Linear Partial Differential Equations

Aheer Srabon

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1 Syllabus

1.1 Description

This course provides students with the basic analytical and computational tools of linear partial differential equations (PDEs) for practical applications in science engineering, including heat/diffusion, wave, and Poisson equations.
Analytics emphasize the viewpoint of linear algebra and the analogy with finite matrix problems including operator adjoints and eigenproblems, series solutions, Green's functions, and separation of variables.
Numerics focus on finite-difference and finite-element techniques to reduce PDEs to matrix problems, including stability and convergence analysis and implicit/explicit time-stepping.
Julia programming language is introduced and used in homework for simple examples. Julia is a high-level, high-performance dynamic language for technical computing, with syntax that is familiar to users of other technical computing environments. It provides a sophisticated compiler, distributed parallel execution, numerical accuracy, and an extensive mathematical function library.

1.2 Recommended books

- Computational Science and Engineering by Strang, Gilbert.
- Introduction to Partial Differential Equations by Olver, Peter.

1.3 Lecture plan

- Overview of linear PDEs and analogies with matrix algebra
- Poisson's equation and eignefunctions in 1D: Fourier sine series
- Finite difference methods and accuracy
- Discrete vs. continuous Laplacians: Symmetry and dot products
- Diagonalizability of infinite-dimensional Hermitian operators
- Start with a truly discrete (finite dimensional) system, and then derive the continuum PDE model as a limit or approximation
- Start in 1D with the "Sturm-Liouville operator", generalize Sturm-Liouville operators to multiple dimensions

- Music and wave equations, separation of variables, in time and space
- Separation of variables in cylindrical geometries: Bessel functions
- General Dirichlet and Neumann boundary conditions
- Multidimensional finite differences
- Kronecker products
- The min-max theorem
- Green's functions with Dirichlet boundaries
- Reciprocity and positivity of Green's functions
- Delta functions and distributions
- Green's function of Δ^2 in 3D for infinite space, the method of images
- The method of images, interfaces, and surface integral equations
- Green's functions in inhomogeneous media: Integral equations and Born approximations
- Dipole sources and approximations, Overview of time-dependent problems
- Time-stepping and stability: Definitions, Lax equivalence
- Von Neumann analysis and the heat equation
- Algebraic properties of wave equations and unitary time evolution, Conservation of energy in a stretched string
- Staggered discretizations of wave equations
- Traveling waves: D'Alembert's solution
- Group-velocity derivation and dispersion.
- Material dispersion and convolutions
- General topic of waveguides, Superposition of modes, Evanscent modes
- Waveguide modes, Reduced eigenproblem
- Guidance, reflection, and refraction at interfaces between regions with different wave speeds
- Numerical examples of total internal reflection
- Perfectly matched layers (PML)
- Perturbation theory and Hellman-Feynman theorem
- Finite element methods: Introduction
- Galerkin discretization
- Convergence proof for the finite-element method, Boundary conditions and the finite-element method
- Finite-element software
- Symmetry and linear PDEs

2 Lecture notes

2.1 L 01 - Overview of linear PDEs and analogies with matrix algebra

A few important PDEs are,

- Poisson's equation
- Laplace's equation
- Heat/diffusion equation
- Scalar wave equation

and many many others ...

- Maxwell (electromagnetism)
- Navier-Stokes (fluids)
- Schrodinger (quantum mechanics)
- Black-Scholes

	constant coefficients = 1	variable coefficients = c($\mathbf{x}$)
Poisson's equation:	$\Delta^2 u = f$ **example:** f = charge density, u = - electric potential **example:** f = heat source / sink rate u = steady-state temperature **example:** f = solute source / sink rate u = steady state concentration **example:** $f \sim$ force on stretched string/drum u = steady-state displacement	$\Delta \cdot (c\Delta u) = f$ c = permittivity ε c = thermal conductivity c = diffusion coefficient $c \sim$ "springy-new"
Laplace's equation:	$\Delta^2 u = 0$ **examples:** as for Poisson, but no sources	$\Delta \cdot (c\Delta u) = 0$
Heat / diffusion equation:	$\frac{\partial u}{\partial t} = \Delta^2 u$ **examples:** u = temperature u = solute concentration	$\frac{\partial^2 u}{\partial t^2} = \Delta \cdot (c\Delta u)$ c = thermal conductivity c = diffusion coefficient
Scalar wave equation:	$\frac{\partial^2 u}{\partial t^2} = \Delta^2 u$ **examples:** u = displacement of stretched string / drum u= density of gas / fluid	$\frac{\partial^2 u}{\partial t^2} = \Delta \cdot (c\Delta u)$ c^2 = 1 / wave speed

Difference between 18.06 and 18.303 is outlined. As I didn't take 18.06, I wouldn't know.

2.2 L 02 - Poisson's equation and eignefunctions in 1D: Fourier sine series

2.2.1 Orthogonality of two functions

Two real functions $f(x)$ and $g(x)$ are orthogonal if for any interval (a, b) in both of their domain, the following is true,

$$\int_a^b f(x)g(x)\,dx = 0 \tag{1}$$

The dot product of two functions is defined as,

$$f(x) \cdot g(x) = \int_{-\infty}^{\infty} f(x)g(x)\,dx \tag{2}$$

If the functions are complex functions, the dot product (or inner product) of the two functions are defined as,

$$\langle f|g\rangle = \int_{-\infty}^{-\infty} \overline{f(x)}g(x)\,dx \tag{3}$$

Equations 2 and 3 need not to be from $-\infty$ to ∞. The limit can be any interval that is being considered. It can be shown than for two sine function $sin(n\pi x)$ and $sin(m\pi x)$, they are orthogonal if $n \neq m$. They are not orthogonal otherwise.

2.2.2 Fourier sine series

The Fourier sine series for a function $f(x)$ defined on $x \in [0, 1]$ writes $f(x)$ as,

$$f(x) = \sum_{n=1}^{\infty} b_n sin(n\pi x) \tag{4}$$

b_n can be computed by multiplying both sides of equation 4 by $sin(m\pi x)$ and then integrating both sides from 0 to 1.

$$b_m = 2\int_0^1 f(x)sin(m\pi x)\,dx \tag{5}$$

Fourier claimed (without proof) that *any* function $f(x)$ can be expanded in terms of sines in this way, even discontinuous functions! That is, these sine functions form an *orthogonal basis* for "all" functions! But this turned out to be false. Rather Carleson (1966) and Hunt (1968) proved that: any function $f(x)$ where $\int(|f(x)|)^p\,dx$ is finite for some $p > 1$ has a Fourier series that converges *almost* everywhere to $f(x)$ (except at isolated points). At points where $f(x)$ has a jump discontinuity, the Fourier series converges to the midpoint of the jump. Except for crazy *divergent functions* or the function values exactly at points of discontinuity, Fourier's remarkable claim is essentially true.
for $f(x) = 1$,

$$b_n = \begin{cases} \frac{4}{n\pi} & \text{n odd} \\ 0 & \text{n even} \end{cases} \tag{6}$$

For triangular function $f(x) = \frac{1}{2} - |x - \frac{1}{2}|$, $b_{m,\text{odd}} = \frac{4}{(m\pi)^2}(-1)^{\frac{m-1}{2}}$. And the Fourier sine series of that triangular function is,

$$f(x) = \frac{4}{\pi^2} sin(\pi x) - \frac{4}{(3\pi)^2} sin(3\pi x) + \frac{4}{(5\pi)^2} sin(5\pi x) + ... \tag{7}$$