CHAPTER 3

Lines and planes in space

3.1 Planes in space

We shall use the language of the vectors to introduce the basic concepts of solid analytic geometry. We assume that a fixed Cartesian coordinate system in space defined by the origin O and the triad $\{\overrightarrow{i}, \overrightarrow{j}, \overrightarrow{k}\}$ has been chosen. Every point M has a position vector $\overrightarrow{r}$; the components of $\overrightarrow{r}$ are the coordinates of M, that is, we have $M(x, y, z)$ and $\overrightarrow{r} = x\overrightarrow{i} + y\overrightarrow{j} + z\overrightarrow{k}$.

1) Plane determined by a point and a normal vector Let $M_0(x_0, y_0, z_0)$ be a point in space and let $\overrightarrow{n} = a\overrightarrow{i} + b\overrightarrow{j} + c\overrightarrow{k}$, $\overrightarrow{n} \neq 0$. Let P be the plane that passes through M_0 and is perpendicular to $\overrightarrow{n}$.

Let $M(x, y, z)$ be an arbitrary point of P. Then $\overrightarrow{r} - \overrightarrow{r_0}$ is perpendicular to $\overrightarrow{n}$, that is $\overrightarrow{n}(\overrightarrow{r} - \overrightarrow{r_0}) = 0$.

Since $\overrightarrow{r}-\overrightarrow{r_0}=(x-x_0)\overrightarrow{i}+(y-y_0)\overrightarrow{j}+(z-z_0)\overrightarrow{k}$, we obtain

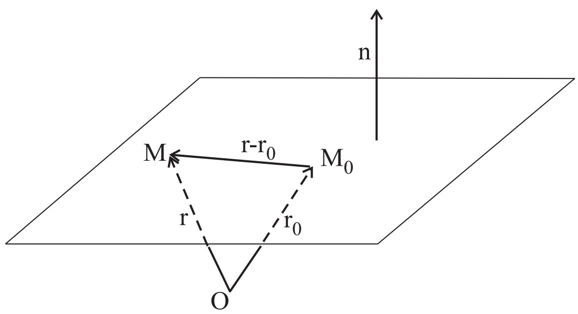
$$a(x-x_0)+b(y-y_0)+c(z-z_0)=0.$$

This is the equation of the plane P. If we denote $d=-ax_0-by_0-cz_0$, then it reads:

$$ax+by+cz+d=0.$$

This is the general form of the equation of a plane. The vector $\overrightarrow{n}=a\overrightarrow{i}+b\overrightarrow{j}+c\overrightarrow{k}$ is called *normal* to the plane.

In particular, the plane xOy passes through the origin and $\overrightarrow{k}$ is normal to it. Hence we can take $x_0=y_0=z_0,\ a=b=0,\ c=1$.
Therefore the equation of the plane xOy is simply $z=0$.



2) Plane determined by three non-collinear points. Let $M_i(x_i,y_i,z_i),\ i=1,2,3$ be three non-collinear points and let P be the plane determined by them. Let $M(x,y,z)$ be an arbitrary point of P. Then M,M_1,M_2,M_3 are coplanar and hence

$$\begin{vmatrix} 1 & x & y & z \\ 1 & x_1 & y_1 & z_1 \\ 1 & x_2 & y_2 & z_2 \\ 1 & x_3 & y_3 & z_3 \end{vmatrix} = 0.$$

This is the equation of the plane P.

3) Plane determined by a point and two non-collinear vectors.

Let P be the plane that passes through a given point $M_0(x_0, y_0, z_0)$ and is parallel to two non-collinear given vectors $\overrightarrow{v_i} = x_i \overrightarrow{i} + y_i \overrightarrow{j} + z_i \overrightarrow{k}$, $i = 1, 2$.

