CHAPTER 4

Linear spaces

4.1 The definition of a linear space

Let K be the field of real numbers or the field of complex numbers.

Definition 4.1 A set V is called a *linear space* (or a *vector space*) over the field K if it satisfies the following conditions:

I) There exists an internal binary operation on V, called addition and denoted by $+$, such that $(V, +)$ is a commutative group.

II) There exists an external binary operation called scalar multiplication, in which each element $k \in K$ can be combined with each element $v \in V$ to give an element $kv \in V$, and such that, for all $k, l \in K$ and $x, y \in V$,

1) $k(x + y) = kx + ky$
2) $(k + l)x = kx + lx$
3) $(kl)x = k(lx)$
4) $1x = x$.

We must be careful to distinguish between the two types of elements: those belonging to V called *vectors*, and those belonging to K called *scalars*.

Example 4.1.1 1) The set $\mathcal{V}_3$ of the vectors in space with the usual definitions of addition and multiplication by a real number, forms a linear space over the field $\mathbb{R}$.

2) Let $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$ $(x_i, y_i \in K)$ be two elements of K^n (the set of n-tuples of elements of K). The addition $x + y$ and scalar multiplication λx $(\lambda \in K)$ may be defined by

$$x + y = (x_1 + y_1, \ldots, x_n + y_n)$$

$$\lambda x = (\lambda x_1, \ldots, \lambda x_n)$$

With these operations it is easily verified that K^n is a linear space over the field K.

3) An obvious generalization of the previous example is the set $\mathcal{M}_{n,m}(K)$ with the usual definitions of addition of matrices and multiplication of a matrix by an element of K.

4) Let S be any set and $F = \{f | f : S \longrightarrow K\}$.
With the usual definitions of addition of functions and multiplication of a function by a number, F is a linear space over K.

We see that the structure of linear space appears in various and quite natural situations.

The first theorem gives a number of elementary deductions from the definition of a linear space. We must be careful to distinguish between 0, the zero of K, and 0, the zero vector of V.

Theorem 4.2 *In any linear space V over K we have*

(i) $0v = 0$*;*

(ii) $k0 = 0$;

(iii) $(-1)v = -v$,

for all $v \in V$ *and* $k \in K$. *(*$-v$ *is the negative of* v *in the group* $(V, +)$*).*

Proof.

(i) Since $0v = (0 + 0)v = 0v + 0v$, we infer that $0v = 0$.

(ii) $k0 = k(0 + 0) = k0 + k0$, hence $k0 = 0$.

(iii) $v + (-1)v = 1v + (-1)v = [1 + (-1)]v = 0v = 0$, therefore $(-1)v = -v$. □

Theorem 4.3 *(a) If* $k \in K, v \in V$ *and* $kv = 0$, *then either* $k = 0$ *or* $v = 0$.

(b) If $lv = kv$ *and* $v \neq 0$, *then* $l = k$.

(c) If $kv = kw$ *and* $k \neq 0$, *then* $v = w$.

Proof.

(a) Suppose that $k \neq 0$. Then there exists $k^{-1} \in K$. We have $k^{-1}(kv) = k^{-1} \cdot 0$, hence $(k^{-1}k)v = 0$. It follows that $1v = 0$ and finally $v = 0$, q.e.d.

(b) $lv = kv$ implies $(l - k)v = 0$. Since $v \neq 0$ we may apply (a) and deduce $l - k = 0$, that is, $l = k$.

(c) is left to the reader. □

4.2 Linear subspaces

Let V be a linear space over K. A non-empty subset W of V is called a *linear subspace* (or a *vector subspace*) of V if

$kx + ly \in W$ for all $k, l \in K$ and $x, y \in W$.

Let us remark that this condition is equivalent to the following two conditions:

(1) $x + y \in W$ for all $x, y \in W$

(2) $kx \in W$ for all $k \in K$ and $x \in W$.

Any linear subspace W contains the vector 0; indeed, for any $v \in W$ we have $0v \in W$ and hence $0 \in W$.

Example 4.2.1 (1) $\{0\}$ and V are linear subspaces of V. These two subspaces are called *improper subspaces* of V; all other subspaces are *proper subspaces.*

(2) $\{a\overrightarrow{i} \mid a \in \mathbb{R}\}$ and $\{a\overrightarrow{i} + b\overrightarrow{j} \mid a, b \in \mathbb{R}\}$ are linear subspaces of $\mathcal{V}_3$.

