RIGID-BODY PHYSICS

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1. Introduction

These course notes will explain in details the physics behind rigid-body motion. Previously, we treated bodies as dimensionless "particles". At time t, they existed in pointwise position $\mathbf{x}(t)$ with velocity $\mathbf{v}(t)$. Net force $\mathbf{F}(t)$ induced acceleration $\mathbf{a}(t)$, where $\mathbf{F}(t) = m\mathbf{a}(t)$, with m as the mass of the particle. For readability, note that we mark vector quantities with bold notation: $\mathbf{v} \in \mathbb{R}^3 = (v_1, v_2, v_3)^T$, where v_1 (for instance) in regular notation is a scalar. Also note that we use vectors as column vectors (and so transpose makes them row vectors) unless otherwise stated.

A particle with velocity $\mathbf{v}$ carries momentum $\mathbf{p} = m\mathbf{v}$. Momentum, as a vector quantity, is conserved in any given axis system of reference on which external forces are not operated. It can change by impulses $\Delta\mathbf{p}$ made instantaneously, or the action of a force in a given time interval: $\Delta p = \int_{t_0}^{t_1} \mathbf{F}(t)dt$. In fact, perfect instantaneous impulses do not exist in practice; they are abstractions of large forces acting in very short time intervals.

2. Rotation and Orientation

In the following we show how the linear terms generalize to rigid bodies. A rigid body is a collection of points, often a continuum, where the distance $|\mathbf{r}_{ij}| = |\mathbf{x}_j - \mathbf{x}_i|$ between every two points remains constant through time. Nevertheless, the actual vector $\mathbf{r}_{ij} = \mathbf{x}_j - \mathbf{x}_i$ changes. Since its length remains constant, $\mathbf{r}_{ij}$ change only by *rotation*.

Suppose that we have a reference axis system R_0, where the rows of R_0 are the axes. We measure $\mathbf{r}_{ij}(0)$ in this coordinates system for some arbitrary time t_0. The two points rotate through time, to obtain $\mathbf{r}_{ij}(t) = R(t)\mathbf{r}_{ij}(0)$. It is clear then that $R(t)$ uniquely defines $\mathbf{r}_{ij}$, given the original vector $\mathbf{r}_{ij}(0)$. That is, $R(t)$ is the relative *orientation* of $\mathbf{r}_{ij}$ with relation to the system of reference. Since the body is rigid, the specific identity of $\mathbf{x}_i$ and $\mathbf{x}_j$ is irrelevant; $R(t)$ is the same for the entire body.

Orientation is then the *angular* analogue of position in the *linear* (sometimes: *translational*) physics of single particles. Like position, orientation is measured relatively to the original system of reference R_0. For instance, in the way that the *object coordinate system* is often defined in graphics, the object does not seem to rotate at all.

2.1. Rotation matrices are problematic.

Rotation in $\mathbb{R}^3$ only has 3 degrees of freedom. That is in contract to the fact that a rotation matrix has 9 coefficients. The redundancy is apparent by its structure: consider the columss as $R(t) = \begin{pmatrix}\mathbf{R}_1 & \mathbf{R}_2 & \mathbf{R}_3\end{pmatrix}$. $\mathbf{R}_1$ has 2 degrees of freedom as a unit-length vector.

$\mathbf{R}_2$, which is orthogonal to $\mathbf{R}_1$, has then only 1 degree of freedom left. Finally, $\mathbf{R}_3 = \mathbf{R}_1 \times \mathbf{R}_2$. It then seems inefficient and problematic to maintain and update rotation matrices.

One alternative representation is that of *axis-angle*. A single vector $\boldsymbol{\theta} \in \mathbb{R}^3$ represents rotation around a unit-length axis $\hat{e} = \frac{\boldsymbol{\theta}}{|\boldsymbol{\theta}|}$ and the angle $|\boldsymbol{\theta}|$, so that $\boldsymbol{\theta} = |\boldsymbol{\theta}|\,\hat{e}$. Note that this factorization is only unique up to mutual sign, which geometrically means that it represents both a rotation around a given axis, and rotation in the opposite direction around the opposite axis (producing the same result).

