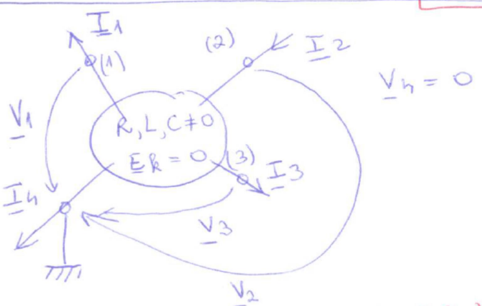
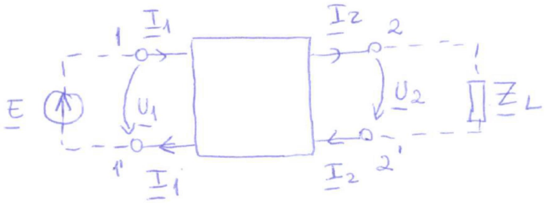
Curs 10

5. TWO-PORT NETWORKS (CUADRIPOLI)



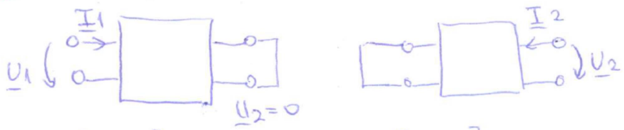


5.1. Equations (Ecuațiile cuadripolilor)

a) Fundamental parameters (Ecuațiile fundamentale ale cuadripolilor)

$$\begin{cases} \underline{U}_1 = \underline{A}\cdot\underline{U}_2 + \underline{B}\cdot\underline{I}_2 \\ \underline{I}_1 = \underline{C}\cdot\underline{U}_2 + \underline{D}\cdot\underline{I}_2 \end{cases} \Rightarrow \underline{A} = \left(\frac{\underline{U}_1}{\underline{U}_2}\right)_{\underline{I}_2=0} = \frac{1}{\left(\frac{\underline{U}_2}{\underline{U}_1}\right)_{\underline{I}_2=0}};\ \underline{B} = \frac{1}{\left(\frac{\underline{I}_2}{\underline{U}_1}\right)_{\underline{U}_2=0}};\ \underline{C} = \frac{1}{\left(\frac{\underline{U}_2}{\underline{I}_1}\right)_{\underline{I}_2=0}};\ \underline{D} = \left(\frac{\underline{I}_1}{\underline{I}_2}\right)_{\underline{U}_2=0}$$

A two-port network is said to be reciprocal if the ratio of the excitation to the response is invariant to an interchange of the position of the excitation and response in the network (Un cuadripol este reciproc dacă raportul dintre excitație și răspunsul la aceasta rămâne constant la repoziționarea excitației și răspunsului în cuadripol).



$$\left[\frac{\underline{I}_1}{\underline{U}_1}\right]_{\underline{U}_2=0} = \left[\frac{\underline{I}_2}{\underline{U}_2}\right]_{\underline{U}_1=0} \Rightarrow \boxed{\underline{A}\cdot\underline{D} - \underline{B}\cdot\underline{C} = 1}$$

Reciprocity condition
Condiția de reciprocitate

b) Impedance parameters (Ecuațiile cuadripolilor în parametri impedanță)

$$\begin{cases} \underline{U}_1 = \underline{Z}_{11}\cdot\underline{I}_1 + \underline{Z}_{12}\cdot\underline{I}_2 \\ \underline{U}_2 = \underline{Z}_{21}\cdot\underline{I}_1 + \underline{Z}_{22}\cdot\underline{I}_2 \end{cases}$$

$$\underline{Z}_{11} = \left(\frac{\underline{U}_1}{\underline{I}_1}\right)_{\underline{I}_2=0} = \frac{\underline{A}}{\underline{C}};\quad \underline{Z}_{21} = \left(\frac{\underline{U}_2}{\underline{I}_1}\right)_{\underline{I}_2=0} = \frac{1}{\underline{C}};\quad \underline{Z}_{22} = \left(\frac{\underline{U}_2}{\underline{I}_2}\right)_{\underline{I}_1=0} = -\frac{\underline{D}}{\underline{C}}$$

$$\underline{Z}_{12} = \left(\frac{\underline{U}_1}{\underline{I}_2}\right)_{\underline{I}_1=0} = -\frac{\underline{A}\cdot\underline{D} - \underline{B}\cdot\underline{C}}{\underline{C}}$$

$$\begin{cases} \underline{U}_1 = \underline{A}\cdot\underline{U}_2 + \underline{B}\cdot\underline{I}_2 \mid :\underline{I}_2 \Rightarrow \frac{\underline{U}_1}{\underline{I}_2} = \underline{A}\cdot\left(-\frac{\underline{D}}{\underline{C}}\right) + \underline{B} \\ 0 = \underline{C}\cdot\underline{U}_2 + \underline{D}\cdot\underline{I}_2 \Rightarrow \frac{\underline{U}_2}{\underline{I}_2} = -\frac{\underline{D}}{\underline{C}} \end{cases}$$

c) Admittance parameters (Ecuațiile cuadripolilor în parametri admitanță)

$$\begin{cases} \underline{I}_1 = \underline{Y}_{11} \cdot \underline{U}_1 + \underline{Y}_{12} \cdot \underline{U}_2 \\ \underline{I}_2 = \underline{Y}_{21} \cdot \underline{U}_1 + \underline{Y}_{22} \cdot \underline{U}_2 \end{cases} \Rightarrow \underline{Y}_{11} = \left(\frac{\underline{I}_1}{\underline{U}_1}\right)_{\underline{U}_2=0} = \frac{\underline{D}}{\underline{B}} ; \underline{Y}_{21} = \left(\frac{\underline{I}_2}{\underline{U}_1}\right)_{\underline{U}_2=0} = \frac{1}{\underline{B}}$$

$$\underline{Y}_{22} = \left(\frac{\underline{I}_2}{\underline{U}_2}\right)_{\underline{U}_1=0} = -\frac{\underline{A}}{\underline{B}} ; \underline{Y}_{12} = \left(\frac{\underline{I}_1}{\underline{U}_2}\right)_{\underline{U}_1=0} = -\frac{\underline{A}\underline{D} - \underline{B}\underline{C}}{\underline{B}}$$

$$\begin{cases} 0 = \underline{A} \cdot \underline{U}_2 + \underline{B} \cdot \underline{I}_2 \\ \underline{I}_1 = \underline{C} \cdot \underline{U}_2 + \underline{D} \cdot \underline{I}_2 \mid \underline{U}_2 \end{cases} \Rightarrow \frac{\underline{I}_2}{\underline{U}_2} = -\frac{\underline{A}}{\underline{B}}$$

$$\Rightarrow \frac{\underline{I}_1}{\underline{U}_2} = \underline{C} - \underline{D} \cdot \frac{\underline{A}}{\underline{B}}$$

Reciprocity condition (condiția de reciprocitate):

