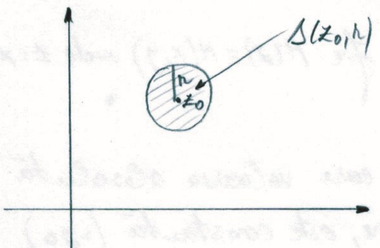
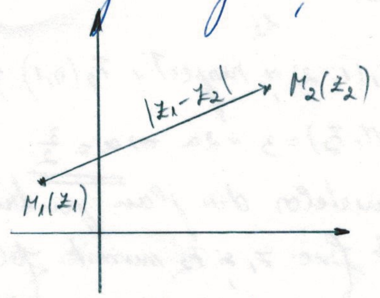
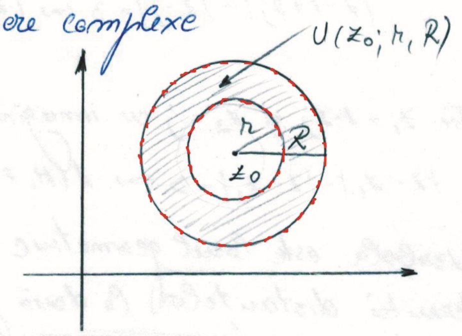
Seminarul 1

Numere complexe. Operații cu numere complexe







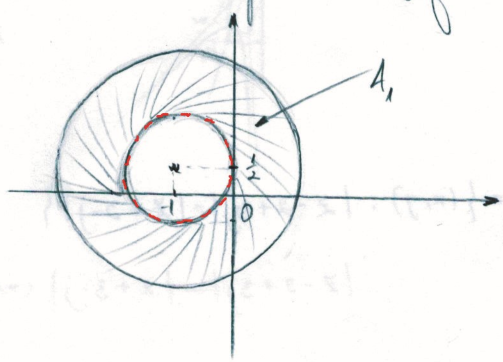
Fie $M_1(z_1)$ și $M_2(z_2)$. Atunci $d(M_1, M_2) = |z_1 - z_2| = M_1M_2$

$\Delta(z_0, r) = \{z \in \mathbb{C} : |z - z_0| < r\}$ $\qquad \Gamma(z_0; r) = \{z \in \mathbb{C} : |z - z_0| = r\}$ ← frontiera discului $\bar{\Delta}(z_0, r)$

$\bar{\Delta}(z_0, r) = \{z \in \mathbb{C} : |z - z_0| \le r\}$ $\qquad U(z_0; r, R) = \{z \in \mathbb{C} : r < |z - z_0| < R\}$

① Fie $z = x + j \cdot y \in \mathbb{C}$ și $M(x, y) = M(z)$. Să se determine următoarele mulțimi din $\mathbb{C}$.

$A_1 = \{(x, y) : 2 < |2z + 2 - j| \le 4\}$ ← imaginea geometrică a numărului complex $z = x + jy$



$2 < |2z + 2 - j| \le 4 \Leftrightarrow$

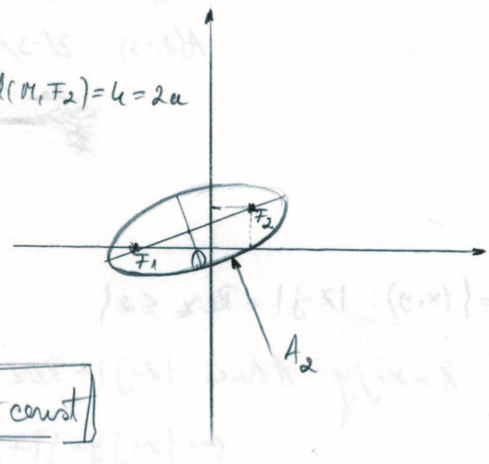
$\Leftrightarrow 2 < |2(z - (-1 + \frac{1}{2}j))| \le 4 \quad |:2$

$\Leftrightarrow 1 < |z - \underbrace{(-1 + \frac{1}{2}j)}_{z_0}| \le 2 \Leftrightarrow z \in U(z_0; 1, 2) \cup \Gamma(z_0; 2)$

$$\Downarrow$$

$$\underline{A_1 = U(z_0; 1, 2) \cup \Gamma(z_0; 2)}$$

$A_2 = \{(x, y) : |z + 2| + |z - 1 - j| = 4\}$



$|z + 2| + |z - 1 - j| = 4 \Leftrightarrow |z - (-2)| + |z - (1 + j)| = 4 \Leftrightarrow d(M, F_1) + d(M, F_2) = 4 = 2a$

Fie $M(z)$ $\qquad F_1(-2, 0) \equiv F_1(-2)$ $\qquad F_2(1, 1) \equiv F_2(1 + j)$

Obs Elipsa este locul geometric al punctelor din plan pentru care suma distanțelor la două puncte fixe F_1 și F_2 numite focare este constantă $(= 2a)$ $\boxed{MF_1 + MF_2 = 2a - \text{const}}$

$(E): \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $a^2 = b^2 + c^2$

În cazul nostru, $\underline{a = 2}$, $\underline{c} = \frac{F_1F_2}{2} = \frac{\sqrt{(-3)^2 + (-1)^2}}{2} = \frac{\sqrt{10}}{2} \longrightarrow b^2 = 4 - \frac{10}{4} = \frac{6}{4} \Rightarrow \underline{b = \frac{\sqrt{6}}{2}}$

Așadar, A_2 este elipsa cu focarele F_1 și F_2 și cu semiaxa focarelor a de lungime 2.

$A_3 = \{(x,y) : |z-1+2j| - |z-j| = 3\}$

$$|z-1+2j|-|z-j|=3 \Leftrightarrow |z-(\underbrace{1-2j}_{z_1})|-|z-\underbrace{j}_{z_2}|=3$$

Fie $z_1 = 1-2j$ și $z_2 = j$ cu imaginile $F_1(1,-2)$ și respectiv $F_2(0,1)$ și fie $M(z) \equiv M(x,y)$ unde $z=x+jy$

Atunci $|z-z_1|-|z-z_2|=3 \Leftrightarrow d(M,F_1)-d(M,F_2)=3=2a \Rightarrow \underline{\underline{a=\frac{3}{2}}}$

__Obs__ Hiperbola este locul geometric al punctelor din plan pentru care valoarea absolută a diferenței distanțelor la două puncte fixe F_1 și F_2, numite focare, este constantă $(=2a)$

$$\boxed{||MF_1-MF_2|=2a|} \text{ const.}$$

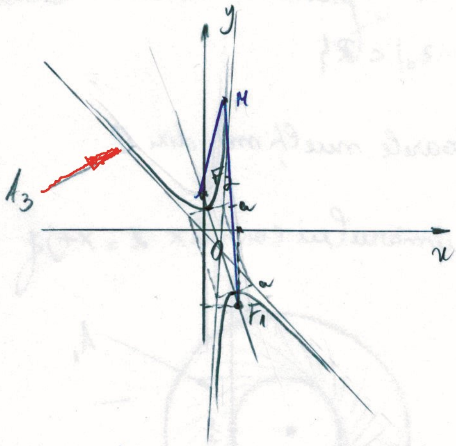
$(H): \frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, $\underline{c^2=a^2+b^2}$

$F_1F_2=\sqrt{1^2+(-3)^2}=\sqrt{10}=2c \Rightarrow c=\frac{\sqrt{10}}{2}\simeq 1,58$

$c^2=a^2+b^2 \Leftrightarrow \frac{10}{4}=\frac{9}{4}+b^2$

$\Downarrow$

$b^2=\frac{1}{4} \Rightarrow \underline{\underline{b=\frac{1}{2}}}$



A_3 - o ramură a hiperbolei cu focarele F_1 și F_2 având semiaxa focarelor de lungime $a=\frac{3}{2}$

$A_4 = \{(x,y) : |z-2+3j| = |z+3-j|\}$

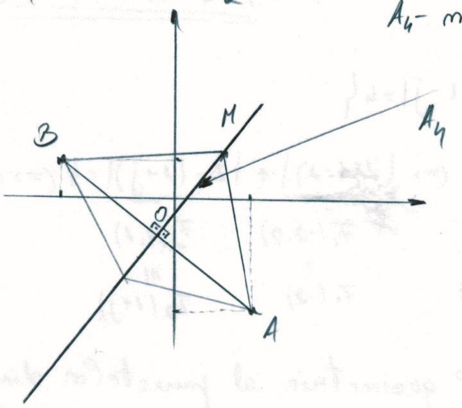
$$|z-2+3j|=|z+3-j| \Leftrightarrow |z-(\underbrace{2-3j}_{z_1})|=|z-(\underbrace{-3+j}_{z_2})|$$

Fie $M(z)$ și $A(z_1)$, $B(z_2)$

$ \equiv \quad\quad \equiv$

$A(2,-3) \quad B(-3,1)$

A_4 - mediatoarea segmentului AB cu $A(2,-3)$ și $B(-3,1)$



$A_5^* = \{(x,y) : |z-j| + \text{Re}\, z \le 2\}$

Fie $z=x+jy$. Atunci $|z-j|+\text{Re}\,z \le 2 \Leftrightarrow$

$\Leftrightarrow |x+jy-j|+x \le 2$

$\Leftrightarrow |x+j(y-1)| \le 2-x$

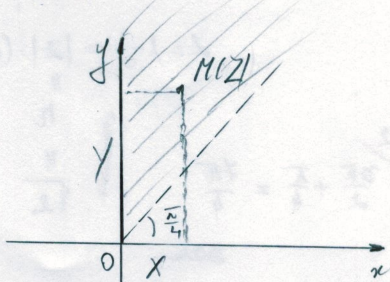
$\Leftrightarrow \sqrt{x^2+(y-1)^2} \le 2-x \quad |()^2 \quad (2-x \ge 0 \Leftrightarrow x \le 2)$

$\Leftrightarrow \cancel{x^2}+(y-1)^2 \le \cancel{x^2}-4x+4$

$\Leftrightarrow y^2-2y+4x-3 \le 0$

$\Leftrightarrow x \le \frac{1}{4}(-y^2+2y+3) \longrightarrow A_5 = \{(x,y) : x \le \frac{1}{4}(-y^2+2y+3), x \le 2\}$

$A_6 = \{(x,y) : \frac{\pi}{4} < \arg \frac{j+z}{j-z} \le \frac{\pi}{2}\}$



Fie $Z = \frac{j+z}{j-z}$, $z = x+jy$

$Z = X+jY$

$$\frac{\pi}{4} < \arg Z \le \frac{\pi}{2} \Leftrightarrow \begin{cases} X \ge 0 \\ Y > 0 \\ Y > X \end{cases} \Leftrightarrow \begin{cases} Y > X \\ X \ge 0 \end{cases} \quad (1)$$

$$Z = \frac{j+z}{j-z} = \frac{x+jy+j}{j-x-jy} = \frac{x+j(y+1)}{-x+j(1-y)} = \frac{(x+j(y+1))(-x-j(1-y))}{(-x+j(1-y))(-x-j(1-y))} = \frac{-x^2 - jx(1-y) - jx(y+1) + (y+1)(1-y)}{x^2+(1-y)^2}$$

$$= \frac{-x^2-2jx+1-y^2}{x^2+(1-y)^2} = \underbrace{\frac{-x^2-y^2+1}{x^2+(1-y)^2}}_{X} - \underbrace{\frac{2x}{x^2+(1-y)^2}}_{Y} \cdot j$$

$x^2+(1-y)^2 \ne 0 \Leftrightarrow x \ne 0$ și $y \ne 1$ $\Leftrightarrow (x,y) \ne (0,1)$

Obținem $(1) \Leftrightarrow \begin{cases} -2x > -x^2-y^2+1 \\ -x^2-y^2+1 \ge 0 \\ (x,y) \ne (0,1) \end{cases} \Leftrightarrow A_6 = \{(x,y) : -2x > -x^2-y^2+1,\ -x^2-y^2+1 \ge 0\}$

② Să se determine următorii radicali complecși:

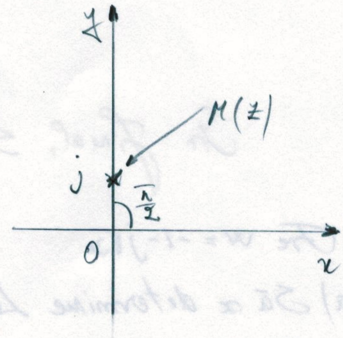
a) $\sqrt[4]{j}$

Obs: Fie $z = x+jy \in \mathbb{C}$. Atunci $r = |z| = \sqrt{x^2+y^2}$, iar pentru $z \ne 0$

$$t = \arg z = \begin{cases} \operatorname{arctg} \frac{y}{x}, & x>0, y \ge 0 \text{ (cadranul I)} \\ \frac{\pi}{2}, & x=0, y>0 \text{ (semiaxa pozitivă } Oy) \\ \operatorname{arctg} \frac{y}{x} + \pi, & x<0, y \in \mathbb{R} \text{ (cadranele II și III)} \\ \frac{3\pi}{2}, & x=0, y<0 \text{ (semiaxa negativă } Oy) \\ \operatorname{arctg} \frac{y}{x} + 2\pi, & x>0, y<0 \text{ (cadranul IV)} \end{cases}$$

Obs: Dacă $z = r\cdot(\cos t + j\sin t) = r\cdot e^{jt}$, atunci

$$\underline{\sqrt[n]{z}} = \{\sqrt[n]{r}\cdot(\cos\frac{t+2k\pi}{n} + j\sin\frac{t+2k\pi}{n}) : 0 \le k \le n-1\}$$
$$= \{\sqrt[n]{r}\cdot e^{j(t+2k\pi)/n} : 0 \le k \le n-1\}$$



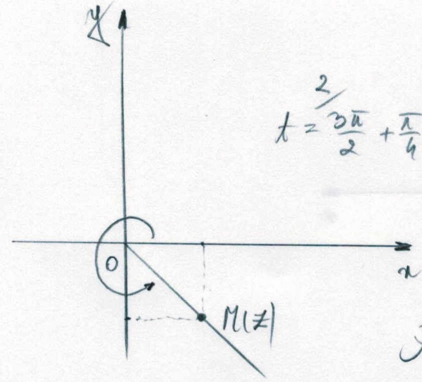
În cazul nostru, $z = j = \underset{r}{1}\cdot(\cos\frac{\pi}{2} + j\sin\underset{t}{\frac{\pi}{2}}) = e^{\frac{\pi}{2}j}$

$t = \arg z = \frac{\pi}{2}$ ($M \in$ semiaxei pozitive Oy)

$n=4$

$$\sqrt[4]{j} = \{\sqrt[4]{1}\cdot e^{j(\frac{\pi}{2}+2k\pi)/4} : k = \overline{0,3}\}$$

b) $\sqrt[50]{1-j}$



$z = 1-j = |z|\cdot(\cos t + j\sin t) = \sqrt{2}\left(\cos\frac{7\pi}{4} + j\sin\frac{7\pi}{4}\right) = \sqrt{2}\cdot e^{\frac{7\pi}{4}j}$

$|z| = r = \sqrt{2}$

$t = \frac{3\pi}{2} + \frac{\pi}{4} = \frac{7\pi}{4}$

sau

$t = \operatorname{arctg}\frac{y}{x} = \operatorname{arctg}(-1) + 2\pi = 2\pi - \frac{\pi}{4} = \frac{7\pi}{4}$, $n = 50$

În final,

$$\sqrt[50]{1-j} = \left\{\sqrt[50]{\sqrt{2}}\cdot e^{j\left(\frac{7\pi}{4}+2k\pi\right)/50} ;\ k=\overline{0,49}\right\} = \left\{2^{\frac{1}{100}}\cdot e^{j\cdot\left(\frac{7\pi}{200}+\frac{k\pi}{25}\right)} ;\ k=\overline{0,49}\right\}$$

(3) Să se rezolve în $\mathbb{C}$ ecuațiile:

a) $z^6 - z^3 + 1 - j = 0$ (temă)

b) $z^8 + (2j-1)z^4 - j - 1 = 0$ (1)

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b) Fie $z^4 = u$. Atunci (1) $\Leftrightarrow u^2 + (2j-1)u - j - 1 = 0$

$$\Delta = (2j-1)^2 + 4(j+1) = \cancel{-4} - \cancel{4j} + 1 + \cancel{4j} + \cancel{4} = 1$$

$$u_{1,2} = \frac{1-2j \pm 1}{2} \quad u_1 = 1-j, \quad u_2 = -j$$

Pentru $u_1 = 1-j \Rightarrow z^4 = 1-j$, $1-j = \sqrt{2}\left(\cos\frac{7\pi}{4} + j\sin\frac{7\pi}{4}\right) = \sqrt{2}e^{\frac{7\pi}{4}j}$

$$z \in \sqrt[4]{1-j} = \left\{\sqrt[8]{2}\cdot e^{j\left(\frac{7\pi}{4}+2k\pi\right)/4} : k=\overline{0,3}\right\} = \left\{\sqrt[8]{2}\cdot e^{j\left(\frac{7\pi}{16}+\frac{k\pi}{2}\right)} : k=\overline{0,3}\right\}$$

Pentru $u_2 = -j \Rightarrow z^4 = -j$, $-j = \cos\frac{3\pi}{2} + j\sin\frac{3\pi}{2} = e^{\frac{3\pi}{2}j}$

$$z \in \sqrt[4]{-j} = \left\{e^{j\left(\frac{3\pi}{2}+2k\pi\right)/4} : k=\overline{0,3}\right\} = \left\{e^{j\left(\frac{3\pi}{8}+\frac{k\pi}{2}\right)} : k=\overline{0,3}\right\}$$

În final, $S = \left\{\sqrt[8]{2}\cdot e^{j\left(\frac{7\pi}{16}+\frac{k\pi}{2}\right)} : k=\overline{0,3}\right\} \cup \left\{e^{j\left(\frac{3\pi}{8}+\frac{k\pi}{2}\right)} : k=\overline{0,3}\right\}$

(4) Fie $w = -1 - j\sqrt{3}$

a) Să se determine $\operatorname{Log} w$ ($\operatorname{Ln} w$)

b) $\operatorname{Re}\operatorname{Log} w$, $\operatorname{Im}\operatorname{Log} w$ ($\operatorname{Re}\operatorname{Ln} w$, $\operatorname{Im}\operatorname{Ln} w$)

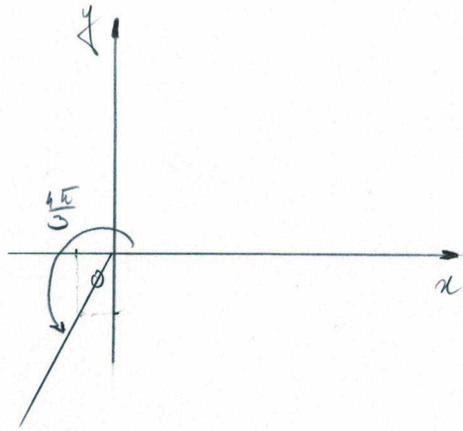
Obs: $\operatorname{Log} w = \operatorname{Ln} w = \{\ln|w| + j(\arg w + 2k\pi) : k \in \mathbb{Z}\}$

$w = |w|\cdot e^{j\cdot\arg w}$

- Logaritmul complex al numărului complex $w = |w|\cdot e^{jt} = |w|(\cos t + j\cdot\sin t)$, unde $t = \arg w$ are în $\mathbb{C}$ o infinitate de elemente (determinări)
- Logaritmul numărului complex 0 nu se definește!!!

• Numărul complex $\ln|w| + j\arg w$ ← s.n. determinare principală a logaritmului complex (se obține pentru $k=0$)

i.e. $\ln w = \ln|w| + j\cdot\arg w,\ w \in \mathbb{C}^*$



a) $\operatorname{Ln} w = \{\ln|w| + j(\arg w + 2k\pi) : k \in \mathbb{Z}\}$ $w = -1 - j\sqrt{3}$, $|w| = 2$

$\arg w = \operatorname{arctg}\sqrt{3} + \pi = \operatorname{arctg}\frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} + \pi = \frac{\pi}{3} + \pi = \frac{4\pi}{3}$

$\operatorname{Ln} w = \{\ln 2 + j\cdot(\frac{4\pi}{3} + 2k\pi) : k \in \mathbb{Z}\}$

b) $\ln w = \ln 2 + \frac{4\pi}{3} j$ ← determinarea principală

$\operatorname{Re}\ln w = \ln 2 \quad \operatorname{Im}\ln w = \frac{4\pi}{3}$

⑤ Fie w - determinarea principală a puterii complexe $(j\sqrt{3}-1)^{-j}$. Să se determine $\frac{\operatorname{Re} w}{\operatorname{Im} w}$.

=

$(j\sqrt{3}-1)^{-j} = e^{\operatorname{Ln}(j\sqrt{3}-1)^{-j}} = e^{-j\cdot\operatorname{Ln}(j\sqrt{3}-1)}$

Fie $z = -1 + j\sqrt{3} \Rightarrow |z| = 2$ și $\arg z = -\operatorname{arctg}\sqrt{3} + \pi = -\frac{\pi}{3} + \pi = \frac{2\pi}{3}$

$\operatorname{Ln}(j\sqrt{3}-1) = \{\ln 2 + j(\frac{2\pi}{3} + 2k\pi) : k \in \mathbb{Z}\}$

$\downarrow$

$\ln(j\sqrt{3}-1) = \ln 2 + \frac{2\pi}{3} j$

Atunci $w = e^{-j\cdot\ln(j\sqrt{3}-1)} = e^{-j\cdot(\ln 2 + \frac{2\pi}{3} j)}$

$= e^{\frac{2\pi}{3} - \ln 2\, j} = e^{\frac{2\pi}{3}} \cdot e^{-\ln 2\cdot j}$

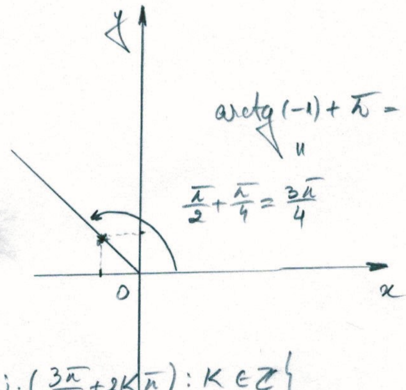
$= e^{\frac{2\pi}{3}} \cdot (\cos(\ln 2) - j\sin(\ln 2))$

În final, $\operatorname{Re} w = e^{\frac{2\pi}{3}} \cdot \cos(\ln 2)$

și

$\operatorname{Im} w = -e^{\frac{2\pi}{3}} \cdot \sin(\ln 2)$

$\Rightarrow \frac{\operatorname{Re} w}{\operatorname{Im} w} = -\operatorname{ctg}(\ln 2)$



$\operatorname{arctg}(-1) + \pi = \pi - \frac{\pi}{4} = \frac{\pi}{2} + \frac{\pi}{4} = \frac{3\pi}{4}$

⑥ Fie w - determinarea principală a lui $(j-1)^{1+j\sqrt{3}}$. Să se determine $\frac{\operatorname{Re} w}{\operatorname{Im} w}$. (temă)

=

$(j-1)^{1+j\sqrt{3}} = e^{\operatorname{Ln}(j-1)^{1+j\sqrt{3}}} = e^{(1+j\sqrt{3})\cdot\operatorname{Ln}(j-1)}$

$|-1+j| = \sqrt{2}$

$\operatorname{Ln}(-1+j) = \{\ln\sqrt{2} + j\cdot(\frac{3\pi}{4} + 2k\pi) : k \in \mathbb{Z}\}$

$\ln(-1+j) = \ln\sqrt{2} + \frac{3\pi}{4} j = \frac{1}{2}\ln 2 + \frac{3\pi}{4} j$

Atunci $w = e^{(1+j\sqrt{3})\cdot\ln(j-1)} =$

$= e^{(1+j\sqrt{3})\cdot(\frac{1}{2}\ln 2 + \frac{3\pi}{4} j)} = e^{\frac{1}{2}\ln 2 + \frac{3\pi}{4} j + \frac{\sqrt{3}}{2}\ln 2\, j - \frac{3\sqrt{3}\pi}{4}} =$

$= e^{\frac{1}{2}\ln 2 - \frac{3\sqrt{3}}{4}\pi} \cdot e^{(\frac{3\pi}{4} + \frac{\sqrt{3}}{2}\ln 2)\cdot j} = e^{\frac{1}{2}\ln 2 - \frac{3\sqrt{3}}{4}\pi} \cdot (\cos(\frac{3\pi}{4} + \frac{\sqrt{3}}{2}\ln 2) + j\cdot \sin(\frac{3\pi}{4} + \frac{\sqrt{3}}{2}\ln 2))$

Obținem $\operatorname{Re} w = e^{\frac{1}{2}\ln 2 - \frac{3\sqrt{3}}{4}\pi} \cdot \cos(\frac{3\pi}{4} + \frac{\sqrt{3}}{2}\ln 2)$

și

$\operatorname{Im} w = e^{\frac{1}{2}\ln 2 - \frac{3\sqrt{3}}{4}\pi} \cdot \sin(\frac{3\pi}{4} + \frac{\sqrt{3}}{2}\ln 2)$

$\Rightarrow \frac{\operatorname{Re} w}{\operatorname{Im} w} = \operatorname{ctg}(\frac{3\pi}{4} + \frac{\sqrt{3}}{2}\ln 2)$

Seminarul 2

Funcții olomorfe

Def 1 Fie $G \subseteq \mathbb{C}$ o mulțime deschisă, $z_0 \in G$ și $f: G \to \mathbb{C}$

• Dacă $\exists \lim\limits_{z \to z_0} \dfrac{f(z) - f(z_0)}{z - z_0} \in \overline{\mathbb{C}}$, atunci ea se numește derivata lui f în z_0.

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = f'(z_0)$$

• Dacă limita $f'(z_0)$ e finită ($f'(z_0) \in \mathbb{C}$) atunci f s.n. monogenă sau derivabilă în z_0.

T1 Fie $G \subseteq \mathbb{C}$ o mulțime deschisă, $z_0 \in G$, $z_0 = x_0 + j y_0$ și $f: G \to \mathbb{C}$, o funcție complexă dată, $f(z) = P(x,y) + j\, Q(x,y)$ unde $z = x + jy$.

• Dacă f monogenă în $z_0 \Rightarrow P$ și Q admit derivate parțiale în (x_0, y_0) și au loc (C-R)

$$\begin{cases} \dfrac{\partial P}{\partial x}(x_0, y_0) = \dfrac{\partial Q}{\partial y}(x_0, y_0) \\[2mm] \dfrac{\partial P}{\partial y}(x_0, y_0) = -\dfrac{\partial Q}{\partial x}(x_0, y_0) \end{cases}$$

cond. de monogenitate Cauchy-Riemann în z_0

Calculul derivatei unei funcții monogene

• Dacă $f: G \subseteq \mathbb{C} \to \mathbb{C}$ e o funcție monogenă în $z_0 = x_0 + j y_0 \in G$, atunci

• $f'(z_0) = \dfrac{\partial P}{\partial x}(x_0, y_0) + j \cdot \dfrac{\partial Q}{\partial x}(x_0, y_0)$

Def 2 O funcție $f: G \to \mathbb{C}$ s.n. olomorfă (întreagă) pe $G \subseteq \mathbb{C}$ (G - deschisă) dacă f este monogenă în fiecare punct $z_0 \in G$.

T2 • Dacă $f(z, \bar{z})$, atunci (C-R) $\Leftrightarrow \dfrac{\partial f}{\partial \bar{z}} = 0$ iar $f'(z_0) = \dfrac{\partial f}{\partial z}(z_0)$

① Să se determine punctele în care $f: \mathbb{C} \to \mathbb{C}$ este monogenă și să se calculeze valoarea derivatei în aceste puncte, unde f e dată de

a) $f(z) = 2z^2 + z \cdot \bar{z} - \bar{z}^2 + 2(1+j)z - \bar{z} + 2 - j = P(x,y) + j\, Q(x,y)$

$f(z) = 2(x+jy)^2 + (x+jy)(x-jy) - (x-jy)^2 + 2(1+j)(x+jy) - (x-jy) + 2 - j =$

$= 2(x^2 + 2xyj - y^2) + x^2 + y^2 - (x^2 - 2xyj - y^2) + 2x + 2yj + 2xj - 2y - x + yj + 2 - j =$

$= \underbrace{2x^2 + x - 2y + 2}_{P(x,y)} + \underbrace{(6xy + 3y + 2x - 1)}_{Q(x,y)} j$

Verificăm cond. (C-R) de monogenitate, scrise într-un punct arbitrar $z = x + jy$.

$$(C\text{-}R)\begin{cases}\dfrac{\partial P}{\partial x}=\dfrac{\partial Q}{\partial y}\\[2mm] \dfrac{\partial P}{\partial y}=-\dfrac{\partial Q}{\partial x}\end{cases} \Leftrightarrow \begin{cases}4x+1=6x+3 \Rightarrow \underline{\underline{x=-1}}\\ -2=-(6y+2)\Rightarrow \underline{\underline{y=0}}\end{cases}$$

Așadar singurul punct în care f este monogenă este $\underline{\underline{z_0=-1}}$ $(-1+0j=z_0)$

Obținem apoi

$$\underline{\underline{f'(z_0)}}=f'(-1)=\frac{\partial P}{\partial x}(-1,0)+j\frac{\partial Q}{\partial x}(-1,0)=\underline{\underline{-3+2j}}$$

sau

Cum $f(z,\bar z)$, conform Teoremei 2, $(C\text{-}R)\Leftrightarrow \dfrac{\partial f}{\partial \bar z}=0 \Leftrightarrow z-2\bar z-1=0$

$$\Leftrightarrow x+jy-2(x-jy)-1=0$$

$$\Leftrightarrow -x-1+3yj=0 \Leftrightarrow \underline{\underline{x=-1}} \text{ și } \underline{\underline{y=0}}$$

Obținem așadar $z_0\underline{\underline{=-1}}$ și $\underline{\underline{f'(z_0)}}=f'(-1)=\dfrac{\partial f}{\partial z}(-1)=\left(4z+\bar z+2(1+j)\right)\Big|_{z=-1}=$

$$=-4-1+2+2j=\underline{\underline{-3+2j}}$$

b) $f(z)=|z|^{10}-\frac{1}{2}\bar z$, $z=x+jy$

$$f(z)=(z\cdot\bar z)^5-\frac{1}{2}\bar z=(x^2+y^2)^5-\frac{1}{2}(x-jy)=\underbrace{(x^2+y^2)^5-\frac{1}{2}x}_{P(x,y)}+\underbrace{\frac{1}{2}y}_{Q(x,y)}\cdot j$$

$z\cdot\bar z=|z|^2$

$$\Rightarrow P(x,y)=(x^2+y^2)^5-\frac{1}{2}x \text{ și } Q(x,y)=\frac{1}{2}y$$

$$(C\text{-}R):\begin{cases}\dfrac{\partial P}{\partial x}=\dfrac{\partial Q}{\partial y}\\[2mm] \dfrac{\partial P}{\partial y}=-\dfrac{\partial Q}{\partial x}\end{cases}\Leftrightarrow\begin{cases}10x(x^2+y^2)^4-\dfrac{1}{2}=\dfrac{1}{2}\Rightarrow 10x(x^2+y^2)^4=1\\ 10y(x^2+y^2)^4=0\Rightarrow y=0 \text{ sau } x^2+y^2=0\end{cases}$$

Dacă $y=0 \Rightarrow 10x^9=1\Rightarrow x=\sqrt[9]{\frac{1}{10}}=\frac{1}{\sqrt[9]{10}}\Rightarrow z_0=\frac{1}{\sqrt[9]{10}}$

Dacă $x^2+y^2=0 \Rightarrow 10x\cdot 0=1$ (imposibil)

Așadar z_0 e singurul punct în care f este monogenă.

$$\underline{\underline{f'(z_0)}}=\frac{\partial P}{\partial x}(x_0,y_0)+j\frac{\partial Q}{\partial x}(x_0,y_0)$$

$$=\frac{\partial P}{\partial x}\left(\frac{1}{\sqrt[9]{10}},0\right)+j\frac{\partial Q}{\partial x}\left(\frac{1}{\sqrt[9]{10}},0\right)=\underline{\underline{\frac{1}{2}}}$$

sau

$$=z\cdot(z\cdot\bar z)^4=z\cdot(|z|^2)^4=z\cdot|z|^8$$

Cum $f(z,\bar z)$, $(C\text{-}R)\Rightarrow \dfrac{\partial f}{\partial \bar z}=0 \Leftrightarrow 5\bar z^5\cdot z^4... $

$$\frac{\partial f}{\partial \bar z}=0 \Leftrightarrow 5z^5\cdot\bar z^4-\frac{1}{2}=0\Leftrightarrow z^5\cdot\bar z^4=\frac{1}{10}$$

$$\Leftrightarrow (x+jy)(x^2+y^2)^4=\frac{1}{10}\Leftrightarrow x(x^2+y^2)^4+j\cdot y(x^2+y^2)^4=\frac{1}{10}$$

$$\Rightarrow x(x^2+y^2)^4=\frac{1}{10} \text{ și } y\cdot(x^2+y^2)^4=0 \Rightarrow \dots \Rightarrow x=\frac{1}{\sqrt[9]{10}} \text{ și } y=0$$

$$\Rightarrow z_0=\frac{1}{\sqrt[9]{10}} \text{ singurul punct de monogenitate}$$

În final, $\underline{f'(z_0)}=\dfrac{\partial f}{\partial z}(z_0)=5\bar z^5\cdot z^4\Big|_{z=z_0}=5\cdot\left(\dfrac{1}{\sqrt[9]{10}}\right)^5\cdot\left(\dfrac{1}{\sqrt[9]{10}}\right)^4=\underline{\underline{\dfrac{1}{2}}}$

② Să se determine $a, b, c, d \in \mathbb{R}$ a.î. $f: \mathbb{C} \to \mathbb{C}$

$$f(z) = ax^2 + bxy + cy^2 + x - y + j\cdot(ax^2 + 2xy + dy^2 + x + ay)$$

să fie olomorfă pe $\mathbb{C}$ și să se scrie $f(z)$ ca funcție de z, $z = x + jy$

$$P(x,y) = ax^2 + bxy + cy^2 + x - y$$
$$Q(x,y) = ax^2 + 2xy + dy^2 + x + ay$$

f olomorfă pe $\mathbb{C}$ $\Leftrightarrow \begin{cases} \dfrac{\partial P}{\partial x} = \dfrac{\partial Q}{\partial y} \\ \dfrac{\partial P}{\partial y} = -\dfrac{\partial Q}{\partial x} \end{cases} \forall x, y \in \mathbb{R}$

$\Leftrightarrow \begin{cases} 2ax + by + 1 = 2x + 2dy + a \\ bx + 2cy - 1 = -(2ax + 2y + 1) \end{cases} \forall x, y \in \mathbb{R}$

$\Rightarrow \begin{cases} 2a = 2 \\ b = 2d \\ a = 1 \\ b = -2a \\ 2c = -2 \end{cases} \Rightarrow \begin{cases} a = 1 \\ b = -2 \\ c = -1 \\ d = -1 \end{cases} \Rightarrow f(z) = x^2 - 2xy - y^2 + x - y + j\cdot(x^2 + 2xy - y^2 + x + y)$

În continuare, funcția olomorfă $f(z)$ se poate determina formal după cum urmează:
Luând $z \in \mathbb{R}$ (punând $y = 0 \Rightarrow z = x \in \mathbb{R}$), obținem

$$f(z) = f(x) = x^2 + x + j\cdot(x^2 + x) = (x^2 + x)(j + 1) = (z^2 + z)(j + 1),\ \forall z \in \mathbb{R}$$

De aici, pe baza principiului identității funcțiilor olomorfe

$$f(z) = (z^2 + z)(1 + j),\ \forall z \in \mathbb{C}$$

• Dacă $f: \mathbb{C} \to \mathbb{C}$ este olomorfă și $f(x) = g(x),\ \forall x \in \mathbb{R}$
$\Downarrow$ cf. pp. id. f-țiilor olomorfe
$f(z) = g(z),\ \forall z \in \mathbb{C}$

③ Să se determine funcția olomorfă $f: \mathbb{C} \to \mathbb{C}$,

$$f(z) = P(x,y) + j\cdot Q(x,y),\quad z = x + jy$$

știind că $\begin{cases} P(x,y) = \operatorname{Re} f(z) = \dfrac{y}{x^2 + y^2} + e^{-y}\cdot\cos x \\ f(j) = 1 + \dfrac{1}{e} \end{cases}$

f-olomorfă $(\mathbb{C}^*)$ $\begin{cases} \dfrac{\partial P}{\partial x} = \dfrac{\partial Q}{\partial y} \\ \dfrac{\partial P}{\partial y} = -\dfrac{\partial Q}{\partial x} \end{cases}$

$$f'(z) = \frac{\partial P}{\partial x} + j\cdot\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial x} - j\cdot\frac{\partial P}{\partial y} =$$

$$= \frac{-2xy}{(x^2 + y^2)^2} - e^{-y}\sin x - j\cdot\left(\frac{x^2 - y^2}{(x^2 + y^2)^2} - e^{-y}\cos x\right)$$

Luând $y = 0 \Rightarrow z = x \in \mathbb{R}$ și $f'(z) = f'(x) = -\sin x - j\cdot\left(\frac{1}{x^2} - \cos x\right) = -\sin z - j\cdot\left(\frac{1}{z^2} - \cos z\right)$, $\forall z \in \mathbb{R}^*$

Cf. pp. id. funcțiilor olomorfe $\Rightarrow f'(z) = -\sin z - j\left(\frac{1}{z^2} - \cos z\right),\ \forall z \in \mathbb{C}^*$

Obținem $f(z) = \int f'(z)\,dz = \cos z - j\left(\left(-\frac{1}{z}\right) - \sin z\right) + c,\ c \in \mathbb{C}$

$\Rightarrow f(z) = \frac{1}{z} j + e^{jz} + C$ $\Big|$ $\Rightarrow 1 + e^{-1} + C = 1 + \frac{1}{e} \Rightarrow C = 0 \Rightarrow f(z) = \frac{1}{z} j + e^{jz},\ z \in \mathbb{C}^*$

$f(j) = 1 + \frac{1}{\ldots}$

④ Să se determine funcția olomorfă $f: \mathbb{C} \to \mathbb{C}$, $f(z) = P(x,y) + j\,Q(x,y)$, $z = x + jy$ în următoarele situații:

a) $P(x,y) + Q(x,y) = e^x[(x+y)\cos y + (x-y)\sin y]$, $f(0) = 1 - j$

$$P(x,y) + Q(x,y) = e^x[(x+y)\cdot\cos y + (x-y)\sin y] \quad \Big| \frac{\partial}{\partial x}(\,)$$

$$\Leftrightarrow \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial x} = e^x[(x+y)\cdot\cos y + (x-y)\cdot\sin y] + e^x[\cos y + \sin y] =$$

$$= e^x[(x+y+1)\cdot\cos y + (x-y+1)\sin y] \quad (1)$$

$$P(x,y) + Q(x,y) = e^x[(x+y)\cos y + (x-y)\sin y] \quad \Big| \frac{\partial}{\partial y}(\,)$$

$$\Leftrightarrow \frac{\partial P}{\partial y} + \frac{\partial Q}{\partial y} = e^x[-x\sin y + \cos y - y\sin y + x\cos y - \sin y - y\cos y]$$

$$= e^x[(x-y+1)\cos y - (x+y+1)\sin y] \quad (2)$$

• f-olomorfă $\to$ P și Q sunt armonice ($\Delta P = 0$ și $\Delta Q = 0$) și au loc condițiile Cauchy-Riemann de monogenitate

$$(C\text{-}R)\begin{cases} \dfrac{\partial P}{\partial x} = \dfrac{\partial Q}{\partial y} \\[2mm] \dfrac{\partial P}{\partial y} = -\dfrac{\partial Q}{\partial x} \end{cases} \quad (3)$$

Adunând rel. (1) și (2), obținem

$$\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial x} + \frac{\partial P}{\partial y} + \frac{\partial Q}{\partial y} = e^x[(x+\cancel{y}+1+x-\cancel{y}+1)\cos y + (\cancel{x}-y+\cancel{1}-\cancel{x}-y-\cancel{1})\sin y]$$

$$\xrightarrow{Cf.(3)} \frac{\partial P}{\partial x} + \cancel{\frac{\partial Q}{\partial x}} - \cancel{\frac{\partial Q}{\partial x}} + \frac{\partial P}{\partial x} = e^x[2(x+1)\cos y - 2y\sin y]$$

$$\to \frac{\partial P}{\partial x} = \frac{e^x}{2}[2(x+1)\cos y - 2y\sin y] = \frac{\partial Q}{\partial y}$$

Scăzând rel. (1) și (2), obținem

$$\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} - \frac{\partial Q}{\partial y} = e^x[(\cancel{x}+y+\cancel{1}-\cancel{x}+y-\cancel{1})\cos y + (x-\cancel{y}+1+x+\cancel{y}+1)\sin y]$$

$$\xrightarrow{Cf.(3)} \cancel{\frac{\partial P}{\partial x}} + \frac{\partial Q}{\partial x} + \frac{\partial Q}{\partial x} - \cancel{\frac{\partial P}{\partial x}} = e^x[2y\cos y + 2(x+1)\sin y]$$

$$\Rightarrow \frac{\partial Q}{\partial x} = \frac{e^x}{2}[2y\cos y + 2(x+1)\sin y] = -\frac{\partial P}{\partial y}$$

Am obținut $\dfrac{\partial P}{\partial x} = \dfrac{e^x}{2}[2(x+1)\cos y - 2y\sin y]$

și

$$\frac{\partial Q}{\partial x} = \frac{e^x}{2}[2y\cos y + 2(x+1)\sin y]$$

Așadar, $f'(z) = \dfrac{\partial P}{\partial x} + j\cdot\dfrac{\partial Q}{\partial x} \to f'(z) = \dfrac{e^x}{2}[2(x+1)\cos y - 2y\sin y] + j\cdot\dfrac{e^x}{2}[2y\cos y + 2(x+1)\sin y]$

Luând $y = 0$, $z = x \in \mathbb{R}$ și $f'(z) = f'(x) = \dfrac{e^x}{\cancel{2}}[\cancel{2}(x+1)]$, $\forall x \in \mathbb{R}$

Cf. pp. id. funcțiilor olomorfe, $\underline{f'(z) = e^z(z+1)}$, $\forall z \in \mathbb{C}$

Obținem $f(z) = \int f'(z)\,dz = \int e^z(z+1)\,dz = \int (e^z)'(z+1)\,dz =$

$= e^z(z+1) - \int e^z\,dz = e^z(z+1) - e^z + C,\ C \in \mathbb{C}$

$= e^z \cdot z + C$

Cum $f(0) = 1-j$ $\Rightarrow C = 1-j$

În final, $f(z) = e^z \cdot z + 1 - j,\ z \in \mathbb{C}$

Probleme suplimentare

① Să se determine punctele de monogenitate pentru $f: \mathbb{C} \to \mathbb{C}$, unde $f(z) = z^2 \cdot \bar{z} + 4j \cdot \operatorname{Re} z$ și să se calculeze $f'(z)$.

