# Lecture #2 Sorting. Heapsort

## Fundamental Algorithms

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# Agenda

1. Lecture #1 review
2. Sorting Problem
3. Heapsort

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# Correctness

- How do we know an algorithm is correct?
- Testing never shows an algorithm is correct. It can only show it is INCORRECT (by finding bugs)
- Absence of evidence != Evidence of absence
- Correctness MUST be proven!
    - if the *pre-conditions* are satisfied, the *post-conditions* will be true when the algorithm *terminates*
    - *partial correctness* = whenever preconditions are satisfied, the postconditions are true
    - *total correctness* = partial correctness + termination condition

# Complexity

- Evaluate **time** and **space** requirements
- **Time** as an estimation of the **amount of work** done
  - As an expression of #*atomic* operations
  - Depends on the *size* of the input data (n)
  - Depends on *case* (best, worst, average to be evaluated)
- **Space** requirements as an expression of **supplementary** memory
  - Need algorithms using *constant extra space*
  - Sometimes, *lgn* extra space is accepted

# Complexity

- Time = amount of work = as a function of $n$ (size of input data)
- We need its asymptotic growth
- Lower bound $\Omega$ depends on the **problem**
- Upper bound $O$ depends on the **algorithm**
- **Efficiency** – compare algorithms (their corresponding $O$ function) among each other – one is more/less efficient
- **Optimality** – $\Omega = O$ in the *worst case* scenario - compare an *algorithm* with the *problem* lower bound

# Stability

- The property of an algorithm to preserve the relative order of equal elements from the input (initial/original data) in the output (final data/result)
- Desired property, when and why?

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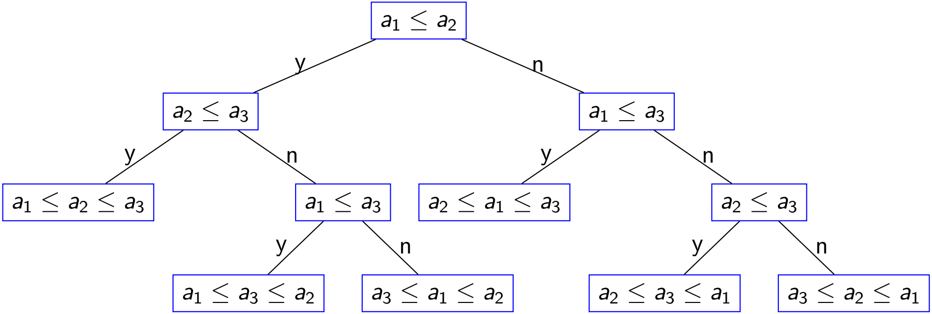
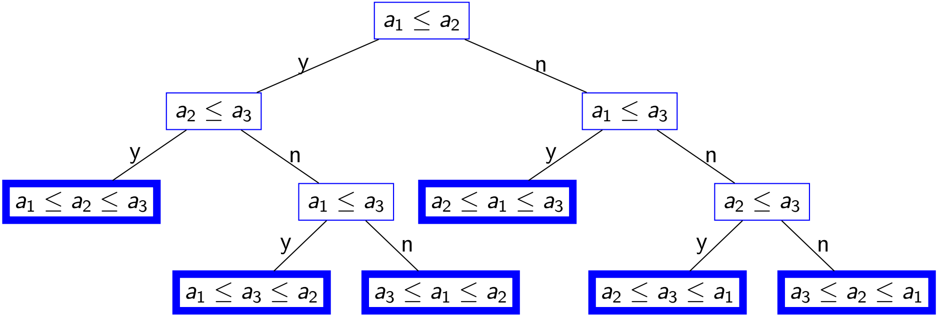


# Sorting Problem Ω

Assume you want to sort a collection of 3 elements: $a_1, a_2$ and $a_3$

- How many comparison *decisions* do you have to make, to cover any of the potential input sequences?
  - How many possibilities (potential inputs) are there?



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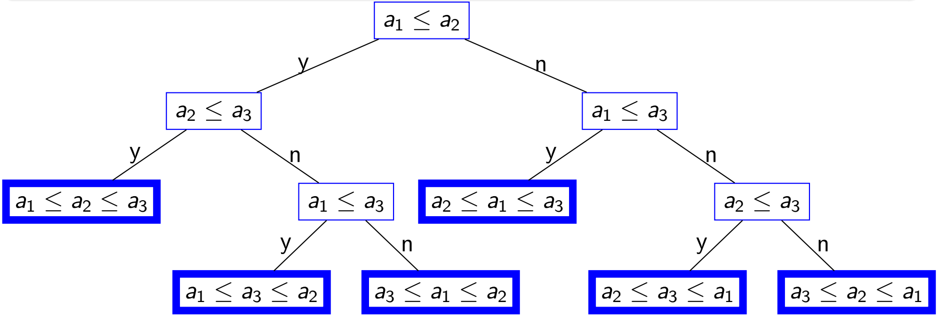


- leaves = each possible answer for any given input
- How many leaves? ($\ell$)

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- Worst case running time $\cong$ what from the tree?

