Linear Algebra Guide

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FALL TERM

One.I.1

1.10 Definition In each row of a system, the first variable with a nonzero coefficient is the row's *leading variable*. A system is in *echelon form* if each leading variable is to the right of the leading variable in the row above it, except for the leading variable in the first row, and any rows with all-zero coefficients are at the bottom.

One.I.1.30 (Proof technique), 1.31 (Proof technique), 1.32 (show), 1.33 (Proof), 1.34 (interesting), 1.35 (interesting),

One.I.2

Vector representation of solution set

One.I.3

I.3 General = Particular + Homogeneous

In the prior subsection the descriptions of solution sets all fit a pattern. They have a vector that is a particular solution of the system added to an unrestricted combination of some other vectors. The solution set from Example 2.13 illustrates.

$$\{\underbrace{\begin{pmatrix}0\\4\\0\\0\\0\end{pmatrix}}_{\substack{\text{particular}\\\text{solution}}}\underbrace{+w\begin{pmatrix}1\\-1\\3\\1\\0\end{pmatrix}+u\begin{pmatrix}1/2\\-1\\1/2\\0\\1\end{pmatrix}}_{\substack{\text{unrestricted}\\\text{combination}}}\mid w,u\in\mathbb{R}\}$$

3.6 Lemma For any homogeneous linear system there exist vectors $\vec{\beta}_1, \ldots, \vec{\beta}_k$ such that the solution set of the system is

$$\{c_1\vec{\beta}_1 + \cdots + c_k\vec{\beta}_k \mid c_1, \ldots, c_k \in \mathbb{R}\}$$

where k is the number of free variables in an echelon form version of the system.

3.7 Lemma For a linear system and for any particular solution $\vec{p}$, the solution set equals $\{\vec{p} + \vec{h} \mid \vec{h}$ satisfies the associated homogeneous system$\}$.

Proof We will show mutual set inclusion, that any solution to the system is in the above set and that anything in the set is a solution of the system.*

For set inclusion the first way, that if a vector solves the system then it is in the set described above, assume that $\vec{s}$ solves the system. Then $\vec{s} - \vec{p}$ solves the associated homogeneous system since for each equation index i,

$$\begin{aligned} &a_{i,1}(s_1 - p_1) + \cdots + a_{i,n}(s_n - p_n) \\ &\qquad = (a_{i,1}s_1 + \cdots + a_{i,n}s_n) - (a_{i,1}p_1 + \cdots + a_{i,n}p_n) = d_i - d_i = 0 \end{aligned}$$

where p_j and s_j are the j-th components of $\vec{p}$ and $\vec{s}$. Express $\vec{s}$ in the required $\vec{p} + \vec{h}$ form by writing $\vec{s} - \vec{p}$ as $\vec{h}$.

For set inclusion the other way, take a vector of the form $\vec{p} + \vec{h}$, where $\vec{p}$ solves the system and $\vec{h}$ solves the associated homogeneous system and note that $\vec{p} + \vec{h}$ solves the given system since for any equation index i,

$$\begin{aligned} &a_{i,1}(p_1 + h_1) + \cdots + a_{i,n}(p_n + h_n) \\ &\qquad = (a_{i,1}p_1 + \cdots + a_{i,n}p_n) + (a_{i,1}h_1 + \cdots + a_{i,n}h_n) = d_i + 0 = d_i \end{aligned}$$

where as earlier p_j and h_j are the j-th components of $\vec{p}$ and $\vec{h}$. QED

The two lemmas together establish Theorem 3.1. Remember that theorem with the slogan, "General = Particular + Homogeneous".

3.10 Corollary Solution sets of linear systems are either empty, have one element, or have infinitely many elements.

Proof We've seen examples of all three happening so we need only prove that there are no other possibilities.

First observe a homogeneous system with at least one non-$\vec{0}$ solution $\vec{v}$ has infinitely many solutions. This is because any scalar multiple of $\vec{v}$ also solves the homogeneous system and there are infinitely many vectors in the set of scalar multiples of $\vec{v}$: if $s, t \in \mathbb{R}$ are unequal then $s\vec{v} \neq t\vec{v}$, since $s\vec{v} - t\vec{v} = (s-t)\vec{v}$ is non-$\vec{0}$ as any non-0 component of $\vec{v}$, when rescaled by the non-0 factor $s-t$, will give a non-0 value.

Now apply Lemma 3.7 to conclude that a solution set

$$\{\vec{p} + \vec{h} \mid \vec{h} \text{ solves the associated homogeneous system}\}$$

is either empty (if there is no particular solution $\vec{p}$), or has one element (if there is a $\vec{p}$ and the homogeneous system has the unique solution $\vec{0}$), or is infinite (if there is a $\vec{p}$ and the homogeneous system has a non-$\vec{0}$ solution, and thus by the prior paragraph has infinitely many solutions). QED

This table summarizes the factors affecting the size of a general solution.

		number of solutions of the homogeneous system	
		one	*infinitely many*
particular solution exists?	*yes*	unique solution	infinitely many solutions
	no	no solutions	no solutions

3.11 Definition A square matrix is *nonsingular* if it is the matrix of coefficients of a homogeneous system with a unique solution. It is *singular* otherwise, that is, if it is the matrix of coefficients of a homogeneous system with infinitely many solutions.

One.II.1 Vectors in Space

Generalizing, a set of the form $\{\vec{p}+t_1\vec{v}_1+t_2\vec{v}_2+\cdots+t_k\vec{v}_k \mid t_1,\ldots,t_k\in\mathbb{R}\}$ where $\vec{v}_1,\ldots,\vec{v}_k\in\mathbb{R}^n$ and $k\leqslant n$ is a k-*dimensional linear surface* (or k-*flat*). For example, in $\mathbb{R}^4$

$$\{\begin{pmatrix}2\\ \pi\\ 3\\ -0.5\end{pmatrix}+t\begin{pmatrix}1\\0\\0\\0\end{pmatrix} \mid t\in\mathbb{R}\}$$

is a line,

$$\{\begin{pmatrix}0\\0\\0\\0\end{pmatrix}+t\begin{pmatrix}1\\1\\0\\-1\end{pmatrix}+s\begin{pmatrix}2\\0\\1\\0\end{pmatrix} \mid t,s\in\mathbb{R}\}$$

is a plane, and

$$\{\begin{pmatrix}3\\1\\-2\\0.5\end{pmatrix}+r\begin{pmatrix}0\\0\\0\\-1\end{pmatrix}+s\begin{pmatrix}1\\0\\1\\0\end{pmatrix}+t\begin{pmatrix}2\\0\\1\\0\end{pmatrix} \mid r,s,t\in\mathbb{R}\}$$

is a three-dimensional linear surface. Again, the intuition is that a line permits motion in one direction, a plane permits motion in combinations of two directions, etc. When the dimension of the linear surface is one less than the dimension of the space, that is, when in $\mathbb{R}^n$ we have an $(n-1)$-flat, the surface is called a *hyperplane*.

A description of a linear surface can be misleading about the dimension. For example, this

$$L = \{ \begin{pmatrix} 1 \\ 0 \\ -1 \\ -2 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \\ 0 \\ -1 \end{pmatrix} + s \begin{pmatrix} 2 \\ 2 \\ 0 \\ -2 \end{pmatrix} \mid t, s \in \mathbb{R} \}$$

is a *degenerate* plane because it is actually a line, since the vectors are multiples of each other and we can omit one.

$$L = \{ \begin{pmatrix} 1 \\ 0 \\ -1 \\ -2 \end{pmatrix} + r \begin{pmatrix} 1 \\ 1 \\ 0 \\ -1 \end{pmatrix} \mid r \in \mathbb{R} \}$$

One.II.2 Length and Angle Measures

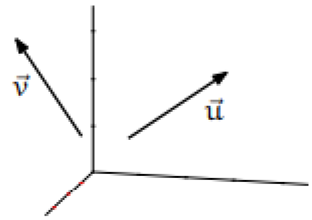
2.1 Definition The *length* of a vector $\vec{v} \in \mathbb{R}^n$ is the square root of the sum of the squares of its components.

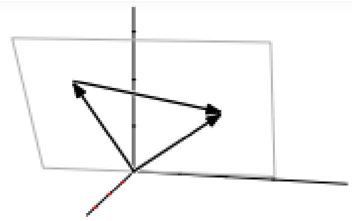
$$|\vec{v}\,| = \sqrt{v_1^2 + \cdots + v_n^2}$$

2.2 Remark This is a natural generalization of the Pythagorean Theorem. A classic motivating discussion is in [Polya].

For any nonzero $\vec{v}$, the vector $\vec{v}/|\vec{v}|$ has length one. We say that the second *normalizes* $\vec{v}$ to length one.

We can use that to get a formula for the angle between two vectors. Consider two vectors in $\mathbb{R}^3$ where neither is a multiple of the other





Apply the Law of Cosines: $|\vec{u}-\vec{v}|^2 = |\vec{u}|^2 + |\vec{v}|^2 - 2\,|\vec{u}|\,|\vec{v}|\cos\theta$ where θ is the angle between the vectors. The left side gives

$$\begin{aligned}(u_1-v_1)^2 &+ (u_2-v_2)^2 + (u_3-v_3)^2 \\ &= (u_1^2 - 2u_1v_1 + v_1^2) + (u_2^2 - 2u_2v_2 + v_2^2) + (u_3^2 - 2u_3v_3 + v_3^2)\end{aligned}$$

while the right side gives this.

$$(u_1^2 + u_2^2 + u_3^2) + (v_1^2 + v_2^2 + v_3^2) - 2\,|\vec{u}|\,|\vec{v}|\cos\theta$$

Canceling squares $u_1^2, \ldots, v_3^2$ and dividing by 2 gives a formula for the angle.

$$\theta = \arccos\left(\frac{u_1v_1 + u_2v_2 + u_3v_3}{|\vec{u}|\,|\vec{v}|}\right)$$

2.3 Definition The *dot product* (or *inner product* or *scalar product*) of two n-component real vectors is the linear combination of their components.

$$\vec{u}\cdot\vec{v} = u_1v_1 + u_2v_2 + \cdots + u_nv_n$$

2.5 Theorem (Triangle Inequality) For any $\vec{u}, \vec{v} \in \mathbb{R}^n$,

$$|\vec{u}+\vec{v}| \leqslant |\vec{u}| + |\vec{v}|$$

with equality if and only if one of the vectors is a nonnegative scalar multiple of the other one.

