CHAPTER 1

Determinants and matrices

The reader is assumed to have some knowledge of the elementary properties of determinants and matrices.

1.1 Laplace's Theorem

Let us consider a determinant D of order n. Let k be an integer, $1 \leq k \leq n$. Consider the rows $i_1, \dots, i_k$ and the columns $j_1, \dots, j_k$. By deleting the other rows and columns we obtain a determinant of order k, called a *minor* of D and denoted by $M^{i_1,\dots,i_k}_{j_1,\dots,j_k}$.

Now, let us delete the rows $i_1, \dots, i_k$ and the columns $j_1, \dots, j_k$; we obtain a determinant of order $n-k$. It is called the *complementary minor* of $M^{i_1,\dots,i_k}_{j_1,\dots,j_k}$ and is denoted by $\widetilde{M}^{i_1,\dots,i_k}_{j_1,\dots,j_k}$. Finally, let us denote $A^{i_1,\dots,i_k}_{j_1,\dots,j_k} = (-1)^{i_1+\cdots+i_k+j_1+\cdots+j_k}\widetilde{M}^{i_1,\dots,i_k}_{j_1,\dots,j_k}$.
$A^{i_1,\dots,i_k}_{j_1,\dots,j_k}$ is called the *cofactor* of $M^{i_1,\dots,i_k}_{j_1,\dots,j_k}$.

Using this notation we shall state (without proof) Laplace's Theorem:

Theorem 1.1 $D = \sum M^{i_1,\dots,i_k}_{j_1,\dots,j_k} A^{i_1,\dots,i_k}_{j_1,\dots,j_k}$, *where:*

1) The indices $i_1, \dots, i_k$ *are fixed*

2) The indices $j_1, \dots, j_k$ *take on all the possible values, such that* $1 \leq j_1 < j_2 < \cdots < j_k \leq n$.

Remark 1.2 a) For $k = 1$, the above formula is the well-known expansion of a determinant using a fixed row.

b) In Theorem 1.1 we have used k fixed rows; a similar result obviously holds by using k fixed columns.

We shall use Laplace's formula in order to prove

Theorem 1.3 *Let* $A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \dots & \dots & \dots \\ a_{n1} & \dots & a_{nn} \end{pmatrix}$, $B = \begin{pmatrix} b_{11} & \dots & b_{1n} \\ \dots & \dots & \dots \\ b_{n1} & \dots & b_{nn} \end{pmatrix}$ *where* a_{ij} *and* b_{ij} *are real or complex numbers. Then* $\det(A \cdot B) = \det A \cdot \det B$.

Proof. Consider the determinant

$$D = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} & 0 & 0 & \dots & 0 \\ a_{21} & a_{22} & \dots & a_{2n} & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} & 0 & 0 & \dots & 0 \\ -1 & 0 & \dots & 0 & b_{11} & b_{12} & \dots & b_{1n} \\ 0 & -1 & \dots & 0 & b_{21} & b_{22} & \dots & b_{2n} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & -1 & b_{n1} & b_{n2} & \dots & b_{nn} \end{vmatrix}$$

Let us expand it with Laplace's formula, by using the first n rows. We obtain $D = \det A \cdot (-1)^{n(n+1)} \det B = \det A \cdot \det B$.

On the other hand, denote $C = A \cdot B$. The entries of the matrix C are $c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$, for $i, j = 1, \dots, n$.

We shall transform D: the purpose is to replace the entries b_{ij} by 0.

1) To the column $n + 1$ we add: column 1 multiplied by b_{11}, column 2 multiplied by $b_{21}, \dots$, column n multiplied by b_{n1}.

2) To the column $n + 2$ we add: column 1 multiplied by b_{12}, column 2 multiplied by $b_{22}, \dots$, column n multiplied by b_{n2}.

...

n) To the column $2n$ we add: column 1 multiplied by $b_{1n}, \ldots,$ column n multiplied by b_{nn}.

