CHAPTER 2

Vectors

2.1 Vectors

A vector $\overrightarrow{v}$ in space is characterized by *magnitude* (denoted by $||\overrightarrow{v}||$), *direction* and *sense*. The vectors are added by either the *triangle law* or the *parallelogram law*.

Vector addition obeys the following postulates:

1. $\overrightarrow{u} + (\overrightarrow{v} + \overrightarrow{w}) = (\overrightarrow{u} + \overrightarrow{v}) + \overrightarrow{w}$ (associative law);
2. $\overrightarrow{u} + \overrightarrow{v} = \overrightarrow{v} + \overrightarrow{u}$ (commutative law);
3. There is a unique vector called the *null vector*, denoted by $\overrightarrow{0}$, such that $\overrightarrow{0} + \overrightarrow{v} = \overrightarrow{v}$ for all $\overrightarrow{v}$;
4. For every vector $\overrightarrow{v}$ there is a unique vector called its *negative* and denoted by $-\overrightarrow{v}$, such that $\overrightarrow{v} + (-\overrightarrow{v}) = \overrightarrow{0}$.

Thus, if we denote by $\mathcal{V}_3$ the set of all vectors in the space, then $(\mathcal{V}_3, +)$ is a *commutative group*. The following postulates hold for the multiplication of vectors by numbers:

5. $1\overrightarrow{a} = \overrightarrow{a}$
6. $s(t\overrightarrow{a}) = (st)\overrightarrow{a}$

7. $(s+t)\overrightarrow{a} = s\overrightarrow{a} + t\overrightarrow{a}$
8. $s(\overrightarrow{a} + \overrightarrow{b}) = s\overrightarrow{a} + s\overrightarrow{b}$

for all $\overrightarrow{a}, \overrightarrow{b} \in \mathcal{V}_3$ and all $s, t \in \mathbb{R}$.

In talking about vectors, numbers are often called *scalars*. The vector $t\overrightarrow{v}$ is called a scalar multiple of the vector $\overrightarrow{v}$.

Consider now the axes Ox, Oy, Oz, mutually perpendicular, forming a right-handed rectangular Cartesian co-ordinate frame. Let $\overrightarrow{i}, \overrightarrow{j}, \overrightarrow{k}$ be the unit vectors for this system. Every vector $\overrightarrow{v}$ can be written, uniquely, in the form $\overrightarrow{v} = a\overrightarrow{i} + b\overrightarrow{j} + c\overrightarrow{k}$, where a, b, c are scalars (called the *components* of $\overrightarrow{v}$). Other important formulas are $||a\overrightarrow{v}|| = |a|||\overrightarrow{v}||$ and $||\overrightarrow{u} + \overrightarrow{v}|| \leq ||\overrightarrow{u}|| + ||\overrightarrow{v}||$ for all $\overrightarrow{u}, \overrightarrow{v} \in \mathcal{V}_3$ and $a \in \mathbb{R}$.

2.2 Scalar product and vector product

One associates with any two vectors $\overrightarrow{a}$ and $\overrightarrow{b}$ a number called their *scalar product* (or inner product) and denoted by $\overrightarrow{a} \cdot \overrightarrow{b}$. The definition reads:

$$\overrightarrow{a} \cdot \overrightarrow{b} = ||\overrightarrow{a}||||\overrightarrow{b}|| \cos\theta, \quad \theta = \text{angle between } \overrightarrow{a} \text{ and } \overrightarrow{b}$$

$$\overrightarrow{a} \cdot \overrightarrow{b} = 0 \text{ if either } \overrightarrow{a} = \overrightarrow{0} \text{ or } \overrightarrow{b} = \overrightarrow{0}.$$

For all $\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c} \in \mathcal{V}_3$ and $s \in \mathbb{R}$ we have

1) $\overrightarrow{a}\,\overrightarrow{b} = \overrightarrow{b}\,\overrightarrow{a}$
2) $\overrightarrow{a}(\overrightarrow{b} + \overrightarrow{c}) = \overrightarrow{a}\,\overrightarrow{b} + \overrightarrow{a}\,\overrightarrow{c}$
3) $(s\overrightarrow{a})\overrightarrow{b} = s(\overrightarrow{a}\,\overrightarrow{b})$
4) $\overrightarrow{a} \cdot \overrightarrow{a} \geq 0$; $\overrightarrow{a} \cdot \overrightarrow{a} = 0 \Longleftrightarrow \overrightarrow{a} = \overrightarrow{0}$.

