Lecture III: Rigid-Body Physics

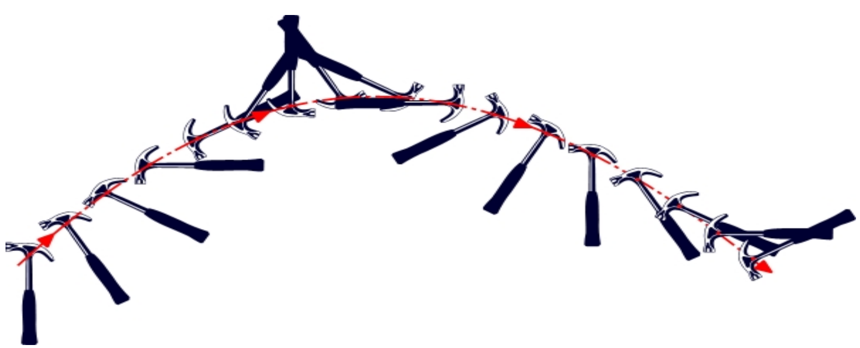
Rigid-Body Motion

- Previously: Point dimensionless objects moving through a trajectory.

- Today: Objects with dimensions, moving as one piece.

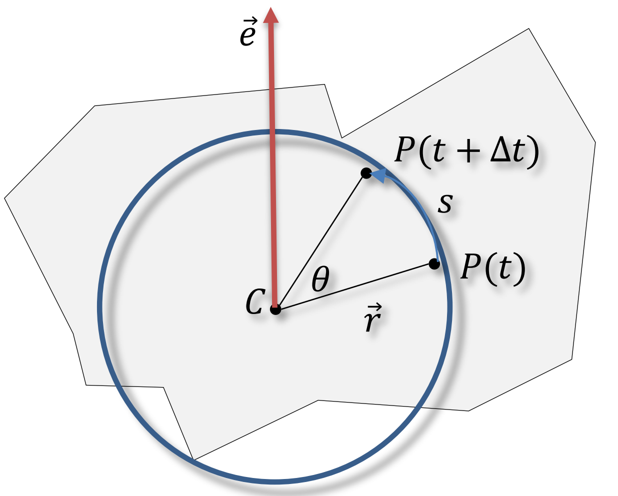
Rigid-Body Kinematics

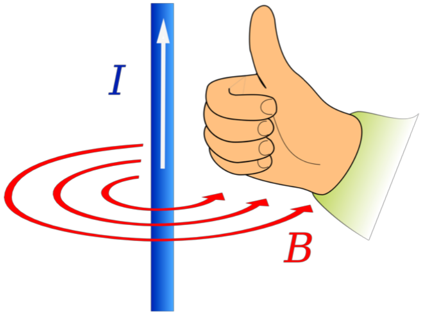
- Objects as sets of points.
- Relative distances between all points are invariant to rigid movement.
- Movement has two components:
  - Linear trajectory of a central point (“translation”).
  - Relative rotation around the point (“rotation”).



Rotational Motion

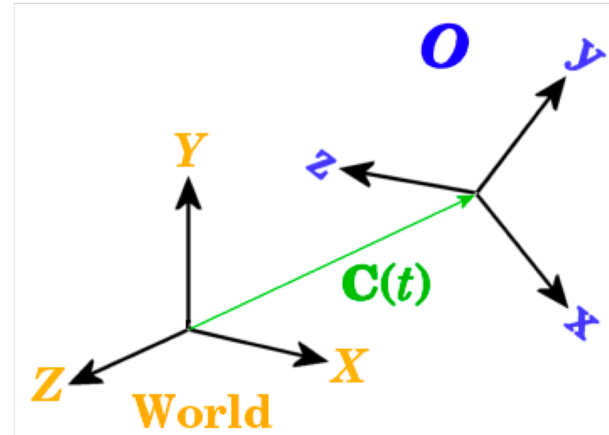
- $\vec{P}$ a point on to the object.
- $\vec{C}$ is the center of rotation.
- Distance vector $\vec{r} = \vec{P} - \vec{C}$.
  - $r = ||\vec{r}||$: distance.
- Object rotates ⇔ P travels along a circular path.
- Unit-length axis of rotation: $\vec{e}$.
  - 2D: $\vec{e}$=$\hat{z}$ (“out” from the screen).
  - “Positive” Rotation: counterclockwise.
    - right-hand rule.





$$\boxed{\vec{r} \times \frac{d\vec{P}}{dt} \parallel \vec{e}}$$

Rigid-Body Kinematics



- Object coordinate system.
- Describing rigid-body motion:
  - Moving origin
  - Rotating orientation.
- Both defined w.r.t. another frame!
  - i.e., the canonical “world coordinates”.

Representing Orientation

- Object axis system (rows):

$$\mathrm{R}_o(t) = \begin{pmatrix} \widehat{x_o}(t) \\ \widehat{y_o}(t) \\ \widehat{z_o}(t) \end{pmatrix}.$$

  - Changes with time t.
  - An orthonormal (rotation matrix).
- Rotations are orientations!
- Canonical ($t = 0$) axes: can be arbitrary.
  - The canonical choice $\mathrm{R}_o(0) = I_{3\times 3}$ is default ("world coordinates").

Axis-Angle Representation

- Every rotation can be represented by an axis $\hat{e}$ and a angle θ.
- Angle-axis → rotation matrix:

$$R = I + \sin\theta\, K + (1 + \cos\theta)K^2$$

- $$\mathrm{K} = [\hat{e}]_\times = \begin{pmatrix} 0 & -\hat{e}_z & \hat{e}_y \\ \hat{e}_z & 0 & -\hat{e}_x \\ -\hat{e}_y & \hat{e}_x & 0 \end{pmatrix}$$

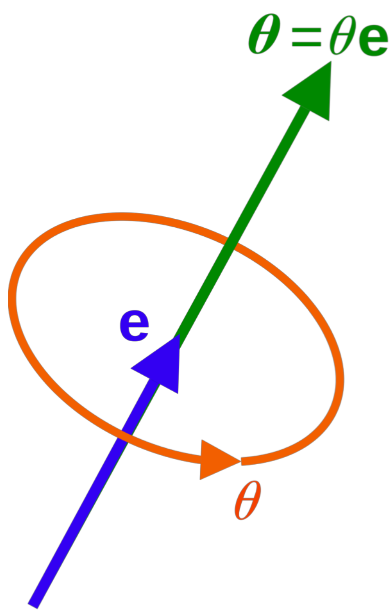
- The cross-product matrix: $Kv = \hat{e} \times v$

- Rotation matrix ← angle axis:

$$\theta = \mathrm{acos}\left({}^{tr(R) - 1}/_{2}\right)$$

$$\hat{e} = \frac{1}{2sin\theta}\begin{pmatrix} R_{32} - R_{23} \\ R_{13} - R_{31} \\ R_{21} - R_{12} \end{pmatrix}$$

- Not unique! Why?