2.2. **Quaternions.** While the axis-angle representation is the most efficient in terms of size, it is not obvious how to apply it to create rotation. We often prefer to work with another representation of orientation, that is on the midway between rotation matrices and axis-angle representations—*quaternions*. A quaternion is defined as $\mathbf{q} = (r, \mathbf{v}) \in \mathbb{R}^4$, where $r \in \mathbb{R}$ is the real part, and $\mathbf{v} \in \mathbb{R}^3$ is the imaginary part of the quaternion. A unit quaternion has $|\mathbf{q}| = \sqrt{r^2 + |\mathbf{v}|^2} = 1$. Quaternions are equipped with a multiplication operator:

$$\mathbf{q}_1 \cdot \mathbf{q}_2 = (r_1 r_2 - \langle \mathbf{v}_1, \mathbf{v}_2 \rangle, r_1 \mathbf{v}_2 + r_2 \mathbf{v}_1 + \mathbf{v}_1 \times \mathbf{v}_2) \tag{1}$$

Note that in general $\mathbf{q}_1 \cdot \mathbf{q}_2 \neq \mathbf{q}_2 \cdot \mathbf{q}_1$, as $\mathbf{v}_1 \times \mathbf{v}_2 = -\mathbf{v}_2 \times \mathbf{v}_1$. They only commute when $\mathbf{v}_1 \parallel \mathbf{v}_2$. Quaternions and their usage comprise a rich theory; we will focus on their classic use as rotation and orientation without providing the proofs for why that works. For rotation axis $\hat{e}$ and angle θ, the quaternion that encodes this rotation is defined as a vector $\mathbf{q} \in \mathbb{R}^4 = \left(cos\left(\frac{\theta}{2}\right), \hat{e} \cdot sin\left(\frac{\theta}{2}\right)\right)$. If we encode positions $\mathbf{x} \in \mathbb{R}^3$ as *imaginary* quaternions $(0, \mathbf{x})$, then a rotated position $\mathbf{x}'$ is parameterized as:

$$\left(0, \mathbf{x}'\right) = \mathbf{q}\,(0, \mathbf{x})\,\mathbf{q}^{-1}, \tag{2}$$

where $\mathbf{q}^{-1} = \frac{(r, -\mathbf{v})}{|\mathbf{q}|^2}$. The quaternion $\overline{\mathbf{q}} = (r, -\mathbf{v})$ is called the *conjugate* of $\mathbf{q}$, and for unit quaternions $\overline{\mathbf{q}} = \mathbf{q}^{-1}$.

The total position of every point on a rigid body is then composed of two elements: a point $\mathbf{c}$ chosen as the center of its rotation (as we will see later, the specific choice is of no consequence), and the orientation $\mathbf{q}$ of the body. With these, the total position of any given point $\mathbf{x}$ with initial orientation $\mathbf{r}(0) = \mathbf{x}(0) - \mathbf{c}(0)$ is:

$$(0, \mathbf{x}(t)) = (0, \mathbf{c}(t)) + \mathbf{q}(t)\,(0, \mathbf{r}(0))\,\mathbf{q}^{-1}(t). \tag{3}$$

Note that the position of a rigid body is parameterized by the set $\{\mathbf{c}(t), \mathbf{q}(t)\}$. As $\mathbf{q}$ has 3 degrees of freedom (being a unit quaternion representing rotation), the number of degrees of freedom for rigid-body total position is 6.

3. Angular Velocity

Having established orientation as the angular analogue of position, we next wish to study the kinematics of rotation. That is, how orientation changes through time, and what is the analogue of linear velocity. We expect any notion of angular velocity to be analogous to that of linear velocity in the sense that it results from the derivative of the orientation, and that for an object rotating in a constant pace (in $\left[\frac{rad}{sec}\right]$) around a fixed axis, we should get constant angular velocity.

