Lecture VII: Soft-Body Physics

Soft Bodies

- Realistic objects are not purely rigid.
  - Good approximation for “hard” ones.
  - …approximation breaks when objects break, or deform.

- Generalization: soft (deformable) bodies
  - Deformed by force: car body, punched or shot at.
  - Prone to stress: piece of cloth, flag, paper sheet.
  - Not solid: snow, mud, lava, liquid.

http://www.games73.com/media.games73.com/files/2012/05/Crysis-3-soft-physics-demo-thumb-610x239.jpg

http://i.huffpost.com/gen/1480563/images/o-DISNEY-facebook.jpg

Grinspun *et al.* “Discrete Shells”

Elasticity

- Forces may cause object deformation.

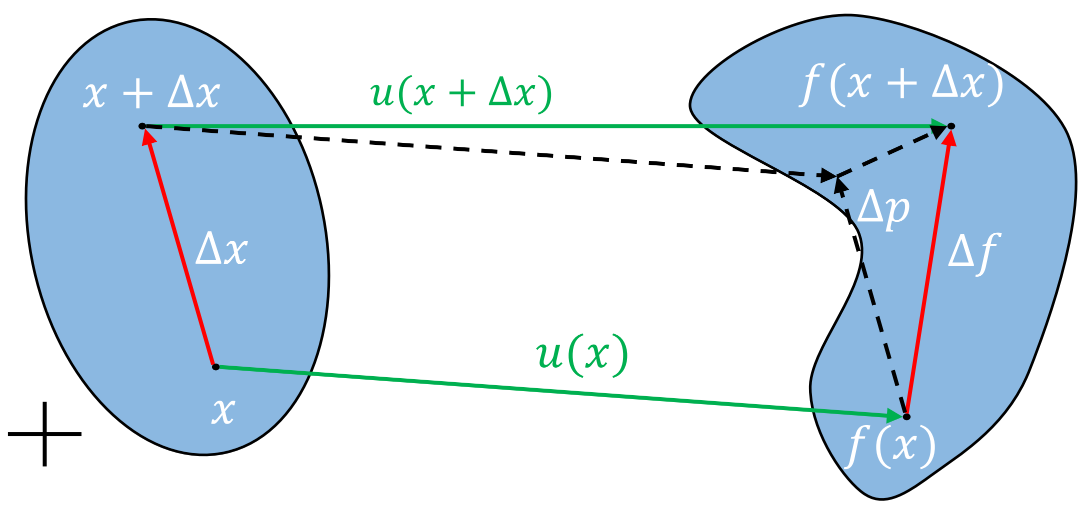
- Elasticity: the tendency of a body to return to its original shape after the forces causing the deformation cease.
  - Rubbers are highly elastic.
  - Metal rods are much less.

http://www.mommypotamus.com/wp-content/uploads/2013/01/homemade-play-dough-recipe-with-natural-dye-6-300x300.jpg

http://www.ibmbigdatahub.com/sites/default/files/elasticity_blog.jpg

Continuum Mechanics

- A deformable object is defined by rest shape and material parameters.
- Deformation map: $f(\vec{x})$ of every material point $\vec{x}$.
- $f: \mathbb{R}^d \rightarrow \mathbb{R}^d$. d : dimension (mostly $d = 2,3$).
- Relative displacement field: $f(\vec{x}) = \vec{x} + u(\vec{x})$.



Local Deformation

- Taylor series:

$$f(\vec{x} + \Delta\vec{x}) \approx f(\vec{x}) + J_f \Delta\vec{x}$$

- 1st-order linear approximation.

- As $f(\vec{x}) = \vec{x} + u(\vec{x})$, we get:

$$\vec{x} + \Delta\vec{x} + u(\vec{x} + \Delta\vec{x}) \approx \vec{x} + u(\vec{x}) + J_f \Delta\vec{x} \Longrightarrow$$
$$u(\vec{x} + \Delta\vec{x}) \approx u(\vec{x}) + \left(J_f - I_{d\times d}\right)\Delta\vec{x}$$

- The Jacobians: $J_f = \left(\frac{\partial f}{\partial x}\right), J_u = \left(\frac{\partial u}{\partial x}\right) = J_f - I_{d\times d}.$

