CHAPTER 5

Inner product spaces

5.1 Inner products

Definition 5.1 An inner product on a real or complex linear space V is any scalar-valued function, defined on V^2 (the set of ordered pairs (x, y) of elements of V) and denoted by $(x|y)$, which satisfies the following three axioms: for all $x, x_1, x_2, y \in V$ and $k_1, k_2 \in K$,

(1) $(x|y) = \overline{(y|x)}$

(2) $(k_1x_1 + k_2x_2|y) = k_1(x_1|y) + k_2(x_2|y)$

(3) $(x|x \geq 0$, and $(x|x) = 0$ if and only if $x = 0$.

In (1) the bar denotes the complex conjugate, and so may be omitted if the vector space is real. Because of (1), $(x|x)$ is real (even if V is a complex vector space) and so the inequality of (3) is meaningful. Corresponding to (2) is the relation

(2') $(x|k_1y_1 + k_2y_2) = \overline{k_1}(x|y_1) + \overline{k_2}(x|y_2)$,

which can be deduced, using (1), from (2) and is equivalent to it. Both (2) and (2') extend, in an obvious manner, to the case where more than two terms occur in either the first or second position in the inner product. We have also $(x|0) = (0|y) = 0$ for all $x, y \in V$.

Example 5.1.1 (1) For $\overrightarrow{u}, \overrightarrow{v} \in \mathcal{V}_3$ define $(\overrightarrow{u}|\overrightarrow{v}) = \overrightarrow{u} \cdot \overrightarrow{v}$. In this way we have an inner product on $\mathcal{V}_3$.

(2) Let $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$. The formula $(x|y) = x_1y_1 + \cdots + x_ny_n$ defines an inner product on $\mathbb{R}^n$, called *the canonical inner product* on $\mathbb{R}^n$.

(3) Let $x = (x_1, \ldots, x_n) \in \mathbb{C}^n, y = (y_1, \ldots, y_n) \in \mathbb{C}^n$.Then $(x|y) = x_1\overline{y_1} + \cdots + x_n\overline{y_n}$ defines the *canonical inner product* on $\mathbb{C}^n$.

(4) Let $C[a, b] = \{f : [a, b] \longrightarrow \mathbb{R} \mid f\text{continuous on } [a, b]\}$. For $f, g \in C[a, b]$, define

$$(f|g) = \int_a^b f(x)g(x)dx$$

Then we have an inner product on $C[a, b]$.

An *inner product space* is any linear space on which an inner product is defined. A finite-dimensional real inner product space is known as a *Euclidean space*; a finite-dimensional complex inner product space is known as a *unitary space*.

Theorem 5.2 *(Schwarz' inequality). Let V be an inner product space and $u, v \in V$. Then*

$$|(u|v)|^2 \leq (u|u)(v|v).$$

Proof. If $v = 0$, the inequality reduces to $0 \leq 0$. So, let $v \neq 0$; then $(v|v) > 0$. We have

(i) $(u - kv|u - kv) \geq 0, \quad \forall k \in K$.

It follows immediately that

(ii) $(u - kv|u) - \overline{k}(u - kv|v) \geq 0,\ \forall k \in K$

For $k_0 = \dfrac{(u|v)}{(v|v)}$ the second inner product equals zero and hence (ii) implies

$$(u|u) - k_0(v|u) \geq 0, \text{ that is,}$$
$$(u|u) - \frac{(u|v)}{(v|v)}(v|u) \geq 0.$$

Since $(u|v)(v|u) = (u|v)\overline{(u|v)} = |(u|v)|^2$ we deduce the desired inequality $|(u|v)|^2 \leq (u|u)(v|v)$. □

5.2 Norm and distance

Definition 5.3 Let V be a linear space over K. A norm on V is any real-valued function defined on V (its value at x being denoted by $||x||$) which satisfies the following axioms:

(1) $||x|| \geq 0$, and $||x|| = 0$ if and only if $x = 0$

(2) $||kx|| = |k|||x||$

(3) $||x_1 + x_2|| \leq ||x_1|| + ||x_2||$ for all $x, x_1, x_2 \in V$ and all $k \in K$.

