• Fie $f \in C[0,1]$, $w:[0,1] \longrightarrow \mathbb{R}_+$ o pondere și $P_m(f)$ polinomul de cea mai bună aproximare în medie pătratică a funcției f.

Să se arate că:

$$\int_0^1 w(x)\, P_m^2(f)(x)\, dx \leq \int_0^1 w(x)\, f^2(x)\, dx$$

REZOLVARE

$P_m(f)$ - este polinomul de cea mai bună aproximare în medie pătratica

$$\underset{(1)}{\Rightarrow} \| P_m(f) - f \|_2 \leq \min_{Q \in \Pi_m} \| f - Q \|_2$$

sau

$$(2) \quad \int_0^1 w(x) \left[f(x) - P_m(f)(x) \right]^2 dx \leq \int_0^1 w(x) \left[f(x) - Q(x) \right]^2 dx \ , \ \forall Q \in \Pi_m$$

$$\int_0^1 w(x)\, f^2(x)\, dx = \int_0^1 w(x) \left[f(x) - P_m(f)(x) + P_m(f)(x) \right]^2 dx$$

$$= \int_0^1 w(x) \left[f(x) - P_m(f)(x) \right]^2 + 2 \int_0^1 w(x) \left[f(x) - P_m(f)(x) \right] \cdot P_m(f)(x)\, dx$$

$$+ \int_0^1 w(x) \left[P_m(f)(x) \right]^2 dx \qquad (3)$$

Considerăm $F(\lambda) = \int_0^1 w(x) \left[f(x) - P_m(f)(x) - \lambda P_m(f)(x) \right]^2 dx$

Avem $F(\lambda) = \int_0^1 w(x) \left[f(x) - P_m(f)(x) - \lambda P_m(f)(x) \right]^2 dx \geq$

(2) cu $Q = P_m(f) + \lambda P_m(f)$

$$\geq \int_0^1 w(x) \left[f(x) - P_m(f)(x) \right]^2 = F(0)$$

$\Rightarrow F(\lambda) \geq F(0)$, $\forall \lambda$ $\Rightarrow$ 0 este PUNCT DE MINIM

$\Rightarrow F'(0) = 0$

Dar $F'(\lambda) = \left(\int_0^1 w(x) \left[f(x) - P_m(f)(x) - \lambda P_m(f)(x) \right]^2 dx \right)'$

$$= \left(\int_0^1 w(x) \left[f(x) - P_m(f)(x) \right]^2 dx - 2\lambda \int_0^1 w(x) \left[f(x) - P_m(f)(x) \right] P_m(f)(x)\, dx + \lambda^2 \int_0^1 w(x) \left(P_m(f)(x) \right)^2 dx \right)'$$

$$= -2 \int_0^1 w(x) \left[f(x) - P_m(f)(x) \right] P_m(f)(x)\, dx + 2\lambda \int_0^1 w(x) \left(P_m(f)(x) \right)^2 dx$$

$F'(0) = 0 \iff -2 \int_0^1 w(x) \left[f(x) - P_m(f)(x) \right] P_m(f)(x)\, dx = 0$

Înlocuim în (3) și obținem:

$$\int_0^1 w(x)\, f^2(x)\, dx = \underbrace{\int_0^1 w(x) \left[f(x) - P_m(f)(x) \right]^2 dx}_{\geq 0} + \int_0^1 w(x) \left(P_m(f)(x) \right)^2 dx$$

$$\Rightarrow \int_0^1 w(x)\, f^2(x)\, dx \geq \int_0^1 w(x) \left(P_m(f)(x) \right)^2 dx$$