$z = x+jy$, $\bar{z} = x - jy$ $\Rightarrow \operatorname{Re} z = x = \frac{z+\bar{z}}{2}$. Obținem $f(z) = z^2 \cdot \bar{z} + 4j \cdot \frac{z+\bar{z}}{2}$

$\Leftrightarrow f(z) = z^2 \cdot \bar{z} + 2(z+\bar{z})j$

Cum $f(z,\bar{z})$, $(C-R) \Leftrightarrow \frac{\partial f}{\partial \bar{z}} = 0 \Leftrightarrow z^2 + 2j = 0 \Leftrightarrow z^2 = -2j = (1-j)^2 \Rightarrow z_{1,2} = \pm(1-j)$

$\Rightarrow z_1 = 1-j$, iar $z_2 = -1+j$ punctele de monogenitate

În final, $f'(z_1) = \frac{\partial f}{\partial z}(z_1) = (\underbrace{2z\cdot\bar{z}}_{|z|^2} + 2j)\Big|_{z=z_1=1-j} = 2\cdot(1-j)(1+j) + 2j = 4+2j$

și

$f'(z_2) = \frac{\partial f}{\partial z}(z_2) = (2z\cdot\bar{z} + 2j)\Big|_{z=z_2=-1+j} = 2(-1+j)(-1-j) + 2j = 4+2j$

② Să se determine punctele din planul complex în care funcția $f: \mathbb{C} \to \mathbb{C}$
$f(z) = z^3 \cdot \bar{z} + 2j \cdot |z|^2$ este monogenă și să se calculeze valoarea derivatei funcției f în aceste puncte.

$f(z) = z^3 \cdot \bar{z} + 2j \cdot z \cdot \bar{z}$ $\quad (z \cdot \bar{z} = |z|^2)$

$(C-R) \Leftrightarrow \frac{\partial f}{\partial \bar{z}} = 0 \Leftrightarrow z^3 + 2z \cdot j = 0 \Leftrightarrow z(z^2 + 2j) = 0 \Rightarrow z_0 = 0$ sau $z^2 = -2j = (1-j)^2$

$\Rightarrow z_{1,2} = \pm(1-j)$

În final, $f'(z_0) = \frac{\partial f}{\partial z}(z_0) = (3z^2\bar{z} + 2\bar{z} \cdot j)\Big|_{z=z_0=0} = 0$

$f'(z_1) = \frac{\partial f}{\partial z}(z_1) = (3z^2 \cdot \bar{z} + 2\bar{z} \cdot j)\Big|_{z=z_1=1-j} = 3\cdot(-2j)(1+j) + 2(1+j)\cdot j$

$= -6j + 6 + 2j - 2 = +4-4j$

$f'(z_2) = \frac{\partial f}{\partial z}(z_2) = (3z^2 \cdot \bar{z} + 2\bar{z} \cdot j)\Big|_{z=z_2=-1+j} = 3\cdot(-2j)(-1-j) + 2(-1-j)\cdot j$

$= 6j - 6 - 2j + 2 = -4+4j$

③ Să se determine funcția olomorfă $f: \mathbb{C} \to \mathbb{C}$, $f(z) = P(x,y) + jQ(x,y)$, $z = x + jy$ dacă

$$P(x,y) - Q(x,y) = \frac{x-y}{2}\ln(x^2+y^2) - (x+y)\,\mathrm{arctg}\,\frac{y}{x} \quad \text{și} \quad f(1) = 2+2j$$

$$P(x,y) - Q(x,y) = \frac{x-y}{2}\ln(x^2+y^2) - (x+y)\cdot\mathrm{arctg}\,\frac{y}{x} \quad \Big|\ \frac{\partial}{\partial x}(\ )$$

$$\frac{\partial P}{\partial x} - \frac{\partial Q}{\partial x} = \frac{1}{2}\ln(x^2+y^2) + \frac{x-y}{\cancel{2}}\cdot\frac{\cancel{2}x}{x^2+y^2} - \mathrm{arctg}\,\frac{y}{x} + (x+y)\cdot\frac{y}{x^2}\cdot\frac{x^2}{x^2+y^2}$$

$$= \frac{1}{2}\ln(x^2+y^2) + \frac{x(x-y)}{x^2+y^2} - \mathrm{arctg}\,\frac{y}{x} + \frac{y(x+y)}{x^2+y^2} \quad (1)$$

$$P(x,y) - Q(x,y) = \frac{x-y}{2}\ln(x^2+y^2) - (x+y)\cdot\mathrm{arctg}\,\frac{y}{x} \quad \Big|\ \frac{\partial}{\partial y}(\ )$$

$$\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial y} = -\frac{1}{2}\ln(x^2+y^2) + \frac{x-y}{\cancel{2}}\cdot\frac{\cancel{2}y}{x^2+y^2} - \mathrm{arctg}\,\frac{y}{x} - (x+y)\cdot\frac{1}{\cancel{x}}\cdot\frac{x^{\cancel{2}}}{x^2+y^2}$$

$$= -\frac{1}{2}\ln(x^2+y^2) + \frac{(x-y)\cdot y}{x^2+y^2} - \mathrm{arctg}\,\frac{y}{x} - \frac{x(x+y)}{x^2+y^2} \quad (2)$$

Din (1)+(2) $\Rightarrow$ $\cancel{\frac{\partial P}{\partial x}} - \frac{\partial Q}{\partial x} + \frac{\partial P}{\partial y} - \cancel{\frac{\partial Q}{\partial y}} = \frac{\cancel{x(x-y)} + \cancel{y(x+y)} + \cancel{y(x-y)} - \cancel{x(x+y)}}{x^2+y^2} - 2\,\mathrm{arctg}\,\frac{y}{x}$

f - olomorfă $\to$ au loc cond. (C-R) $\begin{cases} \frac{\partial P}{\partial x} = \frac{\partial Q}{\partial y} \\ \frac{\partial P}{\partial y} = -\frac{\partial Q}{\partial x} \end{cases}$

Așadar $2\frac{\partial P}{\partial y} = -2\,\mathrm{arctg}\,\frac{y}{x} \Rightarrow \frac{\partial P}{\partial y} = -\,\mathrm{arctg}\,\frac{y}{x}$ (3)

Din (1)-(2) $\Rightarrow$ $\frac{\partial P}{\partial x} - \cancel{\frac{\partial Q}{\partial x}} - \cancel{\frac{\partial P}{\partial y}} + \frac{\partial Q}{\partial y} = \ln(x^2+y^2) + \frac{x^2 \cancel{-xy} \cancel{+xy} + y^2 \cancel{-xy} + y^2 + x^2 \cancel{+xy}}{x^2+y^2}$

$\overset{(C-R)}{\Rightarrow} 2\cdot\frac{\partial P}{\partial x} = \ln(x^2+y^2) + 2 \Rightarrow \frac{\partial P}{\partial x} = \frac{1}{2}\ln(x^2+y^2) + 1$ (4)

Dar. $f'(z) = \frac{\partial P}{\partial x} + j\cdot\frac{\partial Q}{\partial x} \overset{(C-R)}{=} \frac{\partial P}{\partial x} - j\cdot\frac{\partial P}{\partial y} \overset{(3)}{\underset{(4)}{=}} \frac{1}{2}\ln(x^2+y^2) + 1 + j\cdot\mathrm{arctg}\,\frac{y}{x}$, $x \neq 0$

Luând $y = 0 \Rightarrow z = x \in \mathbb{R}^*$ și $f'(z) = f'(x) = \ln x + 1 = \ln z + 1$, $\forall z \in \mathbb{R}^*$

Cf. pp. id. funcțiilor olomorfe $\Rightarrow f'(z) = \ln z + 1$, $\forall z \in \mathbb{C}^*$

Obținem $f(z) = \int f'(z)\,dz = \int(\ln z + 1)\,dz = \int z'(\ln z + 1)\,dz = z(\ln z + 1) - \int z\cdot\frac{1}{z}\,dz =$

$= z(\ln z \cancel{+1}) - \cancel{z} + C$, $C \in \mathbb{C}$

Așadar, $\begin{cases} f(z) = z\ln z + C \\ f(1) = 2+2j \end{cases} \Rightarrow C = 2(1+j) \Rightarrow$ $\underline{f(z) = z\ln z + 2(1+j)}$, $\forall z \in \mathbb{C}^*$

④ Să se determine funcția olomorfă $f: \mathbb{C} \to \mathbb{C}$, $f(z) = P(x,y) + jQ(x,y)$, $z = x + jy$ dacă

$P(x,y) = \mathrm{arctg}\,\frac{y}{x}$, $x \neq 0$, $f(e) = 0$

---

f - olomorfă $\Rightarrow$ (C-R) $\begin{cases} \frac{\partial P}{\partial x} = \frac{\partial Q}{\partial y} \\ \frac{\partial P}{\partial y} = -\frac{\partial Q}{\partial x} \end{cases}$

$$\frac{\partial P}{\partial x}=-\frac{y}{\cancel{x^2}}\cdot\frac{\cancel{x^2}}{x^2+y^2}=-\frac{y}{x^2+y^2}$$

$$\frac{\partial P}{\partial y}=\frac{1}{\cancel{x}}\cdot\frac{x^{\cancel{2}}}{x^2+y^2}=\frac{x}{x^2+y^2}$$

Dar $f'(z)=\frac{\partial P}{\partial x}+j\cdot\frac{\partial Q}{\partial x}\overset{(C-R)}{=}\frac{\partial P}{\partial x}-j\cdot\frac{\partial P}{\partial y}=-\frac{y}{x^2+y^2}-j\cdot\frac{x}{x^2+y^2}$

Luând $y=0 \Rightarrow z=x\in\mathbb{R}^*$ și

$$f'(z)=f'(x)=-\frac{j}{x}=-\frac{j}{z},\ \forall z\in\mathbb{R}^*$$

Cf pp. id. funcțiilor olomorfe obținem că

$$f'(z)=-\frac{j}{z},\ \forall z\in\mathbb{C}^*$$

$$\Downarrow$$

$$f(z)=\int f'(z)\,dz=\int-\frac{j}{z}\,dz=-j\ln z+k,\ k\in\mathbb{C}$$

$$\left.\begin{array}{l}\text{Deci } f(z)=-j\ln z+k,\ k\in\mathbb{C}\\ f(e)=0\end{array}\right\}\to j+k=0\Rightarrow k=j$$

În final, $\underline{f(z)}=-j\ln z+j=\underline{j(1-\ln z)},\ \forall z\in\mathbb{C}^*$

(5) Să se determine funcția olomorfă $f:\mathbb{C}\to\mathbb{C}$, $f(z)=P(x,y)+jQ(x,y)$, $z=x+jy$ unde

$$Q(x,y)=e^x(x\sin y+y\cos y),\ f(1)=e$$

—

f-olomorfă $\Rightarrow$ au loc cond (C-R) $\begin{cases}\frac{\partial P}{\partial x}=\frac{\partial Q}{\partial y}\\ \frac{\partial P}{\partial y}=-\frac{\partial Q}{\partial x}\end{cases}$

$$\frac{\partial Q}{\partial x}=e^x(x\sin y+y\cos y)+e^x(\sin y)=e^x(y\cos y+(x+1)\sin y)$$

$$\frac{\partial Q}{\partial y}=e^x\cdot x\cdot\cos y+e^x\cos y-ye^x\sin y=e^x((x+1)\cos y-y\sin y)$$

$$f'(z)=\frac{\partial P}{\partial x}+j\frac{\partial Q}{\partial x}\overset{(C-R)}{=}\frac{\partial Q}{\partial y}+j\cdot\frac{\partial Q}{\partial x}=e^x((x+1)\cos y-y\sin y)+j\cdot e^x(y\cos y+(x+1)\sin y)$$

Luând $y=0$, $z=x\in\mathbb{R}$ și $f'(z)=e^x\cdot(x+1)=e^z\cdot(z+1)$, $\forall z\in\mathbb{R}$

Cf. pp. id. funcțiilor olomorfe, $f'(z)=e^z\cdot(z+1)$, $\forall z\in\mathbb{C}$

Așadar $f(z)=\int f'(z)\,dz=\int e^z\cdot(z+1)\,dz=\int(e^z)'\cdot(z+1)\,dz=e^z\cdot(z+1)-\int e^z\,dz=$

$$=e^z\cdot(z+\cancel{1})-\cancel{e^z}+k,\ k\in\mathbb{C}\ \to \left.\begin{array}{l}f(z)=e^z\cdot z+k,\ k\in\mathbb{C}\\ f(1)=e\end{array}\right\}\to k+e=e\Rightarrow \underline{k=0}$$

$$\Rightarrow \underline{f(z)=ze^z,\ \forall z\in\mathbb{C}}$$

⑥ Să se determine funcția olomorfă $f:\mathbb{C}\to\mathbb{C}$, $f(z)=P(x,y)+jQ(x,y)$, $z=x+jy$ dacă

$$P(x,y)=\frac{\sin 2x}{\operatorname{ch}2y-\cos 2x},\quad f\left(\frac{\pi}{4}\right)=1$$

—

f olomorfă $\Rightarrow$ au loc cond. (C-R) $\begin{cases}\dfrac{\partial P}{\partial x}=\dfrac{\partial Q}{\partial y}\\[2mm] \dfrac{\partial P}{\partial y}=-\dfrac{\partial Q}{\partial x}\end{cases}$

Obs $\operatorname{sh}x=\dfrac{e^x-e^{-x}}{2}$, $(\operatorname{sh}x)'=\dfrac{e^x+e^{-x}}{2}=\operatorname{ch}x$

$\operatorname{ch}x=\dfrac{e^x+e^{-x}}{2}$, $(\operatorname{ch}x)'=\dfrac{e^x-e^{-x}}{2}=\operatorname{sh}x$

$\operatorname{sh}0=0$, $\operatorname{ch}0=1$

$$\frac{\partial P}{\partial x}=\frac{2\cos 2x(\operatorname{ch}2y-\cos 2x)-2\sin^2 2x}{(\operatorname{ch}2y-\cos 2x)^2}$$

$$\frac{\partial P}{\partial y}=\frac{-2\operatorname{sh}2y\cdot\sin 2x}{(\operatorname{ch}2y-\cos 2x)^2}$$

$$f'(z)=\frac{\partial P}{\partial x}+j\cdot\frac{\partial Q}{\partial x}\overset{(CR)}{=}\frac{2\cos 2x(\operatorname{ch}2y-\cos 2x)-2\sin^2 2x}{(\operatorname{ch}2y-\cos 2x)^2}+j\cdot\frac{2\operatorname{sh}2y\cdot\sin 2x}{(\operatorname{ch}2y-\cos 2x)^2}$$

Luând $y=0$, $z=x\in\mathbb{R}$ și $f'(z)=f'(x)=\dfrac{2\cos 2x(1-\cos 2x)-2\sin^2 2x}{(1-\cos 2x)^2}$, $\forall x\in\mathbb{R}$

$$=\frac{2(\cos 2x-1)}{(1-\cos 2x)^2}=-2\cdot\frac{1}{1-\cos 2x}=-\frac{1}{\sin^2 x}$$

cf pp. id. funcțiilor olomorfe,

$$f'(z)=-\frac{1}{\sin^2 z},\quad z\in\mathbb{C}\setminus\{z\in\mathbb{C}:\sin z=0\}$$

$$f(z)=\int f'(z)\,dz=\int-\frac{1}{\sin^2 z}\,dz=\operatorname{ctg}z+C,\ C\in\mathbb{C}$$

$f\left(\frac{\pi}{4}\right)=1$ $\Rightarrow 1=1+C\Rightarrow C=0$

Așadar, $f(z)=\operatorname{ctg}z$, $z\in\mathbb{C}\setminus\{z\in\mathbb{C}:\sin z=0\}$

⑦ Să se determine funcția olomorfă $f:\mathbb{C}\to\mathbb{C}$, $f(z)=P(x,y)+j\cdot Q(x,y)$, $z=x+jy$ dacă

$$P(x,y)=\frac{x}{2}\ln(x^2+y^2)-y\operatorname{arctg}\frac{y}{x},\quad f(1)=j$$

—

f olomorfă $\to$ au loc cond. (C-R) $\begin{cases}\dfrac{\partial P}{\partial x}=\dfrac{\partial Q}{\partial y}\\[2mm] \dfrac{\partial P}{\partial y}=-\dfrac{\partial Q}{\partial x}\end{cases}$

$$\frac{\partial P}{\partial x}=\frac{1}{2}\ln(x^2+y^2)+\frac{x}{\cancel{2}}\cdot\frac{\cancel{2}x}{x^2+y^2}+\frac{y^2}{x^2}\cdot\frac{x^2}{x^2+y^2}=\frac{1}{2}\ln(x^2+y^2)+1$$

$$\frac{\partial P}{\partial y}=\cancel{\frac{x}{2}\cdot\frac{2y}{x^2+y^2}}-\operatorname{arctg}\frac{y}{x}-\cancel{y\cdot\frac{1}{x}\cdot\frac{x^2}{x^2+y^2}}=-\operatorname{arctg}\frac{y}{x},\ x\neq 0$$

Obținem $f'(z)=\dfrac{\partial P}{\partial x}+j\dfrac{\partial Q}{\partial x}=\dfrac{\partial P}{\partial x}-j\cdot\dfrac{\partial P}{\partial y}=\dfrac{1}{2}\ln(x^2+y^2)+1-j\cdot\operatorname{arctg}\dfrac{y}{x}$, $x\neq 0$

Luând $y=0\Rightarrow z=x\in\mathbb{R}^*$ și $f'(z)=\ln x+1=\ln z+1$ $\forall z\in\mathbb{R}^*$

cf pp. id. funcțiilor olomorfe, $f'(z)=\ln z+1$, $\forall z\in\mathbb{C}^*$

$$f(z)=\int f'(z)\,dz=\int(\ln z+1)\,dz=\int z'(\ln z+1)\,dz=z(\ln z+1)-\int z\cdot\frac{1}{z}\,dz$$

$$=z(\ln z+1)-z+C=z\ln z+C,\ C\in\mathbb{C}$$

$\Rightarrow C=j\Rightarrow f(z)=z\ln z+j$, $z\in\mathbb{C}^*$

Cum $f(1)=j$

Seminarul 3

Funcții olomorfe

Fie $f: \mathbb{C} \to \mathbb{C}$, $f(z) = P(x,y) + j \cdot Q(x,y)$, $z = x + jy$

- f olomorfă $\Rightarrow$ au loc condițiile (C-R) $\begin{cases} \dfrac{\partial P}{\partial x} = \dfrac{\partial Q}{\partial y} \\ \dfrac{\partial P}{\partial y} = -\dfrac{\partial Q}{\partial x} \end{cases}$

- f olomorfă și $f \in C^2(\mathbb{C})$ $\Rightarrow$ P și Q armonice ($\Delta P = 0$ și $\Delta Q = 0$, unde $\Delta = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$)

① Să se determine $f(z)$ olomorfă, dacă

$$|f(z)| = (x^2 + y^2 + 2x + 1) e^{\pi x}, \quad z = x + jy \text{ și } f(-j) = 2j$$

Obs: Procedăm asemănător dacă avem $\arg f(z) = \ldots$

$z = |z| \cdot e^{j \cdot \arg z}$

$f(z) = |f(z)| \cdot e^{j \cdot \arg f(z)}$ $\qquad$ $\text{Ln}\, f(z) = \{ \ln |f(z)| + j(\arg f(z) + 2k\pi);\ k \in \mathbb{Z} \}$

$\ln f(z) = \ln |f(z)| + j \cdot \arg f(z)$

determinarea principală a lui $\text{Ln}\, f(z)$ (se obține pentru $k = 0$)

Obținem $\ln f(z) = \ln\left((x^2 + y^2 + 2x + 1) e^{\pi x}\right) + j \cdot \arg f(z)$

$$= \underbrace{\ln(x^2 + y^2 + 2x + 1) + \pi x}_{P(x,y)} + j \cdot \underbrace{\arg f(z)}_{Q(x,y)}$$

Notăm $g(z) = \ln f(z)$ și obținem $g(z) = P(x,y) + j \cdot Q(x,y)$

Cum f e olomorfă $\Rightarrow$ au loc condițiile (C-R) $\begin{cases} \dfrac{\partial P}{\partial x} = \dfrac{\partial Q}{\partial y} \\ \dfrac{\partial P}{\partial y} = -\dfrac{\partial Q}{\partial x} \end{cases} \Leftrightarrow$

$$\Leftrightarrow \begin{cases} \dfrac{2x + 2}{x^2 + y^2 + 2x + 1} + \pi = \dfrac{\partial Q}{\partial y} \\ \dfrac{2y}{x^2 + y^2 + 2x + 1} = -\dfrac{\partial Q}{\partial x} \end{cases}$$

Calculăm mai întâi

$$g'(z) = \frac{\partial P}{\partial x} + j \cdot \frac{\partial Q}{\partial x} \overset{(C-R)}{=} \frac{\partial P}{\partial x} - j \cdot \frac{\partial P}{\partial y}$$

$$= \frac{2(x+1)}{x^2 + y^2 + 2x + 1} + \pi - \frac{2y}{x^2 + y^2 + 2x + 1} j$$

Luând $y = 0 \Rightarrow z = x \in \mathbb{R}$ și

$$g'(z) = \frac{2(x+1)}{(x+1)^2} + \pi = \frac{2}{z+1} + \pi, \quad \forall z \in \mathbb{R}, z \neq -1$$

Cf pp id funcțiilor olomorfe $\Rightarrow$ $\underline{g'(z) = \pi + \dfrac{2}{z+1}}$, $\forall z \in \mathbb{C} \setminus \{-1\}$

Obținem în final $g(z) = \displaystyle\int g'(z)\, dz = \int \left(\pi + \frac{2}{z+1}\right) dz = \pi z + 2 \ln(z+1) + C$, $C \in \mathbb{C}$, $z \in \mathbb{C} \setminus \{-1\}$

$$\Rightarrow \underline{f(z)} = e^{\pi z + 2 \ln(z+1) + C} = \underline{e^{\pi z} \cdot (z+1)^2 \cdot e^C}, \quad C \in \mathbb{C},\ z \in \mathbb{C} \setminus \{-1\}$$

Cum $f(-j) = 2j \Rightarrow e^{-\pi j} \cdot (-j+1)^2 \cdot e^{c} = 2j \Leftrightarrow$

$\Leftrightarrow [\cos(-\pi) + j\sin(-\pi)] \cdot (-2j) \cdot e^{c} = 2j$

$\Leftrightarrow (-1)\cdot(-2j)\cdot e^{c} = 2j \Leftrightarrow e^{c} = 1 \Rightarrow \underline{c=0}$

Așadar, $\underline{f(z) = e^{\pi z} \cdot (z+1)^2}$

② Fie $f: \mathbb{C} \to \mathbb{C}$, $f(z) = P(x,y) + j\cdot Q(x,y)$, $z = x+jy$

Știind că $Q(x,y) = \operatorname{Im} f = \varphi\left(\frac{x}{y}\right)$, unde $\varphi \in C^2(\mathbb{R})$, să se determine funcția olomorfă f.

Notăm $\frac{x}{y} = u \Rightarrow Q(x,y) = \varphi(u)$

$$\frac{\partial Q}{\partial x} = \frac{\partial \varphi}{\partial u} \cdot \frac{\partial u}{\partial x} = \frac{1}{y} \cdot \frac{\partial \varphi}{\partial u} = \frac{1}{y} \cdot \varphi'(u)$$

$$\frac{\partial Q}{\partial y} = \frac{\partial \varphi}{\partial u} \cdot \frac{\partial u}{\partial y} = -\frac{x}{y^2} \cdot \frac{\partial \varphi}{\partial u} = -\frac{x}{y^2} \cdot \varphi'(u)$$

f olomorfă

$\Downarrow$

$$\Delta Q = 0 \Leftrightarrow \frac{\partial^2 Q}{\partial x^2} + \frac{\partial^2 Q}{\partial y^2} = 0 \quad (1)$$

$$\frac{\partial^2 Q}{\partial x^2} = \frac{1}{y} \cdot \frac{1}{y} \cdot \varphi''(u) \quad (2)$$

$$\frac{\partial^2 Q}{\partial y^2} = \frac{2x}{y^3} \cdot \varphi'(u) - \frac{x}{y^2} \cdot \left(-\frac{x}{y^2}\right) \cdot \varphi''(u) = \frac{2x}{y^3} \cdot \varphi'(u) + \frac{x^2}{y^4} \cdot \varphi''(u) \quad (3)$$

Din (1), (2) și (3) $\Rightarrow \frac{1}{y^2} \cdot \varphi''(u) + \frac{2x}{y^3} \cdot \varphi'(u) + \frac{x^2}{y^4} \cdot \varphi''(u) = 0 \quad | \cdot y^4$

$\Leftrightarrow y^2 \cdot \varphi''(u) + 2xy \cdot \varphi'(u) + x^2 \cdot \varphi''(u) = 0$

$\Leftrightarrow \varphi''(u) \cdot (x^2+y^2) + 2xy \cdot \varphi'(u) = 0 \quad | : y^2$

$\Leftrightarrow \varphi''(u) \cdot (u^2+1) + 2u \cdot \varphi'(u) = 0$

$\Leftrightarrow (\varphi'(u) \cdot (u^2+1))' = 0 \Rightarrow \varphi'(u) \cdot (u^2+1) = c_1, \ \underline{c_1 \in \mathbb{R}}$

$\Downarrow$

$$\varphi'(u) = \frac{c_1}{u^2+1} \Rightarrow \varphi(u) = c_1 \cdot \operatorname{arctg} u + c_2, \quad \underline{c_1, c_2 \in \mathbb{R}}$$

Așadar, $Q(x,y) = c_1 \cdot \operatorname{arctg} \frac{x}{y} + c_2, \quad c_1, c_2 \in \mathbb{R}$

$$f'(z) = \frac{\partial P}{\partial x} + j \cdot \frac{\partial Q}{\partial x} \overset{(C-R)}{=} \frac{\partial Q}{\partial y} + j \cdot \frac{\partial Q}{\partial x} = c_1 \cdot \left(-\frac{x}{\cancel{y^2}}\right) \cdot \frac{\cancel{y^2}}{x^2+y^2} + j \cdot c_1 \cdot \frac{1}{\cancel{y}} \cdot \frac{y^{\cancel{2}}}{x^2+y^2} =$$

$$= \frac{-c_1 x}{x^2+y^2} + j \cdot \frac{c_1 \cdot y}{x^2+y^2}, \quad c_1 \in \mathbb{R}$$

Luând $y = 0 \Rightarrow z = x \in \mathbb{R}$ și

$$f'(z) = -\frac{c_1}{x} = -\frac{c_1}{z}, \quad \forall z \in \mathbb{R}^*$$

Cf. pp. id. funcțiilor olomorfe, $f'(z) = -\frac{c_1}{z}, \ \forall z \in \mathbb{C}^*, \ c_1 \in \mathbb{R}$

În final, $\underline{f(z)} = \int f'(z)\,dz = \int \left(-\frac{c_1}{z}\right) dz = \underline{-c_1 \ln z + c_3}, \ c_1 \in \mathbb{R}, \ c_3 \in \mathbb{C}$

③ Fie $f: \mathbb{C} \to \mathbb{C}$, $f(z) = P(x,y) + j\cdot Q(x,y)$ o funcție olomorfă, $z \in D \subseteq \mathbb{C}$. Dacă P^3 armonică și $P(x,y) \neq 0$, $\forall x \in D$, atunci f este constantă.

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f olomorfă $\to$ P armonică, adică $\Delta P = 0 \Leftrightarrow \dfrac{\partial^2 P}{\partial x^2} + \dfrac{\partial^2 P}{\partial y^2} = 0$ (1)

Cum P^3 armonică $\Rightarrow \dfrac{\partial^2 P^3}{\partial x^2} + \dfrac{\partial^2 P^3}{\partial y^2} = 0$ (2)

$$\frac{\partial P^3}{\partial x} = 3P^2 \cdot \frac{\partial P}{\partial x} \to \frac{\partial^2 P^3}{\partial x^2} = 6P\cdot\left(\frac{\partial P}{\partial x}\right)^2 + 3P^2 \cdot \frac{\partial^2 P}{\partial x^2}$$

$$\frac{\partial P^3}{\partial y} = 3P^2 \cdot \frac{\partial P}{\partial y} \to \frac{\partial^2 P^3}{\partial y^2} = 6P\cdot\left(\frac{\partial P}{\partial y}\right)^2 + 3P^2 \cdot \frac{\partial^2 P}{\partial y^2}$$

Folosindu-ne și de rel. (2), obținem

$$6P\cdot\left[\left(\frac{\partial P}{\partial x}\right)^2 + \left(\frac{\partial P}{\partial y}\right)^2\right] + 3P^2 \cdot \underbrace{\left(\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2}\right)}_{\substack{\overset{(1)}{=} \\ \Delta P = 0}} = 0 \quad | : P \; (P(x,y) \neq 0)$$

$$\to \left(\frac{\partial P}{\partial x}\right)^2 + \left(\frac{\partial P}{\partial y}\right)^2 = 0, \; \forall (x,y) \in D$$

Obținem $\dfrac{\partial P}{\partial x} = \dfrac{\partial P}{\partial y} = 0$, $\forall (x,y) \in D \to$ P constantă (C-R) $\Big\} \Rightarrow$ Q - constantă $\Downarrow$ f - constantă

sau

$$f'(z) = \frac{\partial P}{\partial x} + j\cdot\frac{\partial Q}{\partial x} = 0, \; \forall z \in D$$

$\Rightarrow \exists K \in \mathbb{C}$ a.î. $f(z) = K \to$ f - constantă

④* Să se determine coeficientul de deformare liniară și unghiul de rotație pentru transformarea

$$Z = \frac{2z + 3j}{j(z+1) + 5} \text{ în punctul } z_0 = 2 + j.$$

Obs: • $dl(f; z_0) = |f'(z_0)|$, unde $Z = f(z)$ și $Z' = f'(z)$

deformarea liniară

• $rot(f; z_0) = \arg f'(z_0)$

unghiul de rotație

transformarea este omografică

Obs: Dacă $f(z) = \dfrac{az+b}{cz+d} \to f'(z) = \dfrac{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}{(cz+d)^2}$

$$f'(z) = \frac{2(\cancel{jz} + j + 5) - j\cdot(\cancel{2z} + 3j)}{(j(z+1)+5)^2} = \frac{2j+13}{(j(z+1)+5)^2} = \frac{\begin{vmatrix} 2 & 3j \\ j & 5+j \end{vmatrix}}{(j(z+1)+5)^2} \to f'(z_0) = \frac{13+2j}{(j(3+j)+5)^2} = \frac{13+2j}{(4+3j)^2}$$

$$\to dl(f; z_0) = \left|\frac{13+2j}{(4+3j)^2}\right| = \frac{|13+2j|}{|4+3j|^2} = \frac{\sqrt{173}}{25} \simeq 0{,}5 < 1$$

$\Rightarrow$ avem o contracție

Unghiul de rotație este

$$rot(f; z_0) = \arg\frac{13+2j}{(4+3j)^2} = \left(\arg(13+2j) - 2\arg(4+3j)\right) \bmod 2\pi =$$

$$= \left(\operatorname{arctg}\frac{2}{13} - 2\cdot\operatorname{arctg}\frac{3}{4}\right) \bmod 2\pi$$

5. Să se determine mulțimea de puncte din $\mathbb{C}$ cu proprietatea că în fiecare punct coeficientul de deformare liniară a lui $Z = \frac{jz-1}{5z+j}$ este 1

= $d\ell(f; z) = 1, \ \forall z$

$|f'(z)|$, unde $Z = f(z)$. Cum $f'(z) = \frac{j(5z+j)-5(jz-1)}{(5z+j)^2} = \frac{-1+5}{(5z+j)^2} = \frac{4}{(5z+j)^2}$

Așadar, $|f'(z)| = 1 \Leftrightarrow \frac{4}{|5z+j|^2} = 1 \Leftrightarrow |5z+j|^2 = 4 \quad (5z+j \neq 0 ✓)$

$$|5z+j| = 2 \Leftrightarrow \left|z + \tfrac{1}{5}j\right| = \tfrac{2}{5} \Leftrightarrow z \in \Gamma\left(-\tfrac{j}{5}, \tfrac{2}{5}\right)$$

6. Să se determine mulțimea de puncte din plan cu proprietatea că unghiul de rotație prin transformarea $Z = \frac{2z-j+2}{j-z-1}$ este nul

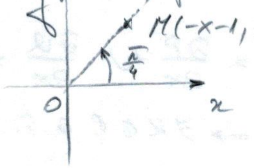
= $Z' = f'(z) = \frac{2(j-z-1)+(2z-j+2)}{(j-z-1)^2} = \frac{2j-\cancel{2z}-\cancel{2}+\cancel{2z}-j+\cancel{2}}{(j-z-1)^2} = \frac{j}{(j-z-1)^2}$

$rot(f; z) = \arg f'(z) = 0 \Leftrightarrow \left(\arg j - 2\arg(j-z-1)\right)_{mod\,2\pi} = 0 \Leftrightarrow$

$\Leftrightarrow \left(\frac{\pi}{2} - 2\arg(j-z-1)\right)_{mod\,2\pi} = 0 \Leftrightarrow \left(\frac{\pi}{4} - \arg(j-z-1)\right)_{mod\,2\pi} = 0 \Leftrightarrow$

$\Leftrightarrow \arg(j-z+1)\big|_{mod\,2\pi} = \frac{\pi}{4}$

$\Leftrightarrow \left(\arg(-x-1+j(1-y))\right)_{mod\,2\pi} = \frac{\pi}{4}$



Obținem $-x-1 = 1-y$, așadar dreapta de ecuație: $x - y + 2 = 0$

7*. Să se determine imaginea lui

$D = \{(x,y) \in \mathbb{R}^2 : x+y = -1\}$ prin transformarea $Z = \frac{z-j}{z+1}$, $f(z) = Z$

= $f(z) = \frac{z-j}{z+1} = Z$ Știm că $x = \frac{z+\bar z}{2}$ și $y = \frac{z-\bar z}{2j}$

Așadar, $x+y = -1 \Leftrightarrow \frac{z+\bar z}{2} + \frac{z-\bar z}{2j} = -1 \quad |\cdot 2j$

$\Leftrightarrow zj + \bar z j + z - \bar z = -2j$

$\Leftrightarrow z(1+j) + \bar z(-1+j) = -2j \quad (1)$

Din $\frac{z-j}{z+1} = Z \Rightarrow z - j = z\cdot Z + Z \Rightarrow z(1-Z) = Z+j \Rightarrow z = \frac{Z+j}{1-Z}$ și $\bar z = \frac{\bar Z - j}{1-\bar Z} \quad (2)$

Din (1) și (2) $\Rightarrow \frac{Z+j}{1-Z}(1+j) + \frac{\bar Z - j}{1-\bar Z}(-1+j) = -2j \quad |\cdot(1-Z)(1-\bar Z)$

$\Leftrightarrow (1-\bar Z)(Z+j)(1+j) + (\bar Z - j)(1-Z)(-1+j) = -2j(1-Z)(1-\bar Z)$

$\Leftrightarrow (Z + j - Z\bar Z - \bar Z j)(1+j) + (\bar Z - Z\bar Z - j + jZ)(-1+j) = -2j(1 - Z - \bar Z + Z\bar Z)$

$\Leftrightarrow \cancel{Z} + j - \cancel{Z\bar Z} - \cancel{\bar Z j} + jZ - \cancel{1} - jZ\bar Z + \cancel{\bar Z} - \bar Z + j\bar Z + \cancel{Z\bar Z} - jZ\bar Z + j + \cancel{1} - jZ - \cancel{Z} =$
$= -2j + 2jZ + 2j\bar Z - \cancel{2jZ\bar Z}$

$\Leftrightarrow 4j = 2j(Z + \bar Z) \Big|_{:4j} \Leftrightarrow \frac{Z+\bar Z}{2} = 1 \Leftrightarrow X = 1$

Prin urmare, dreapta $x+y=-1$ din planul (z) devine dreapta $X=1$ din (Z)

⑧ Să se determine

a) Deformarea liniară și unghiul de rotație al transformării

$$Z = \frac{2z - j + 4}{2 + jz} \quad \text{în punctul } z_0 = j$$

$\Rightarrow d\ell(f; z_0) = |f'(z_0)|$, unde $f'(z) = Z' = \dfrac{2(2+jz) - j(2z-j+4)}{(2+jz)^2} = \dfrac{4 + \cancel{2jz} - \cancel{2jz} - 1 - 4j}{(2+jz)^2} = \dfrac{3-4j}{(2+jz)^2}$

$d\ell(f; j) = \left|\dfrac{3-4j}{1}\right| = \underline{\underline{5}}$ (dilatare)

$f'(z_0) = f'(j) = \dfrac{3-4j}{1}$

$rot(f; z_0) = \arg f'(z_0) = \arg(3-4j) = \underline{\underline{2\pi - \operatorname{arctg} \frac{4}{3}}}$

b) Regiunile din plan care se dilată și cele care se contractă prin transformarea

$$Z = \frac{z+j}{jz+3}$$

$Z' = f'(z) = \dfrac{\cancel{jz} + 3 - \cancel{jz} + 1}{(jz+3)^2} = \dfrac{4}{(jz+3)^2}$

contracție

$|Z'| < 1 \Leftrightarrow \left|\dfrac{4}{(jz+3)^2}\right| < 1 \Leftrightarrow 4 < |jz+3|^2 \Leftrightarrow$

$\Leftrightarrow 2 < |jz+3| \Leftrightarrow 2 < |j\cdot(z-3j)| \Leftrightarrow 2 < |z-3j|$

$\Rightarrow z \in \mathbb{C} \setminus \bar{\Delta}(3j; 2)$

Așadar, mulțimile situate în $\Delta(3j; 2)$ se dilată,
iar cele din exteriorul discului se contractă.

Probleme propuse

① Să se determine coeficientul de deformare liniară și unghiul de rotație pentru transformarea

$$Z = \frac{jz + 1 - j}{z + 1 + 4j} \quad \text{în punctul } z_0 = 3 - j$$

② Să se determine funcția olomorfă $f: \mathbb{C} \to \mathbb{C}$, $f(z) = P(x,y) + j\cdot Q(x,y)$, $z = x + jy$ știind că

a) $Q(x,y) = \psi(x^2+y^2)$, iar $\psi \in C^2(\mathbb{R})$.

b) $P(x,y) = \varphi(x^2-y^2)$, unde $\varphi \in C^2(\mathbb{R})$.

c) $Q(x,y) = \varphi\left(\dfrac{x^2+y^2}{x}\right)$, $x \neq 0$ și $\varphi \in C^2(\mathbb{R})$.