PROOF (We'll use some algebraic properties of dot product that we have not yet checked, for instance that $\vec{u} \cdot (\vec{a} + \vec{b}) = \vec{u} \cdot \vec{a} + \vec{u} \cdot \vec{b}$ and that $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$. See Exercise 18.) Since all the numbers are positive, the inequality holds if and only if its square holds.

$$\begin{aligned}
|\vec{u} + \vec{v}|^2 &\leqslant (\,|\vec{u}\,| + |\vec{v}\,|\,)^2 \\
(\,\vec{u} + \vec{v}\,) \cdot (\,\vec{u} + \vec{v}\,) &\leqslant |\vec{u}\,|^2 + 2\,|\vec{u}\,|\,|\vec{v}\,| + |\vec{v}\,|^2 \\
\vec{u} \cdot \vec{u} + \vec{u} \cdot \vec{v} + \vec{v} \cdot \vec{u} + \vec{v} \cdot \vec{v} &\leqslant \vec{u} \cdot \vec{u} + 2\,|\vec{u}\,|\,|\vec{v}\,| + \vec{v} \cdot \vec{v} \\
2\,\vec{u} \cdot \vec{v} &\leqslant 2\,|\vec{u}\,|\,|\vec{v}\,|
\end{aligned}$$

That, in turn, holds if and only if the relationship obtained by multiplying both sides by the nonnegative numbers $|\vec{u}\,|$ and $|\vec{v}\,|$

$$2\,(\,|\vec{v}\,|\,\vec{u}\,) \cdot (\,|\vec{u}\,|\,\vec{v}\,) \leqslant 2\,|\vec{u}\,|^2\,|\vec{v}\,|^2$$

and rewriting

$$0 \leqslant |\vec{u}\,|^2\,|\vec{v}\,|^2 - 2\,(\,|\vec{v}\,|\,\vec{u}\,) \cdot (\,|\vec{u}\,|\,\vec{v}\,) + |\vec{u}\,|^2\,|\vec{v}\,|^2$$

is true. But factoring shows that it is true

$$0 \leqslant (\,|\vec{u}\,|\,\vec{v} - |\vec{v}\,|\,\vec{u}\,) \cdot (\,|\vec{u}\,|\,\vec{v} - |\vec{v}\,|\,\vec{u}\,)$$

since it only says that the square of the length of the vector $|\vec{u}\,|\,\vec{v} - |\vec{v}\,|\,\vec{u}$ is not negative. As for equality, it holds when, and only when, $|\vec{u}\,|\,\vec{v} - |\vec{v}\,|\,\vec{u}$ is $\vec{0}$. The check that $|\vec{u}\,|\,\vec{v} = |\vec{v}\,|\,\vec{u}$ if and only if one vector is a nonnegative real scalar multiple of the other is easy. QED

Prove generalization of triangle inequality:

One.II.2.28 Yes; we can prove this by induction.

Assume that the vectors are in some $\mathbb{R}^k$. Clearly the statement applies to one vector. The Triangle Inequality is this statement applied to two vectors. For an inductive step assume the statement is true for n or fewer vectors. Then this

$$|\vec{u}_1 + \cdots + \vec{u}_n + \vec{u}_{n+1}| \leqslant |\vec{u}_1 + \cdots + \vec{u}_n| + |\vec{u}_{n+1}|$$

follows by the Triangle Inequality for two vectors. Now the inductive hypothesis, applied to the first summand on the right, gives that as less than or equal to $|\vec{u}_1| + \cdots + |\vec{u}_n| + |\vec{u}_{n+1}|$.

2.6 Corollary (Cauchy-Schwarz Inequality) For any $\vec{u}, \vec{v} \in \mathbb{R}^n$,

$$|\vec{u} \cdot \vec{v}| \leqslant |\vec{u}|\,|\vec{v}|$$

with equality if and only if one vector is a scalar multiple of the other.

Proof The Triangle Inequality's proof shows that $\vec{u} \cdot \vec{v} \leqslant |\vec{u}|\,|\vec{v}|$ so if $\vec{u} \cdot \vec{v}$ is positive or zero then we are done. If $\vec{u} \cdot \vec{v}$ is negative then this holds.

$$|\vec{u} \cdot \vec{v}| = -(\vec{u} \cdot \vec{v}) = (-\vec{u}) \cdot \vec{v} \leqslant |-\vec{u}|\,|\vec{v}| = |\vec{u}|\,|\vec{v}|$$

The equality condition is Exercise 19. QED

2.7 Definition The *angle* between two nonzero vectors $\vec{u}, \vec{v} \in \mathbb{R}^n$ is

$$\theta = \arccos\left(\frac{\vec{u} \cdot \vec{v}}{|\vec{u}|\,|\vec{v}|}\right)$$

(if either is the zero vector then we take the angle to be a right angle).

2.8 Corollary Vectors from $\mathbb{R}^n$ are orthogonal, that is, perpendicular, if and only if their dot product is zero. They are parallel if and only if their dot product equals the product of their lengths.

One.III.1 Reduced Echelon Form

1.3 Definition A matrix or linear system is in *reduced echelon form* if, in addition to being in echelon form, each leading entry is a 1 and is the only nonzero entry in its column.

1.5 Lemma Elementary row operations are reversible.

PROOF For any matrix A, the effect of swapping rows is reversed by swapping them back, multiplying a row by a nonzero k is undone by multiplying by $1/k$, and adding a multiple of row i to row j (with $i \neq j$) is undone by subtracting the same multiple of row i from row j.

$$A \xrightarrow{\rho_i \leftrightarrow \rho_j} \xrightarrow{\rho_j \leftrightarrow \rho_i} A \qquad A \xrightarrow{k\rho_i} \xrightarrow{(1/k)\rho_i} A \qquad A \xrightarrow{k\rho_i + \rho_j} \xrightarrow{-k\rho_i + \rho_j} A$$

(We need the $i \neq j$ condition; see Exercise 18.) QED

1.6 Lemma Between matrices, 'reduces to' is an equivalence relation.

PROOF We must check the conditions (i) reflexivity, that any matrix reduces to itself, (ii) symmetry, that if A reduces to B then B reduces to A, and (iii) transitivity, that if A reduces to B and B reduces to C then A reduces to C.

Reflexivity is easy; any matrix reduces to itself in zero-many operations.

The relationship is symmetric by the prior lemma — if A reduces to B by some row operations then also B reduces to A by reversing those operations.

For transitivity, suppose that A reduces to B and that B reduces to C. Following the reduction steps from $A \to \cdots \to B$ with those from $B \to \cdots \to C$ gives a reduction from A to C. QED

1.7 Definition Two matrices that are interreducible by elementary row operations are *row equivalent*.

One.III.2 Linear Combination Lemma

2.3 Lemma (Linear Combination Lemma) A linear combination of linear combinations is a linear combination.

PROOF Given the set $c_{1,1}x_1 + \cdots + c_{1,n}x_n$ through $c_{m,1}x_1 + \cdots + c_{m,n}x_n$ of linear combinations of the x's, consider a combination of those

$$d_1(c_{1,1}x_1 + \cdots + c_{1,n}x_n) + \cdots + d_m(c_{m,1}x_1 + \cdots + c_{m,n}x_n)$$

where the d's are scalars along with the c's. Distributing those d's and regrouping gives

$$= (d_1c_{1,1} + \cdots + d_mc_{m,1})x_1 + \cdots + (d_1c_{1,n} + \cdots + d_mc_{m,n})x_n$$

which is also a linear combination of the x's. QED

2.5 Lemma In an echelon form matrix, no nonzero row is a linear combination of the other nonzero rows.

2.6 Theorem Each matrix is row equivalent to a unique reduced echelon form matrix.

Two.I.1 Definition of Vector Space

I.1 Definition and Examples

1.1 Definition A *vector space* (over $\mathbb{R}$) consists of a set V along with two operations '+' and '·' subject to the conditions that for all vectors $\vec{v}, \vec{w}, \vec{u} \in V$ and all *scalars* $r, s \in \mathbb{R}$:

(1) the set V is closed under vector addition, that is, $\vec{v} + \vec{w} \in V$

(2) vector addition is commutative, $\vec{v} + \vec{w} = \vec{w} + \vec{v}$

(3) vector addition is associative, $(\vec{v} + \vec{w}) + \vec{u} = \vec{v} + (\vec{w} + \vec{u})$

(4) there is a *zero vector* $\vec{0} \in V$ such that $\vec{v} + \vec{0} = \vec{v}$ for all $\vec{v} \in V$

(5) each $\vec{v} \in V$ has an *additive inverse* $\vec{w} \in V$ such that $\vec{w} + \vec{v} = \vec{0}$

(6) the set V is closed under scalar multiplication, that is, $r \cdot \vec{v} \in V$

(7) scalar multiplication distributes over scalar addition, $(r + s) \cdot \vec{v} = r \cdot \vec{v} + s \cdot \vec{v}$

(8) scalar multiplication distributes over vector addition, $r \cdot (\vec{v} + \vec{w}) = r \cdot \vec{v} + r \cdot \vec{w}$

(9) ordinary multiplication of scalars associates with scalar multiplication, $(rs) \cdot \vec{v} = r \cdot (s \cdot \vec{v})$

(10) multiplication by the scalar 1 is the identity operation, $1 \cdot \vec{v} = \vec{v}$.

1.7 Definition A one-element vector space is a *trivial* space.

Two.I.2 Subspaces and spanning sets

2.1 Definition For any vector space, a *subspace* is a subset that is itself a vector space, under the inherited operations.

2.9 Lemma For a nonempty subset S of a vector space, under the inherited operations the following are equivalent statements.*

(1) S is a subspace of that vector space

(2) S is closed under linear combinations of pairs of vectors: for any vectors $\vec{s}_1, \vec{s}_2 \in S$ and scalars r_1, r_2 the vector $r_1\vec{s}_1 + r_2\vec{s}_2$ is in S

(3) S is closed under linear combinations of any number of vectors: for any vectors $\vec{s}_1, \dots, \vec{s}_n \in S$ and scalars $r_1, \dots, r_n$ the vector $r_1\vec{s}_1 + \dots + r_n\vec{s}_n$ is an element of S.

2.13 Definition The *span* (or *linear closure*) of a nonempty subset S of a vector space is the set of all linear combinations of vectors from S.

$$[S] = \{c_1\vec{s}_1 + \cdots + c_n\vec{s}_n \mid c_1, \ldots, c_n \in \mathbb{R} \text{ and } \vec{s}_1, \ldots, \vec{s}_n \in S\}$$

The span of the empty subset of a vector space is its trivial subspace.