(3) $\{(0, x_2, \dots, x_n) \mid x_2, \dots, x_n \in K\}$ is a linear subspace of K^n.

Let $S \subset V, S \neq \emptyset$. A vector $v \in V$ of the form $v = k_1v_1 + \cdots + k_nv_n$, where $n \in \mathbb{N}^*$, $k_i \in K$ and $v_i \in S$ is called a *linear combination* of elements of S. It is easy to verify that the set of all linear combinations of elements of S is a linear subspace of V, called the subspace *generated* by S.

Theorem 4.4 *Let U and W be linear subspaces of the space V.*

a) $U \cap W$ is a linear subspace of V.

b) The set $U + W = \{u + w \mid u \in U, w \in W\}$ is a linear subspace of V, called the sum of U and W.

The (easy) proof is left to the reader.

4.3 Linear dependence, bases, dimension

A subset X of a linear space V is called a *linearly dependent set* if it contains a finite subset $\{x_1, \ldots, x_r\}(r \geq 1)$ for which there exist scalars $k_1, \ldots, k_r \in K$, not all zero, such that $k_1 x_1 + \cdots + k_r x_r = 0$. Such a linear relation, where not all the k_i are zero, will be called *non-trivial.*

A subset of a linear space is linearly independent if it is not linearly dependent. An alternative definition, equivalent to this is:*A* set X is linearly independent if every linear relation $k_1 x_1 + \cdots + k_r x_r = 0$ $(k_i \in K)$ between the vectors x_i of X has zero coefficients. In other words, every linear relation between the vectors of X is trivial.

Example 4.3.1

1) Every subset $X \subset V$ which contains 0 is linearly dependent.
2) If $v \in V, v \neq 0$, then $\{v\}$ is linearly independent.
3) Let $V = \{f \mid f : \mathbb{R} \to \mathbb{R}\}$. Let $f_i \in V,\ f_i(t) = t^i,\ i = 0, 1, \ldots, n$. Then $\{f_0, f_1, \ldots, f_n\}$ is linearly independent.
4) $\overrightarrow{u}, \overrightarrow{v}, \overrightarrow{w} \in V_3$ are linearly dependent if and only if they are coplanar.

Definition 4.5 Any linearly independent subset of a vector space V, which has the property that it generates V, is called a *basis* of V.

It can be shown that every vector space $V \neq \{0\}$ possesses a basis. Also, if V has a finite basis with r elements, then every basis of V has r elements. We say that the *dimension* of V is r and write $dimV = r$.

If V has no finite bases, it is called infinite-dimensional. In this case we can find arbitrarily large linearly independent finite subsets of V. On the other hand, we write $dim\{0\} = 0$.

Example 4.3.2 1) $\{\vec{i}, \vec{j}, \vec{k}\}$ is a basis of $\mathcal{V}_3$.

2) The vectors $e_1 = (1,0,0,\dots,0), e_2 = (0,1,0,\dots,0),\dots, e_n = (0,\dots,0,1)$ form a basis of K^n, called the *canonical basis* of K^n. Thus, $dimK^n = n$.

3) Let $K_n[X]$ be the linear space of all polynomials of degree $\leq n$, with coefficients in K. A basis of this space is $\{1, X, X^2, \dots, X^n\}$.

4) Let $K[X]$ be the space of all polynomials with coefficients in K. A basis of it is $\{1, X, X^2, \dots, X^n, \dots\}$. Hence $K[X]$ is infinite-dimensional.

Let V be finite-dimensional. It can be shown that if U and W are linear subspaces of V, then

$$dim(U + W) + dim(U \cap W) = dimU + dimW.$$

Theorem 4.6 *Let $T = \{v_1, \dots, v_m\} \subset V$ be a linearly independent set which is not a basis. Then there exists $v \in V$ such that $\{v_1, \dots, v_m, v\}$ is linearly independent.*

Theorem 4.7 *a) Every linearly independent subset of V_n with n elements is a basis of V_n.*

b) Every linearly independent subset of V_n is a part of a basis.

4.4 Coordinates. Change of bases

Let $B = \{b_1, \dots, b_n\}$ be a basis of the n-dimensional linear space V_n over K.