Performing these operations, we obtain

$$D = \begin{vmatrix} a_{11} & \ldots & a_{1n} & c_{11} & \ldots & c_{1n} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{n1} & \ldots & a_{nn} & c_{n1} & \ldots & c_{nn} \\ -1 & \ldots & 0 & 0 & \ldots & 0 \\ 0 & \ldots & 0 & 0 & \ldots & 0 \\ 0 & \ldots & -1 & 0 & \ldots & 0 \end{vmatrix}$$

Now apply Laplace's formula by choosing the last n rows. It follows that $D = (-1)^n(-1)^{1+2+\cdots+2n} \det C = \det C = \det(A \cdot B)$.
Hence $\det(A \cdot B) = D = \det A \cdot \det B$. $\square$

1.2 Vandermonde's determinant

The following determinant of order n:

$$V(a_1, \ldots, a_n) = \begin{vmatrix} 1 & 1 & \ldots & 1 \\ a_1 & a_2 & \ldots & a_n \\ a_1^2 & a_2^2 & \ldots & a_n^2 \\ \ldots & \ldots & \ldots & \ldots \\ a_1^{n-1} & a_2^{n-1} & \ldots & a_n^{n-1} \end{vmatrix}$$

is called the Vandermonde's determinant of the (real or complex) numbers $a_1, \ldots, a_n$. By induction it can be proved that:

$$V(a_1, \ldots, a_n) = \prod_{1 \le i < j \le n} (a_j - a_i)$$

1.3 Circulants

The following determinant is called a circulant:

$$C(a_0, a_1, \ldots, a_{n-1}) = \begin{vmatrix} a_0 & a_1 & a_2 & \ldots & a_{n-1} \\ a_{n-1} & a_0 & a_1 & \ldots & a_{n-2} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_1 & a_2 & a_3 & \ldots & a_0 \end{vmatrix}$$

Let $\epsilon_k = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}, \quad k = 0, 1, \ldots, n-1$. We have $\epsilon_k^n = 1$, $k = 0, 1, \ldots, n-1$. Let us denote $f(x) = a_0 + a_1 x + \cdots + a_{n-1}x^{n-1}$.

Theorem 1.4 $C(a_0, a_1, \ldots, a_{n-1}) = f(\epsilon_0)f(\epsilon_1)\ldots f(\epsilon_{n-1})$.

Proof. We have a wonderful opportunity to emphasize the usefulness of the previous results concerning multiplication of determinants and Vandermonde determinants. In fact, Theorem 1.3 gives us

$$C(a_0, a_1, \ldots, a_{n-1}) \cdot V(\epsilon_0, \epsilon_1, \ldots, \epsilon_{n-1}) =$$

$$= \begin{vmatrix} f(\epsilon_0) & f(\epsilon_1) & \ldots & f(\epsilon_{n-1}) \\ \epsilon_0 f(\epsilon_0) & \epsilon_1 f(\epsilon_1) & \ldots & \epsilon_{n-1} f(\epsilon_{n-1}) \\ \ldots & \ldots & \ldots & \ldots \\ \epsilon_0^{n-1} f(\epsilon_0) & \epsilon_1^{n-1} f(\epsilon_1) & \ldots & \epsilon_{n-1}^{n-1} f(\epsilon_{n-1}) \end{vmatrix} =$$

$$= f(\epsilon_0)f(\epsilon_1)\ldots f(\epsilon_{n-1})V(\epsilon_0, \epsilon_1, \ldots, \epsilon_{n-1}).$$

Since $\epsilon_0, \epsilon_1, \ldots, \epsilon_{n-1}$ are pairwise distinct, we have $V(\epsilon_0, \epsilon_1, \ldots, \epsilon_{n-1}) \neq 0$ and hence $C(a_0, a_1, \ldots, a_{n-1}) = f(\epsilon_0)f(\epsilon_1)\ldots f(\epsilon_{n-1})$. $\square$

1.4 Rank. Elementary transformations.

Let K be the field of real numbers or the field of complex numbers. By $\mathcal{M}_{n,m}(K)$ we shall denote the set of all matrices with n rows, m columns and having entries from K. The number $r \in \mathbb{N}$ is called the rank of the matrix $A \in \mathcal{M}_{n,m}(K)$ if

1) There exists a square submatrix M of A, with r rows and columns, such that $\det M \neq 0$.

