Numerical Calculus

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2025

References

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2. L. Piegl and W. Tiller, *The NURBS Book*, 2nd Ed., Springer-Verlag, Berlin Heidelberg, 1996.
3. D.F. Rogers, *An Introduction to NURBS. With Historical Perspective*, Morgan Kaufmann Publishers, San Francisco, 2001.

Bernstein polynomials (Piegl and Tiller, 1996)

Let $n \in \mathbb{N}$. The *classical nth degree Bernstein polynomials* are given by

$$b_{n,k}(t) = \binom{n}{k} t^k (1-t)^{n-k}, \ \forall\, k = \overline{0, n}, \ \forall\, t \in [0,1]$$

$$n = 0:\ b_{00}(t) = 1$$

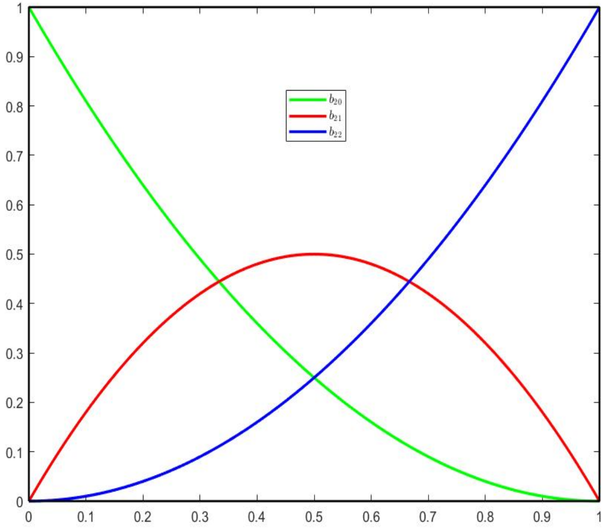
$$n = 1:\ b_{10}(t) = 1 - t,\ b_{11}(t) = t$$

$$n = 2:\ b_{20}(t) = (1-t)^2,\ b_{21}(t) = 2t(1-t),\ b_{22}(t) = t^2$$

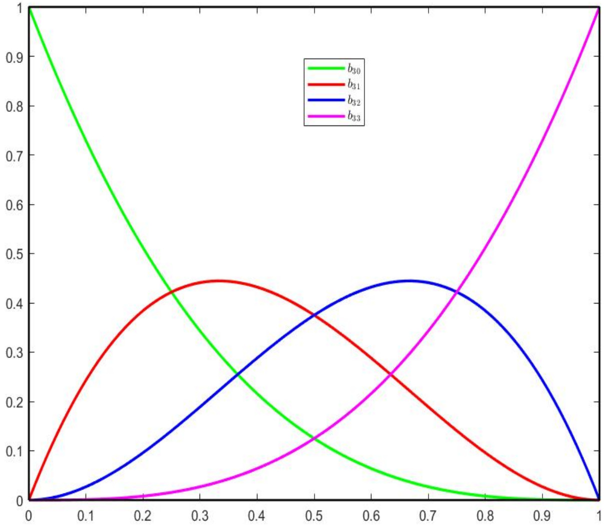
$$n = 3:\ b_{30}(t) = (1-t)^3,\ b_{31}(t) = 3t(1-t)^2,$$

$$b_{32}(t) = 3t^2(1-t),\ b_{33}(t) = t^3$$

Bernstein polynomials



Bernstein polynomials



Bernstein polynomials

We have

$$b_{n,0}(0) = b_{n,n}(1) = 1$$

Nonnegativity

$$b_{n,k}(t) \geq 0,\ \forall\, t \in [0,1],\ \forall\, k = \overline{0,n}$$

Partition of unity

$$\sum_{k=0}^{n} b_{n,k}(t) = \sum_{k=0}^{n} \binom{n}{k} t^k (1-t)^{n-k} =$$

$$= (t + (1-t))^n = 1,\ \forall t \in [0,1]$$

Bernstein polynomials (Piegl and Tiller, 1996; Farin, 2001)