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- $h_T$ ? $\ell$
  - (hint) What is the maximum #leaves (max $\ell$) for a tree of height $h_T$ ?

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  - $h_T > log_2(\ell)$ (why?)

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    $= \Omega(nlgn) \qquad //ignore\ constants$

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|---|---|---|---|---|---|
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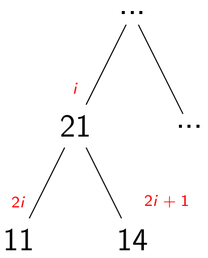

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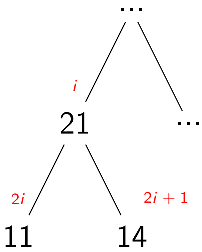

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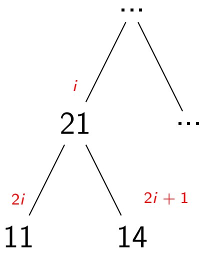

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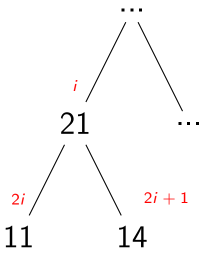
other P.O. relations can be defined



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- Considering the heap property, what consequence (*post condition*) follows?
  - the **root** contains the maximum value

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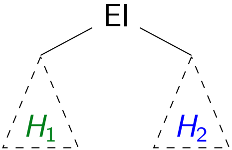
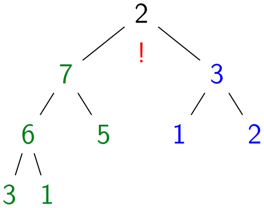
- *Heapsort*
  - Sorts by repeatedly extracting the root of the heap and placing it in the appropriate position of the sorted array

# *Heapify*(Reconstituie heap)

- Pre-condition: 2 heaps ($H_1$, $H_2$)
- Goal: add a single element $El$ s.t. the triple ($El$, $H_1$, $H_2$) generates a new, larger heap ($H$)
- Post-condition: 1 single heap $H$, containing all the elements (from) $El$, $H_1$ and $H_2$
- Strategy: *top–down*, **sink the root** to its correct place in the heap



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$\quad largest = \textsc{index\_of\_max}(H[i], H[left(i)], H[right(i)])$
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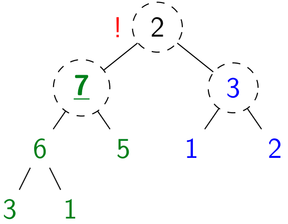

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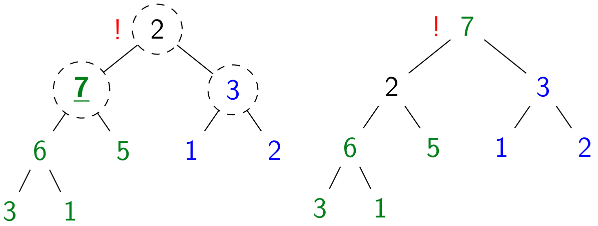

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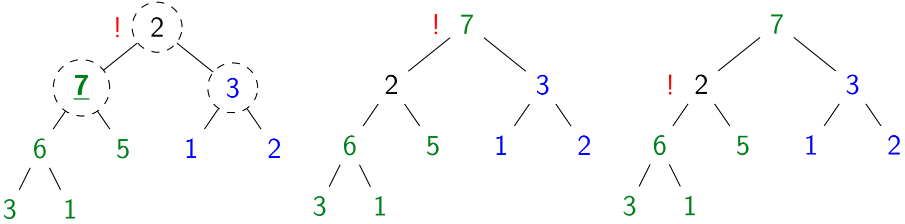

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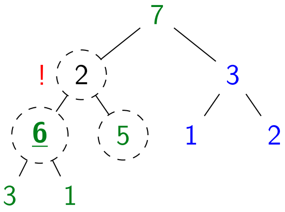

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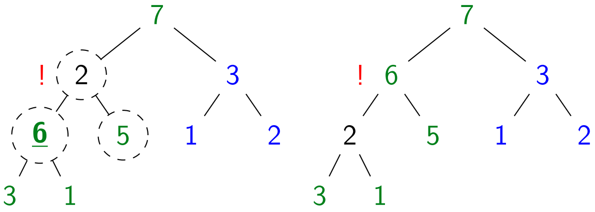

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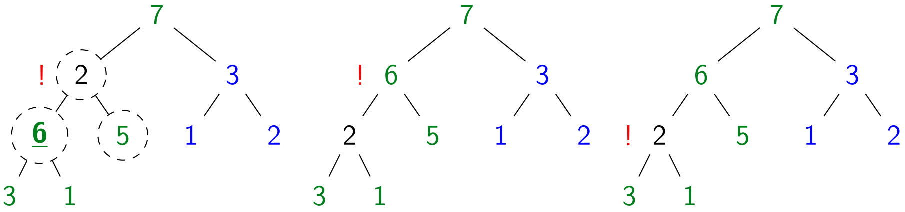

