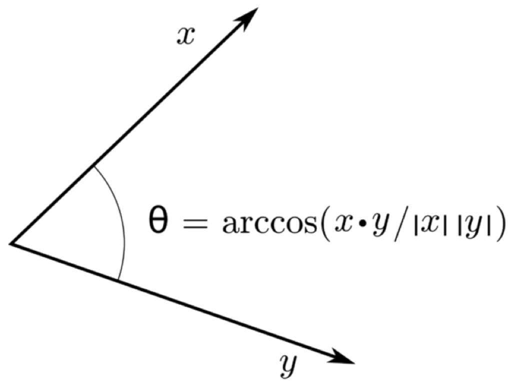
Lecture II: Vector and Multivariate Calculus

Dot Product

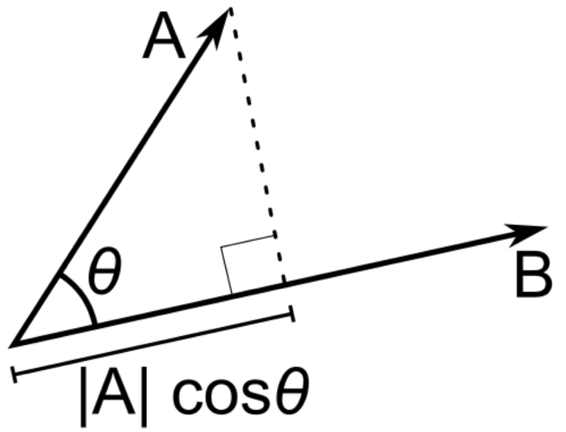
- $\vec{a}, \vec{b} \in \mathbb{R}^n, \vec{a} \cdot \vec{b} = \sum_{i=1}^{n} (a_i \cdot b_i) \in \mathbb{R}$.
- $\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos\theta$.
  - θ **convex** angle between the vectors.
- Squared norm of vector: $|\vec{a}|^2 = \vec{a} \cdot \vec{a}$.
- Alternative notation: $\vec{a} \cdot \vec{b} = \langle \vec{a}, \vec{b} \rangle$
- Matrix multiplication $C = A \cdot B \Leftrightarrow c_{ij} = \langle A_{i,.}, B_{.,j} \rangle$
- Note: $\langle \vec{a}, \vec{a} \rangle$ is always non-negative.
  - $\langle \vec{a}, \vec{b} \rangle$ - measure similarity (angle)
  - $\langle \vec{a}, \vec{a} \rangle$ - measures length.



Dot Product

- A geometric interpretation: the part of $\vec{a}$ which is parallel to a unit vector in the direction of $\vec{b}$.
  - And vice versa!
  - Projected vector: $\overrightarrow{a_\parallel} = \frac{(\vec{a} \cdot \vec{b})}{|\vec{b}|} \vec{b}$.

- The part of $\vec{b}$ orthogonal to $\vec{a}$ has no effect!



Linear Transformations

- Matrix representation: $y = Mx, \mathrm{y} \in \mathbb{R}^m, x \in \mathbb{R}^n, M \in \mathbb{R}^{m \times n}$
- A stack of dot products:

$$y = \begin{pmatrix} \langle M_{1,.}, x \rangle \\ \vdots \\ \langle M_{n,.}, x \rangle \end{pmatrix}$$

- Canonical axis system: $(1,0,0,0, \dots)$ etc.
- When $m = n$, and the matrix is full-rank, it is a change of coordinates.
  - Geometric intuition: representing x with the rows of M, instead of the canonical axis system.

Special Linear Transformations

- Rotation matrix in $\mathbb{R}^2$: $\mathrm{R} = \begin{pmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{pmatrix}$.
- What you learned: rotates a point $p = (x, y)$ by angle θ in CCW direction.
- Alternative interpretation:
  - p is currently represented in the canonical representation.
  - Rp is the representation of p in the basis $(cos\theta, -sin\theta)$ and $(sin\theta, cos\theta)$.
- Watch chalkboard!