Quaternions

- Quaternions: $\boldsymbol{q} = (r, \vec{v})$
  - Real part: scalar r.
  - Imaginary part: vector $\vec{v} \in \mathbb{R}^3$.
- Unit quaternions: $|\boldsymbol{q}|^2 = r^2 + |\vec{v}|^2 = 1$.
- Quaternion multiplication:

$$\boldsymbol{p} \cdot \boldsymbol{q} = \left(r_p r_q - \langle \overrightarrow{v_p}, \overrightarrow{v_q} \rangle, r_q \overrightarrow{v_p} + r_p \overrightarrow{v_q} + \overrightarrow{v_p} \times \overrightarrow{v_q}\right)$$

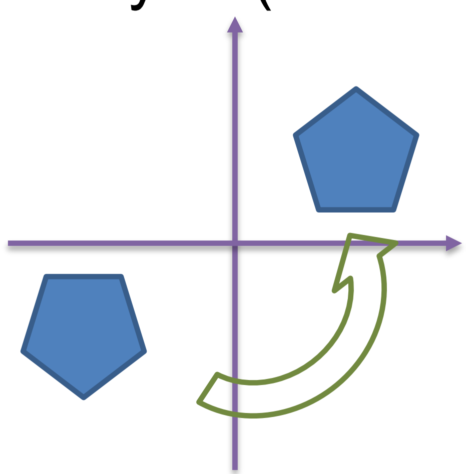
- Not commutative! $\boldsymbol{p} \cdot \boldsymbol{q} \neq \boldsymbol{q} \cdot \boldsymbol{p}$.
- Commutative iff $\overrightarrow{v_p} \times \overrightarrow{v_q} = 0$ (parallel vector parts).
- Inverse and conjugate: $\overline{\boldsymbol{p}} = (r, -\vec{v}), \boldsymbol{p}^{-1} = {}^{\overline{\boldsymbol{p}}}\!/_{|\boldsymbol{p}|^2}$.

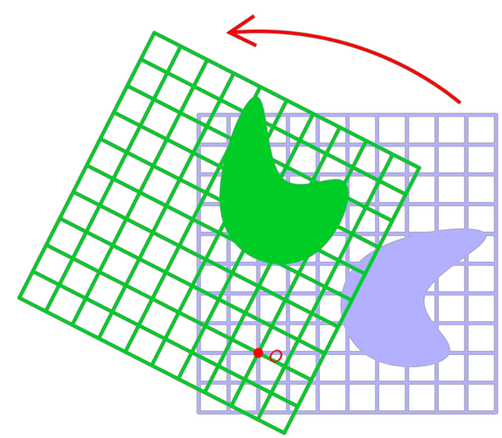
Quaternions as rotations

- Rotations as unit quaternions:
$$\boldsymbol{q} = \left(\cos\frac{\theta}{2}, \sin\frac{\theta}{2}\hat{e}\right)$$
- Vectors $\vec{a} \in \mathbb{R}^3 \Leftrightarrow$ imaginary quaternions $(0, \vec{a})$.
- Rotating vector $\vec{a}$ by θ, $\hat{e}$ into vector $\vec{b}$: $\vec{b}$=$\boldsymbol{q}\vec{a}\boldsymbol{q}^{-1}$.
- Rotation composition: subsequent multiplication:
  - Rotation of $\boldsymbol{q}$ and then $\boldsymbol{p}$: $\vec{c}$=$\boldsymbol{p}\vec{b}\boldsymbol{p}^{-1} = \boldsymbol{pq}\vec{a}\boldsymbol{q}^{-\mathbf{1}}\boldsymbol{p}^{-1}$ $= \boldsymbol{pq}\vec{a}(\boldsymbol{pq})^{-1} = \boldsymbol{s}\vec{a}\boldsymbol{s}^{-1}$
  - Where $\boldsymbol{s} = \boldsymbol{pq}$.
- Quaternions $\Leftrightarrow$ rotation matrices with same axis-angle.
  - Mutual conversion: a bit technical.

Special case: 2D

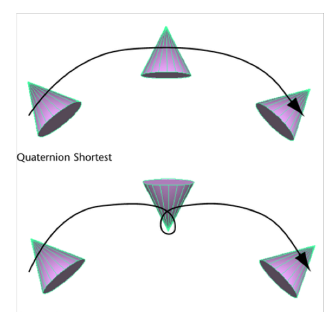
- Axis is always $\hat{e} = \hat{z}$.
- Quaternion $\boldsymbol{q} = \left(\cos\frac{\theta}{2}, \sin\frac{\theta}{2}\hat{z}\right)$ reduces to complex number $\boldsymbol{c} = e^{i\theta/2}$.
- Rotation of (complex) vector: $\vec{b} = \boldsymbol{c}^2\vec{a}$.
- Remember $e^{i\pi} = -1$?
  - Rotation by π! (reflection through point).





Key-framing

- Interpolating between orientation quaternions $\boldsymbol{q}(\mathrm{t})$ and $\boldsymbol{q}(\mathrm{t}+\Delta t)$?
- No single option!
  - Not even in 2D…
- Shortest rotation: $\Delta\boldsymbol{q}(t) = \boldsymbol{q}(\mathrm{t}+\Delta t)\ \boldsymbol{q}(\mathrm{t})^{-1}$
- Can be applied: $\boldsymbol{q}(\mathrm{t}+\Delta t)$ =$\Delta\boldsymbol{q}(t)\boldsymbol{q}(\mathrm{t})$.
- What happens continuously?