Theorem 4.8 *Each $v \in V_n$ can be written uniquely in the form*

$$v = x_1 b_1 + \cdots + x_n b_n$$

with $x_1, \ldots, x_n \in K$. (The scalars $x_1, \ldots, x_n$ are called the coordinates of the vector v relative to the basis B.)

Proof. Let $v \in V_n$. Since B generates V_n, there exist scalars $x_1, \ldots, x_n$ such that $v = x_1 b_1 + \ldots x_n b_n$. We have to prove that they are uniquely determined.

Suppose that $x_1', \ldots, x_n' \in K$ and $v = x_1' b_1 + \cdots + x_n' b_n$. Then it follows $(x_1 - x_1')b_1 + \cdots + (x_n - x_n')b_n = 0$. Since $b_1, \ldots, b_n$ are linearly independent, it follows that $x_1' = x, \ldots, x_n' = x_n$ and the theorem is proved. $\square$

Consider now the above basis B and let $B' = \{b_1', \ldots, b_n'\} \subset V_n$. Then we have $b_j' = \sum\limits_{i=1}^{n} c_{ij} b_i,\ j = 1, \ldots, n$, with $c_{ij} \in K$.

Theorem 4.9 *B' is a basis of V_n if and only if $det(c_{ij}) \neq 0$.*

Proof. Since B' has n elements, the following two statements are equivalent:

(1) B' is a basis

(2) B' is linearly independent

Clearly (2) is equivalent to

(3) $k_1 b_1' + \cdots + k_n b_n' = 0 \Longrightarrow k_1 = \cdots = k_n = 0$.

We have $\sum\limits_{j=1}^{n} k_j b_j' = \sum\limits_{j=1}^{n} k_j \sum\limits_{i=1}^{n} c_{ij} b_i = \sum\limits_{j=1}^{n} \sum\limits_{i=1}^{n} c_{ij} k_j b_i = \sum\limits_{i=1}^{n} \sum\limits_{j=1}^{n} c_{ij} k_j b_i = \sum\limits_{i=1}^{n} \Big(\sum\limits_{j=1}^{n} c_{ij} k_j \Big) b_i$.

Thus the first equality in (3) is equivalent to $\sum\limits_{i=1}^{n}\left(\sum\limits_{j=1}^{n} c_{ij}k_j\right)b_i = 0$, which is equivalent (due to the linear independence of B) to $\sum\limits_{j=1}^{n} c_{ij}k_j = 0,\ i = 1,\dots,n$. Hence (3) is equivalent to

(4) The linear homogeneous system $\sum\limits_{j=1}^{n} c_{ij}k_j = 0,\ i = 1,\dots,n,$ has only the trivial solution.Finally, (4) is equivalent to

(5) $det(c_{ij} \neq 0$

We conclude that (1) and (5) are equivalent and the theorem is proved.

Let us remark that the columns of the matrix $C = (c_{ij}),\ i,j = 1,\dots,n$ are formed with the coordinates of b'_j relative to the basis B. Suppose that C is nonsingular; this means that B' is also a basis of V_n. C is called the *transition matrix* from B to B'.

Let $x \in V_n$. We have $x = \sum\limits_{i=1}^{n} x_i b_i$ and $x = \sum\limits_{j=1}^{n} x'_j b'_j$, with $x_i, x'_j \in K$. Then $x = \sum\limits_{j=1}^{n} x'_j \sum\limits_{i=1}^{n} c_{ij}b_i = \sum\limits_{j=1}^{n}\sum\limits_{i=1}^{n} c_{ij}x'_j b_i = \sum\limits_{i=1}^{n}(\sum\limits_{j=1}^{n} c_{ij}x'_j)b_i$.

Hence $\sum\limits_{i=1}^{n} x_i b_i = \sum\limits_{i=1}^{n}\left(\sum\limits_{j=1}^{n} c_{ij}x'_j\right)b_i$. It follows that

(6) $x_i = \sum\limits_{j=1}^{n} c_{ij}x'_j,\ i = 1,\dots,n.$

We have here the relationship between the coordinates of x relative to the basis B and the coordinates of the same x relative to the basis B'. Let us denote

$$X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \; X' = \begin{pmatrix} x'_1 \\ \vdots \\ x'_n \end{pmatrix}$$

Then (6) is equivalent to $X = CX'$. $\square$

Finally, let us mention-without proof - the following important result.

