Quantum Physics

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1 Linear algebra

Physical quantities can sometimes be discrete, not only continuous.

2 Kets and wave function

Linear algebra and vector spaces are all about structure and patterns, not about the type of objects being used. A particle is represented by a vector in a vector space. A vector in this vector space represent a quantum state. A quantum state is a mathematical object that holds all the physical properties of a particle.
Quantum superposition,

$$|\psi\rangle = \sum c_i \, |E_i\rangle \tag{1}$$

Here, $|\psi\rangle$ is the quantum state of the particle, which is a linear combination of the states ($|E_i\rangle$. The number of states can be infinite. This issue is solved by *Hilbert space* ($\mathcal{H}$).
Although some positions can be more or less likely, position, as a whole, is not discrete (as far as we know). Let's say, the quantum state of a particle is the linear combination of position kets. As position is continuous, the linear combination is the integral from $-\infty$ to $+\infty$.

$$|\psi\rangle = \int_{-\infty}^{+\infty} dx \cdot \psi(x) \cdot |x\rangle \tag{2}$$

So, a wavefunction is a continuous list of coefficients whenever the list of kets in infinite.

3 Hilbert space

We know that, quantum state of a particle is the linear combination of its states where the coefficient function is the wave function of the particle.

$$|\psi\rangle = \sum_{i=1}^{n} c_i \cdot |E_i\rangle \tag{3}$$

But this n can be infinite as well. If n is infinite, things get problematic.

3.1 Problem with infinite sum

Let's consider the basis vector to be the set of all polynomials $1, x, x^2, x^3, \ldots$. Now, let's consider an infinite linear combination of elements of this set (i.e., infinite linear combination of the basis vectors), $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$, which clearly is e^x.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \tag{4}$$

But the problem is, this e^x is outside of the set of basis vectors that constitute it. So, infinitely combining the basis vectors linearly gives us something outside of the set of basis vectors. So, if our quantum state ($|\psi(x)\rangle$) is an infinite superposition of outcome states ($\{|E_i\rangle\}$ or $\{|P_i\rangle\}$), then there's a chance that this quantum state is outside of our vector space, therefore, not a quantum state at all.

3.2 Solution

Let's apply an extra following rule to our vector space,

Every convergent sum of vectors must converge to an element inside of the vector space.

If this rule is applied (along with all the rules of a vector space), then the resulting space is called a *Hilbert* space $\mathcal{H}$. The rigorous definition of a *Hilbert* space is,

Hilbert Space: *A vector space equipped with an* inner product *that is* Cauchy complete.

Cauchy completeness (or complete metric space) is defined as follows,

Every convergent sequence (or sum) of vectors (e.g., partial sums of infinite linear combination) converges to an element inside of the vector space

So,

$$|\psi\rangle \in \mathcal{H} \tag{5}$$

For the time being, inner product can be thought of as being the dot product.

4 Inner product

At its core, an *inner product* takes in two vectors and outputs a number which might be complex,

$$\langle\psi|\phi\rangle = c$$

$\langle\psi|\phi\rangle$ is the inner product of kets. There are a few rules for kets,

$$\begin{aligned}
|\phi\rangle + |\psi\rangle &= |\phi + \psi\rangle \\
a\,|\psi\rangle &= |a\psi\rangle \\
\langle\psi|\phi + \zeta\rangle &= \langle\psi|\phi\rangle + \langle\psi|\zeta\rangle \\
\langle\phi|a\psi\rangle &= a\,\langle\phi|\psi\rangle
\end{aligned}$$

The last two rules state that, the inner product should be linear in the right hand position. Magnitude of a ket is $|\langle\phi|\phi\rangle|$, and it is *positive*.
All the rules stated above are intuitive from dot product. But the *commutative* law does not hold for the inner product, which can be proved through contradiction.
Let's assume that the inner product is indeed commutative and the magnitude is positive.

$$
\begin{aligned}
\langle i\phi|i\phi\rangle &= i\,\langle i\phi|\phi\rangle = i\,\langle\phi|i\phi\rangle \\
&= i^2\,\langle\phi|\phi\rangle = -\,\langle\phi|\phi\rangle
\end{aligned}
$$

Which results in the magnitude of the ket $||i\phi\rangle| = i||\phi\rangle|$. This cannot happen (apparantly, the magnitude should not be complex). That is why, the inner product is not communtative. Rather the following rule applies,

$$
\langle\psi|\phi\rangle = \langle\phi|\psi\rangle^*
$$

which solves the previous problem.

One final rule is, only the zero vector has zero length. So, the inner product definition is as follows,

An inner product $\langle\psi|\phi\rangle$ is a map from vectors to scalars which satisfies the following rules,

$$
\begin{aligned}
\langle\psi|\zeta+\phi\rangle &= \langle\psi|\zeta\rangle + \langle\psi|\phi\rangle \\
\langle\psi|a\phi\rangle &= a\,\langle\psi|\phi\rangle \\
\langle\psi|\phi\rangle &= \langle\phi|\psi\rangle^* \\
\text{for } |\psi\rangle \neq 0, &\langle\psi|\psi\rangle > 0
\end{aligned}
$$