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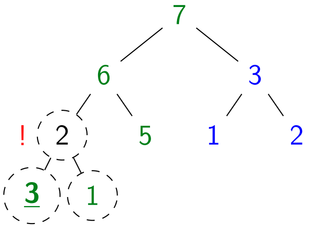

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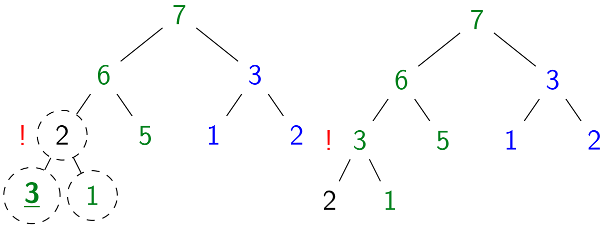

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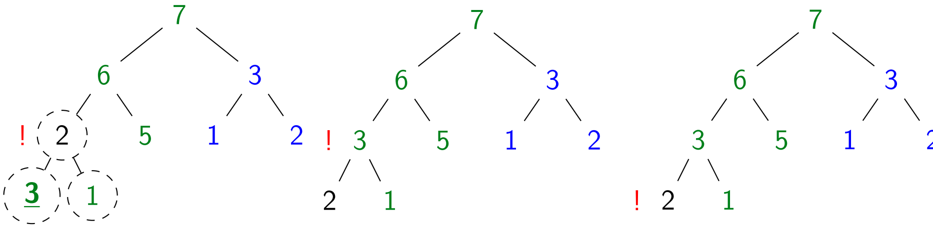

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    - *Exact:* The last row of the tree is *exactly half full*, and we go on that branch (Why?)

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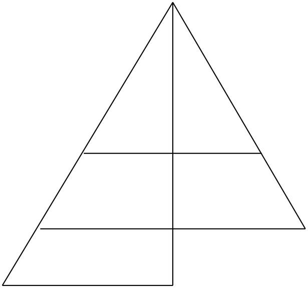

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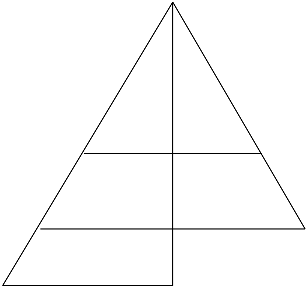

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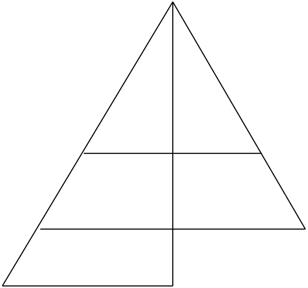

level $h-1$

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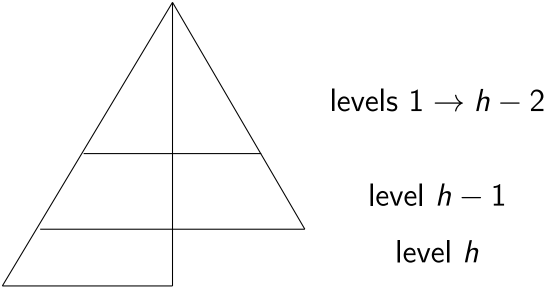
- Worst: The last row of the tree is *exactly half full*, and we go on that branch $->T(n)=?$
  - In complete BT, half of the number of nodes are leaves



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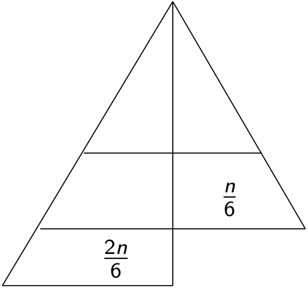


levels $1 \rightarrow h-2$

level $h-1$

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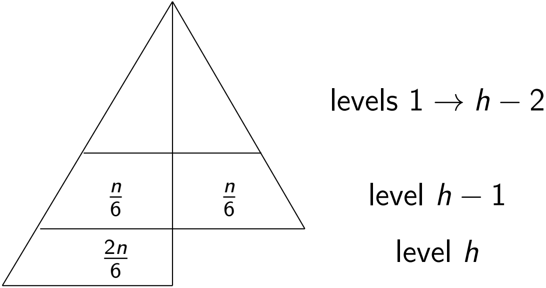


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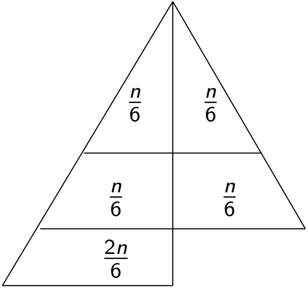


levels $1 \rightarrow h-2$

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- last level and right half of previous level are leaves
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- the rest of the levels contain, in total, as many nodes as level $h-1$

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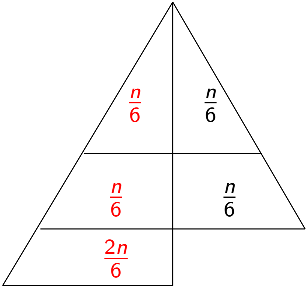