Let $B = \{b_1, \dots, b_n\}$ be a basis of V_n and let $v_1, \dots, v_p \in V$. Write $v_j = \sum_{i=1}^{n} a_{ij} b_i, \; j = 1, \dots, p$, with $a_{ij} \in K$. Consider the matrix

$$A = \begin{pmatrix} a_{11} & \dots & a_{1p} \\ \dots & & \\ a_{n1} & \dots & a_{np} \end{pmatrix}$$

Theorem 4.10 *The dimension of the linear subspace of V_n generated by $\{v_1, \dots, v_p\}$ equals r_A.*

Exercices

4.1 Let $V = \{x \in \mathbb{R} \mid x > 0\}$ be endowed with the internal operation $x \oplus y = xy$. Prove that $(V, \oplus)$ is a linear space over $\mathbb{R}$ with the external operation $\alpha * x = x^\alpha$, for each $x \in V$, $\alpha \in \mathbb{R}$.

4.2 Prove that all square matrices of order n with real elements, form a vector space over the field of real numbers, if the operations involved are addition of matrices and multiplication of a matrix by a scalar. Find the basis and dimension of this space.

4.3 Prove that all polynomials of degree $\leq n$ with real coefficients form a vector space if the operations involved are ordinary addition of polynomials and multiplication of a polynomial by a scalar. Find the basis and dimension of this space.

4.4 Determine which of the following sets are linear subspaces of the corresponding linear spaces.
a) $W_1 = \{(x_1, \ldots, x_n) \mid x_1 + \cdots + x_n = 0\}$, in $\mathbb{R}^n$ over $\mathbb{R}$
b) $W_2 = \{(x_1, \ldots, x_n) \mid x_1 + \cdots + x_n = 1\}$, in $\mathbb{R}^n$ over $\mathbb{R}$
c) $W_3 = \{(x_1, \ldots, x_n) \mid x_i \in \mathbb{Z}, i = 1, \ldots, n\}$, in $\mathbb{R}^n$ over $\mathbb{R}$
d) $W_4 = \{(x, y, z) \mid 2x - 3y + z = 0\}$, in $\mathbb{R}^3$ over $\mathbb{R}$
e) $W_5 = \{(x, y, z) \mid 2x - 3y + z + 6 = 0\}$, in $\mathbb{R}^3$ over $\mathbb{R}$
f) $W_6 = \{(x, y, z) \mid \frac{x}{3} = \frac{y}{-2} = \frac{z}{8}\}$, in $\mathbb{R}^3$ over $\mathbb{R}$
g) $W_7 = \{(x, y, z) \mid \frac{x-1}{3} = \frac{y}{-2} = \frac{z}{8}\}$, in $\mathbb{R}^3$ over $\mathbb{R}$
h) $W_8 = \{f : I \to \mathbb{R} \mid f$ differentiable on $I\}$, in $C(I)$ over $\mathbb{R}$, the space of continuous functions on the interval $I \in \mathbb{R}$
i) $W_9 = \{P \mid P$ is a polynomial of odd degree$\}$, in $\mathbb{R}_n[X]$ over $\mathbb{R}$, the space of polynomials of degree at most n with real coefficients.

4.5 Prove that the following sets of vectors are subspaces in $\mathbb{R}^n$ over $\mathbb{R}$ and find the basis and dimension of each:
a) All n-dimensional vectors with the first and last coordinates equal.
b) All n-dimensional vectors of the form $(\alpha, \beta, \alpha, \beta, \ldots)$, where α and β are any numbers.