Linear precision

$$\sum_{k=0}^{n} b_{n,k}(t)\frac{k}{n} = t, \forall\, t \in [0,1]$$

Symmetry: for any n, the set of polynomials $\{b_{n,k}\}$ is symmetric with respect to $t = \frac{1}{2}$

$$b_{n,k}(t) = b_{n,n-k}(1-t), \forall\, t \in [0,1], \forall\, k = \overline{0,n}$$

$b_{n,k}$ attains exactly one maximum on the interval $[0,1]$, that is, at $t = \frac{k}{n}$

$$\max_{t\in[0,1]} b_{n,k}(t) = b_{n,k}\left(\frac{k}{n}\right), \forall\, k = \overline{0,n}$$

Bernstein polynomials (Piegl and Tiller, 1996)

Recursive definition:

$$b_{n,k}(t) = (1-t)b_{n,k-1}(t) + tb_{n-1,k-1}(t),$$

$$b_{n,k}(t) = 0 \text{ for all } k < 0 \text{ and } k > n$$

Derivatives:

$$b'_{n,k}(t) = n(b_{n-1,k-1}(t) - b_{n-1,k}(t)),$$

$$b_{n-1,-1}(t) = b_{n-1,n}(t) = 0$$

Integrals:

$$\int_0^1 b_{n,k}(t)\,dt = \frac{1}{n+1}, \forall\, k = \overline{0, n}$$

Formulas for Bernstein polynomials (Farin, 2001)

The *power basis* t^k and the *Bernstein basis* $b_{n,k}$ are related by

$$t^k = \sum_{j=k}^{n} \frac{\binom{j}{k}}{\binom{n}{k}} b_{n,j}(t),\ \forall\, t \in [0,1],\ \forall\, k = \overline{0,n}$$

and

$$b_{n,k}(t) = \sum_{j=k}^{n} (-1)^{j-k} \binom{n}{j} \binom{j}{k} t^j, \forall\, t \in [0,1],\ \forall k = \overline{0,n}$$

Product

$$b_{m,i}(t) b_{n,j}(t) = \frac{\binom{m}{i}\binom{n}{j}}{\binom{m+n}{i+j}} b_{m+n,i+j}(t),$$

$$\forall\, t \in [0,1], \forall\, i = \overline{0,m},\ \forall\, j = \overline{0,n}$$

Matrix representation of Bernstein polynomials (Rogers, 2001)

For all $n \in \mathbb{N}$ and $t \in [0,1]$ we have

$$\begin{pmatrix} b_{n,0}(t) \\ b_{n,1}(t) \\ \vdots \\ b_{n,n}(t) \end{pmatrix} = \begin{pmatrix} t^n & t^{n-1} & \dots & 1 \end{pmatrix} M$$

where the individual terms in $M = (m_{ij}) \in \mathcal{M}_{n+1}(\mathbb{R})$ are given by

$$m_{i+1j+1} = \begin{cases} (-1)^{n-i-j}\binom{n}{j}\binom{n-j}{n-i-j}, 0 \le i+j \le n \\ 0, \text{ otherwise} \end{cases} \qquad i,j = \overline{0,n}$$

Bézier Curve (Piegl and Tiller, 1996; Rogers, 2001)

An *nth degree Bézier curve* $\mathcal{C}(t) = (x(t), y(t))$ is defined by

$$\mathcal{C}(t) = \sum_{k=0}^{n} b_{n,k}(t)\boldsymbol{P}_k = \begin{cases} x(t) = \sum_{k=0}^{n} b_{n,k}(t)P_{kx} \\ y(t) = \sum_{k=0}^{n} b_{n,k}(t)P_{ky} \end{cases} \qquad t \in [0, 1]$$

where $\boldsymbol{P}_k(P_{kx}, P_{ky})$ are called *control points*.
The polygon formed by $\{\boldsymbol{P}_0, \boldsymbol{P}_1, \ldots, \boldsymbol{P}_n\}$ is called *control polygon*.

$$\mathcal{C}(0) = \boldsymbol{P}_0, \, \mathcal{C}(1) = \boldsymbol{P}_n$$

The first and last points on the curve are coincident with the first and last points of the control polygon.