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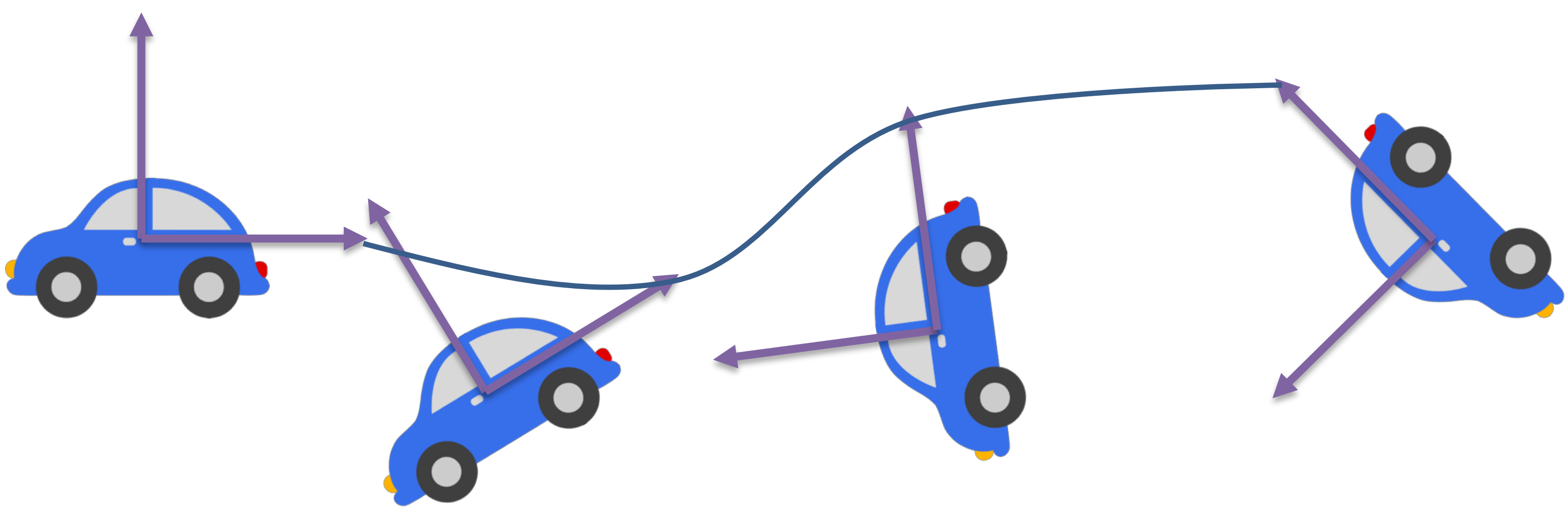
Instantaneous Rotation

- A body goes through orientations $\boldsymbol{q}(t)$ (quaternion).
- The change of orientation, or derivative $\frac{d\boldsymbol{q}}{dt}$:

$$\frac{d\boldsymbol{q}}{dt} = \frac{1}{2}(0, \vec{\omega})\boldsymbol{q}$$

- What is $\vec{\omega}$?

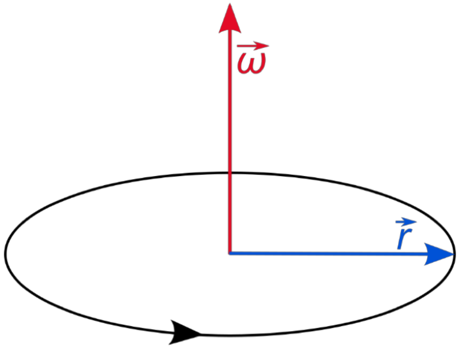


Angular Velocity

- The speed of rotation around the rotation axis:

$$\vec{\omega}(t) = \omega(t)\hat{e}(t).$$

- The angular velocity vector is collinear with the rotation axis:
- unit is $\frac{rad}{sec}$.
- $\hat{e}(t)$: the instantaneous axis of rotation.



Angular Acceleration

- Angular acceleration: the rate of change of the angular velocity:

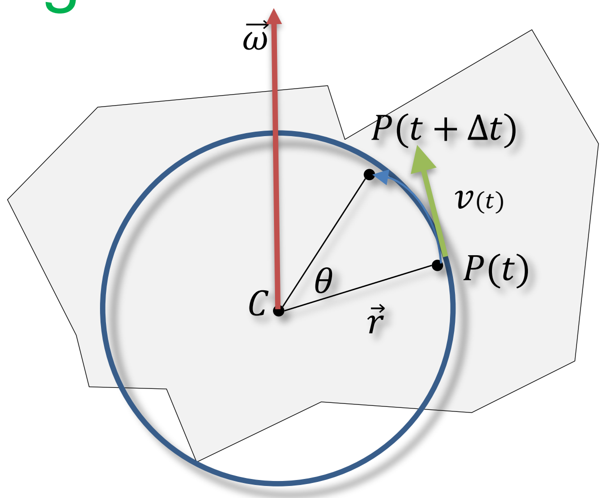
$$\vec{\alpha} = \frac{d\vec{\omega}}{dt}$$

- Paralleling definition of linear acceleration.

- Unit is rad/s^2

Tangential and Angular Velocities

- Every point moves with the same angular velocity.
  - Direction of vector: $\vec{e}$.

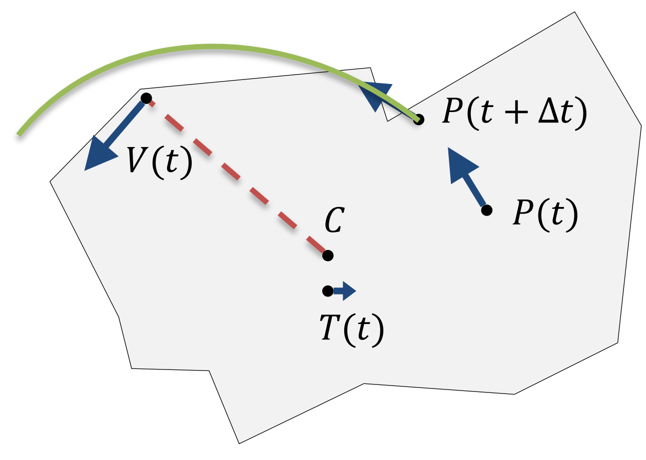


- Tangential velocity vector:

$$\overrightarrow{v^{\perp}} = \vec{\omega} \times \vec{r} = \frac{d\vec{r}}{dt}$$

Or:

$$\vec{\omega} = \frac{\vec{r} \times \vec{v}}{r^2}$$



- $\omega = {}^{v^{\perp}}/_{r}$ (abs. values)

Decomposing Movement

- Define original (object) coordinate system and orientation $\vec{c}(0), \boldsymbol{q}(0)$.
- Position in world coordinates:

$$\vec{x}(t) = \vec{c}(t) + \vec{r}(t) = \vec{c}(t) + \boldsymbol{q}(t)\vec{r}(0)\boldsymbol{q}^{-1}(t).$$

  - $\vec{r}$=$\vec{x}$-$\vec{c}$.
- Combined velocity $\vec{v}(t)$:
  - Linear (translational) component: $\vec{v}^{\parallel}(t) = \frac{d\vec{c}(t)}{dt}$
  - Rotational component: $\vec{v}^{\perp}(t) = \frac{d\vec{r}(t)}{dt} = \vec{\omega}(t)\times\vec{r}(t)$.
- But, are these always distinguishable?
  - Yes!
  - $\vec{r}(t)$ only depends on $\boldsymbol{q}(t)$ and $\vec{r}(0)$ and not on $\vec{c}(t)$.