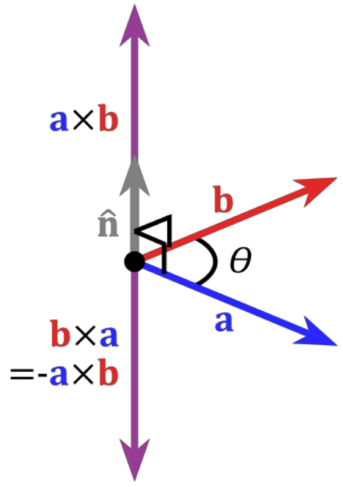
Cross Product

- Typically defined only for $\mathbb{R}^3$.

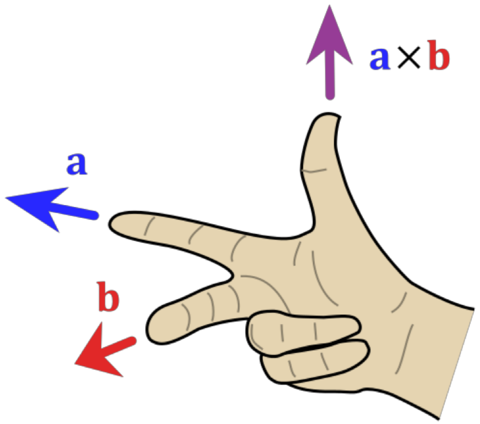
- $\vec{a} \times \vec{b} = \left(a_y b_z - a_z b_y, b_x a_z - b_z a_x, a_x b_y - a_y\right) \in \mathbb{R}^3$.

- Or more generally:

$$\vec{a} \times \vec{b} = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ \hat{x} & \hat{y} & \hat{z} \end{vmatrix}$$



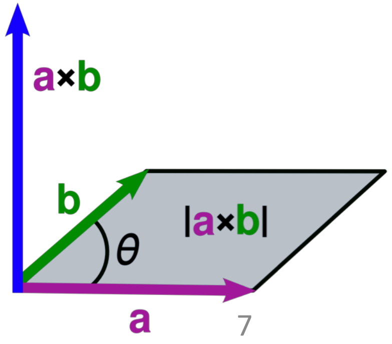
Cross Product



- The result vector is orthogonal to both vectors.
  - Direction: Right-hand rule.
  - Normal to the plane spanned by both vectors.
- Magnitude: $\left|\vec{a}\times\vec{b}\right| = |\vec{a}|\left|\vec{b}\right|\sin\theta.$
  - Parallel vectors ⇔ cross product zero.
- The part of $\vec{b}$ parallel to $\vec{a}$ has no effect on the cross product!
- Geometric interpretation: axis of shortest rotation between $\vec{a}$ and $\vec{b}$.

$$\vec{b} = \mathrm{R}_{\vec{a}\times\vec{b}}(\theta)\vec{a}$$

- Another geometric interpretation: $\left|\vec{a}\times\vec{b}\right| = A_{parallelogram}.$



Symmetric Bilinear Maps

- Also denoted as "2-tensors".
  - Take two vectors into a scalar.
- $M: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $M(\vec{u}, \vec{v}) = c$.
- Symmetry: $M(\vec{u}, \vec{v}) = M(\vec{v}, \vec{u})$
- Linearity: $M(a\vec{u} + b\vec{w}, \vec{v}) = aM(\vec{u}, \vec{v}) + bM(\vec{w}, \vec{v})$.
  - The same for $\vec{v}$ for symmetry.
- Can be represented by symmetric $n \times n$ matrices: $\mathrm{c} = \vec{u}^T M \vec{v}$.

Metric

- M is positive definite if for every $\vec{u}$, $M(\vec{u}, \vec{u}) > 0$.
  - Similarly, negative-definite (< 0), positive semidefinite (≥ 0).
- Interpretation:
  - M is a generalized dot product, or a metric.
  - The original dot product: simply $M = I_{nxn}$.
  - It's only a true metric (=non-negative) if indeed M is PSD.
- Often notated $\left\langle \vec{a}, \vec{b} \right\rangle_M$.
  - Then, $\langle \vec{a}, \vec{a} \rangle_M$ is "the squared length of $\vec{a}$ in the metric of M".