Bézier Curve (Piegl and Tiller, 1996)

The general expression for the derivative of a Bézier curve is

$$\mathcal{C}'(t) = \frac{d}{dt}\Big(\sum_{k=0}^{n} b_{n,k}(t)\boldsymbol{P}_k\Big) = \sum_{k=0}^{n} b'_{n,k}(t)\boldsymbol{P}_k =$$

$$= \sum_{k=0}^{n} n(b_{n-1,k-1}(t) - b_{n-1,k}(t))\boldsymbol{P}_k =$$

$$= n\sum_{k=0}^{n-1} b_{n-1,k}(t)(\boldsymbol{P}_{k+1} - \boldsymbol{P}_k)$$

The derivative of an nth degree Bézier curve is an $(n-1)$th degree Bézier curve.

Bézier Curve (Piegl and Tiller, 1996)

$$\begin{cases} x'(t) = n\sum_{k=0}^{n-1} b_{n-1,k}(t)(P_{k+1x} - P_{kx}) \\ y'(t) = n\sum_{k=0}^{n-1} b_{n-1,k}(t)(P_{k+1y} - P_{ky}) \end{cases}$$

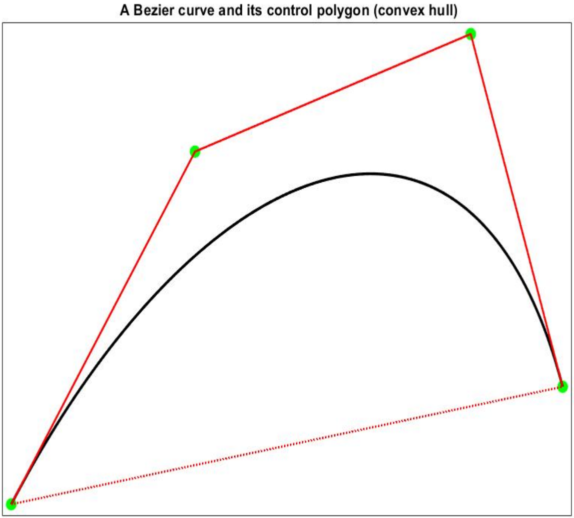
Formulas for the end derivatives

$$\mathcal{C}'(0) = n(\boldsymbol{P}_1 - \boldsymbol{P}_0), \quad \mathcal{C}'(1) = n(\boldsymbol{P}_n - \boldsymbol{P}_{n-1})$$

$$\begin{cases} x'(0) = n(P_{1x} - P_{0x}) \\ y'(0) = n(P_{1y} - P_{0y}) \end{cases} \quad \begin{cases} x'(1) = n(P_{nx} - P_{n-1x}) \\ y'(1) = n(P_{ny} - P_{n-1y}) \end{cases}$$

The tangent directions to the curve at its endpoints are parallel to $\boldsymbol{P}_0\boldsymbol{P}_1$ and $\boldsymbol{P}_{n-1}\boldsymbol{P}_n$.

Convex Hull



Bézier Curve (Rogers, 2001)

Properties of Bézier curves

- The degree of the polynomial defining the curve segment is one less than the number of control polygon points.
- The curve generally follows the shape of the control polygon.
- The curve is contained within the *convex hull* of the control polygon, *i.e.*, within the largest convex polygon defined by the vertices of the control polygon.
- The curve exhibits the *variation-diminishing property*. Basically, this means that the curve does not oscillate about any straight line more often than the control polygon.
- The curve is invariant under an affine transformation.