- $T(n) = T(\frac{2n}{3}) + O(1)$

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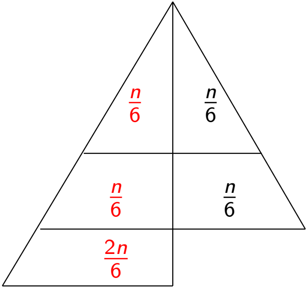


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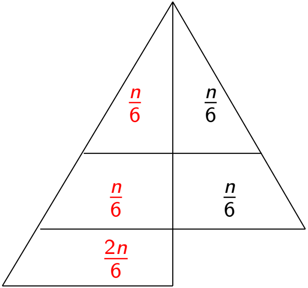


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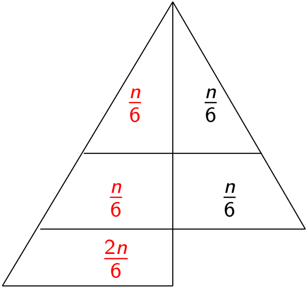


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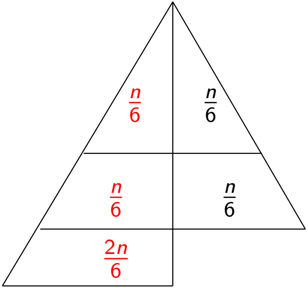


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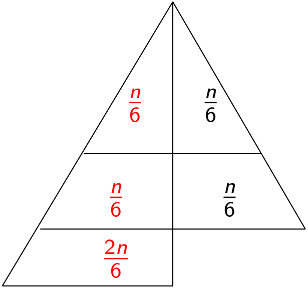


- $T(n) = T(\frac{2n}{3}) + O(1)$
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- (by Master Th., case #2): $T(n) = O(log_{\frac{3}{2}} n) = O(lgn)$

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  - apply *Heapify* to the first non-leaf node (largest index node having at least one child)
  - go to the sibling on the left (of the previously processed element), and to the same, until all elements have been processed (the "last" processed will be the root)

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BUILD-HEAP(H)
  heap_size[H] = |H|
  for i = |H|/2 downto 1 // from the first non-leaf, up to the root
        Heapify(H, i) // put node at index i as root to 2 heaps
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  - This is **bad**! Building the heap already takes $O(nlgn)$!?

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    - So, the total effort for them is: $\frac{1}{2^3} \cdot n \cdot 2$
  - for each subsequent step, the #elements halves, while the #steps required to *Heapify* each increases by 1

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- Running time - exact:

$$T(n) =$$

$$\frac{n}{2} \cdot 0+ \quad // \text{ leaves}$$
$$\frac{n}{2^2} \cdot 1+ \quad // \text{ leaf parents}$$
$$\frac{n}{2^3} \cdot 2+$$
$$\frac{n}{2^4} \cdot 3 + \ldots$$

$$= \sum_0^{[lgn]} [\frac{n}{2^{h+1}}] \cdot O(h)$$

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$$T(n) = \sum_{0}^{[lgn]} \left[\frac{n}{2^{h+1}}\right] \cdot O(h) = \frac{n}{2} \cdot \sum_{0}^{[lgn]} \left[\frac{1}{2^{h}}\right] \cdot h \ (**)$$

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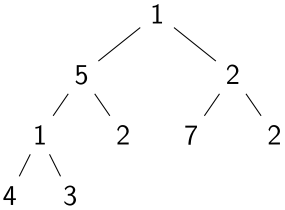
- Replacing now in (**), we get: $T(n) = \frac{n}{2} \cdot 2 = O(n)$

# *Build-Heap* - example



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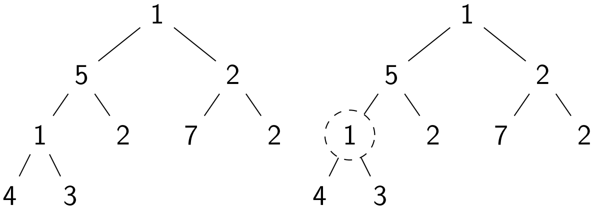

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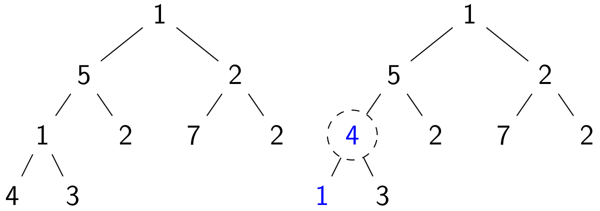

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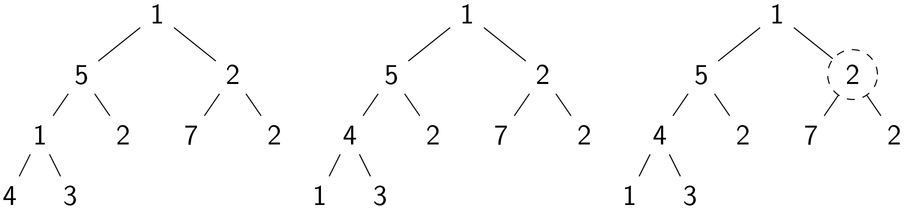