2.15 Lemma In a vector space, the span of any subset is a subspace.

Proof If the subset S is empty then by definition its span is the trivial subspace. If S is not empty then by Lemma 2.9 we need only check that the span $[S]$ is closed under linear combinations of pairs of elements. For a pair of vectors from that span, $\vec{v} = c_1\vec{s}_1 + \cdots + c_n\vec{s}_n$ and $\vec{w} = c_{n+1}\vec{s}_{n+1} + \cdots + c_m\vec{s}_m$, a linear combination

$$\begin{aligned} p\cdot(c_1\vec{s}_1 + \cdots + c_n\vec{s}_n) &+ r\cdot(c_{n+1}\vec{s}_{n+1} + \cdots + c_m\vec{s}_m) \\ &= pc_1\vec{s}_1 + \cdots + pc_n\vec{s}_n + rc_{n+1}\vec{s}_{n+1} + \cdots + rc_m\vec{s}_m \end{aligned}$$

is a linear combination of elements of S and so is an element of $[S]$ (possibly some of the $\vec{s}_i$'s from $\vec{v}$ equal some of the $\vec{s}_j$'s from $\vec{w}$ but that does not matter).
QED

The converse of the lemma holds: any subspace is the span of some set, because a subspace is obviously the span of itself, the set of all of its members. Thus a subset of a vector space is a subspace if and only if it is a span. This fits the intuition that a good way to think of a vector space is as a collection in which linear combinations are sensible.

Two.II Linear Independence

1.2 Lemma Where V is a vector space, S is a subset of that space, and $\vec{v}$ is an element of that space, $[S \cup \{\vec{v}\}] = [S]$ if and only if $\vec{v} \in [S]$.

1.3 Corollary For $\vec{v} \in S$, omitting that vector does not shrink the span $[S] = [S - \{\vec{v}\}]$ if and only if it is dependent on other vectors in the set $\vec{v} \in [S]$.

1.4 Definition In any vector space, a set of vectors is *linearly independent* if none of its elements is a linear combination of the others from the set.* Otherwise the set is *linearly dependent*.

1.5 Lemma A subset S of a vector space is linearly independent if and only if among its elements the only linear relationship $c_1\vec{s}_1 + \cdots + c_n\vec{s}_n = \vec{0}$ is the trivial one, $c_1 = 0, \ldots, c_n = 0$ (where $\vec{s}_i \neq \vec{s}_j$ when $i \neq j$) .

Proof If S is linearly independent then no vector $\vec{s}_i$ is a linear combination of other vectors from S so there is no linear relationship where some of the $\vec{s}$'s have nonzero coefficients.

If S is not linearly independent then some $\vec{s}_i$ is a linear combination $\vec{s}_i = c_1\vec{s}_1 + \cdots + c_{i-1}\vec{s}_{i-1} + c_{i+1}\vec{s}_{i+1} + \cdots + c_n\vec{s}_n$ of other vectors from S. Subtracting $\vec{s}_i$ from both sides gives a relationship involving a nonzero coefficient, the -1 in front of $\vec{s}_i$. QED

1.14 Corollary A set S is linearly independent if and only if for any $\vec{v} \in S$, its removal shrinks the span $[S - \{v\}] \subsetneq [S]$.

Proof This follows from Corollary 1.3. If S is linearly independent then none of its vectors is dependent on the other elements, so removal of any vector will shrink the span. If S is not linearly independent then it contains a vector that is dependent on other elements of the set, and removal of that vector will not shrink the span. QED

So a spanning set is minimal if and only if it is linearly independent.

The prior result addresses removing elements from a linearly independent set. The next one adds elements.

1.15 Lemma Suppose that S is linearly independent and that $\vec{v} \notin S$. Then the set $S \cup \{\vec{v}\}$ is linearly independent if and only if $\vec{v} \notin [S]$.

Proof We will show that $S \cup \{\vec{v}\}$ is not linearly independent if and only if $\vec{v} \in [S]$.

Suppose first that $\vec{v} \in [S]$. Express $\vec{v}$ as a combination $\vec{v} = c_1\vec{s}_1 + \cdots + c_n\vec{s}_n$. Rewrite that $\vec{0} = c_1\vec{s}_1 + \cdots + c_n\vec{s}_n - 1 \cdot \vec{v}$. Since $\vec{v} \notin S$, it does not equal any of the $\vec{s}_i$ so this is a nontrivial linear dependence among the elements of $S \cup \{\vec{v}\}$. Thus that set is not linearly independent.

Now suppose that $S \cup \{\vec{v}\}$ is not linearly independent and consider a nontrivial dependence among its members $\vec{0} = c_1\vec{s}_1 + \cdots + c_n\vec{s}_n + c_{n+1} \cdot \vec{v}$. If $c_{n+1} = 0$ then that is a dependence among the elements of S, but we are assuming that S is independent, so $c_{n+1} \neq 0$. Rewrite the equation as $\vec{v} = (c_1/c_{n+1})\vec{s}_1 + \cdots + (c_n/c_{n+1})\vec{s}_n$ to get $\vec{v} \in [S]$ QED

1.20 Lemma Any subset of a linearly independent set is also linearly independent. Any superset of a linearly dependent set is also linearly dependent.

Proof Both are clear. QED

Two.III Basis and Dimension

=> Overall can prove one is a basis through saying it is both L.I. and spans, or it has same dimension as space and LI, or it has same dimension as space and spans, or only that it UNIQUELY SPANS

1.1 Definition A *basis* for a vector space is a sequence of vectors that is linearly independent and that spans the space.

1.12 Theorem In any vector space, a subset is a basis if and only if each vector in the space can be expressed as a linear combination of elements of the subset in one and only one way.

1.13 Definition In a vector space with basis B the *representation of $\vec{v}$ with respect to* B is the column vector of the coefficients used to express $\vec{v}$ as a linear combination of the basis vectors:

$$\mathrm{Rep}_B(\vec{v}) = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}_B$$

where $B = \langle \vec{\beta}_1, \ldots, \vec{\beta}_n \rangle$ and $\vec{v} = c_1\vec{\beta}_1 + c_2\vec{\beta}_2 + \cdots + c_n\vec{\beta}_n$. The c's are the *coordinates of $\vec{v}$ with respect to* B.

1.18 Lemma Where B is a basis with n elements, for any set of vectors $a_1\vec{v}_1 + \cdots + a_k\vec{v}_k = \vec{0}_V$ if and only if $a_1\mathrm{Rep}_B(\vec{v}_1) + \cdots + a_k\mathrm{Rep}_B(\vec{v}_k) = \vec{0}_{\mathbb{R}^n}$.

Proof Fix a basis $B = \langle \vec{\beta}_1, \ldots, \vec{\beta}_n \rangle$ and suppose

$$\mathrm{Rep}_B(\vec{v}_1) = \begin{pmatrix} c_{1,1} \\ \vdots \\ c_{n,1} \end{pmatrix} \quad \ldots \quad \mathrm{Rep}_B(\vec{v}_k) = \begin{pmatrix} c_{1,k} \\ \vdots \\ c_{n,k} \end{pmatrix}$$

so that $\vec{v}_1 = c_{1,1}\vec{\beta}_1 + \cdots + c_{n,1}\vec{\beta}_n$, etc. Then $a_1\vec{v}_1 + \cdots + a_k\vec{v}_k = \vec{0}$ is equivalent to these.

$$\begin{aligned}\vec{0} &= a_1 \cdot (c_{1,1}\vec{\beta}_1 + \cdots + c_{n,1}\vec{\beta}_n) + \cdots + a_k \cdot (c_{1,k}\vec{\beta}_1 + \cdots + c_{n,k}\vec{\beta}_n) \\ &= (a_1c_{1,1} + \cdots + a_kc_{1,k}) \cdot \vec{\beta}_1 + \cdots + (a_1c_{n,1} + \cdots + a_kc_{n,k}) \cdot \vec{\beta}_n\end{aligned}$$

Obviously the bottom equation is true if the coefficients are zero. But, because B is a basis, Theorem 1.12 says that the bottom equation is true if and only if the coefficients are zero. So the relation is equivalent to this.

$$\begin{aligned}a_1c_{1,1} + \cdots + a_kc_{1,k} &= 0 \\ &\vdots \\ a_1c_{n,1} + \cdots + a_kc_{n,k} &= 0\end{aligned}$$

This is the equivalent recast into column vectors.

$$a_1\begin{pmatrix}c_{1,1}\\ \vdots \\ c_{n,1}\end{pmatrix} + \cdots + a_k\begin{pmatrix}c_{1,k}\\ \vdots \\ c_{n,k}\end{pmatrix} = \begin{pmatrix}0\\ \vdots \\ 0\end{pmatrix}$$

Note that not only does a relationship hold for one set if and only if it holds for the other, but it is the same relationship — the a_i are the same. QED

2.4 Theorem In any finite-dimensional vector space, all bases have the same number of elements.

Proof Fix a vector space with at least one finite basis. Choose, from among all of this space's bases, one $B = \langle \vec{\beta}_1, \ldots, \vec{\beta}_n \rangle$ of minimal size. We will show that any other basis $D = \langle \vec{\delta}_1, \vec{\delta}_2, \ldots \rangle$ also has the same number of members, n. Because B has minimal size, D has no fewer than n vectors. We will argue that it cannot have more than n vectors.

The basis B spans the space and $\vec{\delta}_1$ is in the space, so $\vec{\delta}_1$ is a nontrivial linear combination of elements of B. By the Exchange Lemma, we can swap $\vec{\delta}_1$ for a vector from B, resulting in a basis B_1, where one element is $\vec{\delta}_1$ and all of the $n-1$ other elements are $\vec{\beta}$'s.

The prior paragraph forms the basis step for an induction argument. The inductive step starts with a basis B_k (for $1 \leqslant k < n$) containing k members of D and $n-k$ members of B. We know that D has at least n members so there is a $\vec{\delta}_{k+1}$. Represent it as a linear combination of elements of B_k. The key point: in that representation, at least one of the nonzero scalars must be associated with a $\vec{\beta}_i$ or else that representation would be a nontrivial linear relationship among elements of the linearly independent set D. Exchange $\vec{\delta}_{k+1}$ for $\vec{\beta}_i$ to get a new basis B_{k+1} with one $\vec{\delta}$ more and one $\vec{\beta}$ fewer than the previous basis B_k.