Functions of Several Variables

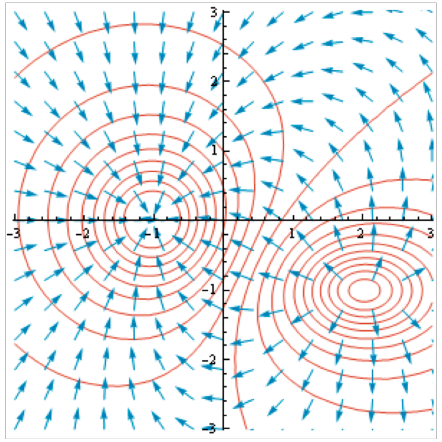
- A single function of several variables:

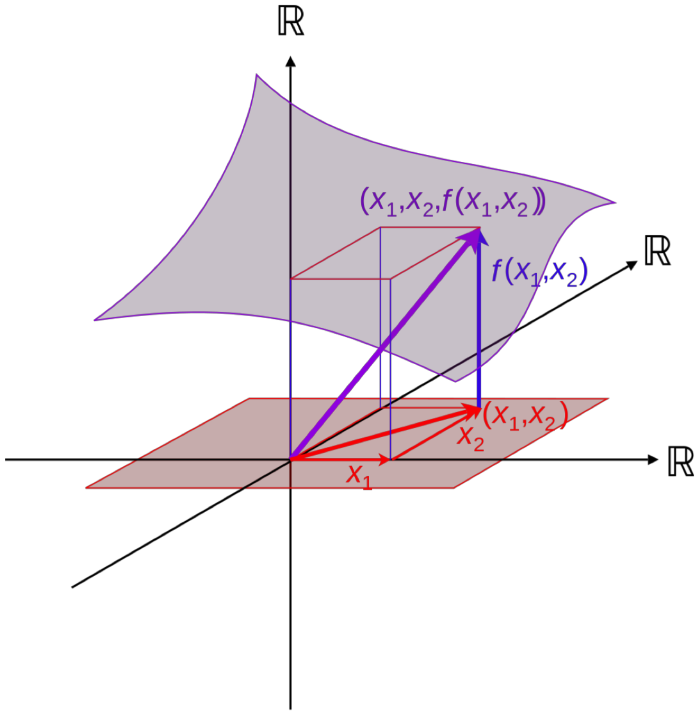
$$f: \mathbb{R}^n \to \mathbb{R}, f(x_1, x_2, \cdots, x_n) = y.$$

- Partial derivative vector, or gradient, is a vector:

$$\nabla f = \left(\frac{\partial y}{\partial x_1}, \cdots, \frac{\partial y}{\partial x_n}\right)$$

- In the direction of steepest ascent.



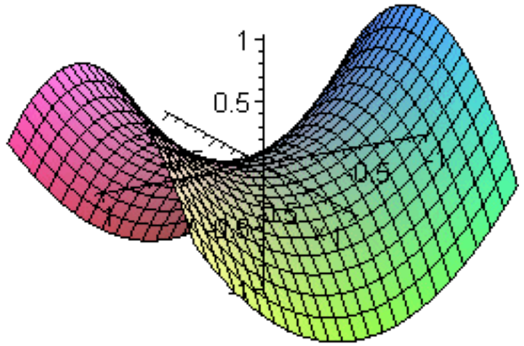


Multi-Valued Functions

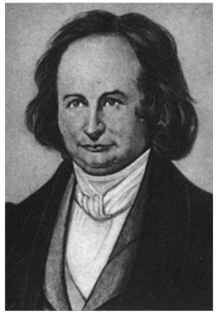
- A vector-valued function of several variables:

$$f : \mathbb{R}^n \to \mathbb{R}^m, f(x_1, x_2, \cdots, x_n) = (y_1, y_2, \cdots, y_m).$$

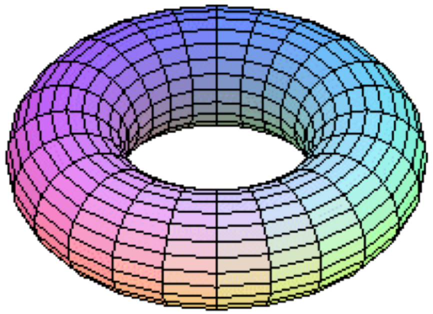
- Can be viewed as a change of coordinates, or a mapping.
  - Recall: Linear functions $\Leftrightarrow \mathbb{R}^{m \times n}$ matrices.



