LAPLACE ELEMENTAR

Rez. ec $x''(t) - x(t) = \sin 3t + \cos 3t$, $x(0)=0$, $x'(0)=2$

=

Obs: $\mathcal{L}\{x'(t)\}(s) = sX(s) - \overset{0}{x(0)}$

$\mathcal{L}\{x''(t)\}(s) = s^2X(s) - s\overset{0}{x(0)} - \overset{2}{x'(0)}$

$\Rightarrow s^2X(s) - 2 - X(s) = \frac{3}{s^2+9} + \frac{s}{s^2+9}$

$X(s)(s^2-1) = 2 + \frac{s+3}{s^2+9} = \frac{2s^2+s+21}{s^2+9}$

$X(s) = \frac{2s^2+s+21}{(s-1)(s+1)(s^2+9)} = \frac{A}{s-1} + \frac{B}{s+1} + \frac{Cs+D}{s^2+9}$

$= \frac{A(s+1)(s^2+9) + B(s-1)(s^2+9) + (Cs+D)(s-1)(s+1)}{(s-1)(s+1)(s^2+9)}$

Pt. $s=1 \Rightarrow 24 = 20A \Rightarrow \underline{A = \frac{6}{5}}$

Pt. $s=-1 \Rightarrow 22 = -20B \Rightarrow \underline{B = -\frac{11}{10}}$

Pt. $s=0 \Rightarrow 21 = 9A - 9B - D \Rightarrow 21 = \overset{2}{\frac{54}{5}} + \frac{99}{10} - D \Rightarrow \underline{D = -\frac{3}{10}}$

Coef. lui $s^3 \Rightarrow 0 = A + B + C \Rightarrow C = -\overset{2}{\frac{6}{5}} + \frac{11}{10} \Rightarrow \underline{C = -\frac{1}{10}}$

Obținem $x(t) = \frac{6}{5}\mathcal{L}^{-1}\{\frac{1}{s-1}\}(t) - \frac{11}{10}\mathcal{L}^{-1}\{\frac{1}{s+1}\}(t) - \frac{1}{10}\mathcal{L}^{-1}\{\frac{s}{s^2+9}\}(t)$

$- \frac{3}{10}\mathcal{L}^{-1}\{\frac{1}{s^2+9}\}(t)$ ...

MODEL TFi

① Să se calculeze spectrul, amplitudinea și faza în frecvență ale semnalului $f:\mathbb{R}\to\mathbb{C}$,

$$f(t)=\frac{2t+j}{(t^2+8t+25)^2}$$

$$=$$

$$\hat{f}(\omega)=\int_{-\infty}^{\infty}\frac{2t+j}{(t^2+8t+25)^2}\cdot e^{-j\omega t}dt,\ \omega\in\mathbb{R}$$

Obs: $\displaystyle\int_{-\infty}^{\infty}\underbrace{\frac{P(x)}{Q(x)}}_{f(x)}e^{j\lambda x}dx=2\pi j\cdot\sum_{\operatorname{Im}x_k>0}\operatorname{Rez}(f(x);x_k)+\pi j\cdot\sum_{\operatorname{Im}b_k=0}\operatorname{Rez}(f(x);b_k)$

Cazul I $\quad -\omega>0\Leftrightarrow\boxed{\omega<0}$

$$\hat{f}(\omega)=\int_{-\infty}^{\infty}\underbrace{\frac{2t+j}{(t^2+8t+25)^2}\cdot e^{-j\omega t}}_{g(t)}dt=2\pi j\cdot\operatorname{Rez}(g;-4+3j)$$

$t^2+8t+25=0$

$\Delta=64-100=(6j)^2$

$t_{1,2}=\dfrac{-8\pm6j}{2}\begin{cases}-4+3j\\-4-3j\end{cases}$ > poli duble

$$\hat{f}(\omega)=2\pi j\cdot\lim_{t\to-4+3j}\left(\cancel{(t+4-3j)^2}\cdot\frac{(2t+j)\cdot e^{-j\omega t}}{\cancel{(t+4-3j)^2}(t+4+3j)^2}\right)'$$

$$=2\pi j\cdot\frac{(2-j\omega(2t+j))\cdot e^{-j\omega t}(t+4+3j)^2-2\cancel{(t+4+3j)}(2t+j)e^{-j\omega t}}{(t+4+3j)^{\cancel{4}\,3}}\Bigg|_{t=-4+3j}$$

$$= 2\pi j \cdot \frac{\left(2 - j\omega(-8+6j+j)\right) \cdot e^{-j\omega(-4+3j)} \cdot \left(\cancel{-4}+3j+\cancel{4}+3j\right)}{\left(\cancel{-4}+3j+\cancel{4}+3j\right)^3} -$$

$$\frac{-2 \cdot (-8+6j+j)\, e^{-j\omega(-4+3j)}}{}$$

$$= 2\cancel{\pi j} \cdot \frac{(2+8\omega j+7\omega) \cdot e^{(3+4j)\omega} \cdot 6j}{6^3 \cdot \cancel{j}^3 \cdot (-1)} - \frac{(-16+14j)\, e^{(3+4j)\omega}}{}$$

$$= -2 \cdot \frac{(12j - 48\omega + 42\omega j + 16 - 14j) \cdot e^{(3+4j)\omega}}{6^3}$$

$$= -\frac{(16 - 48\omega + (42\omega - 2)j)}{108} \cdot e^{(3+4j)\omega} = -\frac{(8 - 24\omega + (21\omega - 1)j)}{54} \cdot e^{(3+4j)\omega}$$

Cazul II $\boxed{\omega > 0}$

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} \frac{2t+j}{(t^2+8t+25)^2} e^{-j\omega t} \quad \frac{t=-x}{dt=-dx} \quad \int_{-\infty}^{\infty} \underbrace{\frac{-2x+j}{(x^2-8x+25)^2} e^{j\omega x}}_{h(x)} dx$$

$$= 2\pi j \cdot \mathrm{Res}(h;\, 4+3j) = \ldots$$

$x^2 - 8x + 25 = 0$

$\Delta = (6j)^2$

$x_{1,2} = \frac{8 \pm 6j}{2} \begin{cases} x_1 = 4+3j \\ x_2 = 4-3j \end{cases}$

Cazul III $\boxed{\omega = 0}$

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} \frac{2t+j}{(t^2+8t+25)^2} dt = 2\pi j \cdot \mathrm{Res}(f;\, -4+3j) = \ldots$$

$$\hat{f}(\omega) = \underbrace{|\hat{f}(\omega)|}_{A(\omega)} \cdot e^{j \cdot \underbrace{\arg \hat{f}(\omega)}_{\phi(\omega)}} \quad ///$$

② Fie $f: \mathbb{R} \to \mathbb{C}$, $f(t) = \dfrac{t}{(3-jt)^5}$

Determinați $E(f)$, $\hat{f}(\omega)$ și $E(\hat{f})$.

$$\underline{\underline{E(f)}} = \int_{-\infty}^{\infty} |f(t)|^2 dt = \int_{-\infty}^{\infty} \frac{t^2}{|3-jt|^{10}} dt = \int_{-\infty}^{\infty} \frac{t^2}{\sqrt{t^2+9}^{\,10}} dt$$

$$= \int_{-\infty}^{\infty} \frac{t^2}{(t^2+9)^5} dt = \int_{-\infty}^{\infty} \frac{t^2+9-9}{(t^2+9)^5} dt = \int_{-\infty}^{\infty} \frac{1}{(t^2+9)^4} dt$$

$$-9 \int_{-\infty}^{\infty} \frac{1}{(t^2+9)^5} dt = J_4 - 9 J_5$$

unde $J_4 = \displaystyle\int_{-\infty}^{\infty} \frac{1}{(t^2+9)^4} dt = 2\pi j \cdot \operatorname{Res}\left(\frac{1}{(t^2+9)^4}; 3j\right)$

$t_{1,2} = \pm 3j$ poli de ordinul 4

$$= 2\pi j \cdot \frac{1}{3!} \lim_{t \to 3j} \left(\cancel{(t-3j)^4} \cdot \frac{1}{\cancel{(t-3j)^4}(t+3j)^4} \right)'''$$

$$= \frac{\pi j}{3} \cdot \lim_{t \to 3j} \left((t+3j)^{-4} \right)''' = \frac{\pi j}{\cancel{3}} \cdot (-4)(-5)(\cancel{-6})^{2} \cdot \frac{1}{(t+3j)^7} \Big|_{t=3j}$$

$$= -40\pi j \cdot \frac{1}{6^7 \cdot j^7} = \underline{\underline{\frac{40\pi}{6^7}}}$$

$$I_5 = \int_{-\infty}^{\infty} \frac{1}{(t^2+9)^5}\,dt = 2\pi j \cdot \mathrm{Res}\left(\frac{1}{(t^2+9)^5};\ 3j\right)$$

$t_{1,2} = \pm 3j$ poli de ordinul 5

$$I_5 = 2\pi j \cdot \frac{1}{4!} \lim_{t\to 3j} \left(\cancel{(t-3j)^5} \cdot \frac{1}{\cancel{(t-3j)^5}(t+3j)^5} \right)^{(4)}$$

$$= \frac{\pi j}{12} \cdot \lim_{t\to 3j} \left((t+3j)^{-5}\right)^{(4)} = \frac{\pi j}{\cancel{12}_{\,2}} \cdot (-5)\cancel{(-6)}(-7)(-\overset{4}{\cancel{8}}) \cdot \frac{1}{(t+3j)^9}\Big|_{t=3j}$$

$$= 140\pi j \cdot \frac{1}{6^9 \cdot j^9} = \underline{\underline{\frac{140\pi}{6^9}}}$$

$$\underline{\underline{E(f)}} = \frac{40\pi}{6^7} - 9 \cdot \frac{\overset{35}{\cancel{140}}\pi}{\cancel{6^9}_{\,6^7}} = \underline{\underline{\frac{5\pi}{6^7}}}$$

$$\underline{\underline{E(\hat{f})}} = 2\pi \cdot E(f) = \underline{\underline{\frac{10\pi}{6^7}}}$$

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} \frac{t}{(3-jt)^5} \cdot e^{-j\omega t}\,dt$$

$3 - jt = 0 \Leftrightarrow jt = 3 \Leftrightarrow t = \frac{3}{j} = -3j$

<u>Cazul I</u> $\omega < 0$

$t_1 = -3j \quad \mathrm{Im}(t_1) < 0 \Longrightarrow \hat{f}(\omega) = 0$

Cazul II $\omega > 0$

$$\hat{f}(\omega) \underset{dt=-dx}{\overset{t=-x}{=}} \int_{-\infty}^{\infty} \underbrace{\frac{-x}{(3+jx)^5} e^{j\omega x}}_{g(x)} dx = 2\pi j \cdot Rez(g; 3j)$$

$3+jx=0 \Rightarrow x = -\frac{3}{j} = 3j$

pol de ordinul 5

$$\hat{f}(\omega) = 2\pi j \cdot \frac{1}{4!} \lim_{x \to 3j} \left(\cancel{(x-3j)^5} \cdot \frac{-x \cdot e^{j\omega x}}{j^5 \cancel{(x-3j)^5}} \right)^{(4)}$$

$$= -\frac{\pi}{12} \cdot \lim_{x \to 3j} \left(x \cdot e^{j\omega x} \right)^{(4)}$$

unde $\left(x \cdot e^{j\omega x}\right)^{(4)} = \sum_{k=0}^{4} C_4^k (x)^{(k)} \cdot \left(e^{j\omega x}\right)^{(4-k)}$

$$= C_4^0 \cdot x \cdot (j\omega)^4 e^{j\omega x} + C_4^1 \cdot (j\omega)^3 e^{j\omega x}$$

$$= x \cdot \omega^4 e^{j\omega x} - 4\omega^3 e^{j\omega x} \cdot j$$

$$\Rightarrow \underline{\hat{f}(\omega)} = -\frac{\pi}{12} \cdot \left(x \cdot \omega^4 e^{j\omega x} - 4\omega^3 e^{j\omega x} \cdot j \right) \Big|_{x=3j}$$

$$= -\frac{\pi}{12} \cdot \left(3j \cdot \omega^4 \cdot e^{-3\omega} - 4\omega^3 j \cdot e^{-3\omega} \right)$$

$$= \frac{\pi}{12} j \left(3\omega^4 - 4\omega^3 \right) e^{-3\omega} = \underline{\frac{\pi}{12} j \omega^3 (3\omega - 4) e^{-3\omega}}$$

Cazul III $\boxed{\omega = 0}$

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} \frac{t}{(3-jt)^5}\, dt = 0$$

③ Calculați TFi prin sinus pentru semnalul

$$f(t) = \frac{1}{t(t^2+9)^2}, \quad t>0 \text{ pe frecvența } \omega = 2$$

=

$$(\mathcal{F}_s f)(\omega) = \int_0^{\infty} \underbrace{\frac{1}{t(t^2+9)^2}}_{\text{pară}} \cdot \sin \omega t \, dt, \quad \omega > 0$$

$$\hat{f}_s(\omega)$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} \frac{1}{t(t^2+9)^2} \cdot \sin \omega t \, dt = J$$

