Numerical Calculus

Pătrulescu Flavius

Technical University of Cluj-Napoca

2025

References

1. C.D. Meyer, *Matrix Analysis and Applied Linear Algebra*, SIAM, Philadelphia, 2000.
2. A. Quarteroni, R. Sacco, F. Saleri, *Numerical Mathematics*, Springer-Verlag, New-York, 2000.

Vector space (Meyer, 2000)

$(\mathbb{K}, +, \cdot)$ -scalar field, $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, $\mathcal{V} \neq \varnothing$ (vectors)
Vector addition and scalar multiplication:

$$+ : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$$

$$(u, v) \in \mathcal{V} \times \mathcal{V} \to u + v \in \mathcal{V}$$

$$\cdot : \mathbb{K} \times \mathcal{V} \to \mathcal{V}$$

$$(\alpha, v) \in \mathbb{K} \times \mathcal{V} \to \alpha \cdot v \in \mathcal{V}$$

$(\mathcal{V}, +, \cdot)$ *is a vector space over* $\mathbb{K}$

1. $(\mathcal{V}, +)$ is *abelian* group
   - 1.1 $\boldsymbol{x} + \boldsymbol{y} = \boldsymbol{y} + \boldsymbol{x}$, $\forall\, \boldsymbol{x}, \boldsymbol{y} \in \mathcal{V}$
   - 1.2 $(\boldsymbol{x} + \boldsymbol{y}) + \boldsymbol{z} = \boldsymbol{x} + (\boldsymbol{y} + \boldsymbol{z})$, $\forall\, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathcal{V}$
   - 1.3 $\exists!\, \mathbf{0}_{\mathcal{V}} \in \mathcal{V}$ s.t. $\boldsymbol{x} + \mathbf{0}_{\mathcal{V}} = \boldsymbol{x}$, $\forall\, \boldsymbol{x} \in \mathcal{V}$
   - 1.4 $\forall\, \boldsymbol{x} \in \mathcal{V}$, $\exists!\, (-\boldsymbol{x}) \in \mathcal{V}$ s.t. $\boldsymbol{x} + (-\boldsymbol{x}) = \mathbf{0}_{\mathcal{V}}$
2. $(\alpha \cdot \beta) \cdot \boldsymbol{x} = \alpha \cdot (\beta \cdot \boldsymbol{x})$, $\forall\, \alpha, \beta \in \mathbb{K}$, $\boldsymbol{x} \in \mathcal{V}$
3. $\alpha \cdot (\boldsymbol{x} + \boldsymbol{y}) = \alpha \cdot \boldsymbol{x} + \alpha \cdot \boldsymbol{y}$, $\forall\, \alpha \in \mathbb{K}$, $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{V}$
4. $(\alpha + \beta) \cdot \boldsymbol{x} = \alpha \cdot \boldsymbol{x} + \beta \cdot \boldsymbol{x}$, $\forall\, \alpha, \beta \in \mathbb{K}$, $\boldsymbol{x} \in \mathcal{V}$
5. $1 \cdot \boldsymbol{x} = \boldsymbol{x}$, $\forall\, \boldsymbol{x} \in \mathcal{V}$

Examples (Meyer, 2000)

- *Real coordinate space*: $(\mathbb{R}^n, +, \cdot)$, $n \in \mathbb{N}^*$ (over $\mathbb{R}$)

$$\mathbb{R}^n = \{\boldsymbol{x} | \boldsymbol{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, x_i \in \mathbb{R}\}$$

$$\boldsymbol{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \boldsymbol{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \Rightarrow \boldsymbol{x} + \boldsymbol{y} = \begin{pmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{pmatrix}, \alpha\boldsymbol{x} = \begin{pmatrix} \alpha x_1 \\ \vdots \\ \alpha x_n \end{pmatrix}$$

