Transformata Laplace

① $I=\int_0^\infty \frac{\cos 2x}{x^2+1}dx = \frac{\pi}{2e^2}$

$\mathcal{L}\left\{\frac{f(t)}{t}\right\}(s)=\int_s^\infty \mathcal{L}\{f(t)\}(y)\,dy$

$\int_0^\infty \frac{\cos 2t}{t^2+1}e^{-st}dt = \mathcal{L}\left\{\frac{\cos 2t}{t^2+1}\right\}(0)$ $\quad t^2+1 \neq t$ (Nu putem aplica T. int. imag.)

$\mathcal{L}\{f(t)\}(s)=\int_0^\infty f(t)\cdot e^{-st}dt$

Fie $I(t)=\int_0^\infty \frac{\cos tx}{x^2+1}dx$ sau $\underbrace{\int_0^\infty \frac{\cos 2tx}{x^2+1}dx}_{I(t)}$

$t\geq 0$

Observăm că $I=I(2)$ sau $I=I(1)$

$\mathcal{L}\{I(t)\}(s) \overset{def}{=} \int_0^\infty I(t)\cdot e^{-st}dt = \int_0^\infty\left(\int_0^\infty \frac{\cos tx}{x^2+1}dx\right)\cdot e^{-st}dt = \int_0^\infty\left(\int_0^\infty \frac{\cos tx}{x^2+1}\cdot e^{-st}dx\right)dt$

schimbăm ord. de integrare $\int_0^\infty\left(\int_0^\infty \frac{\cos tx}{x^2+1}\cdot e^{-st}dt\right)dx = \int_0^\infty \frac{1}{x^2+1}\underbrace{\left(\int_0^\infty \cos tx\cdot e^{-st}dt\right)}_{\mathcal{L}\{\ \}}dx$

$= \int_0^\infty \frac{1}{x^2+1}\cdot\mathcal{L}\{\cos tx\}(s)dx \overset{a=x}{=} \int_0^\infty \frac{1}{x^2+1}\cdot\frac{s}{s^2+x^2}dx$

$\mathcal{L}\{\cos at\}(s)=\frac{s}{s^2+a^2}$

$\frac{s}{(x^2+1)(x^2+s^2)} = \frac{Ax+B}{x^2+1}+\frac{Cx+D}{x^2+s^2}$ ✓

nu avem termeni cu x $= \frac{A}{x^2+1}+\frac{B}{x^2+s^2}$ ✓

$\frac{s}{(x^2+1)(x^2+s^2)}=\frac{s}{s^2-1}\cdot\left(\frac{1}{x^2+1}-\frac{1}{x^2+s^2}\right)$, $\quad \frac{1}{x^2+1}-\frac{1}{x^2+s^2} = \frac{s^2-1}{(x^2+1)(x^2+s^2)}$

$=\int_0^\infty \frac{s}{s^2-1}\cdot\left(\frac{1}{x^2+1}-\frac{1}{x^2+s^2}\right)dx = \frac{s}{s^2-1}\int_0^\infty\left(\frac{1}{x^2+1}-\frac{1}{x^2+s^2}\right)dx$

$=\frac{s}{s^2-1}\cdot\left(\operatorname{arctg}x\Big|_0^\infty - \frac{1}{s}\operatorname{arctg}\frac{x}{s}\Big|_0^\infty\right) = \frac{s}{s^2-1}\cdot\left(\frac{\pi}{2}-\frac{1}{s}\cdot\frac{\pi}{2}\right) = \frac{\cancel{s}}{\cancel{s^2-1}}\cdot\frac{\pi}{2}\cdot\frac{\cancel{s-1}}{\cancel{s}} = \frac{\pi}{2\cdot(s+1)}$

Obținem $\mathcal{L}\{I(t)\}(s)=\frac{\pi}{2\cdot(s+1)}$ $\quad |\mathcal{L}^{-1}$

$I(t)=\mathcal{L}^{-1}\left\{\frac{\pi}{2\cdot(s+1)}\right\}(t) = \frac{\pi}{2}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}(t) \overset{a=-1}{=} \frac{\pi}{2}\cdot e^{-t}\cdot u(t)$