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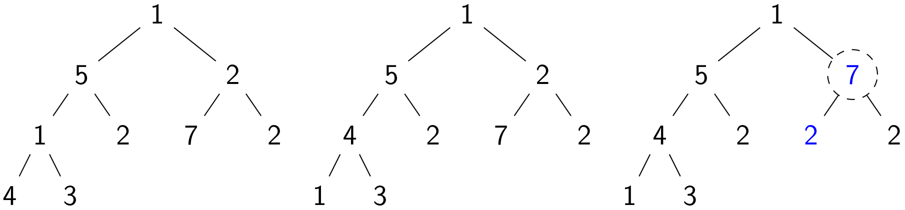

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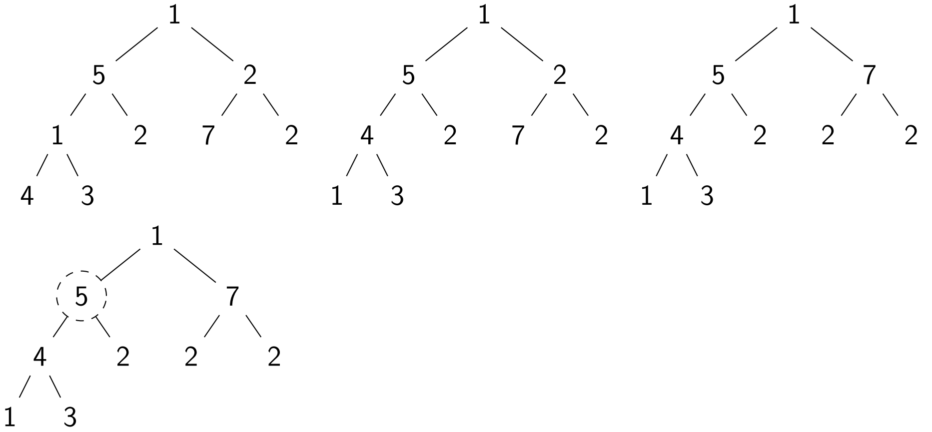

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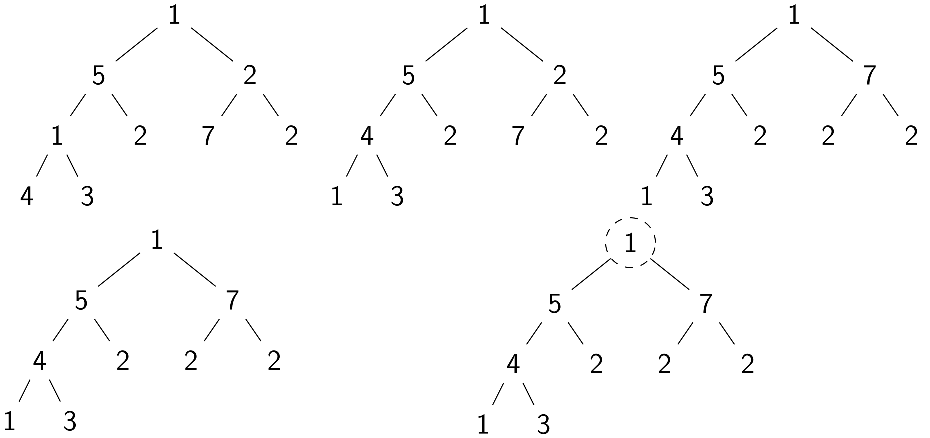

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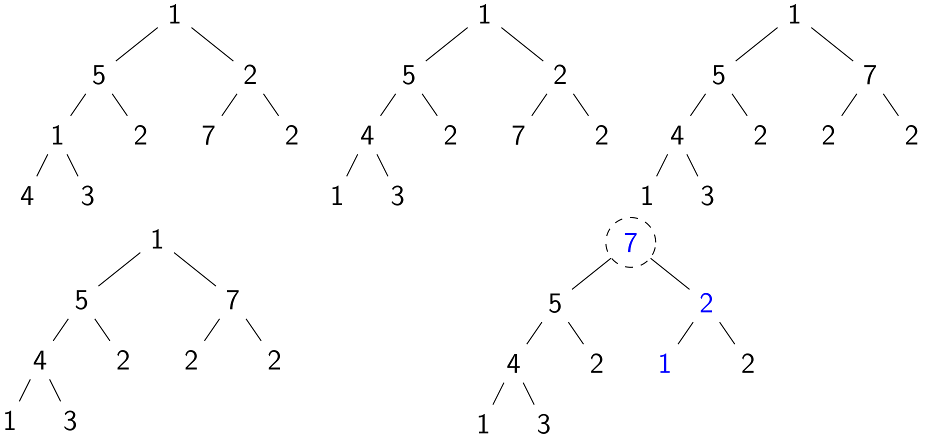