- $(\mathcal{M}_{mn}(\mathbb{R}), +, \cdot)$, $m, n \in \mathbb{N}^*$ (over $\mathbb{R}$)
- $(C[a, b], +, \cdot)$ (over $\mathbb{R}$)

$$C[a, b] = \{f : [a, b] \to \mathbb{R} | f - \textit{continuous}\}$$

$$f, g \in C[a, b] \Rightarrow (f + g)(x) = f(x) + g(x), (\alpha f)(x) = \alpha f(x)$$

Examples (Meyer, 2000)

- $(\mathbb{C}[X], +, \cdot)$ (over $\mathbb{C}$)
- $(\mathbb{R}[X], +, \cdot)$ (over $\mathbb{R}$)
- $(\Pi_n, +, \cdot)$, $n \in \mathbb{N}$ (over $\mathbb{R}$)

$$\Pi_n = \{P \in \mathbb{R}[X] : deg(P) \leqslant n\}$$

- $(\mathcal{L}, +, \cdot)$ (over $\mathbb{R}$)

$$\mathcal{L} = \{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 : x_2 = \alpha x_1\} \subset \mathbb{R}^2, \ \alpha \in \mathbb{R}^*$$

- $(C^1[a, b], +, \cdot)$ (over $\mathbb{R}$)

Subspace (Meyer, 2000)

Let $\mathcal{S}$ be a nonempty subset of a vector space $\mathcal{V}$ over $\mathbb{K}$ (symbolically, $\mathcal{S} \subseteq \mathcal{V}$). If S is also a vector space over $\mathbb{K}$ using the same addition and scalar multiplication operations, then $\mathcal{S}$ is said to be a *subspace* of $\mathcal{V}$.
It's not necessary to check all 10 of the defining conditions in order to determine if a subset is also a subspace-only the closure conditions need to be considered. That is, a nonempty subset $\mathcal{S}$ of a vector space $\mathcal{V}$ is a subspace of $\mathcal{V}$ if and only if

1. $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{S} \Rightarrow \boldsymbol{x} + \boldsymbol{y} \in \mathcal{S}$
2. $\boldsymbol{x} \in \mathcal{S} \Rightarrow \alpha \cdot \boldsymbol{x} \in \mathcal{S}, \forall \alpha \in \mathbb{K}$

Examples (Meyer, 2000)

- Given a vector space $\mathcal{V}$, the set $\mathcal{Z} = \{\mathbf{0}_{\mathcal{V}}\}$ containing only the zero vector is a subspace. Naturally, this subspace is called the *trivial subspace*.
- $\mathcal{L} \subset \mathbb{R}^2$ (the straight lines that pass through the origin) is a *proper* subspace of $(\mathbb{R}^2, +, \cdot)$.
- In $(\mathbb{R}^3, +, \cdot)$ the trivial subspace and lines through the origin are again subspaces, but there is also another one—planes through the origin.
- $(C^1[a, b], +, \cdot)$ is a subspace of $C([a, b], +, \cdot)$
- Π_n is a subspace of $(\mathbb{R}[X], +, \cdot)$, $\forall\, n \in \mathbb{N}^*$

Spanning sets (Meyer, 2000)

For a set of vectors $\mathcal{S} = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n\} \subset \mathcal{V}$ the set of all possible linear combinations of the $\boldsymbol{v}_i$'s is denoted by

$$span(\mathcal{S}) = \{\alpha_1 \boldsymbol{v}_1 + \ldots + \alpha_n \boldsymbol{v}_n | \alpha_i \in \mathbb{K}\}$$

Notice that $span(\mathcal{S})$ is a subspace of $\mathcal{V}$; it is called *space spanned by* $\mathcal{S}$.
If $\mathcal{V}$ is a vector space such that $\mathcal{V} = span(\mathcal{S})$, we say $\mathcal{S}$ is a *spanning set* for $\mathcal{V}$. In other words, $\mathcal{S}$ spans $\mathcal{V}$ whenever each vector in $\mathcal{V}$ is a linear combination of vectors from $\mathcal{S}$.