Let us note that $\overrightarrow{a} \cdot \overrightarrow{a} = ||\overrightarrow{a}||^2$ and $\cos\theta = \dfrac{\overrightarrow{a} \cdot \overrightarrow{b}}{||\overrightarrow{a}|| \cdot ||\overrightarrow{b}||}$. In particular, $\overrightarrow{a} \cdot \overrightarrow{b} = 0 \Longleftrightarrow \overrightarrow{a} \perp \overrightarrow{b}$.

On the other hand, $\overrightarrow{i}\cdot\overrightarrow{i}=\overrightarrow{j}\cdot\overrightarrow{j}=\overrightarrow{k}\cdot\overrightarrow{k}=1$, $\overrightarrow{i}\cdot\overrightarrow{j}=\overrightarrow{j}\cdot\overrightarrow{k}=\overrightarrow{k}\cdot\overrightarrow{i}=0$. Let $\overrightarrow{a},\overrightarrow{b}\in\mathcal{V}_3$, $\overrightarrow{a}=a_1\overrightarrow{i}+a_2\overrightarrow{j}+a_3\overrightarrow{k}$, $\overrightarrow{b}=b_1\overrightarrow{i}+b_2\overrightarrow{j}+b_3\overrightarrow{k}$.

By using the properties of the scalar product, mentioned above, we deduce $\overrightarrow{a}\cdot\overrightarrow{b}=(a_1\overrightarrow{i}+a_2\overrightarrow{j}+a_3\overrightarrow{k})\cdot(b_1\overrightarrow{i}+b_2\overrightarrow{j}+b_3\overrightarrow{k})=a_1b_1\overrightarrow{i}\overrightarrow{i}+a_2b_1\overrightarrow{j}\overrightarrow{i}+a_3b_1\overrightarrow{k}\overrightarrow{i}+a_1b_2\overrightarrow{i}\overrightarrow{j}+a_2b_2\overrightarrow{j}\overrightarrow{j}+a_3b_2\overrightarrow{k}\overrightarrow{j}+a_1b_3\overrightarrow{i}\overrightarrow{k}+a_2b_3\overrightarrow{j}\overrightarrow{k}+a_3b_3\overrightarrow{k}\overrightarrow{k}$.

Thus we have the following important formula:

$$\overrightarrow{a}\cdot\overrightarrow{b}=a_1b_1+a_2b_2+a_3b_3.$$

Combining the previous results we can write:

$$||\overrightarrow{a}||=\sqrt{a_1^2+a_2^2+a_3^2}$$

$$\cos\theta=\frac{a_1b_1+a_2b_2+a_3b_3}{\sqrt{a_1^2+a_2^2+a_3^2}\sqrt{b_1^2+b_2^2+b_3^2}}$$

$$\overrightarrow{a}\perp\overrightarrow{b}\iff a_1b_1+a_2b_2+a_3b_3=0.$$

The vector product of the vectors $\overrightarrow{a}$ and $\overrightarrow{b}$ is the vector, denoted by $\overrightarrow{a}\times\overrightarrow{b}$, characterized by:

1) $||\overrightarrow{a}\times\overrightarrow{b}||=||\overrightarrow{a}||||\overrightarrow{b}||\sin\theta$
2) $\overrightarrow{a}\times\overrightarrow{b}$ is perpendicular to both $\overrightarrow{a}$ and $\overrightarrow{b}$
3) The triad of vectors $\{\overrightarrow{a},\overrightarrow{b},\overrightarrow{a}\times\overrightarrow{b}\}$ is oriented like the triad $\{\overrightarrow{i},\overrightarrow{j},\overrightarrow{k}\}$.