Combined Movement

- Position of every point

Combined velocity $\vec{v}$

$$\vec{p}(t) = \vec{p}_0 + \int_0^t \boxed{\left(\vec{v}^{\perp}(t) + \vec{v}^{\parallel}(t)\right)} dt$$

- $\vec{v}^{\parallel}(t)$: translation of the object axis system.
- $\vec{v}^{\perp}(t)=\vec{\omega}(t)\times\vec{r}(t)$: obj. axis system rotation around its “origin” $\vec{c}(t)$.
- Question: how does the choice of obj. system (axes+origin) matter?

Invariance of Angular Velocity

- $\vec{v}^{\perp}(t) = \frac{d\vec{r}(t)}{dt} = \vec{\omega}(t) \times \vec{r}(t)$
- What is the angular velocity $\vec{\omega}$ for all points of the object?
- The same! Since $\boldsymbol{q}$ is the same:

$$\frac{d\vec{r}(t)}{dt} = \frac{d\boldsymbol{q}(t)}{dt}\vec{r}(0)\boldsymbol{q}^{-1}(t) + \boldsymbol{q}(t)\vec{r}(0)\frac{d\boldsymbol{q}^{-1}(t)}{dt} =$$

$$\frac{1}{2}(0, \vec{\omega})\vec{r}(t) + \frac{1}{2}\vec{r}(t)(0, -\vec{\omega}) = \big(0, \vec{\omega}(t) \times \vec{r}(t)\big)$$

- How does the choice of center $\vec{c}(0)$ matters for $\vec{\omega}$?
  - Or the choice of $\vec{r}(0)$.
  - And why?
- Note: $\vec{\omega}$ is still represented in a (global, stationary, maybe canonical) axis system!
- Another world system = another representation = “different” $\vec{\omega}$.

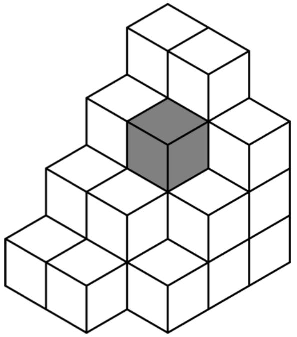
Mass

- The measure of the amount of matter in the volume of an object:

$$m = \int_V \rho \, dV$$

- ρ : the density of each point the object volume V.
- dV: the volume element.

- Equivalently: a measure of resistance to motion or change in motion.



Mass

- For a 3D object, mass is the integral over its volume:

$$m = \int\int\int \rho(x, y, z)\, dx\, dy\, dz$$

- For uniform density (ρ constant):

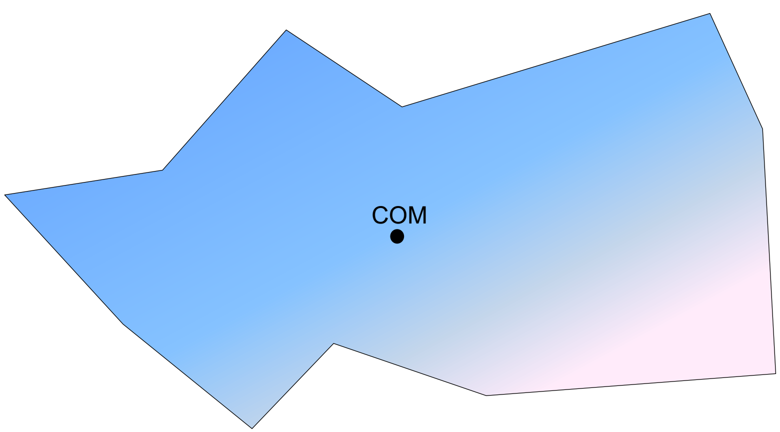
$$m = \rho \cdot V$$

Center of Mass

- The center of mass (COM) is the "average" point of the object, weighted by density:

$$\overrightarrow{COM} = \frac{1}{m} \int_V \rho \cdot \vec{p}\, dV$$

  - p: point coordinates.
- Point of balance for the object.
- Uniform density: COM ⇔ centroid.



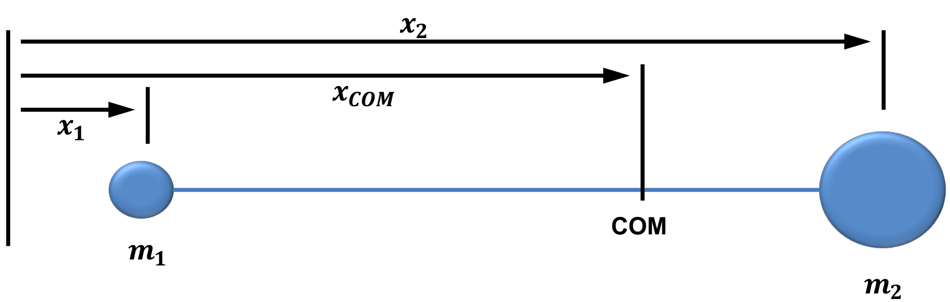
Center of Mass of System

- A system of bodies has a mutual center of mass:

$$COM = \frac{1}{m}\sum_{i=1}^{n} m_i \vec{p}_i$$

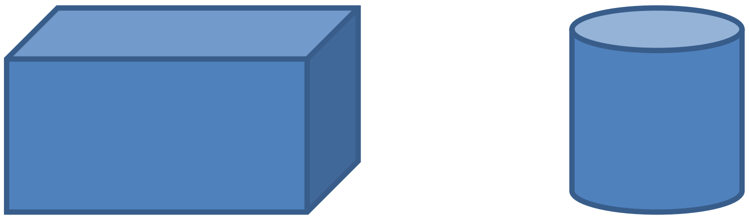
  - m_i: mass of each body.
  - p_i: location of individual COM.
  - $m = \sum_{i=1}^{n} m_i$.
- Example: two spheres in 1D

$$x_{COM} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

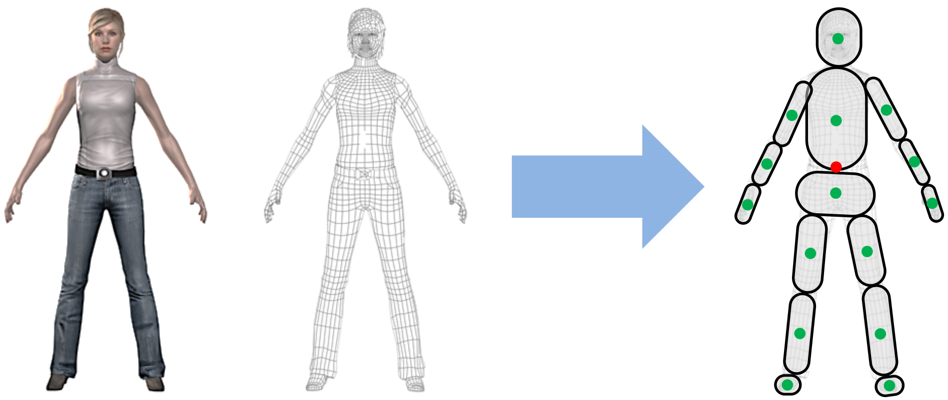


Center of Mass

- Quite easy to determine for primitive shapes



- What about complex surface based models?