Examples (Meyer, 2000)

- The vectors $\left\{ \boldsymbol{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \boldsymbol{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \boldsymbol{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ span $\mathbb{R}^3$
- The units vectors

$$\mathcal{E} = \left\{ \boldsymbol{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \boldsymbol{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \boldsymbol{e}_n = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \right\} \subset \mathbb{R}^n$$

  form a spanning set for $\mathbb{R}^n$.
- The finite set $\{1, x, \ldots, x^n\}$ spans Π_n (polynomials with degree $\leqslant n$) and the infinite set $\{1, x, x^2, \ldots\}$ spans $\mathbb{R}[X]$ or $\mathbb{C}[X]$.

Linear Independence (Meyer, 2000)

A set of vectors $\mathcal{S} = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ is said to be a *linearly independent set* whenever the only solution for the scalars α_i in the homogeneous equation

$$\alpha_1 \boldsymbol{v}_1 + \ldots + \alpha_n \boldsymbol{v}_n = \mathbf{0}_{\mathcal{V}}$$

is the trivial solution $\alpha_1 = \ldots = \alpha_n = 0$. Whenever there is a nontrivial solution for the α's (*i.e.*, at least one $\alpha_i \neq 0$), the set $\mathcal{S}$ is said to be a *linearly dependent set*. In other words, linearly independent sets are those that contain no dependency relationships, and linearly dependent sets are those in which at least one vector is a combination of the others.

Examples (Meyer, 2000)

- The unit vectors $\mathcal{E} = \{\boldsymbol{e}_1, \dots, \boldsymbol{e}_n\} \subset \mathbb{R}^n$ form a linearly independent set.
- *Vandermonde Matrix*

$$V_{m \times n} = \begin{pmatrix} 1 & x_1 & x_1^2 & \dots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \dots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_m & x_m^2 & \dots & x_m^{n-1} \end{pmatrix}$$

The columns in V constitute a linearly independent set whenever $n \leqslant m$.

Examples (Meyer, 2000)

- Let $f_1, f_2, \ldots, f_n \in C^\infty(\mathbb{R})$. *Wronski Matrix*

$$W(x) = \begin{pmatrix} f_1(x) & f_2(x) & \ldots & f_n(x) \\ f_1'(x) & f_2'(x) & \ldots & f_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \ldots & f_n^{(n-1)}(x) \end{pmatrix}$$

  If there is at least one point $x = x_0$ s.t. $W(x_0)$ is nonsingular then $\{f_1, f_2, \ldots f_n\}$ is a linearly independent set.
- The set of polynomials $\{1, x, x^2, \ldots, x^n\}$ is linearly independent in Π_n.

Basis. Dimension (Meyer, 2000)

A linearly independent spanning set for a vector space $\mathcal{V}$ is called a *basis* for $\mathcal{V}$.

Spaces that possess a basis containing an infinite number of vectors are referred to as *infinite-dimensional spaces*, and those that have a finite basis are called *finite-dimensional spaces*.

Although a space $\mathcal{V}$ can have many different bases, the preceding result guarantees that all bases for $\mathcal{V}$ contain the same number of vectors. The dimension of a vector space $\mathcal{V}$ is defined to be

$$dim(\mathcal{V}) = \text{ number of vectors in any basis for } \mathcal{V}$$

Examples (Meyer, 2000)

- The unit vectors $\mathcal{E} = \{\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n\}$ in $\mathbb{R}^n$ are a basis for $\mathbb{R}^n$. This is called the *standard basis* for $\mathbb{R}^n$. $dim(\mathbb{R}^n) = n$
- The set $\{1, x, x^2, \ldots, x^n\}$ is a basis for the vector space Π_n (set of polynomials having degree n or less). $dim(\Pi_n) = n + 1$
- The set $\{1, x, x^2, \ldots\}$ is a basis for the vector space $\mathbb{R}[X]$ (all polynomials). $dim(\mathbb{R}[X]) = \infty$
- The set $\{1, i\}$ $(i^2 = -1)$ is a basis for $\mathbb{C}$. $dim(\mathbb{C}) = 2$

