Laws of Valid Reasoning

Traditional “Laws of Thought”

- Concept that rational thought is constrained to follow certain “laws”
- Three traditional “laws” of thought date to Plato, Aristotle, and others
  - **Identity**: Every object is equal to itself
  - **Non-contradiction**: No proposition is both true and not true
  - **Excluded middle**: Every proposition is either true or not true
- Formal expressions of these laws
  - **Identity**: $\forall x, x = x$. (“For any object *x, x* is equal to itself”)
  - **Non-contradiction:** $\forall x, \neg(x \land \neg x)$. (“For any *proposition*, x, x is either true or not true”)
  - **Excluded middle:** $\forall x, x \lor \neg x$. (For any proposition, P, P is either true or false.)
- These are understood as laws of rational reasoning

The basic idea

- Laws of valid rational thought have to be logically valid
- They have to hold true *independent of any interpretation*
    - E.g., equality: *no matter what x is*, it's equal to itself
    - E.g., non-contradiction: *no matter what proposition x is,* it can't be that it's both true and false
    - E.g., excluded middle: *no matter what proposition x is,* it is either true or it is false
- The logical ∀ (*for all) quantifier* is used to assert a *universal generalization*
    - It gives a name, here x, to some *arbitrary but specific* object
    - It then asserts that some proposition (e.g., x = x) is true of *any* such x
    - E.g., translate to English: $\forall$ *(n : integer), even n* $\lor$ *odd n*
        - We see here the concept of *typed* variables (here n as to be an integer)
        - We also see the concept of a predicate: (even n) is the proposition that *n is even*
- Quantifiers and predicates are central in *predicate (vs propositional) logic*

Inference Rules

- Inference rules are valid rules of deductive reasoning
- Today we recognize a broader range of such rules
- An inference rule is of the form
    - *if you accept propositions, …, to be true,*
    - *then you can deduce that the proposition, …, is true*
- For example,
    - if you accept that x is true, and you accept that y is true,
    - then you can validly deduce that $(x \land y)$ is true
- We can write this as follows: $x, y \vdash x \land y$
    - $\vdash$, the *turnstile* character, is pronounced *entails*
    - *So: x (being true) and y (being true) entails x ∧ y (being true)*
    - If you know or assume x : true, and y : true, you can deduce *x ∧ y* : true
    - The notation, x : true, represents what we will call a *truth judgment*

Natural deduction

- Natural deduction is the process of reasoning about the truth, or not, of a given proposition by deduction using the rules of inference of a given logic
- For the logics we'll study, these basic rules are organized around connectives
- Suppose * is any logical operator, such as *and, or, not,* or *implies*
- If we want to deduce the truth of a proposition with * as its main connective then we use an inference rule called an *introduction rule* for *
  - Example, $x, y \vdash x \land y$ is the introduction rule for $\land$. We will call this rule $\land$*-introduction*
- If we want to decide the truth of some proposition *from* (or *given the truth of*) a proposition with * as its main connective, then we use an *elimination* rule for *
  - Example/exercise: there are two elimination rules for $\land$. Complete their statements here:
    - $x \land y \vdash$ ___ ($\land$*-elimination-left)*
    - $x \land y \vdash$ ___ ($\land$*-elimination-right)*

Display (vertical) notation for inference rules

$$\frac{x, \quad y}{x \wedge y} \qquad \textit{(∧ introduction)}$$

$$\frac{x \wedge y}{x} \qquad \textit{(∧ elimination left)}$$

$$\frac{x \wedge y}{y} \qquad \textit{(∧ elimination right)}$$

Looking ahead: a theorem to prove

To prove: $\land$ *is associative*. What this claim means is that, for any propositions X, Y, and Z, the proposition, *$(X \land Y) \land Z$, is equivalent to $X \land (Y \land Z)$. Note the difference in grouping*. By *equivalent to*, we mean that each implies the other. A full proof is built from two smaller proofs, one in each direction. We will prove the forward direction: *$(X \land Y) \land Z \rightarrow X \land (Y \land Z)$*. Bear in mind that this proposition asserts that *if* *$(X \land Y) \land Z$* is accepted as true, then *from* this fact one can deduce that *$X \land (Y \land Z)$* must also be true. Notably it does not assert that *$(X \land Y) \land Z$ is true*.

So how can we get from assuming that *$(X \land Y) \land Z$ is true* to showing that, in the context of this assumption, *$X \land (Y \land Z)$ must be true?* Let’s see.