③ Să se determine mulțimea de puncte din planul complex cu proprietatea că în fiecare punct al acestei mulțimi coeficientul de deformare liniară al transformării

$$Z = \frac{2z + j}{3z + 2j} \quad \text{este egal cu } 1.$$

Seminarul 4

Funcții olomorfe. Integrala în complex

① Să se rezolve în $\mathbb{C}$ ecuațiile:

a) $\underbrace{\cos z - j\sin z}_{= e^{-jz}} = 2j \quad (1)$

sau $\cos z - j\sin z = e^{-jz} = 2j \Rightarrow -jz \in \mathrm{Ln}\, 2j$

$\Leftrightarrow -jz \in \ln 2 + j\cdot\left(\frac{\pi}{2} + 2\pi\mathbb{Z}\right) \,|\cdot j$

$\Leftrightarrow z \in \ln 2\cdot j - \left(\frac{\pi}{2} + 2\pi\mathbb{Z}\right)$

=

Obs: $\cos z = \dfrac{e^{jz} + e^{-jz}}{2} \left(= \dfrac{\cos z + \cancel{j\sin z} + \cos z - \cancel{j\sin z}}{2}\right)$ $\qquad e^{jz} = \cos z + j\sin z$

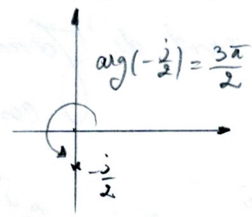
$\sin z = \dfrac{e^{jz} - e^{-jz}}{2j} \left(= \dfrac{\cancel{\cos z} + j\sin z - \cancel{\cos z} + j\sin z}{2j}\right)$ $\qquad e^{-jz} = \cos(-z) + j\sin(-z) = \cos z - j\sin z$

Ec. (1) $\Leftrightarrow \dfrac{e^{jz} + e^{-jz}}{2} - \cancel{j}\cdot\dfrac{e^{jz} - e^{-jz}}{2\cancel{j}} = 2j$ $\quad$ Notând $e^{jz} = u \rightarrow e^{-jz} = \frac{1}{u}$, obținem

$\Leftrightarrow \dfrac{u + \frac{1}{u}}{2} - \dfrac{u - \frac{1}{u}}{2} = 2j \,|\cdot 2 \Leftrightarrow \cancel{u} + \frac{1}{u} - \cancel{u} + \frac{1}{u} = 4j \Leftrightarrow \frac{2}{u} = 4j \Leftrightarrow u = \frac{1}{2j} = \frac{j}{2j^2} = -\frac{j}{2}$

Deci $e^{jz} = -\frac{j}{2} \rightarrow jz \in \mathrm{Ln}\left(-\frac{j}{2}\right) \,|\cdot j$

$\Leftrightarrow -z \in j\,\mathrm{Ln}\left(-\frac{j}{2}\right) \Leftrightarrow z \in -j\cdot\mathrm{Ln}\left(-\frac{j}{2}\right)$



$\arg\left(-\frac{j}{2}\right) = \frac{3\pi}{2} \quad \left|-\frac{j}{2}\right| = \frac{1}{2}$

$-\frac{j}{2} = \left|-\frac{j}{2}\right|\underbrace{\left(\cos\frac{3\pi}{2} + j\cdot\sin\frac{3\pi}{2}\right)}_{e^{\frac{3\pi}{2}j}} = \frac{1}{2}\cdot e^{\frac{3\pi}{2}\cdot j}$

Așadar, $\mathrm{Ln}\left(-\frac{j}{2}\right) = \left\{\ln\frac{1}{2} + j\cdot\left(\frac{3\pi}{2} + 2k\pi\right) : k\in\mathbb{Z}\right\} = -\ln 2 + j\left(\frac{3\pi}{2} + 2\pi\mathbb{Z}\right)$

În final, $z \in -j\cdot\mathrm{Ln}\left(-\frac{j}{2}\right) \Leftrightarrow z \in -j\left[-\ln 2 + j\cdot\left(\frac{3\pi}{2} + 2\pi\mathbb{Z}\right)\right]$

$\Leftrightarrow z \in j\cdot\ln 2 + \frac{3\pi}{2} + 2\pi\mathbb{Z}$

$\rightarrow S = \frac{3\pi}{2} + 2\pi\mathbb{Z} + j\cdot\ln 2$. Observăm că $S \cap \mathbb{R} = \emptyset$

$= \left(2\pi - \frac{\pi}{2}\right) + 2\pi\mathbb{Z} + j\cdot\ln 2 = j\cdot\ln 2 - \left(\frac{\pi}{2} + 2\pi\mathbb{Z}\right)$

b) $\mathrm{sh}\, z + 3\,\mathrm{ch}\, z = j \quad (2)$

=

Obs: $\mathrm{sh}\, z = \dfrac{e^{z} - e^{-z}}{2}$; $\mathrm{ch}\, z = \dfrac{e^{z} + e^{-z}}{2}$

$(2) \Leftrightarrow \dfrac{e^{z} - e^{-z}}{2} + 3\cdot\dfrac{e^{z} + e^{-z}}{2} = j$. Notând $e^{z} = u$, obținem

$\dfrac{u - \frac{1}{u}}{2} + 3\cdot\dfrac{u + \frac{1}{u}}{2} = j \,|\cdot 2 \Leftrightarrow u - \frac{1}{u} + 3u + \frac{3}{u} = 2j \,|\cdot u$

$\Leftrightarrow u^2 - 1 + 3u^2 + 3 = 2uj \Leftrightarrow 4u^2 - 2uj + 2 = 0 \Leftrightarrow 2u^2 - uj + 1 = 0$

$\Delta = (-j)^2 - 8 = -9 = (3j)^2$

$u_{1,2} = \dfrac{j \pm 3j}{4}$ $\quad u_1 = j$, $\quad u_2 = -\dfrac{j}{2}$

$u_1 = j \Rightarrow e^z = j$ și deci $z \in \operatorname{Ln} j$

$u_2 = -\frac{j}{2} \Rightarrow e^z = -\frac{j}{2}$ și deci $z \in \operatorname{Ln}\left(-\frac{j}{2}\right)$ $\qquad j = e^{\frac{\pi}{2}j}$

Așadar, $z \in \operatorname{Ln} j \cup \operatorname{Ln}\left(-\frac{j}{2}\right)$ unde

$$\operatorname{Ln} j = \left\{ \ln|j| + j\cdot\left(\frac{\pi}{2} + 2k\pi\right) : k \in \mathbb{Z} \right\} = \ln 1 + j\cdot\left(\frac{\pi}{2} + 2\pi\mathbb{Z}\right) = j\cdot\left(\frac{\pi}{2} + 2\pi\mathbb{Z}\right)$$

$$\operatorname{Ln}\left(-\frac{j}{2}\right) = \left\{ \ln\left|-\frac{j}{2}\right| + j\cdot\left(\frac{3\pi}{2} + 2k\pi\right) : k \in \mathbb{Z} \right\} = -\ln 2 + j\cdot\left(\frac{3\pi}{2} + 2\pi\mathbb{Z}\right)$$

$$-\frac{j}{2} = \frac{1}{2}\cdot e^{\frac{3\pi}{2}}$$

Prin urmare,

$$z \in j\left(\frac{\pi}{2} + 2\pi\mathbb{Z}\right) \cup \left(-\ln 2 + j\cdot\left(\frac{3\pi}{2} + 2\pi\mathbb{Z}\right)\right)$$

② Să se determine numerele întregi $m, n \in \mathbb{Z}$ a.î.

$$|\cos z|^2 = m\cdot \operatorname{ch}^2 y + n \sin^2 x, \quad \text{unde } z = x + jy$$

$$\cos z = \cos(x+jy) = \cos x \cdot \cos jy - \sin x \cdot \sin jy \quad (1)$$

Ținând cont de formula

$$\cos w = \frac{e^{jw} + e^{-jw}}{2} \text{ și } \sin w = \frac{e^{jw} - e^{-jw}}{2j}, \ \forall w \in \mathbb{C}$$

obținem:

$$\cos jy = \frac{e^{j^2 y} + e^{-j^2 y}}{2} \text{ și } \sin jy = \frac{e^{j^2 y} - e^{-j^2 y}}{2j}$$

$$\frac{e^{y} + e^{-y}}{2} = \operatorname{ch} y \qquad \frac{e^{-y} - e^{y}}{2j} = j\cdot\frac{e^{y} - e^{-y}}{2} = j \operatorname{sh} y$$

Relația (1) $\Rightarrow \cos z = \cos x \cdot \operatorname{ch} y - j\cdot \sin x \cdot \operatorname{sh} y$

$$\Rightarrow \underline{|\cos z|^2} = |\cos x \cdot \operatorname{ch} y - j\cdot \sin x \cdot \operatorname{sh} y|^2 = \cos^2 x \cdot \operatorname{ch}^2 y + \sin^2 x \cdot \operatorname{sh}^2 y =$$

$$= (1 - \sin^2 x)\cdot \operatorname{ch}^2 y + \sin^2 x \cdot \operatorname{sh}^2 y = \operatorname{ch}^2 y + \sin^2 x \cdot \underbrace{(\operatorname{sh}^2 y - \operatorname{ch}^2 y)}_{-1} =$$

$$= \underline{\operatorname{ch}^2 y - \sin^2 x}$$

$$\operatorname{ch}^2 y - \operatorname{sh}^2 y = \frac{e^{2y} + e^{-2y} + 2 - e^{2y} - e^{-2y} + 2}{4} = 1.$$ Obținem așadar $\underline{m = 1}$ și $\underline{n = -1}$

③ Fie $D = \{ z \in \mathbb{C} \mid |z| \leq 1, \ \arg z \in (0, \frac{\pi}{10}] \}$

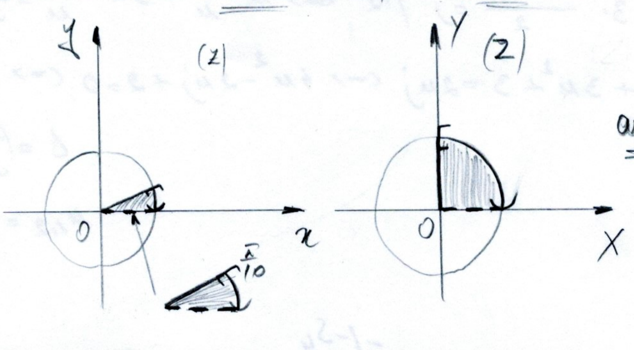
Să se determine imaginile prin transformările

a) $Z = z^5$

b) $Z = z^{15}$

c) $Z = z^{25}$

a) $|z| \leq 1 \Rightarrow \underline{|Z| = |z^5| = |z|^5 \leq 1}$

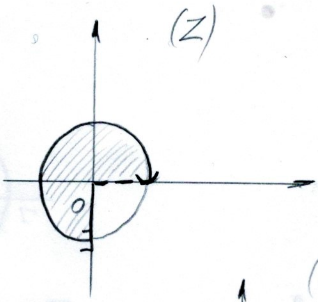


$\underline{\arg Z} = \arg(z^5) =$

$= (5\cdot \arg z) \bmod 2\pi$

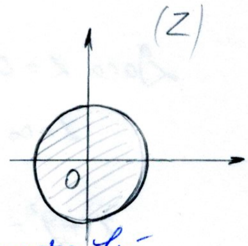
$\underline{\in (0, \frac{\pi}{2}]}$

b) $\underline{|Z|} = |z^{15}| = |z|^{15} \le 1$



$\underline{\arg Z} = \arg(z^{15}) = (15 \arg z) \bmod 2\pi \in \left(0, \frac{15\pi}{10}\right] = \left(0, \frac{3\pi}{2}\right]$

c) $\underline{|Z|} = |z^{25}| = |z|^{25} \le 1$



$\arg Z = \arg(z^{25}) = (25 \arg z) \bmod 2\pi \in \left(0, \frac{5\pi}{2}\right] \bmod 2\pi \Rightarrow \arg Z \in [0, 2\pi)$

④ Să se determine imaginea discului unitate $\Delta(0,1)$ prin transformarea omografică

$$Z = \frac{jz+2}{2z-j}$$

$\bar{\Delta}(0,1) = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \le 1\} = \{z \in \mathbb{C} : |z| \le 1\}$

$\Delta(0,1) = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 < 1\} = \{z \in \mathbb{C} : |z| < 1\}$

Mai întâi vom determina imaginea frontierei discului $\bar{\Delta}(0,1)$ și anume imaginea cercului $\Gamma(0,1) = \{(x,y) \in \mathbb{R}^2 : x^2+y^2=1\} = \{z \in \mathbb{C} : |z|=1\}$

Dacă $\left.\begin{array}{l} |z|=1 \\ z = x+jy \end{array}\right\} \Rightarrow x^2+y^2=1 \qquad Z = X+jY$

$$Z = \frac{jz+2}{2z-j} \overset{z=x+jy}{=} \frac{j\cdot(x+jy)+2}{2(x+jy)-j} = \frac{2-y+jx}{2x+j(2y-1)} \cdot \frac{2x-j(2y-1)}{} = \frac{4x-\cancel{2xy}+\cancel{2xy}-x}{(2x)^2+(2y-1)^2} + j\cdot\frac{(2x^2-4y+2+2y^2-y)}{(2x)^2+(2y-1)^2}$$

$$Z = \frac{3x}{4x^2+4y^2-4y+1} + j\cdot\frac{2x^2+2y^2-5y+2}{4x^2+4y^2-4y+1} \underset{x^2+y^2=1}{=} \underbrace{\frac{3x}{5-4y}}_{X} + j\cdot\underbrace{\frac{4-5y}{5-4y}}_{Y}$$

$$X = \frac{3x}{5-4y} \Rightarrow 3x = X\cdot(5-4y) \quad (1)$$

$$Y = \frac{4-5y}{5-4y} \Rightarrow Y(5-4y) = 4-5y \Leftrightarrow 5Y-4yY = 4-5y \Leftrightarrow y(4Y-5) = 5Y-4 \Rightarrow \underline{y = \frac{5Y-4}{4Y-5}} \quad (2)$$

Înlocuind (2) în (1), obținem

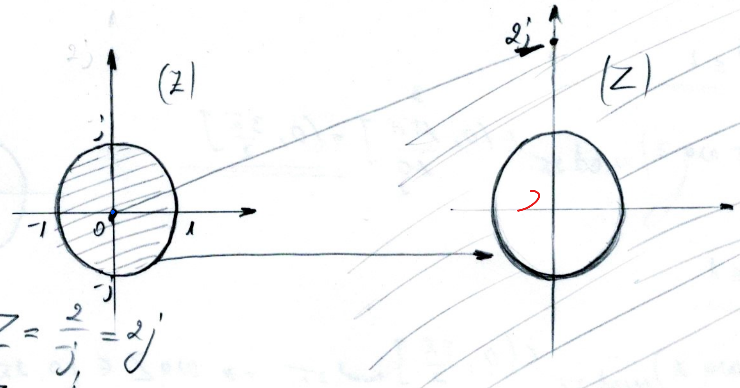
$$3x = X\cdot\left(5-\frac{20Y-16}{4Y-5}\right) = X\cdot\frac{\cancel{20Y}-25-\cancel{20Y}+16}{4Y-5} = -\frac{9X}{4Y-5}$$

$$\Rightarrow \underline{x = \frac{3X}{5-4Y}}$$

Apoi, $x^2+y^2=1 \Leftrightarrow \frac{9X^2}{(5-4Y)^2} + \frac{25Y^2-40Y+16}{(5-4Y)^2} = 1 \quad |\cdot(5-4Y)^2$

$$\Leftrightarrow 9X^2+25Y^2-\cancel{40Y}+16 = 25-\cancel{40Y}+16Y^2$$

$$\Leftrightarrow 9X^2+9Y^2=9 \Leftrightarrow \underline{X^2+Y^2=1}$$



Dacă $z=0 \to Z=\frac{2}{j}=2j$

$z=\infty \to Z=\frac{j}{2}$

$x^2+y^2=1 \longrightarrow X^2+Y^2=1$

Cum transformarea omografică este o transformare continuă, iar $z=0 \longrightarrow Z=2j$ (în afara discului)
rezultă că $\Delta(0,1) \xrightarrow{Z}$ exteriorul discului unitate din sistemul XOY
din sistemul xOy

⑤ Fie $z=\sin\left(\frac{\pi}{3}+j\cdot\ln 5\right)$. Să se calculeze $\frac{\operatorname{Im} z}{\operatorname{Re} z}$.

I

Obs. $\sin w=\frac{e^{jw}-e^{-jw}}{2j}$, $w\in\mathbb{C}$

Așadar,
$$z=\frac{e^{j\left(\frac{\pi}{3}+j\cdot\ln 5\right)}-e^{-j\left(\frac{\pi}{3}+j\cdot\ln 5\right)}}{2j}=\frac{e^{\frac{\pi}{3}j-\ln 5}-e^{-\frac{\pi}{3}j+\ln 5}}{2j}=$$

$$=\frac{e^{-\ln 5}\cdot\left(\cos\frac{\pi}{3}+j\sin\frac{\pi}{3}\right)-e^{\ln 5}\cdot\left(\cos\left(-\frac{\pi}{3}\right)+j\sin\left(-\frac{\pi}{3}\right)\right)}{2j}=$$

$$=\frac{\frac{1}{5}\cdot\left(\frac{1}{2}+\frac{\sqrt{3}}{2}j\right)-5\left(\frac{1}{2}-\frac{\sqrt{3}}{2}j\right)}{2j}=\frac{\frac{1}{10}+\frac{\sqrt{3}}{10}j-\frac{5}{2}+\frac{5\sqrt{3}}{2}j}{2j}=$$

$$=\frac{-\frac{24}{10}+\frac{26}{10}\sqrt{3}j}{2j}=-j\cdot\left(-\frac{12}{10}+\frac{13}{10}\sqrt{3}j\right)=\frac{13}{10}\sqrt{3}+\frac{6}{5}j$$

$\frac{1}{j}=-j$

În final,
$$\frac{\operatorname{Im} z}{\operatorname{Re} z}=\frac{\frac{6}{5}}{\frac{13\sqrt{3}}{10}}=\frac{6}{5}\cdot\frac{10}{13\sqrt{3}}=\frac{12\sqrt{3}}{39}$$

sau

II

$$z=\sin\left(\frac{\pi}{3}+j\cdot\ln 5\right)=\sin\frac{\pi}{3}\cdot\cos(j\cdot\ln 5)+\sin(j\cdot\ln 5)\cdot\cos\frac{\pi}{3}=\ldots$$

unde $\cos(j\cdot\ln 5)=\frac{e^{j^2\ln 5}+e^{-j^2\ln 5}}{2}=\frac{e^{-\ln 5}+e^{\ln 5}}{2}=\frac{\frac{1}{5}+5}{2}$

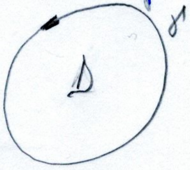
iar

$$\sin(j\cdot\ln 5)=\frac{e^{j^2\ln 5}-e^{-j^2\ln 5}}{2j}=\frac{\frac{1}{5}-5}{2j}$$

⑥ Să se calculeze

$$I = \int_{|z-1| = 2\sqrt{2}} \frac{e^{z^2} + 2z \cdot \sin(z^2+1) + 1}{(\operatorname{sh} z + \operatorname{ch} z + 1)^2}\, dz$$

Teorema Cauchy - Goursat



Fie γ - o curbă simplă, închisă și netedă (sau netedă pe porțiuni) și fie $\Delta = \operatorname{Int} \gamma$ (domeniul mărginit de γ)

Dacă f - olomorfă pe Δ ($f \in \mathcal{H}(\Delta)$) și f - continuă pe γ ($f \in C(\gamma)$), atunci

$$\underline{\int_{\gamma} f(z)\, dz = 0}$$

În cazul problemei noastre,

$$\gamma = \{ z \in \mathbb{C} : |z-1| = 2\sqrt{2} \} = \Gamma(1; 2\sqrt{2}) \quad (\gamma \text{ este un cerc})$$

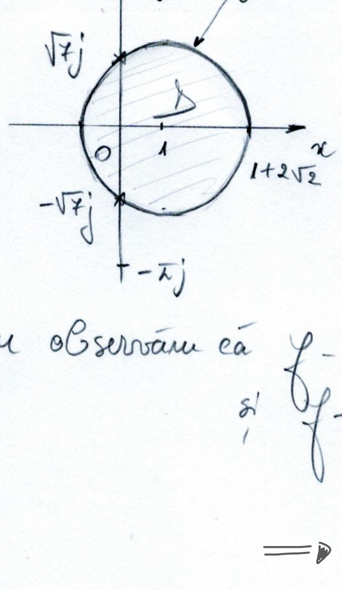
Obs: γ - poate fi o elipsă, un dreptunghi, un triunghi sau orice altă curbă închisă și netedă sau netedă pe porțiuni

$$\Delta = \operatorname{Int}(\gamma) = \{ z \in \mathbb{C} : |z-1| < 2\sqrt{2} \} = \Delta(1; 2\sqrt{2})$$

$$\operatorname{sh} z + \operatorname{ch} z + 1 = 0 \iff \frac{e^z - e^{-z}}{2} + \frac{e^z + e^{-z}}{2} + 1 = 0 \qquad -1 = 1 \cdot (\cos\pi + j\sin\pi) = e^{\pi j}$$

$$\iff e^z = -1 \Rightarrow z \in \operatorname{Ln}(-1) = \{ \ln 1 + j \cdot (\pi + 2k\pi) : k \in \mathbb{Z} \} = j(\pi + 2\pi\mathbb{Z})$$

$$\iff z \in \{ \dots, -\pi j, \pi j, 3\pi j, \dots \}$$



$$|z-1| = 2\sqrt{2} \iff |x + jy - 1| = 2\sqrt{2}$$

$$\iff \sqrt{(x-1)^2 + y^2} = 2\sqrt{2}$$

$$\iff (x-1)^2 + y^2 = 8$$

Dacă $x = 0 \Rightarrow y^2 = 7 \Rightarrow y = \pm\sqrt{7}$

Din desen observăm că f - olomorfă pe $\Delta = \operatorname{Int}(\gamma) = \Delta(1; 2\sqrt{2})$ ($f \in \mathcal{H}(\Delta(1; 2\sqrt{2}))$) și f - continuă pe $\gamma = \Gamma(1; 2\sqrt{2})$ ($f \in C(\Gamma(1; 2\sqrt{2}))$)

$$\Longrightarrow \underline{\int_{\gamma} f(z)\, dz = 0}$$

Probleme propuse

① Fie $z = \operatorname{tg}\left(\frac{3\pi}{4} + j \cdot \ln 2\right)$. Să se calculeze $\frac{\operatorname{Re} z}{\operatorname{Im} z}$.

② Să se determine partea reală și partea imaginară pentru:

a) $z = \cos\left(\frac{\pi}{2} + j \ln 3\right)$

b) $z = \operatorname{sh}\left(1 - j \cdot \frac{\pi}{2}\right)$

③ Să se rezolve în $\mathbb{C}$ ecuațiile:

a) $\sin z - \cos z = 2$

b) $2j \operatorname{ch} z + j \operatorname{sh} z + 1 = 0$

c) $4 \operatorname{ch} z + \operatorname{sh} z = j$

④ Să se demonstreze că

$$\operatorname{Im}(\operatorname{ch} z) = \operatorname{sh} x \cdot \sin y, \quad \text{unde } z = x + jy$$

Seminarul 5

Serii de puteri. Teorema reziduurilor

Def Fie $z_0 \in \mathbb{C}$ fixat. S.n. serie Laurent centrată în z_0 orice serie de funcții de forma

$$\sum_{n=-\infty}^{\infty} a_n (z-z_0)^n = \underbrace{\dots + \frac{a_{-n}}{(z-z_0)^n} + \dots + \frac{a_{-1}}{z-z_0}}_{\sum_{n=1}^{\infty} \frac{a_{-n}}{(z-z_0)^n}} + \underbrace{a_0 + a_1(z-z_0) + \dots + a_n(z-z_0)^n + \dots}_{\sum_{n=0}^{\infty} a_n (z-z_0)^n}, \quad a_n \in \mathbb{C}, \ n \in \mathbb{Z}$$

$\sum_{n=1}^{\infty} \frac{a_{-n}}{(z-z_0)^n}$ – partea principală $\qquad \sum_{n=0}^{\infty} a_n (z-z_0)^n$ – partea tayloriană

Serii Taylor importante

(1) Seria geometrică

$$\frac{1}{1-z} = 1 + z + z^2 + \dots = \sum_{n=0}^{\infty} z^n, \quad \forall z \in \Delta(0;1) \ (|z|<1)$$

Punând $z := -z$, obținem

$$\frac{1}{1+z} = 1 - z + z^2 - \dots = \sum_{n=0}^{\infty} (-1)^n \cdot z^n, \quad \forall z \in \mathbb{C} \text{ a.î. } |z|<1$$

(2) Serii exponențiale, circulare, hiperbolice

$$e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \dots = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \quad \forall z \in \mathbb{C}$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{z^{2n+1}}{(2n+1)!}, \quad \forall z \in \mathbb{C}$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \dots = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{z^{2n}}{(2n)!}, \quad \forall z \in \mathbb{C}$$

$$\operatorname{sh} z = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \dots = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}, \quad \forall z \in \mathbb{C}$$

$$\operatorname{ch} z = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \dots = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}, \quad \forall z \in \mathbb{C}$$

(3) Seria logaritmică

Dacă $f(z) = \ln(1+z)$ este ramura uniformă în $\Delta(0;1)$ a funcției $F(z) = \operatorname{Ln}(1+z)$, iar $f(0) = 0$, atunci

$$\ln(1+z) = \underset{f(0)}{\overset{\shortparallel}{0}} + z - \frac{z^2}{2} + \frac{z^3}{3} - \dots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \cdot z^n, \quad \forall z \in \Delta(0,1) \ (|z|<1)$$

(4) Seria binomială

Dacă $\alpha \in \mathbb{C}$ și $f(z) = (1+z)^\alpha$ este ramura uniformă în $\Delta(0;1)$ a funcției $F(z) = (1+z)^\alpha$, iar $f(0) = 1$, atunci

$$(1+z)^\alpha = \underset{f(0)}{\overset{\shortparallel}{1}} + \sum_{n=1}^{\infty} \frac{\alpha(\alpha-1)\cdot\dots\cdot(\alpha-n+1)}{n!} z^n, \quad \forall z \in \Delta(0;1) \ (|z|<1)$$

① Să se dezvolte în serie de puteri ale lui z în jurul lui $z_0=0$

$$f(z) = \ln \frac{2j+z}{2j-z}, \text{ unde } f(0) = 2\pi j$$

$$\left.\begin{array}{l} 2j - z \neq 0 \Leftrightarrow z \neq 2j \\ 2j + z \neq 0 \Leftrightarrow z \neq -2j \end{array}\right\} \to \Delta = \mathbb{C} \setminus \{\pm 2j\}$$

puncte singulare

domeniul de definiție

Obs: Dacă $g(z) = \ln(1+z)$, atunci $\ln(1+z) = g(0) + z - \frac{z^2}{2} + \ldots = g(0) + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \cdot z^n$, $|z|<1$

Cum $f(z) = \ln \frac{1+\frac{z}{2j}}{1-\frac{z}{2j}} = \ln \frac{1-\frac{zj}{2}}{1+\frac{zj}{2}} = \ln\left(1-\frac{zj}{2}\right) - \ln\left(1+\frac{zj}{2}\right)$,

Obținem

$$\underline{f(z)} = \underset{f(0)}{2\pi j} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \cdot \left(-\frac{zj}{2}\right)^n - \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \cdot \left(\frac{zj}{2}\right)^n, \quad \left|\frac{zj}{2}\right| < 1 \Leftrightarrow \underline{|z|<2}$$

$$= 2\pi j + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \cdot \left(\frac{zj}{2}\right)^n \left[(-1)^n - 1\right]$$

$$= 2\pi j + \sum_{n=1}^{\infty} \frac{(-1)^{2n-1}}{2n} \cdot \left(\frac{zj}{2}\right)^{2n} \underbrace{\left[(-1)^{2n} - 1\right]}_{0} + \sum_{n=0}^{\infty} \frac{(-1)^{2n}}{2n+1} \cdot \left(\frac{zj}{2}\right)^{2n+1} \cdot \underbrace{\left[(-1)^{2n+1} - 1\right]}_{=-2}$$

$$\sum_{n=1}^{\infty} a_n z^n = \sum_{n=1}^{\infty} a_{2n} z^{2n} + \sum_{n=0}^{\infty} a_{2n+1} z^{2n+1}$$

$$= 2\pi j + \sum_{n=0}^{\infty} \frac{1}{2n+1} \cdot \frac{z^{2n+1}}{\cancel{2^{2n+1}}_{\,2^{2n}}} \cdot (-1)^n \cdot j \cdot (\cancel{-2})^{(-1)} = 2\pi j + \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2n+1} \cdot 2^{-2n} \cdot z^{2n+1} j, \quad |z|<2$$

$$j^{2n+1} = (j^2)^n \cdot j = (-1)^n \cdot j$$

② Să se dezvolte în serie Laurent și să se determine tipul punctelor singulare coresp. și $Rez(f; z_0)$ unde:

a) $f(z) = \frac{e^{z^2}-1}{z^2}$, $z_0 = 0$, $0 < |z| < +\infty$

b) $f(z) = (z-1) \cdot \cos \frac{1}{z-2}$, $z_0 = 2$, $0 < |z-2| < +\infty$

z_0 - pol de ordinul k

c) $f(z) = \frac{2 \sin^2 z}{z^5}$, $z_0 = 0$, $0 < |z| < +\infty$

Obs. • Fie z_0 - un punct singular izolat pentru funcția $f: \{0 < |z-z_0| < r\} \to \mathbb{C}$.

Atunci

• z_0 este punct singular aparent dacă $\exists \lim_{z \to z_0} f(z)$ și e finită $\longleftrightarrow$ dezvoltarea în serie Laurent nu are parte principală, adică

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n, \quad \forall z \in \{0 < |z-z_0| < r\}$$

• z_0 este punct singular de tip pol dacă $\exists \lim_{z \to z_0} |f(z)| = +\infty$ $\longleftrightarrow$ dezv. în serie Laurent are parte principală finită

$$\underline{f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + a_{-1} \cdot \frac{1}{z-z_0} + \ldots + \frac{a_{-k}}{(z-z_0)^k}}$$

• z_0 este punct singular esențial dacă $\nexists \lim_{z \to z_0} f(z)$ $\longleftrightarrow$ dezvoltarea în serie Laurent are parte principală infinită

• Un punct $z_0 \in D$ s.n. punct ordinar pentru f dacă $\exists\, \Delta(z_0; r) \subseteq D$, $r>0$ a.î. f să fie olomorfă pe $\Delta(z_0, r)$.

Obs 2 • $Rez(f; z_0) = a_{-1}$ ← coeficientul lui $\frac{1}{z-z_0}$ din dezvoltarea în serie Laurent a lui f în jurul lui z_0

Așadar:

• Reziduul funcției f într-un punct ordinar (sau singular aparent) este 0. (În aceste situații seria Laurent a lui f în jurul lui z_0 se reduce doar la partea sa tayloriană, a_{-1} fiind 0) $Rez(f; z_0) = 0$ ← punct ordinar sau singular aparent

• Dacă z_0 - pol de ordinul $n \in \mathbb{N}^*$, atunci

$$Rez(f; z_0) = \frac{1}{(n-1)!} \cdot \lim_{z\to z_0} \left[(z-z_0)^n \cdot f(z)\right]^{(n-1)}$$

• Dacă $z_0 \in \mathbb{C}$ - pol simplu, atunci

$$Rez(f; z_0) = \lim_{z\to z_0} (z-z_0)\cdot f(z)$$

sau

dacă $f(z) = \frac{g(z)}{h(z)}$, g și h olomorfe într-o vecinătate a lui z_0 cu $g(z_0) \neq 0$, $h(z_0) = 0$ și $h'(z_0) \neq 0$, atunci

$$Rez(f; z_0) = \frac{g(z_0)}{h'(z_0)}$$

• $Rez(f; \infty) = -Rez\left[\frac{1}{z^2} \cdot f\left(\frac{1}{z}\right); 0\right]$

Obs 3

Dacă $z_1, z_2, \dots, z_n \in \mathbb{C}$ și f - olomorfă pe $\mathbb{C} \setminus \{z_1, z_2, \dots z_n\}$, atunci

$$\sum_{k=1}^{n} Rez(f; z_k) + Rez(f; \infty) = 0 \quad (*)$$

Obs 4 • Reziduul funcției f într-un punct singular esențial se calculează fie identificând coeficientul lui $\frac{1}{z-z_0}$ din dezv. în serie Laurent a lui f în jurul lui z_0, fie folosind formula (*) din Obs 3.

• Pentru reziduul funcției f în punctele singulare esențiale nu avem formulă specială de calcul!!

a) $f(z) = \frac{e^{z^2}-1}{z^2}$, $z_0 = 0$ Obs: $e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \dots = \sum_{n=0}^{\infty} \frac{z^n}{n!}$

Așadar $f(z) = \frac{1}{z^2}\left(\cancel{1} + z^2 + \frac{z^4}{2!} + \dots \cancel{-1}\right) = 1 + \frac{z^2}{2!} + \frac{z^4}{3!} + \dots = \sum_{n=1}^{\infty} \frac{z^{2n-2}}{n!}$

Cum seria are doar parte Tayloriană (nu are parte principală) z_0 - punct singular aparent

Putem de altfel observa și că

$$\lim_{z\to 0} f(z) = \lim_{z\to 0} \frac{e^{z^2}-1}{z^2} = 1 \text{ (este finită)}$$

Așadar, $Rez(f;0)=0$

b) $f(z) = (z-1)\cdot \cos\frac{1}{z-2}$, $z_0=2$

Dacă notăm $z-2=u$ $\Rightarrow f(z)=g(u)=(u+1)\cdot\cos\frac{1}{u}$ $\quad Rez(f;2)=Rez(g;0)$

$z=u+2$

A dezvolta funcția f în jurul lui $z_0=2$ e totuna cu a dezvolta funcția g în jurul lui $z_1=0$

Obs: $\cos z = 1-\frac{z^2}{2!}+\frac{z^4}{4!}-\ldots=\sum_{n=0}^{\infty}(-1)^n\cdot\frac{z^{2n}}{(2n)!}$

Așadar,

$$g(u)=(u+1)\cdot\left(1-\frac{1}{2!u^2}+\frac{1}{4!u^4}-\frac{1}{6!u^6}+\ldots\right)=$$

$$=u-\frac{1}{2!u}+\frac{1}{4!u^3}-\frac{1}{6!u^5}+\ldots+$$

$$+1-\frac{1}{2!u^2}+\frac{1}{4!u^4}-\frac{1}{6!u^6}+\ldots$$

$$=\underbrace{\ldots-\frac{1}{2!u^2}-\frac{1}{2!u}}+1+u$$

partea principală a seriei Laurent conține o infinitate de termeni

$\Rightarrow z_1=0$ este un punct singular esențial pentru g

În concluzie $Rez(f;2)=-\frac{1}{2!}$ (coef lui $\frac{1}{u}$)

Obs: Se poate observa de asemenea că $\nexists \lim_{z\to 2}(z-1)\cdot\cos\frac{1}{z-2}$

Obs: Când întâlnim în probleme

$e^{\frac{1}{z}}$, $\sin\frac{1}{z}$, $\cos\frac{1}{z}$, $z=0$ este punct singular esențial

deoarece $\nexists \lim_{z\to 0} e^{\frac{1}{z}}$, $\nexists \lim_{z\to 0}\sin\frac{1}{z}$, $\nexists\lim_{z\to 0}\cos\frac{1}{z}$.

c) $f(z)=\frac{2\sin^2 z}{z^5}$, $z_0=0$

$\cos z = 1-\frac{z^2}{2!}+\frac{z^4}{4!}-\ldots$

$$f(z)=\frac{2\cdot\frac{1-\cos 2z}{2}}{z^5}=\frac{1-\cos 2z}{z^5}=\frac{1}{z^5}-\frac{1}{z^5}\left[1-\frac{(2z)^2}{2!}+\frac{(2z)^4}{4!}-\frac{(2z)^6}{6!}+\ldots\right]$$

$$=\underbrace{\frac{2}{z^3}-\frac{2}{3}\cdot\frac{1}{z}}+\frac{4}{45}\cdot z-\ldots \qquad Rez(f;0)=-\frac{2}{3}$$

Partea principală conține un nr. finit de termeni

$z_0=0$ - pol de ordinul 3

③ Fie $f: \mathbb{C} \setminus \{-1; j\} \to \mathbb{C}$, $f(z) = \frac{1}{z+1} \cdot sh\left(\frac{1}{z-j}\right)$

a) Să se determine punctele singulare ale lui f

b) Să se calculeze $Rez(f; 1+j)$, $Rez(f; -1)$, $Rez(f; -j)$, $Rez(f; j)$ şi $Rez(f; +\infty)$

c) $\int_{|z-j|=2^n} f(z)dz$ pentru $n \in \{-1, 1\}$.

---

a) $z+1=0 \Leftrightarrow z_1=-1$ - pol de ordinul 1

Obs: Funcţiile raţionale nu au alte singularităţi decât poli sau singularităţi eliminabile

• Dacă $f(z) = \frac{1}{(z-z_0)^n}$, atunci $z=z_0$ - pol de ordinul n

Obs: Dacă $g(z) = \frac{sh\left(\frac{1}{z-j}\right)}{(z+1)^2}$, atunci $z=-1$ - pol de ordinul 2

Dacă $g(z) = \frac{e^z}{(z+1)^2 \sin \pi z} = \frac{e^z}{(z+1)^3} \cdot \frac{z+1}{\sin \pi z}$ finită $\Rightarrow z=-1$ - pol de ordinul 3

$$\lim_{z \to -1} \frac{z+1}{\sin \pi z} \overset{l'H}{=} \lim_{z\to -1} \frac{1}{\pi \cos \pi z} = \frac{1}{\pi \cdot (-1)} \text{ finită}$$

Cum $sh\left(\frac{1}{z-j}\right) = \frac{e^{\frac{1}{z-j}} - e^{-\frac{1}{z-j}}}{2}$ şi $\nexists \lim_{z\to j} e^{\frac{1}{z-j}}$, $z_2 = j$ - punct singular esenţial

$\forall z_0 \in \mathbb{C} \setminus \{-1, j\}$ z_0 - punct ordinar (funcţia f este olomorfă în z_0)

Prin urmare, $Rez(f; 1+j) = Rez(f; -j) = 0$, deoarece $z_3 = 1+j$ şi $z_4 = -j$ - puncte ordinare pentru f

Obs: $z=z_0$ - pol de ordinul $n \Rightarrow Rez(f; z_0) = \frac{1}{(n-1)!} \cdot \lim_{z\to z_0} \left[(z-z_0)^n \cdot f(z)\right]^{(n-1)}$

În cazul nostru, $Rez(f, -1) = \lim_{z\to -1} \cancel{(z+1)} \cdot \frac{1}{\cancel{z+1}} \cdot sh\left(\frac{1}{z-j}\right) = sh \frac{1}{-1-j} = sh \frac{-1+j}{2}$

$Rez(f, j)$ = coeficientul lui $\frac{1}{z-j}$ $\quad$ Notând $z-j=u \Rightarrow f(z) = g(u) = \frac{1}{u+1+j} \cdot sh \frac{1}{u}$
$\quad z = u+j$

$\parallel$

coeficientul lui $\frac{1}{u}$ din dezvoltarea în serie Laurent a lui g în jurul lui 0.

Cum $sh\, z = z + \frac{z^3}{3!} + \ldots$, obţinem $sh \frac{1}{u} = \frac{1}{u} + \frac{1}{3! \cdot u^3} + \frac{1}{5! \cdot u^5} + \ldots$

iar $\frac{1}{u+1+j} = \frac{1}{(1+j)(1+\frac{u}{1+j})} = \frac{1}{1+j} \cdot \frac{1}{1+\frac{u(1-j)}{2}} = \frac{1-j}{2} \cdot \frac{1}{1-\frac{j-1}{2}u} =$

$= \frac{1-j}{2} \cdot \left(1 + \frac{j-1}{2}u + \left(\frac{j-1}{2}\right)^2 \cdot u^2 + \left(\frac{j-1}{2}\right)^3 \cdot u^3 + \ldots\right)$

$\frac{1}{1-z} = 1 + z + z^2 + \ldots$

Așadar,

$$g(u)=\frac{1-j}{2}\cdot\left(1+\frac{j-1}{2}u+\frac{(j-1)^2}{4}u^2+\frac{(j-1)^3}{8}u^3+\dots\right)\left(\frac{1}{u}+\frac{1}{3!\,u^3}+\frac{1}{5!\,u^5}+\dots\right)$$

$$=\frac{1-j}{2}\cdot\frac{1}{u}+\dots+\frac{1-j}{2}\cdot\frac{(j-1)^2}{4}\cdot u^2\cdot\frac{1}{3!\,u^3}+\dots+\frac{1-j}{2}\cdot\frac{(j-1)^4}{16}\cdot u^4\cdot\frac{1}{5!\,u^5}+\dots$$

$$\Rightarrow \underbrace{Rez(g,0)}_{Rez(f;j)}=\frac{1-j}{2}\left(1+\left(\frac{j-1}{2}\right)^2\cdot\frac{1}{3!}+\left(\frac{j-1}{2}\right)^4\cdot\frac{1}{5!}+\dots\right)=\frac{1-j}{2}\cdot\sum_{n=0}^{\infty}\left(\frac{j-1}{2}\right)^{2n}\cdot\frac{1}{(2n+1)!}$$

$$=-\sum_{n=0}^{\infty}\left(\frac{j-1}{2}\right)^{2n+1}\cdot\frac{1}{(2n+1)!}=-\operatorname{sh}\frac{j-1}{2}=\operatorname{sh}\frac{1-j}{2}$$

$$Rez(f;\infty)=-Rez\left(\frac{1}{z^2}f\left(\frac{1}{z}\right);0\right)=$$

$$=-Rez\left(\frac{1}{z^2}\cdot\frac{1}{\frac{1}{z}+1}\cdot\operatorname{sh}\frac{\frac{1}{z}}{\frac{1}{z}-j};0\right)=-Rez\Bigg(\underbrace{\frac{1}{z^2}\cdot\frac{z}{z+1}\cdot\operatorname{sh}\frac{z}{1-jz}}_{g(z)};0\Bigg)=-Rez\,(g(z);0)$$

$$g(z)=\frac{\operatorname{sh}\frac{z}{1-jz}}{z}\cdot\frac{1}{z+1}=\underbrace{\frac{\operatorname{sh}\frac{z}{1-jz}}{\frac{z}{1-jz}}}_{\xrightarrow[z\to 0]{}1}\cdot\frac{1-jz}{z+1}\Rightarrow \lim_{z\to 0}g(z)=1\text{ - finită}\rightarrow 0\text{ - punct singular aparent pentru } g$$

$$\Downarrow$$

$$Rez(g;0)=0$$

Așadar, $Rez(f;\infty)=-Rez(g,0)=\underline{\underline{0}}$

Obs: $z=j$ fiind un punct singular esențial, am calculat $Rez(f;j)$ folosindu-ne de dezvoltarea în serie Laurent în jurul lui j.

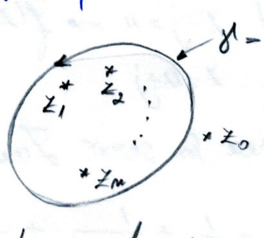
Altfel, $Rez(f;j)$ îl puteam calcula folosind formula:

$$Rez(f;-1)+Rez(f;j)+Rez(f;\infty)=0$$

Ca verificare, într-adevăr

$$\operatorname{sh}\frac{j-1}{2}+\operatorname{sh}\frac{1-j}{2}+0=0 \checkmark$$

c) $\int_{|z-j|=2^n} f(z)\,dz$, pentru $n\in\{-1,1\}$



γ - curbă închisă, netedă (sau netedă pe porțiuni)

Teorema reziduurilor

Dacă f - olomorfă pe

$Int\,\gamma\setminus\{z_1,\dots,z_n\}$, unde z_k - puncte singulare izolate pentru f (poli de orice ordin sau puncte singulare esențiale)

și f - continuă pe γ, atunci

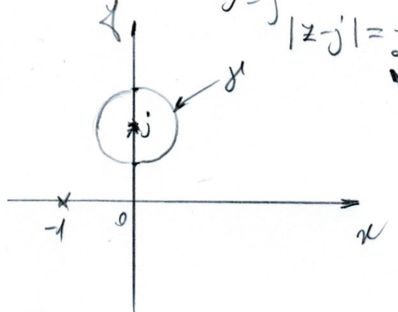
$$\int_\gamma f(z)\,dz=2\pi j\sum_{k=1}^{n}Rez(f;z_k)$$

Obs: Cu ajutorul T. reziduurilor se calculează integrale pe curbe închise, netede sau netede pe porțiuni (de ex γ poate fi cerc, elipsă, triunghi, ...)

Singurele puncte care contează sunt punctele singulare din interiorul curbei. La noi, z_0 nu intră în calculul integralei.

