TOPIC:- INVERSE LAPLACE TRANSFORMATION USING CONVOLUTION THEOREM.

The Inverse Laplace Transform using the Convolution Theorem is a powerful method to find the inverse Laplace transform of the product of two Laplace transforms. Here's a breakdown of the process:

Laplace Transform Basics

- *If you have a function $f(t)$, its Laplace transform is:-*
  $L\{f(t)\}=F(s)=\int_0^{\infty} f(t)\, e^{-st} dt$
- *To reverse this process (i.e., to find $F(s)$= from $f(t)$), you need to compute the Inverse Laplace Transform:*

$L^{-1}\{F(s)\} = f(t)$.

Convolution Theorem:-

The convolution theorem in Laplace transforms states that the inverse Laplace transform of the product of two functions in the Laplace domain is equal to the convolution of their inverse Laplace transforms in the time domain.

The convolution theorem is a powerful tool for solving inverse Laplace transforms of complex functions. It is often used in engineering applications.

To find the inverse laplace transform using convolution theorem :-

Steps:-

1)Split given F(s) into 2 parts $F_1(s)$and $F_2(s)$.

2)Take the L^{-1} $Of\, F_1(s)$and $F_2(s)$.

Then,

$L^{-1}\{F_1(s)\} = F_1(t)$.

$L^{-1}\{F_2(s)\} = F_2(t)$.

3)Using the formula of covolution theorem:-

$$L^{-1}\{F_1(s) * F_2(s)\} = \int_0^t F_1(u) * F_2(t-u) du$$

Example 1. ***Q1. By convolution theorem ,find L^{-1} of $\left[\frac{1}{(s^2+4s+13)^2}\right]$.***

Ans.

$$F(s) = L^{-1}\left[\frac{1}{(s^2+4s+1\ \ )^2}\right] = L^{-1}\left[\frac{1}{(s+2)^2+3^2}\right]^2 = e^{-2t}\cdot L^{-1}\left[\frac{1}{(s^2+3^2)^2}\right]$$

$F_1(s)=\frac{1}{s^2+3^2}$ $\qquad$ $F_2(s)=\frac{1}{(s^2+3^2)^2}$

$L^{-1}\{F_1(s)\}=\frac{1}{3}\sin 3t$ $\qquad$ $L^{-1}\{F_2(s)\}=\frac{1}{3}\sin 3t$

Using the formula of covolution theorem:-

$$L^{-1}\{F_1(s) * F_2(s)\} = \int_0^t F_1(u) * F_2(t-u)du$$

$$L^{-1}[F(s)] = \frac{1}{9}\int_0^t \sin 3u \cdot \sin 3(t-u)\, du$$

$$= -\frac{1}{18}\int_0^t [\cos 3t - \cos(6u-3t)]\, du$$

$$= \frac{1}{18}\left[u\cos 3t - \frac{\sin(6u-3t)}{6}\right]_0^t$$

$$= \frac{1}{18}\left[t\cos 3t - \frac{\sin 3t}{6}\right] = \frac{1}{18}\left[\frac{\sin 3t}{3} - t\cos 3t\right]$$

$$L^{-1}\left[\frac{1}{(s^2+4s+13)^2}\right] = e^{-2t}\left[\frac{\sin 3t}{3} - t\cos 3t\right]$$

Example 2. ***By convolution theorem ,find L^{-1} of $\left[\frac{(s+2)^2}{\left(s^2+4s+8\right)^2}\right]$***

Ans:-

$$L^{-1}\left[\frac{(s+2)^2}{(s^2+4s+8)^2}\right] = L^{-1}\left[\frac{(s+2)^2}{[(s+2)^2+2^2]^2}\right] = e^{-2t}L^{-1}\left[\frac{s^2}{(s^2+2^2)^2}\right]$$