Let $M(x, y, z)$ be an arbitrary point of P. Then the vectors $\overrightarrow{r} - \overrightarrow{r_0}, \overrightarrow{v_1}, \overrightarrow{v_2}$ are coplanar, that is $(\overrightarrow{r} - \overrightarrow{r_0}, \overrightarrow{v_1}, \overrightarrow{v_2}) = 0$. Thus the equation of the plane P can be written in the form:

$$\begin{vmatrix} x - x_0 & y - y_0 & z - z_0 \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = 0.$$

4) An important result is the following:

The equation of a plane passing through the line of intersection of the planes

(1) $a_1x + b_1y + c_1z + d_1 = 0$

(2) $a_2x + b_2y + c_2z + d_2 = 0$

is of the form

(3) $\quad a_1x + b_1y + c_1z + d_1 + \lambda(a_2x + b_2y + c_2z + d_2) = 0, \ \lambda \in \mathbb{R}.$

Indeed, (3) is the equation of a plane P. The coordinates of any point of the line verify (1) and (2) - and hence also (3). Thus the line is contained in the plane P.

3.2 Straight lines in space

Consider a direction in space, determined by the vector

$$\overrightarrow{v} = l\,\overrightarrow{i} + m\,\overrightarrow{j} + n\,\overrightarrow{k} \neq \overrightarrow{0}.$$

The numbers (l, m, n) are called the *direction ratios* of this direction. Clearly any other numbers proportional to them are also direction ratios for the same direction.

Now suppose that $\overrightarrow{v}$ is a unit vector, that is, $||\overrightarrow{v}|| = 1$. Then $l^2 + m^2 + n^2 = 1$. On the other hand, $l = \overrightarrow{v} \cdot \overrightarrow{i} = \cos\alpha$, $m = \overrightarrow{v} \cdot \overrightarrow{j} = \cos\beta$, $n = \overrightarrow{v} \cdot \overrightarrow{k} = \cos\gamma$ where α, β, γ are the angles between $\overrightarrow{v}$ and the axes. Hence the direction ratios are now $(\cos\alpha, \cos\beta, \cos\gamma)$. They are called *direction-cosines.*

Since $l^2+m^2+n^2 = 1$, we have $\cos^2\alpha+\cos^2\beta+\cos^2\gamma = 1$.

1) Line determined by a point and a vector.
Consider the line d determined by the point $M_0(x_0, y_0, z_0)$ and the vector $\overrightarrow{v} = l\,\overrightarrow{i} + m\,\overrightarrow{j} + n\,\overrightarrow{k} \neq \overrightarrow{0}$. Let $M(x, y, z)$ be an arbitrary point of d.

The vectors $\overrightarrow{r} - \overrightarrow{r_0}$ and $\overrightarrow{v}$ are collinear, hence $\overrightarrow{r} - \overrightarrow{r_0} = t\overrightarrow{v}$, with $t \in \mathbb{R}$. Thus we obtain the *parametric equations* of the line d:

$$x = x_0 + lt,\ y = y_0 + mt,\ z = z_0 + nt,\ t \in \mathbb{R}.$$

By eliminating the parameter t between these equations, we deduce the canonical equations of d:

$$\frac{x - x_0}{l} = \frac{y - y_0}{m} = \frac{z - z_0}{n}$$

Since $\overrightarrow{v} \neq \overrightarrow{0}$, at least one denominator is nonnull. If a denominator equals 0, the corresponding numerator must also equal 0.

Example 3.2.1 For the x-axis we can take $M_0 = 0$ and $\overrightarrow{v} = \overrightarrow{i}$. Hence $x_0 = y_0 = z_0 = 0,\ l = 1,\ m = n = 0.$

The canonical equations are $\dfrac{x}{1} = \dfrac{y}{0} = \dfrac{z}{0}$. They are equivalent to $\begin{cases} y = 0 \\ z = 0 \end{cases}$.

2) Equations of the line joining the points $M_0(x_0, y_0, z_0)$ and $M_1(x_1, y_1, z_1)$

Let $\overrightarrow{r_0}$ and $\overrightarrow{r_1}$ be the position vectors of these points. Then the line is determined by the point M_0 and the vector $\overrightarrow{r_1} - \overrightarrow{r_0}$. Consequently, we can take $(x_1 - x_0,\ y_1 - y_0,\ z_1 - z_0)$ as direction-ratios.