Dacă $n=-1$, avem de calculat

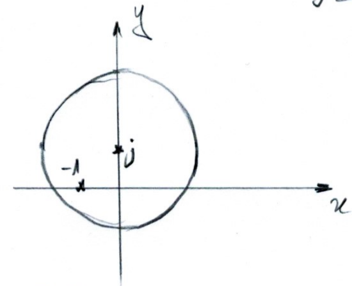
$$I=\int_{|z-j|=\frac{1}{2}} f(z)\,dz$$



$$\underline{\underline{I}}=2\pi j\cdot Rez(f;j)=\underline{\underline{2\pi j\cdot sh\frac{1-j}{2}}}$$

Dacă $n=1$, avem de calculat

$$I=\int_{|z-j|=2} f(z)\,dz$$



$$\underline{\underline{I}}=2\pi j\cdot\left[Rez(f;j)+Rez(f;-1)\right]=2\pi j\left[sh\frac{1-j}{2}-sh\frac{1-j}{2}\right]=0$$

Obs: Din T. Cauchy-Goursat știm că dacă

f-olomorfă pe $\bar{\Delta} \Rightarrow \int_\gamma f(z)\,dz=0$

Aici avem un ex. de funcție care nu este olomorfă pe Int γ și totuși $\int_\gamma f(z)\,dz=0$

Probleme propuse

① Să se dezvolte în serie de puteri ale lui z în jurul lui $z_0=0$

$$f(z)=\ln(1-2z+4z^2)$$

Indicație: $1-(2z)^3=(1-2z)(1-2z+4z^2) \to f(z)=\ln\frac{1-(2z)^3}{1-2z}\ldots$

② Să se dezvolte în serie de puteri ale lui z (în serie Laurent) în jurul lui $z_0=0$

$$f(z)=\ln\frac{1+2jz}{1-2jz},\ \text{unde } f(0)=2\pi j$$

③ Să se dezvolte în serie Laurent și să se determine tipul punctelor singulare corespunzătoare:

a) $f(z)=\dfrac{1-\cos z}{z^7}$, $z_0=0$, $0<|z|<+\infty$

b) $f(z)=z^3\cdot e^{\frac{1}{z}}$, $z_0=0$, $0<|z|<+\infty$

c) $f(z)=z\cdot e^{\frac{1}{z+j}}$, $z_0=-j$, $0<|z+j|<+\infty$

④ Să se calculeze

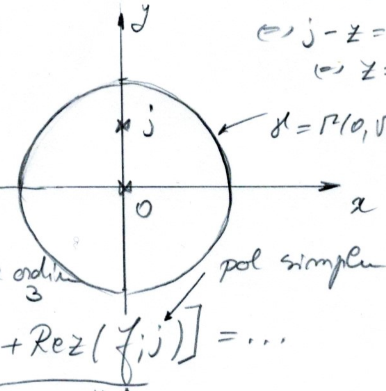
$$I=\int_{|z|=\sqrt{2}}\frac{e^{\pi z}}{z^3(1+jz)}\,dz$$

Indicație

$1+jz=0\ |\cdot j$

$\Leftrightarrow j-z=0$

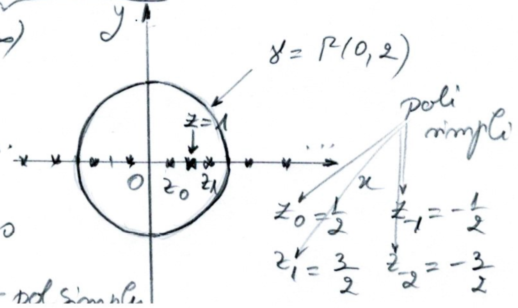
$\Leftrightarrow z=j$



$$I=2\pi j\cdot\underbrace{\left[Rez(f;0)+Rez(f;j)\right]}_{Rez''(f;\infty)}=\ldots$$

⑤ Să se calculeze

$$I=\int_{|z|=2}\frac{\sin\pi z}{(z-1)^2\cdot\cos\pi z}\,dz$$



Indicație

$\cos\pi z=0 \Leftrightarrow \pi z=\frac{\pi}{2}+k\pi \Rightarrow \underline{z_k=k+\frac{1}{2}}$ sau $z-1=0$

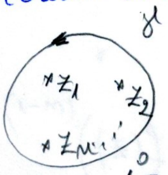
$\Downarrow$

$z=1 \leftarrow$ pol simplu

Seminarul 6

Teorema reziduurilor. Aplicații

Teorema reziduurilor



Dacă γ - curbă închisă netedă (sau netedă pe porțiuni)

1° - f olomorfă pe $Int\,\gamma \setminus \{z_1, \dots, z_n\}$ poli de orice ordin sau puncte singulare esențiale

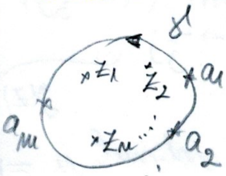
2° - f continuă pe γ

Atunci

$$\int_\gamma f(z)\,dz = 2\pi j \sum_{k=1}^{n} Rez(f; z_k)$$

Obs: γ poate fi un cerc, o elipsă, un triunghi, un dreptunghi sau orice altă curbă închisă netedă (sau netedă pe porțiuni)

Teorema semireziduurilor



Dacă γ - curbă închisă netedă (sau netedă pe porțiuni)

1° - f olomorfă pe $Int\,\gamma \setminus \{z_1, z_2, \dots, z_n\}$ poli de orice ordin sau puncte singulare esențiale

2° - f continuă pe $\gamma \setminus \{a_1, a_2, \dots, a_m\}$ poli simpli

Atunci

$$\int_\gamma f(z)\,dz = 2\pi j \sum_{k=1}^{n} Rez(f, z_k) + \pi j \sum_{k=1}^{m} Rez(f, a_k)$$

① Să se calculeze

$$J = \int_{|3z+j|=1} \underbrace{\frac{e^{\pi j z}}{z(3z+j)^3}}_{f(z)}\,dz$$

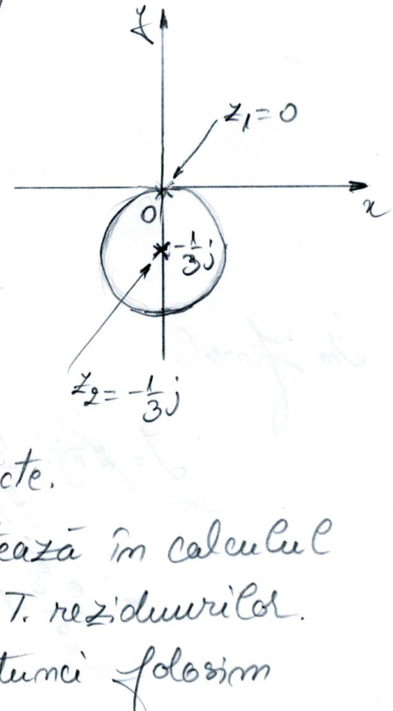
=

$|3z+j| = 1 \iff 3\left|z + \frac{1}{3}j\right| = 1 \iff \left|z + \frac{1}{3}j\right| = \frac{1}{3} \Rightarrow z \in \Gamma\left(-\frac{1}{3}j, \frac{1}{3}\right)$

Căutăm în continuare punctele izolate ale funcției.

Observăm că $z_1 = 0$ - pol de ordinul 1

$3z + j = 0 \Rightarrow z_2 = -\frac{1}{3}j$ - pol de ordinul 3



Folosind Teorema Semireziduurilor, obținem

$$J = 2\pi j \cdot Rez\left(f; -\frac{1}{3}j\right) + \pi j \cdot Rez(f; 0)$$

Obs: După ce găsim punctele singulare ale funcției și identificăm curba γ, ne uităm unde se găsesc aceste puncte.

Doar punctele de pe frontieră sau din interior contează în calculul integralei. Dacă avem puncte doar în interior, folosim T. reziduurilor.

Dacă sunt și pe frontieră și sunt poli simpli, atunci folosim T. semireziduurilor.

$$Rez(f;0)=\frac{e^{\pi j z}}{z'\cdot(3z+j)^3}\Big|_{z=0}=\frac{e^{\pi j z}}{(3z+j)^3}\Big|_{z=0}=\frac{1}{-j}=j$$

$$Rez\left(f;-\frac{1}{3}j\right)=\frac{1}{2!}\lim_{z\to -\frac{1}{3}j}\Big(\underbrace{\left(z+\frac{1}{3}j\right)^3\cdot\frac{e^{\pi j z}}{z(3z+j)^3}}_{g(z)}\Big)''$$

Dacă $z=z_0$ - pol de ordinul $n \Rightarrow Rez(f;z_0)=\frac{1}{(n-1)!}\lim_{z\to z_0}\left((z-z_0)^n\cdot f(z)\right)^{(n-1)}$

$$g(z)=\cancel{\left(z+\frac{1}{3}j\right)^3}\cdot\frac{e^{\pi j z}}{z\cdot 27\cdot\cancel{\left(z+\frac{1}{3}j\right)^3}}=\frac{1}{27}\cdot\frac{e^{\pi j z}}{z}$$

$$g'(z)=\frac{1}{27}\cdot\frac{\pi j\cdot e^{\pi j z}\cdot z-e^{\pi j z}}{z^2}=\frac{1}{27}\cdot\frac{e^{\pi j z}(\pi j z-1)}{z^2}$$

$$g''(z)=\frac{1}{27}\cdot\frac{\left(\pi j\, e^{\pi j z}(\pi j z-1)+\pi j\cdot e^{\pi j z}\right)\cdot z^2-2z\cdot e^{\pi j z}(\pi j z-1)}{z^4}$$

$$=\frac{1}{27}\cdot\frac{-\pi^2 e^{\pi j z}\cdot z^3-2\pi j z^2\cdot e^{\pi j z}+2z\cdot e^{\pi j z}}{z^4}$$

$$=\frac{1}{27}\cdot\frac{e^{\pi j z}(-\pi^2 z^2-2\pi j z+2)}{z^3}$$

Dacă $z=-\frac{1}{3}j \rightarrow z^2=\frac{1}{9}j^2=-\frac{1}{9}$

$\rightarrow z^3=\frac{1}{27}j$

Obținem $Rez\left(f;-\frac{1}{3}j\right)=\frac{1}{2}\cdot g''(z)\Big|_{z=-\frac{1}{3}j}$

$$=\frac{1}{2}\cdot\frac{1}{27}\cdot\frac{e^{\pi j z}\cdot(-\pi^2 z^2-2\pi j z+2)}{z^3}\Big|_{z=-\frac{1}{3}j}$$

$$=\frac{1}{2}\cdot\cancel{\frac{1}{27}}\cdot\frac{e^{\pi j\cdot\left(-\frac{1}{3}j\right)}\cdot\left(-\pi^2\cdot\left(-\frac{1}{9}\right)-2\pi j\cdot\left(-\frac{1}{3}j\right)+2\right)}{\cancel{\frac{1}{27}}j}$$

$$=\frac{1}{2}\cdot\frac{e^{\frac{\pi}{3}}\cdot\left(\frac{\pi^2}{9}-\frac{2\pi}{3}+2\right)}{j}=+\frac{1}{2j}\cdot\frac{e^{\frac{\pi}{3}}}{9}\cdot(\pi^2-6\pi+18)$$

În final,

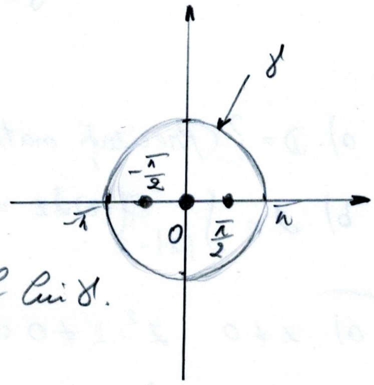
$$I=\cancel{2}\pi j\cdot\frac{e^{\frac{\pi}{3}}}{\cancel{2j}\cdot 9}\cdot(\pi^2-6\pi+18)+\pi j\cdot j=$$

$$=\frac{\pi\cdot e^{\frac{\pi}{3}}}{9}\cdot(\pi^2-6\pi+18)-\pi$$

② * Să se calculeze

$$J = \int_{|z|=\pi} \frac{tg\, z}{z^2}\, dz$$

$\gamma = \{z \in \mathbb{C} : |z| = \pi\} = \Gamma(0, \pi)$



Fie $f(z) = \frac{tg\, z}{z^2} = \frac{\sin z}{z^2 \cdot \cos z}$ $\quad \cos z = 0 \Rightarrow z_k = \frac{(2k+1)\pi}{2},\ k \in \mathbb{Z}$

f - olomorfă pe $\mathbb{C} \setminus \{0, \frac{(2k+1)\pi}{2}\}$ $k \in \mathbb{Z}$

Observăm că doar $z = 0$, $z = -\frac{\pi}{2}$ și $z = \frac{\pi}{2}$ se află în interiorul lui γ.

Așadar, aplicând T. reziduurilor

$$J = 2\pi i \cdot \left(Rez(f; 0) + Rez(f; -\frac{\pi}{2}) + Rez(f; \frac{\pi}{2}) \right)$$

Cum $\lim\limits_{z \to 0} f(z) = \lim\limits_{z \to 0} \frac{\sin z}{z^2 \cdot \cos z} = \lim\limits_{z \to 0} \underbrace{\frac{\sin z}{z}}_{\to 1 \text{ - finită}} \cdot \frac{1}{z^1 \cos z}$ $\Rightarrow z = 0$ pol simplu (și nu dublu)

$$Rez(f; 0) = \frac{1}{(1-1)!} \cdot \lim_{z \to 0} \left((z-0)^1 \cdot f(z) \right)^{(1-1)}$$

$$= \lim_{z \to 0} z \cdot \frac{tg\, z}{z^2} = \lim_{z \to 0} \frac{tg\, z}{z} = 1$$

Punctele $z = -\frac{\pi}{2}$ și $z = \frac{\pi}{2}$ sunt poli de ordinul 1.

$$Rez(f; -\frac{\pi}{2}) = \lim_{z \to -\frac{\pi}{2}} (z + \frac{\pi}{2}) \cdot \frac{\sin z}{z^2 \cdot \cos z} = \lim_{\substack{z' \to 0 \\ z' = z + \frac{\pi}{2}}} z' \cdot \frac{\sin(z' - \frac{\pi}{2})}{(z' - \frac{\pi}{2})^2 \cdot \cos(z' - \frac{\pi}{2})}$$

$$= \lim_{z' \to 0} \underbrace{\frac{z' \cdot (-\cos z')}{\sin z'}}_{\to 1} \cdot \frac{1}{(z' - \frac{\pi}{2})^2} = -\frac{1}{\frac{\pi^2}{4}} = -\frac{4}{\pi^2}$$

sau

$$Rez(f; -\frac{\pi}{2}) = \frac{\sin z}{z^2 \cdot (\cos z)'} \Bigg|_{z = -\frac{\pi}{2}} = \frac{\sin z}{z^2 \cdot (-\sin z)} \Bigg|_{z = -\frac{\pi}{2}} = -\frac{1}{z^2} \Bigg|_{z = -\frac{\pi}{2}} = -\frac{4}{\pi^2}$$

se derivează factorul care anulează polul simplu

În cazul $Rez(f; 0)$ nu se poate aplica deoarece $(z^2)'\big|_{z=0} = 0$ și s anulează numitorul

Analog, $Rez(f; \frac{\pi}{2}) = \frac{\sin z}{z^2 (\cos z)'} \Bigg|_{z = \frac{\pi}{2}} = -\frac{1}{z^2} \Bigg|_{z = \frac{\pi}{2}} = -\frac{4}{\pi^2}$

În final,

$$J = 2\pi i \cdot \left(1 - \frac{4}{\pi^2} - \frac{4}{\pi^2}\right) = 2\pi i \left(1 - \frac{8}{\pi^2}\right)$$

(3) Se consideră funcția $f: D \subseteq \mathbb{C} \to \mathbb{C}$

$$f(z) = \frac{z^{12} \cdot \sin \frac{3\pi j}{z}}{(z^2+2)^5 \cdot (z^2 + jz + 6)}$$

a) $D = ?$ Precizați natura punctelor singulare, inclusiv natura punctului de la ∞.

b) $I = \int_{|z|=2} f(z)\,dz = ?$

---

a) $z \neq 0 \quad z^2+2 \neq 0 \iff z^2 \neq -2 \iff z^2 \neq (\sqrt{2}j)^2 \Rightarrow z \neq \pm\sqrt{2}j$

$z^2 + jz + 6 = 0$

$\Delta = 1 - 24 = -25 = (5j)^2 \Rightarrow z_{1,2} = \frac{-j \pm 5j}{2} \Big\langle \begin{matrix} 2j \\ -3j \end{matrix}$

Observăm că $z_0 = 0$ - punct singular esențial ($\nexists \lim\limits_{z\to 0} \sin \frac{3\pi j}{z}$)

$z_{3,4} = \pm\sqrt{2}j$ - poli de ordinul 5

$z_1 = 2j$ este pol simplu, iar $z_2 = -3j$ este punct singular aparent deoarece

$\left(\lim\limits_{z\to 2j} |f(z)| = +\infty\right) \qquad \left(\lim\limits_{z\to -3j} f(z) = \lim\limits_{z\to -3j} \frac{z^{12} \cdot \sin\frac{3\pi j}{z}}{(z^2+2)^5 \cdot (z-2j)(z+3j)} - \text{finită}\right)$ finită

$$\lim_{z\to -3j} \frac{\sin\frac{3\pi j}{z}}{z+3j} \overset{l'H}{=} \lim_{z\to -3j} -\frac{3\pi j}{z^2} \cdot \underbrace{\cos\frac{3\pi j}{z}}_{\to -1} = \frac{3\pi j}{-9} = -\frac{\pi j}{3} \text{ - finită}$$

Natura punctului de la infinit = natura lui 0 pentru $f\left(\frac{1}{z}\right)$

deoarece $Rez(f; \infty) = -Rez\left(\frac{1}{z^2} f\left(\frac{1}{z}\right); 0\right)$

$$= -Rez\left(\frac{1}{z^2} \cdot \frac{1}{z^{12}} \cdot \frac{\sin 3\pi j z}{\left(\frac{1}{z^2}+2\right)^5 \cdot \left(\frac{1}{z^2} + j \cdot \frac{1}{z} + 6\right)}; 0\right)$$

$$= -Rez\Big(\underbrace{\frac{1}{z^2} \cdot \frac{\sin 3\pi j z}{(2z^2+1)^5 \cdot (6z^2 + jz + 1)}}_{g(z)}; 0\Big)$$

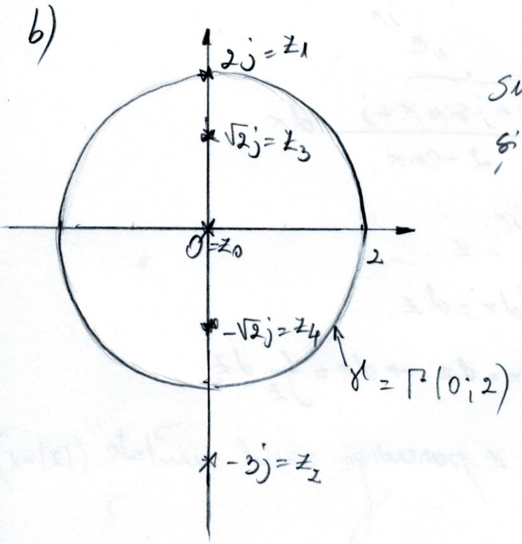
$z = 0$ - pol simplu pentru $g(z)$ deoarece

$$\lim_{z\to 0} g(z) = \lim_{z\to 0} \underbrace{\frac{\sin 3\pi j z}{z}}_{\to 3\pi j \text{ - finită}} \cdot \frac{1}{z \cdot (2z^2+1)^5 \cdot (6z^2 + jz + 1)}$$

$\underline{Rez(f; \infty)} = -Rez(g(z); 0) = -\lim\limits_{z\to 0} z \cdot g(z)$

$$= -\lim_{z\to 0} \cancel{z} \cdot \frac{1}{\cancel{z}} \cdot \underbrace{\frac{\sin 3\pi j z}{z}}_{\to 3\pi j} \cdot \frac{1}{(2z^2+1)^5 \cdot (6z^2 + jz + 1)} = \underline{-3\pi j}$$

b)



Singurele puncte care contează pentru calculul integralei sunt z_0, z_1, z_3 și z_4 (z_2 nu, deoarece nu e nici pe frontieră și nici în interiorul lui γ)

Folosind Teorema semireziduurilor

$$J = 2\pi j\,[Rez(f;0) + Rez(f;\sqrt{2}j) + Rez(f;-\sqrt{2}j)] + \pi j\, Rez(f;2j) \quad (1)$$

Pe de altă parte,

$$Rez(f;0) + Rez(f;\sqrt{2}j) + Rez(f;-\sqrt{2}j) + Rez(f;2j) + Rez(f;-3j) + Rez(f;\infty) = 0 \quad (2)$$

$$\overset{(2)}{\Rightarrow} Rez(f;0) + Rez(f;\sqrt{2}j) + Rez(f;-\sqrt{2}j) = -Rez(f;2j) - Rez(f;-3j) - Rez(f;\infty)$$

Așadar,

$$J = 2\pi j\cdot[-Rez(f;2j) - Rez(f;-3j) - Rez(f;\infty)] + \pi j\cdot Rez(f;2j)$$
$$= -\pi j\cdot[Rez(f;2j) + 2\,Rez(f;-3j) + 2\,Rez(f;\infty)]$$

Obs: Dacă z_0 - pol simplu și $f(z) = \frac{g(z)}{h(z)}$, atunci

$$Rez(f;z_0) = \left.\frac{g(z)}{h'(z)}\right|_{z=z_0}$$

$Rez(f;-3j) = 0$ deoarece $z_2 = -3j$ este punct singular aparent;

$$Rez(f;2j) = \left.\frac{z^{12}\cdot\sin\frac{3\pi j}{z}}{(z^2+2)^5\cdot(z^2+jz+6)'}\right|_{z=2j} = \left.\frac{z^{12}\cdot\sin\frac{3\pi j}{z}}{(z^2+2)^5\cdot(2z+j)}\right|_{z=2j} = \frac{(2j)^{12}\cdot\sin\frac{3\pi}{2}}{-2^5\cdot 5j} = \frac{-2^{12}\cdot j}{2^5\cdot 5} = -\frac{2^7}{5}j$$

Obs

În final, $J = -\pi j\cdot\left(-\frac{2^7}{5}j + 0 - 6\pi j\right) = -\pi\cdot\left(\frac{128}{5} + 6\pi\right)$

(4) Utilizând T. reziduurilor, să se calculeze integrala

$$J = \int_0^{2\pi} \frac{1+\sin x}{2-\cos x}\,dx\,,\quad 2-\cos x \neq 0,\ \forall x\in\mathbb{R}$$

Obs $\int_a^{a+2\pi} R(\sin x, \cos x)\,dx = \int_0^{2\pi} R(\sin x,\cos x)\,dx\,,\ \forall a\in\mathbb{R}$

Dacă am fi avut de calculat $J = \int_{-\pi}^{5\pi} f(x)\,dx$, atunci $J = \int_{-\pi}^{\pi} f(x)\,dx + \int_{\pi}^{3\pi} f(x)\,dx + \int_{3\pi}^{5\pi} f(x)\,dx = 3\int_0^{2\pi} f(x)\,dx$

Asociem integrala $\mathcal{J} = \int_0^{2\pi} \frac{\cos x}{2-\cos x}\,dx$ și

Formăm expresia

$$I + j\cdot J = \int_0^{2\pi} \frac{\cos x + j(1+\sin x)}{2-\cos x}\,dx = \int_0^{2\pi} \frac{\overbrace{\cos x + j\sin x}^{„e^{jx}"} + j}{2-\cos x}\,dx$$

$$= \int_0^{2\pi} \frac{e^{jx}+j}{2-\cos x}\,dx \qquad \text{Notăm } e^{jx} = z$$

$$j\cdot e^{jx}dx = dz$$

Pe de altă parte

$$\cos x = \frac{e^{jx}+e^{-jx}}{2} = \frac{z+\frac{1}{z}}{2} = \frac{z^2+1}{2z} \qquad \Leftrightarrow jz\,dx = dz \to dx = \frac{1}{jz}dz$$

Observăm că atunci când x parcurge intervalul $[0, 2\pi]$, z parcurge cercul unitate. $(|z|=1)$

$$|z| = |e^{jx}| = |\cos x + j\sin x| = \sqrt{\sin^2 x + \cos^2 x} = 1$$

$$\arg z = x \in [0, 2\pi]$$

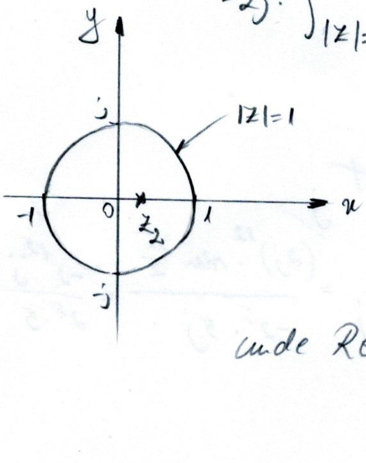
Așadar, $$I + j\cdot J = \int_{|z|=1} \frac{z+j}{2 - \frac{z^2+1}{2z}}\cdot\frac{1}{jz}\,dz = \int_{|z|=1}\frac{z+j}{4z-z^2-1}\cdot\frac{2\cancel{z}}{j\cancel{z}}\,dz =$$

$$= \frac{2}{j}\int_{|z|=1}\frac{z+j}{-z^2+4z-1}\,dz = 2j\int_{|z|=1}\frac{z+j}{z^2-4z+1}\,dz$$

$$z^2-4z+1=0$$
$$\Delta = 16-4 = 12 = (2\sqrt{3})^2$$
$$z_{1,2} = \frac{4\pm 2\sqrt{3}}{2} \quad z_1 = 2+\sqrt{3}, \quad z_2 = 2-\sqrt{3}$$

poli simpli

$$= 2j\cdot\int_{|z|=1}\underbrace{\frac{z+j}{(z-(2+\sqrt{3}))(z-(2-\sqrt{3}))}}_{f(z)}\,dz$$



Folosind T. reziduurilor,

$$I + j\cdot J = 2j\cdot 2\pi j\cdot \mathrm{Rez}(f; 2-\sqrt{3}) = -4\pi\cdot\mathrm{Rez}(f; 2-\sqrt{3})$$

unde $$\mathrm{Rez}(f; 2-\sqrt{3}) = \frac{z+j}{(z^2-4z+1)'}\Big|_{z=2-\sqrt{3}} = \frac{z+j}{2z-4}\Big|_{z=2-\sqrt{3}} =$$

$$= \frac{2-\sqrt{3}+j}{2(2-\sqrt{3})-4} = \frac{2-\sqrt{3}+j}{-2\sqrt{3}}$$

În final, $$I + j\cdot J = -\cancel{4}^{2}\pi\cdot\frac{2-\sqrt{3}}{-\cancel{2}\sqrt{3}} - \cancel{4}^{2}\pi\cdot\frac{1}{-\cancel{2}\sqrt{3}}\,j = \frac{2\cdot(2-\sqrt{3})\pi}{\sqrt{3}} + \frac{2\pi}{\sqrt{3}}\,j$$

$$\Rightarrow I = \frac{2(2-\sqrt{3})\pi}{\sqrt{3}} \text{ și } J = \frac{2\pi}{\sqrt{3}} = \frac{2\pi\sqrt{3}}{3}$$

Obs Integrala J se putea calcula fără să atașăm integrala I doar că, de obicei, obținem cu această metodă poli de ordin mai mare și chiar mai multe puncte singulare.

Probleme propuse

① Utilizând Teorema reziduurilor, să se calculeze:

a) $$I = \int_{|z|=3} \frac{z^9 \cdot ch\left(\frac{\pi}{z}\right)}{(z^2+3)^5}\, dz$$

b) $$I = \int_{|z+1|=3} \frac{sh(\pi j z)}{(z-1)(z+1)^3}\, dz$$

c) $$I = \int_{|z|=3} \frac{e^{\frac{\pi}{z}}}{(z^3+2)^5 \cdot (z^2+z+1)}\, dz$$

② Utilizând Teorema reziduurilor, să se calculeze

a) $$I = \int_{-\pi}^{5\pi} \frac{2\cos x + 5}{3\sin x + 5}\, dx$$

b) $$I = \int_{-\pi}^{3\pi} \frac{3 + 4\cos x}{5 + 3\sin x}\, dx$$

c) $$I = \int_{0}^{2\pi} \frac{3 + \cos x}{7\sin x + 25}\, dx$$

Seminarul 7

Aplicații ale Teoremei Reziduurilor

① Să se calculeze:

$$I=\int_{-\pi}^{\pi}\frac{3+\cos x}{7\sin x+25}\,dx\ ,\quad 7\sin x+25\neq 0,\ \forall x\in\mathbb{R}$$

Obs: $\int_a^{a+2\pi}R(\sin x,\cos x)\,dx=\int_0^{2\pi}R(\sin x,\cos x)\,dx$

$$I=\int_{-\pi}^{\pi}\frac{3+\cos x}{7\sin x+25}\,dx=\int_0^{2\pi}\frac{3+\cos x}{7\sin x+25}\,dx$$

Asociem integrala $J=\int_0^{2\pi}\frac{\sin x}{7\sin x+25}\,dx$ și formăm expresia

$$I+jJ=\int_0^{2\pi}\frac{3+\overbrace{\cos x+j\sin x}^{=e^{jx}}}{7\sin x+25}\,dx=\int_0^{2\pi}\frac{3+e^{jx}}{7\sin x+25}\,dx$$

Facem substituția $z=e^{jx}$

$dz=je^{jx}dx\rightarrow dx=\frac{1}{jz}dz$

$$\sin x=\frac{e^{jx}-e^{-jx}}{2j}=\frac{z-\frac{1}{z}}{2j}=\frac{z^2-1}{2jz}$$

Observăm că atunci când x parcurge intervalul $[0,2\pi]$, z parcurge cercul unitate $|z|=1$

Obținem așadar,

$$I+jJ=\int_{|z|=1}\frac{z+3}{7\cdot\frac{z^2-1}{2jz}+25}\cdot\frac{1}{jz}\,dz$$

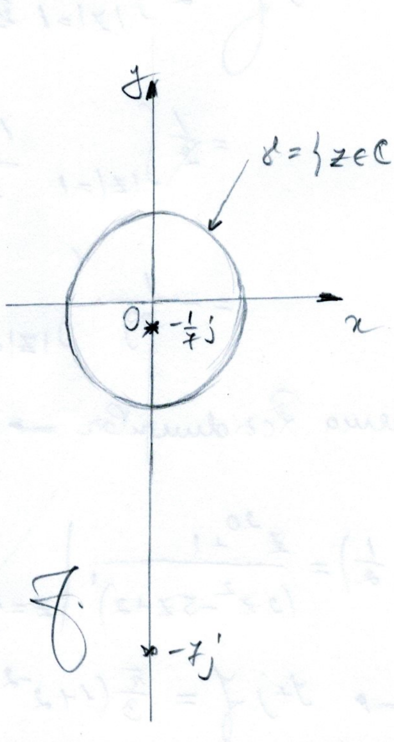
$$=\int_{|z|=1}\frac{z+3}{7z^2+50jz-7}\cdot\frac{\cancel{2jz}}{\cancel{jz}}\,dz$$

$$=2\int_{|z|=1}\underbrace{\frac{z+3}{7z^2+50jz-7}}_{f(z)}\,dz$$

$7z^2+50jz-7=0$

$\Delta=-2500+4\cdot 49=(48j)^2$

$z_{1,2}=\frac{-50j\pm 48j}{14}$, $z_1=-\frac{1}{7}j$, $|z_1|=\frac{1}{7}<1$; $z_2=-7j$, $|z_2|=7>1$



Aplicând Teorema Reziduurilor,

$$I+jJ=2\cdot 2\pi j\cdot \mathrm{Rez}\left(f;-\frac{1}{7}j\right)$$

Observăm că z_1 și z_2 sunt poli simpli pentru f.

$$\operatorname{Rez}\left(f;-\frac{1}{7}j\right)=\frac{z+3}{(7z^2+50jz-7)'}\Bigg|_{z=-\frac{1}{7}j}=\frac{z+3}{14z+50j}\Bigg|_{z=-\frac{1}{7}j}$$

$$=\frac{3-\frac{1}{7}j}{-14\cdot\frac{1}{7}j+50j}=\frac{21-j}{7}\cdot\frac{1}{48j}$$

Așadar, $I+jJ=4\pi j\cdot\dfrac{21-j}{7\cdot 48j} \to I+jJ=\dfrac{21}{7\cdot 12}\pi-\dfrac{\pi}{7\cdot 12}j=\dfrac{\pi}{4}-\dfrac{\pi}{84}j$

În final, $I=\dfrac{\pi}{4}$

② Să se calculeze:

$$I=\int_0^{2\pi}\frac{\cos^2 10x}{5-4\cos x}dx$$

$$I=\int_0^{2\pi}\frac{\frac{1+\cos 20x}{2}}{5-4\cos x}dx=\frac{1}{2}\cdot\int_0^{2\pi}\frac{1+\cos 20x}{5-4\cos x}dx$$

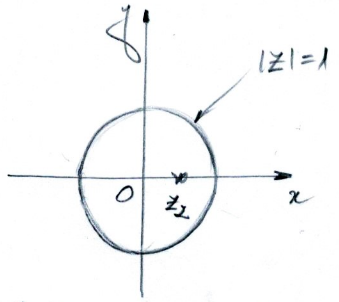
Atașăm integrala $J=\dfrac{1}{2}\displaystyle\int_0^{2\pi}\frac{\sin 20x}{5-4\cos x}dx$ și

formăm expresia $I+jJ=\dfrac{1}{2}\displaystyle\int_0^{2\pi}\frac{\overbrace{1+\cos 20x+j\sin 20x}^{e^{20jx}}}{5-4\cos x}dx$

Facem substituția $e^{jx}=z$ $\quad x\in[0,2\pi]\Rightarrow z\in\mathbb{C},\ |z|=1$

$$j\underbrace{e^{jx}}_{z}dx=dz\to dx=\frac{1}{jz}dz$$

$$\cos x=\frac{e^{jx}+e^{-jx}}{2}=\frac{z+\frac{1}{z}}{2}=\frac{z^2+1}{2z},\quad e^{20jx}=\left(e^{jx}\right)^{20}=z^{20}$$



z_1, z_2 - poli simpli

Așadar, $I+jJ=\dfrac{1}{2}\displaystyle\int_{|z|=1}\frac{1+z^{20}}{5-4\cdot\frac{z^2+1}{2z}}\cdot\frac{1}{jz}dz$

$$=\frac{1}{\cancel{2}}\int_{|z|=1}\frac{1+z^{20}}{-4z^2+10z-4}\cdot\frac{\cancel{2z}}{j\cancel{z}}dz$$

$$=-\frac{1}{2j}\cdot\int_{|z|=1}\underbrace{\frac{z^{20}+1}{2z^2-5z+2}}_{f(z)}dz$$

$2z^2-5z+2=0$

$\Delta=25-16=9\Rightarrow z_{1,2}=\dfrac{5\pm 3}{4}\left\langle\begin{array}{l}z_1=2\\ z_2=\frac{1}{2}\end{array}\right.$

Din Teorema Reziduurilor $\to I+jJ=-\dfrac{1}{\cancel{2j}}\cdot\cancel{2}\pi \cancel{j}\cdot\operatorname{Rez}\left(f;\dfrac{1}{2}\right)=-\pi\operatorname{Rez}\left(f;\dfrac{1}{2}\right)$

$$\operatorname{Rez}\left(f;\frac{1}{2}\right)=\frac{z^{20}+1}{(2z^2-5z+2)'}\Bigg|_{z=\frac{1}{2}}=\frac{z^{20}+1}{4z-5}\Bigg|_{z=\frac{1}{2}}=\frac{1+2^{-20}}{-3}$$

$$\to I+jJ=\frac{\pi}{3}\left(1+2^{-20}\right)\to I=\frac{\pi}{3}\left(1+2^{-20}\right)\text{ și } J=0$$

③ Să se calculeze:

$$\mathcal{I}=\int_{\pi}^{5\pi}(1-\sin x)^{92}\cdot\sin 43x\,dx$$

$$\mathcal{I}=\int_{\pi}^{3\pi}(1-\sin x)^{92}\cdot\sin 43x\,dx+\int_{3\pi}^{5\pi}(1-\sin x)^{92}\cdot\sin 43x\,dx$$

$$=2\int_{0}^{2\pi}(1-\sin x)^{92}\cdot\sin 43x\,dx$$

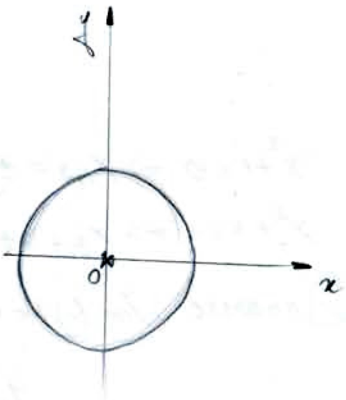
Fie $\mathcal{J}=2\int_{0}^{2\pi}(1-\sin x)^{92}\cdot\cos 43x\,dx$

Formăm expresia $\mathcal{J}+j\cdot\mathcal{I}=2\int_{0}^{2\pi}(1-\sin x)^{92}(\cos 43x+j\cdot\sin 43x)\,dx$

$$=2\int_{0}^{2\pi}(1-\sin x)^{92}\cdot e^{43jx}\,dx$$

Notăm $e^{jx}=z$

$$j\underbrace{e^{jx}}_{z}dx=dz\rightarrow dx=\frac{1}{jz}dz\qquad \sin x=\frac{e^{jx}-e^{-jx}}{2j}=\frac{z^2-1}{2jz}$$



Așadar, $\mathcal{J}+j\cdot\mathcal{I}=2\int_{|z|=1}\left(1-\frac{z^2-1}{2jz}\right)^{92}\cdot z^{43}\cdot\frac{1}{jz}dz$

$$=2\int_{|z|=1}\frac{(-z^2+2jz+1)^{92}}{(2j)^{92}\cdot z^{\cancel{92}\,50}}\cdot\frac{z^{\cancel{43}\,42}}{j\cancel{z}}dz$$

$$=2\int_{|z|=1}\frac{(-z^2+2jz+1)^{92}}{z^{50}}\cdot\frac{\frac{j}{1}}{2^{92}j}dz$$

$j^{92}=(j^4)^{23}=1$

$$=-\frac{j}{2^{91}}\int_{|z|=1}\frac{(-z^2+2jz+1)^{92}}{z^{50}}dz$$

$$=-\frac{j}{2^{91}}\int_{|z|=1}\frac{((jz+1)^2)^{92}}{z^{50}}dz=-\frac{j}{2^{91}}\int_{|z|=1}\underbrace{\frac{(jz+1)^{184}}{z^{50}}}_{f(z)}dz$$

Observăm că $z=0$ este pol de ordinul 50 pentru f

Pe de altă parte, $(jz+1)^{184}=\sum_{k=0}^{184}C_{184}^{k}\cdot(jz)^k\cdot 1\rightarrow f(z)=\frac{1}{z^{50}}\sum_{k=0}^{184}C_{184}^{k}\cdot(jz)^k$

$\mathrm{Rez}(f;0)$
$\shortparallel$
$\rightarrow a_{-1}=C_{184}^{49}\cdot j^{49}=C_{184}^{49}\cdot j$
(coef. lui $\frac{1}{z}$)
$j^{49}=(j^4)^{12}\cdot j=j$

În final, $\mathcal{J}+j\cdot\mathcal{I}=2\pi j\cdot\left(-\frac{j}{2^{91}}\right)\cdot j\cdot C_{184}^{49}$

$$=\frac{\pi j}{2^{90}}\cdot C_{184}^{49}$$

$$\rightarrow \mathcal{I}=\frac{\pi}{2^{90}}\cdot C_{184}^{49}$$

④ Să se calculeze:

$$I = \int_{-\infty}^{\infty} \frac{2z^3+5z^2+z+5}{(z^2+1)^2\cdot(z^2+4)^2}\,dz = \int_{-\infty}^{\infty} \frac{2x^3+5x^2+x+5}{(x^2+1)^2\cdot(x^2+4)^2}\,dx$$

Obs: Dacă $R(x) = \frac{P(x)}{Q(x)}$ este o funcție rațională a.î. grad Q - grad $P \geq 2$ și $Q(x) \neq 0 \ \forall x \in \mathbb{R}$ atunci are loc egalitatea:

$$\int_{-\infty}^{\infty} R(x)\,dx = 2\pi j \cdot \sum_{\operatorname{Im} x_k > 0} \operatorname{Rez}(R; x_k) \quad (*)$$

unde suma din membrul drept se referă la toți polii funcției raționale R situați în semiplanul superior

În cazul nostru, $I = \underbrace{\int_{-\infty}^{\infty} \frac{2x^3+x}{(x^2+1)^2\cdot(x^2+4)^2}\,dx}_{\text{funcție impară}} + \int_{-\infty}^{\infty} \frac{5\cancel{(x^2+1)}}{(x^2+1)^{\cancel{2}}\cdot(x^2+4)^2}\,dx$

$$= 0 + \int_{-\infty}^{\infty} \underbrace{\frac{5}{(x^2+1)(x^2+4)^2}}_{\frac{P(x)}{Q(x)}}\,dx$$

Observăm că grad Q - grad $P \geq 2$ și $Q(x) \neq 0$ $\forall x \in \mathbb{R}$

$x^2+1 = 0 \Rightarrow x_{1,2} = \pm j$ - poli simpli

$x^2+4 = 0 \Rightarrow x_{3,4} = \pm 2j$ - poli dubli

Deoarece $\operatorname{Im}(-j) < 0$ și $\operatorname{Im}(-2j) < 0$, folosindu-ne de $(*)$

$$I = 2\pi j \cdot [\operatorname{Rez}(f; j) + \operatorname{Rez}(f; 2j)]$$

$$\operatorname{Rez}(f; j) = \left.\frac{5}{(x^2+1)'(x^2+4)^2}\right|_{x=j} = \left.\frac{5}{2x\cdot(x^2+4)^2}\right|_{x=j} = \frac{5}{18j}$$

$$\operatorname{Rez}(f; 2j) = \lim_{z\to 2j}\left(\cancel{(x-2j)^2}\cdot\frac{5}{(x^2+1)\cancel{(x-2j)^2}(x+2j)^2}\right)' = \lim_{z\to 2j}\Bigg(\underbrace{\frac{5}{(x^2+1)(x+2j)^2}}_{g(x)}\Bigg)' \overset{(**)}{=}$$

$$g'(x) = \frac{-2x(x+2j)^2\cdot 5 - 2(x^2+1)(x+2j)\cdot 5}{(x^2+1)^2\cdot(x+2j)^4} = \frac{-10x(x+2j) - 10(x^2+1)}{(x^2+1)^2\cdot(x+2j)^3}$$

$$\overset{(**)}{=} \left.\frac{-10x(x+2j) - 10(x^2+1)}{(x^2+1)^2\cdot(x+2j)^3}\right|_{x=2j} = \frac{-20j\cdot 4j - 10\cdot(-3)}{9\cdot(-64j)} = \frac{-110}{576j}$$

În final, $I = 2\pi j \cdot \left[\frac{5}{18j} - \frac{110}{576j}\right] = \frac{5\pi}{9} - \frac{110\pi}{288} = \frac{^{16)}5\pi}{9} - \frac{55\pi}{144} = \frac{25\pi}{144}$

$$\Rightarrow I = \frac{25\pi}{144}$$

⑤ Să se calculeze:

a) $\mathcal{I}_n = \int_0^\infty \frac{1}{(x^2+a^2)^n}\,dx, \quad n \geq 1, \ a > 0$

b) $\mathcal{J} = \int_0^\infty \frac{x^2+5}{(x^2+4)^7}\,dx$

---

a) $\mathcal{I}_n = \int_0^\infty \underbrace{\frac{1}{(x^2+a^2)^n}}_{\text{funcție pară}}\,dx \qquad x^2+a^2=0 \Leftrightarrow x = \pm aj$ - poli de ordinul n

Doar $\operatorname{Im}(aj) > 0$

$$\overset{\|}{\frac{1}{2}\int_{-\infty}^{\infty} \underbrace{\frac{1}{(x^2+a^2)^n}}_{f(x)}\,dx} = \cancel{\frac{1}{2}}\cdot \cancel{2}\pi j \cdot \operatorname{Rez}(f; aj)$$

$$\operatorname{Rez}(f; aj) = \frac{1}{(n-1)!}\cdot \lim_{x\to aj}\left((x-aj)^{\cancel{n}}\cdot \frac{1}{(x-aj)^{\cancel{n}}\cdot (x+aj)^n}\right)^{(n-1)}$$

$$= \frac{1}{(n-1)!}\cdot \lim_{x\to aj}\left((x+aj)^{-n}\right)^{(n-1)} \quad \overset{(*)}{=}$$

$$\left((x+aj)^{-n}\right)' = -n\cdot (x+aj)^{-n-1}$$

$$\left((x+aj)^{-n}\right)'' = -n(-n-1)\cdot (x+aj)^{-n-2}$$

$$\ldots$$

$$\left((x+aj)^{-n}\right)^{(n-1)} = -n\cdot(-n-1)\cdot\ldots\cdot(-n-(n-2))\cdot(x+aj)^{-n-(n-1)}$$

$$= (-1)^{n-1}\cdot n(n+1)\cdot\ldots\cdot(2n-2)\cdot(x+aj)^{-2n+1}$$

$$\overset{(*)}{=} \frac{1}{(n-1)!}\cdot\frac{(-1)^{n-1}\cdot(2n-2)!}{(n-1)!}\cdot(2aj)^{-2n+1} = -\frac{(2n-2)!}{(n-1)!^2}\cdot\frac{1}{(2a)^{2n-1}}\,j$$

$$j^{-2n+1} = \frac{1}{j^{2n-1}} = \frac{j}{j^{2n}} = j\cdot(-1)^n$$

Așadar, $\mathcal{I}_n = \frac{(2n-2)!}{(n-1)!^2}\cdot\frac{1}{(2a)^{2n-1}}\cdot\pi$

b) $\mathcal{J} = \int_0^\infty \frac{x^2+4+1}{(x^2+4)^7}\,dx = \int_0^\infty \frac{1}{(x^2+4)^6}\,dx + \int_0^\infty \frac{1}{(x^2+4)^7}\,dx = \mathcal{I}_6 + \mathcal{I}_7 =$

$$= \frac{10!}{5!^2}\cdot\frac{1}{(2a)^{11}}\,\pi + \frac{12!}{6!^2}\cdot\frac{1}{(2a)^{13}}\cdot\pi = \ldots$$

