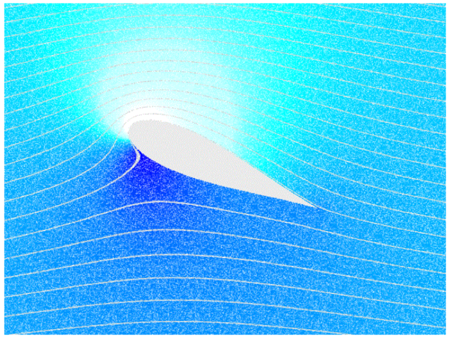
Lecture X: Fluid Physics

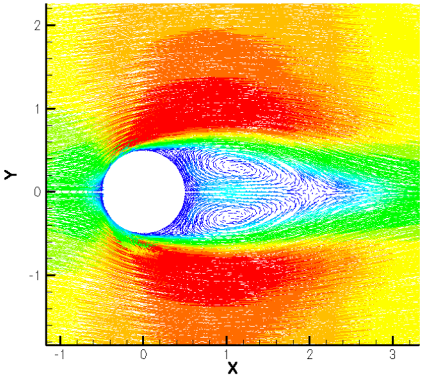
Fluid Motion

- Describes object with no fixed topology
  - Air flow
  - Viscuous fluids
  - Smoke, etc.
- Key descriptor: flow velocity

$$\vec{u} = u(x, t)$$



- Describing the velocity of a “fluid parcel” passing at position x in time t.
- Eulerian description
  - How come?

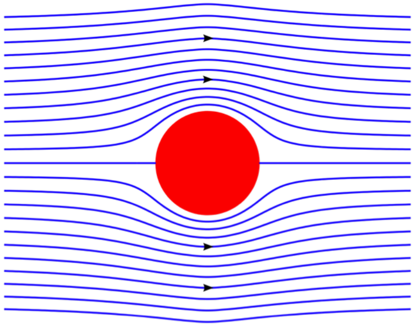


http://cfd.solvcon.net/old/research/cylinder2.gif

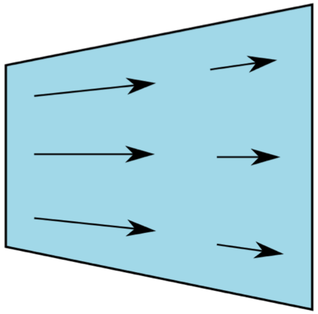
Flow Velocity

- Vector field describing motion
- Steady field: $\frac{\partial u}{\partial t} = 0$
- Incompressible: $\nabla \cdot u = 0$.
- Irrotational (no vortices): $\nabla \times u = 0$

Turbulent with a vortex



Incompressible, irrotational flow



Steady field

Incompressibility Assumption

- No fluid is incompressible
  - Otherwise, no soundwaves!
- Fluid is a non-solid "rigid body".
- Explicit condition:

$$\iint_{\partial\Omega} \vec{u} \cdot \hat{n} = 0$$

- For every (sub) domain Ω with boundary $\partial\Omega$.
- "volume in = volume out".

Incompressibility

- The divergence theorem:

$$\iint_{\partial\Omega} \vec{u} \cdot \hat{n} = \iiint_{\Omega} \nabla \cdot \vec{u} = 0$$

- Since Ω is arbitrary, we have:

$$\nabla \cdot \vec{u} = 0$$

everywhere as our condition for incompressibility!

(Incompressible) Navier-Stokes Equations

- Representing the conservation of mass and momentum for an incompressible fluid ($\nabla \cdot \vec{u} = 0$):

$$\frac{D\vec{u}}{Dt} = \rho(\overbrace{\underbrace{\vec{u}_t}_{\text{Unsteady acceleration}} + \underbrace{\vec{u}\cdot\nabla\vec{u}}_{\text{Convective acceleration}}}^{\text{Material Derivative}}) = \overbrace{\underbrace{\nu\nabla\cdot(\nabla\vec{u})}_{\text{Viscosity}} - \underbrace{\nabla p}_{\text{Pressure gradient}}}^{\text{Divergence of stress}} + \underbrace{\vec{f}}_{\text{External body forces}}$$

  - p: pressure field
  - ν: kinematic viscosity.
  - f: body force per density (usually just gravity ρg).

Material Derivative

- A particle with mass m has velocity $\vec{u}$.