Let $F_1(s) = \frac{s}{s^2+2^2}, \quad F_2(s) = \frac{s}{s^2+2^2}$

$$\therefore L^{-1}F_1(s) = L^{-1}\left[\frac{s}{s^2+2^2}\right] = \cos 2t, \quad L^{-1}F_2(s) = \cos 2t$$

Using the formula of covolution theorem:-

$$L^{-1}\{F_1(s) * F_2(s)\} = \int_0^t F_1(u) * F_2(t-u)du$$

$$\therefore L^{-1}[F(s)] = L^{-1}\left[\left(\frac{s}{s^2+2^2}\right)\left(\frac{s}{s^2+2^2}\right)\right] = \int_0^t \cos 2u \cdot \cos 2(t-u)\, du$$

$$= \frac{1}{2}\int_0^t [\cos 2t + \cos 2(2u-t)]\, du$$

$$= \frac{1}{2}\left[u\cos 2t + \frac{1}{4}\sin 2(2u-t)\right]_0^t$$

$$= \frac{1}{2}\left[t\cos 2t + \frac{1}{4}\sin 2t + \frac{1}{4}\sin 2t\right]$$

$$= \frac{1}{4}[\sin 2t + 2t\cos 2t]$$

$$\therefore L^{-1}\left[\frac{(s+2)^2}{(s^2+4s+8)^2}\right] = \frac{1}{4}e^{-2t}[\sin 2t + 2t\cos 2t]$$

Example 3. ***By convolution theorem ,find*** L^{-1} ***of*** $\frac{s}{s^4+8s^2+16}$

Ans.

$$F(s) = \frac{s}{s^4+8s^2+16} = \frac{s}{(s^2+4)^2} = \frac{s}{s^2+4} \cdot \frac{1}{s^2+4}$$

Let $F_1(s) = \frac{s}{s^2+4}$ and $F_2(s) = \frac{1}{s^2+4}$

$$\therefore L^{-1}F_1(s) = \cos 2t, \quad L^{-1}F_2(s) = \frac{1}{2}\sin 2u$$

Using the formula of covolution theorem:-

$$L^{-1}\{F_1(s) * F_2(s)\} = \int_0^t F_1(u) * F_2(t-u)du$$

$$L^{-1}[F(s)] = \int_0^t \cos 2u \cdot \frac{1}{2}\sin 2(t-u)du$$

$$= \frac{1}{4}\int_0^t [\sin 2t - \sin(4u-2t)]\, du$$

$$= \frac{1}{4}\left[\sin 2t \cdot u + \frac{\cos(4u-2t)}{4}\right]_0^t$$

$$L^{-1}\left[\frac{s}{s^4+8s^2+1}\right] = \frac{1}{4}\left[t \cdot \sin 2t + \frac{\cos 2t}{4} - \frac{\cos 2t}{4}\right] = \frac{1}{4}t\sin 2t$$

Example 4. ***By convolution theorem ,find*** L^{-1} ***of*** $\frac{s^2+5}{(s^2+4s+13)^2}$

Ans.

$$F(s) = \frac{s^2+5}{(s^2+4s+13)^2} = \frac{s^2+5}{[(s+2)^2+3^2]^2}$$

$$F_1(s) = \frac{s+2}{(s+2)^2+9} \text{ and } F_2(s) = \frac{1}{(s+2)^2+9}$$

$$L^{-1}F_1(s) = e^{-2t}\cos(3t), \quad L^{-1}F_2(s) = e^{-2t}\frac{1}{3}\sin(3t)$$