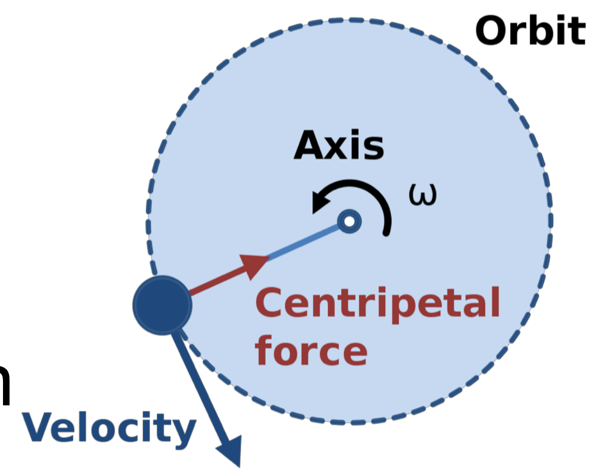


Dynamics

- The centripetal force creates curved motion.

- In the direction of (negative) $\vec{r}$
  - Object is **in orbit.**

- Constant force ⇔ circular rotation with constant tangential velocity.
  - Why?



Tangential & Centripetal Accelerations

- Tangential acceleration $\vec{\alpha}$ holds:

$$\vec{a} = \vec{\alpha} \times \vec{r}$$

- cf. velocity equation $\vec{v} = \vec{\omega} \times \vec{r}$.

- The centripetal acceleration drives the rotational movement:

$$\vec{a}_n = \frac{v^2}{r}\hat{r} = -\omega^2 \vec{r}.$$

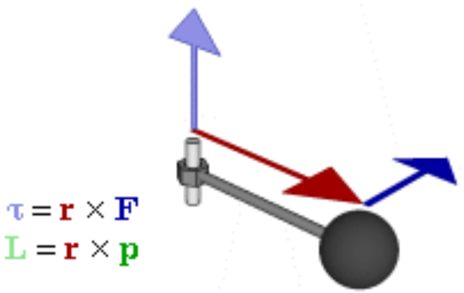
- What is the “centrifugal” force?

Angular Momentum

- Linear motion ➔ linear momentum: $\vec{p} = m\vec{v}$.
- Rotational motion ➔ angular momentum about any fixed relative point (to which $\vec{r}$ is measured):

$$\vec{L} = \int_V (\vec{r} \times \vec{p}) \rho dV$$

  - unit is $N \cdot m \cdot s$
  - $\rho dV = dm$ (mass element)



- Angular momentum is conserved!
  - Just like the linear momentum.
- Caveat: conserved w.r.t. the same axis system.

Angular Momentum

- Plugging in angular velocity:

$$(\vec{r}\times\vec{p})dV = (\vec{r}\times\vec{v})dm = \left(\vec{r}\times(\vec{\omega}\times\vec{r})\right)dm$$

- Integrating, we get:

$$\vec{L} = \int_V \vec{r}\times(\vec{\omega}\times\vec{r})dm$$

- Note: The angular momentum and the angular velocity are not generally collinear!

The inertia Tensor

- Define: $\vec{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $\vec{\omega} = \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$
- For a single rotating body: the angular velocity is constant.
- We get:

$$\vec{L} = \int_V \vec{r}\times(\vec{\omega}\times\vec{r})dm = \int \begin{pmatrix} (y^2+z^2)\omega_x - xy\omega_y - xz\omega_z \\ -yx\omega_x + (z^2+x^2)\omega_y - yz\omega_z \\ -zx\omega_x - zy\omega_y + (x^2+y^2)\omega_z \end{pmatrix} dm =$$

$$\int \begin{pmatrix} (y^2+z^2) & -xy & -xz \\ -yx & (z^2+x^2) & -yz \\ -zx & -zy & (x^2+y^2) \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} dm =$$

$$\begin{bmatrix} \mathbf{I}_{xx} & -\mathbf{I}_{xy} & -\mathbf{I}_{xz} \\ -\mathbf{I}_{yx} & \mathbf{I}_{yy} & -\mathbf{I}_{yz} \\ -\mathbf{I}_{zx} & -\mathbf{I}_{zy} & \mathbf{I}_{zz} \end{bmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}.$$

- Note: replacing integral with a (constant) matrix operating on a vector!

The Inertia Tensor

$$\vec{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

- The inertia tensor I only depends on the geometry of the object and the relative axis system (often, COM with principal axes):

$$\mathbf{I}_{xx} = \int (y^2 + z^2)dm \qquad \mathbf{I}_{xy} = \mathbf{I}_{yx} = \int (xy)dm$$

$$\mathbf{I}_{yy} = \int (z^2 + x^2)dm \qquad \mathbf{I}_{xz} = \mathbf{I}_{zx} = \int (xz)dm$$

$$\mathbf{I}_{zz} = \int (x^2 + y^2)dm \qquad \mathbf{I}_{yz} = \mathbf{I}_{zy} = \int (yz)dm$$

The Inertia Tensor

- Compact form:

$$\mathbf{I} = \begin{bmatrix} \int (y^2 + z^2)dm & -\int (xy)dm & -\int (xz)dm \\ -\int (xy)dm & \int (z^2 + x^2)dm & -\int (yz)dm \\ -\int (xz)dm & -\int (yz)dm & \int (x^2 + y^2)dm \end{bmatrix}$$

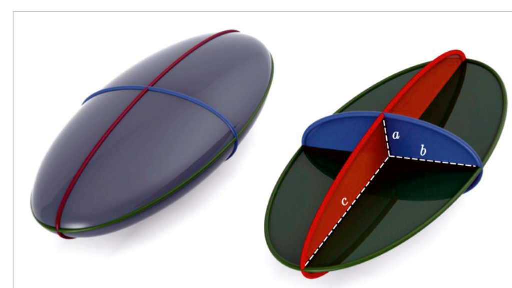
- The diagonal elements are called the (principal) moment of inertia.
- The off-diagonal elements are called products of inertia.