Repeat that until no $\vec{\beta}$'s remain, so that B_n contains $\vec{\delta}_1, \ldots, \vec{\delta}_n$. Now, D cannot have more than these n vectors because any $\vec{\delta}_{n+1}$ that remains would be in the span of B_n (since it is a basis) and hence would be a linear combination of the other $\vec{\delta}$'s, contradicting that D is linearly independent. QED

2.5 Definition The *dimension* of a vector space is the number of vectors in any of its bases.

2.10 Corollary No linearly independent set can have a size greater than the dimension of the enclosing space.

Proof The proof of Theorem 2.4 never uses that D spans the space, only that it is linearly independent. QED

2.12 Corollary Any linearly independent set can be expanded to make a basis.

PROOF If a linearly independent set is not already a basis then it must not span the space. Adding to the set a vector that is not in the span will preserve linear independence by Lemma II.1.15. Keep adding until the resulting set does span the space, which the prior corollary shows will happen after only a finite number of steps. QED

2.13 Corollary Any spanning set can be shrunk to a basis.

PROOF Call the spanning set S. If S is empty then it is already a basis (the space must be a trivial space). If $S = \{\vec{0}\}$ then it can be shrunk to the empty basis, thereby making it linearly independent, without changing its span.

Otherwise, S contains a vector $\vec{s}_1$ with $\vec{s}_1 \neq \vec{0}$ and we can form a basis $B_1 = \langle \vec{s}_1 \rangle$. If $[B_1] = [S]$ then we are done. If not then there is a $\vec{s}_2 \in [S]$ such that $\vec{s}_2 \notin [B_1]$. Let $B_2 = \langle \vec{s}_1, \vec{s_2} \rangle$; by Lemma II.1.15 this is linearly independent so if $[B_2] = [S]$ then we are done.

We can repeat this process until the spans are equal, which must happen in at most finitely many steps. QED

2.14 Corollary In an n-dimensional space, a set composed of n vectors is linearly independent if and only if it spans the space.

PROOF First we will show that a subset with n vectors is linearly independent if and only if it is a basis. The 'if' is trivially true — bases are linearly independent. 'Only if' holds because a linearly independent set can be expanded to a basis, but a basis has n elements, so this expansion is actually the set that we began with.

To finish, we will show that any subset with n vectors spans the space if and only if it is a basis. Again, 'if' is trivial. 'Only if' holds because any spanning set can be shrunk to a basis, but a basis has n elements and so this shrunken set is just the one we started with. QED

✓ 1.34 Let $\langle \vec{\beta}_1, \vec{\beta}_2, \vec{\beta}_3 \rangle$ be a basis for a vector space.

(a) Show that $\langle c_1\vec{\beta}_1, c_2\vec{\beta}_2, c_3\vec{\beta}_3 \rangle$ is a basis when $c_1, c_2, c_3 \neq 0$. What happens when at least one c_i is 0?

(b) Prove that $\langle \vec{\alpha}_1, \vec{\alpha}_2, \vec{\alpha}_3 \rangle$ is a basis where $\vec{\alpha}_i = \vec{\beta}_1 + \vec{\beta}_i$.

Two.III.1.34 **(a)** To show that it is linearly independent, note that if $d_1(c_1\vec{\beta}_1) + d_2(c_2\vec{\beta}_2) + d_3(c_3\vec{\beta}_3) = \vec{0}$ then $(d_1c_1)\vec{\beta}_1 + (d_2c_2)\vec{\beta}_2 + (d_3c_3)\vec{\beta}_3 = \vec{0}$, which in turn implies that each d_ic_i is zero. But with $c_i \neq 0$ that means that each d_i is zero. Showing that it spans the space is much the same; because $\langle\vec{\beta}_1, \vec{\beta}_2, \vec{\beta}_3\rangle$ is a basis, and so spans the space, we can for any $\vec{v}$ write $\vec{v} = d_1\vec{\beta}_1 + d_2\vec{\beta}_2 + d_3\vec{\beta}_3$, and then $\vec{v} = (d_1/c_1)(c_1\vec{\beta}_1) + (d_2/c_2)(c_2\vec{\beta}_2) + (d_3/c_3)(c_3\vec{\beta}_3)$.

If any of the scalars are zero then the result is not a basis, because it is not linearly independent.

(b) Showing that $\langle 2\vec{\beta}_1, \vec{\beta}_1 + \vec{\beta}_2, \vec{\beta}_1 + \vec{\beta}_3\rangle$ is linearly independent is easy. To show that it spans the space, assume that $\vec{v} = d_1\vec{\beta}_1 + d_2\vec{\beta}_2 + d_3\vec{\beta}_3$. Then, we can represent the same $\vec{v}$ with respect to $\langle 2\vec{\beta}_1, \vec{\beta}_1 + \vec{\beta}_2, \vec{\beta}_1 + \vec{\beta}_3\rangle$ in this way $\vec{v} = (1/2)(d_1 - d_2 - d_3)(2\vec{\beta}_1) + d_2(\vec{\beta}_1 + \vec{\beta}_2) + d_3(\vec{\beta}_1 + \vec{\beta}_3)$.

Ex: Prove that the only 3D subspace in R3 is R3

- 3D -> basis B has 3 vectors spans the subspace S
- Also, a set with 3 vectors in R3 is linearly independent iff it spans R3. The set is a basis so it is L.I., so the span of it is R3.
- So S is R3

Induction

Principle of Strong Mathematical Induction

Let $P(n)$ be a property that is defined for integers n, and let a and b be fixed integers with $a \leq b$. Suppose the following two statements are true:

1. $P(a), P(a+1), \ldots,$ and $P(b)$ are all true. (**basis step**)
2. For any integer $k \geq b$, if $P(i)$ is true for all integers i from a through k, then $P(k+1)$ is true. (**inductive step**)

Then the statement

$$\text{for all integers } n \geq a,\ P(n)$$

is true. (The supposition that $P(i)$ is true for all integers i from a through k is called the **inductive hypothesis.** Another way to state the inductive hypothesis is to say that $P(a), P(a+1), \ldots, P(k)$ are all true.)

WINTER TERM

Two.III.3. Vector spaces and Linear Systems

Row Space

3.1 Definition The *row space* of a matrix is the span of the set of its rows. The *row rank* is the dimension of this space, the number of linearly independent rows.

3.2 Example If

$$A = \begin{pmatrix} 2 & 3 \\ 4 & 6 \end{pmatrix}$$

then Rowspace(A) is this subspace of the space of two-component row vectors.

$$\{c_1 \cdot (2 \;\; 3) + c_2 \cdot (4 \;\; 6) \mid c_1, c_2 \in \mathbb{R}\}$$

The second row vector is linearly dependent on the first and so we can simplify the above description to $\{c \cdot (2 \;\; 3) \mid c \in \mathbb{R}\}$.

3.3 Lemma If two matrices A and B are related by a row operation

$$A \xrightarrow{\rho_i \leftrightarrow \rho_j} B \quad \text{or} \quad A \xrightarrow{k\rho_i} B \quad \text{or} \quad A \xrightarrow{k\rho_i + \rho_j} B$$

(for $i \neq j$ and $k \neq 0$) then their row spaces are equal. Hence, row-equivalent matrices have the same row space and therefore the same row rank.

To be related by a row operation means that every row of B is a linear combination of the rows of A.
→ row space(a) = row space(b) if they are related by a row operation (every row of b is a linear combination of the rows of a and vice versa

3.4 Lemma The nonzero rows of an echelon form matrix make up a linearly independent set.

PROOF Lemma One.III.2.5 says that no nonzero row of an echelon form matrix is a linear combination of the other rows. This result restates that using this chapter's terminology. QED

Thus, in the language of this chapter, Gaussian reduction works by eliminating linear dependences among rows, leaving the span unchanged, until no nontrivial linear relationships remain among the nonzero rows. In short, Gauss's Method produces a basis for the row space.

3.5 Example From any matrix, we can produce a basis for the row space by performing Gauss's Method and taking the nonzero rows of the resulting echelon form matrix. For instance,

$$\begin{pmatrix}1&3&1\\1&4&1\\2&0&5\end{pmatrix}\xrightarrow[-2\rho_1+\rho_3]{-\rho_1+\rho_2}\xrightarrow{6\rho_2+\rho_3}\begin{pmatrix}1&3&1\\0&1&0\\0&0&3\end{pmatrix}$$

produces the basis $\langle(1\ \ 3\ \ 1),(0\ \ 1\ \ 0),(0\ \ 0\ \ 3)\rangle$ for the row space. This is a basis for the row space of both the starting and ending matrices, since the two row spaces are equal.

Using this technique, we can also find bases for spans not directly involving row vectors.

EX: if the basis is two, row rank is 2 also for any matrix with that same reduced echelon form.

Column Space

3.6 Definition The *column space* of a matrix is the span of the set of its columns. The *column rank* is the dimension of the column space, the number of linearly independent columns.

Our interest in column spaces stems from our study of linear systems. An example is that this system

$$\begin{aligned} c_1 + 3c_2 + 7c_3 &= d_1 \\ 2c_1 + 3c_2 + 8c_3 &= d_2 \\ c_2 + 2c_3 &= d_3 \\ 4c_1 \qquad\;\; + 4c_3 &= d_4 \end{aligned}$$

has a solution if and only if the vector of d's is a linear combination of the other column vectors,

$$c_1 \begin{pmatrix} 1 \\ 2 \\ 0 \\ 4 \end{pmatrix} + c_2 \begin{pmatrix} 3 \\ 3 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 7 \\ 8 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{pmatrix}$$

meaning that the vector of d's is in the column space of the matrix of coefficients.

The set of all linear combinations of the column is the COLUMN SPACE

Column space has to do with the solutions → the system has a solution if and only if the vector on the right is in the column space of this matrix.

Transpose

3.8 Definition The *transpose* of a matrix is the result of interchanging its rows and columns, so that column j of the matrix A is row j of A^T and vice versa.

So we can summarize the prior example as "transpose, reduce, and transpose back."

We can even, at the price of tolerating the as-yet-vague idea of vector spaces being "the same," use Gauss's Method to find bases for spans in other types of vector spaces.

3.9 Example To get a basis for the span of $\{x^2 + x^4, 2x^2 + 3x^4, -x^2 - 3x^4\}$ in the space $\mathcal{P}_4$, think of these three polynomials as "the same" as the row vectors $(0\ 0\ 1\ 0\ 1)$, $(0\ 0\ 2\ 0\ 3)$, and $(0\ 0\ -1\ 0\ -3)$, apply Gauss's Method

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 0 & 3 \\ 0 & 0 & -1 & 0 & -3 \end{pmatrix} \xrightarrow[\rho_1+\rho_3]{-2\rho_1+\rho_2} \xrightarrow{2\rho_2+\rho_3} \begin{pmatrix} 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and translate back to get the basis $\langle x^2 + x^4, x^4 \rangle$. (As mentioned earlier, we will make the phrase "the same" precise at the start of the next chapter.)