For all $\overrightarrow{a},\overrightarrow{b},\overrightarrow{c}\in\mathcal{V}_3$ and $s\in\mathbb{R}$ we have

I) $\overrightarrow{a}\times\overrightarrow{b}=-\overrightarrow{b}\times\overrightarrow{a}$

II) $(s\overrightarrow{a})\times\overrightarrow{b}=\overrightarrow{a}\times(s\overrightarrow{b})=s(\overrightarrow{a}\times\overrightarrow{b})$

III) $\overrightarrow{a}\times(\overrightarrow{b}+\overrightarrow{c})=\overrightarrow{a}\times\overrightarrow{b}+\overrightarrow{a}\times\overrightarrow{c}$

IV) $\overrightarrow{a}\times\overrightarrow{0}=\overrightarrow{0}$, $\overrightarrow{a}\times\overrightarrow{a}=\overrightarrow{0}$

V) $\overrightarrow{a}\times\overrightarrow{b}=\overrightarrow{0}\iff\overrightarrow{a}||\overrightarrow{b}$

VI) $||\overrightarrow{a} \times \overrightarrow{b}||$ equals the numerical value of the area of the parallelogram constructed on $\overrightarrow{a}$ and $\overrightarrow{b}$.

It is easy to construct the following table:

$\times$	$\overrightarrow{i}$	$\overrightarrow{j}$	$\overrightarrow{k}$
$\overrightarrow{i}$	$\overrightarrow{0}$	$\overrightarrow{k}$	$-\overrightarrow{j}$
$\overrightarrow{j}$	$-\overrightarrow{k}$	$\overrightarrow{0}$	$\overrightarrow{i}$
$\overrightarrow{k}$	$\overrightarrow{j}$	$-\overrightarrow{i}$	$\overrightarrow{0}$

Let $\overrightarrow{a} = a_1\overrightarrow{i} + a_2\overrightarrow{j} + a_3\overrightarrow{k}$, $\overrightarrow{b} = b_1\overrightarrow{i} + b_2\overrightarrow{j} + b_3\overrightarrow{k}$.
Then we can write:

$$\begin{aligned}
&\overrightarrow{a} \times \overrightarrow{b} = (a_1\overrightarrow{i} + a_2\overrightarrow{j} + a_3\overrightarrow{k}) \times (b_1\overrightarrow{i} + b_2\overrightarrow{j} + b_3\overrightarrow{k}) = \\
&= a_1b_1\overrightarrow{i} \times \overrightarrow{i} + a_2b_1\overrightarrow{j} \times \overrightarrow{i} + a_3b_1\overrightarrow{k} \times \overrightarrow{i} + a_2b_2\overrightarrow{j} \times \overrightarrow{j} + \\
&+ a_3b_2\overrightarrow{k} \times \overrightarrow{j} + a_1b_3\overrightarrow{i} \times \overrightarrow{k} + a_2b_3\overrightarrow{j} \times \overrightarrow{k} + a_3b_3\overrightarrow{k} \times \overrightarrow{k} = \\
&= (a_2b_3 - a_3b_2)\overrightarrow{i} + (a_3b_1 - a_1b_3)\overrightarrow{j} + (a_1b_2 - a_2b_1)\overrightarrow{k}.
\end{aligned}$$

Finally we have

$$\overrightarrow{a} \times \overrightarrow{b} = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

This is a remarkable formula! Its simplicity enables us to compute easily the vector product.

2.3 Triple vector product

The vector $\overrightarrow{a} \times (\overrightarrow{b} \times \overrightarrow{c})$ is called the triple vector product of the vectors $\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}$. It has no important geometrical meaning but is expressed by a formula which is of use for applications. To deduce this formula let us choose the Cartesian axes in such a way that the x-axis is directed along the vector $\overrightarrow{b}$ and the y-axis lies in the plane of vectors $\overrightarrow{b}$ and $\overrightarrow{c}$. Clearly we have $\overrightarrow{b} = b_1\overrightarrow{i}$, $\overrightarrow{c} = c_1\overrightarrow{i} + c_2\overrightarrow{j}$, $\overrightarrow{a} = a_1\overrightarrow{i} + a_2\overrightarrow{j} + a_3\overrightarrow{k}$.