Stretch and Compression

- How much an object locally stretches or compresses in each direction.
- New length:

$$|\Delta f|^2 = |f(\vec{x} + \Delta\vec{x}) - f(\vec{x})|^2 \approx \left|J_f \Delta\vec{x}\right|^2$$
$$= \Delta\vec{x}^{T*}\left({J_f}^T J_f\right) * \Delta\vec{x}$$

- Stretch: relative change in length:

$$\frac{|\Delta f|^2}{|\Delta x|^2} \approx \frac{\Delta\vec{x}^T * \left({J_f}^T J_f\right) * \Delta\vec{x}}{\Delta\vec{x}^T * \Delta\vec{x}}$$

http://www.yankodesign.com/images/design_news/2011/06/09/elastic_exerciser.jpg

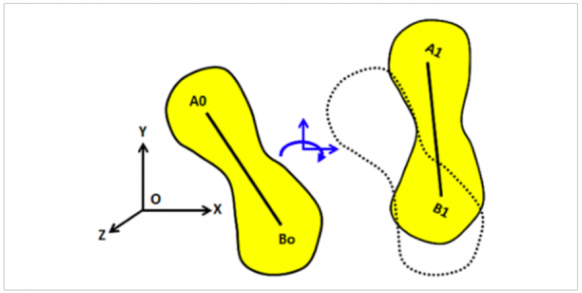
Rigid-Body Deformation

- Transformation:

$$f(\vec{x}) = R\vec{x} + T$$

  - R: rotation (constant)
  - T: translation.

- $J_f = R$, and then ${J_f}^T J_f = I$.
  - No stretch!



http://www.stressebook.com/wp-content/uploads/2015/03/Rigid-Body-Modes-600x300.png

Cauchy-Green Deformation Tensor

- $\left(J_f{}^T J_f\right)_{d\times d}$ is the (right) Cauchy-Green tensor.

- Interpretation: for direction $\hat{d}$, we get:

$$\left|\nabla_{\vec{d}}(f(x))\right|^2 \approx \hat{d}^T \cdot \left(J_f{}^T J_f\right) \cdot \hat{d}$$

- In words: the CG tensor measures the ratio of change in squared length in a direction.

The Green-Lagrange Strain Tensor

- Measures the deviation from rigidity:

$$\mathbf{E}_{3\times 3} = \frac{1}{2}\left({J_f}^T J_f - I\right)$$

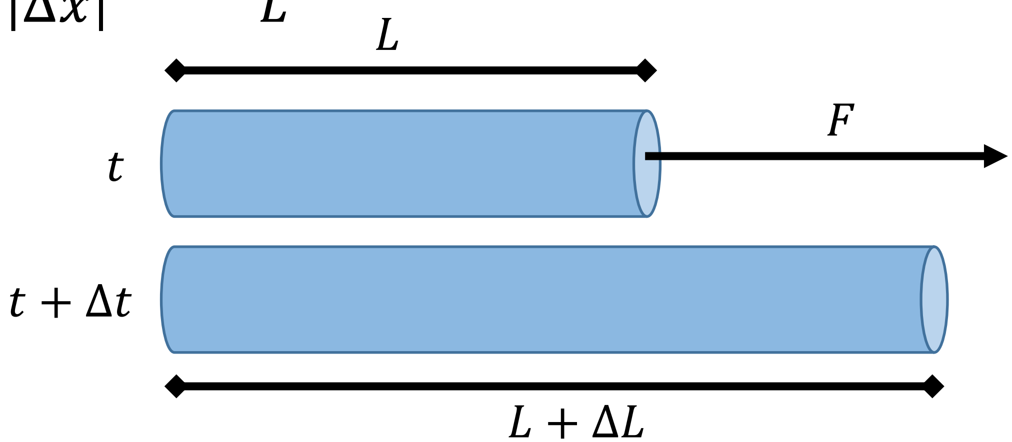
- Gives “0” for rotations.
  - Rather than “1” in the CG tensor.