4.6 Find out if the following matrices are linearly independent in the space $\mathcal{M}_2(\mathbb{R})$, for $a \in \mathbb{R}$:

$$\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 2 & a \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 2 & -1 \end{pmatrix}.$$

4.7 Determine a basis in the linear subspace generated by the set of functions $\{1, \sin^2 x, \cos^2 x, \cos 2x\}$.

4.8 Determine the dimension and a basis for the linear subspace V generated:

a) in $\mathbb{R}^4$ by the vectors: $v_1 = (0, 2, -1, 3)$, $v_2 = (1, 1, 2, -1)$, $v_3 = (2, 5, -2, 3)$ and $v_4 = (-1, 0, 2, 2)$,

b) in $\mathbb{R}^4$ by the vectors: $v_1 = (2, 1, 3, 0)$, $v_2 = (-3, 1, 1, 2)$, $v_3 = (-1, 2, 4, 2)$ and $v_4 = (-1, 0, 2, -2)$,

c) in $\mathbb{R}^3$ by the vectors: $v_1 = (-1, 3, 2)$, $v_2 = (1, 4, 1)$, $v_3 = (0, 1, 2)$.

4.9 Find the dimensions and bases of the linear subspaces spanned (generated) by the following sets of vectors:
a) $a_1 = (1, 0, 0, -1)$, $a_2 = (1, 1, 1, 1)$, $a_3 = (2, 1, 1, 0)$, $a_4 = (1, 2, 3, 4)$ and $a_5 = (0, 1, 2, 3)$.
b) $a_1 = (1, 1, 1, 1, 0)$, $a_2 = (1, 1, -1, -1, -1)$, $a_3 = (2, 2, 0, 0, -1)$, $a_4 = (1, 1, 5, 5, 2)$ and $a_5 = (1, -1, -1, 0, 0)$

4.10 Find the dimensions of the union and intersection of the linear subspaces $S_1 = \text{span}\{a_1, a_2, ..., a_k\}$ and $S_2 = \text{span}\{b_1, b_2, ..., b_m\}$, if:
a) $a_1 = (1, 2, 0, 1)$, $a_2 = (1, 1, 1, 0)$ and $b_1 = (1, 0, 1, 0)$, $b_2 = (1, 3, 0, 1)$
b) $a_1 = (1, 1, 1, 1)$, $a_2 = (1, -1, 1, -1)$, $a_3 = (1, 3, 1, 3)$ and $b_1 = (1, 2, 0, 2)$, $b_2 = (1, 2, 1, 2)$, $b_3 = (3, 1, 3, 1)$.

4.11 Find the bases of the unions and intersections of the linear subspaces $S_1 = \text{span}\{a_1, a_2, ..., a_k\}$ and $S_2 = \text{span}\{b_1, b_2, ..., b_m\}$:
a) $a_1 = (1, 2, 1)$, $a_2 = (1, 1, -1)$, $a_3 = (1, 3, 3)$ and $b_1 = (2, 3, -1)$, $b_2 = (1, 2, 2)$, $b_3 = (1, 1, -3)$.
b) $a_1 = (1, 2, 1, -2)$, $a_2 = (2, 3, 1, 0)$, $a_3 = (1, 2, 2, -3)$ and $b_1 = (1, 1, 1, 1)$, $b_2 = (1, 0, 1, -1)$, $b_3 = (1, 3, 0, -4)$.

4.12 Consider in $\mathbb{R}^3$ the linear subspaces P and Q given by $P : 5x - 2y + z = 0$, $Q : x + y - 3z = 0$. Determine bases in P, Q, $P \cap Q$ and in $\text{sp}(P \cup Q)$.

4.13 Find the coordinates of the vector $v = (-3, 1, 2)$ in the basis $B' = \{(1, -1, 0), (1, 0, -1), (0, 1, -1)\}$.

4.14 Show that the vectors $e_1 = (1,1,1)$, $e_2 = (1,1,2)$, $e_3 = (1,2,3)$ form a basis in $\mathbb{R}^3$ and find the coordinates of the vector $a = (6,2,-7)$ in this basis.

4.15 Show that the vectors $e_1 = (1,2,-1,-2)$, $e_2 = (2,3,0,-1)$, $e_3 = (1,2,1,4)$ and $e_4 = (1,3,-1,0)$ form a basis in $\mathbb{R}^4$ and find the coordinates of the vector $b = (7,14,-1,2)$ in this basis.

4.16 Prove that each of the two sets of vectors is a basis in $\mathbb{R}^3$ and find the relationship between the coordinates of one and the same vector in the two bases:
$a_1 = (1,2,1)$, $a_2 = (2,3,3)$, $a_3 = (3,7,1)$ and $b_1 = (3,1,4)$, $b_2 = (5,2,1)$, $b_3 = (1,1,-6)$.