Thus, the first point for this subsection is that the tools of this chapter give us a more conceptual understanding of Gaussian reduction.

For the second point observe that row operations on a matrix can change its column space.

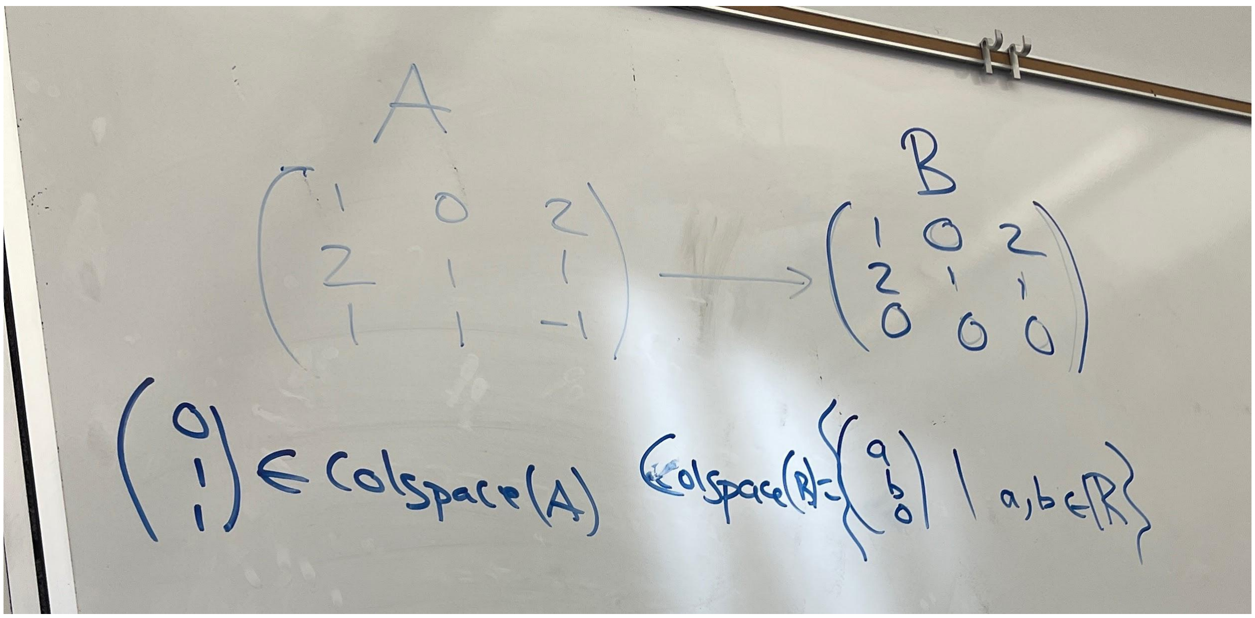
$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \xrightarrow{-2\rho_1+\rho_2} \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$$

The column space of the left-hand matrix contains vectors with a second component that is nonzero but the column space of the right-hand matrix contains only vectors whose second component is zero, so the two spaces are different. This observation makes next result surprising.

→ example: find the basis for the column space of a matrix → transpose so column becomes rows, and row reduces

Row operations MIGHT change COLUMN SPACE, but DON'T change COLUMN RANK, since you can't change the number of linearly independent sets.?

But, it can change the column space because:



→ the colspace of A includes vectors that colspace b does not contain → column space is not contained.

Row Rank = Column Rank

3.10 Lemma Row operations do not change the column rank.

PROOF Restated, if A reduces to B then the column rank of B equals the column rank of A.

This proof will be finished if we show that row operations do not affect linear relationships among columns, because the column rank is the size of the largest set of unrelated columns. That is, we will show that a relationship exists among columns (such as that the fifth column is twice the second plus the fourth) if and only if that relationship exists after the row operation. But this is exactly the first theorem of this book, Theorem One.I.1.5: in a relationship among columns,

$$c_1 \cdot \begin{pmatrix} a_{1,1} \\ a_{2,1} \\ \vdots \\ a_{m,1} \end{pmatrix} + \cdots + c_n \cdot \begin{pmatrix} a_{1,n} \\ a_{2,n} \\ \vdots \\ a_{m,n} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

row operations leave unchanged the set of solutions $(c_1, \ldots, c_n)$. QED

Another way to make the point that Gauss's Method has something to say about the column space as well as about the row space is with Gauss-Jordan reduction. It ends with the reduced echelon form of a matrix, as here.

$$\begin{pmatrix} 1 & 3 & 1 & 6 \\ 2 & 6 & 3 & 16 \\ 1 & 3 & 1 & 6 \end{pmatrix} \longrightarrow \cdots \longrightarrow \begin{pmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Consider the row space and the column space of this result.

The first point made earlier in this subsection says that to get a basis for the row space we can just collect the rows with leading entries. However, because this is in reduced echelon form, a basis for the column space is just as easy: collect the columns containing the leading entries, $\langle \vec{e}_1, \vec{e}_2 \rangle$. Thus, for a reduced echelon

form matrix we can find bases for the row and column spaces in essentially the same way, by taking the parts of the matrix, the rows or columns, containing the leading entries.

Dimensionality Theorem?

Rank is dimensionality of row space = # linearly independent rows
→ r linearly independent rows in the matrix

3.13 Theorem For linear systems with n unknowns and with matrix of coefficients A, the statements

(1) the rank of A is r

(2) the vector space of solutions of the associated homogeneous system has dimension $n - r$

are equivalent.

So if the system has at least one particular solution then for the set of solutions, the number of parameters equals $n - r$, the number of variables minus the rank of the matrix of coefficients.

Proof The rank of A is r if and only if Gaussian reduction on A ends with r nonzero rows. That's true if and only if echelon form matrices row equivalent to A have r-many leading variables. That in turn holds if and only if there are $n - r$ free variables. QED

Full rank = all rows and columns are linearly independent

Equivalence rank and non-singularity statement:

3.14 Corollary Where the matrix A is $n \times n$, these statements

(1) the rank of A is n
(2) A is nonsingular
(3) the rows of A form a linearly independent set
(4) the columns of A form a linearly independent set
(5) any linear system whose matrix of coefficients is A has one and only one solution

are equivalent.

Proof Clearly (1) $\iff$ (2) $\iff$ (3) $\iff$ (4). The last, (4) $\iff$ (5), holds because a set of n column vectors is linearly independent if and only if it is a basis for $\mathbb{R}^n$, but the system

$$c_1 \begin{pmatrix} a_{1,1} \\ a_{2,1} \\ \vdots \\ a_{m,1} \end{pmatrix} + \cdots + c_n \begin{pmatrix} a_{1,n} \\ a_{2,n} \\ \vdots \\ a_{m,n} \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_m \end{pmatrix}$$

has a unique solution for all choices of $d_1, \ldots, d_n \in \mathbb{R}$ if and only if the vectors of a's on the left form a basis. QED

Part 1

EX1

✓ 3.20 Find a basis for the row space of this matrix.

$$\begin{pmatrix} 2 & 0 & 3 & 4 \\ 0 & 1 & 1 & -1 \\ 3 & 1 & 0 & 2 \\ 1 & 0 & -4 & -1 \end{pmatrix}$$

Two.III.3.20 A routine Gaussian reduction

$$\begin{pmatrix} 2 & 0 & 3 & 4 \\ 0 & 1 & 1 & -1 \\ 3 & 1 & 0 & 2 \\ 1 & 0 & -4 & -1 \end{pmatrix} \xrightarrow[-(1/2)\rho_1+\rho_4]{-(3/2)\rho_1+\rho_3} \xrightarrow{-\rho_2+\rho_3} \xrightarrow{-\rho_3+\rho_4} \begin{pmatrix} 2 & 0 & 3 & 4 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & -11/2 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

suggests this basis $\langle (2\ \ 0\ \ 3\ \ 4), (0\ \ 1\ \ 1\ \ -1), (0\ \ 0\ \ -11/2\ \ -3) \rangle$.

Another procedure, perhaps more convenient, is to swap rows first,

$$\xrightarrow{\rho_1 \leftrightarrow \rho_4} \xrightarrow[-2\rho_1+\rho_4]{-3\rho_1+\rho_3} \xrightarrow{-\rho_2+\rho_3} \xrightarrow{-\rho_3+\rho_4} \begin{pmatrix} 1 & 0 & -4 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 11 & 6 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

leading to the basis $\langle (1\ \ 0\ \ -4\ \ -1), (0\ \ 1\ \ 1\ \ -1), (0\ \ 0\ \ 11\ \ 6) \rangle$.

=> to find a basis of a row space, do row reduction

EX2: Rank of a Zero matrix is Zero

EX3:

3.22 Give a basis for the column space of this matrix. Give the matrix's rank.

$$\begin{pmatrix} 1 & 3 & -1 & 2 \\ 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 4 \end{pmatrix}$$

Two.III.3.22 We want a basis for this span.

$$[\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix}] \subseteq \mathbb{R}^3$$

The most straightforward approach is to transpose those columns to rows, use Gauss's Method to find a basis for the span of the rows, and then transpose them back to columns.

$$\begin{pmatrix} 1 & 2 & 0 \\ 3 & 1 & 1 \\ -1 & 1 & 1 \\ 2 & 0 & 4 \end{pmatrix} \xrightarrow[\substack{\rho_1+\rho_3 \\ -2\rho_1+\rho_4}]{-3\rho_1+\rho_2} \xrightarrow[-(4/5)\rho_2+\rho_4]{(3/5)\rho_2+\rho_3} \xrightarrow{-2\rho_3+\rho_4} \begin{pmatrix} 1 & 2 & 0 \\ 0 & -5 & 1 \\ 0 & 0 & 8/5 \\ 0 & 0 & 0 \end{pmatrix}$$

Discard the zero vector as showing that there was a redundancy among the starting vectors, to get this basis for the column space.

$$\langle \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -5 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 8/5 \end{pmatrix} \rangle$$

The matrix's rank is the dimension of its column space, so it is three. (It is also equal to the dimension of its row space.)

EX4:

✓ 3.23 Find a basis for the span of each set.