$\mathcal{L}\{e^{at}\}(s)=\frac{1}{s-a}$ $\quad \mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\}(t)=e^{at}\cdot u(t)$

$\xrightarrow[u(t)=1]{t\geq 0} I(t)=\frac{\pi}{2}\cdot e^{-t}$

$u(t)=\begin{cases}1, & t\geq 0\\ 0, & t<0\end{cases}$

$I=I(2)=\frac{\pi}{2}\cdot e^{-2} = \frac{\pi}{2e^2}$

② Rez. ec.

$x'''(t)-2x''(t)+4x'(t)-8x(t)=e^{-t}\ \Big|\mathcal{L}$ $\quad x(0)=x'(0)=0,\ x''(0)=2$ $\quad \mathcal{L}\{e^{at}\}(s)=\frac{1}{s-a}$

$\Rightarrow \mathcal{L}\{x'''(t)\}(s)-2\mathcal{L}\{x''(t)\}(s)+4\mathcal{L}\{x'(t)\}(s)-8\underbrace{\mathcal{L}\{x(t)\}(s)}_{\overset{not}{=}X(s)} = \mathcal{L}\{e^{-t}\}(s)=\frac{1}{s+1}$ $\quad (*)$

Obs: Notație $\mathcal{L}\{x(t)\}(s)\overset{not}{=}X(s)$

$\mathcal{L}\{x'(t)\}(s)=sX(s)-\underset{0}{x(0)}$ $\quad |\cdot s$ $\quad \Rightarrow \mathcal{L}\{x'(t)\}(s)=sX(s)$

$\mathcal{L}\{x''(t)\}(s)=s^2X(s)-s\underset{0}{x(0)}-\underset{0}{x'(0)}$ $\quad |\cdot s$ $\quad \Rightarrow \mathcal{L}\{x''(t)\}(s)=s^2X(s)$

$\mathcal{L}\{x'''(t)\}(s)=s^3X(s)-s^2\underset{0}{x(0)}-s\underset{0}{x'(0)}-\underset{2}{x''(0)}$ $\quad \Rightarrow \mathcal{L}\{x'''(t)\}(s)=s^3X(s)-2$

$\Rightarrow (*) \Leftrightarrow s^3X(s)-2-2s^2X(s)+4sX(s)-8X(s)=\frac{1}{s+1}$

$X(s)\cdot(\underbrace{s^3-2s^2+4s-8}_{s^2(s-2)+4(s-2)=(s^2+4)(s-2)})=2+\frac{1}{s+1}=\frac{2s+3}{s+1}$

$$\Rightarrow X(s) = \frac{2s+3}{s+1} \cdot \frac{1}{(s-2)(s^2+4)}$$

Descompunem în fracții simple pe $X(s)$

$$X(s) = \frac{2s+3}{(s+1)(s-2)(s^2+4)} = \frac{\overset{(s-2)(s^2+4)}{A}}{s+1} + \frac{\overset{(s+1)(s^2+4)}{B}}{s-2} + \frac{\overset{(s+1)(s-2)}{Cs+D}}{s^2+4}$$

$$\Rightarrow 2s+3 = A(s-2)(s^2+4) + B(s+1)(s^2+4) + (Cs+D)(s+1)(s-2) \quad \forall s \in \mathbb{C}$$

Pt. $s=2 \Rightarrow 7 = B \cdot 3 \cdot 8 \Rightarrow \underline{\underline{B = \frac{7}{24}}}$

Pt. $s=-1 \Rightarrow 1 = A \cdot (-3) \cdot 5 \Rightarrow \underline{\underline{A = -\frac{1}{15}}}$