- The derivatives form a matrix, denoted as the Jacobian:



$$\nabla f = J_f = \begin{pmatrix} \frac{\partial y_1}{\partial x_1} & \vdots & \frac{\partial y_1}{\partial x_n} \\ \cdots & & \cdots \\ \frac{\partial y_n}{\partial x_1} & \vdots & \frac{\partial y_n}{\partial x_n} \end{pmatrix}$$



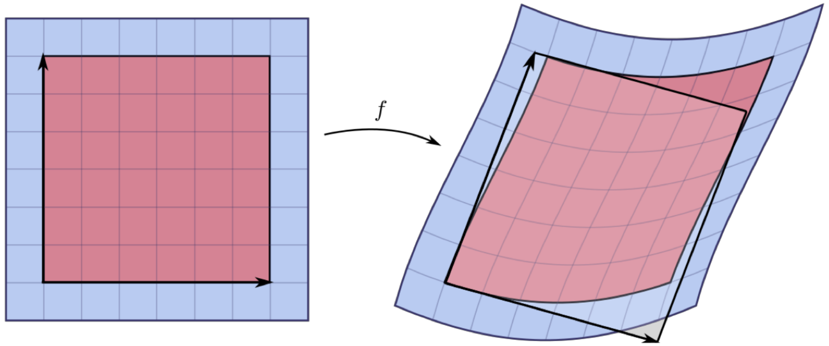
https://www.math.duke.edu/education/ccp/materials/mvcalc/parasurfs/para1.html

$f: \mathbb{R}^n \to \mathbb{R}^m, f(x_1, x_2, \cdots, x_n) = (y_1, y_2, \cdots, y_m)$

Jacobian Measures Deformation

- Consider two points $p, p + \Delta p$, where Δp is very small.
- Taylor series:

$$f(p + \Delta p) \approx f(p) + J_f \Delta p$$



- 1st-order linear approximation.

- Original infinitesimal squared length: $\langle \Delta p, \Delta p \rangle$.
- Target length after map:

$$\langle f(p + \Delta p) - f(p), f(p + \Delta p) - f(p) \rangle \approx$$
$$\langle J_f \Delta p, J_f \Delta p \rangle = \Delta p^T {J_f}^T \cdot J_f \Delta p = \langle \Delta p, \Delta p \rangle_M$$

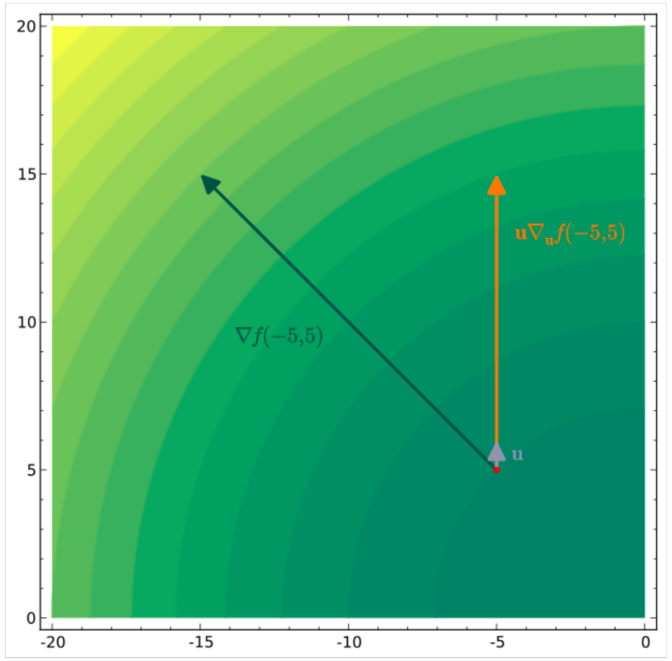
Where $M = \left({J_f}^T \cdot J_f\right)$, a symmetric bilinear form!

- Interpretation: J_f encodes the change of lengths, or deformation.

Directional Derivative

- The change in function u in the (unit) direction $\hat{d}$:
$$\nabla_d u = \langle \nabla u \, , d \rangle$$

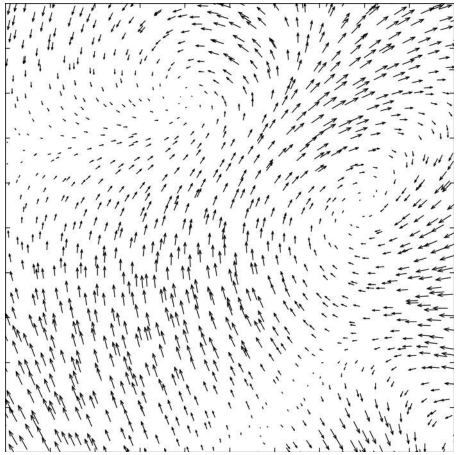
- Formally: a map $\nabla_d : \mathbb{R}^n \to \mathbb{R}$ between direction $\hat{d}$ and scalar $\langle \nabla u \, , d \rangle$
- Biggest change: along the gradient.
- Smallest change: along the level set.