4.17 Let $P_1 = (X-b)(X-c)$, $P_2 = (X-a)(X-c)$, $P_3 = (X-a)(X-b)$ be polynomials from $\mathbb{R}_2[X]$, $a,b,c \in \mathbb{R}$.

a) Determine the condition under which P_1, P_2, P_3 are linearly independent.

b) Considering the condition of (a) satisfied, write the polynomial $P = 1 + X + X^2$ as a linear combination of P_1, P_2 and P_3.

4.18 In the space of polynomials of degree at most two over $\mathbb{R}$, consider the canonical basis $B = \{1, X, X^2\}$ and another basis $B' = \{1, X - a, (X-a)^2\}$, where $a \in \mathbb{R}$.

a) Determine the transition matrix from B to B',

b) Determine the coordinates of the polynomial $f = \alpha + \beta X + \gamma X^2$ in the new basis B'.

4.19 Find the coordinates of the polynomial $f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$ in the following bases:
a) $1, x, x^2, \ldots, x^n$.
b) $1, x-\alpha, (x-\alpha)^2, \ldots, (x-\alpha)^n$.

4.20 Prove that each of the two sets of vectors is a basis in the space of polynomials of degree ≤ 3 with real coefficients and find the transition

matrix between the two bases:
$e_1 = 1$, $e_2 = x$, $e_3 = x^2$ and $e_4 = x^3$ and $e_1^{'} = 1 - x$, $e_2^{'} = 1 + x^2$, $e_3^{'} = x^2 - x$ and $e_4^{'} = x^3 + x^2$

4.21 Find a basis in the real space of the solutions of the following systems:

$$\text{a)} \begin{cases} x + y - z + 2t = 0 \\ x - 2y + t = 0 \end{cases} \qquad \text{b)} \begin{cases} x + y - z + t = 0 \\ x - y + 2z - t = 0 \\ 2x + y - z - t = 0 \end{cases}$$

$$\text{c)} \begin{cases} x + 2y + 4z - 3t = 0 \\ 3x + 5y + 6z - 4t = 0 \\ 3x + 8y + 24z - 19t = 0 \\ 4x + 5y - 2z + 3t = 0 \end{cases} \qquad \text{d)} \begin{cases} x - 2y + z - t = 0 \\ 2x - y + 3z - 3t = 0 \\ x + y + z + t = 0 \\ 2x - y + 2z = 0 \end{cases}$$

4.22 In $\mathbb{R}^3$ consider the subspaces

$$D = \{(x, y, z) \mid \frac{x}{\alpha} = \frac{y}{\beta} = \frac{z}{\gamma},\ \alpha, \beta, \gamma \in \mathbb{R}^*\}$$

and

$$P = \{(x, y, z) \mid ax + by + cz = 0,\ a, b, c \in \mathbb{R}\}.$$

Find the condition wherefore $\mathbb{R}^3 = D \oplus P$.

Solutions

$\boxed{4.1}$ $(V, \oplus)$ is a commutative group. We check also the other axioms, for $x, y \in V$ and $\alpha, \beta \in \mathbb{R}$.

$$\alpha * (x \oplus y) = (xy)^\alpha = x^\alpha y^\alpha = (\alpha * x) \oplus (\alpha * y),$$
$$(\alpha + \beta) * x = x^{\alpha+\beta} = x^\alpha x^\beta = \alpha * x \oplus \beta * x,$$
$$\alpha * (\beta * x) = (\beta * x)^\alpha = (x^\beta)^\alpha = x^{\alpha\beta} = (\alpha\beta) * x,$$
$$1 * x = x^1 = x.$$

$\boxed{\mathbf{4.2}}$ The basis is formed, for example, by the matrices E_{ij} $(i, j = 1, 2, ..., n)$ whose elements in the ith row and the jth column is equal to unity and all other elements are zero. The dimension is n^2.

$\boxed{\mathbf{4.3}}$ The basis is formed, for example, by the polynomials $1, x, x^2, ..., x^n$. The dimension is $n + 1$.

$\boxed{\mathbf{4.4}}$ a) Yes, b) No, c) No, d) Yes, e) No, f) Yes, g) No, h) Yes, i) No.