Pt. $s=0 \Rightarrow 3 = -8A + 4B - 2D$

Coef lui s^3: $0 = A + B + C$ sau $s = \ldots \Rightarrow \ldots$

$$\Rightarrow \underline{\underline{C = -\frac{9}{40}}}, \quad \underline{\underline{D = -\frac{13}{20}}}$$

$$\Rightarrow X(s) = -\frac{1}{15} \cdot \frac{1}{s+1} + \frac{7}{24} \cdot \frac{1}{s-2} + \frac{-\frac{9}{40}s - \frac{13}{20}}{s^2+4}$$

$$\underset{\mathcal{L}\{x(t)\}(s)}{\overset{\parallel}{X(s)}} = -\frac{1}{15} \cdot \frac{1}{s+1} + \frac{7}{24} \cdot \frac{1}{s-2} - \frac{9}{40} \cdot \frac{s}{s^2+4} - \frac{13}{20} \cdot \frac{1}{s^2+4} \quad \Big| \mathcal{L}^{-1}$$

$$x(t) = -\frac{1}{15}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}(t) + \frac{7}{24} \cdot \mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\}(t) - \frac{9}{40} \cdot \mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\}(t) - \frac{13}{20}\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\}(t)$$

$$\left(\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\}(t) = e^{at} \cdot u(t) \quad \mathcal{L}^{-1}\left\{\frac{1}{s^2+a^2}\right\}(t) = \frac{1}{a} \cdot \sin at \cdot u(t) \right.$$
$$\left. \mathcal{L}^{-1}\left\{\frac{s}{s^2+a^2}\right\}(t) = \cos at \cdot u(t) \right)$$

$$\underline{\underline{x(t) = \left(-\frac{1}{15} \cdot e^{-t} + \frac{7}{24} \cdot e^{2t} - \frac{9}{40} \cdot \cos 2t - \frac{13}{20} \cdot \frac{1}{2} \cdot \sin 2t\right) \cdot u(t)}}$$

② Rez. ec.

$$x''(t) + x(t) = \frac{1}{\cos t} \quad \Big|_{\mathcal{L}} \quad x(0) = 1, \quad x'(0) = 2$$

$$\Rightarrow \mathcal{L}\{x''(t)\}(s) + \underset{X(s)}{\underset{\parallel \text{not}}{\mathcal{L}\{x(t)\}(s)}} = \underline{\underline{\mathcal{L}\left\{\frac{1}{\cos t}\right\}(s)}}^{?} \quad (*)$$

Obs $\mathcal{L}\{x'(t)\}(s) = sX(s) - \overset{\overset{1}{\parallel}}{x(0)} \quad | \cdot s$

$$\mathcal{L}\{x''(t)\}(s) = s^2X(s) - s\underset{1}{\underset{\parallel}{x(0)}} - \underset{2}{\underset{\parallel}{x'(0)}} = s^2X(s) - s - 2$$

$$(*) \Leftrightarrow s^2X(s) - s - 2 + X(s) = \mathcal{L}\left\{\frac{1}{\cos t}\right\}(s)$$

$$X(s) \cdot (s^2+1) = s + 2 + \mathcal{L}\left\{\frac{1}{\cos t}\right\}(s) \quad \Big| : (s^2+1)$$

$$\underset{\mathcal{L}\{x(t)\}(s)}{\overset{\parallel}{X(s)}} = \frac{s}{s^2+1} + \frac{2}{s^2+1} + \frac{1}{s^2+1} \cdot \mathcal{L}\left\{\frac{1}{\cos t}\right\}(s) \quad \Big| \mathcal{L}^{-1}$$

$$x(t) = \overset{= \cos t}{\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\}(t)} + 2 \cdot \overset{\overset{\sin t}{\parallel}}{\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}(t)} + \overline{\overline{\mathcal{L}^{-1}\left\{\frac{1}{s^2+1} \cdot \mathcal{L}\left\{\frac{1}{\cos t}\right\}(s)\right\}(t)}}$$

$$\left(\mathcal{L}^{-1}\left\{\frac{s}{s^2+a^2}\right\}(t) = \cos at \cdot u(t) \quad \mathcal{L}^{-1}\left\{\frac{1}{s^2+a^2}\right\}(t) = \frac{1}{a} \cdot \sin at \cdot u(t) \right)$$

$$x(t) = \cos t + 2\sin t + \mathcal{L}^{-1}\left\{\underbrace{\frac{1}{s^2+1}}_{\mathcal{L}\{\ \}} \cdot \mathcal{L}\left\{\frac{1}{\cos t}\right\}(s)\right\}(t)$$