- In deformation field terms ($J_f = J_u + I_{d\times d}$):

$$\mathbf{E} = \frac{1}{2}\left({J_u}^T J_u + J_u + {J_u}^T\right)$$

Strain

- The fractional deformation $\epsilon = \Delta L / L$
  - Dimensionless (a ratio).
  - How much a deformation differs from rigidity in a given direction:
    - Negative: compression
    - Zero: rigid
    - Positive: stretch
- In 1D: $\frac{|\Delta f|}{|\Delta x|} = \frac{\Delta L + L}{L} = 1 + \epsilon$



$$\mathbf{E}_{3\times 3} = \frac{1}{2}\left({J_f}^T J_f - I\right)$$

GL Tensor and Strain

- For Strain $\epsilon = \Delta L / L$ in (unit length) direction $\hat{d}$:

$$\hat{d}^T \mathbf{E} \hat{d} = \epsilon + \frac{1}{2}\epsilon^2$$

- Problem: the GL strain tensor is nonlinear in the deformation f (or the deformation field u).
- Approximation: the infinitesimal strain tensor:

$$\boldsymbol{\varepsilon} = \frac{1}{2}\left({J_u}^T J_u + J_u + {J_u}^T\right) \approx \frac{1}{2}\left(J_u + {J_u}^T\right)$$

Where $\hat{d}^T \mathbf{E} \hat{d} \approx \epsilon$.

The Infinitesimal Strain Tensor

- A.K.A. Cauchy's strain tensor:

$$\boldsymbol{\varepsilon} = \frac{1}{2}\left(J_u + {J_u}^T\right) = \frac{1}{2}\left(J_f + {J_f}^T\right) - I$$

- So that for unit direction $\vec{d}$:

$$\epsilon_d = \hat{d}^T \boldsymbol{\varepsilon} \hat{d}$$

- The Strain tensor is not rotation invariant!
- Good for small deformations $|J_u| \ll 1$.

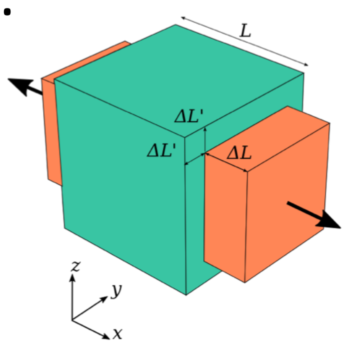
Poisson’s ratio

- Strain in one direction usually causes compression in another.
- Poisson’s ratio: the ratio of transversal to axial strain:

$$\nu = -\frac{d\,[transversal\ strain]}{d\,[axial\ strain]}$$

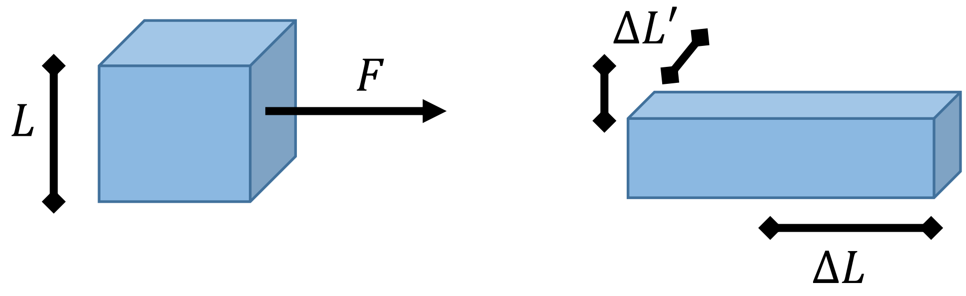
  - Equals 0.5 in perfectly incompressible material.
- If the force is applied along x:

$$\nu = -\frac{d\epsilon_y}{d\epsilon_x} = -\frac{d\epsilon_z}{d\epsilon_x}$$



Poisson's ratio

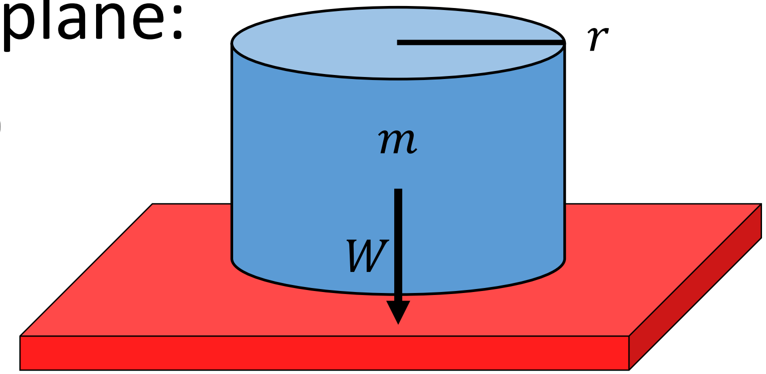
- Example of a cube of size L.
- Average strain in each direction: $\nu \approx \frac{\Delta L'}{\Delta L}$
  - Approximate, because true for small elements and deformation.