$\boxed{\mathbf{4.5}}$ a) The basis is formed, for example, by the vectors $(1, 0, 0, ..., 0, 1)$, $(0, 1, 0, ..., 0, 0)$, $(0, 0, 1, ..., 0, 0)$, ..., $(0, 0, 0, ..., 1, 0)$ and the dimension is $n - 1$.
b) The basis is formed, for example, by the two vectors $(1, 0, 1, 0, ...)$, $(0, 1, 0, 1, ...)$ and the dimension is 2.

$\boxed{\mathbf{4.6}}$ Let $\alpha, \beta, \gamma \in \mathbb{R}$ such that

$$\alpha \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} + \beta \begin{pmatrix} 2 & a \\ 0 & 1 \end{pmatrix} + \gamma \begin{pmatrix} 0 & 1 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

that is $\begin{cases} \alpha + 2\beta = 0 \\ a\beta + \gamma = 0 \\ -\alpha + 2\gamma = 0 \\ \alpha + \beta - \gamma = 0 \end{cases}$. We can notice that if α, β, γ satisfy the first and third equation they also satisfy the last one, so we have the linear homogeneous system $\begin{cases} \alpha + 2\beta = 0 \\ a\beta + \gamma = 0 \\ -\alpha + 2\gamma = 0 \end{cases}$. If the determinant of the system $\begin{vmatrix} 1 & 2 & 0 \\ 0 & a & 1 \\ -1 & 0 & 2 \end{vmatrix} = 2a-2$ is not zero, then the only solution is the trivial one $\alpha = \beta = \gamma = 0$. For $a \neq 1$ the three matrices are linearly independent, and for $a = 1$ they are linearly dependent, for instance $B = 2A + C$.

$\boxed{\textbf{4.7}}$ Since $\sin^2 x = \frac{1}{2} \cdot 1 - \frac{1}{2} \cdot \cos 2x$, $\cos^2 x = \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot \cos 2x$ it means that only two of the elements can be linearly independent. From $\alpha \cdot 1 + \beta \cdot \cos 2x = 0$ follows $\alpha = \beta = 0$ so a basis for the subspace is $\{1, \cos 2x\}$.

$\boxed{\textbf{4.8}}$ a) The 4^{th} order determinant having the four vectors as columns has the value 0, so $\dim(V) < 4$. We can find 3^{rd} order minors that are different from 0, so $\dim(V) = 3$. A basis can be, for instance $\{v_1, v_2, v_3\}$, or $\{v_1, v_2, v_4\}$; b) The rank of the matrix is 3, $\dim(V) = 3$, a basis is for instance $\{v_2, v_3, v_4\}$; c) $\dim(V) = 3$, so the subspace coincides with the whole space $\mathbb{R}^3$.

$\boxed{\textbf{4.9}}$ a) The basis is formed, for example, by the vectors a_1, a_3 and a_4, so the dimension is 3.
b) The basis is formed, for example, by the vectors a_1, a_2 and a_5 and the dimension is 3.

$\boxed{\textbf{4.10}}$ a) The dimensions of the union is 3 and of the intersection is 1. b) The dimensions of the union is 3 and of the intersection is 2.

$\boxed{\textbf{4.11}}$ a) The basis of the union (sum) is formed, for example,

by the vectors a_1, a_2 and b_1 and the basis of the intersection consist of the single vector $x = 2a_1 + a_2 = b_1 + b_2 = (3, 5, 1)$.
b) The basis of the union (sum) is formed, for example, by the vectors a_1, a_2, a_3 and b_2 and the basis of the intersection consist of $b_1 = -2a_1 + a_2 + a_3$ and $b_3 = 5a_1 - a_2 - 2a_3$.

$\boxed{\mathbf{4.12}}$ $P = \{(x, y, z) \in \mathbb{R}^3 \mid 5x - 2y + z = 0\} = \{(x, y, -5x + 2y) \mid x, y \in \mathbb{R}\} = \{x(1, 0, -5) + y(0, 1, 2) \mid x, y \in \mathbb{R}\}$, so $\{(1, 0, -5), (0, 1, 2)\}$ is a basis for P. Similarly, $Q = \mathrm{sp}\{(1, -1, 0), (0, 3, 1)\}$
To find $P \cap Q$ we solve the system $\begin{cases} 5x - 2y + z = 0 \\ x + y - 3z = 0 \end{cases}$ and get
$z = \frac{7}{5}x$, $y = \frac{16}{5}x$, so
$P \cap Q = \mathrm{sp}\{(1, \frac{7}{5}, \frac{16}{5})\} = \mathrm{sp}\{(5, 7, 16)\}$.
$\mathrm{sp}(P \cup Q) = \mathrm{sp}\{(1, 0, -5), (0, 1, 2), (1, -1, 0), (0, 3, 1)\} =$
$$= \mathrm{sp}\{(1, 0, -5), (0, 1, 2), (1, -1, 0)\} = \mathbb{R}^3.$$