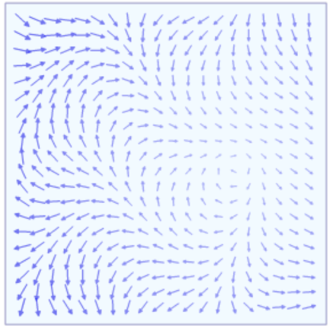
Vector Fields in 3D

- A vector-valued function assigning a vector to each point in space: $g: \mathbb{R}^3 \to \mathbb{R}^3, g(\vec{p}) = \vec{v}$.
- Physics: velocity fields, force fields, advection, etc.
- Special vector fields:
  - Constant
  - Rotational
  - Gradient fields of scalar functions: $\vec{v} = \nabla f$.

http://vis.cs.brown.edu/results/images/Laidlaw-2001-QCE.011.html



Integration over a Curve



- Given a curve $\vec{C}(t) = (x(t), y(t), z(t)), t \in [t_0, t_1]$.
- And a vector field $\vec{v}(x, y, z)$
- The integration of the field on the curve is defined as:

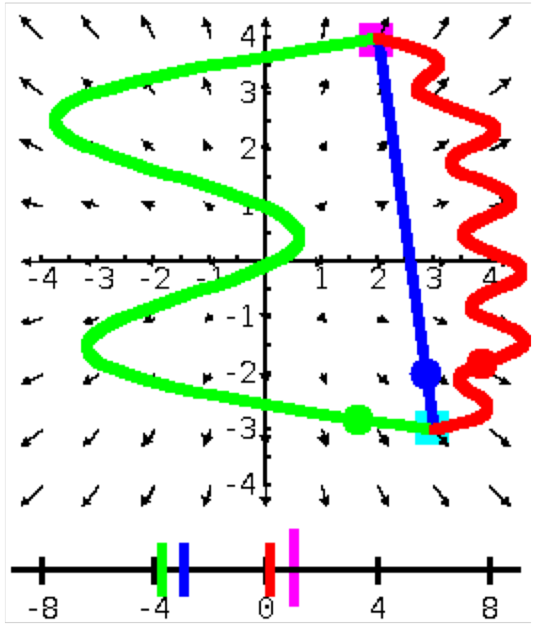
$$\int_C \vec{v} \cdot d\vec{C} = \int_{t_0}^{t_1} \vec{v} \cdot \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right) dt$$

- Interpretation: the “work done“ by $\vec{v}$ along $\vec{C}$.
- Geometric description: summing dot products of vector $\vec{v}$ and infinitesimal curve segments $d\vec{C}$.

Conservative Vector Fields

- A vector field $\vec{v}$ is conservative if there is a scalar function φ so that for every curve $\vec{C}(t), t \in [t_0, t_1]$:

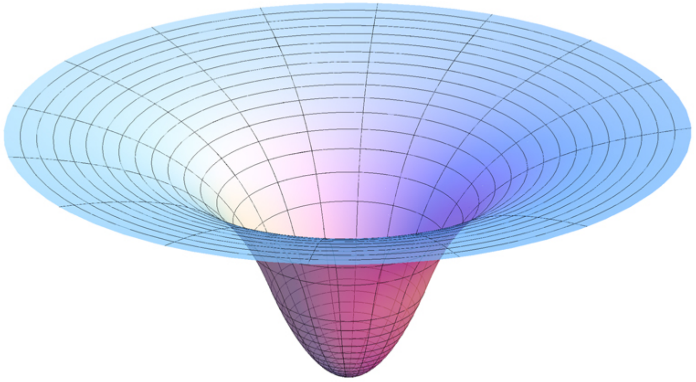
$$\int_C \vec{v} \cdot d\vec{C} = \varphi(t_1) - \varphi(t_0)$$



- Equivalently: if $\vec{v} = \nabla\varphi$.
- The integral is then path independent.