Any linear space on which a norm is defined is known as a *normed vector space.*

Let V be a normed vector space. Define $d : V \times V \to \mathbb{R}$,

$$d(x, y) = ||x - y|| \quad \forall x, y \in V.$$

It is easy to verify that

(4) $d(x, y) \geq 0$, and $d(x, y) = 0$ if and only if $x = y$

(5) $d(x, y) = d(y, x)$

(6) $d(x, y) \leq d(x, z) + d(z, y)$

for all $x, y, z \in V$.

Thus d is a *metric* on V, and V is a *metric space*. The value $d(x, y)$ is called the *distance* between x and y. We refer to $||x||$ as the *length* of the vector x, and call x a *unit vector* if $||x|| = 1$.

The following result is a very important one.

Theorem 5.4 *Every inner product space is a normed space with norm defined by*

$$||x|| = \sqrt{(x|x)}.$$

Proof. Since $(x|x) \geq 0$ for all $x \in V$, $||x|| \geq 0$. Moreover, $||x|| = 0 \Longleftrightarrow (x|x) = 0 \Longleftrightarrow x = 0$ and so axiom (1) from the definition of a norm is satisfied.

Now $||kx|| = \sqrt{(kx|kx)} = \sqrt{k\bar{k}(x|x)} = \sqrt{|k|^2||x||^2} = |k|||x||$, which proves (2).

Finally,

$$\begin{aligned} ||x+y||^2 &= (x+y)|x+y) = (x|x) + (x|y) + (y|x) + (y|y) = \\ &= (x|x) + (x|y) + (\overline{(x|y)} + (y|y) = \\ &= ||x||^2 + 2Re(x|y) + ||y||^2 \\ &\text{(where } Re \text{ signifies the real part)} \\ &\leq ||x||^2 + 2|(x|y)| + ||y||^2 \leq \\ &\leq ||x||^2 + 2\sqrt{(x|x)}\sqrt{(y|y)} + ||y||^2 \\ &\text{by the Schwarz'inequality} \\ &= ||x||^2 + 2||x||||y|| + ||y||^2 = (||x|| + ||y||)^2 \end{aligned}$$

This implies $||x + y \leq ||x|| + ||y||$ and so axiom (3) is also satisfied. $\square$

5.3 Orthonormal bases

Let V be an inner product space. Two vectors $x, y \in V$ are *orthogonal* if $(x|y) = 0$; this definition extends the well-known situation that has appeared in the study of $\mathcal{V}_3$.

A set of vectors $\{x_1, \dots, x_r\} \subset V$ is called *orthonormal* if

$$(x_i|x_j) = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ otherwise.} \end{cases}$$

Thus each x_i is of unit length, and each pair of vectors is orthogonal. Finding an orthonomal set in an inner product space is analogous to choosing a set of mutually perpendicular unit vectors in elementary vector analysis.

Theorem 5.5 *An orthonormal set in an inner product space V is linearly independent.*

Proof. Suppose that $\{x_1, \dots, x_r\}$ is the given orthonormal set and

$$k_1x_1 + \cdots + k_rx_r = 0.$$

Then for each i, $0 = (0|x_i) = (k_1x_1 + \cdots + k_rx_r|x_i) = k_1(x_1|x_i) + \cdots + k_i(x_i|x_i) + \cdots + k_r(x_r|x_i) = k_i$ since $(x_j|x_i) = 0$ unless $j = i$. Thus each coefficient k_i is zero, and so the vectors are linearly independent. □

Let now V_n be an n-dimensional inner product space and $B \subset V_n$ an orthonormal set with n elements. As a consequence of the above theorem we deduce that B is a basis of V_n, called an *orthonormal basis.*