Probleme propuse

1) Să se calculeze următoarele integrale:

a) $I=\int_0^{2\pi}\frac{\sin^2(15x)}{5-3\sin x}\,dx$

b) $I=\int_0^{4\pi}(1-\sin x)^{142}\cdot\sin(71x)\,dx$

c) $I=\int_{-\pi}^{3\pi}(1+\cos x)^{2n}\cdot\cos nx\,dx,\ n\in\mathbb{N}$

d) $I=\int_{-\infty}^{\infty}\frac{x^2+5x+2}{(x^2+1)(x^2+4)^2}\,dx$

e) $I=\int_0^{\infty}\frac{x^2\,dx}{x^4+x^2+1}$

f) $I_n=\int_0^{\infty}\frac{x^2+1}{(x^2+a^2)^{n+1}}\,dx,\ n\in\mathbb{N}^*,\ a>1$

g) $I=\int_{-\infty}^{\infty}\frac{x+3}{(x^2+4jx-5)^2}\,dx$

h) $I=\int_{\pi}^{3\pi}(1+\sin x)^{52}\cdot\sin 23x\,dx$

Seminarul 8

Transformata Fourier integrală

① Să se calculeze spectrul Fourier, amplitudinea și faza în frecvență pentru semnalul $f: \mathbb{R} \to \mathbb{C}$, unde

a) $f(t) = \dfrac{2j}{t^2 - 10t + 29}$

Obs1: Funcția $F: \mathbb{R} \to \mathbb{K}$ asociată unui semnal continual $f \in L^1(\mathbb{R})$ prin

$$\underbrace{\hat{f}(\omega) = (\mathcal{F}f)(\omega) = F(\omega)}_{\text{notații pentru transformata Fourier a semnalului } f(t) \text{ (funcției)}} = \int_{-\infty}^{\infty} f(t) \cdot e^{-j\omega t}\, dt \quad , \forall \omega \in \mathbb{R}$$

Obs2: Calculul integralei de tipul $\int_{-\infty}^{\infty} R(x) e^{j\lambda x} dx$

• Dacă $R(x) = \dfrac{P(x)}{Q(x)}$ este o funcție rațională a.î. grad Q - grad $P \geq 1$ și Q are în $\mathbb{R}$ numai rădăcinile simple $b_1, b_2, \ldots, b_m$, iar $\lambda > 0$, atunci

$$\int_{-\infty}^{\infty} R(x) \cdot e^{j\lambda x} dx = 2\pi j \cdot \sum_{\operatorname{Im} x_k > 0} \operatorname{Rez}\left[R(x) \cdot e^{j\lambda x}; x_k\right] + \pi j \cdot \sum_{k=1}^{m} \operatorname{Rez}\left[R(x) e^{j\lambda x}; b_k\right]$$

ÎN PARALEL

Obs3 Calculul integralei de tipul $\int_{-\infty}^{\infty} R(x) dx$

• Dacă $R(x) = \dfrac{P(x)}{Q(x)}$ este o funcție rațională a.î. grad Q - grad $P \geq 2$ și Q nu are rădăcini reale ($Q(x) \neq 0 \ \forall x \in \mathbb{R}$), atunci

$$\int_{-\infty}^{\infty} R(x)\, dx = 2\pi j \sum_{\operatorname{Im} x_k > 0} \operatorname{Rez}\left[R(x); x_k\right]$$

În cazul nostru,

$$\hat{f}(\omega) = F(\omega) = \int_{-\infty}^{\infty} f(t) \cdot e^{-j\omega t} dt = \int_{-\infty}^{\infty} \frac{2j}{t^2 - 10t + 29} \cdot e^{-j\omega t} dt, \ \omega \in \mathbb{R}$$

Fie funcția $f(t) = \dfrac{2j}{t^2 - 10t + 29}$

$t^2 - 10t + 29 = 0$

$\Delta = 100 - 4 \cdot 29 = 4 \cdot (25 - 29) = -16 = (4j)^2$

t_1 și t_2 - poli simpli

$$t_{1,2} = \frac{10 \pm 2j}{2} \quad \begin{cases} t_1 = 5 + 2j & \operatorname{Im}(t_1) > 0 \\ t_2 = 5 - 2j & \operatorname{Im}(t_2) < 0 \end{cases}$$

Cazul I $\omega < 0 \Rightarrow \lambda = -\omega > 0$

Atunci $f(\omega) = F(\omega) = \int_{-\infty}^{\infty} f(t) \cdot e^{-j\omega t} dt \;\underset{\text{Obs 2}}{\overset{\text{Cf}}{=}}\; 2\pi j \cdot \text{Rez}\left(f(t) \cdot e^{-j\omega t}; 5+2j\right)$

-ω > 0

$$= 2\pi j \cdot \frac{2j \cdot e^{-j\omega t}}{(t^2-10t+29)'}\Big|_{t=5+2j}$$

$$= 2\pi j \cdot \frac{2j \cdot e^{-j\omega t}}{2t-10}\Big|_{t=5+2j}$$

$$= 2\pi j \cdot \frac{2j \cdot e^{-j\omega(5+2j)}}{\cancel{10}+4j-\cancel{10}} = \pi j \cdot e^{2\omega} \cdot e^{-5\omega j}$$

Cazul II $\omega > 0$

Facem substituția $t = -x$ în $F(\omega)$ și obținem

$$f(\omega) = F(\omega) = -\int_{+\infty}^{-\infty} \frac{2j \cdot e^{j\omega x}}{x^2+10x+29} dx = \int_{-\infty}^{\infty} \frac{2j \cdot e^{j\omega x}}{x^2+10x+29} dx$$

$dt = -dx$

ω > 0

$$= 2\pi j \cdot \text{Rez}\left(\frac{2j \cdot e^{j\omega x}}{x^2+10x+29}; -5+2j\right) \overset{(*)}{=}$$

$x^2+10x+29 = 0$

$\Delta = 100 - 4 \cdot 29 = (4j)^2$

$x_{1,2} = \frac{-10 \pm 4j}{2}$ $\left\langle \begin{array}{l} x_1 = -5+2j \\ x_2 = -5-2j \end{array} \right\rangle$ poli simpli

$$\overset{*}{=} 2\pi j \cdot \frac{2j \cdot e^{j\omega x}}{(x^2+10x+29)'}\Big|_{x=-5+2j}$$

$$= 2\pi j \cdot \frac{2j \cdot e^{j\omega x}}{2x+10}\Big|_{x=-5+2j}$$

$$= 2\pi j \cdot \frac{2j \cdot e^{j\omega(-5+2j)}}{-\cancel{10}+4j+\cancel{10}} = \pi j \cdot e^{-2\omega} \cdot e^{-5\omega j}$$

Cazul III $\omega = 0$

$$f(\omega) = F(\omega) = \int_{-\infty}^{\infty} \frac{2j}{t^2-10t+29} dt = 2\pi j \cdot \text{Rez}\left(\frac{2j}{t^2-10t+29}; 5+2j\right)$$

$t^2 - 10t + 29 = 0$

$\Downarrow$

$\left. \begin{array}{l} t_1 = 5+2j \\ t_2 = 5-2j \end{array} \right\rangle$ poli simpli

$$= 2\pi j \cdot \frac{2j}{(t^2-10t+29)'}\Big|_{t=5+2j} = 2\pi j \cdot \frac{2j}{2t-10}\Big|_{t=5+2j}$$

$$= 2\pi j \cdot \frac{2j}{\cancel{10}+4j-\cancel{10}} = \pi j$$

Din I, II și III, rezultă că $f(\omega) = F(\omega) = \pi j \cdot e^{-2|\omega|} \cdot e^{-5\omega j}, \; \forall \omega \in \mathbb{R}$

Obs:

- Funcția $A: \mathbb{R} \to [0,+\infty)$, $A(\omega) = |F(\omega)|$ s.n. amplitudinea în frecvență a semnalului $f(t)$

- Funcția $\phi: \mathbb{R} \setminus \{\omega \in \mathbb{R} : F(\omega) = 0\} \to [0, 2\pi)$,

$\phi(\omega) = \arg F(\omega)$ s.n. faza în frecvență a semnalului $f(t)$

În cazul nostru,

$$A(\omega) = |F(\omega)| = |\pi j \cdot e^{-2|\omega|} \cdot e^{-5\omega j}| = \pi \cdot e^{-2|\omega|}, \ \forall \omega \in \mathbb{R}$$

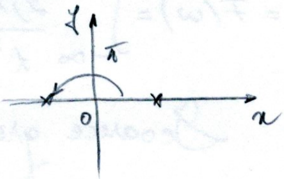
$$|e^{-5\omega j}| = |\cos 5\omega - j \cdot \sin 5\omega| = \sqrt{\sin^2 5\omega + \cos^2 5\omega} = 1, \ \forall \omega \in \mathbb{R}$$

$$\phi(\omega) = \arg F(\omega) = \arg(\pi j \cdot e^{-2|\omega|} \cdot e^{-5\omega j})$$

$$= (\arg \pi j + \arg e^{-2|\omega|} + \arg e^{-5\omega j}) \bmod 2\pi$$

$$= \left(\frac{\pi}{2} + 0 - 5\omega\right) \bmod 2\pi = \left(\frac{\pi}{2} - 5\omega\right) \bmod 2\pi$$

Obs: Dacă $x \in \mathbb{R}^* \Rightarrow \arg x = \begin{cases} 0, & x > 0 \\ \pi, & x < 0 \end{cases}$



$$\arg e^{jx} = x \pmod{2\pi}$$

b) $f(t) = \dfrac{2jt+1}{t^2+8t+20}$

$$\hat{f}(\omega) = F(\omega) = \int_{-\infty}^{\infty} f(t) \cdot e^{-j\omega t} dt = \int_{-\infty}^{\infty} \frac{2jt+1}{t^2+8t+20} \cdot e^{-j\omega t} dt, \quad \omega \in \mathbb{R}$$

Cazul I $\omega < 0 \to \lambda = -\omega > 0$

$$t^2 + 8t + 20 = 0$$

$$\Delta = 64 - 80 = -16 = (4j)^2$$

$$t_{1,2} = \frac{-8 \pm 4j}{2} \quad t_1 = -4+2j, \quad t_2 = -4-2j$$

poli simpli, $\operatorname{Im}(t_1) > 0$

În acest caz,

$$\hat{f}(\omega) = F(\omega) = 2\pi j \cdot \operatorname{Rez}\left(\frac{2jt+1}{t^2+8t+20} \cdot e^{-j\omega t}; -4+2j\right)$$

$$= 2\pi j \cdot \left. \frac{(2jt+1) \cdot e^{-j\omega t}}{(t^2+8t+20)'} \right|_{t=-4+2j}$$

$$= 2\pi j \cdot \frac{(2j \cdot (-4+2j)+1) \cdot e^{-j\omega(-4+2j)}}{2(-4+2j)+8}$$

$$= \cancel{2\pi j} \cdot \frac{(-8j-3) \cdot e^{4\omega j} \cdot e^{2\omega}}{\cancel{2}\cancel{(4j)}}$$

$$= \frac{\pi}{2} \cdot (-8j-3) \cdot e^{4\omega j} \cdot e^{2\omega}$$

Cazul II $\omega > 0$

$$\hat{f}(\omega) = F(\omega) = \int_{-\infty}^{\infty} \frac{2jt+1}{t^2+8t+20} \cdot e^{-j\omega t}\, dt \xrightarrow[dt=-dx]{\substack{\text{Notăm} \\ t=-x}} \int_{-\infty}^{\infty} \frac{-2jx+1}{x^2-8x+20} e^{j\omega x} dx$$

$x^2 - 8x + 20 = 0$

$\Delta = (4j)^2$

$x_{1,2} = \frac{8 \pm 4j}{2}$, $x_1 = 4+2j$, $\operatorname{Im}(x_1) > 0$; $x_2 = 4-2j$

poli simpli

$$\hat{f}(\omega) = F(\omega) = 2\pi j \cdot \operatorname{Rez}\left(\frac{-2jx+1}{x^2-8x+20} e^{j\omega x}; 4+2j\right)$$

$$= 2\pi j \cdot \frac{-2jx+1}{(x^2-8x+20)'} \cdot e^{j\omega x}\Big|_{x=4+2j}$$

$$= 2\pi j \cdot \frac{-2jx+1}{2x-8} \cdot e^{j\omega x}\Big|_{x=4+2j}$$

$$= 2\pi j \cdot \frac{-2j\cdot(4+2j)+1}{2(\cancel{4}+2j)-\cancel{8}} \cdot e^{j\omega(4+2j)}$$

$$= \cancel{2}\pi \cancel{j} \cdot \frac{-8j+5}{\frac{\cancel{4j}}{2}} \cdot e^{4\omega j} \cdot e^{-2\omega} = \frac{\pi}{2}(-8j+5) e^{4\omega j} \cdot e^{-2\omega}$$

Cazul III $\omega = 0$

$$\hat{f}(\omega) = F(\omega) = \int_{-\infty}^{\infty} \frac{2jt+1}{t^2+8t+20} dt = \int_{-\infty}^{\infty} \frac{P(t)}{Q(t)} dt$$

Deoarece grad Q - grad $P = 1\ (\not\geq 2) \to \int_{-\infty}^{\infty} \frac{P(t)}{Q(t)} dt$ - divergentă

$\Downarrow$

$\nexists\ \hat{f}(0)$

Așadar, $\hat{f}(\omega) = \begin{cases} \frac{\pi}{2}(-3-8j) \cdot e^{4\omega j} \cdot e^{2\omega}, & \text{dacă } \omega < 0 \\ \frac{\pi}{2}(5-8j) \cdot e^{4\omega j} \cdot e^{-2\omega}, & \text{dacă } \omega > 0 \end{cases}$

Obs: $|e^{\pm jx}| = 1,\ \forall x \in \mathbb{R}$

$|a \pm bj| = \sqrt{a^2+b^2}$

$$A(\omega) = |\hat{f}(\omega)| = \begin{cases} \frac{\pi}{2} \cdot e^{2\omega} |-3-8j| \cdot \underbrace{|e^{4\omega j}|}_{=1}, & \text{dacă } \omega < 0 \\ \frac{\pi}{2} \cdot e^{-2\omega} \cdot |5-8j| \cdot \underbrace{|e^{4\omega j}|}_{=1}, & \text{dacă } \omega > 0 \end{cases}$$

amplitudinea în frecvență

$$= \begin{cases} \frac{\pi}{2} \cdot e^{2\omega} \cdot \sqrt{73}, & \text{dacă } \omega < 0 \\ \frac{\pi}{2} \cdot e^{-2\omega} \cdot \sqrt{89}, & \text{dacă } \omega > 0 \end{cases}$$

Obs: $\arg e^{jx} = x \pmod{2\pi}$

$x \in \mathbb{R}^* \Rightarrow \arg x = \begin{cases} 0, & x > 0 \\ \pi, & x < 0 \end{cases}$

$$\phi(\omega) = \arg \hat{f}(\omega) = \begin{cases} \underbrace{\arg \frac{\pi}{2}}_{=0} + \underbrace{\arg e^{2\omega}}_{=0} + \arg(-3-8j) + \arg e^{4\omega j} \pmod{2\pi}, & \text{dacă } \omega < 0 \\ \underbrace{\arg \frac{\pi}{2}}_{=0} + \underbrace{\arg e^{-2\omega}}_{=0} + \arg(5-8j) + \arg e^{4\omega j} \pmod{2\pi}, & \text{dacă } \omega > 0 \end{cases}$$

$$= \begin{cases} \operatorname{arctg} \frac{8}{3} + \pi + 4\omega \pmod{2\pi}, & \text{dacă } \omega < 0 \\ -\operatorname{arctg} \frac{8}{5} + 2\pi + 4\omega \pmod{2\pi}, & \text{dacă } \omega > 0 \end{cases}$$

② Să se calculeze transformata Fourier prin cosinus pentru semnalul $f:\mathbb{R}_+ \to \mathbb{K}$,

$$f(t)=\frac{t^2}{t^4-6t^2+25}$$

Obs: $\underbrace{\hat{f}_c(\omega)=(\mathcal{F}_c f)(\omega)=F_c(\omega)}_{\text{notații pentru transformata Fourier integrală prin cosinus a semnalului } f \text{ (TFI)}}=\int_0^\infty f(t)\cdot\cos\omega t\,dt,\ \forall\omega>0$

În cazul nostru,

$$\hat{f}_c(\omega)=\int_0^\infty \underbrace{\frac{t^2\cdot\cos\omega t}{t^4-6t^2+25}}_{\text{pară}}dt,\ \omega>0$$

$$=\frac{1}{2}\int_{-\infty}^{\infty}\frac{t^2\cdot\cos\omega t}{t^4-6t^2+25}dt\ \overset{\text{not.}}{=}\ I.$$

Fie $J=\frac{1}{2}\int_{-\infty}^{\infty}\frac{t^2\cdot\sin\omega t}{t^4-6t^2+25}dt$. Formăm expresia $I+jJ$

Obținem $I+j\cdot J=\frac{1}{2}\int_{-\infty}^{\infty}\frac{t^2(\cos\omega t+j\sin\omega t)}{t^4-6t^2+25}dt=\frac{1}{2}\int_{-\infty}^{\infty}\frac{t^2\cdot e^{j\omega t}}{t^4-6t^2+25}dt$

Cum funcția $g(t)=\frac{t^2\cdot\sin\omega t}{t^4-6t^2+25}$ - impară $(g(-t)=-g(t)) \Rightarrow J=0$

Așadar, $I=\frac{1}{2}\int_{-\infty}^{\infty}\underbrace{\frac{t^2\cdot e^{j\omega t}}{t^4-6t^2+25}}_{h(t)}dt$

Notăm $t^2=u$ și obținem

$t^4-6t^2+25=0 \Leftrightarrow u^2-6u+25=0$

$\Delta=36-100=-64=(8j)^2$

$u_{1,2}=\frac{6\pm 8j}{2}$ $u_1=3+4j$, $u_2=3-4j$

Dacă $u_1=3+4j \Rightarrow t^2=3+4j=(2+j)^2$

$\Downarrow$

$t_{1,2}=\pm(2+j)$, $\operatorname{Im}(2+j)>0$

Dacă $u_2=3-4j \Rightarrow t^2=3-4j=(2-j)^2$

$\Downarrow$

$t_{3,4}=\pm(2-j)$, $\operatorname{Im}(-2+j)>0$

În final,

$I=\frac{1}{\cancel{2}}\cdot\cancel{2}\pi j\cdot[\operatorname{Rez}(h;2+j)+\operatorname{Rez}(h;-2+j)]$, unde

$h(t)=\frac{t^2\cdot e^{j\omega t}}{t^4-6t^2+25}$, $t_1=2+j$ și $t_3=-2+j$ sunt poli simpli

$$\operatorname{Rez}(h;2+j)=\frac{t^2\cdot e^{j\omega t}}{(t^4-6t^2+25)'}\Bigg|_{t=2+j}=\frac{t^2\cdot e^{j\omega t}}{4t^3-12t}\Bigg|_{t=2+j}=\frac{t\cdot e^{j\omega t}}{4t^2-12}\Bigg|_{t=2+j}$$

$$= \frac{(2+j)\cdot e^{j\omega(2+j)}}{4(2+j)^2-12} = \frac{(2+j)\cdot e^{2\omega j-\omega}}{4(\cancel{3}+4j)-\cancel{12}} = \frac{(2+j)\cdot e^{2\omega j-\omega}}{16j}$$

$$Rez(h; -2+j) = \frac{t^2\cdot e^{j\omega t}}{4t^3-12t}\Big|_{t=-2+j} = \frac{t\cdot e^{j\omega t}}{4t^2-12}\Big|_{t=-2+j} = \frac{(-2+j)\cdot e^{j\omega(-2+j)}}{4(-2+j)^2-12}$$

$$= \frac{(-2+j)\cdot e^{-2\omega j-\omega}}{4\cdot(\cancel{3}-4j)-\cancel{12}} = \frac{(-2+j)\cdot e^{-2\omega j-\omega}}{-16j}$$

Mai departe, $\underline{\underline{f_c(\omega)}} = \pi j\cdot\left[\frac{(2+j)e^{-\omega}\cdot e^{2\omega j}}{16j} - \frac{(-2+j)\cdot e^{-\omega}\cdot e^{-2\omega j}}{16j}\right]$

$$= \pi\cdot\left[\frac{2e^{-\omega}\cdot e^{2\omega j}}{16} + j\cdot\frac{e^{-\omega}\cdot e^{2\omega j}}{16} + \frac{2e^{-\omega}\cdot e^{-2\omega j}}{16} - j\cdot\frac{e^{-\omega}\cdot e^{-2\omega j}}{16}\right]$$

$$= \pi\cdot\left[\frac{1}{8}\cdot e^{-\omega}\left(e^{2\omega j}+e^{-2\omega j}\right) + j\cdot\frac{1}{16}e^{-\omega}\left(e^{2\omega j}-e^{-2\omega j}\right)\right]$$

$$= \pi\cdot\left[\frac{1}{4}\cdot e^{-\omega}\cdot\underbrace{\frac{e^{2\omega j}+e^{-2\omega j}}{2}}_{\cos 2\omega} + j^2\cdot\frac{1}{8}\cdot e^{-\omega}\cdot\underbrace{\frac{e^{2\omega j}-e^{-2\omega j}}{2j}}_{\sin 2\omega}\right]$$

$$= \pi\cdot\left[\frac{1}{4}e^{-\omega}\cdot\cos 2\omega - \frac{1}{8}e^{-\omega}\cdot\sin 2\omega\right]$$

$$= \frac{\pi}{8}\cdot e^{-\omega}\left[2\cos 2\omega - \sin 2\omega\right]$$

③ Să se calculeze transformata Fourier prin sinus pentru semnalul $f:(0,+\infty)\to K$,

$$f(t) = \frac{1}{t(t^2+4)^2}$$

$$f_s(\omega) = F_s(\omega) = \int_0^\infty f(t)\sin\omega t\,dt = \int_0^\infty \underbrace{\frac{1}{t(t^2+4)^2}\sin\omega t}_{\text{funcție pară}}\,dt \overset{not}{=} J = \frac{1}{2}\int_{-\infty}^\infty \frac{\sin\omega t}{t(t^2+4)^2}dt$$

Asociem integrala $I = \frac{1}{2}\int_{-\infty}^\infty \underbrace{\frac{\cos\omega t}{t(t^2+4)^2}}_{\text{impară}}dt \Rightarrow I = 0$

și

formăm expresia $\overset{=0}{I} + j\cdot J = \frac{1}{2}\int_{-\infty}^\infty \frac{\cos\omega t + j\sin\omega t}{t(t^2+4)^2}dt = \frac{1}{2}\int_{-\infty}^\infty \frac{e^{j\omega t}}{t(t^2+4)^2}dt$

$$\Rightarrow J = \frac{1}{2j}\int_{-\infty}^\infty \frac{e^{j\omega t}}{t(t^2+4)^2}dt$$

$$J=\frac{1}{2j}\int_{-\infty}^{\infty}\frac{e^{j\omega t}}{t(t^2+4)^2}dt \qquad \text{Fie } g(t)=\frac{e^{j\omega t}}{t(t^2+4)^2}$$

$t(t^2+4)^2=0 \to \underset{\text{pol simplu}}{t_1=0}$ sau $(t^2+4)^2=0 \to t^2=-4=(2j)^2 \to \underset{\text{poli dubli}}{t_{2,3}=\pm 2j}$ Doar $\operatorname{Im}(2j)>0$

Obținem $\hat{f}_s(\omega)=J=\frac{1}{2j}\left[2\pi j\cdot \operatorname{Rez}(g;2j)+\pi j\cdot \operatorname{Rez}(g;0)\right]$, unde

$$\operatorname{Rez}(g;0)=\left.\frac{e^{j\omega t}}{t'\cdot(t^2+4)^2}\right|_{t=0}=\frac{1}{16}$$

$$\operatorname{Rez}(g;2j)=\frac{1}{1!}\lim_{t\to 2j}\left[(t-2j)^2\cdot\frac{e^{j\omega t}}{t\cancel{(t-2j)^2}(t+2j)^2}\right]'=\lim_{t\to 2j}\underbrace{\left(\frac{e^{j\omega t}}{t(t+2j)^2}\right)}_{h(t)}'$$

$$h'(t)=\left(\frac{e^{j\omega t}}{t(t+2j)^2}\right)'=\frac{j\omega\cdot e^{j\omega t}\cdot t\cdot(t+2j)^2-e^{j\omega t}\left((t+2j)^2+2t(t+2j)\right)}{t^2\cdot(t+2j)^4}$$

Așadar, $\operatorname{Rez}(g;2j)=h'(2j)=\dfrac{j\omega\cdot e^{j\omega(2j)}\cdot 2j(2j+2j)^2-e^{j\omega(2j)}\left((4j)^2+4j\cdot(4j)\right)}{(2j)^2\cdot(4j)^4}$

$$=\frac{-2\omega e^{-2\omega}\cdot(4j)^2-2e^{-2\omega}\cdot(4j)^2}{-4\cdot(4j)^2\cdot(4j)^2}=\frac{-\cancel{2}e^{-2\omega}(\omega+1)}{+\cancel{4}_2\cdot 16}=-\frac{e^{-2\omega}}{32}(\omega+1)$$

În final, $\hat{f}_s(\omega)=\frac{1}{2j}\left[-2\pi j\cdot\frac{e^{-2\omega}}{32}(\omega+1)+\frac{\pi j}{16}\right]$

$$=-\pi\cdot\frac{e^{-2\omega}}{32}(\omega+1)+\frac{\pi}{32}=\frac{\pi}{32}\left[1-(\omega+1)e^{-2\omega}\right]$$

Probleme propuse

① Să se calculeze spectrul Fourier, amplitudinea și faza în frecvență pentru fiecare din următoarele semnale $f:\mathbb{R}\to\mathbb{C}$,

a) $f(t)=\dfrac{3t-2}{t^2-2t+10}$ b) $f(t)=\dfrac{2t-j}{t^2+2jt+3}$ c) $f(t)=\dfrac{3jt+2}{jt^2+(3j-2)t+j-3}$

② Să se calculeze $\hat{f}_c(\omega)$, pentru $f:\mathbb{R}_+\to\mathbb{C}$,

$$f(t)=\frac{t^2+1}{(t^2+4)(t^2+a^2)},\ a>0$$

③ Să se calculeze $\hat{f}_s(\omega)$, pentru $f:\mathbb{R}_+\to\mathbb{C}$, $f(t)=\dfrac{1}{t(t^4+4a^4)}$, $a>0$

Seminarul 9

Transformarea Fourier integrală (TFi)

① Să se rezolve ecuațiile integrale Fourier:

a) $\int_0^\infty f(t)\sin\omega t\,dt = e^{-2\omega}$, $\omega > 0$

Obs1: O egalitate de forma $\int_{-\infty}^\infty f(t)\cdot e^{-j\omega t}dt = g(\omega)$, $\omega\in\mathbb{R}$ unde $g\in L^1(\mathbb{R})$ este un semnal dat, iar $f\in L^1(\mathbb{R})$ este semnalul-funcție necunoscut s.n. ecuație integrală Fourier.

Egalitatea se mai scrie sub forma $\mathcal{F}(f;\omega)=g(\omega)$

$\rightarrow f(t) = \mathcal{F}^{-1}(g;t)$, adică $f(t)=\frac{1}{2\pi}\int_{-\infty}^\infty g(\omega)\cdot e^{j\omega t}d\omega$

Obs2: Alte tipuri de ecuații integrale Fourier sunt

$\int_0^\infty f(t)\cos\omega t\,dt = g(\omega)$, $\omega>0$ și $\int_0^\infty f(t)\sin\omega t\,dt = g(\omega)$, $\omega>0$

$f(t) = \mathcal{F}_c^{-1}(g;t) = \frac{2}{\pi}\int_0^\infty g(\omega)\cos\omega t\,d\omega$, $t>0$ $\quad$ $f(t)=\mathcal{F}_s^{-1}(g;t) = \frac{2}{\pi}\int_0^\infty g(\omega)\sin\omega t\,d\omega$, $t>0$

a) $\int_0^\infty f(t)\sin\omega t\,dt = e^{-2\omega} \rightarrow f(t) = \mathcal{F}_s^{-1}(e^{-2\omega};t) = \frac{2}{\pi}\int_0^\infty e^{-2\omega}\cdot\sin\omega t\,d\omega$, $t>0$

Fie $I=\frac{2}{\pi}\int_0^\infty e^{-2\omega}\cdot\sin\omega t\,d\omega$ și $J = \frac{2}{\pi}\int_0^\infty e^{-2\omega}\cdot\cos\omega t\,d\omega$

Formăm expresia $J+j\cdot I = \frac{2}{\pi}\int_0^\infty e^{-2\omega}(\cos\omega t + j\sin\omega t)\,d\omega$

$$= \frac{2}{\pi}\int_0^\infty e^{-2\omega}\cdot e^{j\omega t}d\omega = \frac{2}{\pi}\int_0^\infty e^{\omega(-2+jt)}d\omega$$

$$= \frac{2}{\pi}\cdot\frac{e^{\omega(-2+jt)}}{-2+jt}\Big|_0^\infty = \frac{2}{\pi}\lim_{\omega\to\infty}\frac{e^{\omega(-2+jt)}}{-2+jt} - \frac{2}{\pi}\cdot\frac{1}{-2+jt}\ \left(\cdot\,{-2-jt}\right)$$

$$= \frac{2}{\pi}\cdot 0 - \frac{2}{\pi}\cdot\frac{-2-jt}{t^2+4} = \frac{4}{\pi(t^2+4)} + \frac{2t}{\pi(t^2+4)}j$$

$|e^{\omega jt}| = |\cos(\omega t)+j\sin(\omega t)|=1 \rightarrow \lim_{\omega\to\infty} e^{\omega(-2+jt)} = \lim_{\omega\to\infty} \underbrace{e^{-2\omega}}_{0}\cdot \underbrace{e^{\omega jt}}_{\text{mărginită}} = 0$

Deci $J+jI = \frac{4}{\pi(t^2+4)} + \frac{2t}{\pi(t^2+4)}j \rightarrow I = \frac{2t}{\pi(t^2+4)}$

b) $\int_{-\infty}^{\infty} f(t)\cdot e^{-j\omega t}\,dt = \omega^n\cdot e^{-\omega}\cdot u(\omega),\ \omega\in\mathbb{R}$

=

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\omega^n\cdot e^{-\omega}\cdot u(\omega)\cdot e^{j\omega t}\,d\omega \ \underset{d\omega=-dx}{\overset{\omega=-x}{=\!=}}\ \frac{1}{2\pi}\int_{-\infty}^{\infty}(-x)^n\cdot e^{x}\cdot u(-x)\cdot e^{-jxt}\,dx$$

$$=\frac{(-1)^n}{2\pi}\int_{-\infty}^{\infty}x^n e^x\cdot u(-x)\cdot e^{-jxt}\,dx = \frac{(-1)^n}{2\pi}\cdot \mathcal{F}(x^n e^x\cdot u(-x);t)$$

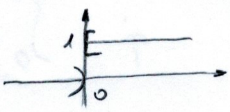
$$= \frac{(-1)^n}{2\pi}\cdot j^n\left(\mathcal{F}(e^x\cdot u(-x);t)\right)^{(n)} \quad (1)$$

$$\boxed{\text{Teorema derivării spectrului} \qquad \mathcal{F}(t^n f(t);\omega) = j^n\left(\mathcal{F}(f;\omega)\right)^{(n)},\ \forall\omega\in\mathbb{R}}$$

Dar, $\mathcal{F}(e^x\cdot u(-x);t) = \int_{-\infty}^{\infty} e^x\cdot u(-x)\cdot e^{-jxt}\,dx = \int_{-\infty}^{\infty} e^{x(1-jt)}\cdot u(-x)\,dx$

$$=\int_{-\infty}^{0} e^{x(1-jt)}\,dx = \frac{e^{x(1-jt)}}{1-jt}\Big|_{-\infty}^{0} = \frac{1}{1-jt} - \lim_{x\to-\infty}\frac{e^{x(1-jt)}}{1-jt} = \frac{1}{1-jt} - 0 = \frac{1}{1-jt} \quad (2)$$

Obs: $u:\mathbb{R}\to\mathbb{R},\ u(t)=\begin{cases}0, & t<0\\ 1, & t\geq 0\end{cases}$



semnalul treaptă unitate

Din (1) și (2) obținem $f(t) = \frac{(-1)^n}{2\pi}\cdot j^n\cdot\underbrace{\left(\frac{1}{1-jt}\right)^{(n)}}_{g(t)}$

Dacă $g(t) = (1-jt)^{-1}$

$g'(t) = (-1)\cdot(-j)\cdot(1-jt)^{-2}$

$g''(t) = (-1)\cdot(-2)\cdot(-j)^2\cdot(1-jt)^{-3}$

...

Inductiv, $g^{(n)}(t) = (-1)\cdot(-2)\cdot\ldots\cdot(-n)\cdot(-j)^n\cdot(1-jt)^{-(n+1)} = (-1)^n\cdot n!\cdot(-1)^n\cdot j^n\cdot\frac{1}{(1-jt)^{n+1}}$

În final, $\underline{f(t)} = \frac{(-1)^n}{2\pi}\cdot\underbrace{j^n\cdot\overset{1}{(-1)^{2n}}\cdot n!\cdot j^n}_{(-1)^n}\cdot\frac{1}{(1-jt)^{n+1}} = \underline{\frac{1}{2\pi}\cdot\frac{n!}{(1-jt)^{n+1}}}$

② Fie $f:\mathbb{R}\to\mathbb{C},\ f(t) = \frac{t}{(2+jt)^4},\ t\in\mathbb{R}$

a) Să se determine energia semnalului f, $E(f) = ?$

b) Să se determine spectrul Fourier, $\hat{f}(\omega) = ?$

c) Să se determine $E(\hat{f})$.

a) $E(f) = \int_{-\infty}^{\infty} |f(t)|^2 dt = \int_{-\infty}^{\infty} \left|\frac{t}{(2+jt)^4}\right|^2 dt = \int_{-\infty}^{\infty} \underbrace{\frac{t^2}{(t^2+4)^4}}_{g(t)} dt$

$$\left|\frac{t}{(2+jt)^4}\right|^2 = \frac{|t|^2}{|2+jt|^8} = \frac{t^2}{(\sqrt{t^2+4})^8} = \frac{t^2}{(t^2+4)^4}$$

$t^2+4=0 \Rightarrow t^2=-4 \Rightarrow t^2=(2j)^2 \Rightarrow t_{1,2}=\pm 2j$ poli de ordinul 4 (Doar $\operatorname{Im}(2j)>0$)

Obținem $E(f) = 2\pi j \cdot \operatorname{Rez}(g;2j)$, unde

$$\operatorname{Rez}(g;2j) = \frac{1}{3!} \lim_{t\to 2j} \left(\cancel{(t-2j)^4} \cdot \frac{t^2}{\cancel{(t-2j)^4}(t+2j)^4}\right)''' = \frac{1}{3!} \cdot \lim_{t\to 2j} \Big(\underbrace{\frac{t^2}{(t+2j)^4}}_{h(t)}\Big)'''$$

Fie $h(t) = \frac{t^2}{(t+2j)^4}$.

$$h'(t) = \frac{2t(t+2j)^4 - 4t^2(t+2j)^3}{(t+2j)^8} = \frac{2t(t+2j)-4t^2}{(t+2j)^5} = \frac{-2t^2+4tj}{(t+2j)^5} = \frac{2t(-t+2j)}{(t+2j)^5}$$

$$h''(t) = \frac{(-4t+4j)(t+2j)^5 - 5\cdot 2t\cdot(-t+2j)\cdot(t+2j)^4}{(t+2j)^{10}} = \frac{4(-t+j)(t+2j)-10t(-t+2j)}{(t+2j)^6}$$

$$= \frac{4(-t^2-tj-2)+10t^2-20tj}{(t+2j)^6} = \frac{6t^2-24tj-8}{(t+2j)^6}$$

$$h'''(t) = \frac{(12t-24j)(t+2j)^6 - 6(t+2j)^5(6t^2-24tj-8)}{(t+2j)^{12}} = \frac{(12t-24j)(t+2j)-6(6t^2-24tj-8)}{(t+2j)^7}$$

Așadar, $\operatorname{Rez}(g;2j) = \frac{1}{6} \cdot \frac{(\cancel{24j-24j})\cdot 4j - 6\cdot(-24+48-8)}{(4j)^7} = \frac{-\cancel{6}\cdot 16}{\cancel{6}\cdot \underset{4^5}{\cancel{4^7}}\cdot(-j)} = \frac{1}{4^5 j}$

$$\Rightarrow E(f) = 2\pi \cancel{j} \cdot \frac{1}{4^5 \cdot \cancel{j}} = \frac{\pi}{512}$$

sau

$$E(f) = \int_{-\infty}^{\infty} \frac{t^2+4-4}{(t^2+4)^4} dt = \int_{-\infty}^{\infty} \frac{1}{(t^2+4)^3} dt - 4\int_{-\infty}^{\infty} \frac{1}{(t^2+4)^4} dt = I_3 - 4I_4, \text{ unde}$$

$I_n = \int_{-\infty}^{\infty} \frac{1}{(t^2+4)^n} dt$ iar $t_{1,2} = \pm 2j$ – poli de ordinul n pentru $h(t) = \frac{1}{(t^2+4)^n}$

...

b)

$$\hat{f}(\omega)=F(\omega)=(\mathcal{F}f)(\omega)=\int_{-\infty}^{\infty} f(t)\cdot e^{-j\omega t}dt=\int_{-\infty}^{\infty}\frac{t}{(2+jt)^4}\cdot e^{-j\omega t}dt$$

Cazul I $\omega<0 \Rightarrow \lambda=-\omega>0$

$$\hat{f}(\omega)=\int_{-\infty}^{\infty}\underbrace{\frac{t}{(2+jt)^4}\cdot e^{-j\omega t}}_{g(t)}dt \qquad 2+jt=0 \mid \cdot j \Leftrightarrow 2j-t=0 \Rightarrow t_1=2j - \text{pol de ordinul } 4$$

$Im(2j)>0$

Obținem,

$$\hat{f}(\omega)=2\pi j\cdot Rez(g;2j), \text{ unde}$$

$$Rez(g;2j)=\frac{1}{3!}\cdot\lim_{t\to 2j}\left((t-2j)^4\cdot\frac{t}{(2+jt)^4}\cdot e^{-j\omega t}\right)'''$$

$$=\frac{1}{6}\cdot\lim_{t\to 2j}\left(\cancel{(t-2j)^4}\cdot\frac{t}{j^4\cdot\cancel{(-2j+t)^4}}\cdot e^{-j\omega t}\right)'''$$

$$=\frac{1}{6}\cdot\lim_{t\to 2j}\underbrace{\left(t\cdot e^{-j\omega t}\right)'''}_{h(t)}$$

Obs:

$$\boxed{\begin{aligned}&(f\cdot g)'''=\sum_{k=0}^{3}C_3^k(f)^{(k)}(g)^{(3-k)}\\&(e^{-j\omega t})'=-j\omega\cdot e^{-j\omega t}\\&(e^{-j\omega t})''=(-j\omega)^2\cdot e^{-j\omega t}\\&(e^{-j\omega t})'''=(-j\omega)^3\cdot e^{-j\omega t}\end{aligned}}$$

$$h'''(t)=(t\cdot e^{-j\omega t})'''$$

$$=C_3^0\cdot(t)^{(0)}\cdot(e^{-j\omega t})'''+C_3^1\cdot t'\cdot(e^{-j\omega t})''$$

$$=t\cdot(-j\omega)^3\cdot e^{-j\omega t}+3\cdot(-j\omega)^2\cdot e^{-j\omega t}$$

$$=tj\omega^3 e^{-j\omega t}-3\omega^2 e^{-j\omega t}$$

$$=\omega^2 e^{-j\omega t}(t\omega j-3)$$

$$\Rightarrow Rez(g;2j)=\frac{1}{6}\cdot\omega^2\cdot e^{-j\omega t}(t\omega j-3)\Big|_{t=2j}$$

$$=\frac{1}{6}\cdot\omega^2\cdot e^{-j\omega\cdot 2j}\cdot(-2\omega-3)$$

$$=-\frac{1}{6}\cdot\omega^2\cdot e^{2\omega}\cdot(2\omega+3) \Rightarrow \hat{f}(\omega)=-\frac{\cancel{2}\pi j}{\cancel{6}_3}\cdot\omega^2 e^{2\omega}\cdot(2\omega+3)$$

$$=-\frac{\pi j}{3}\cdot\omega^2\cdot e^{2\omega}\cdot(2\omega+3)$$

Cazul II $\omega>0 \Rightarrow \lambda=\omega>0$

Facem substituția $t=-x \Rightarrow dt=-dx$

$$\hat{f}(\omega)=\int_{-\infty}^{\infty}\frac{-x}{(2-jx)^4}\cdot e^{j\omega x}dx \qquad 2-jx=0 \mid \cdot j \Rightarrow 2j+x=0 \Rightarrow x_1=-2j$$

Cum $Im(-2j)<0 \Rightarrow \hat{f}(\omega)=0$

Obținem,

$$\hat{f}(\omega)=\begin{cases} -\dfrac{\pi}{3}\,\omega^2\cdot e^{2\omega}\cdot(2\omega+3), & \text{dacă } \omega<0 \\ 0, & \text{dacă } \omega\geq 0 \end{cases}$$

(pentru cazul $\omega=0$ se verifică ușor că $\hat{f}(\omega)=\int_{-\infty}^{\infty}\frac{t}{(2+jt)^4}dt=0$)

c) $E(\hat{f})=\int_{-\infty}^{\infty}|\hat{f}(\omega)|^2 d\omega=\int_{-\infty}^{0}\left|-\frac{\pi}{3}\cdot\omega^2\cdot e^{2\omega}(2\omega+3)\right|^2 d\omega$

$$=\int_{-\infty}^{0}\frac{\pi^2}{9}\cdot\omega^4\cdot e^{4\omega}\cdot(2\omega+3)^2 d\omega \;\underset{d\omega=-dx}{\overset{\omega=-x}{=\!=\!=}}\; \int_{0}^{\infty}\frac{\pi^2}{9}\cdot x^4\cdot e^{-4x}\cdot(-2x+3)^2 dx=\ldots$$

sau

folosim Formula lui Parseval: $E(\hat{f})=2\pi\cdot E(f)$

Așadar $E(\hat{f})=2\pi\cdot\frac{\pi}{512}=\frac{\pi^2}{256}$

③ Fie $f:\mathbb{R}\to\mathbb{R}$, $f(t)=e^{-4t^2}$.

a) Să se determine $\hat{f}(\omega)$ și $E(f)$.

b) Să se determine $\widehat{f^{(10)}}(\omega)\overset{not}{=}\mathcal{F}(f^{(10)};\omega)$

c) Dacă $g(t)=t^2\cdot f(t)$, atunci $\hat{g}(\omega)=?$

---

a) $\hat{f}(\omega)=\mathcal{F}(f;\omega)=\int_{-\infty}^{\infty}f(t)\cdot e^{-j\omega t}dt=\int_{-\infty}^{\infty}e^{-4t^2}\cdot e^{-j\omega t}dt=\int_{-\infty}^{\infty}e^{-4t^2-j\omega t}dt$

$$=\int_{-\infty}^{\infty}e^{-\left(2t+\frac{\omega j}{4}\right)^2-\frac{\omega^2}{16}}dt \;\underset{2dt=dx}{\overset{2t+\frac{\omega j}{4}=x}{=\!=\!=}}\; \int_{-\infty}^{\infty}e^{-x^2}\cdot e^{-\frac{\omega^2}{16}}\cdot\frac{1}{2}dx$$

$$=\frac{e^{-\frac{\omega^2}{16}}}{2}\cdot\underbrace{\int_{-\infty}^{\infty}e^{-x^2}dx}_{\sqrt{\pi}}=\frac{\sqrt{\pi}}{2}\cdot e^{-\frac{\omega^2}{16}}$$

integrala Euler-Poisson $\left(\int_0^{\infty}e^{-x^2}dx=\frac{\sqrt{\pi}}{2}\right)$

$$E(f)=\int_{-\infty}^{\infty}|f(t)|^2dt=\int_{-\infty}^{\infty}e^{-8t^2}dt \;\underset{2\sqrt{2}\,dt=dx}{\overset{2t\sqrt{2}=x}{=\!=\!=}}\; \int_{-\infty}^{\infty}e^{-x^2}\cdot\frac{1}{2\sqrt{2}}dx=\frac{1}{2\sqrt{2}}\int_{-\infty}^{\infty}e^{-x^2}dx$$

$$=\frac{\sqrt{\pi}}{2\sqrt{2}}$$

b) Obs: Teorema derivării semnalului f

$$\mathcal{F}(f^{(n)};\omega)=(j\omega)^n\cdot\mathcal{F}(f;\omega)$$

adică

$$\widehat{f^{(n)}}(\omega)=(j\omega)^n\cdot\hat{f}(\omega)$$

Teorema derivării spectrului (transformatei)

$$\mathcal{F}(t^n f(t);\omega)=j^n\cdot\left(\mathcal{F}(f(t);\omega)\right)^{(n)}$$

adică

$$\widehat{t^n f(t)}(\omega)=j^n\cdot\left(\hat{f}(\omega)\right)^{(n)}$$

Așadar, folosind Teorema derivării semnalului,

$$\widehat{f^{(10)}}(\omega) = (j\omega)^{10} \cdot \hat{f}(\omega) = -\omega^{10} \cdot \frac{\sqrt{\pi}}{2} \cdot e^{-\frac{\omega^2}{16}}$$

c) $g(t) = t^2 \cdot f(t)$

Teorema derivării spectrului

$$\hat{g}(\omega) = \mathcal{F}(t^2 f(t); \omega) = j^2 \cdot (\hat{f}(\omega))'' = -\left(\frac{\sqrt{\pi}}{2} \cdot e^{-\frac{\omega^2}{16}}\right)'' = -\frac{\sqrt{\pi}}{2} \cdot \left(e^{-\frac{\omega^2}{16}}\right)'' = \ldots$$

Probleme propuse

① Să se rezolve ecuația integrală Fourier:

a) $\int_0^\infty f(t) \cdot \cos \omega t \, dt = \frac{\omega^2}{\omega^4 + 64}, \ \omega > 0$

b) $\int_0^\infty f(t) \cdot \sin \omega t \, dt = \frac{1}{\omega(\omega^2 + 2j)^2}, \ \omega > 0$

② Să se calculeze spectrul Fourier pentru $f: \mathbb{R} \to \mathbb{C}$, $f(t) = t^2 \cdot e^{-\frac{t^2}{4}}$

③ Fie $f \in L^1(\mathbb{R})$, $f(t) = e^{-t^2}$. Să se determine TFi (transformata Fourier integrală) pentru $f(t)$, $g(t) = f^{(5)}(t)$ și $h(t) = t^2 f(t)$.