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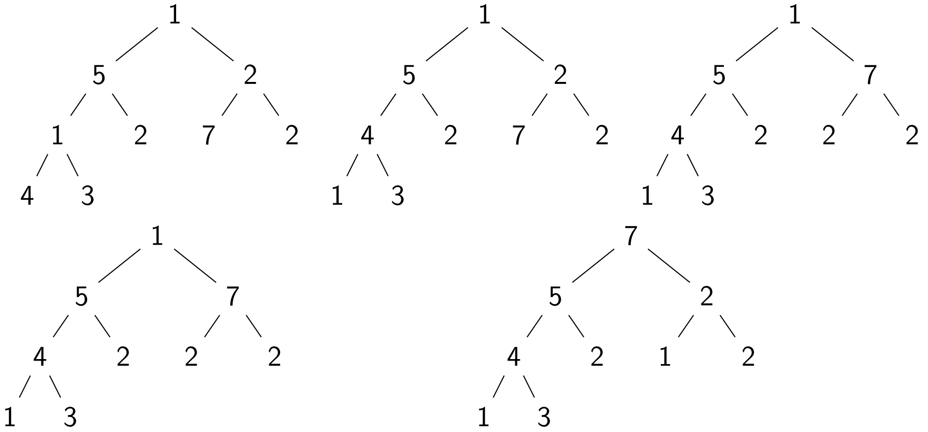

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   - *decrease heap size by 1*
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   - to obtain a heap of size reduced by 1 (the last element in the array is no longer part of the heap)

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   - move the max element in the last position of the array, where it belongs in the ordered array
3 except for the **first** and **last** elements in the array, we have a heap
   - but **last** in the right position in the sorted array, so ...
   - *decrease heap size by 1*
   - *apply Heapify on the* **first** *element, in the new, smaller heap* ...
   - to obtain a heap of size reduced by 1 (the last element in the array is no longer part of the heap)
4 repeat steps 2-3 until the heap size becomes 1

# *Heapsort*

HEAPSORT($H$)

```
BUILD-HEAP(H)
for i = |H| downto 2 // do for all array positions from last to sec.
        H[1] ↔ H[i]// swap root with last element in current heap
        heap_size[H] = heap_size[H] − 1 // decrease heap size
        HEAPIFY(H, 1) // repair heap
```

# *Heapsort* complexity

HEAPSORT($H$)

BUILD-HEAP($H$) // $O(n)$
**for** $i = |H|$ **downto** 2 // $n - 1$ times
    $H[1] \leftrightarrow H[i]$ // $O(1)$
    $heap\_size[H] = heap\_size[H] - 1$ // $O(1)$
    HEAPIFY($H, 1$) // $O(lgn)$

$$T(n) = O(n) + (n-1)[O(1) + O(lgn)]$$
$$T(n) = O(nlgn) = \Omega(nlgn)$$

- *Optimal* algorithm!

# *Heapsort* - example (*Build-Heap* done)

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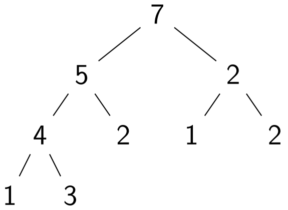


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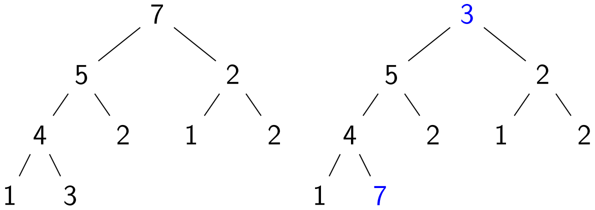


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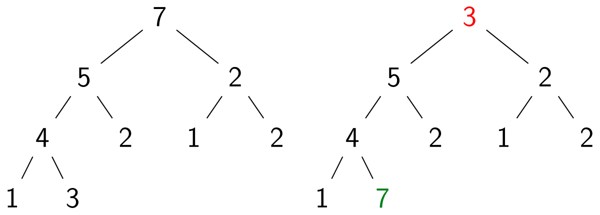


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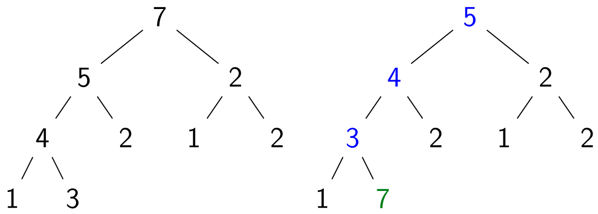


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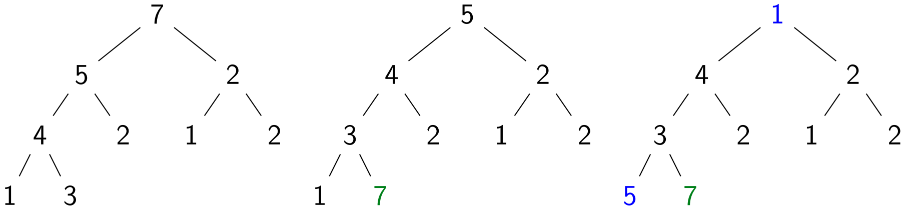