Theorem 5.6 *Let $B = \{b_1, \dots, b_n\}$ be an orthonormal basis of V_n. The coordinates of a vector $v \in V_n$ relative to B are the numbers*

$$(v|b_1), \dots, (v|b_n).$$

Proof. Let $k_1, \dots, k_n \in K$ be the coordinates of v, that is $v = \sum_{i=1}^{n} k_i b_i$. Then $(v|b_j) = (\sum_{i=1}^{n} k_i b_i | b_j) = \sum_{i=1}^{n} k_i (b_i|b_j) = k_j,\ j = 1, \dots, n$. Thus the theorem is proved and we have a very simple procedure for calculating the coordinates of any vector relative to an orthonormal basis. $\square$

Finally, let $B = \{v_1, \dots, v_n\}$ be any basis of V_n. The following procedure enables us to construct an orthonormal basis in V.

Let $x_1 = \dfrac{v_1}{||v_1||}$. Then $\{x_1\}$ is an orthonormal set with one element. Take $x_2 = \dfrac{v_2 - cx_1}{||v_2 - cx_1||}$; note that $v_2 - cx_1 = v_2 - \dfrac{c}{||v_1||} v_1 \neq 0$ for all $c \in K$.
Clearly $||x_2|| = 1$; we shall determine $c \in K$ such that $(x_2|x_1) = 0$. In fact, we find immediately $c = \dfrac{(v_2|x_1)}{(x_1|x_1)}$. So $\{x_1, x_2\}$ is an orthonormal set and $c = (v_2|x_1)$ (since $(x_1|x_1) = 1$).

Now take $x_3 = \dfrac{v_3 - c_1 x_1 - c_2 x_2}{||v_3 - c_1 x_1 - c_2 x_2||}$. As above, we deduce that the set $\{x_1, x_2, x_3\}$ is orthonormal if $c_1 = (v_3|x_1)$ and $c_2 = (v_3|x_2)$.

Proceeding in this way, after n steps we arrive at an orthonormal set $\{x_1, \dots, x_n\}$ with n elements, that is to say, an orthonormal basis of V_n.

The above procedure for constructing an orthonormal basis of V from an arbitrary basis is known as the *Gram-Schmidt orthogonalisation process.*

Definition 5.7 Let W be a linear subspace of the inner product space V. The *orthogonal complement* of W is defined by

$$W^{\perp} = \{v \in V \mid (v|w) = 0, \ \forall\, w \in W\}.$$

Exercices

5.1 Let S be the set of solutions of the following systems and find bases in S and in the orthogonal complement $S^{\perp}$:

a) $\begin{cases} x_1 + x_2 + 2x_3 = 0 \\ 2x_1 + 2x_2 + x_3 = 0 \\ x_1 + x_2 - x_3 = 0 \end{cases}$.

b) $\begin{cases} 2x_1 + x_2 - x_3 + x_4 = 0 \\ x_1 + x_2 + 3x_3 - x_4 = 0 \\ x_2 + 7x_3 - 3x_4 = 0 \end{cases}$.

5.2 Let S be the set of solutions of the system

$$\begin{cases} x + y + t = 0 \\ 2x + y + z - 3v = 0 \\ x - y + 2z - 3t - 6v = 0 \end{cases}.$$

Find an orthonormal basis in S.

5.3 Verify that the following sets of vectors $\{v_1, v_2\}$ are orthogonal and complete them to form orthogonal bases of $\mathbb{R}^4$:

a) $v_1 = (1, 0, -2, 1)$ and $v_2 = (1, 1, 1, 1)$.

b) $v_1 = (1, 0, 2, -1)$ and $v_2 = (1, 2, 0, 1)$.

c) $v_1 = (1, -2, 2, -3)$ and $v_2 = (2, -3, 2, 4)$.

d) $v_1 = (1, 1, 1, 2)$ and $v_2 = (1, 2, 3, -3)$.

5.4 If V and W are linear subspaces of the inner product space U then:
a) $(V+W)^{\perp} = V^{\perp} \cap W^{\perp}$
b) $(V \cap W)^{\perp} = v^{\perp} + W^{\perp}$.