2) If $p > r$, for every submatrix N of A having p rows and columns we have $\det N = 0$.

We shall denote the rank of A by r_A. It can be proved that if $A \in \mathcal{M}_{n,m}(K)$ and $B \in \mathcal{M}_{m,p}(K)$, then

$$r_A + r_B - m \leq r_{AB} \leq \min\{r_A, r_B\}. \tag{1.1}$$

Theorem 1.5 *Let* $A, B \in \mathcal{M}_{n,n}(K), \quad \det A \neq 0$. *Then* $r_{AB} = r_B$.

Proof. Clearly $r_A = n$. By using (1.1) with $m = p = n$ we obtain $r_B \leq r_{AB} \leq r_B$. Hence $r_{AB} = r_B$. □

Definition 1.6 The following operations are called *elementary row transformations on the matrix A:*

1) The interchange of any two rows;

2) The multiplication of a row by any non-zero number;

3) The addition of one row to another.

Similarly we can define the elementary column transformations.

Consider an arbitrary determinant. If it is nonzero, it will be nonzero after performing any elementary transformation; if it is equal to zero, it will remain equal to zero.

We conclude that *the rank of a matrix does not change if we perform any elementary transformation on the matrix.* So we can use elementary transformations in order to compute the rank of a matrix. Namely, given a matrix $A \in \mathcal{M}_{n,m}(K)$, we transform it - by an appropriate succession of elementary transformations - into a matrix B such that

(i) the diagonal entries of B are either 0 or 1, all the $1'$s preceding all the $0'$s on the diagonal,

(ii) all the other entries of B are equal to 0.

Since the rank is invariant under elementary transformations, we have $r_A = r_B$; but r_B is obviously equal to the number of $1'$s on the diagonal. The following example illustrates this method.

$$A = \begin{pmatrix} -2 & -1 & 0 & -5 & -1 \\ 1 & 2 & 6 & -2 & -1 \\ 3 & 1 & -1 & 8 & 1 \\ -1 & 0 & 2 & -4 & -1 \\ -1 & -2 & -7 & 3 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 0 & -5 & -1 \\ -2 & 1 & 6 & -2 & -1 \\ -1 & 3 & -1 & 8 & 1 \\ 0 & -1 & 2 & -4 & -1 \\ 2 & -1 & -7 & 3 & 2 \end{pmatrix} \sim$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -2 & -3 & 6 & -12 & -3 \\ -1 & 1 & -1 & 3 & 0 \\ 0 & -1 & 2 & -4 & -1 \\ 2 & 3 & -7 & 13 & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 4 & 1 \\ 0 & 1 & -1 & 3 & 0 \\ 0 & -1 & 2 & -4 & -1 \\ 0 & 3 & -7 & 13 & 4 \end{pmatrix} \sim$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 4 & 1 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 \end{pmatrix} \sim$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \text{ It follows that } r_A = 3.$$

The following theorem offers a procedure to compute the inverse of a matrix (if this inverse exists).

Theorem 1.7 *If a square matrix is reduced to the identity matrix by a sequence of elementary row operations, the same sequence of elementary row transformations performed on the identity matrix produces the inverse of the given matrix.*

Example 1.4.1 Find the inverse of the matrix $A = \begin{pmatrix} 1 & 1 & 1 \\ 6 & 7 & 6 \\ -1 & 2 & 0 \end{pmatrix}$.

We write the given matrix and the identity:

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 6 & 7 & 6 & 0 & 1 & 0 \\ -1 & 2 & 0 & 0 & 0 & 1 \end{array}\right]$$

Now we perform a succession of elementary *row* transformations in order to transform A into the identity; the same transformations are performed on the identity.