$$\overrightarrow{a} \times \overrightarrow{b} = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ b_1 & 0 & 0 \\ c_1 & c_2 & 0 \end{vmatrix} = b_1 c_2 \overrightarrow{k}$$

$$\overrightarrow{a} \times (\overrightarrow{b} \times \overrightarrow{c}) = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ a_1 & a_2 & a_3 \\ 0 & 0 & b_1 c_2 \end{vmatrix} = a_2 b_1 c_2 \overrightarrow{i} - a_1 b_1 c_2 \overrightarrow{j} =$$

$= (a_1 c_1 + a_2 c_2) b_1 \overrightarrow{i} - a_1 b_1 (c_1 \overrightarrow{i} + c_2 \overrightarrow{j}) =$
$= (\overrightarrow{a}\,\overrightarrow{c})\overrightarrow{b} - (\overrightarrow{a}\,\overrightarrow{b})\overrightarrow{c}$ (check up these formulas!).

Thus we have

$$\overrightarrow{a} \times (\overrightarrow{b} \times \overrightarrow{c}) = (\overrightarrow{a}\,\overrightarrow{c})\overrightarrow{b} - (\overrightarrow{a}\,\overrightarrow{b})\overrightarrow{c}.$$

This final formula no longer contains any components and therefore does not depend on the particular choice of the axes.

2.4 Triple scalar product

The triple scalar product of the vectors $\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}$ is denoted by $(\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c})$ and is defined by $(\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}) = \overrightarrow{a}(\overrightarrow{b} \times \overrightarrow{c})$.
Clearly we have

$$(\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}) = (a_1 \overrightarrow{i} + a_2 \overrightarrow{j} + a_3 \overrightarrow{k}) \cdot \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} =$$

$= a_1(b_2 c_3 - b_3 c_2) + a_2(b_3 c_1 - b_1 c_3) + a_3(b_1 c_2 - b_2 c_1)$.

Finally we have

$$(\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Taking into account this formula, it is easy to prove that

1) $(\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}) = (\overrightarrow{c}, \overrightarrow{a}, \overrightarrow{b}) = (\overrightarrow{b}, \overrightarrow{c}, \overrightarrow{a})$
2) $(\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}) = -(\overrightarrow{a}, \overrightarrow{c}, \overrightarrow{b})$
3) $(s\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}) = s(\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c})$

4) $(\overrightarrow{u}+\overrightarrow{v}, \overrightarrow{b}, \overrightarrow{c}) = (\overrightarrow{u}, \overrightarrow{b}, \overrightarrow{c}) + (\overrightarrow{v}, \overrightarrow{b}, \overrightarrow{c})$

We have also $|(\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c})| = |(\overrightarrow{a}(\overrightarrow{b} \times \overrightarrow{c})|$ = volume of the parallelepiped constructed on $\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}$.
In particular
$(\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}) = 0 \Longleftrightarrow \overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}$ are parallel to the same plane.

Exercices

2.1 Consider a triangle ABC and the heights $AA_1 \perp BC$, $A_1 \in (BC)$, $BB_1 \perp AC$, $B_1 \in (AC)$ with the intersection point H. Prove that $CH \perp AB$.

2.2 Consider four points A, B, C and D in space.
a) Prove that $\overrightarrow{DA} \cdot \overrightarrow{BC} + \overrightarrow{DB} \cdot \overrightarrow{CA} + \overrightarrow{DC} \cdot \overrightarrow{AB} = 0$.
b) If $DA \perp BC$ and $DB \perp CA$ then $DC \perp AB$.

2.3 Let G be the weight center of the triangle ABC.
a) Prove that $\overrightarrow{AG} + \overrightarrow{BG} + \overrightarrow{CG} = 0$.
b) If M is an arbitrary point then $3\overrightarrow{MG} = \overrightarrow{MA} + \overrightarrow{MB} + \overrightarrow{MC}$.

2.4 Let ABC and MNP be two triangles (in the same plane or different planes). Prove that, if $\overrightarrow{AM} + \overrightarrow{BN} + \overrightarrow{CP} = 0$, then the weight centers of the two triangles coincide.

2.5 If $\vec{a} = (3, -1, \alpha)$, $\vec{b} = (0, 1, -2)$ and $\vec{c} = (1, 0, -1)$, determine $\alpha \in \mathbb{R}$ such that the vector $\vec{a} \times (\vec{b} \times \vec{c})$ is parallel to the plane $y0z$.