④ Fie $f: \mathbb{R} \to \mathbb{C}$, $f(t) = (2 - jt)^{-3}$. Să se determine $E(f)$, $\hat{f}$ și $E(\hat{f})$.

Seminarul 10

Transformarea Laplace

Tabel de transformate Laplace

$f(t)$	$\mathcal{L}\{f(t)\}(s)$
e^{at}	$\frac{1}{s-a}$
t^{α}	$\frac{\Gamma(\alpha+1)}{s^{\alpha+1}}$
t^{n}	$\frac{n!}{s^{n+1}}$
1	$\frac{1}{s}$
$\frac{1}{\sqrt{t}}$	$\sqrt{\frac{\pi}{s}}$
$t^{\alpha}\cdot e^{at}$	$\frac{\Gamma(\alpha+1)}{(s-a)^{\alpha+1}}$
$t^{n}\cdot e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
$\cos at$	$\frac{s}{s^2+a^2}$
$\sin at$	$\frac{a}{s^2+a^2}$
$\operatorname{ch} at$	$\frac{s}{s^2-a^2}$
$\operatorname{sh} at$	$\frac{a}{s^2-a^2}$
$t\cdot\cos at$	$\frac{s^2-a^2}{(s^2+a^2)^2}$
$t\cdot\sin at$	$\frac{2as}{(s^2+a^2)^2}$

Formule utile

- Dacă $f\in\mathcal{O}$ (mulțimea funcțiilor original) atunci

$$F(s)=\mathcal{L}\{f(t)\}(s)\overset{def}{=}\int_0^{\infty} f(t)\cdot e^{-st}\,dt$$

s.n. Transformata Laplace a originalului f

- Teorema derivării imaginii

$$\mathcal{L}\{t^n f(t)\}(s)=(-1)^n\cdot\left(\mathcal{L}\{f(t)\}(s)\right)^{(n)}$$

- Teorema integrării imaginii

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\}(s)=\int_s^{\infty}\mathcal{L}\{f(t)\}(y)\,dy$$

- Teorema asemănării

$$\mathcal{L}\{f(at)\}(s)=\frac{1}{a}\cdot\mathcal{L}\{f(t)\}\left(\frac{s}{a}\right),\ \text{pentru } a>0$$

- Transformata Laplace a funcțiilor periodice

Dacă $f\in\mathcal{O}$ și $f\big|_{[0,+\infty)}$ – periodică de perioadă T, atunci

$$\mathcal{L}\{f(t)\}(s)=\frac{1}{1-e^{-sT}}\cdot\int_0^{T} f(t)\cdot e^{-st}\,dt$$

Probleme

① Utilizând definiția și proprietățile transformării Laplace, să se calculeze:

a) $$I=\int_0^{\infty} t\cdot e^{-2t}\cdot\cos^2(3t)\,dt$$

b) $$I=\int_0^{\infty}\frac{3t^2\sqrt{t}+2(\sin^2 t)\cdot e^{t}}{t}\cdot e^{-2t}\,dt$$

---

a) $$I=\int_0^{\infty} t\cdot e^{-2t}\cdot\cos^2(3t)\,dt=\mathcal{L}\{t\cdot\cos^2(3t)\}(2)$$

Dar $\mathcal{L}\{t\cdot\cos^2(3t)\}(s)=(-1)\cdot\left(\mathcal{L}\{\cos^2(3t)\}(s)\right)'$,

T. derivării imaginii $\left(\mathcal{L}\{t^n f(t)\}(s)=(-1)^n\left(\mathcal{L}\{f(t)\}(s)\right)^{(n)}\right)$

unde

$$\mathcal{L}\{\cos^2(3t)\}(s) = \mathcal{L}\left\{\frac{1+\cos 6t}{2}\right\}(s) = \frac{1}{2}\mathcal{L}\{1\}(s) + \frac{1}{2}\mathcal{L}\{\cos 6t\}(s)$$

$$= \frac{1}{2s} + \frac{1}{2}\cdot\frac{s}{s^2+36}$$

$$\left(\mathcal{L}\{\cos^2(3t)\}(s)\right)' = -\frac{1}{2s^2} + \frac{1}{2}\cdot\frac{s^2+36-2s^2}{(s^2+36)^2} = -\frac{1}{2s^2} + \frac{1}{2}\cdot\frac{36-s^2}{(s^2+36)^2}$$

În final,

$$\underline{I} = \mathcal{L}\{t\cdot\cos^2(3t)\}(2) = -\left(-\frac{1}{2s^2} + \frac{1}{2}\cdot\frac{36-s^2}{(s^2+36)^2}\right)\Big|_{s=2}$$

$$= -\left(-\frac{1}{8} + \frac{1}{2}\cdot\frac{36-4}{40\cdot 40}\right) = \frac{1}{8} - \frac{1}{\cancel{2}}\cdot\frac{\cancel{32}^{\cancel{16}^{4}}}{\underset{10}{\cancel{40}}\cdot\underset{10}{\cancel{40}}} = \frac{^{25)}1}{8} - \frac{^{2)}1}{100} = \underline{\underline{\frac{23}{200}}}$$

b) $I = \int_0^\infty \frac{3t^2\sqrt{t} + 2\sin^2 t\, e^t}{t}\cdot e^{-2t}\,dt$

$$= \int_0^\infty 3t\sqrt{t}\cdot e^{-2t}\,dt + 2\int_0^\infty \frac{\sin^2 t}{t}\cdot e^{-t}\,dt$$

$$= \underbrace{3\mathcal{L}\{t\sqrt{t}\}(2)}_{I_1} + \underbrace{2\mathcal{L}\left\{\frac{\sin^2 t}{t}\right\}(1)}_{I_2} = I_1 + I_2, \text{ unde}$$

$\Gamma(p+1) = p\cdot\Gamma(p)$

$$\underline{I_1} = 3\mathcal{L}\{t\sqrt{t}\}(2) = 3\mathcal{L}\{t^{\frac{3}{2}}\}(2) = 3\cdot\frac{\Gamma(\frac{3}{2}+1)}{s^{\frac{3}{2}+1}}\Big|_{s=2} = 3\cdot\frac{\frac{3}{2}\Gamma(\frac{3}{2})}{s^{\frac{5}{2}}}\Big|_{s=2}$$

$$\mathcal{L}\{t^\alpha\}(s) = \frac{\Gamma(\alpha+1)}{s^{\alpha+1}}$$

$\Gamma(\frac{1}{2}) = \sqrt{\pi}$

$$= \frac{9}{2}\cdot\frac{\frac{1}{2}\Gamma(\frac{1}{2})}{s^{\frac{5}{2}}}\Big|_{s=2} = \frac{9}{4}\cdot\frac{\sqrt{\pi}}{s^{\frac{5}{2}}}\Big|_{s=2} = \frac{9\sqrt{\pi}}{4\cdot 2^{\frac{5}{2}}} = \frac{9\sqrt{\pi}}{16\sqrt{2}}$$

$$\underline{\underline{I_2}} = 2\mathcal{L}\left\{\frac{\sin^2 t}{t}\right\}(1) = 2\int_1^\infty \mathcal{L}\{\sin^2 t\}(s)\,ds = 2\int_1^\infty \mathcal{L}\left\{\frac{1-\cos 2t}{2}\right\}(s)\,ds$$

T. integrării imaginii $\left(\mathcal{L}\left\{\frac{f(t)}{t}\right\}(s) = \int_s^\infty \mathcal{L}\{f(t)\}(y)\,dy\right)$

$$= 2\int_1^\infty\left(\frac{1}{2}\mathcal{L}\{1\}(s) - \frac{1}{2}\mathcal{L}\{\cos 2t\}(s)\right)ds$$

$$= 2\int_1^\infty\left(\frac{1}{2s} - \frac{1}{2}\cdot\frac{s}{s^2+4}\right)ds = \cancel{2}\cdot\frac{1}{\cancel{2}}\ln s\Big|_1^\infty - \cancel{2}\cdot\frac{1}{\cancel{2}}\cdot\frac{1}{2}\ln(s^2+4)\Big|_1^\infty$$

$$= \ln\frac{s}{\sqrt{s^2+4}}\Big|_1^\infty = 0 - \ln\frac{1}{\sqrt{5}} = -\ln 1 + \ln\sqrt{5} = \ln\sqrt{5} = \underline{\underline{\frac{1}{2}\ln 5}}$$

În concluzie, $\underline{\underline{I = \frac{9\sqrt{\pi}}{16\sqrt{2}} + \frac{1}{2}\ln 5}}$

② Se consideră funcția

$$f(t) = \sin\sqrt{t},\ t>0$$

a) Să se calculeze $\mathcal{L}\{f(t)\}(s)$

b) $I = \int_0^\infty t\cdot \sin\sqrt{3t}\cdot e^{-2t}dt$

a) Obs: $\sin t = t - \frac{t^3}{3!} + \dots = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{t^{2n+1}}{(2n+1)!}$

Așadar,

$$\sin\sqrt{t} = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{t^{\frac{2n+1}{2}}}{(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{t^{n+\frac{1}{2}}}{(2n+1)!}$$

$$\Rightarrow \mathcal{L}\{\sin\sqrt{t}\}(s) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\cdot \mathcal{L}\{t^{n+\frac{1}{2}}\}(s)$$

Metoda dezvoltării în serie a originalului

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\cdot \frac{\Gamma(n+\frac{3}{2})}{s^{n+\frac{3}{2}}},\text{ unde}$$

$$\mathcal{L}\{t^\alpha\}(s) = \frac{\Gamma(\alpha+1)}{s^{\alpha+1}}$$

$$\Gamma(n+\tfrac{3}{2}) = (n+\tfrac{1}{2})\cdot\Gamma(n+\tfrac{1}{2}) = \dots = (n+\tfrac{1}{2})(n-\tfrac{1}{2})\cdot(n-\tfrac{3}{2})\cdot\dots\cdot\tfrac{1}{2}\cdot\underbrace{\Gamma(\tfrac{1}{2})}_{\text{,,}\sqrt{\pi}}$$

$$\Gamma(p+1) = p\cdot\Gamma(p)$$

$$= \sqrt{\pi}\cdot\frac{1}{2}\cdot\frac{3}{2}\cdot\dots\cdot\frac{2n+1}{2} = \sqrt{\pi}\cdot\frac{(2n+1)!!}{2^{n+1}}$$

$$= \frac{\sqrt{\pi}}{2^{n+1}}\cdot\frac{(2n+1)!}{2\cdot 4\cdot\dots\cdot(2n)} = \frac{\sqrt{\pi}}{2^{n+1}}\cdot\frac{(2n+1)!}{2^n\cdot n!}$$

În final,

$$\mathcal{L}\{\sin\sqrt{t}\}(s) = \sum_{n=0}^{\infty}\frac{(-1)^n}{\cancel{(2n+1)!}}\cdot\frac{\sqrt{\pi}}{2^{n+1}}\cdot\frac{\cancel{(2n+1)!}}{2^n\cdot n!}\cdot\frac{1}{s^n\cdot s\sqrt{s}}$$

$$= \frac{\sqrt{\pi}}{2s\sqrt{s}}\cdot\sum_{n=0}^{\infty}\left(-\frac{1}{4s}\right)^n\cdot\frac{1}{n!} = \frac{\sqrt{\pi}}{2s\sqrt{s}}\cdot e^{-\frac{1}{4s}}$$

$$e^z = \sum_{n=0}^{\infty}\frac{1}{n!}z^n$$

b) $I = \int_0^\infty t\cdot\sin\sqrt{3t}\cdot e^{-2t}dt$

$$= \mathcal{L}\{t\cdot\sin\sqrt{3t}\}(2) = -\left(\mathcal{L}\{\sin\sqrt{3t}\}(s)\right)'\Big|_{s=2}$$

T. derivării imaginii $\left(\mathcal{L}\{t^n f(t)\}(s) = (-1)^n\left(\mathcal{L}\{f(t)\}(s)\right)^{(n)}\right)$

Observăm că $\sin\sqrt{3t} = f(3t)$

Așadar,

$$\mathcal{L}\{\sin\sqrt{3t}\}(s) = \mathcal{L}\{f(3t)\}(s) = \frac{1}{3}\cdot\mathcal{L}\{f(t)\}\left(\frac{s}{3}\right) \overset{a)}{=}$$

T. asemănării $\left(\mathcal{L}\{f(at)\}(s) = \frac{1}{a}\cdot\mathcal{L}\{f(t)\}\left(\frac{s}{a}\right),\ a>0\right)$

$$\overset{a)}{=} \frac{1}{3}\cdot\frac{\sqrt{\pi}}{2\cdot\frac{s}{3}\sqrt{\frac{s}{3}}}\cdot e^{-\frac{3}{4s}}$$

În final,

$$I = -\left(\frac{1}{2}\cdot\frac{\sqrt{3\pi}}{s\sqrt{s}}\cdot e^{-\frac{3}{4s}}\right)'\Bigg|_{s=2} = \ldots$$

③ Să se calculeze $\mathcal{L}\{f(t)\}(s)$, unde $f(t) = \{t\}$, $t>0$

Observăm că $f(t)$ este o funcție periodică de perioadă principală $T=1$

Aplicând formula transformatei Laplace a funcțiilor periodice, obținem

$$\mathcal{L}\{f(t)\}(s) = \frac{1}{1-e^{-sT}}\cdot\int_0^T f(t)\cdot e^{-st}\,dt$$

$$= \frac{1}{1-e^{-s}}\cdot\int_0^1 \{t\}\cdot e^{-st}\,dt = \frac{1}{1-e^{-s}}\cdot\int_0^1 t\cdot e^{-st}\,dt$$

$$= \frac{1}{1-e^{-s}}\cdot\int_0^1\left(\frac{e^{-st}}{-s}\right)'\cdot t\,dt = \frac{1}{1-e^{-s}}\cdot\frac{e^{-st}}{-s}\,t\Bigg|_0^1 + \frac{1}{s(1-e^{-s})}\int_0^1 e^{-st}\,dt$$

$$= -\frac{e^{-s}}{s(1-e^{-s})} + \frac{e^{-st}}{-s^2(1-e^{-s})}\Bigg|_0^1 = -\frac{e^{-s}}{s(1-e^{-s})} - \frac{e^{-s}}{s^2(1-e^{-s})} + \frac{1}{s^2(1-e^{-s})}$$

$$= \frac{1-e^{-s}-se^{-s}}{s^2(1-e^{-s})}$$

④ Se consideră funcția exponențial integral $E_i(t) = \int_t^\infty \frac{e^{-x}}{x}\,dx$, $t>0$

a) Să se calculeze $\mathcal{L}\{E_i(t)\}(s)$

b) $I = \int_0^\infty t\cdot E_i(2t)\cdot e^{-t}\,dt = ?$

a) $$\mathcal{L}\{E_i(t)\}(s) = \int_0^\infty E_i(t)\cdot e^{-st}\,dt = \int_0^\infty\left(\int_t^\infty\frac{e^{-x}}{x}\,dx\right)e^{-st}\,dt$$ Not. $x = tu$, $dx = t\,du$

$$= \int_0^\infty\left(\int_1^\infty\frac{e^{-tu}}{tu}\cdot t\,du\right)\cdot e^{-st}\,dt = \int_0^\infty\left(\int_1^\infty\frac{e^{-tu}}{u}\cdot e^{-st}\,du\right)dt$$

$$= \int_1^\infty\left(\int_0^\infty\frac{e^{-tu}}{u}\cdot e^{-st}\,dt\right)du$$

$f:[a,b]\times[c,d]\to\mathbb{R}$ cont.

Schimbarea ordinii de integrare $\int_a^b\left(\int_c^d f(x,y)\,dy\right)dx = \int_c^d\left(\int_a^b f(x,y)\,dx\right)dy$

$$= \int_1^\infty \frac{1}{u}\,du \underbrace{\int_0^\infty e^{-tu}\cdot e^{-st}\,dt} = \int_1^\infty \frac{1}{u}\cdot \mathcal{L}\{e^{-tu}\}(s)\,du$$

$$= \int_1^\infty \frac{1}{u}\cdot\frac{1}{s+u}\,du = \frac{1}{s}\cdot\int_1^\infty\left(\frac{1}{u}-\frac{1}{s+u}\right)du$$

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$$

$$= \frac{1}{s}\cdot \ln\frac{u}{s+u}\Big|_1^\infty = \frac{1}{s}\cdot\left(\overset{0}{\ln 1} - \ln\frac{1}{s+1}\right) = \underline{\underline{\frac{\ln(s+1)}{s}}}$$

În concluzie,

$$\underline{\underline{\mathcal{L}\{E_i(t)\}(s) = \frac{\ln(s+1)}{s}}}$$

b) $I = \int_0^\infty t\cdot E_i(2t)\cdot e^{-t}\,dt \overset{(*)}{=} \int_0^\infty t\cdot\left(\int_{2t}^\infty \frac{e^{-x}}{x}\,dx\right)e^{-t}\,dt$

$$E_i(t) = \int_t^\infty \frac{e^{-x}}{x}\,dx \longrightarrow E_i(2t) = \int_{2t}^\infty \frac{e^{-x}}{x}\,dx$$

$$\overset{(*)}{=} \mathcal{L}\{t\cdot E_i(2t)\}(1) = -\left(\mathcal{L}\{E_i(2t)\}(s)\right)'\Big|_{s=1}$$

T. derivării imaginii $\left(\mathcal{L}\{t^n f(t)\}(s) = (-1)^n\cdot\left(\mathcal{L}\{f(t)\}(s)\right)^{(n)}\right)$

Dar

$$\left(\mathcal{L}\{E_i(2t)\}(s)\right)' = \frac{1}{2}\left(\mathcal{L}\{E_i(t)\}\left(\frac{s}{2}\right)\right)'$$

T. asemănării $\left(\mathcal{L}\{f(at)\}(s) = \frac{1}{a}\cdot\mathcal{L}\{f(t)\}\left(\frac{s}{a}\right),\ a>0\right)$

$$= +\frac{1}{\cancel{2}}\cdot\left(\frac{\ln\left(\frac{s}{2}+1\right)}{\frac{s}{\cancel{2}}}\right)' = +\frac{\frac{1}{\cancel{2}}\cdot\frac{\cancel{2}}{s+2}\cdot s - \ln\left(\frac{s}{2}+1\right)}{s^2}$$

$$= \frac{\frac{s}{s+2} - \ln\left(\frac{s}{2}+1\right)}{s^2}$$

Așadar,

$$\underline{\underline{I}} = -\left(\frac{\frac{s}{s+2} - \ln\left(\frac{s}{2}+1\right)}{s^2}\right)\Bigg|_{s=1} = -\frac{\frac{1}{3} - \ln\frac{3}{2}}{1} = \underline{\underline{-\frac{1}{3} + \ln\frac{3}{2}}}$$

Probleme propuse

① Se consideră funcția cosinus integral $C_i(t)=\int_t^{\infty}\frac{\cos x}{x}\,dx,\ t>0$

a) Să se demonstreze că $\mathcal{L}\{C_i(t)\}(p)=\frac{\ln(p^2+1)}{2p}$

b) Să se calculeze integrala

$$I=\int_0^{\infty} t\cdot C_i(2t)\cdot e^{-t}\,dt$$

② Să se calculeze $\mathcal{L}\{\sin^2(at)\cdot\sin(bt)\}(p)$ și apoi $\int_0^{\infty}\frac{\sin^2(3t)\cdot\sin 6t}{t}\,dt$.

③ Să se calculeze transformata Laplace pentru fiecare din funcțiile de mai jos:

a) $f(t)=e^{-5t}\cos^2 3t$

b) $f(t)=\dfrac{t^2\,\mathrm{sh}\,2t+2\sin^2 3t}{t}$

c) $f(t)=t\cdot e^{-2t}\cdot\cos^2 2t$

d) $f(t)=\dfrac{e^{-t}\cdot\sin^3 2t}{t}$

Seminarul 11

Transformarea Laplace

Tabel de transformate Laplace

- $\mathcal{L}\{e^{at}\}(s) = \frac{1}{s-a} \iff \mathcal{L}^{-1}\{\frac{1}{s-a}\}(t) = e^{at}\cdot u(t)$
- $\mathcal{L}\{t^n\}(s) = \frac{n!}{s^{n+1}} \iff \mathcal{L}^{-1}\{\frac{1}{s^n}\}(t) = \frac{t^{n-1}}{(n-1)!}\cdot u(t)$
- $\mathcal{L}\{t^n\cdot e^{at}\}(s) = \frac{n!}{(s-a)^{n+1}} \iff \mathcal{L}^{-1}\{\frac{1}{(s-a)^n}\}(t) = \frac{t^{n-1}}{(n-1)!}e^{at}\cdot u(t)$
- $\mathcal{L}\{\cos at\}(s) = \frac{s}{s^2+a^2} \iff \mathcal{L}^{-1}\{\frac{s}{s^2+a^2}\}(t) = \cos at\cdot u(t)$
- $\mathcal{L}\{\sin at\}(s) = \frac{a}{s^2+a^2} \iff \mathcal{L}^{-1}\{\frac{1}{s^2+a^2}\}(t) = \frac{1}{a}\cdot\sin at\cdot u(t)$

① Să se demonstreze egalitatea:

$$I = \int_0^\infty \left(\frac{\sin x}{x}\right)^2 dx = \frac{\pi}{2}$$

Fie $I(t) = \int_0^\infty \frac{\sin^2 tx}{x^2}dx \Longrightarrow I = I(1)$

Aplicând transformarea Laplace, obținem

$$\mathcal{L}\{I(t)\}(s) = \mathcal{L}\left\{\int_0^\infty \frac{\sin^2 tx}{x^2}dx\right\}(s) = \int_0^\infty\left(\int_0^\infty \frac{\sin^2 tx}{x^2}dx\right)e^{-st}dt$$

$$= \int_0^\infty\left(\int_0^\infty \frac{\sin^2 tx}{x^2}\cdot e^{-st}dx\right)dt = \int_0^\infty\left(\int_0^\infty \frac{\sin^2 tx}{x^2}\cdot e^{-st}dt\right)dx$$

$$\int_a^b\left(\int_c^d f(x,y)dy\right)dx = \int_c^d\left(\int_a^b f(x,y)dx\right)dy$$

$$= \int_0^\infty \frac{1}{x^2}\left(\int_0^\infty \sin^2 tx\cdot e^{-st}dt\right)dx = \int_0^\infty \frac{1}{x^2}\cdot\mathcal{L}\{\sin^2 tx\}(s)dx$$

$$= \int_0^\infty \frac{1}{x^2}\cdot\mathcal{L}\left\{\frac{1-\cos 2tx}{2}\right\}(s)dx = \int_0^\infty \frac{1}{x^2}\cdot\left(\frac{1}{2s} - \frac{1}{2}\cdot\frac{s}{s^2+4x^2}\right)dx$$

$$\mathcal{L}\{\cos 2tx\}(s) = \frac{s}{s^2+4x^2}$$

$$= \frac{1}{2}\int_0^\infty\left(\frac{1}{sx^2} - \frac{s}{x^2(s^2+4x^2)}\right)dx$$

$$= \frac{1}{2}\int_0^\infty \left(\cancel{\frac{1}{s x^2}} - \frac{1}{s}\left(\cancel{\frac{1}{x^2}} - \frac{4}{s^2+4x^2}\right)\right) dx = \frac{1}{2}\int_0^\infty \frac{1}{s}\cdot\frac{4}{s^2+4x^2}\,dx$$

$$= \frac{2}{s}\int_0^\infty \frac{1}{s^2+4x^2}\,dx = \frac{\cancel{2}}{s}\int_0^\infty \frac{1}{\cancel{4}_{\,2}\left(x^2+\frac{s^2}{4}\right)}\,dx = \frac{1}{\cancel{2}s}\cdot\frac{\cancel{2}}{s}\cdot \operatorname{arctg}\frac{2x}{s}\Big|_0^\infty$$

$$= \frac{\pi}{2s^2}$$

Am obținut $\mathcal{L}\{J(t)\}(s) = \frac{\pi}{2s^2} \Rightarrow J(t) = \mathcal{L}^{-1}\left\{\frac{\pi}{2s^2}\right\}(t) = \frac{\pi}{2}\cdot t \Rightarrow J = J(1) = \frac{\pi}{2}$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^n}\right\}(t) = \frac{t^{n-1}}{(n-1)!}\cdot u(t)$$

(2) Utilizând metoda (tehnica transformării Laplace) să se rezolve ec.

$$x'''(t) - 2x''(t) + 4x'(t) - 8x(t) = e^{-t}, \quad x(0) = x'(0) = 0,\ x''(0) = 2$$

Etapa 1 Determinăm transformata Laplace a soluției, $X(s) \overset{not}{=} \mathcal{L}\{x(t)\}(s)$

Etapa 2 Determinăm $x(t) = \mathcal{L}^{-1}\{X(s)\}(t)$

Etapa 1 $\quad x'''(t) - 2x''(t) + 4x'(t) - 8x(t) = e^{-t} \;\Big/\; \mathcal{L}\{\ \ \}(s)$

$$\iff \mathcal{L}\{x'''(t)\}(s) - 2\mathcal{L}\{x''(t)\}(s) + 4\mathcal{L}\{x'(t)\}(s) - 8\underbrace{\mathcal{L}\{x(t)\}(s)}_{X(s)} = \mathcal{L}\{e^{-t}\}(s) \quad (*)$$

Obs:
$\mathcal{L}\{x'(t)\}(s) = sX(s) - \overset{0}{x(0)} \quad |\cdot s$

$\mathcal{L}\{x''(t)\}(s) = s^2X(s) - s\overset{0}{x(0)} - \overset{0}{x'(0)} \quad |\cdot s$

$\mathcal{L}\{x'''(t)\}(s) = s^3X(s) - s^2x(0) - sx'(0) - x''(0) = s^3X(s) - 2$

Relația $(*) \iff s^3X(s) - 2 - 2s^2X(s) + 4sX(s) - 8X(s) = \frac{1}{s+1}$

$$\mathcal{L}\{e^{at}\}(s) = \frac{1}{s-a}$$

$$\iff X(s)(s^3 - 2s^2 + 4s - 8) = \frac{1}{s+1} + \overset{s+1}{2}$$

$$\iff X(s)(s^3 - 2s^2 + 4s - 8) = \frac{2s+3}{s+1} \Rightarrow X(s) = \frac{2s+3}{(s+1)(s^3-2s^2+4s-8)}$$

Etapa 2

$$x(t) = \mathcal{L}^{-1}\{X(s)\}(t) = \mathcal{L}^{-1}\left\{\frac{2s+3}{(s+1)(s^3-2s^2+4s-8)}\right\}(t)$$

$$s^3 - 2s^2 + 4s - 8 = s^2(s-2) + 4(s-2) = (s-2)(s^2+4)$$

$$\frac{2s+3}{(s+1)(s^3-2s^2+4s-8)} = \frac{2s+3}{(s+1)(s-2)(s^2+4)} = \frac{\overset{(s-2)(s^2+4)}{A}}{s+1} + \frac{\overset{(s+1)(s^2+4)}{B}}{s-2} + \frac{\overset{(s+1)(s-2)}{Cs+D}}{s^2+4} =$$

$$= \frac{A(s-2)(s^2+4) + B(s+1)(s^2+4) + (Cs+D)(s+1)(s-2)}{(s+1)(s-2)(s^2+4)}$$

$$\rightarrow 2s+3 = A(s-2)(s^2+4) + B(s+1)(s^2+4) + (Cs+D)(s+1)(s-2), \quad \forall s \in \mathbb{R}$$

Dăm valori lui s. Pentru $s=-1 \rightarrow 1 = -15A \Rightarrow A = -\frac{1}{15}$

$$s = 2 \rightarrow 24B = 7 \Rightarrow B = \frac{7}{24}$$

Identificăm coef. lui s^3: $A+B+C=0 \rightarrow -\overset{8/}{\frac{1}{15}} + \overset{5/}{\frac{7}{24}} + C = 0$

$$\rightarrow C = -\frac{27^{(3}}{120} = -\frac{9}{40}$$

Pentru $s=0$ (identificăm termenii liberi)

$$3 = -8A + 4B - 2D$$

$$\rightarrow 3 = -8\cdot\left(-\frac{1}{15}\right) + 4\cdot\frac{7}{24} - 2D$$

$$\rightarrow 2D = \overset{2/}{\frac{8}{15}} + \overset{5/}{\frac{7}{6}} - \overset{30/}{3} \rightarrow 2D = \frac{16+35-90}{30} = -\frac{13}{10}$$

$$\rightarrow D = -\frac{13}{20}$$

Obținem $X(s) = -\frac{1}{15}\cdot\frac{1}{s+1} + \frac{7}{24}\cdot\frac{1}{s-2} + \frac{-\frac{9}{40}s - \frac{13}{20}}{s^2+4}$

$$= -\frac{1}{15}\cdot\frac{1}{s+1} + \frac{7}{24}\cdot\frac{1}{s-2} - \frac{9}{40}\cdot\frac{s}{s^2+4} - \frac{13}{20}\cdot\frac{1}{s^2+4} \quad \Big| \mathcal{L}^{-1}\{\ \}(t)$$

$$\rightarrow \underline{x(t)} = \mathcal{L}^{-1}\{X(s)\}(t) = -\frac{1}{15}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}(t) + \frac{7}{24}\cdot\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\}(t) - \frac{9}{40}\cdot\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\}(t) - \frac{13}{20}\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\}(t)$$

$$= \left[-\frac{1}{15}\cdot e^{-t} + \frac{7}{24}\cdot e^{2t} - \frac{9}{40}\cdot\cos 2t - \frac{13}{40}\cdot\sin 2t\right]\cdot u(t)$$

③ Să se rezolve ecuația diferențială

$$t\cdot x''(t) = (2t-1)\cdot x'(t) - 2x(t), \quad x(0)=1,\ x'(0)=2$$

Etapa 1 Determinăm transformata Laplace a soluției, adică $X(s) \overset{not}{=} \mathcal{L}\{x(t)\}(s)$

$$t\cdot x''(t) = (2t-1)\cdot x'(t) - 2x(t) \quad \Big| \mathcal{L}\{\ \}(s)$$

$$\Rightarrow \mathcal{L}\{t\cdot x''(t)\}(s) = \mathcal{L}\{(2t-1)\cdot x'(t)\}(s) - 2\underbrace{\mathcal{L}\{x(t)\}(s)}_{X(s)} \qquad (*)$$

Teorema derivării imaginii

$$\mathcal{Z}\{t^n f(t)\}(s) = (-1)^n \cdot \left(\mathcal{Z}\{f(t)\}(s)\right)^{(n)}$$

Relația (*) $\Leftrightarrow (-1)\cdot\left(\mathcal{Z}\{x''(t)\}(s)\right)' = 2\,\mathcal{Z}\{t\,x'(t)\}(s) - \mathcal{Z}\{x'(t)\}(s) - 2X(s)$ (**)

Obs $\mathcal{Z}\{x'(t)\}(s) = sX(s) - x(0) \,|\, s = sX(s) - 1$

$$\mathcal{Z}\{x''(t)\}(s) = s^2X(s) - sx(0) - x'(0) = s^2X(s) - s - 2$$

Relația (**) $\Leftrightarrow -\left(s^2X(s) - s - 2\right)' = -2\cdot\left(sX(s) - 1\right)' - sX(s) + 1 - 2X(s)$

T. derivării imaginii

$$\Leftrightarrow -2sX(s) - s^2X'(s) + \cancel{1} = -2X(s) - 2sX'(s) - sX(s) + \cancel{1} - 2X(s)$$

$$\Leftrightarrow X(s)(4-s) = X'(s)(s^2-2s)$$

$$\to X'(s) = \frac{4-s}{s^2-2s}X(s) \to \frac{dX}{ds} = \frac{4-s}{s^2-2s}X \to \frac{dX}{X} = \frac{4-s}{s^2-2s}ds \,\Big|\int \quad (***)$$

Obs: $\dfrac{4-s}{s^2-2s} = \dfrac{4-s}{s(s-2)} = \dfrac{\overset{s-2)}{A}}{s} + \dfrac{\overset{s)}{B}}{s-2} = \dfrac{As-2A+Bs}{s(s-2)} = \dfrac{(A+B)s-2A}{s(s-2)}$

$$\to \begin{cases} A+B=-1 \to \underline{B=1} \\ -2A=4 \to \underline{A=-2} \end{cases}$$

$$\to \frac{4-s}{s^2-2s} = -\frac{2}{s} + \frac{1}{s-2}$$

Integrând relația (***), obținem

$$\ln X = -2\ln s + \ln(s-2) + \ln C$$

$$= \ln\frac{C(s-2)}{s^2} \to X(s) = \frac{C(s-2)}{s^2}$$

Conform Teoremei valorii inițiale, $sX(s) \xrightarrow[s\to\infty]{} x(0)$,

adică $\lim\limits_{s\to\infty} \cancel{s}\cdot\dfrac{C(s-2)}{s^{\cancel{2}}} = 1 \Rightarrow \underline{C=1}$

Obținem așadar $\underline{X(s) = \dfrac{s-2}{s^2}} \to \underline{x(t)} = \mathcal{Z}^{-1}\left\{\dfrac{s-2}{s^2}\right\}(t)$

$$= \mathcal{Z}^{-1}\left\{\frac{1}{s}\right\}(t) - 2\mathcal{Z}^{-1}\left\{\frac{1}{s^2}\right\}(t) = \underline{(1-2t)\cdot u(t)}$$

Verificare $t\cdot 0 = (2t-1)\cdot(-2) - 2(1-2t)$ | $x'(t) = -2$

$\Leftrightarrow \cancel{-4t} + \cancel{2} - \cancel{2} + \cancel{4t} = 0$ ✓ | $x''(t) = 0$

(4) Utilizând metoda transformării Laplace, să se rez. ec.

$$x''(t) + x(t) = \frac{1}{\cos t}, \quad x(0) = 1, \ x'(0) = 2$$

$$x''(t) + x(t) = \frac{1}{\cos t} \quad \Big| \ \mathcal{L}\{\ \}$$

$$\Rightarrow \mathcal{L}\{x''(t)\}(s) + \underbrace{\mathcal{L}\{x(t)\}(s)}_{X(s)} = \mathcal{L}\left\{\frac{1}{\cos t}\right\}(s) \quad (*)$$

Obs

$$\mathcal{L}\{x'(t)\}(s) = sX(s) - x(0) \ |\cdot s = sX(s) - 1$$

$$\mathcal{L}\{x''(t)\}(s) = s^2X(s) - s\,x(0) - x'(0) = s^2X(s) - s - 2$$

Relația $(*) \Leftrightarrow s^2X(s) - s - 2 + X(s) = \mathcal{L}\left\{\frac{1}{\cos t}\right\}(s)$

$$\Leftrightarrow X(s)(s^2+1) = \mathcal{L}\left\{\frac{1}{\cos t}\right\}(s) + s + 2$$

$$\Leftrightarrow X(s) = \frac{1}{s^2+1}\cdot\mathcal{L}\left\{\frac{1}{\cos t}\right\}(s) + \frac{s+2}{s^2+1} \quad \Big| \ \mathcal{L}^{-1}\{\ \}(t)$$

Obținem $x(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\cdot\mathcal{L}\left\{\frac{1}{\cos t}\right\}(s)\right\}(t) + \mathcal{L}^{-1}\left\{\frac{s+2}{s^2+1}\right\}(t)$

$$= \mathcal{L}^{-1}\left\{\mathcal{L}\{\sin t\}(s)\cdot\mathcal{L}\left\{\frac{1}{\cos t}\right\}(s)\right\}(t) + \mathcal{L}^{-1}\left\{\frac{s+2}{s^2+1}\right\}(t)$$

$$\mathcal{L}\{\sin t\}(s) = \frac{1}{s^2+1}$$

$$= \mathcal{L}^{-1}\left\{\mathcal{L}\left\{\frac{1}{\cos t} * \sin t\right\}(s)\right\}(t) + \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\}(t) + 2\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}(t) \quad (1)$$

$$\mathcal{L}\{(f*g)(t)\}(s) = \mathcal{L}\{f(t)\}(s)\cdot\mathcal{L}\{g(t)\}(s) = \mathcal{L}\{(g*f)(t)\}(s)$$

Obs: Prin definiție, $(f * g)(t) = \int_0^t f(x)\cdot g(t-x)\,dx, \ t \geq 0$

produsul de convoluție

Așadar, relația $(1) \Leftrightarrow x(t) = \frac{1}{\cos t} * \sin t + \cos t + 2\cdot\sin t$

Dar $\frac{1}{\cos t} * \sin t = \int_0^t \frac{1}{\cos x}\cdot\sin(t-x)\,dx = \int_0^t \frac{1}{\cos x}\cdot(\sin t\cdot\cos x - \sin x\cdot\cos t)\,dx$

$$= \int_0^t \sin t\,dx - \cos t\cdot\int_0^t \frac{\sin x}{\cos x}\,dx = t\sin t + \cos t\cdot\ln|\cos t| \quad (2)$$

Din (1) și (2) obținem $x(t) = t\sin t + \cos t\cdot\ln|\cos t| + \cos t + 2\sin t$

⑤ Să se rezolve ecuația integro-diferențială de tip convolutiv

$$x''(t)+2\cdot\int_0^t \sin(\tau-t)\cdot x'(\tau)\,d\tau+2x'(t)=\cos t,\ (*)\quad x(0)=2,\ x'(0)=3$$

Obs $(f*g)(t)=\int_0^t f(x)\cdot g(t-x)\,dx=\int_0^t f(\tau)\cdot g(t-\tau)\,d\tau$

Relația $(*) \iff x''(t)-2\int_0^t x'(\tau)\cdot\sin(t-\tau)\,d\tau+2x'(t)=\cos t$

$\iff x''(t)-2x'(t)*\sin t+2x'(t)=\cos t \mid \mathcal{L}\{\ \}(s)$

$\rightarrow \mathcal{L}\{x''(t)\}(s)-2\mathcal{L}\{x'(t)*\sin t\}(s)+2\mathcal{L}\{x'(t)\}(s)=\frac{s}{s^2+1}$

$\iff \mathcal{L}\{x''(t)\}(s)-2\mathcal{L}\{x'(t)\}(s)\cdot\mathcal{L}\{\sin t\}(s)+2\mathcal{L}\{x'(t)\}(s)=\frac{s}{s^2+1}\quad (**)$

Obs: $\mathcal{L}\{x'(t)\}(s)=sX(s)-x(0)=sX(s)-2$

$\mathcal{L}\{x''(t)\}(s)=s^2X(s)-2s-x'(0)=s^2X(s)-2s-3$

$(**) \iff s^2X(s)-2s-3-2(sX(s)-2)\cdot\frac{1}{s^2+1}+2(sX(s)-2)=\frac{s}{s^2+1}$

$\iff X(s)\left(s^2-2s\cdot\frac{1}{s^2+1}+2s\right)=\frac{s}{s^2+1}+2s+3-\frac{4}{s^2+1}+4$

$\iff X(s)\cdot\frac{s^4+s^2-2s+2s^3+2s}{s^2+1}=\frac{s+2s^3+2s+7s^2+7-4}{s^2+1}$

$\iff X(s)\cdot\frac{s^4+2s^3+s^2}{s^2+1}=\frac{2s^3+7s^2+3s+3}{s^2+1}$

$\rightarrow X(s)=\frac{2s^3+7s^2+3s+3}{s^2(s+1)^2}=\frac{A}{s}+\frac{B}{s^2}+\frac{C}{s+1}+\frac{D}{(s+1)^2}$

$=\frac{As(s+1)^2+B(s+1)^2+Cs^2(s+1)+Ds^2}{s^2(s+1)^2}$

$\rightarrow 2s^3+7s^2+3s+3=As(s+1)^2+B(s+1)^2+Cs^2(s+1)+Ds^2\quad \forall s\in\mathbb{R}$

Pentru $s=0\rightarrow \underline{B=3}$

$s=-1\rightarrow \underline{D=-2+7-3+3=5}$

Coef. lui s^3: $2=A+C$

s^2: $7=2A+B+C+D\rightarrow 7=2A+3+C+5\rightarrow 2A+C=-1$ $\}\rightarrow$

$\rightarrow \underline{A=-3}$ și $\underline{C=5}$

Obținem $X(s)=-\frac{3}{s}+\frac{3}{s^2}+\frac{5}{s+1}+\frac{5}{(s+1)^2}\mid \mathcal{L}^{-1}\{\ \}(t)$