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -6 & 1 & 0 \\ 0 & 3 & 1 & 1 & 0 & 1 \\ \hline 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -6 & 1 & 0 \\ 0 & 0 & 1 & 19 & -3 & 1 \\ \hline 1 & 0 & 0 & -12 & 2 & -1 \\ 0 & 1 & 0 & -6 & 1 & 0 \\ 0 & 0 & 1 & 19 & -3 & 1 \end{array}\right]$$

It follows that $A^{-1} = \begin{pmatrix} -12 & 2 & -1 \\ -6 & 1 & 0 \\ 19 & -3 & 1 \end{pmatrix}$

Exercices

1.1 Evaluate the following n^{th} order determinants by reduction to triangular form:

$$\text{a)}\ \begin{vmatrix} 1 & 2 & 3 & \dots & n \\ -1 & 0 & 3 & \dots & n \\ -1 & -2 & 0 & \dots & n \\ .. & .. & \dots & .. & .. \\ -1 & -2 & -3 & \dots & 0 \end{vmatrix}; \qquad \text{b)}\ \begin{vmatrix} b_1 & a_{12} & a_{13} & \dots & a_{1n} \\ b_1 & b_2 & a_{23} & \dots & a_{2n} \\ b_1 & b_2 & b_3 & \dots & a_{3n} \\ .. & .. & \dots & .. & .. \\ b_1 & b_2 & b_3 & \dots & b_n \end{vmatrix};$$

$$\text{c)}\ \begin{vmatrix} 3 & 2 & 2 & \dots & 2 \\ 2 & 3 & 2 & \dots & 2 \\ 2 & 2 & 3 & \dots & 2 \\ .. & .. & \dots & .. & .. \\ 2 & 2 & 2 & \dots & 3 \end{vmatrix}; \qquad \text{d)}\ \begin{vmatrix} 1 & 2 & 3 & \dots & n-2 & n-1 & n \\ 2 & 3 & 4 & \dots & n-1 & n & n \\ 3 & 4 & 5 & \dots & n & n & n \\ .. & .. & .. & \dots & .. & .. & .. \\ n & n & n & \dots & n & n & n \end{vmatrix}.$$

1.2 Calculate the determinant $C(1, 2, \dots, n)$.

1.3 Calculate the determinant $C(C_{n-1}^0, C_{n-1}^1, \dots, C_{n-1}^{n-1})$.

1.4 Calculate the n^{th} order determinant $C(a, b, b, \dots, b)$, with $a, b \in \mathbb{R}$.

1.5 For $a_1, a_2, \dots, a_n \in \mathbf{C}$, $k = 1, \dots, n$, calculate the determinant

$$V_k(a_1, a_2, \dots, a_n) = \begin{vmatrix} 1 & 1 & \dots & 1 \\ a_1 & a_2 & \dots & a_n \\ \dots & \dots & \dots & \dots \\ a_1^{k-1} & a_2^{k-1} & \dots & a_n^{k-1} \\ a_1^{k+1} & a_2^{k+1} & \dots & a_n^{k+1} \\ \dots & \dots & \dots & \dots \\ a_1^n & a_2^n & \dots & a_n^n \end{vmatrix},$$

called the lacunary Vandermonde.

1.6 Prove the following identities without expanding the determinants:

$$\text{a)}\ \begin{vmatrix} 0 & a & b & c \\ a & 0 & c & b \\ b & c & 0 & a \\ c & b & a & 0 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & c^2 & b^2 \\ 1 & c^2 & 0 & a^2 \\ 1 & b^2 & a^2 & 0 \end{vmatrix}; \qquad \text{b)}\ \begin{vmatrix} a & b & c \\ x & y & z \\ \alpha & \beta & \gamma \end{vmatrix} = \begin{vmatrix} a & -b & c \\ -x & y & -z \\ \alpha & -\beta & \gamma \end{vmatrix};$$

c) $\begin{vmatrix} a & b & c \\ p & q & r \\ a\alpha & b\beta & c\gamma \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ bcp & acq & abr \\ \alpha & \beta & \gamma \end{vmatrix}$.