Continuity between Bézier curves (Rogers, 2001)

If one Bézier curve, $\mathcal{C}_1(t)$ of degree n, is defined by vertices $\boldsymbol{P}_k$ and an adjacent Bézier curve $\mathcal{C}_2(t)$ of degree m, by vertices $\boldsymbol{Q}_k$, then the C^1 continuity at the point where the curves join is given by

$$\begin{cases} \mathcal{C}_1(1) = \mathcal{C}_2(0) \\ \mathcal{C}_1'(1) = \alpha \mathcal{C}_2'(0) \end{cases}$$

We obtain

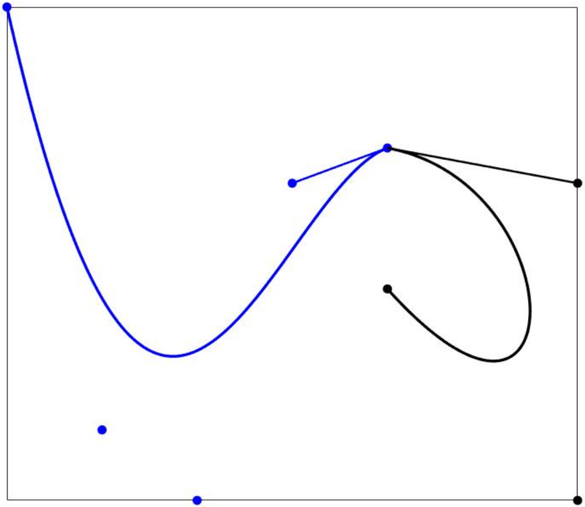
$$\begin{cases} \boldsymbol{Q}_0 = \boldsymbol{P}_n \\ \boldsymbol{Q}_1 - \boldsymbol{Q}_0 = \frac{1}{\alpha}\frac{n}{m}(\boldsymbol{P}_n - \boldsymbol{P}_{n-1}) \end{cases}$$

It implies

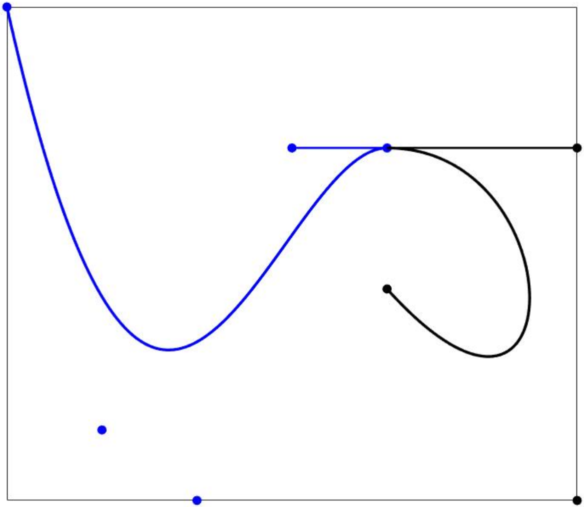
$$\boldsymbol{Q}_1 = \frac{1}{\alpha}\frac{n}{m}(\boldsymbol{P}_n - \boldsymbol{P}_{n-1}) + \boldsymbol{P}_n$$

Thus, the tangent vector directions at the join are the same if the three vertices $\boldsymbol{P}_{n-1}$, $\boldsymbol{P}_n$ and $\boldsymbol{Q}_0$ are collinear.