We consider again a vector $\mathbf{r}(t)$ between any point $\mathbf{x}(t)$ on a rigid body, and the center of its rotation $\mathbf{c}(t)$. We consider the most general rigid movement where the rotation axis and angle are time-dependent. Due to rigidity, we know that $\forall t,\ |\mathbf{r}(t)| = |\mathbf{r}(0)|$. The derivative of this length is then zero, and we get:

(4) $$\frac{d\,|\mathbf{r}(t)|^2}{dt} = 0\frac{d\left(\mathbf{r}^T\cdot\mathbf{r}\right)}{dt} = \frac{d\mathbf{r}^T}{dt}\cdot\mathbf{r} + \mathbf{r}^T\cdot\frac{d\mathbf{r}}{dt} = 0 \Rightarrow \frac{d\mathbf{r}(t)}{dt} \perp \mathbf{r}.$$

The meaning of the result is that the linear velocity $\mathbf{v}^{\perp}(t) = \frac{d\mathbf{r}(t)}{dt}$ of the point, relative to the center of rotation, is always orthogonal to $\mathbf{r}$ itself. As such, we call $\mathbf{v}^{\perp}(t)$ the *tangential velocity* of the point. Tangential velocity is not exactly what we wished for: it is not the same for every point of the body (points that are more distant from the center move faster). We define $\boldsymbol{\omega}(t) = \frac{\mathbf{r}\times\mathbf{v}^{\perp}}{|\mathbf{r}|^2}$ as the *angular velocity* of the point. Angular velocity has the following properties:

(1) it is measured in $\left[\frac{Rad}{sec}\right]$.
(2) $\boldsymbol{\omega}(t)$ is the same for each point of the body, independent of $\mathbf{r}(t)$.
(3) $\boldsymbol{\omega}(t)$ is in the direction of the instantaneous axis of rotation for time t, and evidently orthogonal to both $\mathbf{r}(t)$ and $\mathbf{v}^{\perp}(t)$ from the cross product.
(4) As a consequence, if $\forall t,\ \boldsymbol{\omega}(t) = \boldsymbol{\omega}(0) = |\boldsymbol{\omega}|\,\hat{e}$ (constant through time), then the body rotates with constant rotational speed $|\boldsymbol{\omega}|$ around axis $\hat{e}$. That is, it covers $|\boldsymbol{\omega}|\cdot\Delta t$ radians for time interval Δt.

The only property which is not entirely straightforward is (2), unless we assume the special case of (4). To demonstrate this property, we next show that $\boldsymbol{\omega}(t)$ is in fact purely a function of the orientation $\mathbf{q}(t)$, independent of $\mathbf{r}(t)$. To show this, we need two preliminary identities:

(1) We can form $\boldsymbol{\omega}\times\mathbf{r}$ as a quaternion multiplication: $(0,\boldsymbol{\omega})\cdot(0,\mathbf{r}) = (0,\boldsymbol{\omega}\times\mathbf{r})$.
(2) For a unit quaternion $\mathbf{q}(t)$, there exists an imaginary quaternion $(0,\boldsymbol{\gamma}(t))$ so that $\frac{d\mathbf{q}(t)}{dt} = (0,\boldsymbol{\gamma}(t))\,\mathbf{q}(t)$. The proof is simple (with the same principle as the derivative of $r(t)$), and we omit it for brevity.
(3) As $\frac{d|\mathbf{q}(t)|}{dt} = 0$, we get that $\frac{d\mathbf{q}(t)}{dt}\mathbf{q}(t)^{-1} = -\mathbf{q}(t)\frac{d\mathbf{q}(t)}{dt}^{-1}$.