1.7 Compute the determinants by using Laplace's Rule:

a) $\begin{vmatrix} 1 & 2 & 2 & 1 \\ 0 & 1 & 0 & 2 \\ 2 & 0 & 1 & 1 \\ 0 & 2 & 0 & 1 \end{vmatrix}$; b) $\begin{vmatrix} 2 & 3 & 0 & 0 & 1 & -1 \\ 9 & 4 & 0 & 0 & 3 & 7 \\ 4 & 5 & 1 & -1 & 2 & 4 \\ 3 & 8 & 3 & 7 & 6 & 9 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 3 & 7 & 0 & 0 & 0 & 0 \end{vmatrix}$.

1.8 Calculate the determinant of order $2n$, $D_{2n} = \begin{vmatrix} a & 0 & \dots & 0 & b \\ 0 & a & \dots & b & 0 \\ .. & .. & \dots & .. & .. \\ 0 & b & \dots & a & 0 \\ b & 0 & \dots & 0 & a \end{vmatrix}$.

1.9 Find the inverse of the matrix of order n:

a) $A = \begin{pmatrix} 1 & 1 & ... & 1 & 1 \\ 0 & 1 & ... & 1 & 1 \\ .. & .. & ... & .. & .. \\ 0 & 0 & ... & 1 & 1 \\ 0 & 0 & ... & 0 & 1 \end{pmatrix}$; b) $B = \begin{pmatrix} 0 & 1 & ... & 1 & 1 \\ 1 & 0 & ... & 1 & 1 \\ .. & .. & ... & .. & .. \\ 1 & 1 & ... & 0 & 1 \\ 1 & 1 & ... & 1 & 0 \end{pmatrix}$,

1.10 Find the inverse of the matrix $A = \begin{pmatrix} \hat{2} & \hat{3} & \hat{1} \\ \hat{0} & \hat{1} & \hat{4} \\ \hat{5} & \hat{6} & \hat{2} \end{pmatrix}$ in $\mathbb{Z}_7$.

Solutions

$\boxed{\textbf{1.1}}$ a) $n!$; b) $b_1(b_2-a_{12})(b_3-a_{23})\cdots(b_n-a_{n-1,n})$; c) $1+2n$; d) $(-1)^{n+1}n$.

1.2 $C(1,2,\dots,n) = \prod_{k=0}^{n} P(\varepsilon_k)$, where $\varepsilon_k^n = 1$ and $P(X) = 1 + 2X + 3X^2 + \cdots + nX^{n-1}$. For $\varepsilon_k \neq 1$, we get $P(\varepsilon_k) = \dfrac{n}{\varepsilon_k - 1}$ and $P(1) = \dfrac{n(n+1)}{2}$. $C(1,2,;n) = \dfrac{n^n(n+1)}{2} \prod_{k=1}^{n-1} \dfrac{1}{\varepsilon_k - 1}$. The values ε_k, $k = 1,\dots,n-1$ are the roots of the equation $z^{n-1} + z^{n-2} + \cdots + z + 1 = 0$, so $\prod_{k=1}^{n-1}(z - \varepsilon_k) = z^{n-1} + z^{n-2} + \cdots + z + 1$. Taking $z = 1$, we obtain $\prod_{k=1}^{n-1}(\varepsilon_k - 1) = (-1)^{n-1}n$, so $C(1,2,\dots,n) = (-1)^{n-1}\dfrac{n^{n-1}(n+1)}{2}$.

1.3 $P(X) = C_{n-1}^0 + C_{n-1}^1 X + C_{n-1}^2 X^2 + \cdots + C_{n-1}^{n-1} X^{n-1} = (1+X)^{n-1}$. The determinant has then the value $\prod_{k=1}^{n-1}(1+\varepsilon_k)^{n-1} = [(-1)^n((-1)^n - 1)]^{n+1}$.

1.4 $P(X) = a + bX + bX^2 + \cdots + X^{n-1} = a + b\dfrac{X^n - X}{X - 1}$, for $X \neq 1$, and $P(1) = a + b(n-1)$. $C(a,b,\dots,b) = [a + (n-1)b](a-b)^{n-1}$. The same result can be obtained also directly, using the properties of determinants.