The canonical equations of the line will be

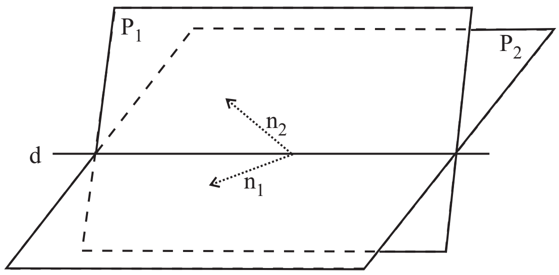
$$\frac{x - x_0}{x_1 - x_0} = \frac{y - y_0}{y_1 - y_0} = \frac{z - z_0}{z_1 - z_0}.$$

3) Line determined by the intersection of two planes

Let d be the intersection of the planes P_1 and P_2. Then the equations of d are

$$\begin{cases} a_1x + b_1y + c_1z + d_1 = 0 \\ a_2x + b_2y + c_2z + d_2 = 0 \end{cases}$$

The normal vectors to P_1, respectively P_2, are $\overrightarrow{n_1} = a_1\overrightarrow{i} + b_1\overrightarrow{j} + c_1\overrightarrow{k}$ and $\overrightarrow{n_2} = a_2\overrightarrow{i} + b_2\overrightarrow{j} + c_2\overrightarrow{k}$.

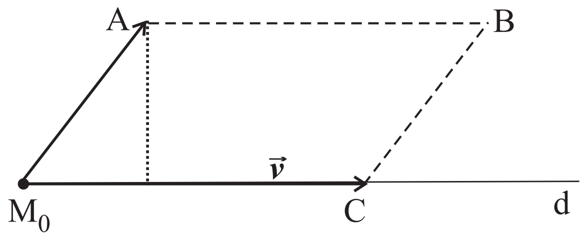


They are both perpendicular to d, so d is parallel to $\overrightarrow{n} = \overrightarrow{n_1} \times \overrightarrow{n_2}$. This enables us to take as direction-ratios of d the components of $\overrightarrow{n}$, that is

$$\left(\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}, \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}, \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \right).$$

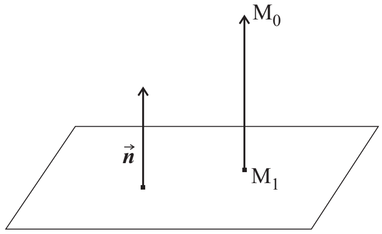
3.3 Distance from a point to a line. Distance from a point to a plane

1) Consider a line d determined by a point M_0 and a vector $\overrightarrow{v}$. The distance from the point A to the line d equals the length of the height of the parallelogram M_0ABC.



Hence $dist(A,d) = \dfrac{||\overrightarrow{v} \times \overrightarrow{M_0A}||}{||\overrightarrow{v}||}$

2) Consider the plane $P : ax + by + cz + d = 0$ and the point $M_0(x_0, y_0, z_0)$. Let M_1 be the projection of M_0 on the plane P.
The vector $\overrightarrow{n} = a\overrightarrow{i} + b\overrightarrow{j} + c\overrightarrow{k}$ is normal to P.



Let (x_1, y_1, z_1) be the coordinates of M_1. Then $ax_1 + by_1 + cz_1 + d = 0$. We have also $\overrightarrow{M_1M_0} = (x_0 - x_1)\overrightarrow{i} + (y_0 - y_1)\overrightarrow{j} + (z_0 - z_1)\overrightarrow{k}$. Therefore

$\overrightarrow{n} \cdot \overrightarrow{M_1M_0} = a(x_0 - x_1) + b(y_0 - y_1) + c(z_0 - z_1) =$
$= ax_0 + by_0 + cz_0 + d - (ax_1 + by_1 + cz_1 + d) =$
$= ax_0 + by_0 + cz_0 + d.$
On the other hand,

$|\overrightarrow{n} \cdot \overrightarrow{M_1M_0}| = ||\overrightarrow{n}|| \cdot ||\overrightarrow{M_1M_0}|| = \sqrt{a^2 + b^2 + c^2}\text{dist}(M_0, P).$

It follows that

$$\mathrm{dist}(M_0, P) = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}.$$

Exercices

3.1 Write the equations of the straight line d that passes through the point $M(3, -1, 0)$ and is parallel to the line $l : \begin{cases} x - 2y + 7 = 0 \\ x + y + z - 6 = 0 \end{cases}$.