General inner product (Meyer, 2000)

$\langle \cdot, \cdot \rangle : \mathcal{V} \times \mathcal{V} \to \mathbb{K}$, $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$

1. $\langle \boldsymbol{x}, \boldsymbol{x} \rangle \in \mathbb{R}, \langle \boldsymbol{x}, \boldsymbol{x} \rangle \geqslant 0$, $\forall\, \boldsymbol{x} \in \mathcal{V}$; $\quad \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0 \Leftrightarrow \boldsymbol{x} = \boldsymbol{0}_{\mathcal{V}}$
2. $\langle \alpha \boldsymbol{x}, \boldsymbol{y} \rangle = \alpha \langle \boldsymbol{x}, \boldsymbol{y} \rangle$, $\forall\, \boldsymbol{x}, \boldsymbol{y} \in \mathcal{V}$, $\alpha \in \mathbb{K}$
3. $\langle \boldsymbol{x} + \boldsymbol{y}, \boldsymbol{z} \rangle = \langle \boldsymbol{x}, \boldsymbol{z} \rangle + \langle \boldsymbol{y}, \boldsymbol{z} \rangle$, $\forall\, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathcal{V}$
4. $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \overline{\langle \boldsymbol{y}, \boldsymbol{x} \rangle}$ (For $\mathbb{K} = \mathbb{R}$: $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \langle \boldsymbol{y}, \boldsymbol{x} \rangle$)

Properties

1. $\langle \boldsymbol{x}, \boldsymbol{y} + \boldsymbol{z} \rangle = \langle \boldsymbol{x}, \boldsymbol{y} \rangle + \langle \boldsymbol{x}, \boldsymbol{z} \rangle$, $\forall\, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathcal{V}$
2. $\langle \boldsymbol{x}, \beta \boldsymbol{y} \rangle = \overline{\beta} \langle \boldsymbol{x}, \boldsymbol{y} \rangle$, $\forall\, \boldsymbol{x}, \boldsymbol{y} \in \mathcal{V}$, $\beta \in \mathbb{K}$
3. $\langle \alpha_1 \boldsymbol{x}_1 + \alpha_2 \boldsymbol{x}_2, \beta_1 \boldsymbol{y}_1 + \beta_2 \boldsymbol{y}_2 \rangle =$
   $= \alpha_1 \overline{\beta}_1 \langle \boldsymbol{x}_1, \boldsymbol{y}_1 \rangle + \alpha_1 \overline{\beta}_2 \langle \boldsymbol{x}_1, \boldsymbol{y}_2 \rangle + \alpha_2 \overline{\beta}_1 \langle \boldsymbol{x}_2, \boldsymbol{y}_1 \rangle + \alpha_2 \overline{\beta}_2 \langle \boldsymbol{x}_2, \boldsymbol{y}_2 \rangle$

Examples (Meyer, 2000)

1. *The standard inner products*

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \boldsymbol{x}^t \boldsymbol{y} = \sum_{i=1}^{n} x_i y_i \text{ (in } \mathbb{R}^n)$$

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \boldsymbol{x}^* \boldsymbol{y} = \sum_{i=1}^{n} \bar{x}_i y_i \text{ (in } \mathbb{C}^n)$$

2. *A-inner product (elliptical inner product):* A non-singular (A^*-conjugate transpose: $a_{ij}^* = \bar{a}_{ji}$)

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle_A = \boldsymbol{x}^* A^* A \boldsymbol{x} \text{ (in } \mathbb{C}^n)$$

3. In $C([a, b])$

$$\langle f, g \rangle_w = \int_a^b w(t) f(t) g(t)\, dt,$$

$w \in C([a, b])$ is a positive function, the *weight* function.