In the following, we will omit "(t)" for clarity of reading. We get:

$$\begin{aligned}\frac{d\mathbf{r}}{dt} &= (0,\omega)\cdot(0,\mathbf{r}) = \frac{d\mathbf{q}}{dt}(0,\mathbf{r}_0)\,\mathbf{q}^{-1} + \mathbf{q}\,(0,\mathbf{r}_0)\,\frac{d\mathbf{q}^{-1}}{dt} \Rightarrow \\ &= \frac{d\mathbf{q}}{dt}\mathbf{q}^{-1}(0,\mathbf{r}) - (0,\mathbf{r})\,\frac{d\mathbf{q}}{dt}\mathbf{q}^{-1} \Rightarrow \\ (0,\boldsymbol{\omega}\times\mathbf{r}) &= (0,\boldsymbol{\gamma})\,(0,\mathbf{r}) - (0,\mathbf{r})\,(0,\boldsymbol{\gamma}) = (0, 2\boldsymbol{\gamma}\times\mathbf{r})\,.\end{aligned} \tag{5}$$

Since $\mathbf{r}$ is arbitrary, and we can choose any point on the rigid body, we get $\boldsymbol{\omega} = 2\boldsymbol{\gamma}$, and consequently:

$$\frac{d\mathbf{q}(t)}{dt} = \frac{1}{2}\,(0,\boldsymbol{\omega})\,\mathbf{q}. \tag{6}$$

And that means that $\boldsymbol{\omega}$ is independent of $\mathbf{r}(t)$, and therefore the same for the entire rigid body.

3.1. **Total velocity.** Given some system of reference (without loss of generality, the canonical system), we can model the movement of any rigid body inside it by decomposing it into two independent components, exactly as for total position:

(1) The *translational* velocity of the center: $\mathbf{v}^{\|}(t)$.
(2) The tangential velocity $\mathbf{v}^{\perp}(t)$ of every other point in the object with relation to the center $c(t)$.

The total velocity of every point $\mathbf{x}(t) = \mathbf{c}(t) + \mathbf{r}(t)$ on the object is then:

$$\overline{\mathbf{v}} = \overline{\mathbf{v}}^{\|}(t) + \mathbf{v}^{\perp}(t) = \overline{\mathbf{v}}^{\|}(t) + \boldsymbol{\omega}(t)\times\mathbf{r}(t). \tag{7}$$

Again, with accordance to total position, the set of two variables (actually functions in time) $\left\{\mathbf{v}^{\|}(t), \boldsymbol{\omega}(t)\right\}$, for a given center $\mathbf{c}(t)$ parameterize the entire movement of the object. How well-defined are these quantities? We already established that ω is only dependent on the orientation of the $\mathbf{r}$ vectors to the center, but how does the choice of the center affects the variables? It is first important to understand what "point on the body" means: it doesn't necessarily mean "in the material of the object". It means a point that, when moving, keeps the same distance to all other point of the body; or more formally, that the $\mathbf{r}$ vectors are always in the same length. We often use the center of mass as a convention, but it is not a must.

We show that the choice of the center, while affecting $\mathbf{v}^{\|}$ and $\mathbf{v}^{\perp}$, does not have any effect on $\boldsymbol{\omega}$. That means that orientation is also independent of translation, and thus well defined. Consider two arbitrary candidate centers $\mathbf{c}_1$ and $\mathbf{c}_2$. Their total velocities are obviously equal; let us see that the angular velocity of the object remains the same regardless of which of these centers we choose. We have that:

$$\begin{aligned}\overline{\mathbf{v}}_1 = \mathbf{v}_1^{\|}, \quad \overline{\mathbf{v}}_2 = \overline{\mathbf{v}}_1^{\|} + \boldsymbol{\omega}_2\times\mathbf{r}_{12},\\ \overline{\mathbf{v}}_1 = \mathbf{v}_2^{\|} + \boldsymbol{\omega}_1\times\mathbf{r}_{21}, \quad \overline{\mathbf{v}}_2 = \mathbf{v}_1^{\|},\end{aligned} \tag{8}$$