3.2 Write the equation of the plane (P) such that:
a) $M(-1, 2 - 3) \in (P)$ and $0z \perp (P)$
b) $M(-1, 2 - 3) \in (P)$ and $0z \parallel (P)$, $0x \parallel (P)$.

3.3 Write the equation of the plane (Q) knowing that it is symmetrical to the plane $(P) : x - 3y + 2z - 1 = 0$ with respect to the point $M(0, -1, 1)$.

3.4 Let $A(2, 2, 2)$, $B(0, 1, 1)$, $C(1, 1, 0)$ and $D(1, 0, 1)$. Find the equations and the length of the height of the tetrahedron $ABCD$ with the basis BCD.

3.5 Let $A(3, -1, 3)$, $B(5, 1, -1)$, $C(0, 4, -3)$. Find the parametric and canonical equations of the lines D_1 and D_2 if:
a) $D_1 = AB$ and $D_2 = BC$
b)D_1 is parallel to AC and passes through B and D_2 is perpendicular to D_1 and passes through C.

3.6 Considering A, B and C from the exercise 3.5, calculate the distances between these three points and find the angles formed by AB, AC and BC.

3.7 Considering A and B from the exercise 3.5, find the equation of a plane with respect to which A and B are symmetrical.

3.8 Find the equation of a plane which passes through the point M and is parallel to the plane (P) if:
a)$M(2,-1,3)$ and $(P): x-3y+5z+2=0$
b)$M(0,-2,4)$ and $(P): 7x+4y-3z-1=0$
c)$M(1,0,-1)$ and $(P): 2y-5x-11z=0$.

3.9 Find the equation of a plane (P) if:
a) $M(2,3,-5) \in (P)$ and $OM \perp (P)$
b) $A(2,1,-6)$ and $B(6,-1,-2)$ are symmetrical about the plane (P).

3.10 Write the equations of three planes that contain $M(3,2,-1)$ and each contains a different coordinate axis.

3.11 Find the equation of a plane which passes through A and is perpendicular to the planes (P_1) and (P_2) if:
a)$A(-1,1,0)$, $(P_1): x-2y+z-5=0$ and $(P_2): y-5z+2=0$
b)$A(1,0,1)$, $(P_1): 3x+y-1=0$ and $(P_2): x+y-z-1=0$

3.12 Write the equations of three planes that contain $A(2,-1,-1)$ and $B(3,1,2)$ and each, is parallel to a different coordinate axis.

3.13 Find the equation of a plane which contains the point A and is perpendicular to AB, if:

a) $A(1,2,-1)$, $B(2,3,5)$
b) $A(1,3,2)$, $B(-3,-1,0)$
c) $A(2,0,1)$, $B(1,1,-1)$.

3.14 A plane cuts, on the coordinate axes, segments equal to $3, 10$ and 5. Find the equation of the plane and the angles formed by the plane and the axes.

3.15 Find the equation of a plane determined by the lines

$$D_1\colon \begin{cases} x+y-3z=0 \\ 2x+3y-z-1=0 \end{cases} \text{ and } D_2\colon \begin{cases} x+5y+4z-3=0 \\ x+2y+2z-1=0 \end{cases}.$$

3.16 Write the equation of a plane which contains $M(-1,1,1)$ and is perpendicular to the line D, if:

a) $D: \dfrac{x-2}{3}=\dfrac{y}{2}=\dfrac{z+1}{-1}$

b) $D: \dfrac{x}{4}=\dfrac{y-2}{-4}=\dfrac{z-3}{5}$

c) D: $\begin{cases} x+y=0 \\ x+y-2z+1=0 \end{cases}.$

3.17 Let D_1, D_2 be two lines parallel to the vectors $d_1=(-1,0,1)$ and $d_2=(1,1,0)$. Find:

a) the angle between D_1 and D_2

b) the parametric equations of the line D_3 perpendicular to D_1 and D_2, which passes through $M(3,2,1)$.