Natural Deduction Proof

Proof: We need to show that <u>*if*</u> $(X \wedge Y) \wedge Z$ is true, then so is $X \wedge (Y \wedge Z)$. *We'll start by <u>assuming</u> that $(X \wedge Y) \wedge Z$ is true. Now let's see what we can deduce with this assumption in our context.*

1. $\underline{(X \wedge Y) \wedge Z} \vdash (X \wedge Y)$ and-elim left gives us $X \wedge Y$: *true*
2. $(X \wedge Y) \wedge Z \vdash Z$ and-elim right gives us Z : *true*
3. $(X \wedge Y) \vdash X$ and-elim left gives us X : *true*
4. $(X \wedge Y) \vdash Y$ and-elim right gives us Y : *true*
5. Y, Z $\vdash (Y \wedge Z)$ and-intro applied to Y, Z
6. X, (Y ∧ Z) $\vdash \underline{X \wedge (Y \wedge Z)}$ and-intro. *QED ("yay!" in mathspeak)*

English language proof

The great thing about an English proof (*proof summary* would be more accurate) is we can skip details that we can reasonably assume readers will understand. Here's an example.

Prove: For any propositions, X, Y, and Z, $(X \land Y) \land Z \to X \land (Y \land Z)$

Proof: To start ,we'll <u>assume</u> (X ∧ Y) ∧ Z. From this we can deduce X, Y, and Z (by applying and-elimination several times). From Y, and Z, we can deduce (Y ∧ Z). (by and-introduction). Next, from X and (Y ∧ Z), we can deduce X ∧ (Y ∧ Z). And so what we have shown is that <u>if</u> (X ∧ Y) ∧ Z is true, *then* so must be X ∧ (Y ∧ Z), which is what we mean by $(X \land Y) \land Z \to X \land (Y \land Z)$. QED.

A bit more reasoning

- On the last pages, we proved a universal generalization: for *any propositions, X, Y,* and *Z,* it is the case that $(X \land Y) \land Z \rightarrow X \land (Y \land Z)$.
- We can write this more concisely as: $\forall$ (X Y Z), $(X \land Y) \land Z \rightarrow X \land (Y \land Z)$
- This is a shorthand for $\forall$ X, $\forall$ Y, $\forall$ Z, $(X \land Y) \land Z \rightarrow X \land (Y \land Z)$
- Clearly the converse is also true: $\forall$ (X Y Z), $X \land (Y \land Z) \rightarrow (X \land Y) \land Z$.
- We can now apply (new information!) the bi-implication introduction rule to conclude $\forall$ (X Y Z), $(X \land Y) \land Z \leftrightarrow X \land (Y \land Z)$. NB: the arrow goes both ways.
- What we have proved is that *no matter what propositions X, Y, and Z are, the two expressions are equivalent: you can always move parentheses to group elements connected by $\land$ without changing the meaning of the expression.*

The power of theorems!

- We now have that ∀ (X Y Z), (*X ∧ Y) ∧ Z* ↔ *X ∧ (Y ∧ Z)* is a ***theorem***
- A theorem is a mathematical statement *that has been proved to be valid*
- The theorem we've proved here shows that *∧ is associative*
  - Theorems are given names to we can refer to them later. Let's call this *∧-associative*
  - Note: Any operator, *, is said to be associative iff (X * Y) * Z = *X * (Y * Z)*
  - Note: we're conflating equivalence of propositions (↔) and equality (=) here – a detail
- Key point: We can now apply a theorem as if it were a new inference rule
  - Suppose we know (SkyBlue *∧ WaterGreen) ∧ EarthBrown*
  - Suppose we want to prove SkyBlue ∧ (*WaterGreen ∧ EarthBrown)*
  - All we have to say is that this is true by applying the theorem that states that *∧ is associative*
- Every time we prove a theorem, it adds a new tool to our reasoning toolkit

Why Natural Deduction Proofs When We Can Use Z3?

- Suppose that X, Y, and Z, are propositional (Boolean) variables
- Now proving $(X \land Y) \land Z \leftrightarrow X \land (Y \land Z)$ means showing that it is *valid*
- We know that we can use Z3 as a SAT solver to check such problems for
  - Satisfiability – there is *at least one* interpretation that makes the formula true (a model)
  - Unsatisfiability – that *no* interpretation makes the formula true
- You can easily check for *validity* (all interpretations are models) by checking to see if the *negation* of a formula is _________. Fill in the blank.
- The logics of SAT and SMT solvers are limited in what they can express
- There are no general-purpose “solvers” for finding models or proving the validity of arbitrary propositions (otherwise we wouldn’t need mathematicians)
- It’s in this context of more expressive logics that proof by natural deduction becomes essential