Where $\boldsymbol{\omega}_1$ is the angular velocity with relation to $\mathbf{c}_1$, and $\boldsymbol{\omega}_2$ for $\mathbf{c}_2$. Subtracting the equations, we get that:

$$\boldsymbol{\omega}_1 \times \mathbf{r}_{12} = \boldsymbol{\omega}_2 \times \mathbf{r}_{12}. \tag{9}$$

Since our choice of $\mathbf{c}_1$ and $\mathbf{c}_2$ was arbitrary, this eventually means that $\boldsymbol{\omega}_1 = \boldsymbol{\omega}_2$. Remember we used a similar technique to show that angular velocity is a function of orientation alone. Both actually mean the same: angular velocity is a (single property) that pertains to the rotation of each point in a rigid body against any other point.

The definition of angular velocity naturally lends to a definition of *angular acceleration*, as $\alpha = \frac{d\omega}{dt}$, measured in $\left[\frac{rad}{sec^2}\right]$.

4. Rigid-Body Dynamics

The force that induces angular motion is called the *centripetal* force. Denoting it as $\mathbf{F}_n$, it is always in the direction of $-\mathbf{r}$. As an example, consider earth's gravity, making satellites move in orbit. That also teaches us that such a force, with a constant magnitude, creates a motion with constant angular velocity in a circle. Consequently, we can also define the *tangential acceleration* $\mathbf{a} = \boldsymbol{\alpha} \times \mathbf{r}$. The centripetal force causes centripetal acceleration $\mathbf{a}_n = -\left|\boldsymbol{\omega}\right|^2 \mathbf{r}$.

5. Angular Momentum

Recall that a dimensionless body with mass m and velocity $\mathbf{v}$ has momentum $\mathbf{p} = m\mathbf{v}$, and that such *linear* momentum is preserved when measured in a a given system of reference, without external forces or impulses. The angular analogue is *angular momentum*. For a body of volume V, a density function ρ, and a choice of center c, producing arm vectors $\mathbf{r}$ for each point, the angular momentum is:

$$L(c) = \int_V (\mathbf{r} \times \mathbf{p})\, \rho dV, \tag{10}$$

where $\mathbf{p}$ is again the linear momentum of each point on the body. The body does not have to be a single, or even a rigid body; this can measure the angular momentum of an entire system of moving bodies, **as long as it's with relation to a given fixed center in the system**. If we measure movement in such a system, then angular momentum is *conserved*, just like linear momentum.

Side note. This brings up a subtle question we did not discuss in class—on purpose. We said that there is a force driving angular motion, which sounds like adding momentum to a system. But from the equation above it is clear that a satellite in a constant orbit has constant angular momentum. How come the centripetal force doesn't add any momentum? the "narrowminded" algebraic answer is that work that a force does to bring an object about distance $\mathbf{d}$ is measured as $\langle \mathbf{F}, \mathbf{d} \rangle$, but that the force is always orthogonal to the angular movement, so the work is algebraically zero! as such, the force is deemed *virtual*. This is actually a deeper subject that touches into the heart of Lagrangian mechanics; we will get a taste of this when we discuss constraints later on in the course.

If we plug in angular velocity instead of the linear momentum $\mathbf{p}$, we get:

$$L=\int_V(\mathbf{r}\times\mathbf{p})\,\rho dV=\int_V(\mathbf{r}\times\mathbf{v})\,dm=\int_V(\mathbf{r}\times(\omega\times\mathbf{r}))\,dm. \tag{11}$$

This brings another important point: the angular momentum $\mathbf{L}$ is not in general collinear to any ω; not even for single object! it is not obvious what direction it is going to take by looking at a complex system.