Norms (Meyer, 2000; Quarteroni *et al.* 2000)

$\| \cdot \| : \mathcal{V} \to \mathbb{R}$

1. $\|\boldsymbol{x}\| \geqslant 0 \quad \forall\, \boldsymbol{x} \in \mathcal{V}, \qquad \|\boldsymbol{x}\| = 0 \Leftrightarrow \boldsymbol{x} = \boldsymbol{0}_{\mathcal{V}}$
2. $\|\alpha \boldsymbol{x}\| = |\alpha|\|\boldsymbol{x}\|, \quad \forall\, \boldsymbol{x} \in \mathcal{V},\, \alpha \in \mathbb{K}$
3. $\|\boldsymbol{x} + \boldsymbol{y}\| \leqslant \|\boldsymbol{x}\| + \|\boldsymbol{y}\|, \quad \forall\, \boldsymbol{x}, \boldsymbol{y} \in \mathcal{V}$ (triangle inequality)

Backward Triangle Inequality. The triangle inequality produces an upper bound for a sum, but it also yields the following lower bound for a difference:

$$\Big| \|\boldsymbol{x}\| - \|\boldsymbol{y}\| \Big| \leqslant \|\boldsymbol{x} - \boldsymbol{y}\|.$$

Continuity. In the case where $\mathcal{V}$ is a finite dimensional space any norm $\| \cdot \|$ defined on $\mathcal{V}$ is a continuous function of its argument, namely, $\forall\, \varepsilon > 0$, $\exists\, C > 0$ s.t. $\|\boldsymbol{x} - \hat{\boldsymbol{x}}\| \leqslant \varepsilon$ then

$$\Big| \|\boldsymbol{x}\| - \|\hat{\boldsymbol{x}}\| \Big| < C\varepsilon,\ \forall\, \boldsymbol{x}, \hat{\boldsymbol{x}} \in \mathcal{V}.$$

p-norm (Meyer, 2000)

For $p \geqslant 1$ the *p-norm* in $\mathbb{R}^n$ or $\mathbb{C}^n$ is defined as

$$\|\boldsymbol{x}\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}}$$

Particular cases

$$p = 1 : \quad \|\boldsymbol{x}\|_1 = \sum_{i=1}^{n} |x_i| \text{ (Minkowski norm)}$$

$$p = 2 : \quad \|\boldsymbol{x}\|_2 = \left(\sum_{i=1}^{n} |x_i|^2 \right)^{\frac{1}{2}} \text{ (euclidean norm)}$$

$$p = \infty : \quad \|\boldsymbol{x}\|_\infty = \max_{i=\overline{1,n}} |x_i| \text{ (Chebyshev norm)}$$

Norm inequalities (Meyer, 2000)

$$\lim_{p\to\infty} \|\boldsymbol{x}\|_p = \|\boldsymbol{x}\|_\infty$$

Hölder inequality: If $p > 1$ and $q > 1$ are real numbers s.t. $1/p + 1/q = 1$ then

$$|\boldsymbol{x}^*\boldsymbol{y}| \leqslant \|\boldsymbol{x}\|_p \|\boldsymbol{y}\|_q$$

C.B.S. inequality (Hölder inequality $p = q = 2$):

$$|\boldsymbol{x}^*\boldsymbol{y}| \leqslant \|\boldsymbol{x}\|_2 \|\boldsymbol{y}\|_2$$

Equality holds if and only if $\boldsymbol{y} = \alpha\boldsymbol{x}$, $\alpha = (\boldsymbol{x}^*\boldsymbol{y})/(\boldsymbol{x}^*\boldsymbol{x})$
Minkowski inequality: for $p > 1$

$$\|\boldsymbol{x} + \boldsymbol{y}\|_p \leqslant \|\boldsymbol{x}\|_p + \|\boldsymbol{y}\|_p$$

Equivalent norms (Quarteroni *et al.* 2000)

Two norms $\|\cdot\|_p$ and $\|\cdot\|_q$ on $\mathcal{V}$ are *equivalent* if there exist two positive constants c_1, c_2 s.t.