The Inertia Tensor

- Equivalently, we separate mass elements to density and volume elements:

$$\mathbf{I} = \int_V \rho(x,y,z) \begin{bmatrix} y^2+z^2 & -xy & -xz \\ -xy & z^2+x^2 & -yz \\ -xz & -yz & x^2+y^2 \end{bmatrix} dx\, dy\, dz$$

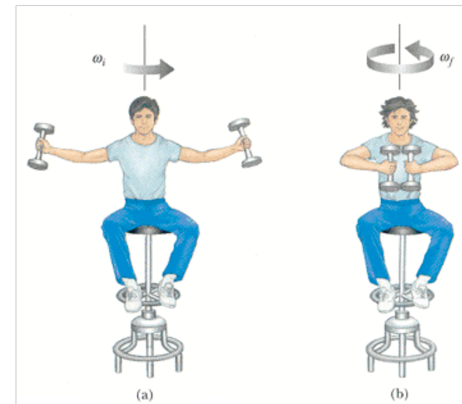
- The diagonal elements: distances to the respective principal axes.
- The non-diagonal elements: products of the perpendicular distances to the respective planes.



Moment of Inertia

- The moment of inertia $\mathbf{I}_e$, with respect to a rotation axis $\vec{e}$, measures how much the mass "spreads out":

$$\mathbf{I}_e = \int_V (r_e)^2 dm$$



  - r_e: perpendicular distance to axis.
  - Through the central rotation origin point.

- Measures ability to resist change in rotational motion.
  - The angular equivalent to mass!

Moment and Tensor

- We have: $(r_e)^2 = |(\vec{e} \times \vec{r})|^2$ for any point $\vec{q}$.
  - Remember: $\vec{r}$ is origin→ $\vec{q}$.
  - Then, $\vec{e} \times \vec{r}$ is closest point on axis → $\vec{q}$.
- We get:

$$\mathbf{I}_e = \int_M |(\vec{e} \times \vec{r})|^2 dm = \vec{e}^T \mathbf{I} \vec{e}$$

- The scalar angular momentum around the axis is then $L_e = \mathbf{I}_e \omega_e$.
  - ω_e is the angular speed around $\vec{e}$.
- Reducible to a planar problem (axis as Z axis).

Change of coordinates

- Suppose we have $\mathbf{I}$, $\vec{\omega}$ in some system.
- How do we find them in a rotated system R?
- $\vec{\omega}_R = R^T\vec{\omega}$.
  - Just a rotation of the axis!

Note: identity matrix $I_{3\times 3}$, not $\mathbf{I}$!

- Insight: $\mathbf{I} = \int_V [|\vec{r}|^2 I_{3\times 3} - \vec{r}\cdot\vec{r}^T]dm$
- As $\vec{r}_R = R^T\vec{r}$:

$$I_R = |\vec{r}|^2 I_{3\times 3} - R^T\vec{r}\cdot\vec{r}^T R =$$
$$= R^T(|\vec{r}|^2 I_{3\times 3} - \vec{r}\cdot\vec{r}^T)R = R^T\mathbf{I}R.$$

- Angular momentum is then:

$$L_R = \mathbf{I}_R\vec{\omega}_R = R^T\mathbf{I}RR^T\vec{\omega} = R^T L$$

- Just the rotated vector!

Warning

- We said that angular velocity is the same regardless of the chosen (moving) center in the object axis system.
- But that is the angular velocity around this center.
- It does not mean that it is the same as the angular velocity around the origin of the world!
- Angular velocity appears even without “rotation”.
- Always know who $\vec{r}$ and $\vec{v}$ are measured against.
  - More geometrically: who is the axis of rotation.
  - Look at chalkboard.

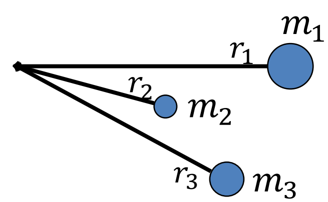
Moment of Inertia

- For a mass point:

  $$I = m \cdot {r_u}^2$$

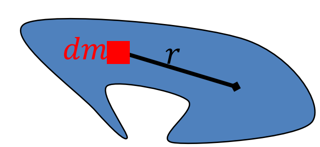
- For a collection of mass points:

  $$I = \sum_i m_i {r_i}^2$$



- For a continuous mass distribution on the plane:

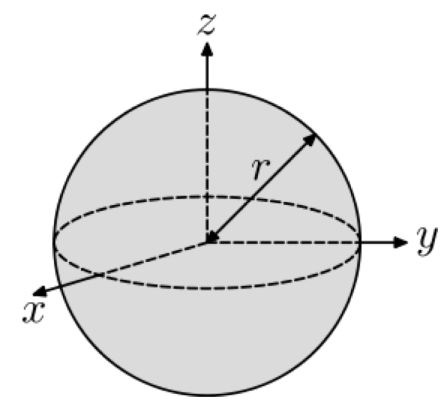
  $$I = \int_M {r_u}^2 \, dm$$



Inertia of Primitive Shapes

- For primitive shapes, the inertia can be expressed with the parameters of the shape

- Illustration on a solid sphere
  - Calculating inertia by integration of thin discs along one axis (*e.g.* z).
  - Surface equation: $x^2 + y^2 + z^2 = R^2$



Inertia of Primitive Shapes

- Distance to axis of rotation is the radius of the disc at the cross section along z: $r^2 = x^2 + y^2 = R^2 - z^2$.

- Summing moments of inertia of small cylinders of inertia $I_Z = \frac{r^2 m}{2}$ along the z-axis:

$$d\mathbf{I}_Z = \frac{1}{2} r^2 dm = \frac{1}{2} r^2 \rho dV = \frac{1}{2} r^2 \rho \pi r^2 dz$$

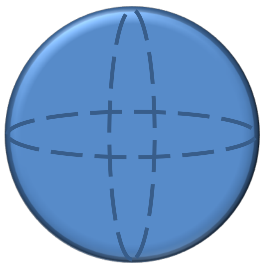
- We get:

$$\mathbf{I}_Z = \frac{1}{2}\rho\pi \int_{-R}^{R} r^4 dz = \frac{1}{2}\rho\pi \int_{-R}^{R} (R^2 - z^2)^2 dz = \frac{1}{2}\rho\pi [R^4 z - 2R^2 z^3/3 + z^5/5]_{-R}^{R} = \rho\pi(1 - 2/3 + 1/5)R^5.$$

- As $m = \rho(4/3)\pi R^3$, we finally obtain: $\mathbf{I}_Z = \frac{2}{5} m R^2$.