1.5 Consider another Vandermonde determinant:

$$V(a_1,\dots,a_n,X) = V(a_1,\dots,a_n)\prod_{k=1}^{n}(X - a_k) =$$
$$= V(a_1,\dots,a_n)(X^n - S_1 X^{n-1} + \cdots + (-1)^{n-k} S_{n-k} X^k + \cdots + (-1)^n S_n),$$

where S_k are the Viéte sums corresponding to the polynomial with the roots $a_1,\dots,a_n$. On the other hand, expanding the same determinant by the last column we get: $V(a_1,\dots,a_n,X) = (-1)^{n+2}V_0(a_1,\dots,a_n) + \cdots + (-1)^{n+2+k}X^k V_k(a_1,\dots,a_n) + \cdots + (-1)^{2n+2}X^n V_n(a_1,\dots,a_n)$. From the two expressions we obtain $V_k(a_1,\dots,a_n) = V(a_1,\dots,a_n)S_{n-k}$.

1.6 a) Multiply the second column of the determinant in the left-hand member of the identity by bc, the third column by ac and the fourth by ab. b) Multiply the second column and the second row by (-1). c) Multiply the second row of the determinant by abc then divide the first column by a, the second by b and the third by c.

$\boxed{\mathbf{1.7}}$ a) 9; b) For example, we expand after the last two rows: 1000.

$\boxed{\mathbf{1.8}}$ Using Laplace's formula with rows n and $n+1$ we get the recurrence relationship $D_{2n} = \begin{vmatrix} a & b \\ b & a \end{vmatrix} (-1)^{n+n+1+n+n+1} D_{2n-2} = (a^2 - b^2) D_{2n-2}$, and by induction $D_{2n} = (a^2 - b^2)^n$.

$\boxed{\mathbf{1.9}}$ a) Subtracting each row from the row above it, follows:

$$
\begin{array}{|cccccc|cccccc|}
\hline
1 & 1 & \dots & 1 & 1 & & 1 & 0 & 0 & \dots & 0 & 0 \\
0 & 1 & \dots & 1 & 1 & & 0 & 1 & 0 & \dots & 0 & 0 \\
.. & .. & \dots & .. & .. & & .. & .. & .. & \dots & .. & .. \\
0 & 0 & \dots & 1 & 1 & & 0 & 0 & 0 & \dots & 1 & 0 \\
0 & 0 & \dots & 0 & 1 & & 0 & 0 & 0 & \dots & 0 & 1 \\
\hline
1 & 0 & \dots & 0 & 0 & & 1 & -1 & 0 & \dots & 0 & 0 \\
0 & 1 & \dots & 0 & 0 & & 0 & 1 & -1 & \dots & 0 & 0 \\
.. & .. & \dots & .. & .. & & .. & .. & .. & \dots & .. & .. \\
0 & 0 & \dots & 1 & 0 & & 0 & 0 & 0 & \dots & 1 & -1 \\
0 & 0 & \dots & 0 & 1 & & 0 & 0 & 0 & \dots & 0 & 1 \\
\hline
\end{array}
$$

b) We can apply the following succession of elementary transformations: add all rows to the first one, multiply row one by $\dfrac{1}{n-1}$, subtract row one from all he other rows, add again all the rows to the first one and finally multiply all the rows (except the first) by -1. The inverse matrix is $B^{-1} = \dfrac{1}{n-1} \begin{pmatrix} 2-n & 1 & \dots & 1 & 1 \\ 1 & 2-n & \dots & 1 & 1 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 1 & \dots & 2-n & 1 \\ 1 & 1 & \dots & 1 & 2-n \end{pmatrix}$.

$\boxed{\mathbf{1.10}}$ $A^{-1} = \begin{pmatrix} \hat{5} & \hat{0} & \hat{1} \\ \hat{5} & \hat{5} & \hat{5} \\ \hat{4} & \hat{6} & \hat{4} \end{pmatrix}$.