Obs $\mathcal{L}\{f(t)\}(s) \cdot \mathcal{L}\{g(t)\}(s) = \mathcal{L}\{f(t) * g(t)\}(s)$ produs de convoluție

unde $f(t) * g(t) = \int_0^t f(x) \cdot g(t-x)\,dx$, $\quad f(t)*g(t) = g(t) * f(t)$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+1} \cdot \mathcal{L}\left\{\frac{1}{\cos t}\right\}(s)\right\}(t) = \mathcal{L}^{-1}\left\{\mathcal{L}\{\sin t\}(s) \cdot \mathcal{L}\left\{\frac{1}{\cos t}\right\}(s)\right\}(t)$$

$$\mathcal{L}\{\sin at\}(s) = \frac{a}{s^2+a^2}$$

$$= \mathcal{L}^{-1}\left\{\mathcal{L}\left\{\sin t * \frac{1}{\cos t}\right\}(s)\right\}(t)$$

$$= \sin t * \frac{1}{\cos t}$$

unde $\sin t * \dfrac{1}{\cos t} \overset{def}{=} \displaystyle\int_0^t \sin x \cdot \frac{1}{\cos(t-x)}\,dx$

$$\sin t * \frac{1}{\cos t} = \frac{1}{\cos t} * \sin t \overset{def}{=} \int_0^t \frac{1}{\cos x} \cdot \sin(t-x)\,dx$$

$$= \int_0^t \frac{\sin t \cdot \cos x - \sin x \cdot \cos t}{\cos x}\,dx$$

$$= \sin t \cdot \int_0^t \frac{\cancel{\cos x}}{\cancel{\cos x}}\,dx + \cos t \int_0^t \frac{-\sin x}{\cos x}\,dx$$

$$= \sin t \cdot x \Big|_0^t + \cos t \cdot \ln|\cos x| \Big|_0^t$$

$$= t \cdot \sin t + \cos t \cdot \ln|\cos t|$$

$$\Longrightarrow x(t) = (\cos t + 2\sin t + t \cdot \sin t + \cos t \cdot \ln|\cos t|) \cdot u(t)$$

(3) Rez. ec.

$$x''(t) + 2\int_0^t \sin(\tau - t) \cdot x'(\tau)\,d\tau + 2x'(t) = \cos t \Big/ \mathcal{L} \qquad x(0) = 2,\ x'(0) = 3$$

$$\mathcal{L}\{x''(t)\}(s) + 2\mathcal{L}\left\{\int_0^t \sin(\tau - t) x'(\tau)\,d\tau\right\}(s) + 2\mathcal{L}\{x'(t)\}(s) = \mathcal{L}\{\cos t\}(s) = \frac{s}{s^2+1} \qquad (*)$$

Obs $\mathcal{L}\{x(t)\}(s) \overset{not}{=} X(s)$

$\mathcal{L}\{x'(t)\}(s) = sX(s) - \underset{}{\overset{2}{x(0)}} \quad \Big| \cdot s \quad = sX(s) - 2$

$\mathcal{L}\{x''(t)\}(s) = s^2X(s) - s\underbrace{x(0)}_{2} - \underbrace{x'(0)}_{3} = s^2X(s) - 2s - 3$

Obs $f(t) * g(t) = \int_0^t f(x) \cdot g(t-x)\,dx \ \underset{dx = d\tau}{\overset{x = \tau}{=}} \ \int_0^t f(\tau) \cdot g(t-\tau)\,d\tau$

