Să se calculeze diferența divizată :

$$\left[-1, 0, \tfrac{1}{2}, 1, \dots, k ; x^n \sin \pi x\right] ,$$

$k \in \mathbb{N}$, $k \geq 2$, $n \geq 0$

<u>Rezolvare :</u>

Vom folosi următorul rezultat teoretic :

$$[x_0, x_1, \dots, x_m ; f] = \sum_{i=0}^{m} \frac{f(x_i)}{\ell'(x_i)} ,$$

unde $\ell(x) = (x-x_0)(x-x_1)\dots(x-x_m)$ este polinomul nodal de grad $m+1$ (are ca rădăcini nodurile $x_0, x_1, \dots, x_m$).

Mai mult,

$$\ell'(x) = (x-x_1)(x-x_2)\dots(x-x_m) + (x-x_0)(x-x_2)\dots(x-x_m) + \dots + (x-x_0)(x-x_1)\dots(x-x_{m-1})$$

În cazul nostru $f(x) = x^n \sin \pi x$

$x_0 = -1$, $x_1 = 0$, $x_2 = \frac{1}{2}$, $x_3 = 1$, ..., $x_{k+1} = k-1$, $x_{k+2} = k$

(avem $k+3$ noduri)

$$\ell(x) = (x+1)(x-0)(x-\tfrac{1}{2})(x-1)\dots(x-k)$$

↳ polinomul nodal de grad $k+3$

$$\left[-1, 0, \tfrac{1}{2}, 1, \dots, k ; x^n \sin \pi x\right] =$$

$$= \frac{(-1)^n \sin \pi(-1)}{\ell'(-1)} + \frac{0^n \cdot \sin \pi \cdot 0}{\ell'(0)} + \frac{\left(\frac{1}{2}\right)^n \cdot \sin \frac{\pi}{2}}{\ell'(\frac{1}{2})} + \frac{1^n \cdot \sin \pi \cdot 1}{\ell'(1)} + \dots + \frac{k^n \sin \pi k}{\ell'(k)} =$$

$$= \left(\frac{1}{2}\right)^n \cdot \frac{\sin \frac{\pi}{2}}{\ell'(\frac{1}{2})} = \frac{1}{2^n} \cdot \frac{1}{\ell'(\frac{1}{2})}$$

Dar $\ell'(\frac{1}{2}) = (\frac{1}{2}+1)(\frac{1}{2}-0)(\frac{1}{2}-1)\dots(\frac{1}{2}-k) = \frac{3}{4} \cdot (-1)^k \cdot \frac{(2k-1)!!}{2^k}$

$$\left[-1, 0, \tfrac{1}{2}, 1, \dots, k ; x^n \sin x\right] = \frac{1}{2^n} \cdot \frac{1}{\frac{3}{4}(-1)^k \cdot \frac{(2k-1)!!}{2^k}} = \frac{4}{3} \cdot \frac{(-1)^k}{(-1)^{2k}} \cdot \frac{1}{2^n} \cdot \frac{2^k}{(2k-1)!!} =$$

$$= (-1)^k \cdot \frac{4}{3} \cdot 2^{k-n} \cdot \frac{1}{(2k-1)!!}$$