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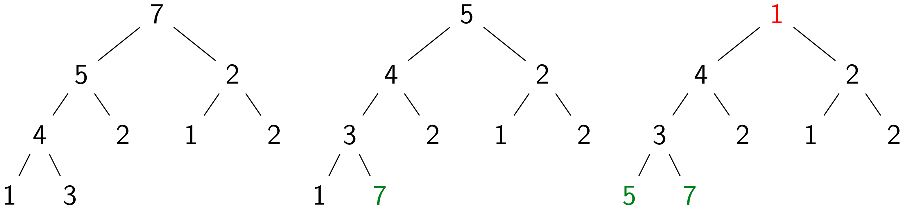


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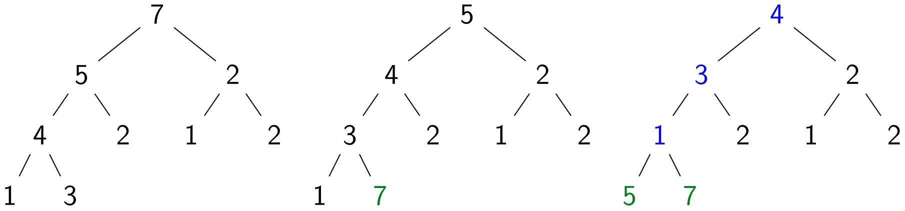


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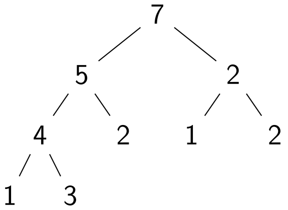


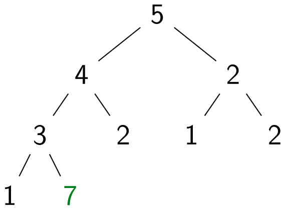


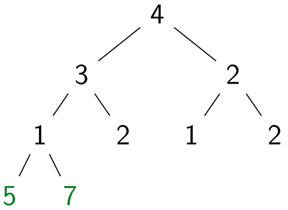


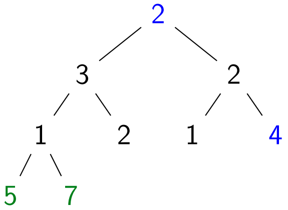


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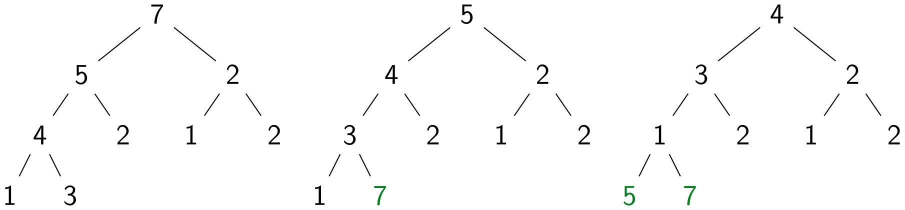

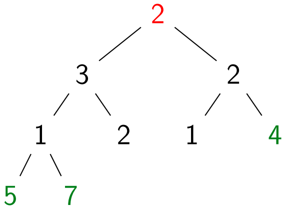

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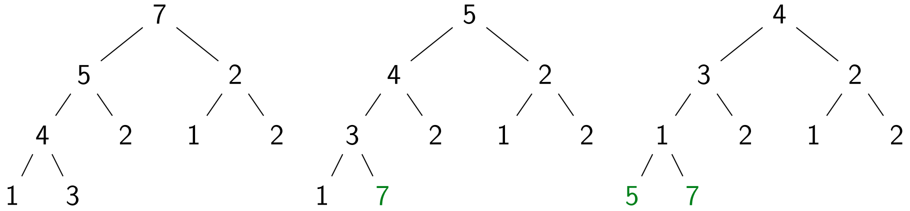


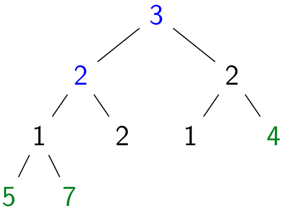


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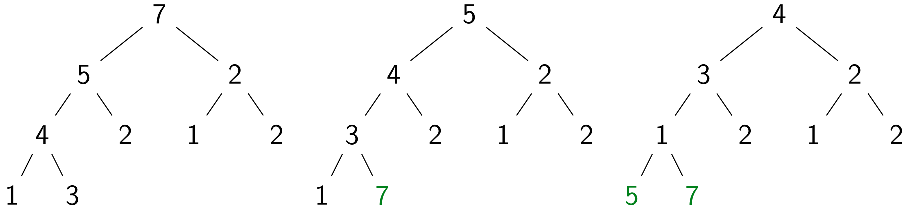