$f(\tau) = x'(\tau) \Rightarrow f(t) = x'(t)$
$g(t-\tau) = \sin(t-\tau) \Rightarrow g(t) = \sin t$

$$\int_0^t \sin(\tau - t) \cdot x'(\tau)\,d\tau = -\int_0^t \sin(t-\tau) \cdot x'(\tau)\,d\tau = -x'(t) * \sin t$$

$$(*) \Leftrightarrow s^2X(s) - 2s - 3 + 2\mathcal{L}\{-x'(t) * \sin t\}(s) + 2sX(s) - 4 = \frac{s}{s^2+1}$$

$$\mathcal{L}\{-x'(t) * \sin t\}(s) = -2 \cdot \underbrace{\mathcal{L}\{x'(t)\}(s)}_{sX(s)-2} \cdot \underbrace{\mathcal{L}\{\sin t\}(s)}_{\frac{1}{s^2+1}}$$

$$(*) \Leftrightarrow s^2 X(s) - 2s - 3 + 2\,\underbrace{\mathcal{L}\{-x'(t) * \sin t\}(s)}_{=\,-2\cdot\mathcal{L}\{x'(t)\}(s)\cdot\mathcal{L}\{\sin t\}(s)} + 2sX(s) - 4 = \frac{s}{s^2+1}$$

$$-2\cdot\underbrace{\mathcal{L}\{x'(t)\}(s)}_{sX(s)-2}\cdot\underbrace{\mathcal{L}\{\sin t\}(s)}_{\frac{1}{s^2+1}}$$

$$(*) \Leftrightarrow s^2X(s) - 2s - 3 - 2(sX(s) - 2)\cdot\frac{1}{s^2+1} + 2sX(s) - 4 = \frac{s}{s^2+1}$$

$$X(s)\cdot\left(s^2 - 2s\cdot\frac{1}{s^2+1} + 2s\right) = \frac{s}{s^2+1} + 2s + 7 - \frac{4}{s^2+1}$$

$$X(s)\cdot\frac{s^4+s^2-\cancel{2s}+2s^3+\cancel{2s}}{s^2+1} = \frac{s+2s^3+2s+7s^2+7-4}{s^2+1} \quad \Big|\,(s^2+1)$$

$$X(s)\cdot\underbrace{(s^4+2s^3+s^2)}_{=\,s^2(s+1)^2} = 2s^3+7s^2+3s+3$$

$$X(s) = \frac{2s^3+7s^2+3s+3}{s^2\cdot(s+1)^2} = \frac{As+B}{s^2} + \frac{Cs+D}{(s+1)^2} \quad \Big|\,\mathcal{L}^{-1} \quad \times$$

$$= \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+1} + \frac{D}{(s+1)^2} \quad ✓$$

$$\mathcal{L}^{-1}\left\{\frac{s}{(s+1)^2}\right\}(t) = ?$$

$$\left(\text{ex: } \frac{s}{(s+1)^3\cdot s^4} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{D}{s^4} + \frac{E}{s+1} + \frac{F}{(s+1)^2} + \frac{G}{(s+1)^3} \ ✓\right)$$

$$\Rightarrow 2s^3+7s^2+3s+3 = As(s+1)^2 + B(s+1)^2 + Cs^2(s+1) + Ds^2 \quad \forall s$$

Pt $s=0 \Rightarrow \underline{3 = B}$

Pt $s=-1 \Rightarrow \underline{5 = D}$

Coef lui s^3: $2 = A + C$

Pt $s=1 \Rightarrow 15 = 4A + 4B + 2C + D$ $\Big\} \Rightarrow \underline{A = -3}$, $\underline{C = 5}$

$$\underbrace{X(s)}_{=\,\mathcal{L}\{x(t)\}(s)} = \frac{-3}{s} + \frac{3}{s^2} + \frac{5}{s+1} + \frac{5}{(s+1)^2} \quad \Big|\,\mathcal{L}^{-1}$$

$$\Rightarrow x(t) = -3\,\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}(t) + 3\,\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}(t) + 5\,\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}(t) + 5\,\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\}(t)$$

$$\left(\mathcal{L}^{-1}\left\{\frac{1}{(s-a)^n}\right\}(t) = \frac{t^{n-1}}{(n-1)!}\cdot e^{at}\cdot u(t)\right)$$

$$\underline{x(t) = \left(-3\cdot 1 + 3\cdot t + 5\cdot e^{-t} + 5\cdot t\cdot e^{-t}\right)\cdot u(t)}$$