Using the formula of covolution theorem:-

$$L^{-1}\{F_1(s) * F_2(s)\} = \int_0^t F_1(u) * F_2(t-u)du$$

$$L^{-1}[F(s)]= \int_0^t e^{-2u}\cos(3u)\cdot e^{-2(t-u)}\frac{1}{3}\sin\big(3(t-u)\big)du$$

$$L^{-1}\left[\frac{s^2+5}{(s^2+4s+1\ \ )^2}\ \right] = \frac{1}{3}\int_0^t e^{-2t}\cos(3u)\sin\big(3(t-u)\big)du$$

$$=\frac{1}{3}\int_0^t e^{-2t}\big(\sin(3t)\cos(3u) - \cos(3t)\sin(3u)\big)du.$$

$$=\frac{1}{3}\ \int_0^t e^{-2t}\cos(3u)\sin(2t)du + \int_0^t e^{-2t}\sin(3u)\cos(3t)\,du$$

$$=\frac{1}{3}\left[\sin(2t)\frac{e^{-2t}}{2}(1-e^{-2t}) + \cos(3t)\frac{3}{(4+9)}(1-e^{-2t})\right]$$

$$L^{-1}\left[\frac{s^2+5}{(s^2+4s+1\ \ )^2}\ \right]\ =\frac{1}{3}\left(\sin(3t)\cdot\frac{e^{-2t}}{2}(1-e^{-2t}) - \cos(3t)\cdot\frac{3}{13}(1-e^{-2t})\right)$$

Example 5. ***By convolution theorem ,find L^{-1} of $\frac{s}{(s^2-a^2)^2}$***

Ans.

$F(s) = \frac{s}{(s^2-a^2)^2}$

$F_1(s) = \frac{s}{s^2-a^2}$ and $\quad F_2(s) = \frac{1}{s^2-a^2}$

$L^{-1}F_1(s) = \cosh(at), \quad L^{-1}F_2(s) = \frac{1}{a}\sinh(at)$

Using the formula of covolution theorem:-

$$L^{-1}\{F_1(s) * F_2(s)\} = \int_0^t F_1(u) * F_2(t-u)du$$

$$L^{-1}\left\{\frac{s}{(s^2-a^2)^2}\right\} = \int_0^t \cosh(au)\cdot\frac{1}{a}\sinh[a(t-u)]\,du$$

$$= \int_0^t \cosh(au)\cdot\frac{1}{a}\sinh[a(t-u)]\,du$$

$$== \frac{1}{2a}\left[u\sinh(at) - \frac{1}{2a}\cosh[a(t-2u)]\right]_0^t$$

$$L^{-1}\left\{\frac{s}{(s^2-a^2)^2}\right\} = \frac{1}{2a}t\sinh(at)$$

Example 6. ***By convolution theorem ,find*** L^{-1} ***of*** $\frac{s^2}{(s^2+a^2)\,(s^2+b^2)}$

Ans.

$$F(s) = \frac{s^2}{(s^2+a^2)\,(s^2+b^2)}$$

$$F_1(s) = \frac{s}{s^2+a^2} \text{ and } \quad F_2(s) = \frac{s}{s^2+b^2}$$

$$L^{-1}F_1(s) = \cos at, \quad L^{-1}F_2(s) = \cos bt$$

Using the formula of covolution theorem:-

$$L^{-1}\{F_1(s) * F_2(s)\} = \int_0^t F_1(u) * F_2(t-u)du$$

$$L^{-1}\left\{\frac{s^2}{(s^2+a^2)\,(s^2+b^2)}\right\} = \int_0^t \cos au \cdot \cos b(t-u)du$$

$$= \frac{1}{2}\int_0^t [\cos[(a-b)u + bt] + \cos[(a+b)u - bt]]\, du$$

$$= \frac{1}{2}\left[\frac{\sin[(a-b)t + bt]}{a-b} + \frac{\sin[(a+b)t - bt]}{a+b}\right]$$

$$= \frac{1}{2}\left[\frac{\sin at}{a-b} + \frac{\sin at}{a+b} + \frac{\sin bt}{a-b} + \frac{\sin bt}{a+b}\right]$$

$$= \frac{1}{2}\left[\frac{2a\sin at - 2b\sin bt}{a^2-b^2}\right]$$

$$L^{-1}\left\{\frac{s^2}{(s^2+a^2)\,(s^2+b^2)}\right\} = \frac{a\sin at - b\sin bt}{a^2-b^2}$$