$\boxed{\mathbf{4.13}}$ The transition matrix from the canonical basis to the basis B' is $\begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix}$. Denoting by a, b, c the coordinates in the new basis we have $\begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and we get $a = -1$, $b = -2$, $c = 0$. Indeed, $v = -(1, -1, 0) - 2(1, 0, -1)$.

$\boxed{\mathbf{4.14}}$ $(15, -5, -4)$.

$\boxed{\mathbf{4.15}}$ $(0, 2, 1, 2)$.

$\boxed{\textbf{4.16}}$ We consider the same vector in the first basis $(\alpha_1, \alpha_2, \alpha_3)$ and in the second basis $(\beta_1, \beta_2, \beta_3)$. Then $\alpha_1 = -27\beta_1 - 71\beta_2 - 41\beta_3$, $\alpha_2 = 9\beta_1 + 20\beta_2 + 9\beta_3$ and $\alpha_3 = 4\beta_1 + 12\beta_2 + 8\beta_3$.

$\boxed{\textbf{4.17}}$ Let $\alpha, \beta, \gamma \in \mathbb{R}$ such that $\alpha P_1 + \beta P_2 + \gamma P_3 = 0$. This means

$$\alpha(X-b)(X-c)+\beta(X-a)(X-c)+\gamma(X-a)(X-b) = 0, \ \forall\, x \in \mathbb{R}.$$

Assigning to X the values a, b or c it follows that $\alpha(a-b)(a-c) = 0$, $\beta(b-a)(b-c) = 0$ and $\gamma(c-a)(c-b) = 0$. If a, b, c are distinct two by two we get $\alpha = \beta = \gamma = 0$, so P_1, P_2, P_3 are linearly independent. If, for instance, $a = b$, for $\alpha = 1$, $\beta = -1$, $\gamma = 0$, we have $P_1 - P_2 = 0$, so they are not linearly independent. The same for $a = c$ or $b = c$. In conclusion the condition of linear independence is $(a-b)(a-c)(b-c) \neq 0$. b) We must determine l, m, n such that $1+X+X^2 = l(X-b)(X-c) + m(X-a)(X-c) + n(X-a)(X-b)$. Assigning to X the values a, b or c we get $l = \dfrac{1+a+a^2}{(a-b)(a-c)}$, $m = \dfrac{1+b+b^2}{(b-a)(b-c)}$, $n = \dfrac{1+c+c^2}{(c-a)(c-b)}$.

$\boxed{\textbf{4.18}}$ a) The transition matrix is $C = \begin{pmatrix} 1 & -a & a^2 \\ 0 & 1 & -2a \\ 0 & 0 & 1 \end{pmatrix}$, b) $f = \alpha + \beta a + \gamma a^2 + (\beta + 2\gamma a)(X-a) + \gamma (X-a)^2$ or $f = f(a) + \dfrac{f'(a)}{1!}(X-a) + \dfrac{f''(a)}{2!}(X-a)^2$.

$\boxed{\textbf{4.19}}$ a) $a_0, a_1, a_2, \ldots, a_n$. b) $f(\alpha)$, $f^{'}(\alpha)$, $f^{''}(\alpha)/2!$, …, $f^{(n)}(\alpha)/n!$.

$\boxed{\mathbf{4.20}}$ $\begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

$\boxed{\mathbf{4.21}}$ a) $(1, 0, -1, -1), (0, 1, 5, 2)$. b) $(2, -7, -4, 1)$. c) $(8, -6, 1, 0)$, $(-7, 5, 0, 1)$. d) $(-10/3, -2/3, 3, 1)$.

$\boxed{\mathbf{4.22}}$ $\alpha a + \beta b + \gamma c \neq 0$.