3.18 Calculate the distance between the point $A(3,-1,1)$ and the line D_1 if:

a) D_1: $\begin{cases} 2x-y+2z-3=0 \\ x-y-3z+2=0 \end{cases}$

b)D_1: $\dfrac{x-1}{4}=\dfrac{y}{-5}=\dfrac{z+2}{3}.$

3.19 Write the equation of a plane which passes through the point $M(1,-1,1)$ and is perpendicular to the line D if:

a) D: $\dfrac{x-3}{2}=\dfrac{y}{3}=\dfrac{z+1}{-1}$

b) D: $\begin{cases} x-z+3=0 \\ 2x-y=0 \end{cases}$

3.20 A plane contains the point $A(1,0,1)$ and the line D. Find the equation of the plane if:

a) D: $\begin{cases} x=2-3t \\ y=4+t \\ z=1-2t \end{cases}$

b) D: $\dfrac{x}{-2}=\dfrac{y-1}{4}=z-5$

c) D: $\begin{cases} x+z+1=0 \\ x-2y+z-3=0 \end{cases}$

3.21 We consider the planes P_1, P_2 and P_3 such that $A(-1,-2,2) \in P_1$ and the vector normal to P_1 is $(1,-2,2)$, the plane P_2 is perpendicular to the line D: $\dfrac{x}{2} = \dfrac{y+7}{-1} = \dfrac{z-1}{-2}$ and contains the point $B(1,1,1)$ and P_3: $2x+2y+z=2$.

1) Find the equations of P_1 and P_2.
2) Show that each of two planes are perpendicular.
3) Find the intersection of the planes.
4) Calculate the distance from $A(2,4,7)$ to P_1.

3.22 Find the equation of a plane which contains the symmetric points of $A(2,3,-1)$, $B(1,2,4)$ and $C(0,1,-1)$ with respect to the plane P : $x-y+2z+2=0$.

3.23 Find the projection of $M(2,1,1)$, on the plane $P : x+y+3z+5=0$ and calculate the distance from M to P.

3.24 Find the equations of two planes P_1 and P_2 if both pass through the line D: $\begin{cases} 2x+y-3z+2=0 \\ 5x+5y-4z+3=0 \end{cases}$, $P_1 \perp P_2$ and P_1 contains $M(4,-3,1)$.

3.25 Find the position of the line D relative to the plane P if:

a) D: $\begin{cases} x=t \\ y=1+2t \\ z=-6t \end{cases}$ and $P : 4x+y+z=4$

b) D: $\begin{cases} x=13+8t \\ y=1+2t \\ z=4+3t \end{cases}$ and $P : 4x+y+z=4$.

3.26 Find the distance between two lines D_1 and D_2 and the equation of the common perpendicular if it exists, for:

a) D_1: $\dfrac{x-1}{-5} = y-2 = z$ D_2: $\begin{cases} x+2z=4 \\ y=0 \end{cases}$

b) D_1: $\dfrac{x}{3} = \dfrac{y-1}{2} = z-5$ and D_2: $\begin{cases} x=1+3t \\ y=2t \\ z=1+t \end{cases}$

c) D_1: $\dfrac{x-1}{3} = \dfrac{y+2}{2} = z - 4$ and D_2: $\begin{cases} x = 1 + t \\ y = 2t - 2 \\ z = 4 + 5t \end{cases}$.