C^0 continuity between Bézier curves



C^1 continuity between Bézier curves



Increasing the flexibility of Bézier curves. Degree elevation (Rogers, 2011)

If a Bézier curve with additional flexibility is required, the degree of the defining polynomial can be increased by increasing the number of control polygon points. For every point on a Bézier curve with n control polygon vertices $\boldsymbol{P}_0, \ldots, \boldsymbol{P}_n$, the same point on a new Bézier curve with $n+1$ control polygon vertices $\boldsymbol{P}_0^*, \ldots, \boldsymbol{P}_{n+1}^*$ is given by

$$\mathcal{C}(t) = \sum_{k=0}^{n} b_{n,k}(t)\boldsymbol{P}_k = \sum_{k=0}^{n+1} b_{n+1,k}(t)\boldsymbol{P}_k^*$$

$$\begin{cases} \boldsymbol{P}_0^* = \boldsymbol{P}_0, \\ \boldsymbol{P}_k^* = \alpha_k \boldsymbol{P}_{k-1} + (1-\alpha_k)\boldsymbol{P}_k,\ \alpha_k = \frac{k}{n+1},\ k = \overline{1, n} \\ \boldsymbol{P}_{n+1}^* = \boldsymbol{P}_n \end{cases}$$

This technique can be applied successively. In the limit, the polygon converges to the curve.

Rational Bézier Curves (Piegl and Tiller, 1996)

Although polynomials offer many advantages, there exist a number of important curve and surface types which cannot be represented precisely using polynomials, *e.g.*, circles, ellipses, hyperbolas, cylinders, cones, spheres, etc.

For example, the unit circle in the xy plane, centered at the origin, cannot be represented using polynomial coordinate functions. In contrast, let us assume that

$$x(t) = a_0 + a_1 t + \ldots + a_n t^n$$

$$y(t) = b_0 + b_1 t + \ldots + b_n t^n$$

Rational Bézier Curves (Piegl and Tiller, 1996)

Then $x^2 + y^2 - 1 = 0$ implies that

$$0 = (a_0 + a_1 t + \dots a_n t^n)^2 + (b_0 + b_1 t + \dots + b_n t^n)^2 - 1 =$$

$$= (a_0^2 + b_0^2 - 1) + 2(a_0 a_1 + b_0 b_1)t +$$

$$+(a_1^2 + 2a_0 a_2 + b_1^2 + 2b_0 b_2)t^2 + \dots +$$

$$+(a_{n-1}^2 + 2a_{n-2} a_n + b_{n-1}^2 + 2b_{n-2} b_n)t^{2n-2} +$$

$$+2(a_n a_{n-1} + b_n b_{n-1})t^{2n-1} + (a_n^2 + b_n^2)t^{2n}$$

This equation must hold for all t, which implies that $a_i = b_i = 0$, $\forall\, i = \overline{1, n}$. Thus, $x(t) = a_0$ and $y(t) = b_0$, which is an obvious contradiction.

Rational Bézier Curves (Piegl and Tiller, 1996)

It is known from classical mathematics that all the conic curves, including the circle, can be represented using *rational functions*, which are defined as the ratio of two polynomials. In fact, they are represented with rational functions of the form

$$x(t) = \frac{X(t)}{W(t)}, \quad y(t) = \frac{Y(t)}{W(t)},$$

where $X(t)$, $Y(t)$ and $W(t)$ are polynomials (each of the coordinate functions has the same denominator).
Circle of radius 1, centered at the origin

$$x(t) = \frac{1 - t^2}{1 + t^2}, \quad y(t) = \frac{2t}{1 + t^2}$$

Rational Bézier Curves (Piegl and Tiller, 1996)

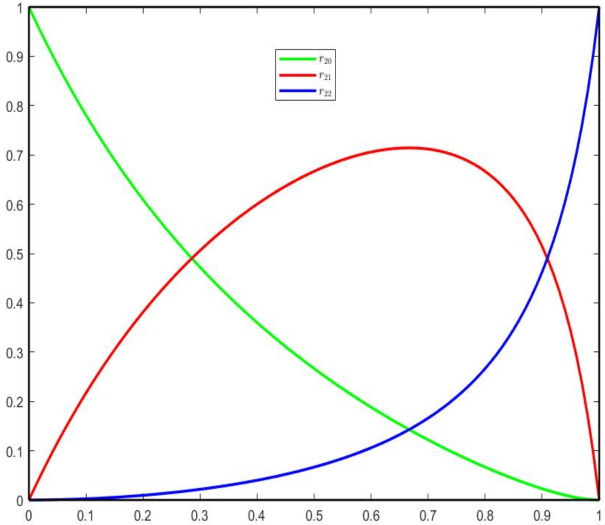
An *nth degree rational Bézier curve* is given by