(a) $\{(1\ \ 3), (-1\ \ 3), (1\ \ 4), (2\ \ 1)\} \subseteq \mathcal{M}_{1\times 2}$

(b) $\{\begin{pmatrix}1\\2\\1\end{pmatrix}, \begin{pmatrix}3\\1\\-1\end{pmatrix}, \begin{pmatrix}1\\-3\\-3\end{pmatrix}\} \subseteq \mathbb{R}^3$

(c) $\{1+x, 1-x^2, 3+2x-x^2\} \subseteq \mathcal{P}_3$

(d) $\{\begin{pmatrix}1&0&1\\3&1&-1\end{pmatrix}, \begin{pmatrix}1&0&3\\2&1&4\end{pmatrix}, \begin{pmatrix}-1&0&-5\\-1&-1&-9\end{pmatrix}\} \subseteq \mathcal{M}_{2\times 3}$

Two.III.3.23 **(a)** This reduction

$$\begin{pmatrix}1&3\\-1&3\\1&4\\2&1\end{pmatrix} \xrightarrow[\substack{-\rho_1+\rho_3\\-2\rho_1+\rho_4}]{\rho_1+\rho_2} \xrightarrow[(5/6)\rho_2+\rho_4]{-(1/6)\rho_2+\rho_3} \begin{pmatrix}1&3\\0&6\\0&0\\0&0\end{pmatrix}$$

gives $\langle(1\ \ 3), (0\ \ 6)\rangle$.

(b) Transposing and reducing

$$\begin{pmatrix}1&2&1\\3&1&-1\\1&-3&-3\end{pmatrix} \xrightarrow[-\rho_1+\rho_3]{-3\rho_1+\rho_2} \begin{pmatrix}1&2&1\\0&-5&-4\\0&-5&-4\end{pmatrix} \xrightarrow{-\rho_2+\rho_3} \begin{pmatrix}1&2&1\\0&-5&-4\\0&0&0\end{pmatrix}$$

and then transposing back gives this basis.

$$\langle\begin{pmatrix}1\\2\\1\end{pmatrix}, \begin{pmatrix}0\\-5\\-4\end{pmatrix}\rangle$$

(c) Notice first that the surrounding space is as $\mathcal{P}_3$, not $\mathcal{P}_2$. Then, taking the first polynomial $1+1\cdot x+0\cdot x^2+0\cdot x^3$ to be "the same" as the row vector $(1\ \ 1\ \ 0\ \ 0)$, etc., leads to

$$\begin{pmatrix}1&1&0&0\\1&0&-1&0\\3&2&-1&0\end{pmatrix} \xrightarrow[-3\rho_1+\rho_3]{-\rho_1+\rho_2} \xrightarrow{-\rho_2+\rho_3} \begin{pmatrix}1&1&0&0\\0&-1&-1&0\\0&0&0&0\end{pmatrix}$$

which yields the basis $\langle 1+x, -x-x^2\rangle$.

(d) Here "the same" gives

$$\begin{pmatrix}1&0&1&3&1&-1\\1&0&3&2&1&4\\-1&0&-5&-1&-1&-9\end{pmatrix} \xrightarrow[\rho_1+\rho_3]{-\rho_1+\rho_2} \xrightarrow{2\rho_2+\rho_3} \begin{pmatrix}1&0&1&3&1&-1\\0&0&2&-1&0&5\\0&0&0&0&0&0\end{pmatrix}$$

leading to this basis.

$$\langle\begin{pmatrix}1&0&1\\3&1&-1\end{pmatrix}, \begin{pmatrix}0&0&2\\-1&0&5\end{pmatrix}\rangle$$

?EX5:

3.28 Show that a linear system with at least one solution has at most one solution if and only if the matrix of coefficients has rank equal to the number of its columns.

Two.III.3.28 We apply Theorem 3.13. The number of columns of a matrix of coefficients A of a linear system equals the number n of unknowns. A linear system with at least one solution has at most one solution if and only if the space of solutions of the associated homogeneous system has dimension zero (recall: in the 'General = Particular + Homogeneous' equation $\vec{v} = \vec{p} + \vec{h}$, provided that such a $\vec{p}$

exists, the solution $\vec{v}$ is unique if and only if the vector $\vec{h}$ is unique, namely $\vec{h} = \vec{0}$). But that means, by the theorem, that $n = r$.

EX6:

3.38 True or false: the column space of a matrix equals the row space of its transpose.

Two.III.3.38 False. The first is a set of columns while the second is a set of rows.

This example, however,

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}, \qquad A^{T} = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$$

indicates that as soon as we have a formal meaning for "the same", we can apply it here:

$$\text{Columnspace}(A) = [\{ \begin{pmatrix} 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \end{pmatrix}, \begin{pmatrix} 3 \\ 6 \end{pmatrix} \}]$$

while

$$\text{Rowspace}(A^{T}) = [\{ (1\ \ 4), (2\ \ 5), (3\ \ 6) \}]$$

are "the same" as each other.

Part 2

EX:

3.31 Show that the set $\{(1, -1, 2, -3), (1, 1, 2, 0), (3, -1, 6, -6)\}$ does not have the same span as $\{(1, 0, 1, 0), (0, 2, 0, 3)\}$. What, by the way, is the vector space?

These are tuples since the commas separate the numbers. In a normal row they don't add commas between numbers.

?Solution

Two.III.3.31 First, the vector space is the set of four-tuples of real numbers, under the natural operations. Although this is not the set of four-wide row vectors, the difference is slight — it is "the same" as that set. So we will treat the four-tuples like four-wide vectors.

With that, one way to see that $(1,0,1,0)$ is not in the span of the first set is to note that this reduction

$$\begin{pmatrix} 1 & -1 & 2 & -3 \\ 1 & 1 & 2 & 0 \\ 3 & -1 & 6 & -6 \end{pmatrix} \xrightarrow[-3\rho_1+\rho_3]{-\rho_1+\rho_2} \xrightarrow{-\rho_2+\rho_3} \begin{pmatrix} 1 & -1 & 2 & -3 \\ 0 & 2 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and this one

$$\begin{pmatrix} 1 & -1 & 2 & -3 \\ 1 & 1 & 2 & 0 \\ 3 & -1 & 6 & -6 \\ 1 & 0 & 1 & 0 \end{pmatrix} \xrightarrow[\substack{-3\rho_1+\rho_3 \\ -\rho_1+\rho_4}]{-\rho_1+\rho_2} \xrightarrow[-(1/2)\rho_2+\rho_4]{-\rho_2+\rho_3} \xrightarrow{\rho_3\leftrightarrow\rho_4} \begin{pmatrix} 1 & -1 & 2 & -3 \\ 0 & 2 & 0 & 3 \\ 0 & 0 & -1 & 3/2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

yield matrices differing in rank. This means that addition of $(1,0,1,0)$ to the set of the first three four-tuples increases the rank, and hence the span, of that set. Therefore $(1,0,1,0)$ is not already in the span.

EX2:

✓ 3.32 Show that this set of column vectors

$$\{\begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} \mid \text{there are } x,\ y, \text{ and } z \text{ such that: } \begin{array}{rcl} 3x+2y+4z & = & d_1 \\ x \qquad\ \ - z & = & d_2 \\ 2x+2y+5z & = & d_3 \end{array}\}$$

is a subspace of $\mathbb{R}^3$. Find a basis.

Since (d1 d2 d3) is a linear combination of the columns → this set is inherently a column space → that means that the set is the span of all columns → that also mean that it is a subspace since a span is a subspace (you can also use non-zero and closed under operations to prove it is a subspace, although that is longer) of coefficients. To find a basis for the column space,

$$\{c_1 \begin{pmatrix}3\\1\\2\end{pmatrix} + c_2 \begin{pmatrix}2\\0\\2\end{pmatrix} + c_3 \begin{pmatrix}4\\-1\\5\end{pmatrix} \mid c_1, c_2, c_3 \in \mathbb{R}\}$$

we eliminate linear relationships among the three column vectors from the spanning set by transposing, reducing,

$$\begin{pmatrix}3&1&2\\2&0&2\\4&-1&5\end{pmatrix} \xrightarrow[-(4/3)\rho_1+\rho_3]{-(2/3)\rho_1+\rho_2} \xrightarrow{-(7/2)\rho_2+\rho_3} \begin{pmatrix}3&1&2\\0&-2/3&2/3\\0&0&0\end{pmatrix}$$

omitting the zero row, and transposing back.

$$\langle \begin{pmatrix}3\\1\\2\end{pmatrix}, \begin{pmatrix}0\\-2/3\\2/3\end{pmatrix} \rangle$$

Transpose !!!

EX3:

3.34 In this subsection we have shown that Gaussian reduction finds a basis for the row space.

(a) Show that this basis is not unique — different reductions may yield different bases.

(b) Produce matrices with equal row spaces but unequal numbers of rows.

(c) Prove that two matrices have equal row spaces if and only if after Gauss-Jordan reduction they have the same nonzero rows.

For a) we can also say that echelon forms are not UNIQUE

Two.III.3.34 **(a)** These reductions give different bases.

$$\begin{pmatrix}1&2&0\\1&2&1\end{pmatrix} \xrightarrow{-\rho_1+\rho_2} \begin{pmatrix}1&2&0\\0&0&1\end{pmatrix} \qquad \begin{pmatrix}1&2&0\\1&2&1\end{pmatrix} \xrightarrow{-\rho_1+\rho_2} \xrightarrow{2\rho_2} \begin{pmatrix}1&2&0\\0&0&2\end{pmatrix}$$

(b) An easy example is this.

$$\begin{pmatrix}1&2&1\\3&1&4\end{pmatrix} \qquad \begin{pmatrix}1&2&1\\3&1&4\\0&0&0\end{pmatrix}$$

This is a less simplistic example.

$$\begin{pmatrix}1&2&1\\3&1&4\end{pmatrix} \qquad \begin{pmatrix}1&2&1\\3&1&4\\2&4&2\\4&3&5\end{pmatrix}$$

(c) Because the row spaces of A and B are equal, the two are row equivalent. Because each row equivalence class contains a unique reduced echelon form (Theorem One.III.2.6), the reduced echelon form of A must equal the reduced echelon form of B.

EX4:

3.37 Show that the rank of a matrix equals the rank of its transpose.

Two.III.3.37 Because the rows of a matrix A are the columns of A^{T} the dimension of the row space of A equals the dimension of the column space of A^{T}. But the dimension of the row space of A is the rank of A and the dimension of the column space of A^{T} is the rank of A^{T}. Thus the two ranks are equal.

EX5:

3.41 An $m \times n$ matrix has *full row rank* if its row rank is m, and it has *full column rank* if its column rank is n.

(a) Show that a matrix can have both full row rank and full column rank only if it is square.

(b) Prove that the linear system with matrix of coefficients A has a solution for any $d_1, \ldots, d_n$'s on the right side if and only if A has full row rank.