6. The Inertia Tensor

Let us consider a single rotating rigid body again, where $\boldsymbol{\omega}$ is constant, and with center c. We show that we can compactify the angular momentum into a more intuitive form. For this, we use the cross-product identity:

$$\mathbf{r}\times(\omega\times\mathbf{r})=|\mathbf{r}|^2\cdot\boldsymbol{\omega}-\langle\boldsymbol{\omega},\mathbf{r}\rangle\mathbf{r}. \tag{12}$$

We can further write this in matrix-vector multiplication form as ($I_{3\times3}$ is the identity matrix):

$$\mathbf{r}\times(\omega\times\mathbf{r})=\left(|\mathbf{r}|^2 I_{3\times3}-\mathbf{r}\cdot\mathbf{r}^T\right)\boldsymbol{\omega}. \tag{13}$$

Integrating over the entire body, while considering that $\boldsymbol{\omega}$ is constant, and therefore can be taken out of the integrand, we get:

$$L=\int_V(\mathbf{r}\times(\omega\times\mathbf{r}))\,\rho dV=\left[\int_V\left(|\mathbf{r}|^2 I_{3\times3}-\mathbf{r}\cdot\mathbf{r}^T\right)dm\right]\boldsymbol{\omega}=\mathbf{I}\boldsymbol{\omega}. \tag{14}$$

Note that the integral is integrating into a 3×3 matrix element-wise, to get a *single* 3×3 matrix $\mathbf{I}$. The matrix $\mathbf{I}$ is called the *inertia tensor*. The inertia tensor only depends on all $\mathbf{r}$ and pointwise mass densities in the system. It is then purely a property of the geometry and the material of the rigid body with relation to the chosen center $\mathbf{c}$. To make the inertia tensor more explicit, suppose that every $\mathbf{r}$ is broken into the individual coordinates (again, in the canonical system chosen) $\begin{pmatrix}x\\y\\z\end{pmatrix}$. Then we have:

$\mathbf{I}_{11}=\int_V\left(y^2+z^2\right)dm \quad \mathbf{I}_{12}=\mathbf{I}_{21}=\int_V(xy)\,dm$

$\mathbf{I}_{22}=\int_V\left(x^2+z^2\right)dm \quad \mathbf{I}_{13}=\mathbf{I}_{31}=\int_V(xz)\,dm$

$\mathbf{I}_{33}=\int_V\left(x^2+y^2\right)dm \quad \mathbf{I}_{23}=\mathbf{I}_{32}=\int_V(yz)\,dm$

6.1. **The moment of inertia.** We next study the case where a specific axis $\hat{e}$ of rotation, passing through $\mathbf{c}$, is given. It is clear that each point $\mathbf{x}(t)$ on the rigid body rotates around this axis in a circular trajectory. The length of the radius vector of this trajectory is given by $|\hat{e} \times \mathbf{r}|$. We would like to measure the ratio of the *scalar* angular momentum $L = \langle \mathbf{L}, \hat{e} \rangle$ relative to this axis, to the angular speed ω around it; that would give us a measure of how much the body *resists* rotation around this axis. We call this ratio the *moment of inertia* $\mathbf{I}_{\hat{e}}$, and then have $L = \mathbf{I}_{\hat{e}} \omega$.

Recall that $\mathbf{L} = \mathbf{I}\omega$, and then we simply get: $L_{\hat{e}} = \langle \hat{e}, \mathbf{I}\hat{e} \, |\omega| \rangle$ where we arrive at

$$\mathbf{I}_{\hat{e}} = \int_V |\hat{e} \times \mathbf{r}|^2 \, dm = \hat{e}^T \mathbf{I} \hat{e}. \tag{15}$$

We see that the inertia tensor $\mathbf{I}$, when applied as a *metric* on the vector $\hat{e}$, provides us with the moment of inertia $\mathbf{I}_{\hat{e}}$. As such, the inertia tensor is the angular analogue to mass; recall that in the linear setting $\mathbf{p} = m\mathbf{v}$, and mass is in fact the ability to resist linear velocity (we use "resist" as in "you need to acquire more momentum by using more force, to get the same velocity"). Since rigid bodies have different sizes in each direction, the shape of the matrix $\mathbf{I}$ tells us of the resistance to rotation against each possible axis of rotation. It is then expected that the resulting moment of inertia $\mathbf{I}_{\hat{e}}$ is bigger where the object is more spread orthogonally to the axis; and indeed we get this, as the $\mathbf{r}$ vectors are bigger in that direction. For perfect spheres, for instance, we get a scalar matrix (scalar $\times$ an identity matrix), giving the same moment of inertia in all directions .