Solutions

$\boxed{3.1}$ We find first the direction vector of of l, for instance $\vec{l} = \vec{n}_1 \times \vec{n}_2$, where $\vec{n}_1 = (1, -2, 0)$ and $\vec{n}_2 = (1, 1, 1)$ are the normals to the planes that determine l. So $\vec{l} = (-2, -1, 3)$ and the equations of the line d are $\dfrac{x-3}{-2} = \dfrac{y+1}{-1} = \dfrac{z}{3}$ or, in another form, $d : \begin{cases} x - 2y - 5 = 0 \\ 3y + z + 3 = 0 \end{cases}$.

$\boxed{3.2}$ a) $0z$ has the direction vector $\vec{k}$ and is normal to the requested plane (P). The equation is $0 \cdot (x+1) + 0 \cdot (y-2) + 1 \cdot (z+3) = 0$, that is $z + 3 = 0$. b) The plane is determined by a point and two vectors, $\begin{vmatrix} x+1 & y-2 & z+3 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} = 0$, that is $y - 2 = 0$. (In fact, the plane is perpendicular to $0y$).

$\boxed{3.3}$ We choose three points that belong to the plane (P), for instance $A(1, 0, 0)$, $B(0, 1, 2)$ and $C(-1, 0, 1)$. We determine their symmetrical points A_1, B_1, C_2 with respect to M, from the fact that M is the middle of the segments $[AA_1]$, $[BB_1]$, $[CC_1]$, getting $A_1(-1, -2, 2)$, $B_1(0, -3, 0)$, $C_1(1, -2, 1)$. The plane (Q) is determined by these three points: $x - 3y + 2z - 9 = 0$.

$\boxed{\mathbf{3.4}}$ The plane BCD has the equation $x+y+z-2=0$, so the normal is $\vec{n}=(1,1,1)$. The equations of the height from A are $x=y$ and $x=z$ and the intersection point between the height and the plane BCD is $H(\dfrac{2}{3},\dfrac{2}{3},\dfrac{2}{3})$. The length of the height is $AH=\frac{4}{3}\sqrt{3}$.

$\boxed{\mathbf{3.5}}$ a) $D_1 = AB = \frac{x-3}{2} = \frac{y+1}{2} = \frac{z-3}{-4}$
b) $D_1 \parallel AC$ means the direction of D_1 is $\bar{D_1} = \bar{AC} = (-3,5,-6)$, then $D_1 : \frac{x-5}{-3} = \frac{y-1}{5} = \frac{z+1}{-6}$.
Let $CM \perp D_1$ and $M(a,b,c) \in D_1$, then $D_2 = CM$. We know D_2 is perpendicular to D_1, so $(-3,5,-6)\cdot(a,b-4,c+3)=0$. Also $M \in D_1 \Leftrightarrow \frac{a-5}{-3} = \frac{b-1}{5} = \frac{c+1}{-6}$ and after finding a, b, and c from this system, we obtain the line $D_2 = CM$.

$\boxed{\mathbf{3.6}}$ $d(A,B) = \sqrt{(5-3)^2+(1+1)^2+(-1-3)^2} = 2\sqrt{6}$, etc.

Let $\alpha = \sphericalangle(AB,AC)$, then we have $\cos\alpha = \frac{\vec{AB}\cdot\vec{AC}}{\|\vec{AB}\|\cdot\|\vec{AC}\|}$, if $\vec{AB}=(2,2,-4)$ and $\vec{AC}=(-3,5,-6)$.

$\boxed{\mathbf{3.7}}$ Consider $M(x,y,z)$ an arbitrary point on the plane P. Then $\|AM\| = \|MB\|$ which implies $P : x+y-2z-2=0$.

$\boxed{\mathbf{3.8}}$ a) Let P_1 be the plane parallel to P, then the vector normal to P_1 is the vector normal to P, $\vec{n}=(1,-3,5)$. The equation of the plane is P_1: $x-2-3(y+1)+5(z-3)=0$.