$\rightarrow \underline{x(t)}=\mathcal{L}^{-1}\{X(s)\}(t)=\underline{(-3+3t+5\cdot e^{-t}+5t\cdot e^{-t})\cdot u(t)}$

⑥ Utilizând metoda transformării Laplace, să se rezolve următorul sistem de ec. dif. cu condiții inițiale date:

$$\begin{cases} x'(t) + 2y(t) = x(t) \\ x''(t) + \cos t = 2(t - y'(t)) \end{cases}, \quad x(0)=0,\ x'(0)=-1,\ y(0)=\frac{1}{2}$$

Fie $X(s) = \mathcal{L}\{x(t)\}(s)$ și $Y(s) = \mathcal{L}\{y(t)\}(s)$

Sistemul operatorial devine

$$\begin{cases} \mathcal{L}\{x'(t)\}(s) + 2Y(s) = X(s) \\ \mathcal{L}\{x''(t)\}(s) + \mathcal{L}\{\cos t\}(s) = 2\mathcal{L}\{t\}(s) - 2\mathcal{L}\{y'(t)\}(s) \end{cases} \quad (1)$$

Obs: $\mathcal{L}\{x'(t)\}(s) = sX(s) - x(0) = sX(s)$ | $\mathcal{L}\{y'(t)\}(s) = sY(s) - y(0) = sY(s) - \frac{1}{2}$

$\mathcal{L}\{x''(t)\}(s) = s^2X(s) - s\,x(0) - x'(0) = s^2X(s) + 1$

Ținând cont de obs., sistemul (1) devine

$$\begin{cases} sX(s) + 2Y(s) = X(s) \\ s^2X(s) + \cancel{1} + \dfrac{s}{s^2+1} = \dfrac{2}{s^2} - 2\left(sY(s) - \cancel{\tfrac{1}{2}}\right) \end{cases}$$

$$\Leftrightarrow \begin{cases} Y(s) = X(s)\cdot\dfrac{1-s}{2} \\ s^2X(s) + \dfrac{s}{s^2+1} = \dfrac{2}{s^2} - 2sY(s) \end{cases} \Rightarrow s^2X(s) + \frac{s}{s^2+1} = \frac{2}{s^2} - \cancel{2}s\cdot X(s)\cdot\frac{1-s}{\cancel{2}}$$

$$\Rightarrow X(s)(\cancel{s^2} + s - \cancel{s^2}) = \frac{2}{s^2} - \frac{s}{s^2+1} \Rightarrow X(s) = \frac{2}{s^3} - \frac{1}{s^2+1}$$

$$\Rightarrow \underline{\underline{x(t)}} = \mathcal{L}^{-1}\{X(s)\}(t) = \underline{\underline{(t^2 - \sin t)\cdot u(t)}}$$

$$Y(s) = \left(\frac{2}{s^3} - \frac{1}{s^2+1}\right)\cdot\frac{1-s}{2} = \left(\frac{2}{s^3} - \frac{1}{s^2+1}\right)\left(\frac{1}{2} - \frac{s}{2}\right) = \frac{1}{s^3} - \frac{1}{s^2} - \frac{1}{2}\cdot\frac{1}{s^2+1} + \frac{1}{2}\cdot\frac{s}{s^2+1}$$

$$\Rightarrow \underline{\underline{y(t)}} = \mathcal{L}^{-1}\{Y(s)\}(t) = \underline{\underline{\left(\frac{t^2}{2} - t - \frac{1}{2}\sin t + \frac{1}{2}\cos t\right)\cdot u(t)}}$$

Probleme propuse

① Să se determine $\mathcal{L}\{\sin^3 t\}(s)$ și să se calculeze integrala $\int_0^\infty \left(\frac{\sin x}{x}\right)^3 dx$

② Să se determine $\mathcal{L}\{\cos^3 t\}(s)$ și să se calculeze integrala $\int_0^\infty \frac{\cos^3 2x}{x^2+1}\, dx$

③ Să se arate că $\int_0^\infty \frac{\cos 2x}{1+x^2}\, dx = \frac{\pi}{2e^2}$

④ Folosind metoda transformării Laplace, să se rezolve ecuația diferențială:

$$t\,x''(t) + 2x'(t) + t\,x(t) = -2\cos t, \qquad x(0)=0,\ x'(0)=1$$

⑤ Să se rezolve ecuația diferențială

$$x''(t) + 4x'(t) + 4x(t) = \frac{e^{-2t}}{t+2}, \qquad x(0)=0,\ x'(0)=2$$

Seminarul 12

Transformarea Laplace

(1) Se consideră funcția cosinus integral $C_i(t) = \int_t^\infty \frac{\cos x}{x}\,dx$, $t>0$.

a) Să se demonstreze că $\mathcal{L}\{C_i(t)\}(s) = \frac{\ln(s^2+1)}{2s}$

b) Să se calculeze integrala

$$J = \int_0^\infty t\cdot C_i(2t)\cdot e^{-t}\,dt$$

a) $\mathcal{L}\{C_i(t)\}(s) = \int_0^\infty C_i(t)\cdot e^{-st}\,dt = \int_0^\infty \left(\int_t^\infty \frac{\cos x}{x}\,dx\right)\cdot e^{-st}\,dt$ Not. $x = tu$, $dx = t\,du$

$$= \int_0^\infty \left(\int_1^\infty \frac{\cos tu}{\cancel{t}u}\,\cancel{t}\,du\right)\cdot e^{-st}\,dt = \int_0^\infty \left(\int_1^\infty \frac{\cos tu}{u}\cdot e^{-st}\,du\right)dt$$

$$= \int_1^\infty \left(\int_0^\infty \frac{\cos tu}{u}\cdot e^{-st}\,dt\right)du$$

Schimbarea ordinii de integrare $\int_a^b\left(\int_c^d f(x,y)\,dy\right)dx = \int_c^d\left(\int_a^b f(x,y)\,dx\right)dy$

$$= \int_1^\infty \frac{1}{u}\,du \int_0^\infty \cos tu\cdot e^{-st}\,dt = \int_1^\infty \frac{1}{u}\cdot \mathcal{L}\{\cos tu\}(s)\,du$$

$$= \int_1^\infty \frac{1}{u}\cdot\frac{s}{s^2+u^2}\,du = \frac{1}{s}\cdot\int_1^\infty\left(\frac{1}{u} - \frac{u}{u^2+s^2}\right)du = \frac{1}{s}\cdot\left(\ln u - \frac{1}{2}\ln(u^2+s^2)\right)\Big|_1^\infty$$

$$= \frac{1}{s}\cdot\ln\frac{u}{\sqrt{u^2+s^2}}\Big|_1^\infty = \frac{1}{s}\cdot\left(\ln 1 - \ln\frac{1}{\sqrt{s^2+1}}\right) = \frac{\ln\sqrt{s^2+1}}{s} = \frac{\ln(s^2+1)}{2s}$$

b) $J = \int_0^\infty t\cdot C_i(2t)\cdot e^{-t}\,dt = \mathcal{L}\{t\cdot C_i(2t)\}(1) = -\left(\mathcal{L}\{C_i(2t)\}(s)\right)'\Big|_{s=1}$

T. derivării imaginii $\left(\mathcal{L}\{t^n f(t)\}(s) = (-1)^n\left(\mathcal{L}\{f(t)\}(s)\right)^{(n)}\right)$

Dar $\mathcal{L}\{C_i(2t)\}(s) = \frac{1}{2}\mathcal{L}\{C_i(t)\}\left(\frac{s}{2}\right) = \frac{1}{2}\cdot\frac{\ln\left(\frac{s^2}{4}+1\right)}{2\cdot\frac{s}{2}} = \frac{\ln\left(\frac{s^2+4}{4}\right)}{2s}$

T. asemănării $\left(\mathcal{L}\{f(at)\}(s) = \frac{1}{a}\mathcal{L}\{f(t)\}\left(\frac{s}{a}\right), a>0\right)$

Așadar,

$$J = -\left(\frac{\ln\left(\frac{s^2+4}{4}\right)}{2s}\right)'\Bigg|_{s=1} = -\frac{\frac{\cancel{2}s}{\cancel{4}}\cdot\frac{4}{s^2+4}\cdot 2s - 2\cdot\ln\left(\frac{s^2+4}{4}\right)}{4s^2}\Bigg|_{s=1}$$

$$= -\frac{\frac{4}{5} - 2\ln\frac{5}{4}}{4} = -\frac{1}{5} + \frac{1}{2}\ln\frac{5}{4}$$

② Să se calculeze $\mathcal{L}\{\sin^2(at)\cdot\sin(bt)\}(s)$ şi apoi $\int_0^\infty \frac{\sin^2(3t)\cdot\sin 6t}{t}\,dt$

• $\mathcal{L}\{\sin^2(at)\cdot\sin(bt)\}(s) = \mathcal{L}\{\frac{1-\cos(2at)}{2}\cdot\sin(bt)\}(s)$

$$= \frac{1}{2}\mathcal{L}\{\sin(bt)\}(s) - \frac{1}{2}\mathcal{L}\{\cos(\overbrace{2at}^{\beta})\cdot\sin(\overbrace{bt}^{\alpha})\}(s)$$

$$= \frac{1}{2}\cdot\frac{b}{s^2+b^2} - \frac{1}{2}\mathcal{L}\{\frac{1}{2}(\sin(2a+b)t + \sin(b-2a)t)\}(s)$$

$\sin\alpha\cdot\cos\beta = \frac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha-\beta)]$

$$= \frac{1}{2}\cdot\frac{b}{s^2+b^2} - \frac{1}{4}\mathcal{L}\{\sin(2a+b)t\}(s) + \frac{1}{4}\mathcal{L}\{\sin(2a-b)t\}(s)$$

$$= \frac{1}{2}\cdot\frac{b}{s^2+b^2} - \frac{1}{4}\cdot\frac{2a+b}{s^2+(2a+b)^2} + \frac{1}{4}\cdot\frac{2a-b}{s^2+(2a-b)^2}$$

$\mathcal{L}\{\sin at\}(s) = \frac{a}{s^2+a^2}$

• $\int_0^\infty \frac{\sin^2(3t)\cdot\sin 6t}{t}\,dt = \mathcal{L}\{\frac{\sin^2(3t)\cdot\sin 6t}{t}\}(0) = \int_0^\infty \mathcal{L}\{\sin^2(3t)\cdot\sin 6t\}(s)\,ds$

T. integrării imaginii $\mathcal{L}\{\frac{f(t)}{t}\}(s) = \int_s^\infty \mathcal{L}\{f(t)\}(y)\,dy$

$$= \int_0^\infty \left(\frac{1}{\cancel{2}}\cdot\frac{\overset{3}{\cancel{6}}}{s^2+36} - \frac{1}{\cancel{4}}\cdot\frac{\overset{3}{\cancel{12}}}{s^2+12^2} + \frac{1}{4}\cdot 0\right)ds$$

$a=3,\ b=6$

$$= \frac{\cancel{3}}{\underset{2}{\cancel{6}}}\cdot\operatorname{arctg}\frac{s}{6}\Big|_0^\infty - \frac{\cancel{3}}{\underset{4}{\cancel{12}}}\cdot\operatorname{arctg}\frac{s}{12}\Big|_0^\infty = \frac{\pi}{4} - \frac{\pi}{8} = \frac{\pi}{8}$$

③ Să se arate că $\int_0^\infty \frac{\cos 2x}{x^2+1}\,dx = \frac{\pi}{2e^2}$

Fie $\mathcal{J}(t) = \int_0^\infty \frac{\cos 2tx}{x^2+1}\,dx \rightarrow \mathcal{J} = \int_0^\infty \frac{\cos 2x}{x^2+1}\,dx = \mathcal{J}(1)$

Aplicând în continuare transformarea Laplace, obţinem

$$\mathcal{L}\{\mathcal{J}(t)\}(s) = \mathcal{L}\left\{\int_0^\infty \frac{\cos 2tx}{x^2+1}\,dx\right\}(s) = \int_0^\infty \left(\int_0^\infty \frac{\cos 2tx}{x^2+1}\,dx\right)\cdot e^{-st}\,dt$$

$$= \int_0^\infty \left(\int_0^\infty \frac{\cos 2tx}{x^2+1}\cdot e^{-st}\,dx\right)dt =$$

$\int_a^b\left(\int_c^d f(x,y)\,dy\right)dx = \int_c^d\left(\int_a^b f(x,y)\,dx\right)dy$

$$= \int_0^\infty \left(\int_0^\infty \frac{\cos 2tx}{x^2+1}\cdot e^{-st}\,dt\right)dx$$

$$=\int_0^\infty \frac{1}{x^2+1}\left(\int_0^\infty \cos 2tx \cdot e^{-st}\,dt\right)dx = \int_0^\infty \frac{1}{x^2+1}\cdot \mathcal{L}\{\cos 2tx\}(s)\,dx$$

$$=\int_0^\infty \frac{1}{x^2+1}\cdot\frac{s}{s^2+4x^2}\,dx = \frac{s}{s^2-4}\cdot\int_0^\infty\left(\frac{1}{x^2+1}-\frac{4}{4x^2+s^2}\right)dx$$

$$\mathcal{L}\{\cos at\}(s)=\frac{s}{s^2+a^2} \qquad \frac{1}{(x^2+1)\cdot(4x^2+s^2)} = \frac{1}{s^2-4}\cdot\frac{1}{x^2+1} - \frac{4}{s^2-4}\cdot\frac{1}{4x^2+s^2}$$

$$=\frac{s}{s^2-4}\cdot\int_0^\infty \frac{1}{x^2+1}\,dx - \frac{s}{s^2-4}\cdot\int_0^\infty\frac{1}{x^2+\left(\frac{s}{2}\right)^2}\,dx$$

$$=\frac{s}{s^2-4}\cdot \operatorname{arctg} x\Big|_0^\infty - \frac{\cancel{s}}{s^2-4}\cdot\frac{2}{\cancel{s}}\cdot\operatorname{arctg}\frac{2x}{s}\Big|_0^\infty$$

$$=\frac{s}{s^2-4}\cdot\frac{\pi}{2}-\frac{2}{s^2-4}\cdot\frac{\pi}{2}=\frac{s-2}{s^2-4}\cdot\frac{\pi}{2}=\frac{\pi}{2(s+2)}$$

Așadar, $\mathcal{L}\{I(t)\}(s)=\frac{\pi}{2(s+2)} \ \Big|\ \mathcal{L}^{-1}\{\ \}(t)$

$$\Rightarrow I(t)=\mathcal{L}^{-1}\left\{\frac{\pi}{2(s+2)}\right\}(t)=\frac{\pi}{2}\cdot e^{-2t} \rightarrow I=I(1)=\frac{\pi}{2e^2}$$

(4) Să se determine $\mathcal{L}\{\sin^3 t\}(s)$ și să se calculeze integrala $\underbrace{\int_0^\infty \left(\frac{\sin x}{x}\right)^3 dx}_{"I"}$

Obs: $\sin 3t = 3\sin t - 4\sin^3 t \rightarrow \sin^3 t = \frac{3\sin t-\sin 3t}{4}$

Așadar,

$$\mathcal{L}\{\sin^3 t\}(s) = \mathcal{L}\left\{\frac{3\sin t - \sin 3t}{4}\right\}(s) = \frac34\,\mathcal{L}\{\sin t\}(s) - \frac14\,\mathcal{L}\{\sin 3t\}(s)$$

$$=\frac34\cdot\frac{1}{s^2+1}-\frac14\cdot\frac{3}{s^2+9}=\frac{3}{\cancel{4}}\cdot\frac{\cancel{8}^{\,2}}{(s^2+1)(s^2+9)}=\frac{6}{(s^2+1)(s^2+9)}$$

$$\mathcal{L}\{\sin at\}(s)=\frac{a}{s^2+a^2}$$

Analog se calculează $\mathcal{L}\{\cos^3 t\}(s)$ folosind formula $\cos 3t = 4\cos^3 t - 3\cos t$

Fie $I(t)=\int_0^\infty \frac{\sin^3 tx}{x^3}\,dx$. Atunci $I=I(1)$

Aplicând transformarea Laplace, obținem

$$\mathcal{L}\{I(t)\}(s) = \mathcal{L}\left\{\int_0^\infty \frac{\sin^3 tx}{x^3}\,dx\right\}(s)=\int_0^\infty\left(\int_0^\infty \frac{\sin^3 tx}{x^3}\,dx\right)\cdot e^{-st}\,dt$$

$$=\int_0^\infty\left(\int_0^\infty \frac{\sin^3 tx}{x^3}\cdot e^{-st}\,dx\right)dt = \int_0^\infty\left(\int_0^\infty\frac{\sin^3 tx}{x^3}\cdot e^{-st}\,dt\right)dx$$

schimbăm ordinea de integrare

$$= \int_0^\infty \frac{1}{x^3}\left(\int_0^\infty \sin^3 tx \cdot e^{-st}\,dt\right)dx = \int_0^\infty \frac{1}{x^3}\cdot \mathcal{L}\{\sin^3 tx\}(s)\,dx$$

$$= \int_0^\infty \frac{1}{x^3}\cdot\frac{1}{x}\cdot\frac{6}{\left(\frac{s^2}{x^2}+1\right)\left(\frac{s^2}{x^2}+9\right)}\,dx$$

T. asemănării

$$\mathcal{L}\{\underbrace{\sin^3 t}_{f(t)}\}(s) = \frac{6}{(s^2+1)(s^2+9)} \quad \text{și } \mathcal{L}\{\underbrace{\sin^3 tx}_{f(tx)}\}(s) = \frac{1}{x}\mathcal{L}\{\sin^3 t\}\left(\frac{s}{x}\right)$$

$$= 6\cdot\int_0^\infty \frac{1}{(x^2+s^2)(9x^2+s^2)}\,dx = \frac{\cancel{6}^{\,3}}{\cancel{8}_{4}s^2}\cdot\int_0^\infty\left(\frac{9}{9x^2+s^2}-\frac{1}{x^2+s^2}\right)dx$$

$$= \frac{3}{4s^2}\cdot\int_0^\infty\left(\frac{1}{x^2+\left(\frac{s}{3}\right)^2}-\frac{1}{x^2+s^2}\right)dx = \frac{3}{4s^2}\cdot\frac{3}{s}\cdot \operatorname{arctg}\frac{3x}{s}\Big|_0^\infty - \frac{3}{4s^2}\cdot\frac{1}{s}\cdot\operatorname{arctg}\frac{x}{s}\Big|_0^\infty$$

$$= \frac{9}{4s^3}\cdot\frac{\pi}{2} - \frac{3}{4s^3}\cdot\frac{\pi}{2} = \frac{\cancel{6}^{\,3}\pi}{\cancel{8}_{4}s^3} = \frac{3\pi}{4s^3}$$

În final, $\mathcal{L}\{J(t)\}(s) = \frac{3\pi}{4s^3} \to J(t) = \mathcal{L}^{-1}\left\{\frac{3\pi}{4s^3}\right\}(t) = \frac{3\pi}{4}\cdot\frac{t^2}{2}\cdot u(t) = \frac{3\pi t^2}{8}\cdot u(t)$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^n}\right\}(t) = \frac{t^{n-1}}{(n-1)!}\cdot u(t)$$

$$J = J(1) = \frac{3\pi}{8}$$

⑤ Să se rezolve ecuația diferențială

$$t\cdot x''(t) + 2x'(t) + t\,x(t) = -2\cos t\,,\quad \text{unde } x(0)=0,\ x'(0)=1$$

Etapa 1 Determinăm transformata Laplace a soluției, $X(s) \overset{not}{=} \mathcal{L}\{x(t)\}(s)$

$$t\cdot x''(t) + 2x'(t) + t\cdot x(t) = -2\cos t \;\Big|\; \mathcal{L}\{\ \}(s)$$

$$\to \mathcal{L}\{t\cdot x''(t)\}(s) + 2\,\mathcal{L}\{x'(t)\}(s) + \mathcal{L}\{t\cdot x(t)\}(s) = -2\,\mathcal{L}\{\cos t\}(s) \quad (*)$$

Obs: $\mathcal{L}\{x'(t)\}(s) = sX(s) - x(0) = sX(s)$

$$\mathcal{L}\{x''(t)\}(s) = s^2X(s) - s\,x(0) - x'(0) = s^2X(s) - 1$$

Folosind Teorema derivării imaginii $\left(\mathcal{L}\{t^n f(t)\}(s) = (-1)^n\left(\mathcal{L}\{f(t)\}(s)\right)^{(n)}\right)$ și Obs, relația $(*)$ devine:

$$-\left(\mathcal{L}\{x''(t)\}(s)\right)' + 2sX(s) - \left(\mathcal{L}\{x(t)\}(s)\right)' = -2\cdot\frac{s}{s^2+1}$$

$$\Leftrightarrow -\left(s^2X(s)-1\right)' + 2sX(s) - X'(s) = -\frac{2s}{s^2+1}$$

$$\Leftrightarrow -\cancel{2sX(s)} - s^2X'(s) + \cancel{2sX(s)} - X'(s) = -\frac{2s}{s^2+1}$$

$$\Leftrightarrow X'(s)(s^2+1) = +\frac{2s}{s^2+1}$$

$$\Leftrightarrow X'(s) = \frac{2s}{(s^2+1)^2} \Rightarrow X(s) = \int \frac{2s}{(s^2+1)^2}\,ds = -\frac{1}{s^2+1} + C$$

Conform Teoremei valorii inițiale, $sX(s) \xrightarrow[s\to\infty]{} x(0)$,

adică $\lim_{s\to\infty} s\cdot\left(-\frac{1}{s^2+1}+C\right) = 0 \Rightarrow C = 0$

Prin urmare,

$$X(s) = -\frac{1}{s^2+1} \Rightarrow x(t) = \mathcal{L}^{-1}\left\{-\frac{1}{s^2+1}\right\}(t) = -\sin t \cdot u(t)$$

Verificare $x(t) = -\sin t$, $x'(t) = -\cos t$, $x''(t) = \sin t$

$$t\,x''(t) + 2x'(t) + t\,x(t) = -2\cos t \Leftrightarrow t\sin t - 2\cos t - t\sin t = -2\cos t \checkmark$$

⑥ Să se rezolve ecuația diferențială

$$x''(t) + 4x'(t) + 4x(t) = \frac{e^{-2t}}{t+2} \quad (*), \text{ unde } x(0)=0,\ x'(0)=2$$

Aplicând transformarea Laplace relației (*), obținem

$$\mathcal{L}\{x''(t)\}(s) + 4\mathcal{L}\{x'(t)\}(s) + 4\underbrace{\mathcal{L}\{x(t)\}(s)}_{\overset{not}{=}X(s)} = \mathcal{L}\left\{\frac{e^{-2t}}{t+2}\right\}(s) \quad (**)$$

Obs: $\mathcal{L}\{x'(t)\}(s) = sX(s) - x(0) = sX(s)$

$\mathcal{L}\{x''(t)\}(s) = s^2X(s) - s\,x(0) - x'(0) = s^2X(s) - 2$

Relația (**) $\Leftrightarrow s^2X(s) - 2 + 4sX(s) + 4X(s) = \mathcal{L}\left\{\frac{e^{-2t}}{t+2}\right\}(s)$

$$\Leftrightarrow X(s)(s+2)^2 = \mathcal{L}\left\{\frac{e^{-2t}}{t+2}\right\}(s) + 2$$

$$\Leftrightarrow X(s) = \frac{1}{(s+2)^2}\cdot\mathcal{L}\left\{\frac{e^{-2t}}{t+2}\right\}(s) + \frac{2}{(s+2)^2} \quad \Big|\ \mathcal{L}^{-1}\{\ \}(t)$$

Obținem $x(t) = \mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2}\cdot\mathcal{L}\left\{\frac{e^{-2t}}{t+2}\right\}(s)\right\}(t) + \mathcal{L}^{-1}\left\{\frac{2}{(s+2)^2}\right\}(t)$

$$= \mathcal{L}^{-1}\left\{\mathcal{L}\{t\cdot e^{-2t}\}(s)\cdot\mathcal{L}\left\{\frac{e^{-2t}}{t+2}\right\}(s)\right\}(t) + 2\cdot t\cdot e^{-2t}$$

$$\mathcal{L}\{t^n e^{at}\}(s) = \frac{n!}{(s-a)^{n+1}} \text{ și } \mathcal{L}^{-1}\left\{\frac{1}{(s-a)^n}\right\}(t) = \frac{t^{n-1}}{(n-1)!}e^{at}\cdot u(t)$$

Obs: Cum $(f*g)(t) = \int_0^t f(x)\cdot g(t-x)\,dx$

și $\mathcal{L}\{(f*g)(t)\}(s) = \mathcal{L}\{f(t)\}(s)\cdot\mathcal{L}\{g(t)\}(s)$, obținem că

$$\mathcal{L}^{-1}\left\{\mathcal{L}\{t\cdot e^{-2t}\}(s)\cdot\mathcal{L}\left\{\frac{e^{-2t}}{t+2}\right\}(s)\right\} = \mathcal{L}^{-1}\left\{\mathcal{L}\left\{t\cdot e^{-2t} * \frac{e^{-2t}}{t+2}\right\}(s)\right\}(t)$$

$$= t\cdot e^{-2t} * \frac{e^{-2t}}{t+2} = \int_0^t x\cdot e^{-2x}\cdot\frac{e^{-2(t-x)}}{t-x+2}\,dx$$

$$= \int_0^t \frac{x \cdot e^{-2t}}{t-x+2}\, dx = e^{-2t} \int_0^t \frac{x-t-2+t+2}{t-x+2}\, dx$$

$$= -e^{-2t} \cdot \int_0^t dx + e^{-2t} \cdot (t+2) \cdot \int_0^t \frac{1}{t-x+2}\, dx$$

$$= -t \cdot e^{-2t} - e^{-2t} \cdot (t+2) \cdot \ln |t-x+2| \Big|_0^t$$

$$= -t \cdot e^{-2t} - e^{-2t} \cdot (t+2) \ln 2 + e^{-2t} \cdot (t+2) \cdot \ln |t+2|$$

$$= -t \cdot e^{-2t} + e^{-2t} (t+2) \cdot \ln \frac{|t+2|}{2}$$

$$\Rightarrow \underline{x(t) = t \cdot e^{-2t} + e^{-2t} \cdot (t+2) \cdot \ln \frac{|t+2|}{2}}$$

Seminarul 13

Transformata Fourier discretă (TFD)

• Def: Fie $N \in \mathbb{N}^*$, $\omega = \omega_N = e^{\frac{2\pi j}{N}}$ și $x:\mathbb{Z}\to K$ o funcție periodică de perioadă N (numită semnal finit de lungime N). Funcția $X:\mathbb{Z}\to K$,

$$X(m)=\sum_{n=0}^{N-1} x(n)\cdot\omega^{-mn}, \quad \forall m\in\mathbb{Z} \quad (\text{sau } 0\le m\le N-1)$$

s.n. transformata Fourier discretă (TFD) a semnalului $x(n)$.

• Notație: $X(m) = (\mathcal{F}_d x)(m) = \mathcal{F}_d\{x(n); m\}$ sau $X(m)=TFD\{x(n); m\}$

• Obs: $X(m+kN)=X(m)$, $\forall m\in\mathbb{Z}$, X fiind o funcție periodică de perioadă N.

De aceea, în def., este suficient să scriem $0\le m\le N-1$ în loc. de $m\in\mathbb{Z}$.

De asemenea, $\omega^N=1$, $\omega^{N/4}=j$, $\omega^{N/2}=-1$, $\omega^{3N/4}=-j$, $\omega^{kN+r}=\omega^r$, $\forall k,r\in\mathbb{Z}$

(1) Să se calculeze $\mathcal{F}_d(x;m)$ pentru următoarele semnale finite:

a) $x(n)=\left(\frac{j-1}{\sqrt2}\right)^n$, $x\in K^{100}$ $(N=100)$

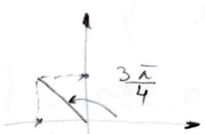
b) $x(n)=\left(\frac{\sqrt3-j}{2}\right)^n$, $x\in K^{180}$ $(N=180)$

Obs: K^N – mulțimea semnalelor finite $x:\mathbb{Z}\to K$ de lungime (perioadă) N.

a) $\mathcal{F}_d(x;m)=X(m)=\sum_{n=0}^{99} x(n)\cdot\omega^{-mn}$, $0\le m\le 99$, unde $\omega=e^{\frac{2\pi j}{100}}$

$$=\sum_{n=0}^{99}\left(\frac{j-1}{\sqrt2}\right)^n\cdot\left(e^{-\frac{2m\pi j}{100}}\right)^n=\sum_{n=0}^{99}\left(\frac{j-1}{\sqrt2}\cdot e^{-\frac{2m\pi j}{100}}\right)^n$$

$$=\sum_{n=0}^{99}\left(e^{\left(\overset{25}{\frac{3\pi}{4}}-\frac{2m\pi}{100}\right)j}\right)^n=\sum_{n=0}^{99}\left(\underbrace{e^{\frac{75-2m}{100}\pi j}}_{y}\right)^n=\sum_{n=0}^{99} y^n=\begin{cases}\dfrac{y^{100}-1}{y-1}, & y\neq 1\\ 100, & y=1\end{cases}$$



$$\frac{j-1}{\sqrt2}=\cos\frac{3\pi}{4}+j\sin\frac{3\pi}{4}=e^{\frac{3\pi}{4}j}$$

$\mathrm{Ln}\, z=\{\ln|z|+j(\arg z+2k\pi): k\in\mathbb{Z}\}$

Dar $y=1 \iff e^{\frac{75-2m}{100}\cdot\pi j}=1 \iff \frac{75-2m}{100}\cdot\pi j\in \mathrm{Ln}\,1=\{j\cdot(0+2k\pi): k\in\mathbb{Z}\}$

$\iff \frac{75-2m}{100}\cdot\pi j\in\{\dots,-4\pi j,-2\pi j,0,2\pi j,4\pi j,\dots\} \quad \Big|\cdot\frac{100}{\pi j}$

$\iff 75-2m\in\{\dots,-400,-200,0,200,400,\dots\}$

$\iff -2m \in \{\dots,-475,-275,-75,125,325,\dots\}$

Dar $m\in\mathbb{N},\ 0\le m\le 99$ $\rightarrow \nexists\, m$ a.î. $e^{\frac{75-2m}{100}\pi j}=1$

Aşadar, $X(m)=\dfrac{y^{100}-1}{y-1}=\dfrac{\left(e^{\frac{75-2m}{100}\pi j}\right)^{100}-1}{e^{\frac{75-2m}{100}\pi j}-1}=\dfrac{e^{(75-2m)\pi j}-1}{e^{\frac{75-2m}{100}\pi j}-1}$

$$=\frac{\cos(75-2m)\pi+j\sin(75-2m)\pi-1}{e^{\frac{75-2m}{100}\pi j}-1}=\frac{\cos\pi+j\sin\pi-1}{e^{\frac{75-2m}{100}\pi j}-1}$$

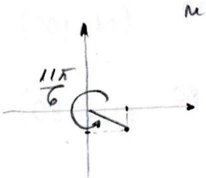
$$=\frac{-2}{e^{\frac{75-2m}{100}\pi j}-1}=\frac{2}{1-e^{\frac{75-2m}{100}\pi j}},\quad 0\le m\le 99 \text{ şi}$$

$X(m+100)=X(m),\ \forall m\in\mathbb{Z}$

b) $X(m)=\sum\limits_{n=0}^{179} x(n)\cdot\omega^{-mn},\quad 0\le m\le 179$, unde $\omega=e^{\frac{2\pi j}{180}}$

$$=\sum_{n=0}^{179}\left(\frac{\sqrt{3}-j}{2}\cdot e^{-\frac{2m\pi}{180}j}\right)^n=\sum_{n=0}^{179}\left(e^{\frac{11\pi}{6}j}\cdot e^{-\frac{2m\pi}{180}j}\right)^n=\sum_{n=0}^{179}\left(e^{\left(\overset{30}{\frac{11}{6}}-\frac{2m}{180}\right)\pi j}\right)^n$$

$\dfrac{\sqrt{3}-j}{2}=e^{\frac{11\pi}{6}j}$



$\arg\left(\frac{\sqrt{3}}{2}-\frac{j}{2}\right)=-\operatorname{arctg}\frac{1}{\sqrt{3}}+2\pi$
$=2\pi-\frac{\pi}{6}=\frac{11\pi}{6}$

$$=\sum_{n=0}^{179}\Big(\underbrace{e^{\frac{330-2m}{180}\pi j}}_{y}\Big)^n=\begin{cases}\dfrac{y^{180}-1}{y-1}, & \text{pt. } y\ne 1\\ 180, & \text{pt. } y=1\end{cases}$$

Dar $y=1 \iff e^{\frac{330-2m}{180}\pi j}=1 \iff \frac{330-2m}{180}\pi j\in \operatorname{Ln}1=\{2k\pi j : k\in\mathbb{Z}\}=\{\dots,-4\pi j,-2\pi j,0,2\pi j,\dots\}$

$\iff 330-2m\in\{\dots,-4\cdot180,-2\cdot180,0,2\cdot180,4\cdot180,\dots\}$

$\iff -2m\in\{\dots -4\cdot180-330,-2\cdot180-330,-330,30,390,\dots\}$

Cum $m\in\mathbb{N},\ 0\le m\le 179$ $\rightarrow$

$\rightarrow$ doar $-2m=-330$ ne convine $\rightarrow$ $m=165$

Pentru $y\ne 1$, $X(m)=\dfrac{\left(e^{\frac{330-2m}{180}\pi j}\right)^{180}-1}{e^{\frac{330-2m}{180}\pi j}-1}=\dfrac{\cos(330-2m)\pi+j\sin(330-2m)\pi-1}{e^{\frac{330-2m}{180}\pi j}-1}$

$\Updownarrow$

$m\ne 165$

$0\le m\le 179$

$$=\frac{\cos0+j\sin0-1}{e^{\frac{330-2m}{180}\pi j}-1}=0$$

În final,

$$\underline{X(m)} = \begin{cases} 0, & 0 \le m \le 179,\ m \ne 165 \\ 180, & m = 165 \end{cases} \quad \text{și } \underline{X(m+180) = X(m)},\ \forall m \in \mathbb{Z}$$

② Se consideră semnalul $x \in K^{62}$, $x(n) = (-j)^n$ pentru $0 \le n \le 61$. $(\underline{N=62})$

a) Să se calculeze suma $s = x(20) + x(62) + x(155) + x(-70)$

b) $E(x) = ?$

Obs: $x(n+62) = x(n)$, $\forall n \in \mathbb{Z}$

c) $X(m) = \mathcal{F}_d(x; m) = ?$

d) $E(X) = ?$

---

a) $\underline{s} = x(20) + x(0) + x(2\cdot 62 + 31) + x(-70+62+62)$

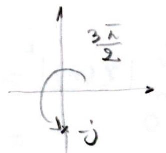
$= x(20) + x(0) + x(31) + x(54) = (-j)^{20} + (-j)^0 + (-j)^{31} + (-j)^{54} = \cancel{1} + 1 + j - \cancel{1} = \underline{\underline{1+j}}$

b) $\underline{E(x)} = \sum_{n=0}^{61} |x(n)|^2 = \sum_{n=0}^{61} |(-j)^n|^2 = \sum_{n=0}^{61} |-j|^{2n} = \underline{62}$

Obs: Dacă $x \in K^N \Rightarrow \underline{E(x) = \sum_{n=0}^{N-1} |x(n)|^2}$

c) $X(m) = \sum_{n=0}^{61} x(n) \cdot \left(e^{-\frac{2m\pi j}{62}}\right)^n = \sum_{n=0}^{61} \left(-j \cdot e^{-\frac{2m\pi j}{62}}\right)^n = \sum_{n=0}^{61} \left(e^{\frac{3\pi}{2}j} \cdot e^{-\frac{2m\pi j}{62}}\right)^n$

$-j = e^{\frac{3\pi}{2}j}$



$$= \sum_{n=0}^{61} \left(e^{\left(\frac{3}{2} - \frac{2m}{62}\right)\pi j}\right)^n = \sum_{n=0}^{61} \Big(\underbrace{e^{\frac{93-2m}{62}\pi j}}_{y}\Big)^n = \begin{cases} \dfrac{y^{62}-1}{y-1}, & \text{pt. } y \ne 1 \\ 62, & \text{pt. } y = 1 \end{cases}$$

$y = 1 \Leftrightarrow e^{\frac{93-2m}{62}\pi j} = 1 \Leftrightarrow \frac{93-2m}{62}\pi j \in \operatorname{Ln} 1 = \{2k\pi j : k \in \mathbb{Z}\} \quad \Big| \cdot \frac{62}{\pi j}$

$\Leftrightarrow 93 - 2m \in \{124k : k \in \mathbb{Z}\}$

$\left.\begin{array}{l} \Leftrightarrow -2m \in \{124k - 93 : k \in \mathbb{Z}\} \\ m \in \mathbb{N};\ 0 \le m \le 61 \end{array}\right\} \Rightarrow \nexists\, m$

$$\frac{y^{62}-1}{y-1} = \frac{\left(e^{\frac{93-2m}{62}\pi j}\right)^{62} - 1}{e^{\frac{93-2m}{62}\pi j} - 1} = \frac{\cos(93-2m)\pi + j\sin(93-2m)\pi - 1}{e^{\frac{93-2m}{62}\pi j} - 1} = \frac{\cos\pi + j\sin\pi - 1}{e^{\frac{93-2m}{62}\pi j} - 1} = \frac{-2}{e^{\frac{93-2m}{62}\pi j} - 1}$$

În final, $\underline{X(m) = \dfrac{-2}{e^{\frac{93-2m}{62}\pi j} - 1}}$, pentru $0 \le m \le 61$ și $\underline{X(m+62) = X(m)},\ \forall m \in \mathbb{Z}$

d) $E(X)=\sum_{m=0}^{61}{}' |X(m)|^2 = \sum_{m=0}^{61}{}' \left| -\dfrac{2}{e^{\frac{93-2m}{62}\pi j}-1}\right|^2 = \ldots$

sau

Formula lui Parseval

$E(X) = N\cdot E(x)$

Folosind formula lui Parseval, obținem

$\underline{E(X) = 62\cdot 62 = 62^2}$

③ Se consideră $x = (\overset{x(0)}{0}, \overset{x(1)}{1+j}, \overset{x(2)}{2+j}, \overset{x(3)}{3+j})^T \in K^4$, $0\le n\le 3$

a) Să se determine $s = x(-1)+x(73)+x(-73)$

b) $E(x) = ?$

c) $X(m) = \mathcal{F}_d(x;m) = ?$ $(X = \mathcal{F}_d x = ?)$

d) $E(X) = ?$ Să se verifice formula lui PARSEVAL.

a) Observăm că $N=4$, $x(0)=0$, $x(1)=1+j$, $x(2)=2+j$ și $x(3)=3+j$

De asemenea $x(n+4)=x(n)$, $\forall n\in\mathbb{Z}$

$$\underline{s} = x(-1)+x(73)+x(-73) = x(-1+4)+x(72+1)+x(-72-1) = x(3)+x(1)+\underbrace{x(-1)}_{x(-1+4)=x(3)}$$

$$= 3+j+1+j+3+j = \underline{\underline{7+3j}}$$

b) $\underline{E(x)} = \sum_{n=0}^{3}{}' |x(n)|^2 = |x(0)|^2+|x(1)|^2+|x(2)|^2+|x(3)|^2$

$$= 0+|1+j|^2+|2+j|^2+|3+j|^2 = 2+5+10 = \underline{\underline{17}}$$

c) $X(m) = \sum_{n=0}^{3}{}' x(n)\cdot\omega^{-mn}$, unde $\omega = e^{\frac{2\pi j}{N}} = e^{\frac{\pi j}{2}} = \cos\frac{\pi}{2}+j\sin\frac{\pi}{2} = j$

$$= \sum_{n=0}^{3}{}' x(n)\cdot j^{-mn} = \ldots$$

$\underline{\underline{-j = \omega^{-1}}} = \frac{1}{j}$

sau

$\boxed{X = W\cdot x}$, unde $W = \begin{pmatrix} 1 & 1 & 1 & 1\\ 1 & -j & -1 & j\\ 1 & -1 & 1 & -1\\ 1 & j & -1 & -j\end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1\\ 1 & \omega^{-1} & \omega^{-2} & \omega^{-3}\\ 1 & \omega^{-2} & \omega^{-4} & \omega^{-6}\\ 1 & \omega^{-3} & \omega^{-6} & \omega^{-9}\end{pmatrix}$

Așadar,

$$\underline{X} = \begin{pmatrix} 1 & 1 & 1 & 1\\ 1 & -j & -1 & j\\ 1 & -1 & 1 & -1\\ 1 & j & -1 & -j\end{pmatrix}\cdot\begin{pmatrix}0\\ 1+j\\ 2+j\\ 3+j\end{pmatrix} = \begin{pmatrix} 1+j+2+j+3+j\\ -j+\cancel{1}-2-j+3j-\cancel{1}\\ -1-j+2+\cancel{j}-3-\cancel{j}\\ \cancel{j}-\cancel{1}-2-\cancel{j}-3j+\cancel{1}\end{pmatrix} = \underline{\underline{\begin{pmatrix}6+3j\\ -2+j\\ -2-j\\ -2-3j\end{pmatrix}}} = \begin{pmatrix}X(0)\\ X(1)\\ X(2)\\ X(3)\end{pmatrix} = (X(0), X(1), X(2), X(3))^T$$

d) $\underline{E(X)} = \sum_{m=0}^{3} |X(m)|^2 = |X(0)|^2 + |X(1)|^2 + |X(2)|^2 + |X(3)|^2 = 45+5+5+13 = \underline{68}$

Verificarea formulei lui PARSEVAL:

$$E(X) = N \cdot E(x)$$

$$\Leftrightarrow 68 = 4 \cdot 17 \checkmark$$

④ Se consideră $X = (1, 0, 2-j, 3+4j)^T \in K^4$ $(N=4)$

$X(0)\ X(1)\ X(2)\ X(3)$

a) Să se calculeze $\sigma = X(-5) + X(55) + X(-155)$

b) $E(X) = ?$

c) $x = \mathcal{F}_d^{-1} X = ?$

d) $E(x) = ?$ Să se verifice formula lui PARSEVAL.

a) $\underline{\sigma} = X(-5) + X(55) + X(-155) = X(-5+4+4) + X(52+3) + X(-155+156)$

$= X(3) + X(3) + X(1) = 3+4j + 3+4j + 0 = \underline{6+8j}$

b) $\underline{E(X)} = \sum_{m=0}^{3} |X(m)|^2 = 1+0+5+25 = \underline{31}$

c) $\boxed{x = \frac{1}{N} \overline{W} X}$, unde $W = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{pmatrix}$, iar $\overline{W} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{pmatrix}$

Așadar,

$$\underline{x} = \frac{1}{4} \cdot \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 2-j \\ 3+4j \end{pmatrix} = \frac{1}{4} \cdot \begin{pmatrix} 1+2-j+3+4j \\ 1-2+j-3j+4 \\ \cancel{1+2}-j-\cancel{3}-4j \\ 1-2+j+3j-4 \end{pmatrix} = \frac{1}{4} \cdot \begin{pmatrix} 6+3j \\ 3-2j \\ -5j \\ -5+4j \end{pmatrix} = \begin{pmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{pmatrix}$$

d) $E(x) = \sum_{m=0}^{3} |x(m)|^2 = \frac{45+13+25+41}{16} = \frac{124}{16} = \frac{31}{4}$

Verificarea formulei lui Parseval:

$$E(X) = N \cdot E(x)$$

$$\Leftrightarrow 31 = 4 \cdot \frac{31}{4} \checkmark$$

5) Fie $x \in K^{20}$, $x(n) = 2jn+1$, $0 \le n \le 19$ $\quad (x(n+20)=x(n),\ \forall n\in\mathbb{Z})$ $\quad N=20$

a) Să se calculeze $s = x(20)+x(101)+x(-101)$

b) $E(x) = ?$

c) $X = \mathcal{F}_d\, x = ?$

d) $E(X) = ?$

---

a) $s = x(20)+x(101)+x(-101) = x(0)+x(1)+\underset{x(20-1)=x(19)}{x(-1)} = 1+2j+1+38j+1 = 3+40j$

b) $E(x) = \sum_{n=0}^{19} |2jn+1|^2 = \sum_{n=0}^{19} \sqrt{4n^2+1}^{\,2} = \sum_{n=0}^{19}(4n^2+1) = 4\sum_{n=0}^{19} n^2 + \sum_{n=0}^{19} 1$

$$= \cancel{4}^{\,2}\cdot\frac{n(n+1)(2n+1)}{\cancel{6}_{\,3}}\Bigg|_{n=19} + 20 = \frac{2\cdot 19\cdot 20\cdot \cancel{39}^{\,13}}{\cancel{3}} + 20 = 20(38\cdot 13+1) = 9900$$

c) $X(m) = \sum_{n=0}^{19} x(n)\cdot \omega^{-mn}$, unde $\omega = e^{\frac{2\pi j}{N}} = e^{\frac{\pi j}{10}} \Rightarrow \omega^{20} = e^{2\pi j} = \cos 2\pi + j\sin 2\pi = 1$

$$= \sum_{n=0}^{19}(2jn+1)\cdot\omega^{-mn} = \sum_{n=0}^{19}(2jn+1)\cdot\Big(\underbrace{e^{-\frac{m\pi j}{10}}}_{q}\Big)^n = \sum_{n=0}^{19}(2jn+1)\cdot q^n = \underbrace{2j\cdot\sum_{n=0}^{19} n q^n}_{X_1(m)} + \underbrace{\sum_{n=0}^{19} q^n}_{X_2(m)}$$

unde

$$X_1(m) = 2j\cdot\sum_{n=0}^{19} n q^n = 2qj\cdot(1+2q+3q^2+\dots+19q^{18})$$

$$= 2qj\cdot(q+q^2+q^3+\dots+q^{19})' \overset{\text{pentru } q\neq 1}{=} 2qj\cdot\left(q\cdot\frac{q^{19}-1}{q-1}\right)'$$

$$= 2qj\cdot\left(\frac{q^{20}-q}{q-1}\right)' = 2qj\cdot\frac{(20q^{19}-1)(q-1)-(q^{20}-q)}{(q-1)^2}$$

$$= 2qj\cdot\frac{19q^{20}-20q^{19}+1}{(q-1)^2} = 2j\cdot\frac{19q^{21}-20q^{20}+q}{(q-1)^2} \overset{q^{20}=e^{-2m\pi j}=1}{=} 2j\cdot\frac{19q-20+q}{(q-1)^2}$$

$$= 2j\cdot\frac{20\cancel{(q-1)}}{(q-1)^{\cancel{2}}} = \frac{40j}{q-1}, \text{ dacă } q\neq 1$$

Dacă $q=1 \Leftrightarrow e^{-\frac{m\pi j}{10}} = 1 \Rightarrow -\frac{m\pi j}{10} \in \operatorname{Ln} 1\ \{2k\pi j : k\in\mathbb{Z}\}\ \Big|\cdot\frac{10}{\pi j}$

$$\left.\begin{array}{l} \Rightarrow -m \in \{20k : k\in\mathbb{Z}\} \\ 0\le m\le 19,\ m\in\mathbb{N}\end{array}\right\} \Rightarrow m=0$$

În acest caz, $X_1(m) = 2j\cdot\sum_{n=0}^{19} n = 2j\cdot\frac{19\cdot 20}{2} = 380j$

Așadar, $X_1(m) = \begin{cases} \dfrac{40j}{q-1}, & 1\le m\le 19 \\ 380j, & m=0 \end{cases}$

$$X_2(m)=\sum_{n=0}^{19} q^n = 1+q+\dots+q^{19} = \begin{cases} \dfrac{q^{20}-1}{q-1}, & \text{pentru } q\neq 1 \\ 20, & \text{pentru } q=1 \end{cases} = \begin{cases} 0, & 1\le m\le 19 \\ 20, & m=0 \end{cases}$$

În final,

$$X(m)=X_1(m)+X_2(m)=\begin{cases} \dfrac{40j}{q-1}, & 1\le m\le 19 \\ 20+380j, & m=0 \end{cases} = \begin{cases} \dfrac{40j}{e^{-\frac{m\pi j}{10}}-1}, & 1\le m\le 19 \\ 20+380j, & m=0 \end{cases}$$

și $X(m+20)=X(m),\ \forall m\in\mathbb{Z}$

d) Folosind formula lui Parseval

$$E(X)=N\cdot E(x)$$

$\Leftrightarrow E(X)=20\cdot 9900$, obținem $E(X)=198.000$

Probleme propuse

① Fie $x=(1,0,j,2+j)^T\in K^4$ și $X=\mathcal{F}_d\, x$.
Să se determine $x(-15)$, energia $E(x)$ și $X(3)$.