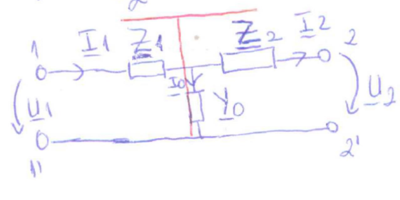
$$\boxed{\underline{A} \cdot \underline{D} - \underline{B} \cdot \underline{C} = 1} \text{ or / sau } \boxed{\underline{Z}_{12} = -\underline{Z}_{21}} \text{ or / sau } \boxed{\underline{Y}_{12} = -\underline{Y}_{21}}$$

Symmetry: A two-port network is said to be symmetrical if the input and output ports can be interchanged without altering the port voltages and currents (Un cuadripol este simetric dacă bornele de intrare se pot interschimba cu cele de ieșire fără ca tensiunile și curenții circuitului să se modifice).

$$\boxed{\underline{A} = \underline{D}} \text{ or / sau } \boxed{\underline{Z}_{11} = -\underline{Z}_{22}} \text{ or / sau } \boxed{\underline{Y}_{11} = -\underline{Y}_{12}}$$

5.2 Equivalent two-port networks

a) "T" Equivalent „T" schematic (Schema echivalentă în „T")



$$\begin{cases} \underline{U}_1 = \underline{A} \cdot \underline{U}_2 + \underline{B} \cdot \underline{I}_2 \\ \underline{I}_1 = \underline{C} \cdot \underline{U}_2 + \underline{D} \cdot \underline{I}_2 \end{cases}$$

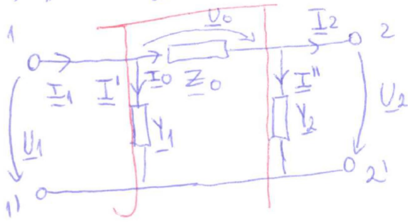
$$\underline{I}_1 = \underline{I}_0 + \underline{I}_2 = \underline{Y}_0 (\underline{Z}_2 \underline{I}_2 + \underline{U}_2) + \underline{I}_2 = \underbrace{\underline{Y}_0}_{\underline{C}} \underline{U}_2 + \underbrace{(1 + \underline{Z}_2 \underline{Y}_0)}_{\underline{D}} \cdot \underline{I}_2$$

$$\underline{U}_1 = \underline{Z}_1 \cdot \underline{I}_1 + \underline{Z}_2 \cdot \underline{I}_2 + \underline{U}_2 = \underline{Z}_1 \cdot [\underline{Y}_0 \cdot \underline{U}_2 + (1 + \underline{Z}_2 \underline{Y}_0) \underline{I}_2] + \underline{Z}_2 \underline{I}_2 + \underline{U}_2$$

$$\underline{U}_1 = \underbrace{(1 + \underline{Z}_1 \cdot \underline{Y}_0)}_{\underline{A}} \cdot \underline{U}_2 + \underbrace{(\underline{Z}_1 + \underline{Z}_2 + \underline{Z}_1 \underline{Z}_2 \underline{Y}_0)}_{\underline{B}} \cdot \underline{I}_2$$

$$\Rightarrow \begin{cases} \underline{A} = 1 + \underline{Z}_1 \cdot \underline{Y}_0 \\ \underline{B} = \underline{Z}_1 + \underline{Z}_2 + \underline{Z}_1 \underline{Z}_2 \underline{Y}_0 \\ \underline{C} = \underline{Y}_0 \\ \underline{D} = 1 + \underline{Z}_2 \cdot \underline{Y}_0 \end{cases} \quad \begin{cases} \underline{Y}_0 = \underline{C} \\ \underline{Z}_1 = \frac{\underline{A} - 1}{\underline{C}} \\ \underline{Z}_2 = \frac{\underline{D} - 1}{\underline{C}} \end{cases}$$

b) „Π" Equivalent „Π" schematic (Schema echivalentă în „Π")



$$U_1 = U_0 + U_2 = Z_0 \cdot I_0 + U_2 = Z_0(I_2 + Y_2 \cdot U_2) + U_2$$

$$U_1 = \underbrace{(1 + Z_0 \cdot Y_2)}_{A} \cdot U_2 + \underbrace{Z_0}_{B} \cdot I_2$$

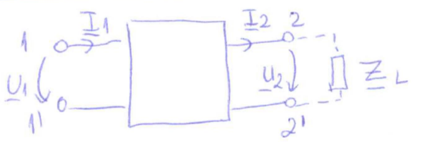
$$I_1 = I' + I_0 = U_1 \cdot Y_1 + I_2 + Y_2 \cdot U_2 = Y_1\left[(1+Z_0 Y_2) U_2 + Z_0 I_2\right] + I_2 + Y_2 U_2$$

$$I_1 = \underbrace{(Y_1 + Y_2 + Y_1 Y_2 Z_0)}_{C} \cdot U_2 + \underbrace{(1 + Y_1 Z_0)}_{D} \cdot I_2$$

$$\begin{cases} A = 1 + Z_0 Y_2 \\ B = Z_0 \\ C = Y_1 + Y_2 + Y_1 Y_2 Z_0 \\ D = 1 + Z_0 \cdot Y_1 \end{cases} \Rightarrow \begin{cases} Y_1 = \dfrac{D-1}{B} \\ Y_2 = \dfrac{A-1}{B} \\ Z_0 = B \end{cases}$$

5.3 Finding the A, B, C, D parameters (Determinarea parametrilor A, B, C, D)

$$\begin{cases} U_1 = A \cdot U_2 + B \cdot I_2 \\ I_1 = C \cdot U_2 + D \cdot I_2 \end{cases}$$



$$\Rightarrow \begin{cases} U_1 = U_{10} + U_{1sc} \\ I_1 = I_{10} + I_{1sc} \end{cases}$$

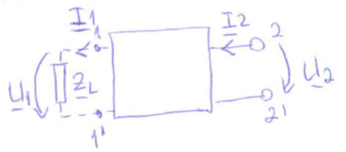
If $Z_L \to \infty \Rightarrow I_2 = 0$ (OPEN CIRCUIT) (mers în gol)

If $Z_L \to 0 \Rightarrow U_2 = 0$ (SHORT CIRCUIT) (scurt circuit)

$$\left.\begin{matrix} U_{10} = A \cdot U_2 \\ I_{10} = C \cdot U_2 \end{matrix}\right\} \Rightarrow Z_{10} = \frac{U_{10}}{I_{10}} = \frac{A}{C}$$

$$\left.\begin{matrix} U_{1sc} = B \cdot I_2 \\ I_{1sc} = D \cdot I_2 \end{matrix}\right\} Z_{1sc} = \frac{U_{1sc}}{I_{1sc}} = \frac{B}{D}$$

$$\begin{cases} U_2 = D \cdot U_1 + B \cdot I_1 \\ I_2 = C \cdot U_1 + A \cdot I_1 \end{cases}$$



$$\begin{cases} U_2 = U_{20} + U_{2sc} \\ I_2 = I_{20} + I_{2sc} \end{cases}$$

If $Z_L \to \infty \Rightarrow I_1 = 0$

$$\begin{cases} U_{20} = D \cdot U_1 \\ I_{20} = C \cdot U_1 \end{cases} \Rightarrow Z_{20} = \frac{U_{20}}{I_{20}} = \frac{D}{C}$$