Fie $I = \frac{1}{2} \int_{-\infty}^{\infty} \frac{1}{t(t^2+9)^2} \cos \omega t \, dt$

$$I + j \cdot J = \frac{1}{2} \int_{-\infty}^{\infty} \underbrace{\frac{e^{j\omega t}}{t(t^2+9)^2}}_{g(t)} dt$$

$t_1 = 0 \in \mathbb{R}$ pol simplu $\quad t_{2,3} = \pm 3j$ poli duble $\quad \operatorname{Im}(3j) > 0$

$$= \frac{1}{2} \cdot \left(2\pi j \cdot \operatorname{Res}(g; 3j) + \pi j \cdot \operatorname{Res}(g; 0) \right)$$

unde

$$\operatorname{Rez}(g; 3j) = \lim_{t \to 3j} \left(\cancel{(t-3j)^2} \cdot \frac{e^{j\omega t}}{t \cancel{(t-3j)^2} (t+3j)^2} \right)'$$

$$= \frac{j\omega \cdot e^{j\omega t} \cdot t(t+3j)^{\cancel{2}} - e^{j\omega t}\left((t+3j)^{\cancel{2}} + 2t\cancel{(t+3j)}\right)}{t^2 (t+3j)^{\cancel{4}\,3}} \Bigg|_{t=3j}$$

$$= \frac{(j\omega t(t+3j) - t - 3j - 2t) \cdot e^{j\omega t}}{t^2 \cdot (t+3j)^3} \Bigg|_{t=3j}$$

$$= \frac{(-3\omega \cdot 6j - 6j - 6j) \cdot e^{-3\omega}}{(-9) \cdot 6^3 \cdot j^3} = -\frac{\cancel{6j}(3\omega+2) \cdot e^{-3\omega}}{9 \cdot \cancel{6^3 \cdot j}\; 6^2}$$

$$= -\frac{(3\omega+2) \cdot e^{-3\omega}}{324}$$

$$\operatorname{Rez}(g; 0) = \frac{e^{j\omega t}}{t^1 (t^2+9)^2} \Bigg|_{t=0} = \frac{1}{81}$$

$$\rightarrow I + j \cdot J = -\frac{(3\omega+2) \cdot e^{-3\omega}}{324} \pi j + \frac{\pi j}{162}$$

$$\rightarrow J = \frac{2\pi - (3\omega+2) \cdot e^{-3\omega} \pi}{324} = \hat{f}_s(\omega)$$

$$\hat{f}_s(2) = \frac{2\pi - 8 \cdot e^{-6} \pi}{324} = \frac{(1 - 4e^{-6})\pi}{162}$$

④ Fie $f: \mathbb{R} \to \mathbb{C}$

$$f(t) = \int_{-\infty}^{\infty} \frac{1}{(t^2+x^2-2tx+4)(x^2+2jx+3)^2}\,dx$$

Determinați $a \in \mathbb{R}$ a.î.

$$f(t) = \frac{1}{t^2+a} * \frac{1}{(t^2+2jt+3)^2}$$ și calculați $\hat{f}(a)$.

⑤ Fie $f, g, h: \mathbb{R} \to \mathbb{R}$, $f(t) = e^{-25t^2}$,

$g(t) = f^{(6)}(t)$ și $h(t) = t^2 f(t)$. Calculați $E(f)$, $\hat{f}(1)$, $\hat{g}(1)$ și $\hat{h}(1)$.

⑥ Fie $g: \mathbb{R} \to \mathbb{R}$, $g(\omega) = e^{-4\omega^2}$

și ecuația $\underbrace{\int_{-\infty}^{\infty} f(t)\cdot e^{-j\omega t}\,dt}_{\hat{f}(\omega)} = g(\omega)$, $f \in L^1(\mathbb{R})$

Calculați $E(g)$, $E(f)$ și $f(1)$.

PARSEVAL:

$E(\hat{f}) = 2\pi \cdot E(f)$

$E(g) = 2\pi \cdot E(\hat{f})$

$(\mathcal{F}f)(\omega) = g(\omega) \Rightarrow f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(\omega)\cdot e^{j\omega t}\,d\omega$

$\hat{f}(\omega) = g(\omega)$!

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-4\omega^2}\cdot e^{j\omega t}\,d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-(4\omega^2 - j\omega t)}\,d\omega$$

$$\left(4\omega^2 - j\omega t = \left(2\omega - \frac{jt}{4}\right)^2 + \frac{t^2}{16}\right)$$

$$2ab = j\omega t$$

$$4\cancel{\omega} b = j\cancel{\omega} t \Rightarrow b = \frac{jt}{4}$$

$$\underline{\underline{f(t)}} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\left(2\omega - \frac{jt}{4}\right)^2 - \frac{t^2}{16}}\, d\omega$$

$$= \frac{1}{2\pi}\cdot e^{-\frac{t^2}{16}}\int_{-\infty}^{\infty} e^{-\left(2\omega - \frac{jt}{4}\right)^2} d\omega \qquad \frac{2\omega - \frac{jt}{4} = x}{2\,d\omega = dx}$$

$$= \frac{1}{2\pi}\cdot e^{-\frac{t^2}{16}}\int_{-\infty}^{\infty} e^{-x^2}\cdot\frac{1}{2}\,dx$$

$$= \frac{1}{4\pi}\cdot e^{-\frac{t^2}{16}}\cdot\underbrace{\int_{-\infty}^{\infty} e^{-x^2}dx}_{\sqrt{\pi}} = \underline{\underline{\frac{1}{4\sqrt{\pi}}\cdot e^{-\frac{t^2}{16}}}}$$

$$\Rightarrow f(t) = \frac{1}{4\sqrt{\pi}}\cdot e^{-\frac{t^2}{16}}$$

$$E(g) = \int_{-\infty}^{\infty}|g(\omega)|^2 d\omega = \int_{-\infty}^{\infty}\left|e^{-4\omega^2}\right|^2 d\omega = \int_{-\infty}^{\infty} e^{-8\omega^2} d\omega$$

$$\frac{2\sqrt{2}\,\omega = x}{2\sqrt{2}\,d\omega = dx} \qquad \int_{-\infty}^{\infty} e^{-x^2}\frac{1}{2\sqrt{2}}\,dx = \frac{\sqrt{\pi}}{2\sqrt{2}} = \frac{\sqrt{2\pi}}{4}$$

$$E(f) = \int_{-\infty}^{\infty}|f(t)|^2 dt = \int_{-\infty}^{\infty}\frac{1}{16\pi}\cdot e^{-\frac{t^2}{8}}dt = \frac{1}{16\pi}\int_{-\infty}^{\infty} e^{-\frac{t^2}{8}}dt$$

$$\frac{\frac{t}{2\sqrt{2}} = x}{\frac{1}{2\sqrt{2}}\,dt = dx} \qquad \frac{1}{\cancel{16}\,8\pi}\int_{-\infty}^{\infty} e^{-x^2}\cdot\cancel{2}\sqrt{2}\,dx = \frac{\sqrt{2\pi}}{8\pi} \qquad \left(\text{Verificare: } \frac{\sqrt{2\pi}}{4} = 2\pi\cdot\frac{\sqrt{2\pi}}{4\pi}\right) \checkmark$$

(7) Rezolvați ecuația integrală Fourier

$$\int_0^\infty f(t)\cdot\cos\omega t\,d\omega = \underbrace{\frac{1}{(\omega^2+9)^2\cdot(\omega^2+1)}}_{g(\omega)}, \quad f\in L^1(0,\infty),\ \omega>0$$

$$(\mathcal{F}_c f)(\omega) = g(\omega) \quad \Big|\ \mathcal{F}_c^{-1}$$

$$\Rightarrow f(t) = \frac{2}{\pi}(\mathcal{F}_c^{-1} g)(t) = \frac{2}{\pi}\int_0^\infty \underbrace{\frac{\cos\omega t}{(\omega^2+9)^2\cdot(\omega^2+1)}}_{\text{para}}\,d\omega,\ t>0$$

$$= \frac{\cancel{2}}{\pi}\cdot\frac{1}{\cancel{2}}\int_{-\infty}^\infty \frac{\cos\omega t}{(\omega^2+9)^2\cdot(\omega^2+1)}\,d\omega = J$$

Fie $f = \frac{1}{\pi}\int_{-\infty}^\infty \frac{\sin\omega t}{(\omega^2+9)^2\cdot(\omega^2+1)}\,d\omega$

$$\Rightarrow J + j\cdot f = \frac{1}{\pi}\int_{-\infty}^\infty \underbrace{\frac{e^{j\omega t}}{(\omega^2+9)^2(\omega^2+1)}}_{h(\omega)}\,d\omega$$

$(\omega^2+9=0 \Rightarrow \omega_{1,2}=\pm 3j$ poli duble$)$ $\quad$ $(\omega^2+1=0 \Rightarrow \omega_{3,4}=\pm j$ poli simpli$)$

$$= \frac{1}{\cancel{\pi}}\,2\cancel{\pi}j\cdot\left(\mathrm{Res}(h;j) + \mathrm{Res}(h;3j)\right)$$

$$= 2j\left(\frac{e^{j\omega t}}{(\omega^2+9)^2\cdot 2\omega}\Big|_{\omega=j} + \lim_{\omega\to 3j}\left(\cancel{(\omega-3j)^2}\cdot\frac{e^{j\omega t}}{\cancel{(\omega-3j)^2}(\omega+3j)^2(\omega^2+1)}\right)'\right)$$

$$2j\cdot\left(\frac{e^{-t}}{64\cdot 2j}+\frac{jte^{j\omega t}(\omega+3j)^{2}(\omega^{2}+1)-e^{j\omega t}\left(2(\omega+3j)(\omega^{2}+1)+2\omega(\omega+3j)^{2}\right)}{(\omega+3j)^{4}\cdot(\omega^{2}+1)^{2}}\right)\Bigg|_{\omega=3j}$$

$$=\frac{e^{-t}}{64}+\cancel{2j}\cdot\frac{jte^{-3t}\cdot 6j\cdot(-8)-e^{-3t}(-16+6j\cdot 6j)}{6^{3}j^{\cancel{3}}(-8)^{2}}$$

$$=\frac{e^{-t}}{64}-\frac{+48t+52}{6^{3}\cdot\cancel{64}_{32}}\cdot e^{-3t}\,(-1)$$

$$\Rightarrow J=\frac{e^{-t}}{64}-\frac{12t+13}{6^{3}\cdot\cancel{32}_{8}}\cdot e^{-3t}\quad\Rightarrow J=\frac{e^{-t}}{64}-\frac{(12t+13)e^{-3t}}{1728}\ \checkmark$$

MODEL LAPLACE

(1) Rez. ec.

$$x''(t)+4x(t)=\frac{1}{\cos^2 t+3\sin^2 t} \qquad x(0)=2,\ x'(0)=8$$

$$= \quad x''(t)+4x(t)=\frac{1}{1+2\sin^2 t}=\frac{1}{1+2\cdot\frac{1-\cos 2t}{2}}=\frac{1}{2-\cos 2t}$$

$$\mathcal{L}\{x''(t)\}(s)+4X(s)=\mathcal{L}\left\{\frac{1}{2-\cos 2t}\right\}(s)$$