6.2. **Change of coordinates.** It is often comfortable to compute the inertia tensor in a given orientation of the object; for instance, when it has some symmetry that makes the integration reducible to simple formulas. The most classic case is that of an ellipsoid, where the inertia tensor is best computed in a coordinate system aligned with its *principal* axes. How do we then transform the inertia tensor to any given system of reference? Let us consider a change of coordinates by a rotation matrix R. Then, the vectors of $\boldsymbol{\omega}$ and $\mathbf{L}$ become $\boldsymbol{\omega}_R = R^T \boldsymbol{\omega}$ and $L_R = R^T \mathbf{L}$. We then get:

$$\begin{aligned} \mathbf{L}_R = \mathbf{I}_R \boldsymbol{\omega}_R \Rightarrow \\ R^T \mathbf{L} = \mathbf{I}_R \cdot R^T \boldsymbol{\omega} \Rightarrow \\ \mathbf{L} = R \mathbf{I}_R R^T \omega = \mathbf{I} \boldsymbol{\omega}, \end{aligned} \tag{16}$$

and consequently $\mathbf{I}_R = R^T \mathbf{I} R$.

6.3. **"Angular" does not imply "rotation".** We keep using examples where a single body is rotated around a center on it, because this is an introductory way to demonstrate angular quantities. However, angular momentum, velocity, etc. do not necessarily correlate with what we perceive as "rotation". In fact, for every given center and system of reference, and every movement of body (rigid or otherwise), every point has a radius $\mathbf{r}(t)$ and instantaneous angular velocity $\boldsymbol{\omega}(t) = \frac{\mathbf{r} \times \overline{\mathbf{v}}}{|\mathbf{r}|^2}$, where $\overline{\mathbf{v}}(t)$ is the total velocity of the point (and thus only the part that is orthogonal to $\mathbf{r}$, the tangential velocity, would contribute in this formula). When the chosen center $\mathbf{c}$ is not on the body, the magnitude $|\mathbf{r}|^2$ may change. But the entire formulation is still well-defined and angular momentum is preserved!

Let us consider as a motivating example a single particle moving in a linear trajectory $\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}_0$, where $\mathbf{r}_0$ and $\mathbf{v}_0$ are constant, and the center as the canonical origin $\mathbf{0}$. The angular velocity at each point in time is $\boldsymbol{\omega} = \frac{(\mathbf{r}_0 + t\mathbf{v}_0)\times\mathbf{v}_0}{|r(t)|^2} = \frac{\mathbf{r}_0\times\mathbf{v}_0}{|r(t)|^2}$. We can see it is not a pure rotation since the denominator is time dependent. The angular momentum is however $\mathbf{L}(t) = (\mathbf{r}_0 + t\mathbf{v}_0) \times \mathbf{v}_0 = \mathbf{r}_0 \times \mathbf{v}_0$ which is time-independent, and therefore conserved.

To make matters worse, for a complex system with time-varying $\mathbf{r}(t)$ we also get time varying inertia tensors. Moreover, it is not clear how to form the inertia tensor at all under any center that is not on the body. As we see in the next section, for any given point in time, we can transfer the inertia tensor from a comfortable axis system in which it was computed, to another around a different center.