2.6 Find the angle between
a) the vector $\vec{a} = \frac{\sqrt{3}}{2}\vec{i} + \frac{1}{2}\vec{k}$ and the axis Ox
b) $\vec{AB}$ and $\vec{AC}$ where $A(3, 1, -2)$, $B(2, 1, -1)$ and $C(3, 0, -1)$.

2.7 Let $\vec{a} = 3\vec{i} - \vec{j} + 2\vec{k}$ and $\vec{b} = \vec{j} - 2\vec{k}$. Determine the height of the parallelogram with the edges $\vec{a}$ and $\vec{b}$, considering $\vec{a}$ as the basis.

2.8 Determine the vector $\vec{w}$ such that $\|\vec{w}\|=2$, $\vec{w}$ is perpendicular on the axis Oz and makes a 45° angle with the positive direction of Ox.

2.9 Let $\vec{a}=\vec{i}+\vec{j}+\vec{k}$ and $\vec{b}=2\vec{i}-\vec{j}$, $\vec{c}=\vec{j}+3\vec{k}$. Determine the height of the parallelepiped with the edges $\vec{a}$, $\vec{b}$, $\vec{c}$ considering the parallelogram with edges $\vec{a}$, $\vec{b}$ as basis.

2.10 Prove the identity of Lagrange $\|\vec{a}\times\vec{b}\|^2+(\vec{a}\cdot\vec{b})^2=\|\vec{a}\|^2\|\vec{b}\|^2$, for any vectors $\vec{a},\vec{b}$.

2.11 Prove that $(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d})=(\vec{a}\cdot\vec{c})(\vec{b}\cdot\vec{d})-(\vec{a}\cdot\vec{d})(\vec{b}\cdot\vec{c})$, for any vectors $\vec{a},\vec{b},\vec{c},\vec{d}$.

2.12 Let $\vec{a},\vec{b}$ and $\vec{c}$ be three non-coplanar vectors, making two by two, angles of measures α,β,γ. Prove that, if

$$\vec{a}\times(\vec{a}\times\vec{b})+\vec{b}\times(\vec{b}\times\vec{c})+\vec{c}\times(\vec{c}\times\vec{a})=0$$

then $\cos\alpha\cos\beta\cos\gamma=1$.

Solutions

$\boxed{2.1}$ Use, for instance the fact that $\overrightarrow{AH}\cdot\overrightarrow{BC}=0$, $\overrightarrow{BH}\cdot\overrightarrow{AC}=0$, $\overrightarrow{AH}=\overrightarrow{AC}+\overrightarrow{CH}$, $\overrightarrow{BH}=\overrightarrow{BC}+\overrightarrow{CH}$.

$\boxed{2.2}$ a) Using the triangle rule we get that $\overrightarrow{BC}=\overrightarrow{BD}+\overrightarrow{DC}=\overrightarrow{DC}-\overrightarrow{DB}$, $\overrightarrow{CA}=\overrightarrow{DA}-\overrightarrow{DC}$, $\overrightarrow{AB}=\overrightarrow{DB}-\overrightarrow{DA}$ and the equality follows. b) Follows directly from a).

$\boxed{2.3}$ a) Let A_1 be the middle of (BC). Use the relations $\overrightarrow{BG}=\overrightarrow{BA_1}+\overrightarrow{A_1G}$, $\overrightarrow{CG}=\overrightarrow{CA_1}+\overrightarrow{A_1G}$, $\overrightarrow{AG}=2\overrightarrow{GA_1}$.

$\boxed{2.4}$ $\overrightarrow{G_1G_2}=\overrightarrow{G_1A}+\overrightarrow{AM}+\overrightarrow{MG_2}$, $\overrightarrow{G_1G_2}=\overrightarrow{G_1B}+\overrightarrow{BN}+\overrightarrow{NG_2}$, $\overrightarrow{G_1G_2}=\overrightarrow{G_1C}+\overrightarrow{CP}+\overrightarrow{PG_2}$. Add the three relations and use the previous exercise.