$$\mathcal{C}(t) = \frac{\sum_{k=0}^{n} b_{n,k}(t) w_k \boldsymbol{P}_k}{\sum_{k=0}^{n} b_{n,k}(t) w_k} = \sum_{k=0}^{n} \underbrace{\frac{b_{n,k}(t) w_k}{\sum_{k=0}^{n} b_{n,k}(t) w_k}}_{=r_{n,k(t)}} \boldsymbol{P}_k = \sum_{k=0}^{n} r_{n,k}(t) \boldsymbol{P}_k$$

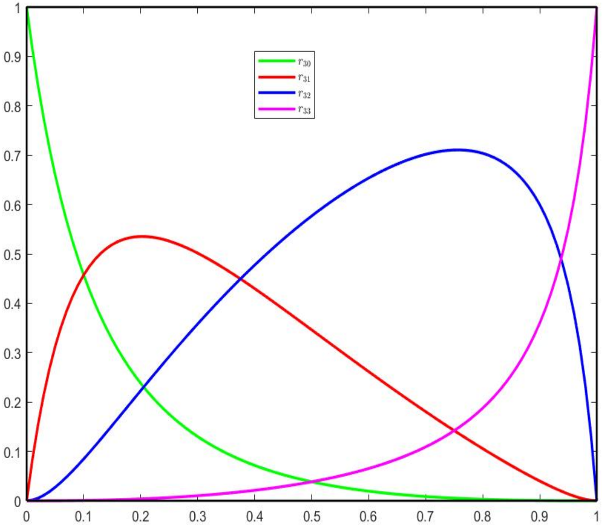
where w_k are scalars, called *weights*.

$$r_{n,k}(t) = \frac{b_{n,k}(t) w_k}{\sum_{k=0}^{n} b_{n,k}(t) w_k}, \quad t \in [0, 1]$$

Rational basis functions $(n = 2,\ w_0 = 4,\ w_1 = 5,\ w_2 = 1)$



Rational basis functions ($n = 3$, $w_0 = 1$, $w_1 = 3$, $w_2 = 5$, $w_3 = 1$)



Rational basis functions (Piegl and Tiller, 1996)

The *rational basis functions*, $r_{n,k}$, have the following properties

$r_{n,k}(t) \geq 0$, $\forall\, k, n$ and $0 \leq t \leq 1$ (nonnegativity)

$\sum_{k=0}^{n} r_{n,k}(t) = 1$ (partition of unity)

$r_{n,0}(0) = r_{n,n}(1) = 1$

$r_{n,k}$ attains exactly one maximum on the interval $[0, 1]$

if $w_k = 1$ for all k, then $r_{n,k}(t) = b_{k,n}(t)$ for all k, *i.e.*, $b_{n,k}$ are a special case of $r_{n,k}$

Rational basis functions (Piegl and Tiller, 1996)

The rational Bézier curves have the following geometric properties

- *convex hull property*: the curves are contained in the convex hull of their defining control points
- *transformation invariance*: rotations, translations and scalings are applied to the curve by applying them to the control points
- *endpoint interpolation*: $\mathcal{C}(0) = \boldsymbol{P}_0$ and $\mathcal{C}(1) = \boldsymbol{P}_n$
- the kth derivative at $t = 0$ ($t = 1$) depends on the first (last) $k + 1$ control points and weights;
- *variation diminishing property* (the curve does not oscillate about any straight line more often than the control polygon)

Rational Bézier curve ($w_0 = 1,\ w_1 = 1,\ w_2 = 2$)