Example 7. ***By convolution theorem ,find*** L^{-1} ***of*** $\frac{1}{(s^2+1)(s^2+9)}$

Ans.

$$F(s) = \frac{1}{(s^2+1)(s^2+9)}$$

$$F_1(s) = \frac{1}{s^2+1} \text{ and } \quad F_2(s) = \frac{1}{s^2+9}$$

$$L^{-1}F_1(s) = \sin t, \quad L^{-1}F_2(s) = \frac{1}{3}\sin 3t$$

Using the formula of covolution theorem:-

$$L^{-1}\{F_1(s) * F_2(s)\} = \int_0^t F_1(u) * F_2(t-u)du$$

$$L^{-1}\left\{\frac{1}{(s^2+1)(s^2+9)}\right\} = \int_0^t \sin u \cdot \frac{1}{3}\sin 3(t-u)\, du.$$

$$= -\frac{1}{6}\int_0^t \cos\left[(1-3)u+3t\right] - \cos[(4u-3t)]du$$

$$= -\frac{1}{6}\int_0^t \cos(3t-2u) - \cos(4u-3t)du$$

$$= \frac{1}{6}\left[\frac{\sin(3t-2u)}{-2} + \frac{\sin(4u-3t)}{4}\right]_0^t$$

$$= -\frac{1}{6}\left[\frac{-\sin t}{2} \cdot \frac{\sin t}{4} - \left(\frac{-\sin 3}{2} \cdot \frac{\sin(-3t)}{4}\right)\right] = \frac{1}{6}\left[\frac{\sin t \cdot \sin 3t}{4} - \frac{\sin 3t \cdot \sin(-3t)}{2}\right]$$

$$L^{-1}\left\{\frac{1}{(s^2+1)(s^2+9)}\right\} = \frac{1}{24}[3\sin t - \sin 3t]$$

Example 8. ***By convolution theorem ,find L^{-1} of $\frac{1}{(s-4)^2(s+3)}$***

Ans.

$$F(s) = \frac{1}{(s-4)^2(s+3)}$$

$$F_1(s) = \frac{1}{s+3} \quad \text{and} \quad F_2(s) = \frac{1}{(s-4)^2}$$

$$L^{-1}F_1(s) = e^{-3t}, \quad L^{-1}F_2(s) = e^{4t}[t]$$

Using the formula of covolution theorem:-

$$L^{-1}\{F_1(s) * F_2(s)\} = \int_0^t F_1(u) * F_2(t-u)du$$

$$L^{-1}\left\{\frac{1}{(s-4)^2(s+3)}\right\} = \int_0^t e^{-3u} \cdot e^{4(t-u)}(t-u)du$$

$$=\int_0^t e^{(4-3)u}\,(t-u)du$$

$$= e^{4t}\int_0^t e^{-7u}\,(t-u)du$$

$$=e^{4t}\left[\frac{e^{-7u}}{-7}(t-u) + \frac{1}{49}e^{-7u}\right]_0^t$$

$$=e^{4t}\left[\frac{t}{7} + \frac{e^{-7t}}{49} - \frac{1}{49}\right]$$

$$L^{-1}\left\{\frac{1}{(s-4)^2(s+3)}\right\} = e^{4t}\left[\frac{t}{7} + \frac{e^{-7t}-1}{49}\right]$$

Example 9. ***By convolution theorem ,find*** L^{-1} ***of*** $\frac{1}{(s-a)(s+a)^2}$

Ans.