Conservative Vector Fields

- Physical interpretation: the vector field $\vec{v}$ is the result of a potential φ.
- Example: the work (potential energy) W by gravity force $\vec{G} = -\nabla W$ is only dependent of the height gained\lost.
  - Question: What the scalar potential W for a gravity field?
- Corollary: the integral of a conservative vector field over a closed curve is zero!



Gradients and Potential Energy

- Potential energy is a model that describes the potential for acceleration.
- An object with potential energy E is operated upon with force $-\nabla E$.
  - Why negative?
- Example:

spring energy: $E = K(l - l_0)^2$,

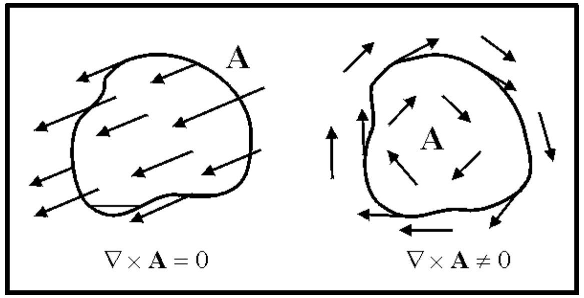
force: $\mathrm{F} = -\nabla E = -K(l - l_0)$

The Curl (Rotor) Operator

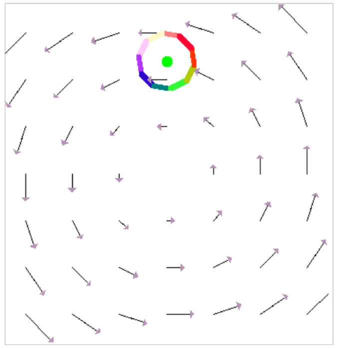
- Definition: $\nabla\times\vec{v} = \left(\partial/_{\partial x}, \partial/_{\partial y}, \partial/_{\partial z}\right)\times\vec{v}$.
- Produces a vector field from a vector field.
- Geometric intuition: $\nabla\times\vec{v}$ encodes local rotation (**vorticity**) that the vector field (as a force) induces locally on the point.
- Integral definition:

$$(\nabla\times\vec{v})\cdot\hat{n} = \lim_{A\to 0}\frac{1}{|A|}\oint_C \vec{v}\cdot d\vec{C}$$

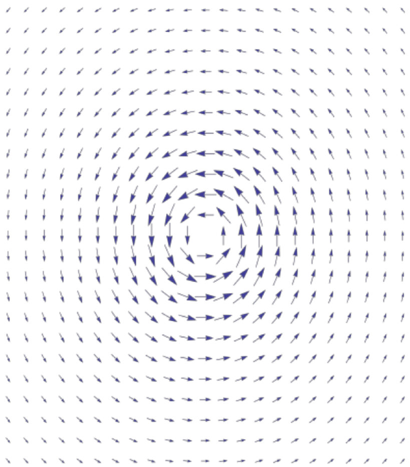
  - C : infinitesemal curve around the point
  - A : area it encompasses.
  - $\hat{n}$: rotation axis around which we measure rotation.



http://www.chabotcollege.edu/faculty/shildreth/physics/gifs/curl.gif



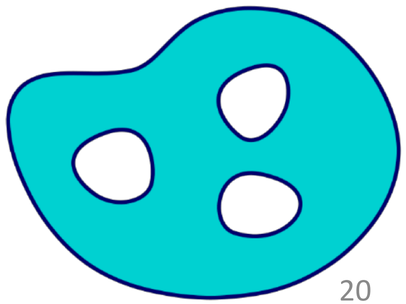
Irrotational Fields



- Fields where $\nabla \times \vec{v} = 0$.
  - Also denoted Curl-free.
- Conservative fields ➔ irrotational.
  - as for every scalar φ:

$$\nabla \times \nabla \varphi = 0$$

- It is evident from the integral definition: $\lim_{A \to 0} \frac{1}{|A|} \oint_C \vec{v} \cdot d\vec{C}$.
- Is irrotational ➔ Conservative fields also correct?
- Only for simply-connected domains!