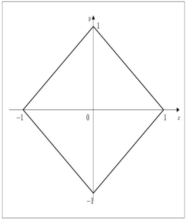
$$c_1\|\boldsymbol{x}\|_q \leqslant \|\boldsymbol{x}\|_p \leqslant c_2\|\boldsymbol{x}\|_q.$$

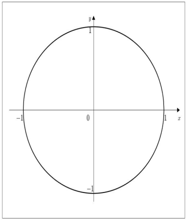
In finite-dimensional spaces all norms are equivalent.
In the case $\mathcal{V} = \mathbb{R}^n$ we have $\forall\, \boldsymbol{x} \in \mathbb{R}^n$

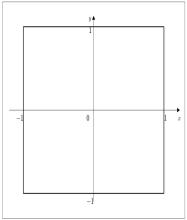
$$\begin{aligned}
&a)\ \|\boldsymbol{x}\|_2 \leqslant \|\boldsymbol{x}\|_1 \leqslant \sqrt{n}\|\boldsymbol{x}\|_2\\
&b)\ \|\boldsymbol{x}\|_\infty \leqslant \|\boldsymbol{x}\|_2 \leqslant \sqrt{n}\|\boldsymbol{x}\|_\infty\\
&c)\ \|\boldsymbol{x}\|_\infty \leqslant \|\boldsymbol{x}\|_1 \leqslant n\|\boldsymbol{x}\|_\infty
\end{aligned}$$

Unit p-spheres (Meyer, 2000)

$$\mathcal{S}_1 = \{\boldsymbol{x} : \|\boldsymbol{x}\|_1 = 1\},\ \mathcal{S}_2 = \{\boldsymbol{x} : \|\boldsymbol{x}\|_2 = 1\},\ \mathcal{S}_\infty = \{\boldsymbol{x} : \|\boldsymbol{x}\|_\infty = 1\}$$







Norms in Inner-Product Spaces (Meyer, 2000)

If $\mathcal{V}$ is an inner-product space with an inner product $\langle \cdot, \cdot \rangle$, then

$$\|\boldsymbol{x}\| = \sqrt{\langle \boldsymbol{x}, \boldsymbol{x} \rangle}, \ \forall \boldsymbol{x} \in \mathcal{V}$$

defines a norm on $\mathcal{V}$.
General C.B.S. inequality

$$|\langle \boldsymbol{x}, \boldsymbol{y} \rangle| \leqslant \|\boldsymbol{x}\| \|\boldsymbol{y}\|.$$

Equality holds if and only if $\boldsymbol{y} = \alpha \boldsymbol{x}$, $\alpha = \langle \boldsymbol{x}, \boldsymbol{y} \rangle / \|\boldsymbol{x}\|^2$.

Examples (Meyer, 2000)

- In $C[a,b]$

$$\|f\|_2 = \sqrt{\langle f, f\rangle} = \left(\int_a^b |f(t)|^2\, dt\right)^{\frac{1}{2}}.$$

- *A-norm (elliptical norm)* $A \in \mathcal{M}_n(\mathbb{C})$, $\det(A) \neq 0$

$$\|\boldsymbol{x}\|_A = \sqrt{\langle \boldsymbol{x}, \boldsymbol{x}\rangle_A} = \sqrt{\boldsymbol{x}^* A^* A \boldsymbol{x}} = \|A\boldsymbol{x}\|_2.$$

- In $\mathbb{R}^n$ the standard inner product generates the euclidean norm.

$$\sqrt{\langle \boldsymbol{x}, \boldsymbol{x}\rangle} = \sqrt{\boldsymbol{x}^t \boldsymbol{x}} = \left(\sum_{i=1}^n x_i^2\right)^{\frac{1}{2}} = \|\boldsymbol{x}\|_2.$$

Parallelogram Identity (Meyer, 2000)

For a given norm $\|\cdot\|$ on a vector space $\mathcal{V}$, there exists an inner product on $\mathcal{V}$ such that $\langle\cdot,\cdot\rangle = \|\cdot\|^2$ if and only if the *parallelogram identity*

$$\|\boldsymbol{x}+\boldsymbol{y}\|^2 + \|\boldsymbol{x}-\boldsymbol{y}\|^2 = 2(\|\boldsymbol{x}\|^2 + \|\boldsymbol{y}\|^2)$$

holds for all $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{V}$.