5.5 Let $\mathbb{R}^4$ be the inner product space with the canonical inner product. Apply the Gram-Schmidt orthogonalization to construct orthogonal bases for the subspaces spanned by the following sets of vectors:
a) $(1,2,2,-1)$, $(1,1,-5,3)$, $(3,2,8,-7)$.
b) $(1,1,-1,-2)$, $(5,8,-2,-3)$, $(3,9,3,8)$.

5.6 Find an orthonormal basis for the subspace spanned by the vectors $v_1 = (1,-1,1,-1)$, $v_2 = (5,1,1,1)$, $v_3 = (-3,-3,1,-3)$.

5.7 Show that the vectors $(1,0,1)$, $(1,1,0)$ and $(0,1,1)$ form a basis of $\mathbb{R}^3$ and find an orthonormal basis of this space, by using the Gram-Schmidt process.

5.8 For $f,g \in C[1,e]$ denote

$$(f|g) = \int_1^e f(x)g(x)(\ln x)\,\mathrm{d}x.$$

a) Prove that this defines an inner product in $C[1,e]$.
b) Find the norm of $f(x) = x$.
c) Find the polynomials of degree 1 which are orthogonal on the constant functions.

5.9 Let $p,q \in \mathbb{R}_2[X]$, $p = a_1X^2 + b_1X + c_1$, $q = a_2X^2 + b_2X + c_2$. Define

$$(p|q) = a_1a_2 + b_1b_2 + c_1c_2.$$

a) Prove that this defines an inner product in $\mathbb{R}_2[X]$.
b) Let $p_1 = 3X^2 + 2X + 1$, $p_2 = -X^2 + 2X + 1$, $p_3 = 3X^2 + 2X + 5$, $p_4 = 3X^2 + 5X + 2$. Find $p \in \mathbb{R}_2[X]$ which is equidistant with respect to p_1, p_2, p_3 and p_4. Find also the common distance.

5.10 Prove Pythagoras' Theorem: If V is an inner product space and $x, y \in V$ are orthogonal, then

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2.$$

Solutions

$\boxed{\mathbf{5.1}}$ a) The system has the determinant zero, and since $\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} \neq 0$, we choose x_2, x_3 the primary unknowns and $x_1 = \alpha$ the secondary unknown. We get $x_2 = -\alpha$, $x_3 = 0$ so $S = \{(\alpha, -\alpha, 0) \mid \alpha \in \mathbb{R}\}$ with $\{(1, -1, 0)\}$ a basis. The orthogonal complement is $S^\perp = \{(a, b, c) \mid (a, b, c) \perp (1, -1, 0)\}$. We obtain $a - b = 0$, so $S = \{(a, a, c) \mid a, c \in \mathbb{R}\} = \{a(1, 1, 0) + c(0, 0, 1) \mid a, c \in \mathbb{R}\}$. A basis in $S^\perp$ is $\{(1, 1, 0), (0, 0, 1)\}$.
b) A basis in S is, for example $\{(4, -7, 1, 0), (-2, 3, 0, 1)\}$ and a basis in $S^\perp$ is $\{(1, 0, -4, 2), (0, 1, 7, -3)\}$.

$\boxed{\mathbf{5.2}}$ The solution set of the system is $S = \{(-\alpha+\beta+3\gamma, \alpha - 2\beta - 3\gamma, \alpha, \beta, \gamma) \mid \alpha, \beta \in \mathbb{R}\}$, with a basis $\{v_1, v_2, v_3\}$, where $v_1 = (-1, 1, 1, 0, 0)$, $v_2 = (1, -2, 0, 1, 0)$, $v_3 = (3, -3, 0, 0, 1)$. To get an orthonormal basis we use the Gram-Schmidt procedure. First, $x_1 = v_1$. Then $x_2 = v_2 - c_1 x_1 = (0, -1, 1, 1, 0)$ $\left(c_1 = \dfrac{(v_2|x_1)}{(x_1|x_1)} = -1\right)$. Finally, $x_3 = v_3 - c_1 x_1 - c_2 x_2 = (1, 0, 1, -1, 1)$. This basis is orthogonal, in order to get an orthonormal one, we divide each vector to its own norm: $x_1' = \dfrac{1}{\sqrt{3}}(-1, 1, 1, 0, 0)$, $x_2' = \dfrac{1}{\sqrt{3}}(0, -1, 1, 1, 0)$ and $x_3' = \dfrac{1}{2}(1, 0, 1, -1, 1)$.