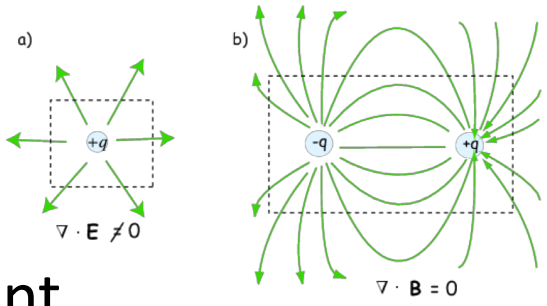


Divergence

- Definition: $\nabla \cdot \vec{v} = \left({}^{\partial}/_{\partial x}, {}^{\partial}/_{\partial y}, {}^{\partial}/_{\partial z}\right) \cdot \vec{v}$.
- Produces a scalar value from a vector field.
- Geometric intuition:$\nabla \cdot \vec{v}$ encodes local change in density induced by vector field as a flux.
- Integral definition:

$$\nabla \cdot \vec{v} = \lim_{V \to \{p\}} \frac{1}{|V|} \iint_{S(V)} (\vec{v} \cdot \hat{n}) dS$$

http://magician.ucsd.edu/essentials/WebBookse8.html

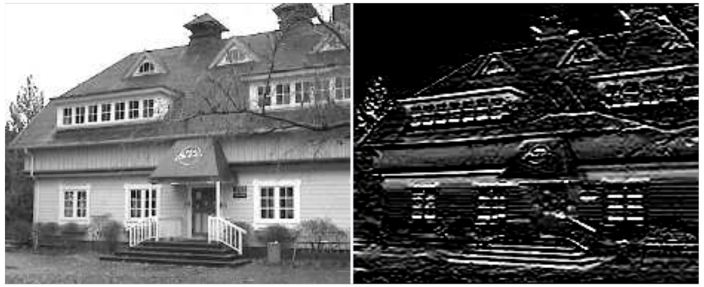


- $S(V)$ is the surface of an infinitesimal volume around the point.
- $\hat{n}$ is the outward local normal.
- Divergence-free fields ($\nabla \cdot \vec{v} = 0$) are called solenoidal.

Laplacian

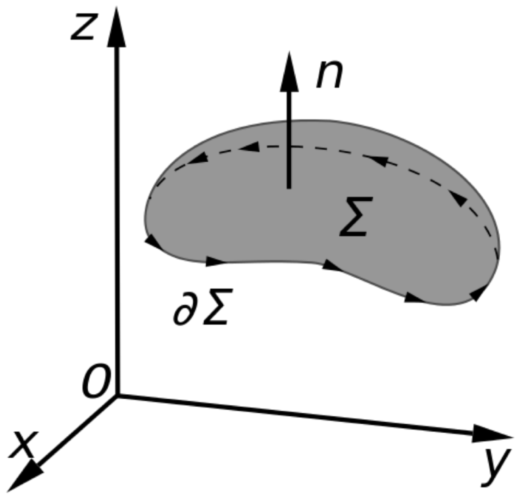
- The divergence of the gradient of a scalar field:

$$\Delta\varphi = \nabla^2\varphi = \nabla\cdot(\nabla\varphi).$$



- Produces a scalar value from a scalar field.
- Geometric intuition: Measuring how much a function is diffused or similar to the average of its surrounding.
  - Found in heat and wave equations.
  - Used extensively in signal processing, e.g. for denoising.

Stokes Theorem



- A more general form of the idea of “conservative fields”
- The modern definition:

$$\int_{V} dw = \int_{\partial V} w$$

- Geometric interpretation: Integrating the differential of a field inside a domain ⇔ integrating the field on the boundary.
- Colloquial definition: the change inside is exactly the sum of “ingoing” and “outgoing” through the boundary.

Stokes Theorem

- Generalizes many classical results.
- Integrating along a curve: $\int_C \nabla\varphi \cdot d\vec{C} = \varphi(t_1) - \varphi(t_0)$.
  - Special case: Fundamental theory of calculus: $\int_{x_0}^{x_1} F'(x)dx = F(x_1) - F(x_0)$.
- Kelvin-Stokes Theorem:

$$\oint_{\partial S} \vec{v} \cdot d\vec{C} = \iint_S (\nabla \times \vec{v})dS.$$

- Divergence theorem:

$$\iiint_V (\nabla \cdot \vec{v})dV = \oiint_{\partial V} (\vec{v} \cdot \hat{n})dS$$

- …and many similar more.