(c) Prove that a homogeneous system has a unique solution if and only if its matrix of coefficients A has full column rank.

(d) Prove that the statement "if a system with matrix of coefficients A has any solution then it has a unique solution" holds if and only if A has full column rank.

Two.III.3.41 **(a)** Row rank equals column rank so each is at most the minimum of the number of rows and columns. Hence both can be full only if the number of rows equals the number of columns. (Of course, the converse does not hold: a square matrix need not have full row rank or full column rank.)

(b) If A has full row rank then, no matter what the right-hand side, Gauss's Method on the augmented matrix ends with a leading one in each row and none of those leading ones in the furthest right column (the "augmenting" column). Back substitution then gives a solution.

On the other hand, if the linear system lacks a solution for some right-hand side it can only be because Gauss's Method leaves some row so that it has all zeroes on the left of the "augmenting" bar and has a nonzero entry on the right. Thus, if A does not have a solution for some right-hand sides, then A does not have full row rank because some of its rows have been eliminated.

(c) The matrix A has full column rank if and only if its columns form a linearly independent set. That's equivalent to the existence of only the trivial linear relationship among the columns, so the only solution of the system is where each variable is 0.

(d) The matrix A has full column rank if and only if the set of its columns is linearly independent, and so forms a basis for its span. That's equivalent to the existence of a unique linear representation of all vectors in that span. That proves it, since any linear representation of a vector in the span is a solution of the linear system.

EX6:

3.43 What is the relationship between $\text{rank}(A)$ and $\text{rank}(-A)$? Between $\text{rank}(A)$ and $\text{rank}(kA)$? What, if any, is the relationship between $\text{rank}(A)$, $\text{rank}(B)$, and $\text{rank}(A+B)$?

Two.III.3.43 Clearly $\text{rank}(A) = \text{rank}(-A)$ as Gauss's Method allows us to multiply all rows of a matrix by -1. In the same way, when $k \neq 0$ we have $\text{rank}(A) = \text{rank}(kA)$.

Addition is more interesting. The rank of a sum can be smaller than the rank of the summands.

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} -1 & -2 \\ -3 & -4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

The rank of a sum can be bigger than the rank of the summands.

$$\begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

But there is an upper bound (other than the size of the matrices). In general, $\text{rank}(A+B) \leqslant \text{rank}(A) + \text{rank}(B)$.

To prove this, note that we can perform Gaussian elimination on $A+B$ in either of two ways: we can first add A to B and then apply the appropriate sequence of reduction steps

$$(A+B) \xrightarrow{\text{step}_1} \cdots \xrightarrow{\text{step}_k} \text{echelon form}$$

or we can get the same results by performing step_1 through step_k separately on A and B, and then adding. The largest rank that we can end with in the second

case is clearly the sum of the ranks. (The matrices above give examples of both possibilities, $\text{rank}(A+B) < \text{rank}(A)+\text{rank}(B)$ and $\text{rank}(A+B) = \text{rank}(A)+\text{rank}(B)$, happening.)

Three.I Isomorphisms

DEFINITION

1.3 Definition An *isomorphism* between two vector spaces V and W is a map $f\colon V \to W$ that

(1) is a correspondence: f is one-to-one and onto;*

(2) *preserves structure:* if $\vec{v}_1, \vec{v}_2 \in V$ then

$$f(\vec{v}_1 + \vec{v}_2) = f(\vec{v}_1) + f(\vec{v}_2)$$

and if $\vec{v} \in V$ and $r \in \mathbb{R}$ then

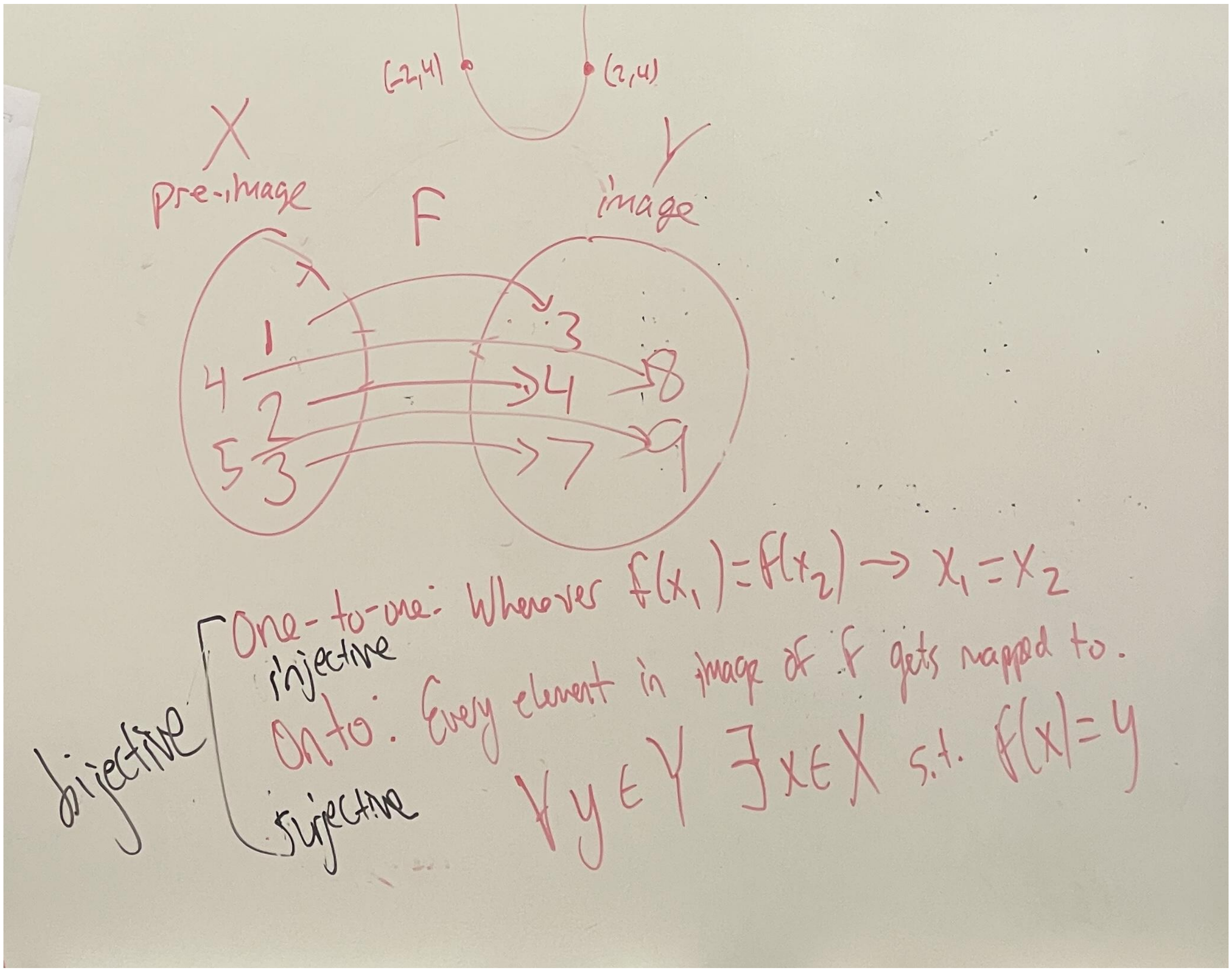
$$f(r\vec{v}) = rf(\vec{v})$$

(we write $V \cong W$, read "V is isomorphic to W", when such a map exists).

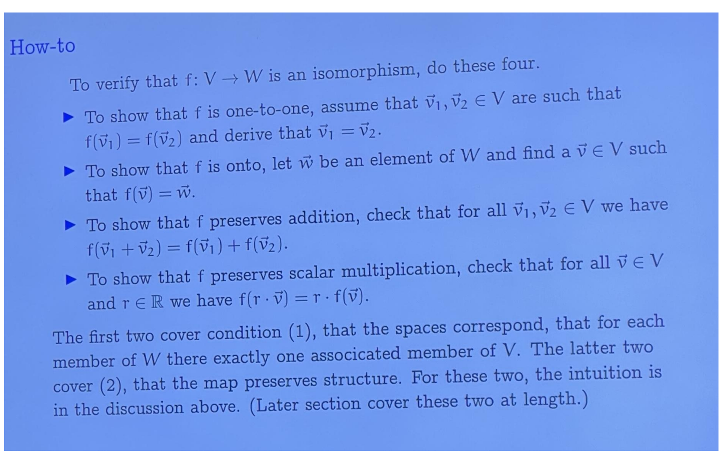
"Morphism" means map, so "isomorphism" means a map expressing sameness.

One to one and onto? What does onto mean

- Isomorphism is a function between vector space that maps values and preserves structure?
- Preserves Basis If we have a basis in one space that is an isomorphism then it is a basis in the other space.
- Independent Subspaces
- Same Dimensions
- Inverse of isomorphism is also isomorphic
- Isomorphism is an equivalence relation between vector spaces.
- Isomorphism maps a zero vector to a zero vector (preserves zero vector)



PROOF



How-to

To verify that $f\colon V \to W$ is an isomorphism, do these four.

- To show that f is one-to-one, assume that $\vec{v}_1, \vec{v}_2 \in V$ are such that $f(\vec{v}_1) = f(\vec{v}_2)$ and derive that $\vec{v}_1 = \vec{v}_2$.
- To show that f is onto, let $\vec{w}$ be an element of W and find a $\vec{v} \in V$ such that $f(\vec{v}) = \vec{w}$.
- To show that f preserves addition, check that for all $\vec{v}_1, \vec{v}_2 \in V$ we have $f(\vec{v}_1 + \vec{v}_2) = f(\vec{v}_1) + f(\vec{v}_2)$.
- To show that f preserves scalar multiplication, check that for all $\vec{v} \in V$ and $r \in \mathbb{R}$ we have $f(r \cdot \vec{v}) = r \cdot f(\vec{v})$.

The first two cover condition (1), that the spaces correspond, that for each member of W there exactly one associcated member of V. The latter two cover (2), that the map preserves structure. For these two, the intuition is in the discussion above. (Later section cover these two at length.)

EXAMPLE:

1.4 Example The vector space $G = \{c_1 \cos\theta + c_2 \sin\theta \mid c_1, c_2 \in \mathbb{R}\}$ of functions of θ is isomorphic to $\mathbb{R}^2$ under this map.

$$c_1 \cos\theta + c_2 \sin\theta \overset{f}{\longmapsto} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

We will check this by going through the conditions in the definition. We will first verify condition (1), that the map is a correspondence between the sets underlying the spaces.