If $Z_L \to 0 \Rightarrow U_1 = 0$

$$\left.\begin{matrix} U_{2sc} = B \cdot I_1 \\ I_{2sc} = A \cdot I_1 \end{matrix}\right\} \Rightarrow Z_{2sc} = \frac{U_{2sc}}{I_{2sc}} = \frac{B}{A}$$

$\Rightarrow$

$Z_{10} = \dfrac{A}{C}$	$Z_{1sc} = \dfrac{B}{D}$
$Z_{20} = \dfrac{D}{C}$	$Z_{2sc} = \dfrac{B}{A}$

$$\Rightarrow Z_{10} \cdot Z_{2sc} = Z_{1sc} \cdot Z_{20} = \frac{B}{C}$$

$$Z_{10} = Z_{10} \cdot e^{j\varphi_{10}} = \frac{U_{10}}{I_{10}} \cdot e^{j\varphi_{10}}$$

$$P_{10} = U_{10} \cdot I_{10} \cdot \cos\varphi_{10} \Rightarrow \cos\varphi_{10} = \frac{P_{10}}{U_{10} \cdot I_{10}} \Rightarrow \varphi_{10}$$

$$\Rightarrow \underline{A} = \underline{C} \cdot \underline{Z}_{10}$$
$$\underline{B} = \underline{C} \cdot \underline{Z}_{10} \cdot \underline{Z}_{2sc} = \underline{C} \cdot \underline{Z}_{20} \cdot \underline{Z}_{1sc}$$
$$\underline{D} = \underline{C} \cdot \underline{Z}_{20}$$

From $\underline{A}\,\underline{D} - \underline{B}\cdot\underline{C} = 1 \Rightarrow \underline{C}^2 \cdot \underline{Z}_{10} \cdot \underline{Z}_{20} - \underline{C}^2 \underline{Z}_{10} \cdot \underline{Z}_{2sc} = 1$

or (sau) $\underline{C}^2 \cdot \underline{Z}_{10} \cdot \underline{Z}_{20} - \underline{C}^2 \cdot \underline{Z}_{20} \cdot \underline{Z}_{1sc} = 1$

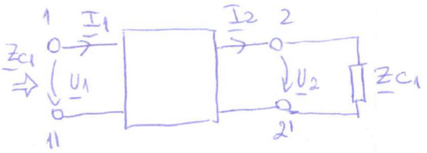
$$\Rightarrow \underline{C} = \pm \frac{1}{\sqrt{\underline{Z}_{20}(\underline{Z}_{10} - \underline{Z}_{1sc})}} = \pm \frac{1}{\sqrt{\underline{Z}_{10}(\underline{Z}_{20} - \underline{Z}_{2sc})}}$$

5.4 Iterative impedances (Impedanțe caracteristice)

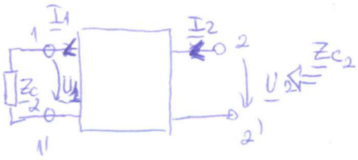
If a value of the output impedance $\underline{Z}_C$ exists, for which the input impedance to a two-port is also $\underline{Z}_C$, then this impedance is known as iterative impedance:

$$\boxed{\underline{Z}_C = \frac{\underline{U}_1}{\underline{I}_1} = \frac{\underline{U}_2}{\underline{I}_2}}$$

Dacă există o valoare $\underline{Z}_C$ a impedanței de ieșire a unui cuadripol pentru care impedanța de intrare e tot $\underline{Z}_C$, această se numește impedanța caracteristică.



$$\begin{cases} \underline{U}_1 = \underline{A}\cdot\underline{U}_2 + \underline{B}\cdot\underline{I}_2 \\ \underline{I}_1 = \underline{C}\cdot\underline{U}_2 + \underline{D}\cdot\underline{I}_2 \end{cases} \Rightarrow \underline{Z}_{C_1} = \frac{\underline{U}_1}{\underline{I}_1} = \frac{\underline{A}\cdot\underline{U}_2 + \underline{B}\cdot\underline{I}_2}{\underline{C}\cdot\underline{U}_2 + \underline{D}\cdot\underline{I}_2} \Rightarrow$$



$$\Rightarrow \underline{Z}_{C_1} = \frac{\underline{A}\cdot\underline{Z}_{C_1} + \underline{B}}{\underline{C}\cdot\underline{Z}_{C_1} + \underline{D}}$$

$$\Rightarrow \underline{C}\cdot\underline{Z}_{C_1}^2 + (\underline{D} - \underline{A})\cdot\underline{Z}_{C_1} - \underline{B} = 0$$

$$\begin{cases} \underline{Z}_{C_1} = \frac{1}{2\underline{C}} \cdot \left[(\underline{A} - \underline{D}) \pm \sqrt{(\underline{D} - \underline{A})^2 + 4\underline{B}\,\underline{C}}\right] \\ \underline{Z}_{C_2} = \frac{1}{2\underline{C}} \left[(\underline{D} - \underline{A}) \pm \sqrt{(\underline{A} - \underline{D})^2 + 4\underline{B}\,\underline{C}}\right] \end{cases}$$

If the two-port is symmetrical (dacă cuadripolul este simetric)

$$\underline{A} = \underline{D} \Rightarrow \boxed{\underline{Z}_{C_1} = \underline{Z}_{C_2} = \pm\sqrt{\frac{\underline{B}}{\underline{C}}} = \underline{Z}_C}$$

If $\underline{B} = \underline{C} = 0$ (particular case – caz particular) $\Rightarrow$ "T" $\underline{Z}_1 = \underline{Z}_2 = \underline{Z}$; "π" $\underline{Y}_1 = \underline{Y}_2 = \underline{Y}$

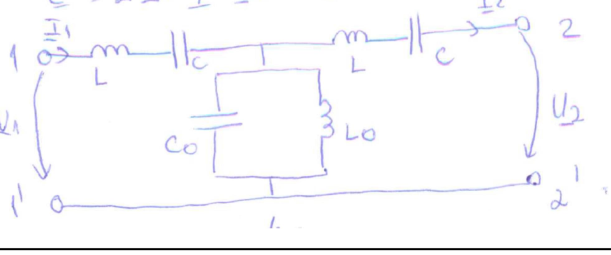
"T" – cuadripol $\Rightarrow \underline{B} = \underline{Z}_1 + \underline{Z}_2 + \underline{Z}_1\underline{Z}_2\cdot\underline{Y}_0 = 2\underline{Z} + \underline{Z}^2\,\underline{Y}_0 = 0 \Rightarrow \underline{Z} = 0$