Stress

- Magnitude of applied force per area of application.
  - large value ⇔ force is large or surface area is small
- Pressure measure $\vec{\sigma}$.
- Unit: Pascal: $[Pa] = [\frac{N}{m^2}]$
- Example: gravity stress on plane:

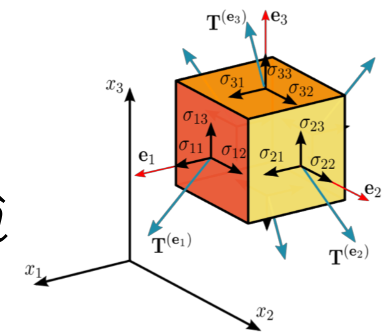
$$\vec{\sigma} = m\vec{g}/(\pi r^2)$$



The Linear Stress Tensor

- Measuring stress for each (unit) direction $\hat{n}$ in an infinitesimal volume element:

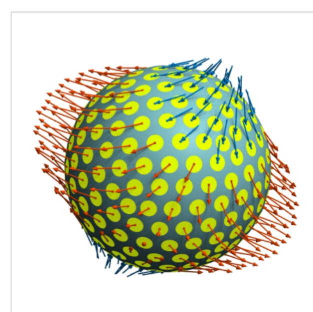
$$\sigma(\hat{n}) = \begin{pmatrix} \sigma_{xx} & \sigma_{yx} & \sigma_{zx} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{zy} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{pmatrix} \hat{n} = \boldsymbol{\sigma}\hat{n}$$

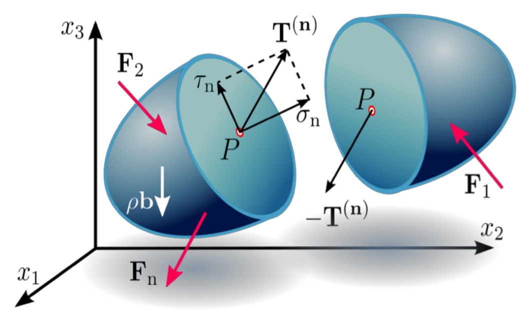


- Note that $\boldsymbol{\sigma}\hat{n}$ is not necessarily parallel to $\hat{n}$!
- $\boldsymbol{\sigma}\hat{n} = \langle \boldsymbol{\sigma}\hat{n}, \hat{n} \rangle \hat{n} + \tau$

normal stress

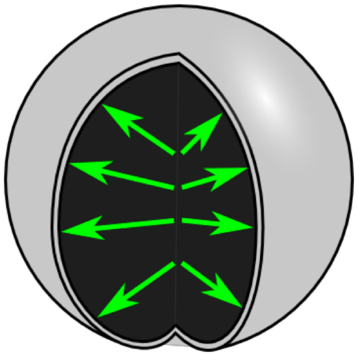
shear stress





Body Material

- The amount of stress to produce a strain is a property of the material.
- Isotropic materials: same in all directions.
- Modulus: a ratio of stress to strain.
  - Usually in a linear direction, along a planar region or throughout a volume region.
    - Young’s modulus, Shear modulus, Bulk modulus
  - Describing the material reaction to stress.



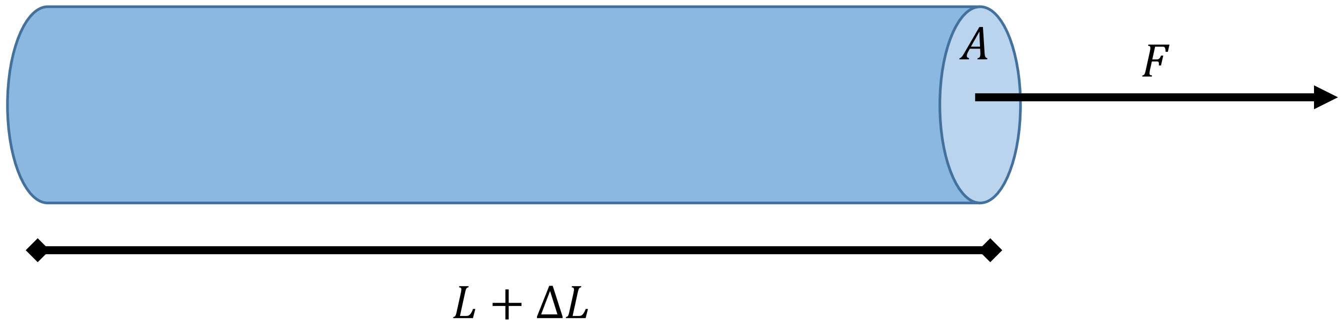
http://openalea.gforge.inria.fr/doc/vplants/mechanics/doc/_build/html/_images/system_geometry7.png