Obs $\mathcal{L}\{x'(t)\}(s)=sX(s)-\underset{2}{x(0)}$

$$\mathcal{L}\{x''(t)\}(s)=s^2X(s)-s\underset{2}{x(0)}-\underset{8}{x'(0)}$$

$$\rightarrow s^2X(s)-2s-8+4X(s)=\mathcal{L}\left\{\frac{1}{2-\cos 2t}\right\}(s)$$

$$X(s)(s^2+4)=2s+8+\mathcal{L}\left\{\frac{1}{2-\cos 2t}\right\}(s)$$

$$X(s)=\frac{2s}{s^2+4}+\frac{8}{s^2+4}+\frac{1}{s^2+4}\cdot\mathcal{L}\left\{\frac{1}{2-\cos 2t}\right\}(s)$$

$$x(t)=\mathcal{L}^{-1}\left\{\frac{2s}{s^2+4}\right\}(t)+\mathcal{L}^{-1}\left\{\frac{8}{s^2+4}\right\}(t)+\mathcal{L}^{-1}\left\{\mathcal{L}\left\{\frac{\sin 2t}{2}\right\}\mathcal{L}\left\{\frac{1}{2-\cos 2t}\right\}\right\}(s)$$

$$x(t)=2\cdot\cos 2t+\overset{4}{8}\cdot\frac{1}{2}\sin 2t+\frac{1}{2}\sin 2t * \frac{1}{2-\cos 2t}$$

unde $\sin 2t * \frac{1}{2-\cos 2t}=\int_0^t \sin 2x\cdot\frac{1}{2-\cos 2(t-x)}\,dx$

$$= \int_0^t \frac{1}{2-\cos 2x} \cdot \sin 2(t-x)\,dx = \int_0^t \frac{\sin 2t \cos 2x - \sin 2x \cos 2t}{2-\cos 2x}\,dx$$

$$= \sin 2t \int_0^t \frac{\cos 2x}{2-\cos 2x}\,dx - \frac{\cos 2t}{2}\int_0^t \frac{2\sin 2x}{2-\cos 2x}\,dx$$

$$= \sin 2t \cdot J_1 - \cos 2t \cdot \frac{1}{2} \ln|2-\cos 2x| \Big|_0^t = J_1 \sin 2t - \frac{\cos 2t}{2} \cdot \ln|2-\cos 2t|$$

unde $J_1 = \int_0^t \frac{\cos 2x}{2-\cos 2x}\,dx = \int_0^{tg\,t} \frac{1-u^2}{1+u^2} \cdot \frac{1}{2-\frac{1-u^2}{1+u^2}} \cdot \frac{1}{1+u^2}\,du$

$$\cos 2x = \frac{1-tg^2 x}{1+tg^2 x} \qquad tg\,x = u$$

$$(1+u^2)dx = du \Rightarrow dx = \frac{1}{1+u^2}\,du$$

$$= \int_0^{tg\,t} \frac{1-u^2}{1+u^2} \cdot \frac{1}{3u^2+1}\,du = \int_0^{tg\,t} \frac{-1}{1+u^2}\,du + \frac{2}{3}\int_0^{tg\,t} \frac{du}{u^2+\frac{1}{3}} \overset{(*)}{=}$$

$$\frac{1-u^2}{(1+u^2)(1+3u^2)} = \frac{A}{1+u^2} + \frac{B}{1+3u^2} = \frac{A+3Au^2+B+Bu^2}{(1+u^2)(1+3u^2)}$$

$$\Rightarrow \begin{cases} 1 = A+B \\ -1 = 3A+B \end{cases} \Rightarrow \underline{\underline{B=2}}$$

$$(-)$$

$$2 = -2A \Rightarrow \underline{\underline{A=-1}}$$

$$= -\operatorname{arctg} u \Big|_0^{\operatorname{tg} t} + \frac{2}{3}\cdot\sqrt{3}\cdot \operatorname{arctg} \sqrt{3}u \Big|_0^{\operatorname{tg} t}$$

$$= -t + \frac{2\sqrt{3}}{3} \operatorname{arctg}(\sqrt{3}\operatorname{tg} t)$$

$\Rightarrow x(t) = \dots$

② Rez. ec.

$$x''(t) - 8x'(t) + 16x(t) = \frac{e^{4t}}{2t+1}, \quad x(0)=0,\ x'(0)=1$$

=

Obs $\mathcal{L}\{x'(t)\}(s) = sX(s) - \overset{=0}{x(0)}$

$\mathcal{L}\{x''(t)\}(s) = s^2X(s) - s\underset{=0}{x(0)} - \underset{=1}{x'(0)}$

$$\Rightarrow s^2X(s) - 1 - 8\cdot sX(s) + 16X(s) = \mathcal{L}\left\{\frac{e^{4t}}{2t+1}\right\}(s)$$

$$X(s)(s^2 - 8s + 16) = 1 + \mathcal{L}\left\{\frac{e^{4t}}{1+2t}\right\}(s)$$

$$X(s) = \frac{1}{(s-4)^2} + \frac{1}{(s-4)^2}\cdot \underbrace{\mathcal{L}\left\{\frac{e^{4t}}{1+2t}\right\}(s)}_{\mathcal{L}\{t\cdot e^{4t}\}(s)}$$

$$x(t) = \mathcal{L}^{-1}\left\{\frac{1}{(s-4)^2}\right\}(t) + \mathcal{L}^{-1}\left\{t\cdot e^{4t} * \frac{e^{4t}}{1+2t}\right\}(t)$$

$$x(t) = t\cdot e^{4t} + t\cdot e^{4t} * \frac{e^{4t}}{1+2t}$$

unde $\underline{\underline{t\cdot e^{4t} * \frac{e^{4t}}{1+2t}}} = \int_0^t x\cdot e^{4x}\cdot \frac{e^{4(t-x)}}{1+2(t-x)}\,dx$

$$= \int_0^t \frac{(t-x)\cdot e^{4(t-x)}}{1+2x}\cdot e^{4x}\,dx$$

$$= e^{4t}\int_0^t \frac{t-x}{1+2x}\,dx$$

$$= \frac{e^{4t}}{2}\int_0^t \frac{2t-2x}{1+2x}\,dx = -\frac{e^{4t}}{2}\int_0^t \frac{2x+1-1-2t}{1+2x}\,dx$$

$$= -\frac{e^{4t}}{2}\cdot\int_0^t dx + \frac{e^{4t}}{2}(1+2t)\int_0^t \frac{1}{1+2x}\,dx$$

$$= -\frac{e^{4t}}{2}\cdot t + \frac{e^{4t}}{4}(1+2t)\cdot \ln(1+2t)$$

$$\underline{\underline{= \frac{e^{4t}}{4}\left(-2t + (1+2t)\ln|1+2t|\right)}}$$

$$\Rightarrow x(t) = \frac{e^{4t}}{4}\left(2t + (1+2t)\ln(1+2t)\right)$$

③ Fie $f(t) = \int_0^t e^{x+t}(x-t)^2 \cdot (\cos(2t-2x))^{(5)} dx$

Calculați $\mathcal{L}\{f(t)\}(3)$

$$f(t) = \int_0^t e^{t-x} \cdot e^{2x} \cdot (t-x)^2 (\cos 2(t-x))^{(5)} dx$$

$$= \int_0^t \underbrace{e^{2x}}_{f_1(x)} \cdot \underbrace{e^{t-x}(t-x)^2 \cdot (\cos 2(t-x))^{(5)}}_{g_1(t-x)} dx = e^{2t} * \left(e^t \cdot t^2 (\cos 2t)^{(5)}\right)$$

④ Calculați

$$J = \int_0^\infty \frac{t^2 \cos 3t + 4\sin^2 t \cdot \cos t}{t} \cdot e^{-t} dt$$

$$= \underbrace{J = \int_0^\infty t \cdot \cos 3t \cdot e^{-t} dt}_{J_1} + 4 \underbrace{\int_0^\infty \frac{\sin^2 t \cdot \cos t}{t} \cdot e^{-t} dt}_{J_2}$$

unde $J_1 = \mathcal{L}\{t \cdot \cos 3t\}(1) = \frac{1-9}{(1+9)^2} = -\frac{8}{100} = -\frac{2}{25}$

$$\mathcal{L}\{t \cdot \cos at\}(s) = \frac{s^2 - a^2}{(s^2+a^2)^2}$$

$$J_2 = \mathcal{L}\left\{\frac{\sin^2 t \cdot \cos t}{t}\right\}(1) = \int_1^\infty \mathcal{L}\{\sin^2 t \cdot \cos t\}(s)\, ds$$

$$= \int_1^\infty \mathcal{L}\left\{ \frac{1-\cos 2t}{2} \cdot \cos t \right\}(s)\, ds = \frac{1}{2} \int_1^\infty \mathcal{L}\{\cos t\}(s)\, ds$$

$$- \frac{1}{2} \int_1^\infty \mathcal{L}\{\cos 2t \cdot \cos t\}(s)\, ds = \frac{1}{4} \int_1^\infty \frac{2s}{s^2+1}\, ds -$$

$$- \frac{1}{2} \int_1^\infty \mathcal{L}\left\{ \frac{\cos 3t + \cos t}{2} \right\}(s)\, ds = \frac{1}{4} \ln(s^2+1) \Big|_1^\infty -$$

$$\left(\cos a \cdot \cos b = \frac{\cos(a+b) + \cos(a-b)}{2} \right)$$

$$- \frac{1}{8} \int_1^\infty \left(\frac{2s}{s^2+9} + \frac{2s}{s^2+1} \right) ds = \left[\frac{1}{4} \ln(s^2+1) - \frac{1}{8} \left(\ln (s^2+9)(s^2+1) \right) \right]_1^\infty$$

$$= \frac{1}{4} \ln \frac{s^2+1}{\sqrt{s^2+9} \cdot \sqrt{s^2+1}} \Bigg|_1^\infty = \frac{1}{4} \cdot \overset{0}{\ln 1} - \frac{1}{4} \ln \frac{2}{\sqrt{10} \cdot \sqrt{2}}$$

$$= -\frac{1}{4} \ln \frac{1}{\sqrt{5}} = \underline{\underline{\frac{1}{8} \ln 5}}$$

En final, $\underline{\underline{I = -\frac{2}{25} + \frac{1}{2} \ln 5}}$

(5) Fie $f(t)=\int_0^{\sqrt{t}} x^9 e^{-2x^2}dx$

Calculați $\mathcal{L}\{f'(t)\}(s)$, $\mathcal{L}\{f(t)\}(s)$, $\int_0^{\infty} t\cdot e^{-3t} f(5t)\,dt$

și $\mathcal{L}\{e^{-4t} t f(t)\}(1)$

$$f'(t)=(\sqrt{t})'\cdot(\sqrt{t})^9\cdot e^{-2t}=\frac{1}{2\sqrt{t}}\cdot t^{\frac{9}{2}}\cdot e^{-2t}=\frac{1}{2}t^4\cdot e^{-2t}$$

$$\mathcal{L}\{f'(t)\}(s)=\frac{1}{2}\mathcal{L}\{t^4\cdot e^{-2t}\}(s)=\frac{4!}{(s+2)^5}\cdot\frac{1}{2}=\frac{12}{(s+2)^5} \quad (1)$$

$$\mathcal{L}\{t^n e^{at}\}(s)=\frac{n!}{(s-a)^{n+1}}$$

Fie $\mathcal{L}\{f(t)\}(s)=F(s)$

Pe de altă parte, $\mathcal{L}\{f'(t)\}(s)=sF(s)-\overset{=0}{f(0+0)}$
(din T. deriv. originalului)

$$\Leftrightarrow \frac{12}{(s+2)^5}=sF(s)\Rightarrow F(s)=\frac{12}{s(s+2)^5} \quad (2)$$

$$I_1=\int_0^{\infty} t\cdot e^{-3t}\cdot f(5t)\,dt=\mathcal{L}\{t\cdot f(5t)\}(3)$$

$$=-\left(\mathcal{L}\{f(5t)\}(s)\right)'\Big|_{s=3}$$

T. deriv. imag.

unde $\mathcal{L}\{f(5t)\}(s)=\frac{1}{5}\mathcal{L}\{f(t)\}\left(\frac{s}{5}\right)=\frac{1}{5}\cdot\frac{12}{\frac{s}{5}\left(\frac{s}{5}+2\right)^5}$

$$= \frac{12\cdot 5^5}{s(s+10)^5}$$

$$\Rightarrow \underline{\underline{J_1}} = -12\cdot 5^5 \left(\frac{1}{s(s+10)^5}\right)'\Bigg|_{s=3}$$

$$= +12\cdot 5^5 \cdot \frac{(s+10)^{\cancel{5}} + 5s\cancel{(s+10)^4}}{s^2(s+10)^{\cancel{10}\,6}}\Bigg|_{s=3}$$

$$= \overset{4}{\cancel{12}}\cdot 5^5 \cdot \frac{13+15}{\underset{3}{\cancel{9}}\cdot 13} = 4\cdot 5^5 \cdot \frac{28}{3\cdot 13} = \underline{\underline{\frac{350.000}{39}}}$$

$$J_2 = \mathcal{L}\{e^{-4t}\, t\, f(t)\}(s) = \mathcal{L}\{t\cdot f(t)\}(s+4)$$

T. deplasării imaginii

$$\mathcal{L}\{e^{at} f(t)\}(s) = \mathcal{L}\{f(t)\}(s-a)$$

$$= -\left(\mathcal{L}\{f(t)\}(s)\right)'\Big|_{s=s+4} = -12\left(\frac{1}{s(s+2)^5}\right)'\Bigg|_{s=s+4}$$

$$= +12\cdot \frac{(s+2)^{\cancel{5}} + 5s\cancel{(s+2)^4}}{s^2(s+2)^{\cancel{10}\,6}}\Bigg|_{s=s+4}$$

$$= 12\cdot \frac{s+6+5(s+4)}{(s+4)^2\cdot(s+6)^6} = 12\cdot \frac{6s+26}{(s+4)^2(s+6)^6} = 24\,\frac{3s+13}{(s+4)^2(s+6)^6}$$

$$\Rightarrow \underline{\underline{\mathcal{L}\{e^{-4t}\, t\, f(t)\}(1)}} = 24\,\frac{16}{25\cdot 7^6} = \underline{\underline{\frac{384}{25\cdot 7^6}}}$$

$$\mathcal{L}\{e^{-4t}\cdot t f(t)\}(s) = -\left(\mathcal{L}\{f(t)\cdot e^{-4t}\}(s)\right)'$$

$$= -\left(\mathcal{L}\{f(t)\}(s+4)\right)' = -12\cdot\left(\frac{1}{(s+4)(s+6)^5}\right)'$$

$$= +12\cdot\frac{(s+6)^{\cancel{5}} + 5(s+4)(s+6)^4}{(s+4)^2\cdot(s+6)^{\cancel{10}\,6}} = 12\cdot\frac{s+6+5s+20}{(s+4)^2\cdot(s+6)^6}$$

$$= 12\cdot\frac{6s+26}{(s+4)^2\cdot(s+6)^6} = 24\cdot\frac{3s+13}{(s+4)^2\cdot(s+6)^6}$$

$$\Rightarrow \underline{\mathcal{L}\{e^{-4t}\cdot t\cdot f(t)\}(1)} = 24\cdot\frac{16}{25\cdot 7^6} = \underline{\underline{\frac{384}{25\cdot 7^6}}}$$