The parallelogram identity doesn't hold for p-norms when $p \neq 2$.

$$\boldsymbol{x} = \boldsymbol{e}_1,\ \boldsymbol{y} = \boldsymbol{e}_2 \Rightarrow \begin{cases} \|\boldsymbol{e}_1+\boldsymbol{e}_2\|_p^2 + \|\boldsymbol{e}_1-\boldsymbol{e}_2\|_p^2 = 2^{\frac{p+2}{2}} \\ 2(\|\boldsymbol{e}_1\|_p^2 + \|\boldsymbol{e}_2\|_p^2) = 4 \end{cases}$$

Orthogonality. Angles (Meyer, 2000)

In an inner-product space $\mathcal{V}$, two vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{V}$ are said to be *orthogonal* (to each other) whenever $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = 0$, and this is denoted by writing $\boldsymbol{x} \perp \boldsymbol{y}$.
For $\mathbb{R}^n$ with the standard inner product, $\boldsymbol{x} \perp \boldsymbol{y} \Leftrightarrow \boldsymbol{x}^t \boldsymbol{y} = 0$.
For $\mathbb{C}^n$ with the standard inner product, $\boldsymbol{x} \perp \boldsymbol{y} \Leftrightarrow \boldsymbol{x}^* \boldsymbol{y} = 0$.

In a real inner-product space $\mathcal{V}$, the radian measure of the *angle* between nonzero vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{V}$ is defined to be the number $\theta \in [0, \pi]$ such that

$$\cos\theta = \frac{\langle \boldsymbol{x}, \boldsymbol{y} \rangle}{\|\boldsymbol{x}\| \|\boldsymbol{y}\|}.$$

Orthonormal Sets (Meyer, 2000)

The set $\mathcal{B} = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n\} \in \mathcal{V}$ is called *orthogonal set* whenever $\boldsymbol{u}_i \perp \boldsymbol{u}_j, \forall\, i \neq j$. Moreover, if $\|\boldsymbol{u}_i\| = 1, \forall\, i$ then it is called *orthonormal set*. In other words

$$\langle \boldsymbol{u}_i, \boldsymbol{u}_j \rangle = \begin{cases} 1, \; i = j \\ 0, \; i \neq j \end{cases}$$

Every orthonormal set is linearly independent.

Every orthonormal set of n vectors from an n-dimensional space $\mathcal{V}$ is an orthonormal basis for $\mathcal{V}$.

Examples (Meyer, 2000)

- The standard basis $\mathcal{E} = \{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n\}$ in $\mathbb{R}^n$ is orthonormal.
- Let $L^1(-\pi, \pi)$ the space of real-valued functions that are integrable on the interval $(-\pi, \pi)$ and where the inner product is given by

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(t)g(t)\,dt.$$

The set

$$\mathcal{B}' = \{1, \cos t, \cos 2t, \ldots, \sin t, \sin 2t, \ldots\}$$

is a set of mutually orthogonal functions. Normalizing each function produces the orthonormal set

$$\mathcal{B} = \left\{\frac{1}{\sqrt{2\pi}}, \frac{\cos t}{\sqrt{\pi}}, \frac{\cos 2t}{\sqrt{\pi}}, \ldots, \frac{\sin t}{\sqrt{\pi}}, \frac{\sin 2t}{\sqrt{\pi}}, \ldots\right\}$$

Gram-Schmidt orthogonalization procedure (Meyer,2000)

If $\mathcal{O} = \{\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n\}$ is a basis for a general inner-product space $\mathcal{V}$ then the Gram-Scmidt sequence $\mathcal{B} = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n\}$ defined by

$$\boldsymbol{u}_1 = \frac{\boldsymbol{u}_1}{\|\boldsymbol{u}_1\|}, \; \boldsymbol{u}_k = \frac{\boldsymbol{x}_k - \sum_{i=1}^{k-1} \langle \boldsymbol{u}_i, \boldsymbol{x}_k \rangle \boldsymbol{u}_i}{\|\boldsymbol{x}_k - \sum_{i=1}^{k-1} \langle \boldsymbol{u}_i, \boldsymbol{x}_k \rangle \boldsymbol{u}_i\|}$$

is an orthonormal basis for $\mathcal{V}$.