$$F(s) = \frac{1}{(s-a)(s+a)^2}$$

$F_1(s) = \frac{1}{s-a}$ and $\quad F_2(s) = \frac{1}{(s+a)^2}$

$$L^{-1}F_1(s) = e^{at}, \quad L^{-1}F_2(s) = te^{-a}$$

Using the formula of covolution theorem:-

$$L^{-1}\{F_1(s) * F_2(s)\} = \int_0^t F_1(u) * F_2(t-u)du$$

$$L^{-1}\left\{\frac{1}{(s-a)(s+a)^2}\right\} = \int_0^t e^{au}\, e^{-a(t-u)}(t-u)du$$

$$= \int_0^t e^{au}\, e^{-at}e^{au}(t-u)du$$

$$= e^{-a} \int_0^t e^{2a} \ (t-u)du = e^{-at}\left[\frac{e^{2au}}{2a}(t-u) - \frac{e^{2au}}{4a^2}\right]_0^t$$

$$= e^{-at}\left[0 + \frac{e^{2a}}{4a^2}\left(\frac{t}{2a} + \frac{1}{4a^2}\right)\right]$$

$$= \frac{1}{4a^2}[e^{at} - 2ate^{-a} \ + e^{-a} \ ]$$

$$\mathcal{L}^{-1}\left[\frac{1}{(s-a)(s+a)^2}\right] = \frac{1}{4a^2}[e^{at} - 2ate^{-a} \ + e^{-at}]$$

Example 10. ***By convolution theorem ,find L^{-1} of $\frac{1}{(s+3)(s-2)^4}$***

Ans.

$$F(s) = \frac{1}{(s+3)(s-2)^4}$$

$$F_1(s) = \frac{1}{s+3} \text{and} \quad F_2(s) = \frac{1}{(s-2)^4}$$

$$L^{-1}F_1(s) = e^{-3t}, \quad L^{-1}F_2(s) = e^{2t}\frac{t^3}{6}$$

Using the formula of covolution theorem:-

$$L^{-1}\{F_1(s) * F_2(s)\} = \int_0^t F_1(u) * F_2(t-u)du$$

$$L^{-1}\left\{\frac{1}{(s+3)(s-2)^4}\right\} = \int_0^t e^{-3u}\, e^{2(t-u)}\frac{(t-u)^3}{6}du$$

$$= \int_0^t e^{-5u}\frac{(t-u)^3}{6}du$$

$$= e^{2t}\left[\frac{(t-u)^3}{6}\cdot\frac{e^{-5u}}{5} - \frac{(t-u)^2}{2}\cdot\frac{e^{-5u}}{25} + (t-u)(-1)\cdot\frac{e^{-5u}}{125} + (-1)\cdot\frac{e^{-5u}}{625}\right]_0^t$$

$$L^{-1}\left\{\frac{1}{(s+3)(s-2)^4}\right\} = e^{2t}\left[\frac{e^{-5t}}{625}\left(-\frac{t^3}{30} + \frac{t^2}{50} - \frac{t}{125} + \frac{1}{625}\right)\right] + \frac{e^{-3t}}{625}\left(t^3 - \frac{t^2}{2} + t - \frac{1}{6}\right)$$

Example 11. ***By convolution theorem ,find L^{-1} of*** $\frac{s^2+s}{(s^2+1)(s^2+2s+2)}$

Ans.

$F(s) = \frac{s^2+s}{(s^2+1)(s^2+2s+2)}$

$F_1(s) = \frac{s}{s^2+1}$ and $F_2(s) = \frac{s+1}{s^2+2s+2}$

$L^{-1}F_1(s) = \text{cost}\,, \quad L^{-1}F_2(s) = e^{-t}\text{cost}$

Using the formula of covolution theorem:-

$$L^{-1}\{F_1(s) * F_2(s)\} = \int_0^t F_1(u) * F_2(t-u)du$$

$$L^{-1}\left\{\frac{s(s+1)}{(s^2+1)(s^2+2s+2)}\right\} = \int_0^t e^{-u}\cos u \cdot \cos(t-u)du$$

$$= \int_0^t e^{-u}\left[\frac{1}{2}\Big(\cos(u+t-u) + \cos\big(u-(t-u)\big)\Big)\right]du$$