⑥ Calculati

$$\int_0^\infty x\cdot\left(\frac{\sin 5x}{x}\right)^3 dx = \int_0^\infty \frac{\sin^3 5x}{x^2}\,dx$$

Fie $J(t) = \int_0^\infty \frac{\sin^3 tx}{x^2}\,dx$, $t \geq 0$, $J = J(5)$

$$\mathcal{L}\{J(t)\}(s) = \int_0^\infty \left(\int_0^\infty \frac{\sin^3 tx}{x^2}\,dx\right)\cdot e^{-st}\,dt$$

$$= \int_0^\infty \left(\int_0^\infty \frac{\sin^3 tx}{x^2}\cdot e^{-st}\,dt\right)dx = \int_0^\infty \frac{1}{x^2}\,\mathcal{L}\{\sin^3 tx\}(s)\,dx$$

$$= \int_0^\infty \frac{1}{\cancel{x^2}}\cdot\frac{1}{\cancel{x}}\cdot\frac{6x^{\cancel{4}}}{(x^2+s^2)(9x^2+s^2)}\,dx = 6\int_0^\infty \frac{x}{(x^2+s^2)(9x^2+s^2)}\,dx$$

$$\left(\mathcal{L}\{\sin^3 t\}(s) = \frac{6}{(s^2+1)(s^2+9)}\quad \mathcal{L}\{\sin^3 tx\}(s) = \frac{1}{x}\mathcal{L}\{\sin^3 t\}\left(\frac{s}{x}\right) =\right.$$

$$= \frac{1}{x} \cdot \frac{6x^4}{(s^2+x^2)(s^2+9x^2)} \Big) \quad (\text{Rez 08, pb 2})$$

$$\left(\frac{x}{(x^2+s^2)(9x^2+s^2)} = \frac{-x}{8s^2} \cdot \left(\frac{1}{x^2+s^2} - \frac{9}{9x^2+s^2} \right) \right)$$

$$\underline{\underline{\mathcal{L}\{y(t)\}(s)}} = \frac{-\cancel{6}^{\,3}}{\cancel{8}_{\,4}\, s^2} \int_0^\infty \left(\frac{x}{x^2+s^2} - \frac{9x}{9x^2+s^2} \right) dx$$

$$= -\frac{3}{8s^2} \ln \frac{x^2+s^2}{9x^2+s^2} \Big|_0^\infty$$

$$= -\frac{3}{8s^2} \cdot \ln \frac{1}{9} = \underline{\underline{\frac{3}{4s^2} \ln 3}} \quad \Big| \mathcal{L}^{-1}$$

$$\rightarrow y(t) = \frac{3}{4} \cdot \ln 3 \cdot t \cdot u(t)$$

$$\underline{\underline{y}} = \frac{3}{4} \cdot 5 \cdot \ln 3 = \underline{\underline{\frac{15}{4} \ln 3}} \checkmark$$

(7) Rez ec. în cadrul teoriei distribuțiilor:

$$x^{(4)}(t) + 8x'''(t) + 25x''(t) = \delta^{(4)}(t) + 2\delta'(t)$$

unde $x(0) = x'(0) = x''(0) = x'''(0) = 0$

distribuție (impulsul lui Dirac)

$\underline{\text{CG}}$: $\mathcal{L}\{\delta(t)\}(s) = 1 \Leftrightarrow \mathcal{L}^{-1}\{1\}(t) = \delta(t)$, $\delta(t) = \begin{cases} \infty, & t=0 \\ 0, & t \neq 0 \end{cases}$

$\underline{\mathcal{L}\{\delta^{(n)}(t)\}(s) = s^n} \Leftrightarrow \mathcal{L}^{-1}\{s^n\}(t) = \delta^{(n)}(t)$

Obs: $\mathcal{L}\{x'(t)\}(s) = sX(s) - \underset{0}{x(0)}$ $\quad | \cdot s$

$\mathcal{L}\{x''(t)\}(s) = s^2X(s) - s\underset{0}{x(0)} - \underset{0}{x'(0)}$

$\mathcal{L}\{x'''(t)\}(s) = s^3X(s) - s^2\underset{0}{x(0)} - s\underset{0}{x'(0)} - \underset{0}{x''(0)}$

$\mathcal{L}\{x^{(4)}(t)\}(s) = s^4X(s) - s^3\underset{0}{x(0)} - s^2\underset{0}{x'(0)} - s\underset{0}{x''(0)} - \underset{0}{x'''(0)}$

$\Rightarrow s^4X(s) + 8\cdot s^3X(s) + 25s^2X(s) = s^4 + 2s$

$X(s)\cdot s^2(s^2+8s+25) = s^4+2s$

$$X(s) = \frac{\cancel{s}(s^3+2)}{s^{\cancel{2}}(s^2+8s+25)} = \frac{s^3+2}{s(s^2+8s+25)}$$

$$= \frac{s^3+8s^2+25s-8s^2-25s+2}{s(s^2+8s+25)}$$

$$= 1 - \frac{8s^2+25s-2}{s(s^2+8s+25)}$$

$$\frac{8s^2+25s-2}{s(s^2+8s+25)} = \frac{A}{s} + \frac{Bs+C}{s^2+8s+25}$$

$$= \frac{As^2+8As+25A+Bs^2+Cs}{s(s^2+8s+25)}$$

$$\Rightarrow \begin{cases} 8 = A+B \\ 25 = 8A+C \\ -2 = 25A \end{cases} \Rightarrow \underline{\underline{A = -\frac{2}{25}}} \Rightarrow \underline{\underline{B}} = 8 + \frac{2}{25} = \underline{\underline{\frac{202}{25}}}$$

$$\underline{\underline{C}} = 25 + 8\cdot\frac{2}{25} = \frac{625+16}{25} = \underline{\underline{\frac{641}{25}}}$$

$$x(t) = \mathcal{L}^{-1}\{X(s)\}(t) = \mathcal{L}^{-1}\{1\}(t) - \mathcal{L}^{-1}\left\{\frac{202}{25}\cdot\frac{1}{s} + \right.$$

$$\left. + \frac{\frac{202}{25}s + \frac{641}{25}}{s^2+8s+25}\right\}(t)$$

$$= \delta(t) - \frac{202}{25}\cdot u(t) + \frac{1}{25}\cdot \underbrace{\mathcal{L}^{-1}\left\{\frac{202s+641}{s^2+8s+25}\right\}(t)}$$

unde $\mathcal{L}^{-1}\left\{\frac{202s+641}{s^2+8s+25}\right\}(t) = \mathcal{L}^{-1}\left\{\underbrace{\frac{202(s+4)-167}{(s+4)^2+9}}_{F(s+4)}\right\}(t)$

$$= e^{-4t}\cdot\mathcal{L}\left\{\frac{202s-167}{s^2+9}\right\}(t)$$

$$\mathcal{L}\{F(s-a)\}(t) = e^{at}\mathcal{L}\{F(s)\}(t), \quad F(s) = \frac{202s-167}{s^2+9}$$

$$= e^{-4t}\cdot\left(202\cdot\cos 3t - \frac{167}{3}\cdot\sin 3t\right)\cdot u(t)$$

8) Rez. ec. integro-diferențială de tip convolutiv:

$$x''(t) + 12\int_0^t e^{8(t-\tau)}x''(\tau)\,d\tau = \cos t, \quad x(0)=x'(0)=0$$

$$= \int_0^t e^{8(t-\tau)}x''(\tau)\,d\tau = x''(t) * e^{8t}$$

$$f(t)*g(t) = \int_0^t f(x)\cdot g(t-x)\,dx = \int_0^t f(\tau)\cdot g(t-\tau)\,d\tau$$

$$\mathcal{L}\{x''(t)\}(s) + 12\,\mathcal{L}\{x''(t)\}(s)\cdot\mathcal{L}\{e^{8t}\}(s) = \mathcal{L}\{\cos t\}(s)$$

<u>Obs</u>: $\mathcal{L}\{x'(t)\}(s) = sX(s) - \overset{0}{x(0)}\,|\cdot s$

$\mathcal{L}\{x''(t)\}(s) = s^2X(s) - s\,\overset{0}{x(0)} - \overset{0}{x'(0)}$

$$\Rightarrow s^2X(s) + 12s^2X(s)\cdot\frac{1}{s-8} = \frac{s}{s^2+1}$$

$$s^2X(s)\overset{s-8}{\left(1+\frac{12}{s-8}\right)} = \frac{s}{s^2+1}$$

$$\Leftrightarrow s^2X(s)\cdot\frac{s+4}{s-8} = \frac{s}{s^2+1} \Rightarrow X(s) = \frac{s-8}{s(s+4)(s^2+1)}$$

$$\frac{s-8}{s(s+4)(s^2+1)} = \frac{A}{s} + \frac{B}{s+4} + \frac{Cs+D}{s^2+1}$$

$$= \frac{A(s+4)(s^2+1) + Bs(s^2+1) + (Cs+D)s(s+4)}{s(s+4)(s^2+1)}$$

Pt. $s=0 \Rightarrow -8 = 4A \Rightarrow \underline{\underline{A=-2}}$

Pt. $s=-4 \Rightarrow -12 = 17\cdot(-4)\cdot B \Rightarrow B = \frac{\cancel{12}^{\,3}}{17\cdot\cancel{4}} \Rightarrow \underline{\underline{B=\frac{3}{17}}}$

Pt. $s=1 \Rightarrow -7 = 10A + 2B + 5(C+D)$

Coef lui s^3: $0 = A + B + C$

$$0 = -2 + \frac{3}{17} + C \Rightarrow \underline{C} = \overset{\frac{17}{}}{2} - \frac{3}{17} = \underline{\underline{\frac{31}{17}}}$$

$$-7 = -20 + \frac{6}{17} + \frac{155}{17} + 5\Delta$$

$$13 = \frac{161}{17} + 5\Delta \quad | \cdot 17$$

$$5\Delta = \frac{47}{17} \Rightarrow \underline{\underline{\Delta = \frac{47}{85}}}$$

$$x(t) = \left(-2 + \frac{3}{17} \cdot e^{-4t} + \frac{31}{17} \cdot \cos t + \frac{47}{85} \sin t\right) \cdot u(t)$$

9. (i) Calculați $\mathcal{L}\{\cos^3(at)\}(s)$

(ii) Calculați

$$\int_0^\infty \frac{\sqrt{7}e^{-t} + \cos^3(2t) - \cos^3(8t)}{t} \, dt$$

(i) $\mathcal{L}\{\cos^3(at)\}(s) = \mathcal{L}\left\{\frac{\cos 3at + 3\cos at}{4}\right\}(s)$

$$\cos 3t = 4\cos^3 t - 3\cos t \Rightarrow \cos^3 t = \frac{\cos 3t + 3\cos t}{4}$$

$$= \frac{1}{4}\mathcal{L}\{\cos 3at\}(s) + \frac{3}{4}\mathcal{L}\{\cos at\}(s)$$

$$= \frac{1}{4} \cdot \frac{s}{s^2 + 9a^2} + \frac{3}{4} \cdot \frac{s}{s^2 + a^2}$$