$\boxed{\mathbf{3.9}}$ c) Consider $A(a,0,0) \in Ox$ and $B(0,0,a) \in Oz$, $\|OA\| =$

$\|OB\|$. We write the plane P in two ways:

$$(AM_1M_2) : \begin{vmatrix} x & y & z & 1 \\ a & 0 & 0 & 1 \\ 3 & 2 & 1 & 1 \\ 6 & 6 & 8 & 1 \end{vmatrix} = 0, \qquad (BM_1M_2) : \begin{vmatrix} x & y & z & 1 \\ 0 & 0 & a & 1 \\ 3 & 2 & 1 & 1 \\ 6 & 6 & 8 & 1 \end{vmatrix} = 0$$

and obtain $P : 2x - 5y + 2z + 2 = 0$

3.10 $(MOx) : \begin{vmatrix} x & y & z \\ 1 & 0 & 0 \\ 3 & 2 & -1 \end{vmatrix} = 0$ and obtain $(MOx) : y + 2z = 0$, etc.

3.11 a) The normals $\vec{n}_1$ and $\vec{n}_2$ of the planes P_1 and P_2 are parallel to the plane P, so we get $P : \begin{vmatrix} x+1 & y-1 & z \\ 1 & -2 & 1 \\ 0 & 1 & -5 \end{vmatrix} = 0$, $9x + 5y + z + 4 = 0$.

3.12 $P_1 : \begin{vmatrix} x-3 & y-1 & z-2 \\ 1 & 0 & 0 \\ 1 & 2 & 3 \end{vmatrix} = 0$, etc.

3.13 a) The normal of the plane is the vector $\vec{AB} = (1, 1, 6)$, so the equation of the plane, which contains A, is $x - 1 + y - 2 + 6(z + 1) = 0$.

3.14 The equation of the plane is P: $10x + 3y + 6z - 30 = 0$. Consider the normal to the plane $\vec{n}$, and for the coordinates axes we have the unit vectors $\vec{i}$, $\vec{j}$, $\vec{j}$. By denoting $\alpha = \sphericalangle(\vec{i}, \vec{n})$, we have $\sin\alpha = \dfrac{\vec{i}\cdot\vec{n}}{\left\|\vec{i}\right\|\cdot\|\vec{n}\|} = \dfrac{10}{\sqrt{145}}$, etc.

$\boxed{\mathbf{3.15}}$ $P: x+2y+2z-1=0$

$\boxed{\mathbf{3.16}}$ c) The direction of the line D is

$$d = \left(\begin{vmatrix} 1 & 0 \\ 1 & -2 \end{vmatrix}, \begin{vmatrix} 0 & 1 \\ -2 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}\right) = (-2, 2, 0)$$

and this line is normal to the plane. The equation of the plane is $P: x-y+2=0$

$\boxed{\mathbf{3.17}}$ a) $\alpha = 2\pi/3$
b)

$$\bar{d}_1 \times \bar{d}_2 = \begin{vmatrix} i & j & k \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = -i+j-k.$$

Then the equation of the line is D: $\begin{cases} x = 3-t \\ y = 2+t \\ z = 1-t \end{cases}$

$\boxed{\mathbf{3.18}}$ a) $\vec{d_1} = (-5,-8,1)$, so $\left\|\vec{d_1}\right\| = 3\sqrt{10}$, then the distance is calculated from $d(A, D_1) = \frac{\left\|\vec{MA}\times\vec{d_1}\right\|}{\left\|\vec{d_1}\right\|}$.

$\boxed{\mathbf{3.19}}$ b) $P: x+2y+z=0$

$\boxed{\mathbf{3.20}}$ c) The pencil of planes which pass through D is $x-2y+z-3+\lambda(x+z+1)=0$ and we need the plane which contains A, so, $\lambda = 1/3$ then the plane is $P: 4x-6y+4z-8=0$.

$\boxed{\textbf{3.21}}$ 1) $P_1 : x - 2y + 2z - 7 = 0$, $P_2 : 2x - y - 2z + 1 = 0$.
2) We verify that the scalar product between the normals of two planes is zero.
3) $P_1 \cap P_2 \cap P_3 = M(1, -1, 2)$.
4) $d(A, P_1) = 1/3$.