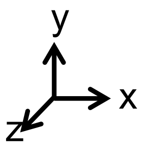
Inertia of Primitive Shapes

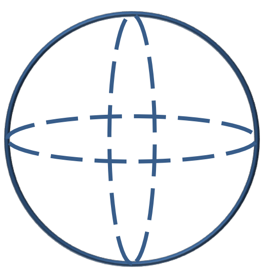
- Solid sphere, radius r and mass m:



$$\mathbf{I} = \begin{bmatrix} \frac{2}{5}mr^2 & 0 & 0 \\ 0 & \frac{2}{5}mr^2 & 0 \\ 0 & 0 & \frac{2}{5}mr^2 \end{bmatrix}$$

- Hollow sphere, radius r and mass m:

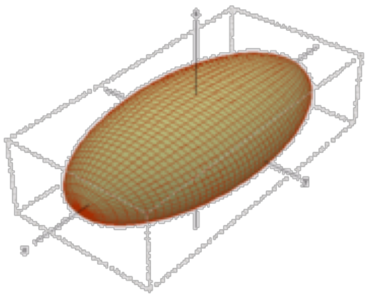




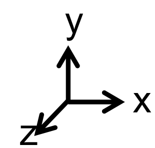
$$\mathbf{I} = \begin{bmatrix} \frac{2}{3}mr^2 & 0 & 0 \\ 0 & \frac{2}{3}mr^2 & 0 \\ 0 & 0 & \frac{2}{3}mr^2 \end{bmatrix}$$

Inertia of Primitive Shapes

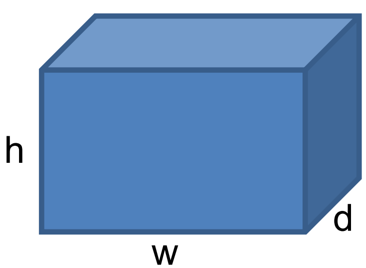
- Solid ellipsoid, semi-axes a, b, c and mass m:



$$\mathbf{I} = \begin{bmatrix} \frac{1}{5}m(b^2+c^2) & 0 & 0 \\ 0 & \frac{1}{5}m(a^2+c^2) & 0 \\ 0 & 0 & \frac{1}{5}m(a^2+b^2) \end{bmatrix}$$



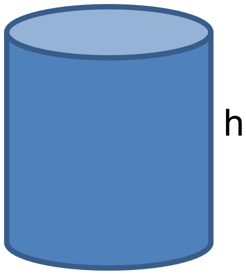
- Solid box, width w, height h, depth d and mass m:



$$\mathbf{I} = \begin{bmatrix} \frac{1}{12}m(h^2+d^2) & 0 & 0 \\ 0 & \frac{1}{12}m(w^2+d^2) & 0 \\ 0 & 0 & \frac{1}{12}m(w^2+h^2) \end{bmatrix}$$

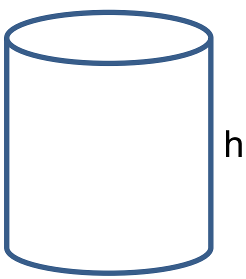
Inertia of Primitive Shapes

- Solid cylinder, radius r, height h and mass m:



$$\mathbf{I} = \begin{bmatrix} \frac{1}{12}m(3r^2+h^2) & 0 & 0 \\ 0 & \frac{1}{12}m(3r^2+h^2) & 0 \\ 0 & 0 & \frac{1}{2}mr^2 \end{bmatrix}$$

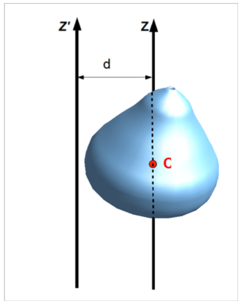
- Hollow cylinder, radius r, height h and mass m:



$$\mathbf{I} = \begin{bmatrix} \frac{1}{12}m(6r^2+h^2) & 0 & 0 \\ 0 & \frac{1}{12}m(6r^2+h^2) & 0 \\ 0 & 0 & mr^2 \end{bmatrix}$$

Parallel-Axis Theorem

- We can compute the moment of inertia around an axis $\vec{z}$ going through the COM.
- How to efficiently calculate it for any parallel axis $\vec{z}'$?
- parallel axis theorem:



$$\mathbf{I}_{z'} = \mathbf{I}_z + md^2$$

  - d is the distance between the axes.
- Summary: it is easy to find the moment of inertia for every axis and center from the "built-in" one.

Parallel-Axis Theorem

- More generally, for point displacements: (d_x, d_y, d_z)

$$I_{xx} = \int (y^2 + z^2) dm + m{d_x}^2 \qquad I_{xy} = \int (xy) dm + m d_x d_y$$

$$I_{yy} = \int (z^2 + x^2) dm + m{d_y}^2 \qquad I_{xz} = \int (xz) dm + m d_x d_z$$

$$I_{zz} = \int (x^2 + y^2) dm + m{d_z}^2 \qquad I_{yz} = \int (yz) dm + m d_y d_z$$

Torque

- A force $\vec{F}$ applied at a distance r from the origin.

- Tangential part causes tangential acceleration:

$$\vec{F}_{\perp} = m \cdot \vec{a}_{\perp}$$

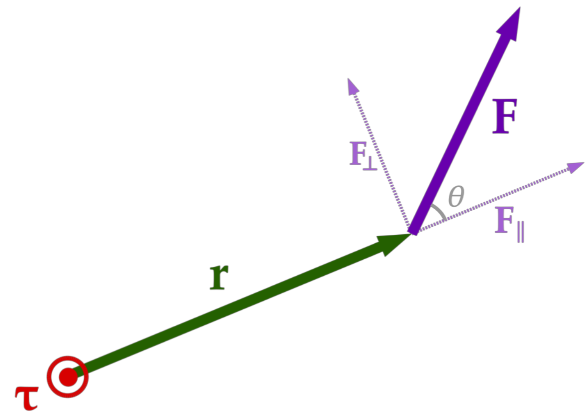
- The torque $\vec{\tau}$ is defined as:

$$\vec{\tau} = \vec{r} \times \vec{F}$$

- So we get

$$\tau = m \cdot (r \cdot \alpha) \cdot r = mr^2\alpha.$$

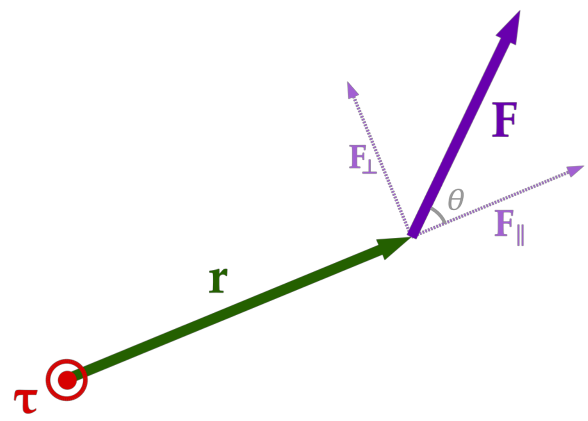
  - unit is $N \cdot m$
  - Induces the rotation of the system.