$\underline{C} = \underline{Y}_0 = 0$

"π" two port $\Rightarrow \underline{B} = \underline{Z}_0 = 0$

$\underline{C} = 2\underline{Y} + \underline{Y}^2\cdot\underline{Z}_0 \Rightarrow \underline{Y} = 0$

$\Rightarrow$ "T"



$$\boxed{\begin{array}{c} \omega L = \frac{1}{\omega C} \\ \frac{1}{\omega C_0} = \omega L_0 \end{array}} \begin{array}{l} \Rightarrow \underline{Z} = 0 \\ \underline{Y}_0 = 0 \end{array}$$

5.5. The propagation constant (constanta de propagare)

Can be computed for two-ports that are:
Poate fi determinată pentru cuadripoli:

- Symmetrical $\underline{A} = \underline{D}$
- Reciprocal $\underline{A}\underline{D} - \underline{B}\underline{C} = 1$
- Load impedance = Iterative impedance = $\underline{Z}_c = \frac{\underline{U}_2}{\underline{I}_2}$
- Impedanța de sarcină = imped. caracteristică

$$\begin{cases} \underline{U}_1 = \underline{A}\cdot\underline{U}_2 + \underline{B}\cdot\underline{I}_2 = \underline{U}_2\left(\underline{A} + \underline{B}\cdot\frac{\underline{I}_2}{\underline{U}_2}\right) = \underline{U}_2\left(\underline{A} + \underline{B}\cdot\frac{1}{\underline{Z}_c}\right) = \underline{U}_2\left(\underline{A} + \sqrt{\underline{B}\cdot\underline{C}}\right) \\ \underline{I}_1 = \underline{C}\cdot\underline{U}_2 + \underline{D}\cdot\underline{I}_2 = \underline{I}_2\left(\underline{C}\cdot\frac{\underline{U}_2}{\underline{I}_2} + \underline{D}\right) = \underline{I}_2\left(\underline{C}\cdot\underline{Z}_c + \underset{\underline{A}}{\underline{D}}\right) = \underline{I}_2\left(\underline{A} + \sqrt{\underline{B}\underline{C}}\right) \end{cases}$$

$$\Rightarrow \underline{\gamma} = \ln\frac{\underline{U}_1}{\underline{U}_2} = \ln\frac{\underline{I}_1}{\underline{I}_2} = \ln\left(\underline{A} + \sqrt{\underline{B}\cdot\underline{C}}\right) \text{ - propagation constant}$$

$$\underline{\gamma} = \alpha + j\cdot\beta$$

β → phase constant (constanta de fază)
α → damping constant (constanta de atenuare)

$$e^{\underline{\gamma}} = \frac{\underline{U}_1}{\underline{U}_2}\ ;\ \underline{\gamma} = \ln\left(\frac{U_1}{U_2}\cdot e^{j\beta}\right) = \underbrace{\ln\frac{U_1}{U_2}}_{\alpha} + j\cdot\beta \quad \begin{cases} \alpha = \ln\frac{U_1}{U_2} \\ \beta = \gamma_{U_1} - \gamma_{U_2} \end{cases}$$

$$\frac{U_1}{U_2} = e^{\alpha} \Rightarrow \begin{cases} \alpha > 0 \Rightarrow U_2 < U_1 \text{ - signal damped (semnal atenuat)} \\ \alpha = 0 \Rightarrow U_2 = U_1 \text{ - the signal passes unchanged (semnal nemodificat)} \\ \alpha < 0 \Rightarrow U_2 > U_1 \text{ - voltage amplification (semnal amplificat)} \end{cases}$$

$$\left.\begin{matrix} \underline{A}\cdot\underline{D} - \underline{B}\cdot\underline{C} = 1 \\ \underline{A} = \underline{D} \end{matrix}\right\} \Rightarrow \underline{A}^2 - \underline{B}\cdot\underline{C} = 1 \Rightarrow \underbrace{\left(\underline{A} - \sqrt{\underline{B}\cdot\underline{C}}\right)}_{e^{-\underline{\gamma}}}\underbrace{\left(\underline{A} + \sqrt{\underline{B}\cdot\underline{C}}\right)}_{e^{\underline{\gamma}}} = 1$$

$$\Rightarrow \begin{cases} \underline{A} + \sqrt{\underline{B}\cdot\underline{C}} = e^{\underline{\gamma}} \\ \underline{A} - \sqrt{\underline{B}\cdot\underline{C}} = e^{-\underline{\gamma}} \end{cases} \Rightarrow \begin{cases} \underline{A} = \underline{D} = \frac{e^{\underline{\gamma}} + e^{-\underline{\gamma}}}{2} = ch\,\underline{\gamma} \\ \sqrt{\underline{B}\cdot\underline{C}} = \frac{e^{\underline{\gamma}} - e^{-\underline{\gamma}}}{2} = sh\,\underline{\gamma} \\ \sqrt{\frac{\underline{B}}{\underline{C}}} = \underline{Z}_c \end{cases} \Rightarrow \begin{cases} \underline{A} = \underline{D} = ch\,\underline{\gamma} \\ \underline{B} = \underline{Z}_c\, sh\,\underline{\gamma} \\ \underline{C} = \frac{1}{\underline{Z}_c}\, sh\,\underline{\gamma} \end{cases}$$

$$\Rightarrow \text{Ec. fundamentale } \begin{cases} \underline{U}_1 = ch\,\underline{\gamma}\cdot\underline{U}_2 + \underline{Z}_c\cdot sh\,\underline{\gamma}\cdot\underline{I}_2 \\ \underline{I}_1 = \frac{1}{\underline{Z}_c}\cdot sh\,\underline{\gamma}\cdot\underline{U}_2 + ch\,\underline{\gamma}\cdot\underline{I}_2 \end{cases}$$

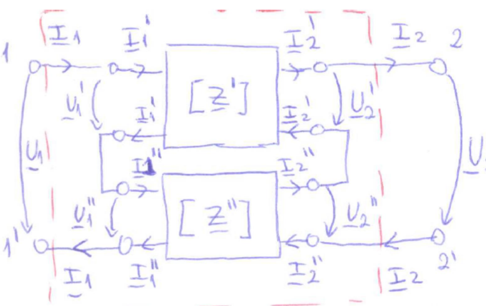
For
$$\begin{cases} \underline{I}_2 = 0 \text{ (open circuit)} \rightarrow \underline{Z}_{10} = \frac{\underline{U}_{10}}{\underline{I}_{10}} = \frac{\underline{Z}_c}{th\,\underline{\gamma}} \\ \underline{U}_2 = 0 \text{ (short circuit)} \rightarrow \underline{Z}_{1sc} = \frac{\underline{U}_{1sc}}{\underline{I}_{1sc}} = \underline{Z}_c\cdot th\,\underline{\gamma} \end{cases} \Rightarrow \begin{cases} \underline{Z}_c = \sqrt{\underline{Z}_{10}\cdot\underline{Z}_{1sc}} \\ th\,\underline{\gamma} = \sqrt{\frac{\underline{Z}_{1sc}}{\underline{Z}_{10}}} \end{cases}$$