② Fie $X=(0,1+j,2j,3+j)^T\in K^4$ și $x=\mathcal{F}_d^{-1}(X)$
Să se determine $X(71)+X(-110)$, $x(0)$, $E(X)$ și să se verifice formula lui Parseval.

③ Fie semnalul $x\in K^{18}$, $x(n)=3jn+2$, $0\le n\le 17$ și fie $X=\mathcal{F}_d\, x$.
Să se determine $x(-16)+x(92)$, energia $E(x)$ și $X(9)$.

④ Fie $x=(j,1+j,2+j,3+j)^T\in K^4$ și $X=\mathcal{F}_d\, x$.
Să se determine $x(-9)+x(154)$, energia $E(x)$, $X(3)$ și să se verifice formula lui Parseval

Transformarea Z

• Fiind dat un semnal $x \in S_d^+$ (șirurile standard), notăm cu U- mulțimea de convergență a seriei Laurent

$$\sum_{n=0}^{\infty} \frac{x(n)}{z^n}$$

Obs: • S_d - mulț. șirurilor $x: \mathbb{Z} \to K$
$x(n) = x_n \in K, \forall n \in \mathbb{Z}$
• $S_d^+ = \{x \in S_d : x(n) = 0, \forall n < 0\}$

• Funcția $X: U \to \mathbb{C}$

$$X(z) = \sum_{n=0}^{\infty} \frac{x(n)}{z^n}, \ z \in U$$ s.n. transformata Z

not ‖ $\mathcal{Z}\{x(n)\}(z)$

Obs: • $\delta_k(n) = \begin{cases} 1, & n = k \\ 0, & n \neq k \end{cases}$

s.n. impulsul lui Dirac

• Teorema asemănării

$$\mathcal{Z}\{a^n x(n)\}(z) = \mathcal{Z}\{x(n)\}\left(\frac{z}{a}\right), \ a \in K^*$$

• Teorema înmulțirii cu n

$$\mathcal{Z}\{n\, x(n)\}(z) = -z \cdot \left(\mathcal{Z}\{x(n)\}(z)\right)'$$

Formule

• $\mathcal{Z}\{\delta_k(n)\}(z) = z^{-k} \to \mathcal{Z}^{-1}\{z^k\}(n) = \delta_{-k}(n)$

• $\mathcal{Z}\{a^n u(n)\}(z) = \dfrac{z}{z-a} \Rightarrow \mathcal{Z}^{-1}\left\{\dfrac{z}{z-a}\right\}(n) = a^n u(n)$

• $\mathcal{Z}\{n a^n u(n)\}(z) = \dfrac{az}{(z-a)^2} \to \mathcal{Z}^{-1}\left\{\dfrac{z}{(z-a)^2}\right\}(n) = n \cdot a^{n-1} \cdot u(n)$

• $\mathcal{Z}\{\sin an \cdot u(n)\}(z) = \dfrac{z \sin a}{z^2 - 2z\cos a + 1}$

• $\mathcal{Z}\{\cos an \cdot u(n)\}(z) = \dfrac{z(z-\cos a)}{z^2 - 2z\cos a + 1}$

• $\mathcal{Z}\{u(n)\}(z) = \dfrac{z}{z-1}$

• $\mathcal{Z}\{n \cdot u(n)\}(z) = \dfrac{z}{(z-1)^2}$

• $\mathcal{Z}\{n^2 \cdot u(n)\}(z) = \dfrac{z(z+1)}{(z-1)^3}$

① Utilizând transformarea Z, să se calculeze suma pt. fiecare din urm. serii numerice:

a) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n(2n^2+7)}{3^n}$ b) $\displaystyle\sum_{n=1}^{\infty} \frac{n \cdot \cos\frac{2n\pi}{3} + (-1)^n}{2^n}$

a) $\displaystyle s = \sum_{n=0}^{\infty} \frac{(-1)^n(2n^2+7)}{3^n} = \mathcal{Z}\{(-1)^n(2n^2+7)\}(3) \overset{\text{T. asemănării}}{=} \mathcal{Z}\{2n^2+7\}(-3) = 2\mathcal{Z}\{n^2\}(-3) + 7 \cdot \mathcal{Z}\{1\}(-3)$

$$= \left(2 \cdot \frac{z(z+1)}{(z-1)^3} + 7 \cdot \frac{z}{z-1}\right)\Bigg|_{z=-3} = 2 \cdot \frac{(-3)\cdot(-2)}{(-4)^3} + 7 \cdot \frac{-3}{-4} = -\frac{3}{16} + \frac{21}{4} = \frac{81}{16} = \left(\frac{3}{2}\right)^4$$

b) $\displaystyle s = \sum_{n=0}^{\infty} \frac{n \cdot \cos\frac{2n\pi}{3} + (-1)^n}{2^n} - \frac{(-1)^0}{2^0} = \mathcal{Z}\{n \cdot \cos\frac{2n\pi}{3} + (-1)^n\}(2) - 1 \overset{\text{T. înmulțirii cu n + T. asemănării}}{=} -z\left(\mathcal{Z}\{\cos\frac{2n\pi}{3}\}(z)\right)'\Big|_{z=2} +$

$$+ \mathcal{Z}\{1\}(-2) - 1 = -\frac{z}{2} \cdot \frac{(4z+1)(z^2+z+1) - z(2z+1)^2}{(z^2+z+1)^2}\Bigg|_{z=2} + \frac{z}{z-1}\Big|_{z=-2} - 1 \overset{(*)}{=}$$

Dar

$$\left(\mathcal{Z}\{\cos\frac{2n\pi}{3}\}(z)\right)' = \left(\frac{z(z-\cos\frac{2\pi}{3})}{z^2 - 2z\cos\frac{2\pi}{3} + 1}\right)' = \left(\frac{z\cdot(z+\frac{1}{2})}{z^2+z+1}\right)' = \frac{1}{2}\cdot\left(\frac{z(2z+1)}{z^2+z+1}\right)' = \frac{1}{2}\cdot\frac{(4z+1)(z^2+z+1) - z(2z+1)^2}{(z^2+z+1)^2}$$

$\cos\frac{2\pi}{3} = \cos(\pi - \frac{\pi}{3}) = -\frac{1}{2}$

$$\overset{(*)}{=} -\frac{9\cdot 7 - 25\cdot 2}{49} + \frac{2}{3} - 1 = -\frac{13}{49} - \frac{1}{3} = -\frac{88}{147}$$

② Utilizând metoda transformării Z, să se rezolve ec. cu diferențe finite

a) $2x(n+2)-7x(n+1)+3x(n)=3^n\cdot u(n)$ $(*)$; $x(0)=0$, $x(1)=2$

Obs: $x(n)=x_n \to (*) \Leftrightarrow 2x_{n+2}-7x_{n+1}+3x_n=3^n$, unde $n\geq 0$, $x_0=0$, $x_1=2$.

relație de recurență liniară neomogenă

Etapa 1: Determinăm $X(z)=\mathcal{Z}\{x(n)\}(z)$

$$2x(n+2)-7x(n+1)+3x(n)=3^n\cdot u(n) \quad \Big|\ \mathcal{Z}\{\}(z)$$

$$\to 2\mathcal{Z}\{x(n+2)\}(z)-7\mathcal{Z}\{x(n+1)\}(z)+3\cdot\mathcal{Z}\{x(n)\}(z)=\mathcal{Z}\{3^n\cdot u(n)\}(z) \quad (1)$$

Obs: $\mathcal{Z}\{x(n+1)\}(z)=zX(z)-z\,x(0) \,|\cdot z\ = zX(z)$

$\mathcal{Z}\{x(n+2)\}(z)=z^2X(z)-z^2x(0)-zx(1)=z^2X(z)-2z$

$2z^2-7z+3=0$

$\Delta=49-24=25$

$z_{1,2}=\frac{7\pm5}{4}$, $z_1=3$, $z_2=\frac{1}{2}$

Relația (1) $\Leftrightarrow 2(z^2X(z)-2z)-7zX(z)+3X(z)=\frac{z}{z-3}$

$$\Leftrightarrow X(z)(2z^2-7z+3)=\frac{z}{z-3}+4z=\frac{z+4z(z-3)}{z-3}=\frac{4z^2-11z}{z-3}$$

$$\Leftrightarrow X(z)\cdot 2\left(z-\tfrac{1}{2}\right)(z-3)=\frac{4z^2-11z}{z-3}$$

$$\Leftrightarrow X(z)=\frac{4z^2-11z}{(2z-1)(z-3)^2}$$

Etapa 2 Determinăm $x(n)=\mathcal{Z}^{-1}\{X(z)\}(n)$

Apoi considerăm $\frac{X(z)}{z}$ și descompunem în fracții simple

$$\frac{X(z)}{z}=\frac{4z-11}{(2z-1)(z-3)^2}=\frac{\frac{4z-11}{2}}{\left(z-\frac{1}{2}\right)(z-3)^2}=\frac{A}{z-\frac{1}{2}}+\frac{B}{z-3}+\frac{C}{(z-3)^2}$$

Obținem $\frac{4z-11}{2}=A(z-3)^2+B(z-3)\left(z-\frac{1}{2}\right)+C\left(z-\frac{1}{2}\right)$

Pentru $z=3 \to \frac{1}{2}=\frac{5}{2}C \to C=\frac{1}{5}$

$z=\frac{1}{2} \to -\frac{9}{2}=\frac{25}{4}A \to A=-\frac{18}{25}$

Coef. lui z^2: $0=A+B \to B=-A=\frac{18}{25}$

Prin urmare, $\frac{X(z)}{z}=-\frac{18}{25}\cdot\frac{1}{z-\frac{1}{2}}+\frac{18}{25}\cdot\frac{1}{z-3}+\frac{1}{5}\cdot\frac{1}{(z-3)^2}$

$$\to X(z)=-\frac{18}{25}\cdot\frac{z}{z-\frac{1}{2}}+\frac{18}{25}\cdot\frac{z}{z-3}+\frac{1}{5}\cdot\frac{z}{(z-3)^2} \quad \Big|\ \mathcal{Z}^{-1}\{\}$$

În final, $x(n)=\left(-\frac{18}{25}\cdot\left(\frac{1}{2}\right)^n+\frac{18}{25}\cdot 3^n+\frac{1}{5}\cdot n\cdot 3^{n-1}\right)\cdot u(n)$

③ Utilizând metoda transformării z, să se rez. ec. cu dif. finite

$x(n+2)\cdot x^3(n+1) = 2x^4(n), \quad x(0)=2, \quad x(1)=8, \quad x \in S_d^+$

$$x(n+2)\cdot x^3(n+1) = 2x^4(n) \quad \Big| \log_2()$$

$$\Leftrightarrow \log_2 x(n+2) + 3\log_2 x(n+1) = 1 + 4\log_2 x(n)$$

Notăm cu $y(n) = \log_2 x(n)$ și obținem

$$y(n+2) + 3y(n+1) = 1 + 4y(n) \quad \Big|, \; y(0)=1, \; y(1)=3 \quad \Big| \mathcal{Z}\{\}$$

$$\Rightarrow \mathcal{Z}\{y(n+2)\}(z) + 3\mathcal{Z}\{y(n+1)\}(z) = \mathcal{Z}\{1\}(z) + 4\mathcal{Z}\{y(n)\}(z) \quad (1)$$

Notăm cu $Y(z) = \mathcal{Z}\{y(n)\}(z)$ și observăm că

$$\mathcal{Z}\{y(n+1)\}(z) = z\,Y(z) - z\cdot y(0) = zY(z) - z$$

$$\mathcal{Z}\{y(n+2)\}(z) = z^2Y(z) - z^2y(0) - z\,y(1) = z^2Y(z) - z^2 - 3z$$

Ecuația (1) devine

$$z^2Y(z) - z^2 - 3z + 3zY(z) - 3z = \frac{z}{z-1} + 4\cdot Y(z)$$

$$\Leftrightarrow Y(z)(z^2+3z-4) = \frac{z}{z-1} + z^2 + 3z + 3z$$

$$= \frac{z}{z-1} + z^2 + 6z = \frac{z + z^3 - z^2 + 6z^2 - 6z}{z-1}$$

$$= \frac{z^3+5z^2-5z}{z-1}$$

$$\Leftrightarrow Y(z)(z+4)(z-1) = \frac{z(z^2+5z-5)}{z-1}$$

$$\Leftrightarrow Y(z) = \frac{z(z^2+5z-5)}{(z+4)(z-1)^2} \quad \Leftrightarrow \frac{Y(z)}{z} = \frac{z^2+5z-5}{(z+4)(z-1)^2} = \frac{A}{z+4} + \frac{B}{z-1} + \frac{C}{(z-1)^2}$$

$$= \frac{A(z-1)^2 + B(z-1)(z+4) + C(z+4)}{(z+4)(z-1)^2}$$

Pentru $z=1 \Rightarrow 1 = 5C \Rightarrow C = \frac{1}{5}$

$z=-4 \Rightarrow -9 = 25A \Rightarrow A = -\frac{9}{25}$

Coef. lui z^2: $1 = A + B \Rightarrow B = 1 - A = 1 + \frac{9}{25} = \frac{34}{25}$

Obținem $\frac{Y(z)}{z} = -\frac{9}{25}\cdot\frac{1}{z+4} + \frac{34}{25}\cdot\frac{1}{z-1} + \frac{1}{5}\cdot\frac{1}{(z-1)^2} \Rightarrow Y(z) = -\frac{9}{25}\cdot\frac{z}{z+4} + \frac{34}{25}\cdot\frac{z}{z-1} + \frac{1}{5}\cdot\frac{z}{(z-1)^2}$

$$\Rightarrow y(n) = \mathcal{Z}^{-1}\{y(n)\}(z) = \left(-\frac{9}{25}\cdot(-4)^n + \frac{34}{25}\cdot 1^n + \frac{1}{5}\cdot n\right)\cdot u(n) \quad \Rightarrow x(n) = 2^{y(n)}$$

(4) Utilizând metoda transformării Z, să se rezolve urm. ecuație cu diferențe finite, unde $x \in S_d^+$

$$9(n+2)(n+3)x(n)+(n+1)(n+2)\cdot x(n+2)+6(n+1)(n+3)x(n+1)=$$
$$=(-2)^n(n+1)(n+2)(n+3) \quad \Big| \; n\geq 0,\; x(0)=3,\; x(1)=6$$
$$\Big| : (n+1)(n+2)(n+3)$$

Obținem $9\cdot\frac{x(n)}{n+1}+\frac{x(n+2)}{n+3}+6\cdot\frac{x(n+1)}{n+2}=(-2)^n$

Notând cu $y(n)=\frac{x(n)}{n+1}$, $n\geq 0$, rel. de mai sus devine

$$9\cdot y(n)+y(n+2)+6y(n+1)=(-2)^n \quad \Big| \; \left(y(0)=\frac{x(0)}{1}=3,\; y(1)=\frac{x(1)}{2}=3\right)$$
$$\Big| \; Z\{\}$$

$$\iff 9\cdot Z\{y(n)\}(z)+Z\{y(n+2)\}(z)+6Z\{y(n+1)\}(z)=Z\{(-2)^n\}(z)$$

$$\iff \underline{9\cdot Y(z)}+\underline{z^2\cdot Y(z)-3z^2-3z}+\underline{6zY(z)-18z}=\frac{z}{z+2} \quad (1)$$

Obs: $Z\{y(n+1)\}(z)=z\cdot Y(z)-z\cdot y(0)=zY(z)-3z$

$Z\{y(n+2)\}(z)=z^2Y(z)-z^2\cdot y(0)-z\cdot y(1)=z^2\cdot Y(z)-3z^2-3z$

Relația (1) $\iff Y(z)(z^2+6z+9)=\frac{z}{z+2}+3z^2+21z$

$$\iff Y(z)(z+3)^2=\frac{z+3z^3+6z^2+21z^2+42z}{z+2}$$

$$\iff Y(z)=\frac{3z^3+27z^2+43z}{(z+2)(z+3)^2}$$

Aplicăm metoda descompunerii în fracții simple pentru $\frac{Y(z)}{z}$

$$\frac{Y(z)}{z}=\frac{3z^2+27z+43}{(z+2)(z+3)^2}=\frac{A}{z+2}+\frac{B}{z+3}+\frac{C}{(z+3)^2}=\dots=\frac{1}{z+2}+\frac{2}{z+3}+\frac{11}{(z+3)^2}$$

$$\rightarrow Y(z)=\frac{z}{z+2}+2\cdot\frac{z}{z+3}+11\cdot\frac{z}{(z+3)^2}\rightarrow y(n)=Z^{-1}\left\{\frac{z}{z+2}\right\}(n)+2\cdot Z^{-1}\left\{\frac{z}{z+3}\right\}(n)+11\cdot Z^{-1}\left\{\frac{z}{(z+3)^2}\right\}(n)$$

$$y(n)=\left((-2)^n+2\cdot(-3)^n+11n\cdot(-3)^{n-1}\right)\cdot u(n)$$

În final obținem $x(n)=(n+1)\left[(-2)^n+2(-3)^n+11n\cdot(-3)^{n-1}\right]\cdot u(n)$

(5) Se consideră SLDIT (S_d, S_d, L), $L(x)=h*x$. Să se determine intrarea $x\in S_d$ în fiecare din situațiile de mai jos, știind că ieșirea $y=L(x)\in S_d$ și răspunsul impuls $h=L(\delta)$ sunt date.

$$h=\delta_{-2}+2\cdot\delta_{-1}-3\delta, \quad y(n)=(-1)^n 3^n u(n)$$

sisteme liniare discrete invariante în timp

Din $y=L(x)\rightarrow h*x=y$. Aplicând transformarea z, obținem

$$\mathcal{Z}\{(h*x)(n)\}(z) = \mathcal{Z}\{y(n)\}(z) \overset{not}{=} Y(z)$$

$$\underbrace{\mathcal{Z}\{h(n)\}(z)}_{\overset{\|not}{H(z)}} \cdot \underbrace{\mathcal{Z}\{x(n)\}(z)}_{\overset{\|not}{X(z)}} = Y(z)$$

Așadar, $X(z) = \dfrac{Y(z)}{H(z)} = \dfrac{\mathcal{Z}\{(-1)^n 3^n u(n)\}(z)}{\mathcal{Z}\{\delta_{-2}(n) + 2\delta_{-1}(n) - 3\delta(n)\}(z)} =$

$$= \frac{\mathcal{Z}\{3^n u(n)\}(-z)}{z^2+2z-3} = \frac{\mathcal{Z}\{1\}\left(-\frac{z}{3}\right)}{z^2+2z-3} = \frac{\frac{-\frac{z}{3}}{-\frac{z}{3}-1}}{z^2+2z-3} = \frac{z}{z+3}\cdot\frac{1}{z^2+2z-3}$$

$\mathcal{Z}\{\delta_k(n)\}(z) = z^{-k}$, $\mathcal{Z}\{\delta(n)\}(z) = 1$

Obținem $\dfrac{X(z)}{z} = \dfrac{1}{(z+3)(z^2+2z-3)} = \dfrac{1}{(z-1)(z+3)^2} = \dfrac{A}{z-1} + \dfrac{B}{z+3} + \dfrac{C}{(z+3)^2} =$

$$= \ldots = \frac{1}{16}\cdot\frac{1}{z-1} - \frac{1}{16}\cdot\frac{1}{z+3} - \frac{1}{4}\cdot\frac{1}{(z+3)^2}$$

$$\Rightarrow x(n) = \frac{1}{16}\cdot\mathcal{Z}^{-1}\left\{\frac{z}{z-1}\right\}(n) - \frac{1}{16}\cdot\mathcal{Z}^{-1}\left\{\frac{z}{z+3}\right\}(n) - \frac{1}{4}\mathcal{Z}^{-1}\left\{\frac{z}{(z+3)^2}\right\}(n)$$

$$= \left[\frac{1}{16} - \frac{1}{16}\cdot(-3)^n - \frac{n}{4}\cdot(-3)^{n-1}\right]\cdot u(n)$$

Probleme propuse

① Utilizând transformarea Z, să se calculeze suma pt. fiecare din următoarele serii numerice:

a) $\displaystyle\sum_{n=1}^{\infty} \frac{n\cdot\sin\frac{n\pi}{6}}{3^n}$ b) $\displaystyle\sum_{n=2}^{\infty} \frac{n^2}{2^n(n+1)}$ c) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+2n+5}{3^n}$

② Utilizând metoda transformării Z, să se rezolve urm. ecuații cu diferențe finite, unde $x \in S_d^+$:

a) $x(n+2) + 3x(n+1) + 2x(n) = (-5)^n$, $n \geq 0$, $x(0) = 2$, $x(1) = -1$.

b) $x(n+2)\cdot x^2(n+1) = e\cdot x^3(n)$, $n \geq 0$, $x(0) = x(1) = 1$

c) $x(n+1)\cdot x^2(n-1) = 2x^2(n)\cdot x(n-2)$, $n \geq 2$, $x(0) = x(1) = 8$, $x(2) = 4$

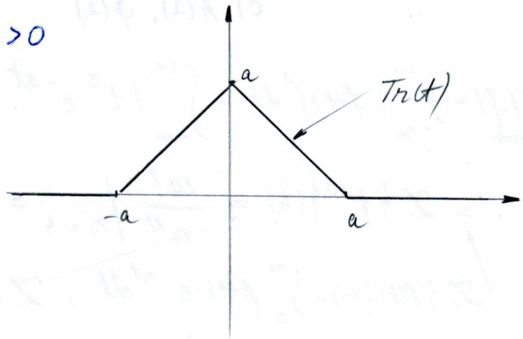
d) $x(n+2)\cdot x^3(n+1) = 9\cdot x^4(n)$, $n \geq 0$, $x(0) = 1$, $x(1) = 3$

Seminarul 14

Recapitulare

① Să se calculeze spectrul Fourier, amplitudinea și faza în frecvență pentru „fereastra triunghiulară"

$$f(t) = \begin{cases} a - |t|, & |t| \le a \\ 0, & |t| > a \end{cases} \qquad a > 0$$



$$\hat{f}(\omega) = F(\omega) = \int_{-\infty}^{\infty} f(t) \cdot e^{-j\omega t} dt$$

$$= \int_{-a}^{a} (a - |t|) \cdot e^{-j\omega t} dt$$

$$= \int_{-a}^{0} (a - |t|) \cdot e^{-j\omega t} dt + \int_{0}^{a} (a - |t|) \cdot e^{-j\omega t} dt$$

$$= \int_{-a}^{0} (a + t) \cdot e^{-j\omega t} dt + \int_{0}^{a} (a - t) \cdot e^{-j\omega t} dt = \ldots$$

— SAU —

$$\hat{f}(\omega) = F(\omega) = \int_{-a}^{a} (a - |t|) \cdot e^{-j\omega t} dt = \int_{-a}^{a} (a - |t|) \cdot (\cos\omega t - j\sin\omega t) dt$$

$$= \underbrace{\int_{-a}^{a} (a - |t|) \cdot \cos\omega t \, dt}_{\text{funcție pară}} - j \cdot \underbrace{\int_{-a}^{a} (a - |t|) \cdot \sin\omega t \, dt}_{\text{funcție impară}}^{=0}$$

Dacă $\omega \neq 0$

$$= 2\int_{0}^{a} (a - |t|) \cdot \cos\omega t \, dt = 2\int_{0}^{a} (a - t) \cdot \cos\omega t \, dt = 2\int_{0}^{a} (a - t) \cdot \left(\frac{\sin\omega t}{\omega}\right)' dt$$

$$= 2(a - t) \cdot \frac{\sin\omega t}{\omega}\Big|_{0}^{a} + \frac{2}{\omega}\int_{0}^{a} \sin\omega t \, dt = 0 - \frac{2}{\omega} \cdot \frac{\cos\omega t}{\omega}\Big|_{0}^{a}$$

$$= -\frac{2\cos\omega t}{\omega^2}\Big|_{0}^{a} = -\frac{2}{\omega^2}\cos a\omega + \frac{2}{\omega^2} = \frac{2}{\omega^2}(1 - \cos a\omega)$$

$$= \frac{2}{\omega^2} \cdot 2 \cdot \sin^2\frac{a\omega}{2} = \frac{4\sin^2\frac{a\omega}{2}}{\omega^2} = \frac{\sin^2\frac{a\omega}{2}}{\left(\frac{a\omega}{2}\right)^2} \cdot a^2$$

Dacă $\omega = 0$, $\hat{f}(\omega) = 2\int_{0}^{a} (a - t) dt = 2 \cdot \left(at - \frac{t^2}{2}\right)\Big|_{0}^{a} = 2\left(a^2 - \frac{a^2}{2}\right) = a^2$

Obs: $sa(t) = \begin{cases} \frac{\sin t}{t}, & t \neq 0 \\ 1, & t = 0 \end{cases}$

funcția sinus atenuat

În final, ținând cont de obs

$$\hat{f}(\omega) = a^2 \cdot sa^2\left(\frac{a\omega}{2}\right)$$

$A(\omega) = |\hat{f}(\omega)| = a^2 \cdot sa^2\left(\frac{a\omega}{2}\right)$, iar $\phi(\omega) = \arg \hat{f}(\omega) = 0$

② Fie $f:\mathbb{R}\to\mathbb{R}$,

$$f(t)=t^5\cdot e^{-2t}\cdot u(t)$$

Să se calculeze: a) $E(f)$

b) $\hat{f}(\omega)$

c) $A(2),\ \phi(2)$

Obs: $u(t)=\begin{cases}1, & t\geq 0\\ 0, & t<0\end{cases}$

a) $$E(f)=\int_{-\infty}^{\infty}|f(t)|^2\,dt=\int_{-\infty}^{\infty}|t^5\cdot e^{-2t}\cdot u(t)|^2\,dt=\int_{-\infty}^{\infty}t^{10}\cdot e^{-4t}\cdot u(t)\,dt=\int_0^{\infty}t^{10}\cdot e^{-4t}\,dt$$

$$=\mathcal{L}\{t^{10}\}(4)=\frac{10!}{s^{11}}\Big|_{s=4}=\frac{10!}{4^{11}}$$

$\mathcal{L}\{f(t)\}(s)=\int_0^{\infty}f(t)\cdot e^{-st}\,dt$, $\mathcal{L}\{t^n\}(s)=\frac{n!}{s^{n+1}}$

b) $$\hat{f}(\omega)=F(\omega)=\int_{-\infty}^{\infty}f(t)\cdot e^{-j\omega t}\,dt$$

$$\hat{f}(\omega)=\mathcal{F}(f(t);\omega)=\mathcal{F}(t^5\cdot e^{-2t}\cdot u(t);\omega)=j^5\cdot\left(\mathcal{F}(e^{-2t}\cdot u(t);\omega)\right)^{(5)}, \text{ unde}$$

T. derivării spectrului $\left(\mathcal{F}(t^n\cdot f(t);\omega)=j^n\cdot\left(\mathcal{F}(f(t);\omega)\right)^{(n)}\right)$

$$\mathcal{F}(e^{-2t}\cdot u(t);\omega)=\int_{-\infty}^{\infty}e^{-2t}\cdot u(t)\cdot e^{-j\omega t}\,dt=\int_0^{\infty}e^{-2t}\cdot e^{-j\omega t}\,dt=\int_0^{\infty}e^{-t(2+j\omega)}\,dt$$

$$=\frac{e^{-t(2+j\omega)}}{-(2+j\omega)}\Big|_0^{\infty}=\frac{1}{2+j\omega}$$

Aşadar, $\hat{f}(\omega)=j\cdot\left(\frac{1}{2+j\omega}\right)^{(5)}=\dots$

= SAU =

$\mathcal{L}\{t^n\}(s)=\frac{n!}{s^{n+1}}$

$$\hat{f}(\omega)=\int_{-\infty}^{\infty}t^5\cdot e^{-2t}\cdot u(t)\cdot e^{-j\omega t}\,dt=\int_0^{\infty}t^5\cdot e^{-(2+j\omega)t}\,dt=\mathcal{L}\{t^5\}(2+j\omega)=\frac{5!}{(2+j\omega)^6}=\frac{5!(2-j\omega)^6}{(\omega^2+4)^6}$$

c) $A(\omega)=|\hat{f}(\omega)|$

$$A(2)=|\hat{f}(2)|=\left|\frac{5!}{(2+2j)^6}\right|=\frac{5!}{(2\sqrt{2})^6}=\frac{5!}{2^9}=\frac{15}{2^6}=\frac{15}{64}$$

$$\phi(\omega)=\arg\hat{f}(\omega)=\left(\arg\frac{5!}{(\omega^2+4)^6}+6\arg(2-j\omega)\right)\bmod 2\pi=(6\arg(2-j\omega))\bmod 2\pi$$

$$\phi(2)=(6\cdot\arg(2-2j))\bmod 2\pi=\left(6\cdot(-\operatorname{arctg}1+2\pi)\right)\bmod 2\pi=\left(6\cdot\left(2\pi-\frac{\pi}{4}\right)\right)\bmod 2\pi$$

$$=\left(6\cdot\frac{7\pi}{4}\right)\bmod 2\pi=\left(\frac{21\pi}{2}\right)\bmod 2\pi=\frac{\pi}{2}$$

sau

$$\phi(2)=(\arg 5!-6\cdot\arg(2+2j))\bmod 2\pi=\left(-6\cdot\frac{\pi}{4}\right)\bmod 2\pi=\frac{\pi}{2}$$

(3) Să se calculeze spectrul Fourier pentru $f: \mathbb{R} \to \mathbb{C}$, unde

$$f(t) = t^2 \cdot e^{-\frac{t^2}{4}}$$

=

$$\hat{f}(\omega) = F(\omega) = \mathcal{F}(t^2 \cdot e^{-\frac{t^2}{4}}; \omega) = j^2 \cdot \left(\mathcal{F}(e^{-\frac{t^2}{4}}; \omega)\right)'' \quad (1)$$

T. derivării spectrului $\mathcal{F}(t^n f(t); \omega) = j^n \cdot \left(\mathcal{F}(f(t); \omega)\right)^{(n)}$

unde

$$\mathcal{F}(e^{-\frac{t^2}{4}}; \omega) = \int_{-\infty}^{\infty} e^{-\frac{t^2}{4}} \cdot e^{-j\omega t} dt = \int_{-\infty}^{\infty} e^{-\frac{t^2}{4} - j\omega t} dt = \int_{-\infty}^{\infty} e^{-\left(\frac{t^2}{4} + j\omega t\right)} dt$$

$$= \int_{-\infty}^{\infty} e^{-\left(\frac{t}{2} + j\omega\right)^2 - \omega^2} dt = e^{-\omega^2} \cdot \int_{-\infty}^{\infty} e^{-\left(\frac{t}{2}+j\omega\right)^2} dt \quad \frac{\frac{t}{2} + j\omega = u}{\frac{1}{2} dt = du}$$

$$= 2 e^{-\omega^2} \cdot \int_{-\infty}^{\infty} e^{-u^2} du = 2\sqrt{\pi} \cdot e^{-\omega^2} \quad (2)$$

$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$, $\int_{0}^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$ (integrala Euler-Poisson)

Din (1) și (2)

$$\hat{f}(\omega) = -2\sqrt{\pi} \cdot \left(e^{-\omega^2}\right)'' = -2\sqrt{\pi} \cdot \left(-2\omega \cdot e^{-\omega^2}\right)' = 4\sqrt{\pi} \cdot \left(\omega \cdot e^{-\omega^2}\right)'$$

$$= 4\sqrt{\pi} \cdot e^{-\omega^2} (1 - 2\omega^2)$$

(4) Să se rezolve ecuația integrală Fourier

$$\int_0^{\infty} f(t) \sin \omega t \, dt = \underbrace{\omega^2 \cdot e^{-3\omega}}_{g(\omega)}, \quad \omega > 0$$

=

$$f(t) = (\mathcal{F}_s^{-1} g)(t) = \frac{2}{\pi} \cdot \int_0^{\infty} g(\omega) \cdot \sin \omega t \, d\omega = \frac{2}{\pi} \cdot \int_0^{\infty} \omega^2 \cdot e^{-3\omega} \sin \omega t \, d\omega$$

Așadar, $f(t) = \frac{2}{\pi} \int_0^{\infty} \omega^2 \cdot e^{-3\omega} \sin \omega t \, d\omega$

Considerăm funcția $h(t) = \frac{2}{\pi} \int_0^{\infty} \omega^2 \cdot e^{-3\omega} \cos \omega t \, d\omega$ și formăm expresia

$$h(t) + j \cdot f(t) = \frac{2}{\pi} \int_0^{\infty} \omega^2 \cdot e^{-3\omega} \cdot (\cos \omega t + j \sin \omega t) d\omega = \frac{2}{\pi} \int_0^{\infty} \omega^2 \cdot e^{-3\omega} \cdot e^{j\omega t} d\omega$$

$$= \frac{2}{\pi} \int_0^{\infty} \omega^2 \cdot e^{-(3 - jt)\omega} d\omega = \frac{2}{\pi} \mathcal{L}\{\omega^2\}(3 - jt)$$

$\mathcal{L}\{f(t)\}(s) = \int_0^{\infty} f(t) \cdot e^{-st} dt = \int_0^{\infty} f(\omega) \cdot e^{-t\omega} d\omega$

$$= \frac{2}{\pi} \cdot \frac{2!}{(3 - jt)^3} = \frac{4}{\pi (3 - jt)^3} =$$

$\mathcal{L}\{t^n\}(s) = \frac{n!}{s^{n+1}}$

$$= \frac{4(3 + jt)^3}{\pi(9 + t^2)^3}$$

Am obținut

$$h(t)+j\,f(t)=\frac{4\cdot(3+jt)^3}{\pi\cdot(9+t^2)^3}=\frac{4}{\pi(9+t^2)^3}\cdot\left[27+27jt-9t^2-jt^3\right]$$

$$=\frac{4}{\pi(9+t^2)^3}\cdot\left[(27-9t^2)+j\cdot(27t-t^3)\right]$$

$$\rightarrow f(t)=\frac{4(27t-t^3)}{\pi(9+t^2)^3}$$

5 Să se calculeze: $I=\int_0^\infty \frac{\cos^2(at)-\cos^2(bt)}{t}\,dt,\quad a,b\in\mathbb{R},\ a\neq b$

$$I=\int_0^\infty \frac{\cos^2(at)-\cos^2(bt)}{t}\,dt \overset{def}{=} \mathcal{L}\left\{\frac{\cos^2(at)-\cos^2(bt)}{t}\right\}(0)$$

$$\left(\mathcal{L}\left\{\frac{f(t)}{t}\right\}(s)=\int_s^\infty \mathcal{L}\{f(t)\}(y)\,dy\right)$$

$$=\int_0^\infty \mathcal{L}\{\cos^2(at)-\cos^2(bt)\}(s)\,ds \qquad (1)$$

Dar $\mathcal{L}\{\cos^2(at)\}(s)=\mathcal{L}\left\{\frac{1+\cos(2at)}{2}\right\}(s)=\frac{1}{2s}+\frac{1}{2}\cdot\frac{s}{s^2+4a^2}$ (2). Analog $\mathcal{L}\{\cos^2(bt)\}(s)$...

Din (1) și (2) obținem

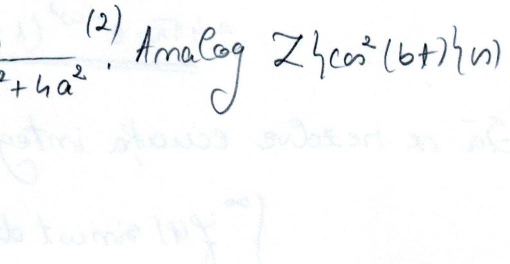
$$I=\frac{1}{2}\int_0^\infty\left(\cancel{\frac{1}{s}}+\frac{s}{s^2+4a^2}-\cancel{\frac{1}{s}}-\frac{s}{s^2+4b^2}\right)ds$$

$$=\frac{1}{4}\cdot\int_0^\infty\left(\frac{2s}{s^2+4a^2}-\frac{2s}{s^2+4b^2}\right)ds=\frac{1}{4}\left(\ln|s^2+4a^2|-\ln|s^2+4b^2|\right)\Big|_0^\infty$$

$$=\frac{1}{4}\ln\frac{s^2+4a^2}{s^2+4b^2}\Big|_0^\infty=-\frac{1}{4}\ln\frac{4a^2}{4b^2}=-\frac{1}{4}\ln\left(\frac{a}{b}\right)^2$$

6 Să se calculeze transformata Laplace pentru

$$f(t)=\begin{cases} e^{2-t}, & 0\le t<3\\ 0, & 3\le t<5\\ f(t-5), & t\ge 5\end{cases}$$



$$f(t)\overset{?}{=}f(t+5)$$

Cum
$$f(t+5)=\begin{cases} e^{2-t-5}, & 0\le t<3\\ 0, & 3\le t<5\\ f(t), & t+5\ge 5\end{cases}=\begin{cases} e^{-3-t}, & -5\le t<-2\\ 0, & -2\le t<0\\ f(t), & t\ge 0\end{cases}$$

f - funcție periodică de perioadă principală $\underline{T=5}$

Așadar, $\mathcal{L}\{f(t)\}(s) = \dfrac{1}{1-e^{-sT}} \cdot \displaystyle\int_0^T f(t) \cdot e^{-st} dt$

$$= \frac{1}{1-e^{-5s}} \cdot \int_0^5 f(t) \cdot e^{-st} dt = \frac{1}{1-e^{-5s}} \int_0^3 e^{2-t-st} dt$$

$$= \frac{1}{1-e^{-5s}} \int_0^3 e^{2-t(1+s)} dt = \frac{1}{1-e^{-5s}} \cdot \left. \frac{e^{2-t(1+s)}}{-(1+s)} \right|_0^3$$

$$= -\frac{1}{1-e^{-5s}} \cdot \frac{e^{2-3(1+s)}}{1+s} + \frac{1}{1-e^{-5s}} \cdot \frac{e^2}{1+s} = \frac{1}{(1+s)(1-e^{-5s})}\left[e^2 - e^{-1-3s}\right]$$