Young's Modulus

- Defined as the ratio of linear stress to linear strain:

$$Y = \frac{linear\ stress}{linear\ strain} = \frac{F/A}{\Delta L/L}$$

- Example:

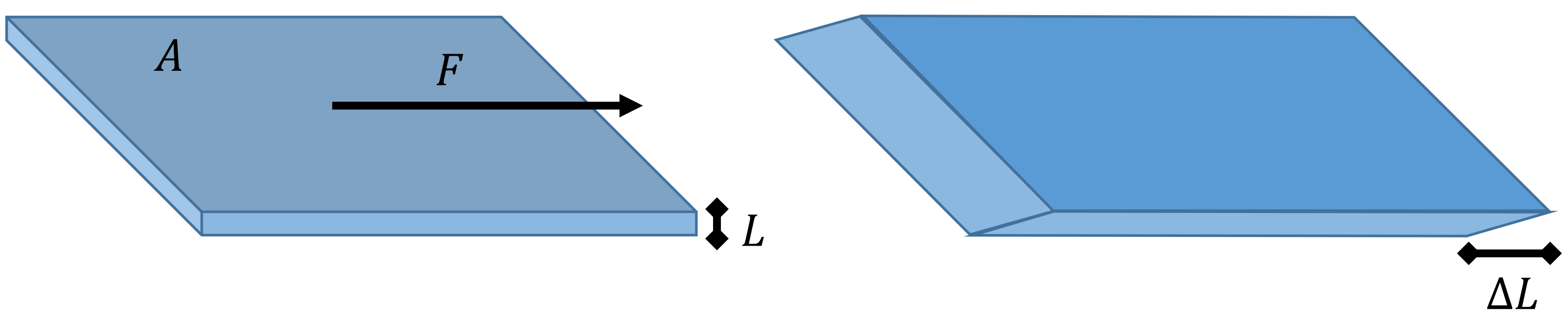


Shear modulus

- The ratio of planar stress to planar strain:

$$S = \frac{planar\ stress}{planar\ strain} = \frac{F/A}{\Delta L/L}$$

- Example:

