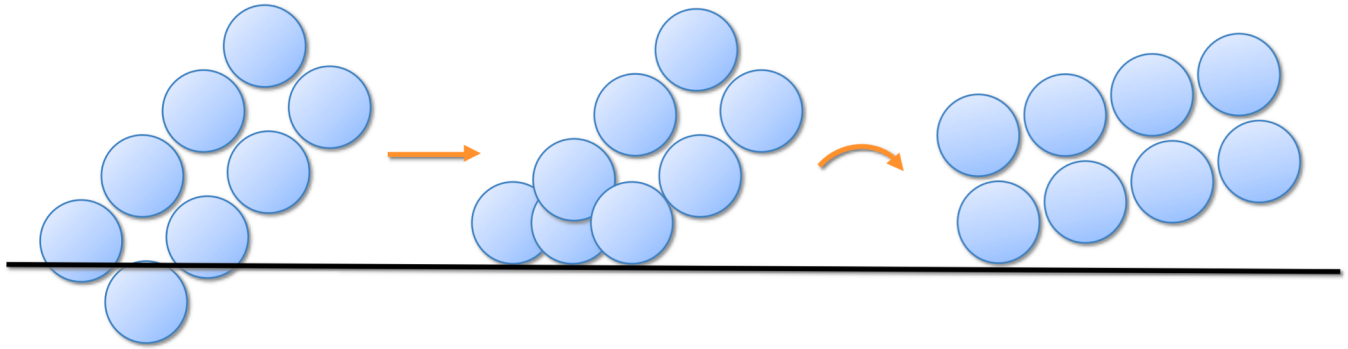
Directional Rejection



- Penetration normal $\vec{n}_{ij}$
  - From the geometry of the mesh
- Center-to-center vector $\vec{x}_{ij}$
- Penetration depth $\mathrm{d} = \left(r_i + r_j\right) - \left|\vec{x}_{ij}\right|$.
- Using alternative normal:
- $n_{ij}^{*} = \begin{cases} \vec{x}_{ij} - 2\left\langle \vec{x}_{ij}, \vec{n}_{ij} \right\rangle \vec{n}_{ij} & \left\langle \vec{x}_{ij}, \vec{n}_{ij} \right\rangle < 0 \\ \vec{n}_{ij} & else \end{cases}$

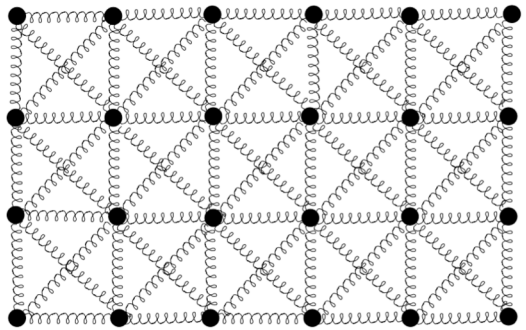
Rigid-Body Handling

- Can be done the usual way with rigidity constraints.
- Alternative: estimate COM $\vec{c}$ and average orientation R for all particles of same affiliation
- Update all particles of affiliation rigidly.
- $\vec{c}$: the usual weighted average.
- How to find R?



Mass-Spring System

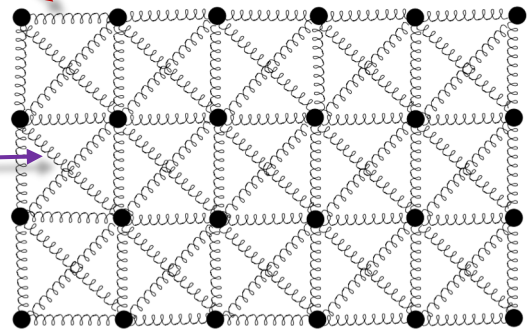
- Object consists of point masses $m_i, i = 1 \cdots n$
- Connected by a network of massless springs.
- System state: positions x_i and velocities v_i.
- Point force sum f_i:
  - External forces (*e.g.* gravity, friction).
  - Spring connections with neighbors.



http://steffe.se/wp-content/uploads/2011/08/osten.png

Mass-Spring System

- Mass points are initially regularly spaced in a 3D lattice.
- The edges are connected by structural springs.
  - resist longitudinal deformations
- Opposite corner mass points are connected by shear springs.
  - resist shear deformations.
- The rest lengths define the rest shape of the object.



Mass-Spring System

- The force acting on mass point i generated by the spring connecting i and j is

$$f_i = K_i\left(\left|\vec{x}_{ij}\right| - l_{ij}\right)\frac{\vec{x}_{ij}}{\left|\vec{x}_{ij}\right|}$$

  - $\vec{x}_{ij} = x_j - x_i$.
  - K_i: stiffness of the spring.
  - l_{ij}: rest length.
- To simulate dissipation of energy, a damping force is added:

$$f_i = K_i\left\langle \vec{v}_{ij}, \vec{x}_{ij} \right\rangle \frac{\vec{x}_{ij}}{\left|\vec{x}_{ij}\right|^2}$$

Mass-Spring System

"Artistic Simulation of Curly Hair" Disney\Pixar ©

- Pro: intuitive and simple to implement.
- Con: Not accurate and does not necessarily converges to correct solution.
  - depends on the mesh resolution and topology
  - …and the choice of spring constants.
- Can be good enough for games, especially cloth animation
  - For possible strong stretching resistance and weak bending resistance.

Fast Simulation of Mass-Spring Systems [Liu *et al.* 2013]