$\boxed{\textbf{3.22}}$ Consider A', B', C' the symmetrical points of A, B, and C with respect to the plane P. We take $C_0 = CC' \cap P$, $C_0 \in P$ and obtain $C_0(1/6, 5/6, -2/3)$ after we solve the system obtained from the equations of the line $CC' : \frac{x}{1} = \frac{y-1}{-1} = \frac{z+1}{2}$ and the plane P. Then we find the coordinates of C' by knowing $\|CC_0\| = \left\|C_0C'\right\|$. By using the same procedure we find A' and B'.

$\boxed{\textbf{3.23}}$ Consider $M' \in P$ the projection of M on P, $MM' : x - 2 = y - 1 = \frac{z-1}{3}$, then $M'(1, 0, -2)$ and $d(M', P) = 11/\sqrt{11}$.

$\boxed{\textbf{3.24}}$ The pencil of planes passing through D is
$P_\mu : 2x + y - 3z + 2 + \mu(5x + 5y - 4z + 3) = 0$. $M \in P_1$ and $P_1 \subset P_\mu \Longrightarrow M \in P_\mu$ which gives us $\mu = -1$, so $P_1 : 3x + 4y - z + 1 = 0$.
Let $P_2 : ax + by + cz + d = 0$, from $P_1 \perp P_2$ we have $3a + 4b - c = 0$ and considering also $P_2 \subset P_\mu$ we obtain the relations $a = 2 + 5\mu, b = 1 + 5\mu, c = -3 - 4\mu$. Then, for $\mu = -1/3$, we obtain $P_2 : x - 2y - 5z + 3 = 0$.

$\boxed{\textbf{3.25}}$ a) Consider the system $\begin{cases} 2x = y - 1 \\ -6x = z \\ 4x + y + z = 4 \end{cases}$

which have the determinant zero and the rank of the corresponding matrix 2 and observe the system is inconsistent, so $D \parallel P$. This could be, also, observed if we check that the normal to the plane is perpendicular to the line D.

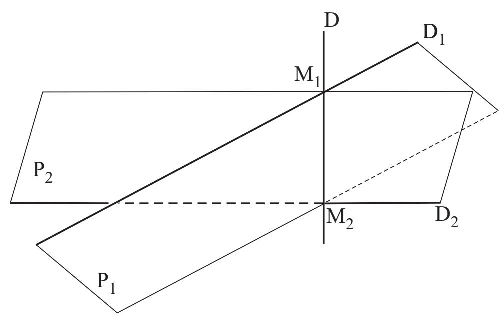
$\boxed{\mathbf{3.26}}$ a) The direction of the common perpendicular of D_1 and D_2 is

$$\vec{d} = (-5,1,1)\times(2,0,-1) = \begin{vmatrix} i & j & k \\ -5 & 1 & 1 \\ 2 & 0 & -1 \end{vmatrix} = -i - 3j - 2k = (-1,-3,-2)$$

The equation of the plane which contains D_1 and D is

$$P_1 : \begin{vmatrix} x-1 & y-2 & z \\ -5 & 1 & 1 \\ -1 & -3 & -2 \end{vmatrix} = 0 \Leftrightarrow x - 11y + 16z + 21 = 0.$$

Similarly, the equation of the plane which contains D_2 and D is $P_2 : -3x + 5y - 6z + 12 = 0$. The common perpendicular is $D : \begin{cases} P_1 \\ P_2 \end{cases}$.



The distance is $d(D_1, D_2) = \|M_1M_2\|$, where $\{M_1\} = D_1 \cap D$ and $\{M_2\} = D_2 \cap D$.

b) $D_1 \parallel D_2$, let $A_1 \in D_1$ and $A_2 \in D_2$, then

$$d(D_1, D_2) = d(A_1, D_2) = \frac{\left\|\vec{d_2} \times \vec{A_1A_2}\right\|}{\left\|\vec{d_2}\right\|}.$$

c) $D_1 \cap D_2 = (1, -2, 4)$, so the distance is zero.