- Newton's 2nd law:

$$m\frac{D\vec{u}}{Dt} = \vec{F}$$

- What are the forces acting on the particle?
- Simplest component: gravity $\vec{F}_g = \mathrm{m}\vec{g}$.

Pressure

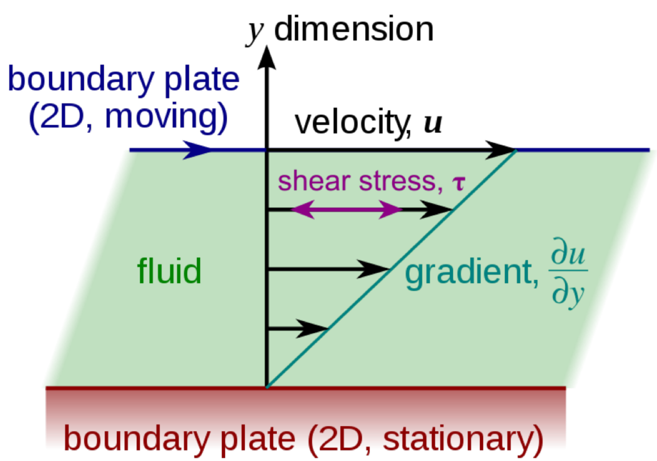
- The particle receives pressure from all directions in a fluid.
- What causes pressure? Incompressibility!
  - Pressure balances flow that creates compression.
- Algebraically:
  - $\nabla \cdot \vec{u} = 0$ is a constraint
  - Pressure is the Lagrange multiplier λ!
- Suppose particle volume V.
- Integrated (scalar!) pressures over particle : p.
- Net force enacted by pressure: $\vec{F}_{pre} = -V\nabla p$.

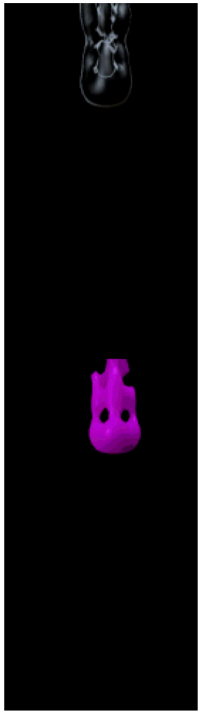
Viscosity

- Resistance to deformation by shear stress.
- Expressed by coefficient ν:

$$\frac{F}{A} = \nu \frac{\partial u}{\partial y}$$

- Higher ν: more pressure required for shearing!
  - Viscid fluids.





Viscosity

- Another interpretation: how much the flow of a particle different from the flow of its neighbors.
- Laplacian: $\nu\nabla \cdot (\nabla\vec{u}) = \nu\Delta\vec{u}$.
  - ν: dynamic viscosity.
- ν is defined per volume unit, so to get force:
- $\vec{F}_{vis} = V \cdot [\nu\nabla \cdot (\nabla\vec{u})]$
- What is the Laplacian of a vector?
  - For each coordinate independently.

Putting it All Together

- The Navier-Stokes equation in particle form:

$$m\frac{D\vec{u}}{Dt} = \vec{F}_{vis} + \vec{F}_{pre} + \vec{F}_{ext} = V\nu\nabla\cdot(\nabla\vec{u}) - V\nabla p + m\vec{g}$$

- In the limit (infinitesimal particles): $\frac{m}{V} \rightarrow \rho$.
- dividing by V we get:

$$\rho\frac{D\vec{u}}{Dt} = \nu\nabla\cdot(\nabla\vec{u}) - \nabla p + \rho\vec{g}$$

- What is the material derivative $\frac{D\vec{u}}{Dt}$?

Material Derivative

- The connecting thread between Lagrangian and Eulerian approaches.
- Reminder:
  - Lagrangian: tracking elements as they move (particles, mesh vertices).
  - Eulerian: tracking movement in a fixed point in space (temperature on grid, amount of material in a cell).
- Lagrangian description:
  - Suppose a set of particles, each with quantity q.
  - The particle is moving with velocity $\vec{u}(t) = \frac{d\vec{x}(t)}{dt}$
- Eulerian description:
  - $q(t, \vec{x})$: the quantity of the particle that passes through fixed point $\vec{x}$ at time t.
- How to tie both narratives?

Material Derivative

- Material (or total) derivative:

$$\frac{Dq}{Dt} = \frac{d}{dt} q(t, \vec{x}(t))$$

- How much q changes for the moving particle.

$$\frac{\partial}{\partial t} q(t, \vec{x})$$

- How much q changes at a fixed point $\vec{x}$.