5.6. Interconnection of two-port networks (Conexiunile cuadripolilor)

$\boxed{A}$ Series connection (conexiunea serie)

$$\begin{bmatrix} \underline{U}_1 \\ \underline{U}_2 \end{bmatrix} = [\underline{Z}]\cdot\begin{bmatrix} \underline{I}_1 \\ \underline{I}_2 \end{bmatrix} \left(\text{From } \begin{cases} \underline{U}_1 = \underline{Z}_{11}\cdot\underline{I}_1 + \underline{Z}_{12}\cdot\underline{I}_2 \\ \underline{U}_2 = \underline{Z}_{21}\cdot\underline{I}_1 + \underline{Z}_{22}\cdot\underline{I}_2 \end{cases}\right);\ [\underline{Z}] = [\underline{Z}'] + [\underline{Z}'']$$

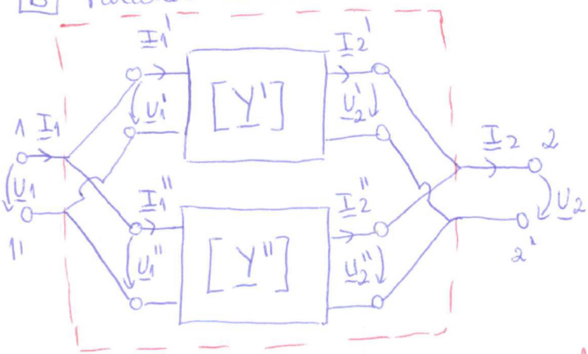
Serie!



$$\begin{cases} \underline{I}_1 = \underline{I}_1' = \underline{I}_1'' \\ \underline{I}_2 = \underline{I}_2' = \underline{I}_2'' \end{cases} \text{ and (și) } \begin{cases} \underline{U}_1 = \underline{U}_1' + \underline{U}_1'' \\ \underline{U}_2 = \underline{U}_2' + \underline{U}_2'' \end{cases}$$

$$\boxed{[\underline{Z}] = [\underline{Z}'] + [\underline{Z}'']}$$

B Parallel connection (conexiunea paralel)

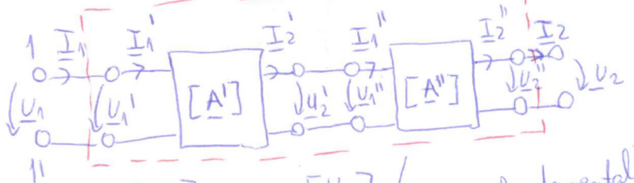


$$\begin{cases} \underline{U}_1 = \underline{U}_1' = \underline{U}_1'' \\ \underline{U}_2 = \underline{U}_2' = \underline{U}_2'' \end{cases} \text{ and (și) } \begin{cases} \underline{I}_1 = \underline{I}_1' + \underline{I}_1'' \\ \underline{I}_2 = \underline{I}_2' + \underline{I}_2'' \end{cases}$$

$$\begin{bmatrix} \underline{I}_1 \\ \underline{I}_2 \end{bmatrix} = [\underline{Y}] \cdot \begin{bmatrix} \underline{U}_1 \\ \underline{U}_2 \end{bmatrix} \left(\text{From } \begin{cases} \underline{I}_1 = \underline{Y}_{11} \underline{U}_1 + \underline{Y}_{12} \cdot \underline{U}_2 \\ \underline{I}_2 = \underline{Y}_{21} \underline{U}_1 + \underline{Y}_{22} \underline{U}_2 \end{cases} \right)$$

$$\Rightarrow \boxed{[\underline{Y}] = [\underline{Y}'] + [\underline{Y}'']}$$

C Cascade (conexiunea în cascadă)

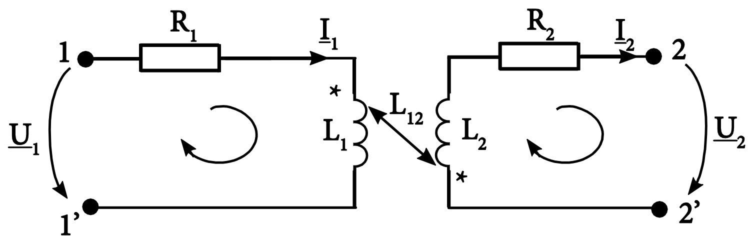


$$\begin{cases} \underline{U}_1 = \underline{U}_1' \\ \underline{U}_2' = \underline{U}_1'' \\ \underline{U}_2'' = \underline{U}_2 \end{cases} \qquad \begin{cases} \underline{I}_1 = \underline{I}_1' \\ \underline{I}_2' = \underline{I}_1'' \\ \underline{I}_2'' = \underline{I}_2 \end{cases}$$

$$\begin{bmatrix} \underline{U}_1 \\ \underline{I}_1 \end{bmatrix} = [\underline{A}] \begin{bmatrix} \underline{U}_2 \\ \underline{I}_2 \end{bmatrix} \left(\text{From fundamental equations} \right) \Rightarrow \boxed{[\underline{A}] = [\underline{A}'] \cdot [\underline{A}'']}$$

<u>Exemple:</u>

1. Pentru cuadripolul din figură, să se determine parametrii impedanţă, $[\underline{Z}]$, şi constantele fundamentale $\underline{A}$, $\underline{B}$, $\underline{C}$, $\underline{D}$.



<u>Soluţie:</u>

Aplicând teorema a doua a lui Kirchhoff pentru cele două ochiuri de circuit rezultă sistemul:

$$\begin{cases} \underline{U}_1 = \underline{I}_1 \cdot (R_1 + j\omega L_1) + \underline{I}_2 \cdot j\omega L_{12} \\ -\underline{U}_2 = \underline{I}_2 \cdot (R_2 + j\omega L_2) + \underline{I}_1 \cdot j\omega L_{12} \; / \cdot (-1) \end{cases} \Rightarrow \begin{cases} \underline{U}_1 = \underline{I}_1 \cdot (R_1 + j\omega L_1) + \underline{I}_2 \cdot j\omega L_{12} \\ \underline{U}_2 = -\underline{I}_2 \cdot (R_2 + j\omega L_2) - \underline{I}_1 \cdot j\omega L_{12} \end{cases}$$