$$= \frac{1}{2}\int_0^t e^{-u}\left[\text{cost} + \cos(2u-t)\right]du = \frac{1}{2}\left[\int_0^t e^{-u}\,\text{cost}du + \int_0^t e^{-u}\cos(2u-t)du\right]$$

$$= \frac{1}{2}\left[\text{cost}\int_0^t e^{-u}\,du + \int_0^t e^{-u}\cos(2u-t)du\right]$$

$$= \frac{1}{2}\left[\text{cost}\left(\frac{e^{-u}}{-1}\right)_0^t + \frac{e^{-u}}{(1-1)^2+(2)^2} \times [(-1)\cos(2u-t) + 2\sin(2u-t)]_0^t\right]$$

$$= \frac{1}{2}\left[(-e^{-t}+1)\text{cost} + \frac{e^{-t}}{5}\left[(-\text{cost}+2\text{sint}) - \frac{1}{5}(\text{cost}-2\text{sint})\right]\right]$$

$$L^{-1}\left\{\frac{s(s+1)}{(s^2+1)(s^2+2s+2)}\right\}$$

$$=\frac{1}{2}\left[-e^{-t}\mathrm{cos}t+\mathrm{cos}t+\frac{e^{-t}}{5}(-\mathrm{cos}t+2\mathrm{sin}t)-\frac{1}{5}(\mathrm{cos}t-2\mathrm{sin}t)\right]$$

Example 12. ***By convolution theorem ,find L^{-1} of*** $\frac{1}{s^4+13s^2+36}$

Ans.

$$F(s) = \frac{1}{s^4+13s^2+3} = \frac{1}{(s^2+4)} + \frac{1}{(s^2+9)}$$

$$F_1(s) = \frac{1}{(s^2+4)} \text{and} \quad F_2(s) = \frac{1}{(s^2+9)}$$

$$L^{-1}F_1(s) = \frac{1}{2}\sin 2t, \quad L^{-1}F_2(s) = \frac{1}{3}\sin 3t$$

Using the formula of covolution theorem:-

$$L^{-1}\{F_1(s) * F_2(s)\} = \int_0^t F_1(u) * F_2(t-u)du$$

$$L^{-1}\left\{\frac{1}{s^4+13s^2+36}\right\} = \int_0^t \frac{1}{2}\sin 2u \cdot \frac{1}{3}\sin 3(t-u)du$$

$$= \frac{1}{6}\int_0^t \sin 2u \sin 3(t-u)du$$

$$= \frac{1}{6} \times \frac{1}{2}\int_0^t [\cos(2u-3t+3u) - \cos(2u+3t-3u)]\,du$$

$$= \frac{1}{12}\int_0^t [\cos(5u-3t) - \cos(3t-u)]\,du$$

$$= \frac{1}{12}\left[\frac{\sin(5u-3t)}{5} - \frac{\sin(3t-u)}{1}\right]_0^t = \frac{1}{12}\left[\frac{\sin(5u-3t)}{5} + \sin(3t-u)\right]_0^t$$

$$= \frac{1}{12}\left[\left(\frac{\sin 2t}{5} + \sin 2t\right) - \left(\frac{-\sin(3t)}{5} + \sin 3t\right)\right]$$

$$= \frac{1}{12}\left[\left(\frac{\sin 2t + 5\sin 2t}{5}\right) - \left(\frac{-\sin 3t + 5\sin 3t}{5}\right)\right]$$

$$= \frac{1}{12}\left[\frac{6\sin 2t}{5} - \frac{4\sin 3t}{5}\right] = \frac{1}{10}\left[\frac{3\sin 2t - 2\sin 3t}{5}\right]$$

$$L^{-1}\left\{\frac{1}{s^4 + 13s^2 + 36}\right\} = \frac{1}{30}[3\sin 2t - 2\sin 3t]$$

Thank you!!