6.4. **The parallel-axis theorem.** Given the moment of inertia computed around a center $\mathbf{c}_1$ (at a given point in time), and for given axis $\hat{e}$, we would like to obtain the moment of inertia around another center $\hat{c}_2$ with the same axis, for a body of mass m. Without loss of generality, we assume that the displacement $\mathbf{d}_{12} = \mathbf{c}_2 - \mathbf{c}_1$ is orthogonal to $\hat{e}$; that is, $\langle \mathbf{d}_{12}, \hat{e}\rangle$=0. we get that every $\mathbf{r}_2 = \mathbf{r}_1 + \mathbf{d}_{12}$. Since we have the orthogonality, that directly means that $\mathbf{r}_{\hat{e},1} + \mathbf{d}_{12} = \mathbf{r}_{\hat{e},2}$ ($\mathbf{r}_{\hat{e}}$ is the part of the vector $\mathbf{r}$ that is between the point and the closest point on the axis). From this, we get that:

$$\mathbf{I}_{\hat{e},2} = \mathbf{I}_{\hat{e},1} + m\left|\mathbf{d}_{12}\right|^2. \tag{17}$$

In light of this, our modus operandi, to when we want to compute the angular momentum or moment of inertia of a complex system around an arbitrary center, is to first compute the moment of inertia comfortably for any single body, transform them locally to the axes system of reference around the respective local centers, and then translate all the moments of inertia to a common center using the parallel-axis theorem.

7. Forces and Torques

For any given system, angular momentum $\mathbf{L}$ is conserved, unless there are external forces in action. Let us see how such external forces induce change of movement in the angular setting. Recall that the application of force $\mathbf{F}(t) = m\mathbf{a}(t)$ induces change of linear momentum in a time interval: $\int_{t_0}^{t}\mathbf{F}(t)dt = m\int_{t_0}^{t}\mathbf{a}(t)dt = m\Delta\mathbf{v} = \Delta\mathbf{p}$. A change of momentum can also appear as an impulse, where $\Delta\mathbf{p}$ is given instantaneously.

A force applied at a point on an object with vector $\mathbf{r}$ to the center $\mathbf{c}$ produces a *torque*, defined as $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$. Torque is the angular analogue of force for all purposes, having the following properties:

- The torque $\boldsymbol{\tau}$ and the angular acceleration $\boldsymbol{\alpha}$ are related by $\boldsymbol{\tau} = \mathbf{I}\boldsymbol{\alpha}$, analogously to Newton's second law $\mathbf{F} = m\mathbf{a}$ in the linear setting.
- The *net* Torque of an object (sum of all torques) is the derivative of angular momentum:
$$\frac{d\mathbf{L}}{dt} = \frac{d\mathbf{r}}{dt} \times \mathbf{p} + \mathbf{r} \times \frac{d\mathbf{p}}{dt} = \mathbf{v} \times m\mathbf{v} + \mathbf{r} \times \mathbf{F} = 0 + \boldsymbol{\tau}.$$
Recall that this is the case in the linear setting, where $\frac{d\mathbf{p}}{dt} = \mathbf{F}$.
- It is possible to apply torque impulses ΔL which are $\tau\Delta t$ for $\Delta t \to 0$.

Naturally, a force induces zero torque if it is made in parallel to the vector $\mathbf{r}$. The dependence of torque on the radius vectors is intuitive: we create more rotation by applying a force tangentially to an object, rather than directly to its center. Recall however that, like angular momentum, the concept of torque is algebraically abstract, and not necessarily creating any visual rotation; it depends how we define the center with relation to a moving body.

8. Rotational Kinetic Energy

The rotational kinetic energy is defined as $E_r = \frac{1}{2}\boldsymbol{\omega}^T \mathbf{I}\boldsymbol{\omega}$, echoing the linear kinetic energy $E_k = \frac{1}{2}m\,|\mathbf{v}|^2$. It is naturally dependent on the system of reference, rather than a "natural property" of the object; it is the same for all types of energies. In a closed system, it is conserved in the same manner:

$$E_k(t+\Delta t) + E_r(t+\Delta t) + E_p(t+\Delta t) + E_0 = E_k(t) + E_r(t) + E_p(t), \tag{18}$$

where E_p is any potential energy, and E_0 is energy lost to heat and sound.