Ţinând cont de parametrii impedanţă, ecuaţiile cuadripolului se rescriu:

$$\begin{cases} \underline{U}_1 = \underline{Z}_{11} \cdot \underline{I}_1 + \underline{Z}_{12} \cdot \underline{I}_2 \\ \underline{U}_2 = \underline{Z}_{21} \cdot \underline{I}_1 + \underline{Z}_{22} \cdot \underline{I}_2 \end{cases}$$

$$\Rightarrow \underline{Z}_{11} = R_1 + j\omega L_1; \quad \underline{Z}_{12} = j\omega L_{12}; \quad \underline{Z}_{21} = -j\omega L_{12}; \quad \underline{Z}_{22} = -\left(R_2 + j\omega L_2\right);$$

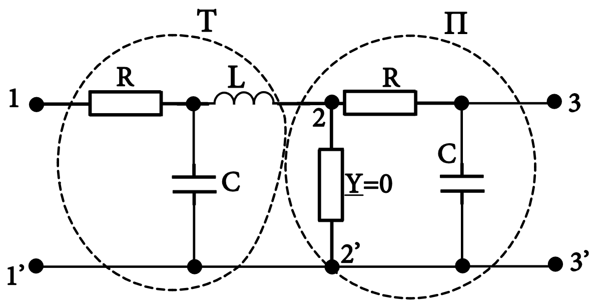
Se observă faptul că este îndeplinită condiția de reciprocitate pentru acest cuadripol: ($\underline{Z}_{12} = -\underline{Z}_{21}$)

Expresiile lui $\underline{A}$, $\underline{B}$, $\underline{C}$, $\underline{D}$ în funcție de parametrii impedanță se determină cu formulele: $\underline{A} = \frac{\underline{Z}_{11}}{\underline{Z}_{21}}$; $\underline{B} = \frac{\underline{Z}_{12}^2 - \underline{Z}_{11} \cdot \underline{Z}_{21}}{\underline{Z}_{21}}$; $\underline{C} = \frac{1}{\underline{Z}_{21}}$; $\underline{D} = -\frac{\underline{Z}_{22}}{\underline{Z}_{21}}$.

Astfel se obțin:

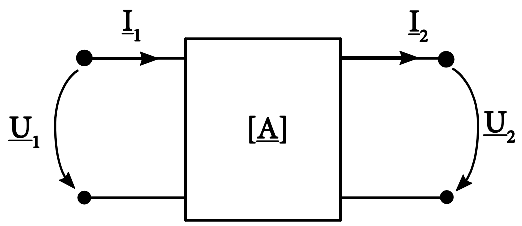
$$\underline{A} = -\frac{R_1 + j\omega L_1}{j\omega L_{12}}; \underline{B} = -\left(R_1 + j\omega L_1 - j\omega L_{12}\right); \underline{C} = -\frac{1}{j\omega L_{12}}; \underline{D} = \frac{R_2 + j\omega L_2}{j\omega L_{12}}$$

2. Se cere schema echivalentă în π pentru cuadripolul din figură dacă $\omega^2 LC = 1$ $\left(\omega L = \frac{1}{\omega C}\right)$.



Soluție

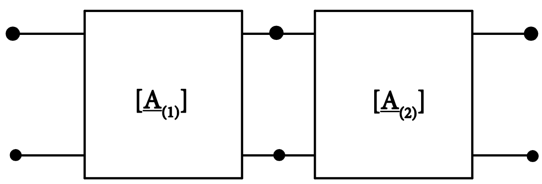
Orice cuadripol poate fi reprezentat ca în figura de mai jos:



Acesta este caracterizat de constantele sale fundamentale (grupate în matricea [$\underline{A}$]), determinate conform relației:

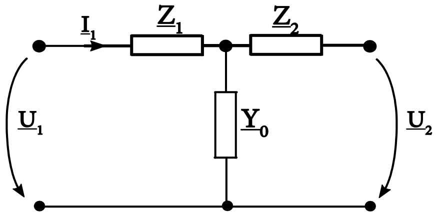
$$\begin{bmatrix} \underline{U}_1 \\ \underline{I}_1 \end{bmatrix} = \underbrace{\begin{bmatrix} \underline{A} & \underline{B} \\ \underline{C} & \underline{D} \end{bmatrix}}_{[\underline{A}]} \cdot \begin{bmatrix} \underline{U}_2 \\ \underline{I}_2 \end{bmatrix}.$$

Problema noastră poate fi deci privită ca o conexiune în lanț între doi cuadripoli, caracterizați de matricile [$\underline{A}_{(1)}$] și [$\underline{A}_{(2)}$]:



Cuadripolul echivalent va putea fi determinat, în final, din relaţia: $\left[\underline{A}\right]=\left[\underline{A}_{(1)}\right]\cdot\left[\underline{A}_{(2)}\right]$

Primul cuadripol (în „T”):



Elementele schemei echivalente în „T” sunt:

$$\underline{Z}_1=\frac{\underline{A}-1}{\underline{C}},\ \underline{Z}_2=\frac{\underline{D}-1}{\underline{C}},\ \underline{Y}_0=\underline{C};$$

unde: $\underline{Z}_1=\mathrm{R},\ \underline{Z}_2=j\omega L,\ \underline{Y}_0=j\omega C$.

Ca urmare, constantele fundamentale ale acestui cuadripol se determină din relaţiile:

$$\begin{cases}\underline{A}=1+\underline{Z}_1\cdot\underline{Y}_0\\ \underline{B}=\underline{Z}_1+\underline{Z}_2+\underline{Z}_1\cdot\underline{Z}_2\cdot\underline{Y}_0\\ \underline{C}=\underline{Y}_0\\ \underline{D}=1+\underline{Z}_2\cdot\underline{Y}_0\end{cases}$$

obţinându-se:

$$\underline{A}_1=1+\underline{Z}_1\cdot\underline{Y}_0=1+\mathrm{R}\cdot j\omega C$$

$$\underline{B}_1=\underline{Z}_1+\underline{Z}_2+\underline{Z}_1\cdot\underline{Z}_2\cdot\underline{Y}_0=\mathrm{R}+j\omega L+j\omega L\cdot\mathrm{R}\cdot j\omega C=\mathrm{R}+j\omega L+j^2\cdot\mathrm{R}=j\omega L$$

$$\underline{C}_1=\underline{Y}_0=j\omega C$$

$$\underline{D}_1=1+\underline{Z}_2\underline{Y}_0=1+j\omega L\cdot j\omega C=0.$$

Deoarece este îndeplinită condiţia $\underline{A}\underline{D}-\underline{B}\underline{C}=0-j^2=1$, cuadripolul este reciproc. Dar fiindcă $\underline{A}\neq\underline{D}$, cuadripolul nu este simetric!