Newton's Second Law

- The law $\vec{F} = m \cdot \vec{a}$ has an equivalent formulation with the inertia tensor and torque:

$$\vec{\tau} = \mathbf{I}\vec{\alpha}$$



- Force ⇔ linear acceleration
- Torque ⇔ angular acceleration

Torque and Angular Momentum

- Reminder: linear: $\vec{F} = \frac{d\vec{p}}{dt}$ ($\vec{p}$: linear momentum).
- Similarly with torque and angular momentum:

$$\frac{d\vec{L}}{dt} = \frac{d\vec{r}}{dt}\times\vec{p} + \vec{r}\times\frac{d\vec{p}}{dt} = \vec{v}\times m\vec{v} + \vec{r}\times\vec{F} = 0 + \vec{\tau}$$

- Momentum-torque relation:

$$\frac{d\vec{L}}{dt} = \frac{d(\mathbf{I}\vec{\omega})}{dt} = \mathbf{I}\frac{d\vec{\omega}}{dt} = \mathbf{I}\vec{\alpha} = \vec{\tau}$$

- Force ⇔ derivative of linear momentum.

- Torque ⇔ derivative of angular momentum.

Rotational Kinetic Energy

- Translating energy formulas to rotational motion.

- The rotational kinetic energy is defined as:

$$E_{Kr} = \frac{1}{2}\vec{\omega}^T \cdot \mathbf{I} \cdot \vec{\omega}$$

Conservation of Mechanical Energy

- Adding rotational kinetic energy

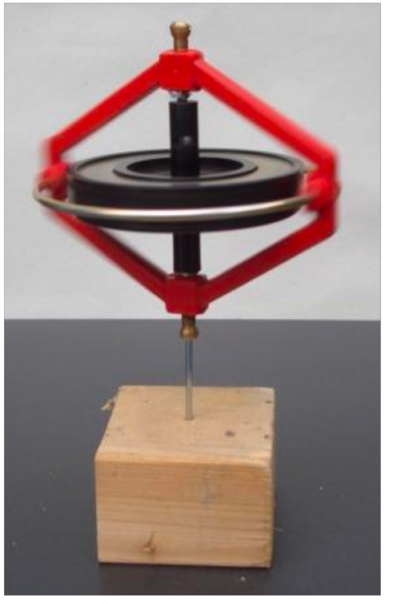
$$E_{Kt}(t+\Delta t) + E_P(t+\Delta t) + E_{Kr}(t+\Delta t)$$
$$= E_{Kt}(t) + E_P(t) + E_{Kr}(t) + E_O$$

- E_{Kt} is the translational kinetic energy.
- E_P is the potential energy.
- E_{Kr} is the rotational kinetic energy.
- E_O the “lost” energies (surface friction, air resistance etc.).

Torque Impulse

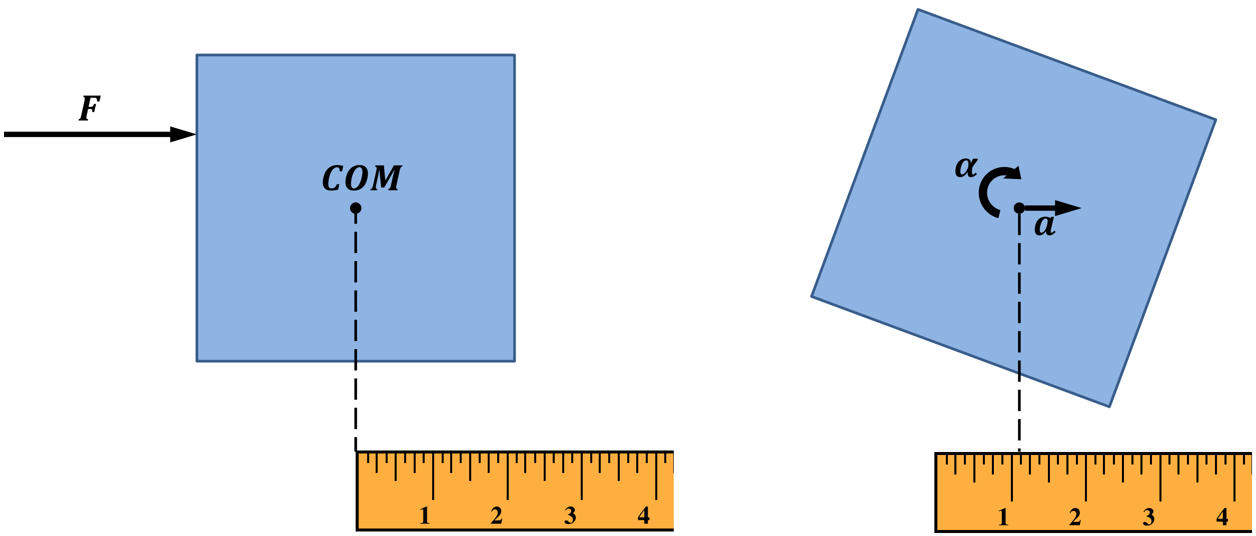
- We may apply off-center forces for a very short amount of time.
  - Or as a collision.
- torque impulse → instantaneous change in angular momentum, *i.e.* in angular velocity:

$$\tau\Delta t = \Delta L$$



Rigid Body Forces

- A force can be applied anywhere on the object, producing also a rotational motion.
- Q: how come two non-rotating objects can cause a rotation?



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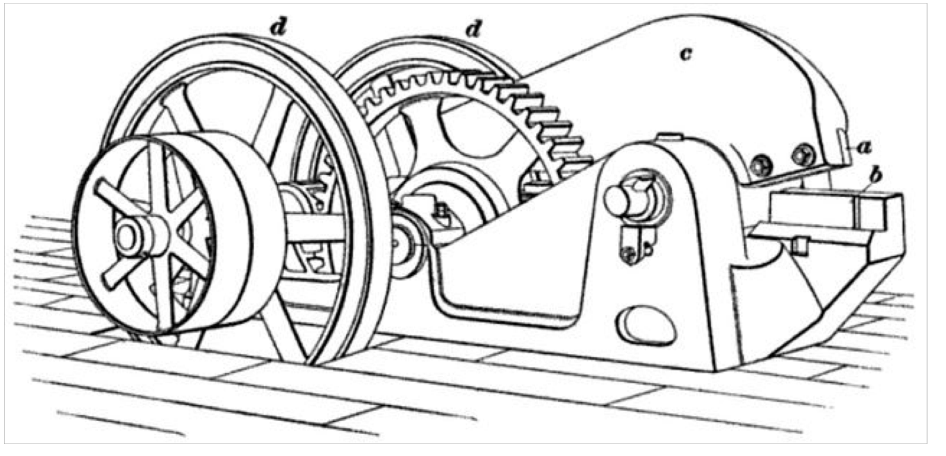
Complex Objects

- When an object consists of multiple primitive shapes:
  - Calculate the individual moment inertia of each shape around a the prescribed axis in the same coordinates system, and their individual origins.
  - Use parallel axis theorem to transform to inertia to unified object coordinates.
  - Add the moments together.



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