General Matrix Norms (Meyer, 2000)

$\| \cdot \| : \mathcal{M}_n(\mathbb{C}) \to \mathbb{R}$

1. $\|A\| \geqslant 0, \|A\| = 0 \Leftrightarrow A = O_n$
2. $\|\alpha A\| = |\alpha| \|A\|, \forall\, \alpha \in \mathbb{C}$
3. $\|A + B\| \leqslant \|A\| + \|B\|$
4. $\|AB\| \leqslant \|A\| \|B\|$

Examples: Frobenius norm

$$\|A\|_F = \left(\sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|^2 \right)^{\frac{1}{2}} = \left(\sum_{i=1}^{n} \|A(i,:)\|_2^2 \right)^{\frac{1}{2}} = \left(\sum_{j=1}^{n} \|A(:,j)\|_2^2 \right)^{\frac{1}{2}}$$

Induced Matrix Norms (Meyer, 2000)

A vector norm $\|\cdot\|_v$ on $\mathbb{C}^n$ induces a matrix norm on $\mathcal{M}_n(\mathbb{C})$

$$|||A||| = \max_{\boldsymbol{x}\neq\boldsymbol{0}_{\mathbb{C}^n}} \frac{\|A\boldsymbol{x}\|_v}{\|\boldsymbol{x}\|_v} = \max_{\|\boldsymbol{x}\|_v=1} \|A\boldsymbol{x}\|_v,\ A \in \mathcal{M}_n(\mathbb{C})$$

Examples:

$$|||A|||_1 = \max_{\|\boldsymbol{x}\|_1=1} \|A\boldsymbol{x}\|_1 = \max_{1\leqslant j\leqslant n} \sum_{i=1}^{n} |a_{ij}|$$

$$|||A|||_\infty = \max_{\|\boldsymbol{x}\|_\infty=1} \|A\boldsymbol{x}\|_\infty = \max_{1\leqslant i\leqslant n} \sum_{j=1}^{n} |a_{ij}|$$

$$|||A|||_2 = \max_{\|\boldsymbol{x}\|_2=1} \|A\boldsymbol{x}\|_2 = \sqrt{\rho(A^*A)}$$

Induced Matrix Norms (Meyer, 2000)

$$\|\|I_n\|\| = 1, \; \forall \, n \in \mathbb{N}$$

It's apparent that an induced matrix norm is compatible with its underlying vector norm in the sense that

$$\|A\boldsymbol{x}\|_v \leqslant \|\|A\|\| \|\boldsymbol{x}\|_v$$

When A is nonsingular

$$\|\|A^{-1}\|\| = \frac{1}{\min\limits_{\|\boldsymbol{x}\|_v = 1} \|A\boldsymbol{x}\|_v}$$

Induced Matrix Norms (Meyer, 2000)

$$\|\!|A|\!\|_2 \leqslant \|A\|_F \leqslant \sqrt{n}\|\!|A|\!\|_2$$

$$\frac{1}{\sqrt{n}}\|\!|A|\!\|_\infty \leqslant \|\!|A|\!\|_2 \leqslant \sqrt{n}\|\!|A|\!\|_\infty$$

$$\frac{1}{\sqrt{n}}\|\!|A|\!\|_1 \leqslant \|\!|A|\!\|_2 \leqslant \sqrt{n}\|\!|A|\!\|_1$$

$$\max_{1\leqslant i,j\leqslant n} |a_{ij}| \leqslant \|\!|A|\!\|_2 \leqslant n \max_{1\leqslant i,j\leqslant n} |a_{ij}|$$