$\boxed{\mathbf{2.5}}$ $\vec{b}\times\vec{c} = -\vec{i}-2\vec{j}-\vec{k} = (-1,-2,-1)$, $\vec{a}\times(\vec{b}\times\vec{c}) = (1+2\alpha, 3-\alpha, -7)$. If the vector is parallel to the plane $y0z$, then it is perpendicular on the axis $0x$, that is the dot product is zero. This gives $1+2\alpha = 0$, so $\alpha = -\frac{1}{2}$.

$\boxed{\mathbf{2.6}}$ a) $\cos\alpha = \dfrac{\vec{a}\cdot\vec{j}}{\|\vec{a}\|} = \dfrac{\sqrt{3}}{2}$, so $\alpha = \dfrac{\pi}{6}$. b) $\vec{AB} = -\vec{i}+\vec{k}$, $\vec{AC} - \vec{j}+\vec{k}$, $\alpha = \dfrac{\pi}{3}$.

$\boxed{\mathbf{2.7}}$ The area of the parallelogram is $\|\vec{a}\times\vec{b}\| = 6\vec{j}+3\vec{k} = 3\sqrt{5}$. On the other hand, $area = h\|\vec{a}\|$, so we get $h = \frac{3\sqrt{5}}{\sqrt{14}}$.

$\boxed{\mathbf{2.8}}$ If $\vec{w} = a\vec{i}+b\vec{j}+c\vec{k}$, from $\vec{w}\perp\vec{k}$, we get $c = 0$. We have $\vec{w}\cdot\vec{i} = \|\vec{w}\|\cos\pi/4 = \sqrt{2}$. On the other hand, $\vec{w}\cdot\vec{i} = a$, so $a = \sqrt{2}$. Finally, since $\|\vec{w}\| = \sqrt{a^2+b^2+c^2} = \sqrt{2}$, it follows that $b = \sqrt{2}$ or $b = -\sqrt{2}$.

$\boxed{\mathbf{2.9}}$ The volume of the parallelepiped is given by the mixed product $volume = |(\vec{a},\vec{b},\vec{c})| = 7$. The area of the basis is $area = \|\vec{a}\times\vec{b}\| = \sqrt{14}$. The height is $h = \dfrac{7}{\sqrt{14}}$.

$\boxed{\mathbf{2.10}}$ Denoting by α the angle formed by the two vectors, we have $\|\vec{a}\times\vec{b}\| = \|\vec{a}\|\|\vec{b}\|\sin\alpha$, $\vec{a}\cdot\vec{b} = \|\vec{a}\|\|\vec{b}\|\cos\alpha$ and the identity follows immediately.

$\boxed{\mathbf{2.11}}$ We use the properties of the triple product:

$$(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d}) = (\vec{a}\times\vec{b},\vec{c},\vec{d}) = (\vec{d},\vec{a}\times\vec{b},\vec{c}) = \vec{d}\cdot((\vec{a}\times\vec{b})\times\vec{c}) =$$
$$= -\vec{d}((\vec{c}\cdot\vec{b})\vec{a}-(\vec{c}\cdot\vec{a})\vec{b}) = \vec{d}\cdot(\vec{c}\cdot\vec{a})\vec{b} - \vec{d}\cdot(\vec{c}\cdot\vec{b})\vec{a}.$$

$\boxed{\mathbf{2.12}}$ We get $(\vec{a}\cdot\vec{b}-\vec{c}\cdot\vec{c})\vec{a}+(\vec{b}\cdot\vec{c}-\vec{a}\cdot\vec{a})\vec{b}+(\vec{c}\cdot\vec{a}-\vec{b}\cdot\vec{b})\vec{c} = 0$. Since the three vectors are non-coplanar, this means $\vec{a}\cdot\vec{b} = \vec{c}\cdot\vec{c}$, $\vec{b}\cdot\vec{c} = \vec{a}\cdot\vec{a}$, $\vec{c}\cdot\vec{a} = \vec{b}\cdot\vec{b}$. It follows that $\|\vec{a}\|\|\vec{b}\|\cos\alpha = \|\vec{c}\|^2$, $\|\vec{b}\|\|\vec{c}\|\cos\beta = \|\vec{a}\|^2$ and $\|\vec{c}\|\|\vec{a}\|\cos\gamma = \|\vec{b}\|^2$. Multiplying the last three relationships we get now the desired equality.