Material Derivative

- The chain rule:

$$\frac{Dq}{Dt} = \frac{d}{dt} q\big(t, \vec{x}(t)\big) = \frac{\partial q}{\partial t} + \nabla q \cdot \frac{d\vec{x}}{dt} = \frac{\partial q}{\partial t} + \nabla q \cdot \vec{u}$$

- $\frac{\partial q}{\partial t}$: How much q changes at the fixed point $\vec{x}$.
- $\nabla q \cdot \vec{u}$: how much of the change is due to advection (transfer of material).
  - Also: convection, transport.
- When q doesn't change for a particle, we get the advection equation:

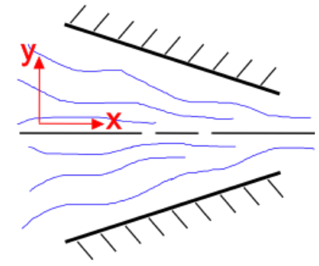
$$\frac{\partial q}{\partial t} + \nabla q \cdot \vec{u} = 0$$

Material Derivative

- The change in the velocity of the fluid parcel passing at position x in time t.

$$\frac{D\vec{u}}{dt} = \vec{u}_t + \vec{u} \cdot \nabla\vec{u}$$

- $\vec{u}_t$: unsteady acceleration.
    - How much velocity changes in fixed x over time.
- $\vec{u} \cdot \nabla\vec{u}$: convective acceleration.
    - How much velocity changes due to movement along trajectory.



http://www.continuummechanics.org/images/continuity/converging_nozzle.png

Inviscid Flow

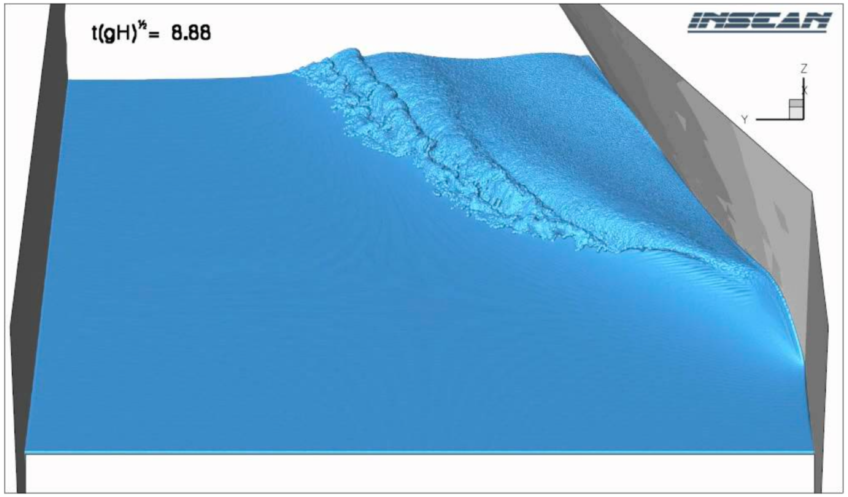
- We often drop viscosity altogether.
- Getting the Euler equations:

$$m\frac{D\vec{u}}{Dt} = \vec{F}_{pre} + \vec{F}_{ext} = -V\nabla p + m\vec{g}$$

- "numerical dissipation": errors that mimic physical viscosity enough for the visual effet.
  - Numerical dissipation often >> true physical viscosity!

Boundary Conditions

- Solid walls: no velocity through:
$$\vec{u} \cdot \hat{n} = 0$$
  - What about pressure?
- For viscid fluid, there is also tangential "stick".
  - "no-slip" conditions: $\vec{u} = 0$



https://i.ytimg.com/vi/7tAoJ4RarAw/maxresdefault.jpg

Water Surface

- Actually a meeting between two fluids (water and air).
- We can assume $p_{AIR} = 0$
- Model: water tries to minimize surface area.

https://abm-website-assets.s3.amazonaws.com/laboratoryequipment.com/s3fs-public/legacyimages/061813_IoW.jpg

Water Surface

- Pressure jump: $[p] = \gamma\kappa$
- γ: surface tension coefficient
- κ: surface mean curvature.
- Mean curvature: measure of area gradient.

$$2\kappa\hat{n} = \lim_{A \to 0} \frac{\nabla A}{A}$$