Matricea constantelor fundamentale pentru acest cuadripol este:

$$\left[\underline{A}_{(1)}\right]=\begin{bmatrix}\underline{A}_1 & \underline{B}_1\\ \underline{C}_1 & \underline{D}_1\end{bmatrix}=\begin{bmatrix}1+\mathrm{R}\cdot j\omega C & j\omega L\\ j\omega C & 0\end{bmatrix}.$$

Pentru cuadripolul al doilea, elementele schemei echivalente în „π” sunt:

$$\underline{Y}_1=\frac{\underline{D}-1}{\underline{B}},\ \underline{Y}_2=\frac{\underline{A}-1}{\underline{B}},\ \underline{Z}_0=\underline{B}\text{, unde: }\underline{Z}_0=\mathrm{R},\ \underline{Y}_1=j\omega C,\ \underline{Y}_2=0\ .$$

Drept urmare, constantele fundamentale se determină cu relaţiile:

$$\begin{cases}\underline{A}=1+\underline{Z}_0\cdot\underline{Y}_2\\ \underline{B}=\underline{Z}_0\\ \underline{C}=\underline{Y}_1+\underline{Y}_2+\underline{Y}_1\cdot\underline{Y}_2\cdot\underline{Z}_0\\ \underline{D}=1+\underline{Z}_0\cdot\underline{Y}_1\end{cases}\quad\text{obţinându-se:}\quad\begin{cases}\underline{A}_2=1+\mathrm{R}\cdot j\omega C\\ \underline{B}_2=\mathrm{R}\\ \underline{C}_2=j\omega C\\ \underline{D}_2=1\end{cases}.$$

Matricea constantelor fundamentale pentru acest cuadripol este:

$$\left[\mathrm{A}_{(2)}\right]=\begin{bmatrix}\underline{A}_2 & \underline{B}_2\\ \underline{C}_2 & \underline{D}_2\end{bmatrix}=\begin{bmatrix}1+\mathrm{R}\cdot j\omega C & \mathrm{R}\\ j\omega C & 1\end{bmatrix}$$

Revenind acum la cuadripolul echivalent, matricea constantelor fundamentale ale acestuia este:

$$\left[\mathrm{A}_{(ech)}\right]=\left[\mathrm{A}_{(1)}\right]\cdot\left[\mathrm{A}_{(2)}\right]=\begin{bmatrix}1+\mathrm{R}\cdot j\omega C & j\omega L\\ j\omega C & 0\end{bmatrix}\cdot\begin{bmatrix}1+\mathrm{R}\cdot j\omega C & \mathrm{R}\\ j\omega C & 1\end{bmatrix}=$$

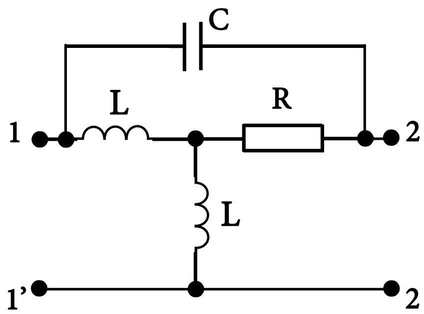
$$=\begin{bmatrix}\left(1+\mathrm{R}\cdot j\omega C\right)^2+j^2\omega^2LC & \mathrm{R}\left(1+\mathrm{R}\cdot j\omega C\right)+j\omega L\\ j\omega C\left(1+\mathrm{R}\cdot j\omega C\right) & \mathrm{R}\cdot j\omega C\end{bmatrix}=$$

$$=\left[\begin{bmatrix}\mathrm{R}\cdot\omega C\left(2j-\mathrm{R}\cdot\omega C\right)=\underline{A}_{ech} & \mathrm{R}+\mathrm{R}^2\cdot j\omega C+\dfrac{j}{\omega C}=\underline{B}_{ech}\\ j\omega C-\omega^2C^2\mathrm{R}=\underline{C}_{ech} & \mathrm{R}\cdot j\omega C=\underline{D}_{ech}\end{bmatrix}\right]$$

Parametrii schemei echivalente în „ π ” se determină în final cu formulele:

$$\underline{Y}_1^{ech} = \frac{\underline{D}_{ech} - 1}{\underline{B}_{ech}} \qquad \underline{Y}_2^{ech} = \frac{\underline{A}_{ech} - 1}{\underline{B}_{ech}} \qquad \underline{Z}_0^{ech} = \underline{B}_{ech}$$

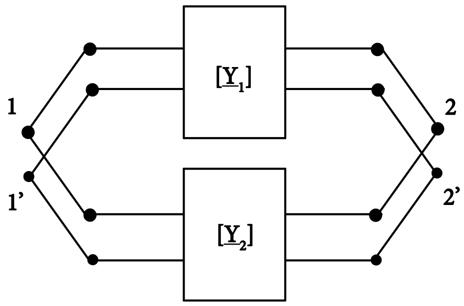
3. Se dă cuadripolul:



Să se determine elementele schemei echivalente în „T” ştiind că $\omega L = \frac{1}{\omega C} = R$.

Soluţie:

Privim cuadripolul ca doi cuadripoli conectaţi în paralel:

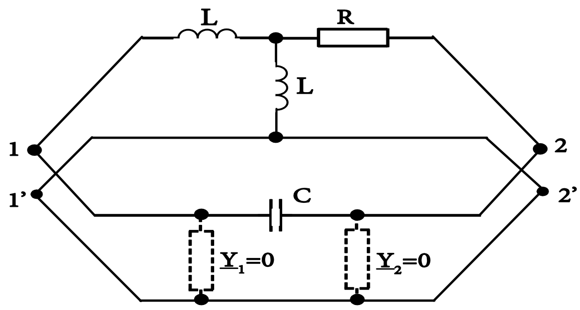


Fiecare dintre aceşti doi cuadripoli este caracterizat de ecuaţia matricială: $\begin{bmatrix} \underline{I}_1 \\ \underline{I}_2 \end{bmatrix} = \begin{bmatrix} \underline{Y}_{11} & \underline{Y}_{12} \\ \underline{Y}_{21} & \underline{Y}_{22} \end{bmatrix} \cdot \begin{bmatrix} \underline{U}_1 \\ \underline{U}_2 \end{bmatrix}$;

unde: $\underline{Y}_{11} = \frac{\underline{D}}{\underline{B}}$; $\underline{Y}_{12} = -\frac{\underline{A} \cdot \underline{D} - \underline{B} \cdot \underline{C}}{\underline{B}}$; $\underline{Y}_{21} = \frac{1}{\underline{B}}$; $\underline{Y}_{22} = -\frac{\underline{A}}{\underline{B}}$.

Condiţia de reciprocitate pentru aceşti cuadripoli este: $\underline{Y}_{21} = -\underline{Y}_{12}$.