$\boxed{\textbf{5.3}}$ a) Is clear that $(v_1|v_2) = 0$. We need two more vectors to form a basis. Let $v = (a, b, c, d)$. From $(v_1|v) = 0$ and $(v_2|v) = 0$ we have $\begin{cases} a - 2c + d = 0 \\ a + b + c + d = 0 \end{cases}$ so $a = 2c - d$, $b = -3c$. Choosing $c = 0$, $d = 1$ we get $v_3 = (-1, 0, 0, 1)$. Now v_4 has to be orthogonal also on v_3, that gives $c = d$. Choosing $c = d = 1$ we have $v_4 = (1, -3, 1, 1)$. Obviously, the solution is not unique.
b) For example, they may be completed by adjoining the vectors $v_3 = (1, -1, 0, 1)$ and $v_4 = (-1, 0, 1, 1)$.
c) $v_3 = (2, 2, 1, 0)$ and $v_4 = (5, -2, -6, -1)$.
d) $v_3 = (1, -2, 1, 0)$ and $v_4 = (25, 4, -17, -6)$.

$\boxed{\textbf{5.4}}$ a) Let $x \in (V + W)^{\perp}$. Then, for any $v \in V$ and $w \in W$, $(x|v + w) = 0$. Taking $w = 0$ follows that $(x|v) = 0$, for any $v \in V$, that is $x \in V^{\perp}$. Taking $v = 0$ follows $x \in W^{\perp}$. So $(V + W)^{\perp} \subset V^{\perp} \cap W^{\perp}$. Conversely, let $x \in V^{\perp} \cap W^{\perp}$. Let $y = v + w \in V + W$. Then $(x|y) = (x|v) + (x|w) = 0$ so $x \in (V + W)^{\perp}$. b) In the relation (a) we replace V by $V^{\perp}$ and W by $W^{\perp}$. We get $(V^{\perp} + W^{\perp})^{\perp} = (V^{\perp})^{\perp} \cap (W^{\perp})^{\perp}$ that is $(V^{\perp} + W^{\perp})^{\perp} = V \cap W$ and further $V^{\perp} + W^{\perp} = (V \cap W)^{\perp}$.

$\boxed{\textbf{5.5}}$ a) $(1, 2, 2, -1), (2, 3, -3, 2), (2, -1, -1, -2)$.
b) $(1, 1, -1, -2)$, $(2, 5, 1, 3)$.

$\boxed{\textbf{5.6}}$ A basis of the generated subspace is v_1, v_2. Applying the orthogonalisation, we obtain the orthogonal basis $u_1 = (1, -1, 1, -1)$ and $u_2 = (4, 2, 0, 2)$. An orthonormal basis is w_1, w_2, where $w_1 = u_1 / \|u_1\| = 1/2(1, -1, 1, -1)$ and $w_2 =$

$u_2 / \|u_2\| = 1/\sqrt{6}(2, 1, 0, 1)$.

$\boxed{\mathbf{5.7}}$ An orthonormal basis is formed by the three vectors $1/\sqrt{2}(1, 0, 1)$, $1/\sqrt{6}(1, 2, -1)$ and $1/\sqrt{3}(-1, 1, 1)$.

$\boxed{\mathbf{5.8}}$ b) $\|f\| = \dfrac{1}{3}\sqrt{2e^3 + 1}$. c) $p(x) = a\left(x - \dfrac{e^2 + 1}{4}\right)$, $a \in \mathbb{R}$.

$\boxed{\mathbf{5.9}}$ b) $p = X^2 + 3X + 3$. The common distance is 3.