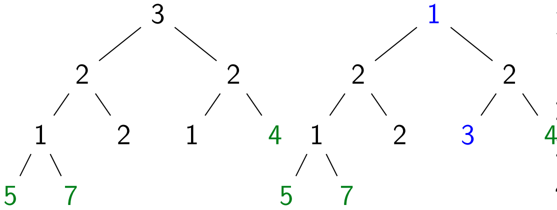


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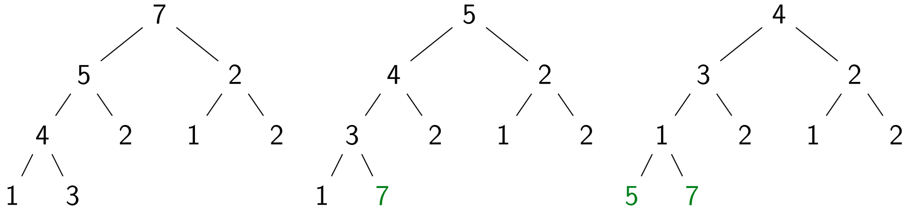


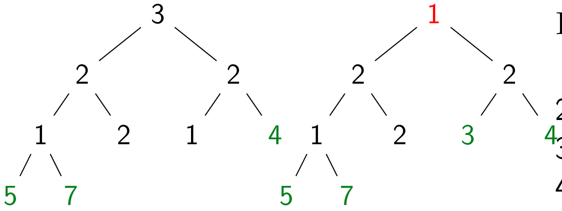


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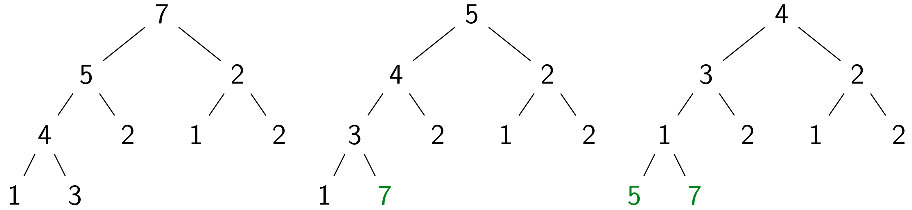


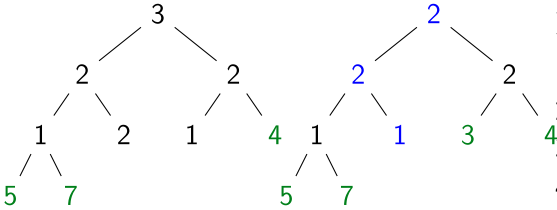


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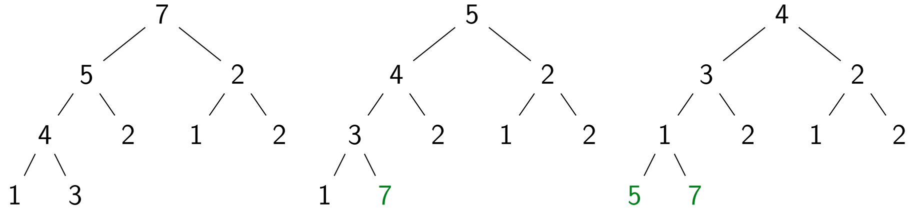


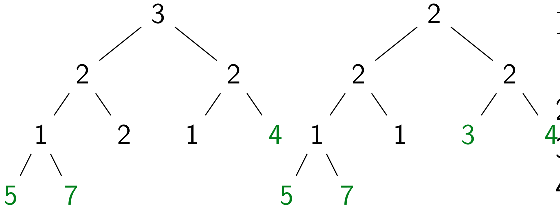


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  - insert: *HEAP-PUSH*, a.k.a. *MAX-HEAP-INSERT*
    - add one item to the heap

# Heap as a Data Structure - *POP-HEAP*

POP-HEAP($H$)

```
if heap_size[H] < 1 // empty heap
        return
max = H[1] // save root (max)
H[1] = H[heap_size[H]] // move bottom element to root
heap_size[H] = heap_size[H] − 1 // decrease heap size
HEAPIFY(H,1) // push element in root position down, to restore he
return max // return max element
```

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$\text{PUSH-HEAP}(H, key)$

$heap\_size[H] = heap\_size[H] + 1$ // increase heap size
$H[heap\_size[H]] = key$
$i = heap\_size[H]$
**while** $i > 1$ and $H[\text{PARENT}(i)] < H[i]$
    $H[i] \leftrightarrow H[\text{PARENT}(i)]$
    $i = \text{PARENT}(i)$

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  - $T(n) = 2 \cdot 1 + 4 \cdot 2 + 8 \cdot 3 + ... + \frac{n}{2} \cdot lgn$

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# Building the heap - comparison

| Approach | Bottom-up | Top-down |
|---|---|---|
| Approach for 1 elem | sinks the root | from root, add new leaf |
| Approach for all elems | from leaves towards root | bubbles a leaf |
| Complexity | $O(n)$ | $O(n \cdot lgn)$ |
| Advantage | faster | handle variable dimension |
| Usage | sorting | priority queues |

# *Heapsort* – Conclusions

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- *Quicksort* is faster in practice! (even if not optimal, in classic approach)
- Good *Quicksort* implementations are (considered) optimal

# Bibliography


- Cormen, Thomas H., et al., *"Introduction to algorithms."*, MIT press, 2009, chap. 6 and 8.1 (sorting lower bound)