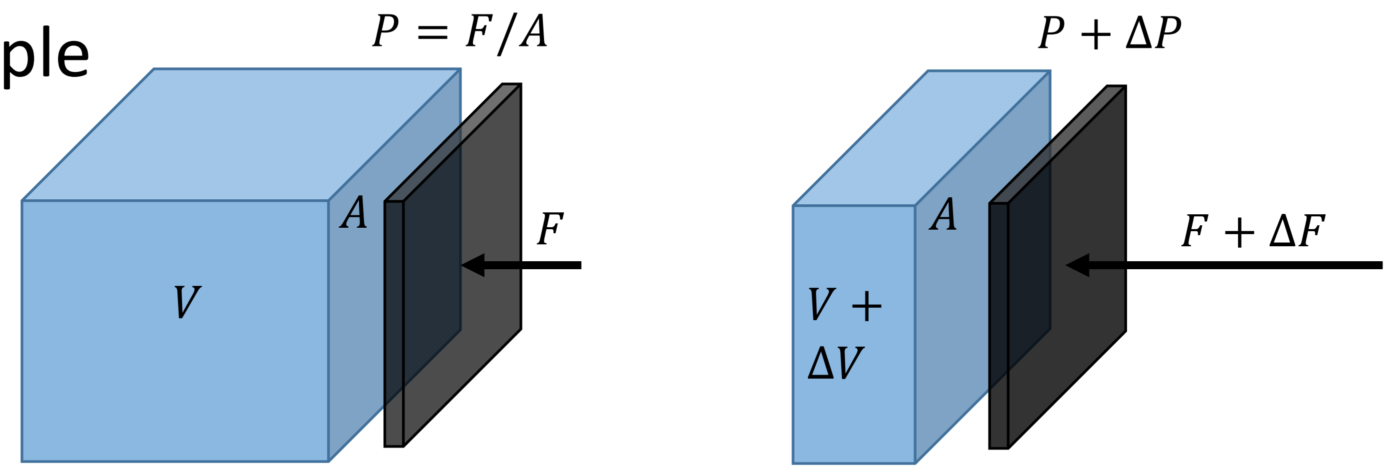


Bulk modulus

- The ratio of volume stress to volume strain (inverse of compressibility):

$$B = \frac{volume\ stress}{volume\ strain} = \frac{\Delta P}{\Delta V / V}$$

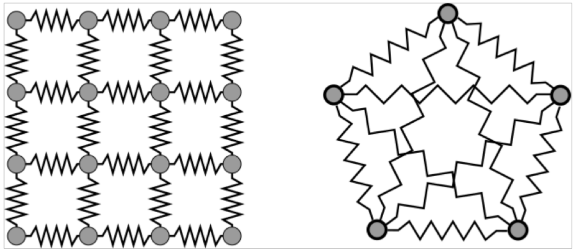
- Example



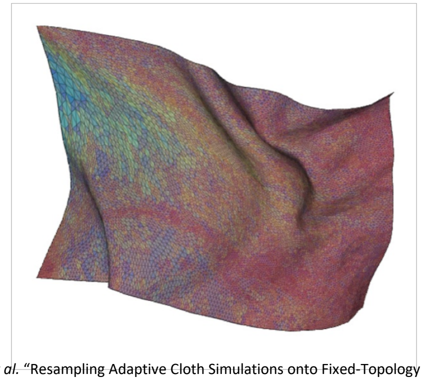
Linear Elasticity

- Stress and strain are related by Hooke's law
  - Remember $F = -k\Delta x$?
- Reshape tensors to vector form:
- $\bar{\sigma} = \left(\sigma_{xx}, \sigma_{xy}, \cdots, \sigma_{zz}\right)$, and similarly for $\bar{\epsilon}$.
- Then the stiffness tensor $\mathbf{C}_{9x9}$ holds:

$$\bar{\sigma} = \mathbf{C}\bar{\epsilon}$$



https://people.eecs.berkeley.edu/~sequin/CS184/TOPICS/SpringMass/Spring_mass_2D.GIF



Brown *et al.* "Resampling Adaptive Cloth Simulations onto Fixed-Topology Meshes"

Hyperelastic Materials

- Seek to return to their “rest shape”.
  - Have a potential deformation energy
- Spring energy: $E_s = \frac{1}{2}k||x - x_0||^2$.
- Underlying assumption: deformation energy is not path-dependent!

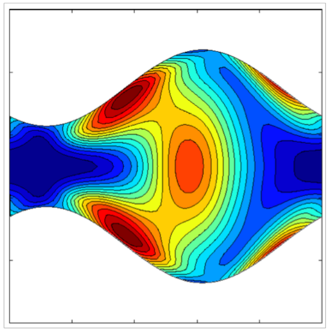
Linear Elasticity Energy

- One Possibility is: $E = \frac{1}{2}\int_T \langle \bar{\sigma}, \bar{\epsilon} \rangle dV$.
  - Possibilities depend on the type of Stress\Strain tensors to use.
  - This one is popular for linear elasticity with FEM.
- We get:

$$E = \frac{1}{2}\int_T \bar{\epsilon}\mathbf{C}\bar{\epsilon}dV$$

Dynamic Elastic Materials

- For every point q, The PDE is given by
$$\rho * \frac{d^2\vec{u}}{dt^2} = \nabla \cdot \sigma + \vec{F}$$
  - ρ: the density of the material.
  - $\nabla \cdot \sigma = (\partial/\partial x, \partial/\partial y, \partial/\partial z) * \sigma$ is the divergence of the stress tensor (modeling internal forces).
  - F: external body forces (per point)
- Generalized Newton's 2nd law!
  - Remember $F = ma$?
  - Similar, in elasticity language.



http://www.cims.nyu.edu/cmcl/ComplexFluids/Images/OB-Pump.png