(ii) $J = \mathcal{L}\left\{\frac{\sqrt{t}\, e^{-t} + \cos^3(2t) - \cos^3(8t)}{t}\right\}(0) =$

$$= \int_0^\infty \mathcal{L}\{\sqrt{t}\,e^{-t} + \cos^3(2t) - \cos^3(8t)\}(s)\,ds$$

$$\left(\mathcal{L}\{\sqrt{t}\cdot e^{-t}\}(s) = \mathcal{L}\{\sqrt{t}\}(s+1) = \mathcal{L}\{t^{\frac{1}{2}}\}(s+1) = \frac{\Gamma(1+\frac{1}{2})}{s^{\frac{3}{2}}}\Big|_{s=s+1}\right.$$

$$= \frac{\frac{1}{2}\sqrt{\pi}}{(s+1)\sqrt{s+1}} = \frac{\sqrt{\pi}}{2}\cdot(s+1)^{-\frac{3}{2}}$$

$$\mathcal{L}\{\cos^3(2t)\}(s) = \frac{1}{4}\cdot\frac{s}{s^2+36} + \frac{3}{4}\cdot\frac{s}{s^2+4}$$

$$\left.\mathcal{L}\{\cos^3(8t)\}(s) = \frac{1}{4}\cdot\frac{s}{s^2+9\cdot 64} + \frac{3}{4}\cdot\frac{s}{s^2+64}\right)$$

$$\underline{\underline{J}} = \int_0^\infty \left(\frac{\sqrt{\pi}}{2}\cdot(s+1)^{-\frac{3}{2}} + \frac{1}{4}\cdot\frac{s}{s^2+36} + \frac{3}{4}\cdot\frac{s}{s^2+4} - \frac{1}{4}\cdot\frac{s}{s^2+9\cdot 64} - \frac{3}{4}\cdot\frac{s}{s^2+64}\right)ds$$

$$= \frac{\sqrt{\pi}}{\cancel{2}}\cdot\frac{(s+1)^{-\frac{1}{2}}}{-\frac{1}{\cancel{2}}}\Bigg|_0^\infty + \left(\frac{1}{8}\ln(s^2+36) + \frac{3}{8}\ln(s^2+4) - \frac{1}{8}\cdot\ln(s^2+576)\right.$$
$$\left. - \frac{3}{8}\ln(s^2+64)\right)\Bigg|_0^\infty$$

$$= -\sqrt{\pi}\cdot\frac{1}{\sqrt{s+1}}\Bigg|_0^\infty + \frac{1}{8}\cdot\ln\frac{(s^2+36)\cdot(s^2+4)^3}{(s^2+576)\cdot(s^2+64)^3}\Bigg|_0^\infty$$

$$= +\sqrt{\pi} - \frac{1}{8}\ln\frac{\cancel{36}\cdot 4^{\cancel{3}\,4}}{\underset{\cancel{16}}{\cancel{576}}\cdot 64^3} = \sqrt{\pi} - \frac{1}{8}\ln\left(\frac{2}{8^3}\right)^2$$

$$= \sqrt{\pi} - \frac{1}{4}\ln\frac{1}{256} = \underline{\underline{\sqrt{\pi} + \ln 4}} \checkmark$$

$$\approx 3{,}15$$

(10) Rezolvați ecuația integrală Fourier

$$\int_0^\infty f(t)\cdot \sin\omega t\,dt = \frac{1}{\omega(\omega^2+4)^2}, \quad \omega>0$$

$t>0$

$$= (F_s f)(\omega) = \underbrace{\frac{1}{\omega(\omega^2+4)^2}}_{g(\omega)} \Rightarrow f(t) = \frac{2}{\pi}\int_0^\infty \underbrace{\frac{\sin\omega t}{\omega(\omega^2+4)^2}}_{\text{pară}}\,d\omega,$$

$$f(t) = \frac{1}{\cancel{2}}\cdot\frac{\cancel{2}}{\pi}\int_{-\infty}^{\infty}\frac{\sin\omega t}{\omega(\omega^2+4)^2}\,d\omega$$

Fie $h(t) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\cos\omega t}{\omega(\omega^2+4)^2}\,d\omega$

Formăm expresia $h(t)+j\cdot f(t) = \frac{1}{\pi}\int_{-\infty}^{\infty}\underbrace{\frac{e^{j\omega t}}{\omega(\omega^2+4)^2}}_{h(\omega)}\,d\omega$

$\underline{\underline{\omega_0 = 0}}$ $\in\mathbb{R}$ – pol simplu $\quad \underline{\underline{\omega_{1,2}}} = \pm 2j$ poli dubli

$$h(t)+j\cdot f(t) = \frac{1}{\cancel{\pi}}\cdot 2\cancel{\pi}j\cdot \operatorname{Rez}(h;2j) + \frac{1}{\cancel{\pi}}\cdot\cancel{\pi}j\cdot\operatorname{Rez}(h;0)$$

$$= 2j\cdot\lim_{\omega\to 2j}\left(\frac{e^{j\omega t}}{\omega(\omega+2j)^2}\right)' + j\cdot\frac{e^{j\omega t}}{(\omega^2+4)^2}\Big|_{\omega=0}$$

$$= 2j\cdot\frac{jt\cdot e^{j\omega t}\,\omega(\omega+2j)^{\cancel{2}} - e^{j\omega t}\left((\omega+2j)^{\cancel{2}} + 2\omega\cancel{(\omega+2j)}\right)}{\omega^2(\omega+2j)^{\cancel{4}\,3}}\Bigg|_{\omega=2j} + j\cdot\frac{1}{16} = 2j\cdot\frac{-2t\cdot 4j - 8j}{(+4)^{\cancel{3}\,3}}\cdot e^{-2t} + j\cdot\frac{1}{16}$$

$$= \frac{-(t+1)}{16}j\cdot e^{-2t} + \frac{j}{16} \Rightarrow \underline{\underline{f(t) = \frac{-(t+1)e^{-2t}+1}{16}}}$$ ✓

(11) Rezolvați ec integrală Fourier

$$\int_0^\infty f(t)\cdot\cos\omega t\,dt = \underbrace{e^{-\frac{\omega}{2}}}_{g(\omega)}, \quad \omega>0$$

$$= (\mathcal{F}_c f)(\omega) = g(\omega) \Rightarrow f(t) = \frac{2}{\pi}\int_0^\infty \underbrace{e^{-\frac{\omega}{2}}\cos\omega t}_{\text{nu este pară}}\,d\omega, \quad t>0$$

Fie $h(t) = \frac{2}{\pi}\int_0^\infty e^{-\frac{\omega}{2}}\sin\omega t\,d\omega$

Formăm expresia $f(t)+j\cdot h(t) = \frac{2}{\pi}\int_0^\infty e^{-\frac{\omega}{2}}\cdot e^{j\omega t}\,d\omega$

$$= \frac{2}{\pi}\int_0^\infty e^{-(\frac{1}{2}-jt)\omega}\,d\omega$$

$$= \frac{2}{\pi}\,\mathcal{L}\{1\}\left(\frac{1}{2}-jt\right)$$

$$= \frac{2}{\pi}\cdot\frac{1}{\frac{1}{2}-jt} = \frac{4}{\pi}\cdot\frac{1}{1-2jt}$$

$$= \frac{4(1+2jt)}{4t^2+1}\cdot\frac{1}{\pi}$$

$$\Rightarrow \underline{f(t) = \frac{4}{4t^2+1}\cdot\frac{1}{\pi}}$$

sau

$$\underline{f(t)} = \frac{2}{\pi}\,\mathcal{L}\{\cos\omega t\}\left(\frac{1}{2}\right) = \frac{2}{\pi}\cdot\frac{s}{s^2+t^2}\Big|_{s=\frac{1}{2}} = \frac{2}{\pi}\cdot\frac{\frac{1}{2}}{\frac{1}{4}+t^2}$$

$$= \underline{\frac{4}{\pi}\cdot\frac{1}{1+4t^2}}$$

Ex.3 Fie $f:[0,+\infty)\to\mathbb{R}$ $f(t)=\int_0^{\sqrt{t}} x^5 e^{-2x^2}dx$

1) Calculați $\mathcal{L}\{f'(t)\}(s)$

2) Calculați $I=\int_0^{\infty} t\cdot e^{-t} f(t)\,dt$

=

1) $f'(t)=(\sqrt{t})'\cdot(\sqrt{t})^5\cdot e^{-2(\sqrt{t})^2}=\frac{1}{2\cancel{\sqrt{t}}}\cdot t^2\cancel{\sqrt{t}}\cdot e^{-2t}=\frac{t^2}{2}e^{-2t}$

$\mathcal{L}\{f'(t)\}(s)=\frac{1}{2}\mathcal{L}\{t^2e^{-2t}\}(s)=\frac{1}{2}\cdot\frac{2!}{(s+2)^3}=\frac{1}{(s+2)^3}$ (1)

$\mathcal{L}\{t^n e^{at}\}(s)=\frac{n!}{(s-a)^{n+1}}$

2) $I=\mathcal{L}\{t f(t)\}(1)=-\big(\underbrace{\mathcal{L}\{f(t)\}(s)}_{F(s)}\big)'\Big|_{s=1}=-F'(1)$ (2)

Din Teorema derivării originalului

$\Rightarrow \mathcal{L}\{f'(t)\}(s)=sF(s)-\underset{0}{f(0+0)}$

$\Rightarrow F(s)=\frac{1}{s(s+2)^3}$ (3)

Din (2) și (3) $\Rightarrow \underline{I}=-\left(\frac{1}{s(s+2)^3}\right)'\Big|_{s=1}=+\frac{(s+2)^{\cancel{3}}+3s\cancel{(s+2)^2}}{s^2(s+2)^{\cancel{6}\,4}}\Big|_{s=1}$

$=\frac{3+3}{3^4}=\frac{6}{3^4}=\underline{\underline{\frac{2}{27}}}$

Ex 4 (temă)

Fie $g:[0,\infty)\to\mathbb{R}$, $g(t)=\int_0^{\sqrt{t}} x^6 e^{-3x^2}dx$

1) Calculați $\mathcal{L}\{g'(t)\}(s)$

2) Calculați $I=\int_0^{\infty} t\cdot g(t)\cdot e^{-6t}dt$

Ex 5 (temă)

Calculați $I=\int_0^{\infty}\frac{\cos^3 t-\cos^3 2t}{t}dt$

Obs

$\cos 3x = 4\cos^3 x - 3\cos x \to \cos^3 x = \frac{3}{4}\cos x + \frac{1}{4}\cos 3x$

$\sin 3x = 3\sin x - 4\sin^3 x \to \sin^3 x = \frac{3}{4}\sin x - \frac{1}{4}\sin 3x$

Ex 6 (temă)

Calculați $I=\int_0^{\infty}\frac{\cos^2(3x)}{x^2+4}dx$

Ex 7

1) Să se arate că $\mathcal{L}\{\ln t\}(s)=\frac{\Gamma'(1)-\ln s}{s}=-\frac{\gamma+\ln s}{s}$

unde $\Gamma(a)=\int_0^{\infty}x^{a-1}e^{-x}dx$, $a>0$ (Euler)

și $\gamma=\lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n}-\ln n\right)\simeq 0{,}57$ constanta lui Euler

2) Să se calculeze $I=\int_0^{\infty} t\ln(2t)\cdot e^{-3t}dt$

$$\mathcal{L}\{t^\alpha\}(s) = \frac{\Gamma(\alpha+1)}{s^{\alpha+1}}, \quad \alpha > -1$$

$$\Leftrightarrow \int_0^\infty t^\alpha \cdot e^{-st}\,dt = \frac{\Gamma(\alpha+1)}{s^{\alpha+1}} \quad \Big|\,(\ )'_\alpha$$

$$\Rightarrow \int_0^\infty t^\alpha (\ln t)\cdot e^{-st}\,dt = \frac{\Gamma'(\alpha+1)\cdot s^{\alpha+1} - s^{\alpha+1}\cdot(\ln s)\cdot\Gamma(\alpha+1)}{s^{2\alpha+2}}$$

$$\Rightarrow \mathcal{L}\{t^\alpha \ln t\}(s) = \frac{\Gamma'(\alpha+1) - (\ln s)\cdot\Gamma(\alpha+1)}{s^{\alpha+1}}$$

Pentru $\alpha = 0 \Rightarrow \underline{\underline{\mathcal{L}\{\ln t\}(s)}} = \frac{\Gamma'(1) - \Gamma(1)\ln s}{s} = \underline{\underline{-\frac{\gamma + \ln s}{s}}}$

$$\left(\underline{\Gamma(n) = (n-1)!},\ \Gamma(a) = \int_0^\infty x^{a-1}e^{-x}\,dx = \mathcal{L}\{t^{a-1}\}(1)\right)$$

$$\Gamma(1) = \int_0^\infty e^{-x}\,dx = -e^{-x}\Big|_0^\infty = 1$$

$$\Rightarrow \gamma'(1) = -\gamma$$

2) $I = \mathcal{L}\{t\cdot\ln(2t)\}(3) = -\left(\mathcal{L}\{\ln(2t)\}(s)\right)'\Big|_{s=3}$

$$= -\left(\frac{1}{2}\mathcal{L}\{\ln t\}\left(\frac{s}{2}\right)\right)'\Big|_{s=3} = +\frac{1}{\cancel{2}}\cdot\left(\frac{\gamma + \ln\frac{s}{2}}{\frac{s}{\cancel{2}}}\right)'\Big|_{s=3}$$

$$= \left(\frac{\gamma + \ln s - \ln 2}{s}\right)'\Big|_{s=3} = \frac{\frac{1}{s}\cdot s - \gamma - \ln s + \ln 2}{s^2}\Big|_{s=3}$$

$$= \frac{1 - \gamma - \ln 3 + \ln 2}{9} = \frac{1}{9}\left(1 - \gamma - \ln\frac{3}{2}\right)$$

Ex 8

Rezolvați ecuația integrală Fourier

$$\int_0^\infty f(t)\sin\omega t\,dt = \underbrace{\frac{1}{\sqrt{\omega}}}_{g(\omega)}, \quad \omega>0$$