Primul cuadripol (în „T”):



Elementele schemei echivalente în „T” se identifică sub forma:

$$\underline{Z}_1 = j\omega L,\ \underline{Z}_2 = R,\ \underline{Y}_0 = \frac{1}{j\omega L};$$

iar constantele fundamentale sunt:

$$\begin{cases} \underline{A}_1 = 1 + \underline{Z}_1 \cdot \underline{Y}_0 = 1 + j\omega L \cdot \dfrac{1}{j\omega L} = 2; \\ \underline{B}_1 = \underline{Z}_1 + \underline{Z}_2 + \underline{Z}_1 \cdot \underline{Z}_2 \cdot \underline{Y}_0 = j\omega L + 2R = R(2+j); \\ \underline{C}_1 = \underline{Y}_0 = \dfrac{1}{j\omega L} = \dfrac{1}{jR}; \\ \underline{D}_1 = 1 + \underline{Z}_2 \cdot \underline{Y}_0 = 1 + R \cdot \dfrac{1}{j\omega L} = 1 - j\ . \end{cases}$$

Matricea admitanţă: $\left[Y_{(1)}\right] = \begin{bmatrix} \underline{Y}_{11} & \underline{Y}_{12} \\ \underline{Y}_{21} & \underline{Y}_{22} \end{bmatrix}$ se determină imediat cu relaţiile:

$$\underline{Y}_{11} = \frac{\underline{D}_1}{\underline{B}_1} = \frac{1-j}{R\cdot(2+j)};\quad \underline{Y}_{12} = -\frac{2(1-j) - \dfrac{2+j}{j}}{R(2+j)} = -\frac{2-2j+2j-1}{R(2+j)} = -\frac{1}{R(2+j)};$$

$$\underline{Y}_{21} = \frac{1}{\underline{B}_1} = \frac{1}{R(2+j)};\qquad \underline{Y}_{22} = -\frac{\underline{A}_1}{\underline{B}_1} = -\frac{2}{R(2+j)}.$$

Deci : $\left[\underline{Y}_{(1)}\right] = \begin{bmatrix} \dfrac{1-j}{R(2+j)} & -\dfrac{1}{R(2+j)} \\ \dfrac{1}{R(2+j)} & -\dfrac{2}{R(2+j)} \end{bmatrix}.$

Elementele schemei echivalente în „π”pentru al doilea cuadripol se identifică sub forma: $\underline{Y}_1 = 0,\ \underline{Y}_2 = 0;\ \underline{Z}_0 = \dfrac{1}{j\omega C}$; iar constantele fundamentale sunt:

$$\begin{cases} \underline{A}_2 = 1 + \underline{Z}_0 \cdot \underline{Y}_2 = 1 \\ \underline{B}_2 = \underline{Z}_0 = \dfrac{1}{j\omega C} = -jR \\ \underline{C}_2 = \underline{Y}_1 + \underline{Y}_2 + \underline{Y}_1 \cdot \underline{Y}_2 \cdot \underline{Z}_0 = 0 \\ \underline{D}_2 = 1 + \underline{Z}_0 \cdot \underline{Y}_1 = 1 \end{cases}$$

Matricea admitanţă se calculează direct şi este: $\left[\underline{Y}_{(2)}\right] = \begin{bmatrix} \dfrac{j}{R} & -\dfrac{j}{R} \\ \dfrac{j}{R} & -\dfrac{j}{R} \end{bmatrix}$

Se observă că sunt îndeplinite condiţiile:

$\underline{Y}_{12} = -\underline{Y}_{21}$ (reciprocitate) şi

$\underline{Y}_{22} = -\underline{Y}_{11}$ (simetrie).

Cei doi cuadripoli fiind conectaţi în paralel, matricea admitanţă a cuadripolului echivalent se obţine cu relaţia:

$$[\underline{Y}]=\left[\underline{Y}_{(1)}\right]+\left[\underline{Y}_{(2)}\right]=\begin{bmatrix}\dfrac{(1-j)+j(2+j)}{R(2+j)} & \dfrac{-1-j(2+j)}{R(2+j)}\\[2mm] \dfrac{1+j(2+j)}{R(2+j)} & \dfrac{-2-j(2+j)}{R(2+j)}\end{bmatrix}=\begin{bmatrix}\dfrac{j}{R(2+j)} & \dfrac{-2j}{R(2+j)}\\[2mm] \dfrac{2j}{R(2+j)} & \dfrac{-2j-1}{R(2+j)}\end{bmatrix}.$$

Deci: $\underline{Y}_{11}^{tot}=\dfrac{j}{R(2+j)}$; $\underline{Y}_{12}^{tot}=\dfrac{-2j}{R(2+j)}$; $\underline{Y}_{21}^{tot}=\dfrac{2j}{R(2+j)}$; $\underline{Y}_{22}^{tot}=\dfrac{-2j-1}{R(2+j)}$.

De asemenea, constantele fundamentale ale cuadripolului echivalent sunt:

$$\underline{A}^{tot}=-\frac{\underline{Y}_{22}^{tot}}{\underline{Y}_{21}^{tot}}=\frac{2j+1}{R(2+j)}\cdot\frac{R(2+j)}{2j}=\frac{2j+1}{2j}=1+\frac{1}{2j}=1-j\frac{1}{2}\ ;$$

$$\underline{B}^{tot}=\frac{1}{\underline{Y}_{21}^{tot}}=\frac{R(2+j)}{2j}=\frac{R}{j}+\frac{1}{2}=\frac{1}{2}-jR\ ;$$

$$\underline{C}^{tot}=\frac{-\underline{Y}_{11}^{tot}\cdot\underline{Y}_{22}^{tot}+\underline{Y}_{12}^{tot}\cdot\underline{Y}_{21}^{tot}}{\underline{Y}_{21}^{tot}}=\frac{\dfrac{(2j+1)j}{\left(R(2+j)\right)^2}-\dfrac{(2j)^2}{\left(R(2+j)\right)^2}}{\dfrac{2j}{R(2+j)}}=$$

$$=\frac{2j^2+j-4j^2}{\left(R(2+j)\right)^2}\cdot\frac{R(2+j)}{2j}=\frac{2+j}{R(2+j)}\cdot\frac{1}{2j}=-j\cdot\frac{1}{2R};$$

$$\underline{D}^{tot}=\frac{\underline{Y}_{11}^{tot}}{\underline{Y}_{21}^{tot}}=\frac{j}{R(2+j)}\cdot\frac{R(2+j)}{2j}=\frac{1}{2}.$$

În final, găsim elementele cuadripolului echivalent în „T” ($\underline{Z}_1,\underline{Z}_2,\underline{Y}_0$):

$$\underline{Y}_0=\underline{C}^{tot}=-j\frac{1}{2R};\qquad \underline{Z}_1=\frac{\underline{A}^{tot}-1}{\underline{Y}_0}=\frac{1-j\frac{1}{2}-1}{-j\frac{1}{2R}}=\frac{1}{2}\cdot 2R=R;$$

$$\underline{Z}_2=\frac{\underline{D}^{tot}-1}{\underline{Y}_0}=\frac{\frac{1}{2}-1}{-j\frac{1}{2R}}=-jR.$$

Observaţie:

Aceeaşi schemă poate fi considerată ca fiind echivalentă cu doi cuadripoli conectaţi în serie, având structura din figura de mai jos. Problema poate fi astfel rezolvată utilizând parametrii impedanţă în locul parametrilor admitanţă, utilizaţi în schema paralel.