$=$

$(\mathcal{F}_s f)(\omega) = g(\omega)$,

$\Rightarrow f(t) = (\mathcal{F}_s^{-1} g)(t) = \frac{2}{\pi}\int_0^\infty g(\omega)\sin\omega t\,d\omega$

$$f(t) = \frac{2}{\pi}\int_0^\infty \frac{\sin\omega t}{\sqrt{\omega}}\,d\omega \quad (1)$$

Fie $h(t) = \frac{2}{\pi}\int_0^\infty \frac{\cos\omega t}{\sqrt{\omega}}\,d\omega$ (2)

Formăm expresia $h(t) - j f(t) = \frac{2}{\pi}\int_0^\infty \frac{e^{-j\omega t}}{\sqrt{\omega}}\,d\omega = \frac{2}{\pi}\int_0^\infty \frac{e^{-\omega(jt)}}{\sqrt{\omega}}\,d\omega$

Obs.: $\mathcal{L}\{f(t)\}(s) = \int_0^\infty f(t)e^{-st}\,dt$

$\mathcal{L}\{r(\omega)\}(s) = \int_0^\infty r(\omega)\cdot e^{-s\omega}\,d\omega$ $\quad \mathcal{L}\{\frac{1}{\sqrt{t}}\}(s) = \sqrt{\frac{\pi}{s}}$

Deci $h(t) - j f(t) = \frac{2}{\pi}\cdot\mathcal{L}\{\frac{1}{\sqrt{\omega}}\}(jt) = \frac{2}{\pi}\cdot\sqrt{\frac{\pi}{s}}\Big|_{s=jt}$

$= \frac{2}{\sqrt{\pi}}\cdot\frac{1}{\sqrt{jt}} = \frac{2}{\sqrt{\pi t}}\cdot j^{-\frac{1}{2}} \overset{(2)}{=} \frac{2}{\sqrt{\pi t}}\cdot\frac{\sqrt{2}}{2}(1-j) = \frac{\sqrt{2}}{\sqrt{\pi t}}(1-j)$

unde $j^{-\frac{1}{2}} = e^{-\frac{1}{2}\log j} = e^{-\frac{1}{2}(\ln|j| + j\cdot\underbrace{\arg j}_{=\frac{\pi}{2}})} = e^{-\frac{\pi}{4}j} = \cos\frac{\pi}{4} - j\sin\frac{\pi}{4}$

$= \frac{\sqrt{2}}{2} - j\cdot\frac{\sqrt{2}}{2}$ (2)

$z^\alpha = e^{\alpha\log z}$

def. pp.

$\Rightarrow \boxed{f(t) = h(t) = \sqrt{\frac{2}{\pi t}},\ t>0}$

Ex 9

Rezolvați ecuația $\int_0^\infty f(t)\cos\omega t\,dt = \omega^2 e^{-\omega}$, $\omega > 0$

$(\mathcal{F}_c f)(\omega) = g(\omega) = \omega^2 e^{-\omega} \Rightarrow f(t) = (\mathcal{F}_c^{-1} g)(t) = \frac{2}{\pi}\int_0^\infty \omega^2 e^{-\omega}\cos\omega t\,d\omega$

Fie $h(t) = \frac{2}{\pi}\int_0^\infty \omega^2 e^{-\omega}\sin\omega t\,d\omega$

Formăm expresia $f(t) + j\,h(t) = \frac{2}{\pi}\int_0^\infty \omega^2 e^{\omega(-1+jt)}\,d\omega$

$$= \frac{2}{\pi}\mathcal{L}\{\omega^2\}(1-jt)$$

$$= \frac{2}{\pi}\cdot\frac{2!}{s^3}\Big|_{s=1-jt}$$

$$= \frac{4}{\pi}\cdot\frac{1}{\overset{(1+jt)^3}{(1-jt)^3}} = \frac{4(1+jt)^3}{\pi(1+t^2)^3}$$

$\Rightarrow f(t) = \frac{4}{\pi}\cdot\frac{1-3t^2}{(1+t^2)^3}$

Ex 4 (temă)

Fie $g:[0,+\infty)\to\mathbb{R}$, $g(t)=\int_0^{\sqrt{t}} x^6\cdot e^{-3x^2}dx$

1) Calculați $\mathcal{L}\{g'(t)\}(s)$

2) Calculați $I=\int_0^{\infty} t\cdot g(t)\cdot e^{-6t}dt$

—

1) $g'(t)=(\sqrt{t})'\cdot(\sqrt{t})^6\cdot e^{-3t}=\frac{1}{2\sqrt{t}}\cdot t^3\cdot e^{-3t}=\frac{1}{2}\cdot t^{\frac{5}{2}}\cdot e^{-3t}$

$\mathcal{L}\{g'(t)\}(s)=\frac{1}{2}\mathcal{L}\{t^{\frac{5}{2}}\cdot e^{-3t}\}(s)=\frac{1}{2}\mathcal{L}\{t^{\frac{5}{2}}\}(s+3)$

$\mathcal{L}\{e^{at}f(t)\}(s)=\mathcal{L}\{f(t)\}(s-a)$
(T. deplasării imaginii)

$=\frac{1}{2}\cdot\frac{\Gamma(\frac{5}{2}+1)}{(s+3)^{\frac{5}{2}+1}}$

$=\frac{1}{2}\cdot\frac{\frac{5}{2}\Gamma(\frac{5}{2})}{(s+3)^{\frac{7}{2}}}=\frac{1}{2}\cdot\frac{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma(\frac{1}{2})}{(s+3)^{\frac{7}{2}}}=\frac{15}{16}\cdot\frac{\sqrt{\pi}}{(s+3)^{\frac{7}{2}}}$

$\mathcal{L}\{g'(t)\}(s)=sG(s)-g(0+0)$

$=s\cdot G(s)=\frac{15}{16}\cdot\frac{\sqrt{\pi}}{(s+3)^{\frac{7}{2}}}\longrightarrow \underset{\mathcal{L}\{g(t)\}(s)}{\overset{\shortparallel}{G(s)}}=\frac{15\sqrt{\pi}}{16s\cdot(s+3)^{\frac{7}{2}}}$

2) $I=\mathcal{L}\{t\cdot g(t)\}(6)=-(\mathcal{L}\{g(t)\}(s))'\Big|_{s=6}$

$=-(G(s))'\Big|_{s=6}=+\frac{15\sqrt{\pi}}{16}\cdot\frac{(s+3)^{\frac{7}{2}}+\frac{7}{2}s(s+3)^{\frac{5}{2}}}{s^2\cdot(s+3)^7}\Big|_{s=6}=\frac{25}{2^5\cdot 3^9}\cdot\sqrt{\pi}$

Transformata Laplace

(1) a) Să se arate că

$$\mathcal{L}\left\{\int_0^\infty \frac{\cos(tx)}{x^2+1}\,dx\right\}(s) = \frac{\pi}{2(s+1)}$$

$x(0) = x'(0) = 0$

b) Res. ec.

$$x''(t) - 2x'(t) + x(t) = \frac{2}{\pi}\int_0^\infty \frac{\cos(tx)}{x^2+1}\,dx$$

a) $$\mathcal{L}\left\{\overbrace{\int_0^\infty \frac{\cos(tx)}{x^2+1}\,dx}^{x(t)}\right\}(s) = \int_0^\infty \left(\int_0^\infty \frac{\cos tx}{x^2+1}\cdot e^{-st}\,dx\right)dt$$

$$= \int_0^\infty \left(\int_0^\infty \frac{\cos tx\, e^{-st}}{x^2+1}\,dt\right)dx = \int_0^\infty \frac{1}{x^2+1}\left(\int_0^\infty \cos tx\cdot e^{-st}\,dt\right)dx$$

$$= \int_0^\infty \frac{1}{x^2+1}\,\mathcal{L}\{\cos tx\}(s)\,dx = \int_0^\infty \frac{1}{x^2+1}\cdot\frac{s}{x^2+s^2}\,dx$$

$$= \frac{s}{s^2-1}\int_0^\infty \left(\overbrace{\frac{1}{x^2+1}}^{x^2+s^2} - \overbrace{\frac{1}{x^2+s^2}}^{x^2+1}\right)dx = \frac{s}{s^2-1}\left(\operatorname{arctg} x\Big|_0^\infty - \frac{1}{s}\operatorname{arctg}\frac{x}{s}\Big|_0^\infty\right)$$

$$= \frac{s}{s^2-1}\left(\frac{\pi}{2} - \frac{\pi}{2s}\right) = \frac{s}{\underbrace{s^2-1}_{s+1}}\cdot\frac{s-1}{2s}\cdot\pi = \frac{\pi}{2(s+1)}$$

$$x(t) = \frac{\pi}{2}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}(t) = \frac{\pi}{2}\cdot e^{-t}\cdot u(t)$$

b) Ob. $\mathcal{L}\{x'(t)\}(s) = sX(s) - \overset{=0}{x(0)}$

$$\mathcal{L}\{x''(t)\}(s) = s^2X(s) - s\underset{=0}{x(0)} - \underset{=0}{x'(0)}$$

$$\Rightarrow s^2X(s) - 2sX(s) + X(s) = \frac{2}{\cancel{\pi}} \cdot \frac{\cancel{\pi}}{2(s+1)}$$

$$X(s)(s^2-2s+1) = \frac{1}{s+1}$$

$$X(s) = \frac{1}{(s+1)(s-1)^2} = \frac{A}{s+1} + \frac{B}{s-1} + \frac{C}{(s-1)^2}$$

$$\Rightarrow 1 = A(s-1)^2 + B(s+1)(s-1) + C(s+1)$$

Pt. $s=1 \Rightarrow 1 = 2C \Rightarrow C = \frac{1}{2}$

Pt. $s=-1 \Rightarrow 1 = 4A \Rightarrow A = \frac{1}{4}$

Coef. lui s^2: $0 = A + B \Rightarrow B = -\frac{1}{4}$

$$x(t) = \left(\frac{1}{4} \cdot e^{-t} - \frac{1}{4} \cdot e^{t} + \frac{1}{2} t \cdot e^{t}\right) u(t)$$

② $f(t) = \int_0^t e^{-2x} (x-t)^2 (\sin(t-x))^{(4)} dx$

Calculat $\mathcal{L}\{f(t)\}(s)$

$$f(t) = e^{-2t} * (t^2 \cdot (\sin t)^{(4)})$$

$$\mathcal{L}\{f(t)\}(s) = \frac{1}{s+2}\Big|_{s=1} \cdot \mathcal{L}\{t^2 \cdot \sin t\}(s)\Big|_{s=1}$$

unde $\mathcal{L}\{t^2 \cdot \sin t\}(s) = (-1)^2 \cdot (\mathcal{L}\{\sin t\}(s))'' = \left(\frac{1}{s^2+1}\right)''$

$$= \left(\frac{-2s}{(s^2+1)^2}\right)' = -2 \cdot \frac{(s^2+1)^{\cancel{2}} - 4s^2\cancel{(s^2+1)}}{(s^2+1)^{\cancel{4}\,3}} \Rightarrow \mathcal{L}\{f(t)\}(s) = \frac{1}{3} \cdot \frac{1}{2} = \underline{\underline{\frac{1}{6}}}$$

Problème MSi

Model TFi

① Se consideră semnalul $f: \mathbb{R} \to \mathbb{C}$,

$$f(t) = \frac{t}{(2+jt)^5}.$$

Determinați $E(f)$, spectrul $\hat{f}(\omega)$ și $E(\hat{f})$.

$$E(f) = \int_{-\infty}^{\infty} |f(t)|^2 dt = \int_{-\infty}^{\infty} \frac{t^2}{|2+jt|^{10}} dt$$

$$= \int_{-\infty}^{\infty} \frac{t^2}{\sqrt{t^2+4}^{\,10}} dt = \int_{-\infty}^{\infty} \frac{t^2}{(t^2+4)^5} dt$$

$$= \int_{-\infty}^{\infty} \frac{t^2+4-4}{(t^2+4)^5} dt = \underbrace{\int_{-\infty}^{\infty} \frac{\cancel{t^2+4}}{(t^2+4)^{\cancel{5}\,4}} dt}_{J_4} - 4 \underbrace{\int_{-\infty}^{\infty} \frac{dt}{(t^2+4)^5}}_{J_5}$$

$$\Rightarrow E(f) = J_4 - 4J_5$$

unde

$$J_4 = \int_{-\infty}^{\infty} \frac{dt}{(t^2+4)^4} = 2\pi j \cdot \mathrm{Res}\left(\frac{1}{(t^2+4)^4}; 2j\right)$$

$t_{1,2} = \pm 2j$ poli de ordinul 4

$$I_4 = 2\pi j \cdot \frac{1}{3!} \lim_{t \to 2j} \left(\cancel{(t-2j)^4} \cdot \frac{1}{\cancel{(t-2j)^4}(t+2j)^4} \right)'''$$

$$= \frac{\pi j}{3} \cdot \lim_{t \to 2j} \left((t+2j)^{-4} \right)'''$$

$$\left((t+2j)^{-4} \right)' = -4 \cdot (t+2j)^{-5}$$

$$(\quad)'' = (-4)\cdot(-5)\cdot(t+2j)^{-6}$$

$$(\quad)''' = (-4)\cdot(-5)\cdot(-6)\cdot(t+2j)^{-7}$$

$$\Rightarrow I_4 = \frac{\pi j}{3} \cdot \lim_{t \to 2j} \frac{(-4)\cdot(-5)\cdot(-6)}{(t+2j)^7} = -\frac{\pi \cancel{j}}{3} \cdot \frac{\cancel{4} \cdot 5 \cdot 6}{4^6 \cancel{4} \cdot \cancel{j^7}}$$

$$j^6 = -1$$

$$\underline{\underline{I_4}} = \frac{5 \cdot \overset{2}{\cancel{6}} \pi}{\cancel{3} \cdot 4^6} = \frac{10\pi}{4^6} = \underline{\underline{\frac{5\pi}{2048}}}$$

$$I_5 = \int_{-\infty}^{\infty} \frac{1}{(t^2+4)^5} dt$$

$t_{1,2} = \pm 2j$ poli de ordinul 5

$$\underline{\underline{I_5}} = 2\pi j \cdot \frac{1}{4!} \lim_{t \to 2j} \left(\cancel{(t-2j)^5} \cdot \frac{1}{\cancel{(t-2j)^5}(t+2j)^5} \right)^{(4)}$$

$$= \frac{\pi j}{12} \cdot \lim_{t \to 2j} \left((t+2j)^{-5} \right)^{(4)}$$

$$= \frac{\pi j}{12} \cdot (-5)\cdot(-6)\cdot(-7)\cdot(-8) \cdot \frac{1}{(t+2j)^9} \Big|_{t=2j}$$

$$= \frac{\pi \cancel{j}}{\cancel{12}} \cdot \frac{5\cdot \cancel{6}\cdot 7\cdot \cancel{8}^{4}}{4^9 \cdot \cancel{j^9}} = \frac{35\pi}{4^8} = \underline{\underline{\frac{35\pi}{65536}}}$$

$$\Longrightarrow \underline{\underline{E(f)}} = \frac{10\pi}{4^6} - \cancel{4} \cdot \frac{35\cdot\pi}{4^{\cancel{8}\,7}}$$

$$= \frac{10\pi}{4^6} - \frac{35\pi}{4^7}$$

$$= \frac{40\pi - 35\pi}{4^7} = \underline{\underline{\frac{5\pi}{4^7}}}$$

Din Formula lui PARSEVAL

$$\Longrightarrow E(\hat{f}) = 2\pi \cdot E(f)$$

$$E(\hat{f}) = 2\pi \cdot \frac{5\pi}{4^7}$$

$$\underline{\underline{E(\hat{f}) = \frac{10\pi^2}{4^7}}}$$

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} \underbrace{\frac{t}{(2+jt)^5} \cdot e^{-j\omega t}}_{g(t)} dt$$

<u>Cazul I</u> $\omega < 0$

$2 + jt = 0 \quad / \cdot (-j)$

$-2j + t = 0 \rightarrow t_1 = 2j \qquad Im(2j) > 0$

pol de ordinul 5

$$\hat{f}(\omega) = 2\pi j \cdot Rez(g; 2j)$$

$$= 2\pi j \cdot \frac{1}{4!} \cdot \lim_{t \to 2j} \left(\frac{\cancel{(t-2j)^5} \cdot t \cdot e^{-j\omega t}}{(2+jt)^5} \right)^{(4)}$$

$$j^5 \left(\frac{2}{j} + t\right)^5 = j \cancel{(t-2j)^5}$$

$$= \frac{\pi j}{12} \cdot \left(\frac{t \cdot e^{-j\omega t}}{j} \right)^{(4)} \Bigg|_{t=2j}$$

$$= \frac{\pi}{12} \cdot \left(t \cdot e^{-j\omega t} \right)^{(4)} \Bigg|_{t=2j}$$

$$= \frac{\pi}{12} \cdot \sum_{k=0}^{4} C_4^k \, t^{(k)} \cdot \left(e^{-j\omega t}\right)^{(4-k)} \Bigg|_{t=2j}$$

$$(f \cdot g)^{(n)} = \sum_{k=0}^{n} C_n^k \cdot f^{(k)} \cdot g^{(n-k)}$$

$$= \frac{\pi}{12}\left(C_4^0 \cdot t \cdot (e^{-j\omega t})^{(4)} + C_4^1 \cdot t' \cdot (e^{-j\omega t})^{'''} + 0\right)\Big|_{t=2j}$$

$$= \frac{\pi}{12} \cdot \left(t \cdot (e^{-j\omega t})^{(4)} + 4 \cdot (e^{-j\omega t})^{'''}\right)\Big|_{t=2j}$$

$$\left(\begin{aligned} &(e^{-j\omega t})' = (-j\omega) \cdot e^{-j\omega t} \\ &(\quad)'' = (-j\omega)^2 \cdot e^{-j\omega t} \\ &(\quad)''' = (-j\omega)^3 \cdot e^{-j\omega t} \\ &(\quad)^{(4)} = (-j\omega)^4 \cdot e^{-j\omega t} \end{aligned} \right)$$

$$= \frac{\pi}{12} \cdot \left(t \cdot (-j\omega)^4 \cdot e^{-j\omega t} + 4 \cdot (-j\omega)^3 \cdot e^{-j\omega t}\right)\Big|_{t=2j}$$

$$= \frac{\pi}{12} \cdot \left(2j \cdot \omega^4 \cdot e^{2\omega} + 4j \cdot \omega^3 \cdot e^{2\omega}\right)$$

$$= \frac{\pi}{\cancel{12}_{6}} \cdot \cancel{2}j \cdot \omega^3 \cdot e^{2\omega} \cdot (\omega + 2)$$

$$= \frac{\pi j}{6} \cdot \omega^3 \cdot (\omega + 2) \cdot e^{2\omega} = \hat{f}(\omega)$$

Cazul II $\omega > 0$

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} \frac{t}{(2+jt)^5} \cdot e^{\overset{\lambda<0}{-j\omega} t} dt$$

$$\frac{t=-x}{dt=-dx} \int_{+\infty}^{-\infty} \frac{-x}{(2-jx)^5} \cdot e^{j\omega x} (-dx)$$

$$= \int_{-\infty}^{\infty} \frac{-x}{(2-jx)^5} \cdot e^{j\omega x} dx$$

$2-jx=0 \quad | \cdot j$

$2j+x=0 \Rightarrow x_1=-2j \quad Im(-2j)<0$

pol de ordinul 5

$\Rightarrow \hat{f}(\omega)=0$

Cazul III $\omega=0$

$$\hat{f}(\omega)=\hat{f}(0)=\int_{-\infty}^{\infty} \frac{t}{(2+jt)^5} dt = 2\pi j \cdot Rez(f; 2j)$$

$t_1=2j$ pol de ordinul 5

$$= 2\pi j \cdot \frac{1}{4!} \lim_{t \to 2j} \left(\cancel{(t-2j)^5} \cdot \frac{t}{(2+jt)^5} \right)^{(4)}$$

$$\overset{\shortparallel}{j^5 \cancel{(t-2j)^5}}$$

$$= \frac{\pi j}{12} \cdot \frac{1}{j^5} \cdot t^{(4)} \Big|_{t=2j} = 0$$

$$\Rightarrow \hat{f}(\omega) = \begin{cases} \frac{\pi j}{6} \cdot \omega^3 \cdot (\omega+2) \cdot e^{2\omega}, & \omega < 0 \\ 0, & \omega \geq 0 \end{cases}$$

---

② Să se calculeze spectrul, amplitudinea și faza în frecvență ale semnalului $f: \mathbb{R} \to \mathbb{C}$

$$f(t) = \frac{j-t}{(t^2+8t+20)^2}, \text{ pentru frecvența } \omega = 10$$

$$\hat{f}(\omega) \overset{def}{=} \int_{-\infty}^{\infty} f(t) \cdot e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} \frac{j-t}{(t^2+8t+20)^2} \cdot e^{-j\omega t} dt$$

$\lambda = -10 < 0$

$$\hat{f}(10) = \int_{-\infty}^{\infty} \frac{j-t}{(t^2+8t+20)^2} \cdot e^{-j\cdot 10\cdot t}\, dt$$

$\lambda = 10 > 0$

$$\underset{dt=-dx}{\overset{t=-x}{=}} \int_{+\infty}^{-\infty} \frac{j+x}{(x^2-8x+20)^2} \cdot e^{j\cdot 10\cdot x} \cdot (-dx)$$

$$= \int_{-\infty}^{\infty} \underbrace{\frac{j+x}{(x^2-8x+20)^2} \cdot e^{10jx}}_{g(x)} \cdot dx$$

$x^2-8x+20$

$\Delta = 64-80 = -16 = (4j)^2$

$\mathrm{Im}(x_1) > 0$

$x_{1,2} = \frac{8 \pm 4j}{2}$ $\begin{cases} x_1 = 4+2j \\ x_2 = 4-2j \end{cases}$ poli doppi

$\Rightarrow \hat{f}(10) = 2\pi j \cdot \mathrm{Res}(g; 4+2j)$

$$= 2\pi j \cdot \frac{1}{1!} \lim_{x \to 4+2j} \left(\frac{\cancel{(x-4-2j)^2} \cdot (j+x) \cdot e^{10jx}}{\cancel{(x-4-2j)^2} (x-4+2j)^2} \right)'$$

$$= 2\pi j \cdot \left(\frac{(j+x) \cdot e^{10jx}}{(x-4+2j)^2} \right)' \Bigg|_{x=4+2j}$$

$$= 2\pi j \cdot \left(\frac{e^{10jx} + (j+x)\cdot 10j \cdot e^{10jx} \cdot (x-4+2j)^{\cancel{2}}}{(x-4+2j)^{\cancel{4}\,3}} - \frac{2\cancel{(x-4+2j)}\cdot(j+x)\cdot e^{10jx}}{} \right)\Bigg|_{x=4+2j}$$

$$= 2\pi j \cdot e^{10jx} \cdot \frac{(1+(j+x)\cdot 10j)(x-4+2j) - 2(j+x)}{(x-4+2j)^3}\Bigg|_{x=4+2j}$$

$$= 2\pi j \cdot e^{10j(4+2j)} \cdot \frac{(1+(4+3j)\cdot 10j)\cdot 4j - 2\cdot(4+3j)}{4^3 \cdot \cancel{j^3}} \qquad j^2 = -1$$

$$= -2\pi \cdot e^{10(-2+4j)} \cdot \frac{(-29+40j)\cdot 4j - 8 - 6j}{4^3}$$

$$= -2\pi \cdot e^{10(-2+4j)} \cdot \frac{-116j - 160 - 8 - 6j}{4^3}$$

$$= -2\pi \cdot e^{20(-1+2j)} \cdot \frac{-168 - 122j}{4^3}$$

$$= +\cancel{4}\pi \cdot e^{20(-1+2j)} \cdot \frac{84+61j}{4^{\cancel{3}\,2}}$$

$$= \frac{\pi}{16} \cdot e^{-20} \cdot (\cos 40 + j \cdot \sin 40)(84+61j) = \vec{f}(10)$$

$$z = |z| \cdot e^{j \cdot \arg z}$$

$$\hat{f}(\omega) = \underbrace{|\hat{f}(\omega)|}_{A(\omega)} \cdot e^{j \cdot \underbrace{\arg \hat{f}(\omega)}_{\phi(\omega)}}$$

$A(\omega)$ pt. $\omega = 10$ este $A(10)$

$$A(10) = |\hat{f}(10)| = \left| \frac{\pi}{16} \cdot \underset{>0}{e^{-20}} (\cos 40 + j \sin 40)(84 + 61j) \right|$$

$$= \frac{\pi}{16} \cdot e^{-20} \cdot \underbrace{|\cos 40 + j \sin 40|}_{=1} \cdot |84 + 61j|$$

$$= \frac{\pi}{16} \cdot e^{-20} \cdot \sqrt{84^2 + 61^2}$$

$$\phi(10) = \arg \hat{f}(10)$$

$$= \arg\left(\frac{\pi}{16} \cdot e^{-20} \cdot e^{40j} \cdot (84 + 61j)\right)$$

$$= \left(\arg\left(\frac{\pi}{16} \cdot \underset{>0}{e^{-20}}\right) + \arg(e^{40j}) + \arg(84 + 61j)\right)_{\mod 2\pi}$$

$$= \left(0 + 40 + \operatorname{arctg} \frac{61}{84}\right)_{\mod 2\pi} = 40 - 12\pi + \operatorname{arctg} \frac{61}{84}$$

$\arg e^{jx} = x \pmod{2\pi}$, $\arg e^{40j} = 40 \pmod{2\pi}$
$= 40 - 12\pi$

③ Se consideră semnalul $f: \mathbb{R} \to \mathbb{C}$

$$f(t) = \int_{-\infty}^{\infty} \frac{(t-x)\cdot e^{-4x^2}}{(t^2+x^2-2tx+4)^2}\,dx$$

Determinați $a \in \mathbb{R}$ a.î. $f(t) = \frac{t}{(t^2+a)^2} * e^{-4t^2}$

OBS: Produsul de convoluție la TF-i!

$$f_1(t) * g_1(t) = \int_{-\infty}^{\infty} f_1(x)\cdot g_1(t-x)\,dx$$

$$f(t) = \int_{-\infty}^{\infty} e^{-4x^2}\cdot \frac{t-x}{((t-x)^2+4)^2}\,dx$$

$$= e^{-4t^2} * \frac{t}{(t^2+4)^2} = \frac{t}{(t^2+4)^2} * e^{-4t^2}$$

produsul de conv. este comutativ

$\Rightarrow a = 4$

Deci $f(t) = \frac{t}{(t^2+4)^2} * e^{-4t^2}$

$$\hat{f}(\omega) = \mathcal{F}\left\{\frac{t}{(t^2+4)^2} * e^{-4t^2}\right\}(\omega)$$

$$= \mathcal{F}\left\{\frac{t}{(t^2+4)^2}\right\}(\omega) \cdot \mathcal{F}\{e^{-4t^2}\}(\omega)$$

$$\left(\mathcal{F}\{f*g\}(\omega) = \mathcal{F}\{f\}(\omega) \cdot \mathcal{F}\{g\}(\omega)\right)$$

$$\Rightarrow \hat{f}(\lambda) = \mathcal{F}\left\{\frac{t}{(t^2+4)^2}\right\}(\lambda) \cdot \mathcal{F}\{e^{-4t^2}\}(\lambda)$$

unde

$$\mathcal{F}\left\{\frac{t}{(t^2+4)^2}\right\}(\lambda) = \int_{-\infty}^{\infty} \frac{t}{(t^2+4)^2} \cdot e^{-j\cdot\lambda\cdot t}\, dt$$

$$\underset{dt=-dx}{\overset{t=-x}{=}} \int_{+\infty}^{-\infty} \frac{-x}{(x^2+4)^2} \cdot e^{jx} \cdot (-dx)$$

$$= \int_{-\infty}^{\infty} \frac{-x}{(x^2+4)^2} \cdot e^{jx}\, dx$$

$$\left(x^2+4=0 \Rightarrow x^2 = -4 = (2j)^2 \quad x_{1,2} = \pm 2j \text{ poli duoli}\right)$$

$$= -2\pi j \cdot \operatorname{Res}\left(\frac{x \cdot e^{jx}}{(x^2+4)^2}, 2j\right)$$

$$= -2\pi j \cdot \lim_{x \to 2j} \left(\cancel{(x-2j)^2} \cdot \frac{x \cdot e^{jx}}{\cancel{(x-2j)^2} \cdot (x+2j)^2}\right)'$$

$$= -2\pi j \cdot \lim_{x \to 2j} \left(\frac{x \cdot e^{jx}}{(x+2j)^2} \right)'$$

$$= -2\pi j \cdot \frac{(e^{jx} + jxe^{jx})(x+2j)^{\cancel{2}} - 2\cancel{(x+2j)} \cdot x \cdot e^{jx}}{(x+2j)^{\cancel{4}\,3}} \Big|_{x=2j}$$

$$= -2\pi j \cdot e^{jx} \cdot \frac{(1+jx)(x+2j) - 2x}{(x+2j)^3} \Big|_{x=2j}$$

$$= -2\pi j \cdot e^{-2} \cdot \frac{-1 \cdot 4j - 4j}{4^3 \cdot \cancel{j^3}\, j^2} = 2\pi \cdot e^{-2} \cdot \frac{-8j}{4^3} = -\frac{\pi e^{-2} j}{4}$$

iar

$$\mathcal{F}\{e^{-4t^2}\}(x) = \int_{-\infty}^{\infty} e^{-4t^2} \cdot e^{-jxt}\, dt = \int_{-\infty}^{\infty} e^{-(4t^2 + jxt)}\, dt \overset{*}{=}$$

$$\left(4t^2 + jxt = \left(2t + \frac{j}{4}\right)^2 + \frac{1}{16} \right)$$

$$2 \cdot 2\cancel{t} \cdot b = j\cancel{t} \rightarrow b = \frac{j}{4}$$

$$\overset{*}{=} \int_{-\infty}^{\infty} e^{-\left(\left(2t + \frac{j}{4}\right)^2 + \frac{1}{16}\right)}\, dt = e^{-\frac{1}{16}} \int_{-\infty}^{\infty} e^{-\left(2t + \frac{j}{4}\right)^2}\, dt$$

$$\frac{2t + \frac{j}{4} = x}{2\,dt = dx} \quad e^{-\frac{1}{16}} \int_{-\infty}^{\infty} e^{-x^2} \cdot \frac{1}{2}\, dx = \frac{1}{2} e^{-\frac{1}{16}} \underbrace{\int_{-\infty}^{\infty} e^{-x^2}\, dx}_{\sqrt{\pi}}$$

$$= \frac{\sqrt{\pi}}{2} \cdot e^{-\frac{1}{16}}$$

$$\Rightarrow \underline{\underline{\hat{f}(1)}} = -\frac{\pi e^{-2}}{4} j \cdot \frac{\sqrt{\pi}}{2} \cdot e^{-\frac{1}{16}}$$

$$= \underline{\underline{-\frac{\pi\sqrt{\pi}}{8} \cdot e^{-(2+\frac{1}{16})} j}}$$

(4) Fie $f(t)$ soluția ecuației integrale Fourier

$$\int_0^\infty f(t)\sin(\omega t)dt = \frac{1}{\omega(4\omega^2+1)^2}, \quad \omega > 0$$

Determinați $f(2\pi)$.

$$f(t) = \frac{2}{\pi}\int_0^\infty \underbrace{\frac{1}{\omega(4\omega^2+1)^2} \cdot \sin(\omega t)}_{\text{pară}} d\omega, \quad t > 0$$

$$= \frac{1}{\cancel{2}} \cdot \frac{\cancel{2}}{\pi}\int_{-\infty}^\infty \frac{1}{\omega(4\omega^2+1)^2} \cdot \sin(\omega t)d\omega$$

Fie $h(t) = \frac{1}{\pi}\int_{-\infty}^\infty \underbrace{\frac{1}{\omega(4\omega^2+1)^2} \cdot \cos(\omega t)}_{\text{impară}} d\omega = 0$

$$\underset{0}{\underset{\shortparallel}{h(t)}} + j \cdot f(t) = \frac{1}{\pi}\int_{-\infty}^\infty \frac{1}{\omega(4\omega^2+1)^2} \cdot e^{j\omega t} d\omega$$

$\lambda > 0$

$$\Rightarrow f(t) = \frac{1}{\pi j} \int_{-\infty}^{\infty} \underbrace{\frac{1}{\omega(4\omega^2+1)^2} \cdot e^{j\omega t}}_{g(\omega)} \, d\omega$$

$\underline{\underline{\omega_0 = 0}} \in \mathbb{R}$ pol simple

$4\omega^2 + 1 = 0 \Rightarrow \omega^2 = -\frac{1}{4} = \left(\frac{1}{2} j\right)^2$

$\underline{\underline{\omega_{1/2} = \pm \frac{1}{2} j}}$ poli doppi

$$\Rightarrow f(t) = \frac{1}{\pi j} \left(2\pi j \cdot \mathrm{Res}\left(g; \tfrac{1}{2} j\right) + \pi j \cdot \mathrm{Res}(g; 0) \right)$$

$$= 2\,\mathrm{Res}\left(g; \tfrac{1}{2} j\right) + \mathrm{Res}(g; 0)$$

$$= 2 \cdot \lim_{\omega \to \frac{1}{2} j} \left(\cancel{\left(\omega - \tfrac{1}{2} j\right)^2} \cdot \frac{e^{j\omega t}}{\underset{2}{\cancel{4}}\,\omega \cancel{\left(\omega - \tfrac{1}{2} j\right)^2} \left(\omega + \tfrac{1}{2} j\right)^2} \right)'_{\omega}$$

$$+ \left. \frac{e^{j\omega t}}{\omega^1 (4\omega^2+1)^2} \right|_{\omega = 0}$$

$$= \frac{1}{2} \left(\frac{e^{j\omega t}}{\omega \left(\omega + \frac{1}{2} j\right)^2} \right)' \Bigg|_{\omega = \frac{1}{2} j} + 1$$

$= \ldots$

/ /

⑤ Fie semnalele

$$f, g : \mathbb{R} \to \mathbb{R},\ f(t) = e^{-t^2+2t}$$

$$g(t) = f^{(7)}(t)$$

$$h(t) = t \cdot f(t)$$

Să se calculeze $E(f)$, $\hat{f}(\omega)$, $E(\hat{f})$, $\hat{g}(\omega)$, $\hat{h}(\omega)$ și faza în frecvență a semnalului $f(t)$ pt. $\omega = 5$

$$E(f) \overset{def}{=} \int_{-\infty}^{\infty} |f(t)|^2 dt = \int_{-\infty}^{\infty} |e^{-t^2+2t}|^2 dt$$

$$= \int_{-\infty}^{\infty} e^{-2t^2+4t} dt = \int_{-\infty}^{\infty} e^{-(2t^2-4t)} dt \overset{*}{=}$$

$$\left(2t^2 - \underset{\shortparallel}{4t} = (\sqrt{2}t - \sqrt{2})^2 - 2 \right.$$

$$\left. 2 \cdot \sqrt{2}\cancel{t} \cdot b = 4\cancel{t} \Rightarrow b = \frac{4}{2\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2} \right)$$

$$\overset{*}{=} \int_{-\infty}^{\infty} e^{-((\sqrt{2}t-\sqrt{2})^2-2)} dt = e^2 \int_{-\infty}^{\infty} e^{-(\sqrt{2}t-\sqrt{2})^2} dt$$

$$\underline{\underline{\sqrt{2}t - \sqrt{2} = x}} \quad e^2 \int_{-\infty}^{\infty} e^{-x^2} \cdot \frac{1}{\sqrt{2}} dx = \frac{e^2}{\sqrt{2}} \underbrace{\int_{-\infty}^{\infty} e^{-x^2} dx}_{\sqrt{\pi}}$$

$$\sqrt{2}\, dt = dx$$

$$= e^2 \cdot \sqrt{\frac{\pi}{2}} = E(f)$$

Cf formulei lui PARSEVAL,

$$E(\hat{f}) = 2\pi \cdot E(f)$$

$$= 2\pi \cdot e^2 \sqrt{\frac{\pi}{2}} = \pi\sqrt{2\pi} \cdot e^2$$

$$g(t) = f^{(7)}(t)$$

$$\hat{g}(\omega) = \mathcal{F}\{f^{(7)}(t); \omega\} = (j\omega)^7 \cdot \hat{f}(\omega)$$

T derivării semnalului

$$\mathcal{F}\{f^{(n)}; \omega\} = (j\omega)^n \cdot \mathcal{F}\{f; \omega\}$$

$$= -j\omega^7 \cdot \hat{f}(\omega)$$

unde

$$\hat{f}(\omega) \overset{def}{=} \int_{-\infty}^{\infty} e^{-t^2+2t} \cdot e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} e^{-(t^2+(-2+j\omega)t)} dt \overset{*}{=}$$

$$\left(t^2 + \underbrace{(-2+j\omega)t}_{\substack{\parallel \\ 2\cdot t\cdot b}} = \left(t + \frac{-2+j\omega}{2}\right)^2 - \left(\frac{-2+j\omega}{2}\right)^2 \right)$$

$$2\cdot t\cdot b \Rightarrow b = \frac{-2+j\omega}{2}$$

$$\underline{\underline{(*)}} \int_{-\infty}^{\infty} e^{-\left(\left(t+\frac{-2+j\omega}{2}\right)^2 - \left(\frac{-2+j\omega}{2}\right)^2\right)} dt$$

$$= e^{\left(\frac{-2+j\omega}{2}\right)^2} \cdot \int_{-\infty}^{\infty} e^{-\left(t+\frac{-2+j\omega}{2}\right)^2} dt$$

$$\underline{\underline{\begin{matrix} t + \frac{-2+j\omega}{2} = x \\ dt = dx \end{matrix}}} \quad e^{\left(\frac{-2+j\omega}{2}\right)^2} \cdot \int_{-\infty}^{\infty} e^{-x^2} dx = \underline{\underline{\sqrt{\pi}\cdot e^{\left(\frac{-2+j\omega}{2}\right)^2}}} = \hat{f}(\omega)$$

$$\Rightarrow \underline{\hat{g}(\omega) = -j\omega \cdot \sqrt{\pi} \cdot e^{\left(\frac{-2+j\omega}{2}\right)^2}}$$

$$h(t) = t\cdot f(t)$$

$$\hat{h}(\omega) = \mathcal{F}\{t f(t); \omega\} = j\cdot(\hat{f}(\omega))'$$

T. deriv. spectrum:

$$\mathcal{F}\{t^n f(t); \omega\} = j^n \cdot (\hat{f}(\omega))^{(n)}$$

$$= j\sqrt{\pi}\cdot\left(e^{\left(\frac{-2+j\omega}{2}\right)^2}\right)' = \ldots$$