To establish that f is one-to-one we must prove that $f(\vec{a}) = f(\vec{b})$ only when $\vec{a} = \vec{b}$. If

$$f(a_1 \cos\theta + a_2 \sin\theta) = f(b_1 \cos\theta + b_2 \sin\theta)$$

then by the definition of f

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

from which we conclude that $a_1 = b_1$ and $a_2 = b_2$, because column vectors are equal only when they have equal components. Thus $a_1 \cos\theta + a_2 \sin\theta = b_1 \cos\theta + b_2 \sin\theta$, and as required we've verified that $f(\vec{a}) = f(\vec{b})$ implies that $\vec{a} = \vec{b}$.

*More information on correspondences is in the appendix.

To prove that f is onto we must check that any member of the codomain $\mathbb{R}^2$ is the image of some member of the domain G. So, consider a member of the codomain

$$\begin{pmatrix} x \\ y \end{pmatrix}$$

and note that it is the image under f of $x \cos\theta + y \sin\theta$.

Next we will verify condition (2), that f preserves structure. This computation shows that f preserves addition.

$$\begin{aligned} f\big((a_1 \cos\theta + a_2 \sin\theta) + (b_1 \cos\theta + b_2 \sin\theta)\big) \\ &= f\big((a_1 + b_1)\cos\theta + (a_2 + b_2)\sin\theta\big) \\ &= \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \end{pmatrix} \\ &= \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \\ &= f(a_1 \cos\theta + a_2 \sin\theta) + f(b_1 \cos\theta + b_2 \sin\theta) \end{aligned}$$

The computation showing that f preserves scalar multiplication is similar.

$$\begin{aligned} f\big(r \cdot (a_1 \cos\theta + a_2 \sin\theta)\big) &= f(\, r a_1 \cos\theta + r a_2 \sin\theta\,) \\ &= \begin{pmatrix} r a_1 \\ r a_2 \end{pmatrix} \\ &= r \cdot \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \\ &= r \cdot f(a_1 \cos\theta + a_2 \sin\theta) \end{aligned}$$

With both (1) and (2) verified, we know that f is an isomorphism and we can say that the spaces are isomorphic $G \cong \mathbb{R}^2$.

EXAMPLE OF PROOF (ADVANCED Example)

Example Consider these two vector spaces (under the natural operations)

$$V = \{\begin{pmatrix} a & b \\ c & 0 \end{pmatrix} \mid a, b, c \in \mathbb{R}\} \qquad W = \{(x \;\; y \;\; z) \mid x, y, z \in \mathbb{R}\}$$

and consider this function.

$$\begin{pmatrix} a & b \\ c & 0 \end{pmatrix} \stackrel{f}{\longmapsto} (b \;\; 2a \;\; a+c)$$

Here is an example of the map's action.

$$\begin{pmatrix} 3 & 2 \\ 1 & 0 \end{pmatrix} \stackrel{f}{\longmapsto} (2 \;\; 6 \;\; 4)$$

We will verify that f is an isomorphism.

To show that f is one-to-one, suppose that

$$f(\begin{pmatrix} a_1 & b_1 \\ c_1 & 0 \end{pmatrix}) = f(\begin{pmatrix} a_2 & b_2 \\ c_2 & 0 \end{pmatrix})$$

Then $(b_1 \;\; 2a_1 \;\; a_1 + c_1) = (b_2 \;\; 2a_2 \;\; a_2 + c_2)$. The first entries give that $b_1 = b_2$, the second entries that $a_1 = a_2$, and with that the third entries give that $c_1 = c_2$.

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & 0 \end{pmatrix} = \begin{pmatrix} a_2 & b_2 \\ c_2 & 0 \end{pmatrix}$$

To show that f is onto, consider a member of W.

$$\vec{w} = (x \;\; y \;\; z)$$

We must find a $\vec{v}$ so that $f(\vec{v}) = \vec{w}$. The map sends the upper right entry of the input to the first entry of the output, so the upper right of $\vec{v}$ is x. Similarly, the upper left of $\vec{v}$ is $(1/2)y$. With that, the lower left is $z - (1/2)y$.

$$(x \;\; y \;\; z) = f(\begin{pmatrix} y/2 & x \\ z - y/2 & 0 \end{pmatrix})$$

To show that f preserves addition, assume

$$f(\begin{pmatrix} a_1 & b_1 \\ c_1 & 0 \end{pmatrix} + \begin{pmatrix} a_2 & b_2 \\ c_2 & 0 \end{pmatrix}) = f(\begin{pmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & 0 \end{pmatrix})$$

which equals $(b_1 + b_2 \;\; 2(a_1 + a_2) \;\; (a_1 + a_2) + (c_1 + c_2))$. In turn, that equals this.

$$(b_1 \;\; 2a_1 \;\; a_1 + c_1) + (b_2 \;\; 2a_2 \;\; a_2 + c_2) = f(\begin{pmatrix} a_1 & b_1 \\ c_1 & 0 \end{pmatrix}) + f(\begin{pmatrix} a_2 & b_2 \\ c_2 & 0 \end{pmatrix})$$

Preservation of scalar multiplication is similar.

$$\begin{aligned} f(r \cdot \begin{pmatrix} a & b \\ c & 0 \end{pmatrix}) &= f(\begin{pmatrix} ra & rb \\ rc & 0 \end{pmatrix}) \\ &= (rb \;\; 2ra \;\; ra + rc) \\ &= r \cdot (b \;\; 2a \;\; a + c) \\ &= r \cdot f(\begin{pmatrix} a & b \\ c & 0 \end{pmatrix}) \end{aligned}$$

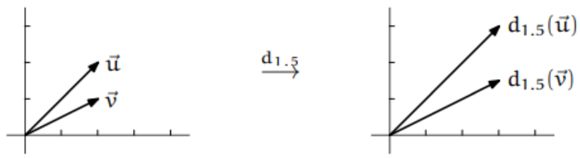
Every space is isomorphic to itself under identity map.

1.6 Example Every space is isomorphic to itself under the identity map. The check is easy.

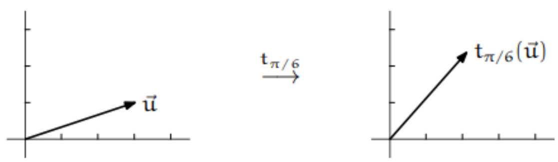
Automorphism

1.7 Definition An *automorphism* is an isomorphism of a space with itself.

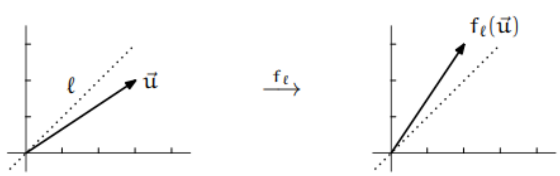
1.8 Example A *dilation* map $d_s\colon \mathbb{R}^2 \to \mathbb{R}^2$ that multiplies all vectors by a nonzero scalar s is an automorphism of $\mathbb{R}^2$.



Another automorphism is a *rotation* or *turning map*, $t_\theta\colon \mathbb{R}^2 \to \mathbb{R}^2$ that rotates all vectors through an angle θ.



A third type of automorphism of $\mathbb{R}^2$ is a map $f_\ell\colon \mathbb{R}^2 \to \mathbb{R}^2$ that *flips* or *reflects* all vectors over a line ℓ through the origin.



Checking that these are automorphisms is Exercise 33.

Isomorphism maps zero vector to zero vector.

1.10 Lemma An isomorphism maps a zero vector to a zero vector.

Proof Where $f\colon V \to W$ is an isomorphism, fix some $\vec{v} \in V$. Then $f(\vec{0}_V) = f(0 \cdot \vec{v}) = 0 \cdot f(\vec{v}) = \vec{0}_W$. QED

f(v) is a vector in w → 0 * f(v) = zero vector in w

⇒ ZERO vector is PRESERVED in isomorphism

Equivalent Statements

1.11 Lemma For any map $f\colon V \to W$ between vector spaces these statements are equivalent.

(1) f preserves structure

$$f(\vec{v}_1 + \vec{v}_2) = f(\vec{v}_1) + f(\vec{v}_2) \quad \text{and} \quad f(c\vec{v}) = c\, f(\vec{v})$$

(2) f preserves linear combinations of two vectors

$$f(c_1\vec{v}_1 + c_2\vec{v}_2) = c_1 f(\vec{v}_1) + c_2 f(\vec{v}_2)$$

(3) f preserves linear combinations of any finite number of vectors

$$f(c_1\vec{v}_1 + \cdots + c_n\vec{v}_n) = c_1 f(\vec{v}_1) + \cdots + c_n f(\vec{v}_n)$$

PROOF Since the implications (3) $\Longrightarrow$ (2) and (2) $\Longrightarrow$ (1) are clear, we need only show that (1) $\Longrightarrow$ (3). So assume statement (1). We will prove (3) by induction on the number of summands n.

The one-summand base case, that $f(c\vec{v}_1) = c\, f(\vec{v}_1)$, is covered by the second clause of statement (1).

For the inductive step assume that statement (3) holds whenever there are k or fewer summands. Consider the $k+1$-summand case. Use the first half of (1)

to break the sum along the final '+'.

$$f(c_1\vec{v}_1 + \cdots + c_k\vec{v}_k + c_{k+1}\vec{v}_{k+1}) = f(c_1\vec{v}_1 + \cdots + c_k\vec{v}_k) + f(c_{k+1}\vec{v}_{k+1})$$

Use the inductive hypothesis to break up the k-term sum on the left.

$$= f(c_1\vec{v}_1) + \cdots + f(c_k\vec{v}_k) + f(c_{k+1}\vec{v}_{k+1})$$

Now the second half of (1) gives

$$= c_1\, f(\vec{v}_1) + \cdots + c_k\, f(\vec{v}_k) + c_{k+1}\, f(\vec{v}_{k+1})$$

when applied $k+1$ times. QED

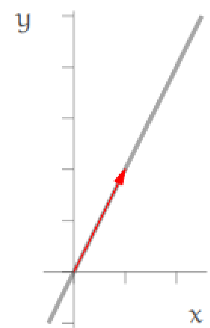
We often use item (2) to simplify the verification that a map preserves structure.

Finally, a summary. In the prior chapter, after giving the definition of a vector space, we looked at examples and noted that some spaces seemed to be essentially the same as others. Here we have defined the relation '$\cong$' and have argued that it is the right way to precisely say what we mean by "the same" because it preserves the features of interest in a vector space—in particular, it preserves linear combinations. In the next section we will show that isomorphism is an equivalence relation and so partitions the collection of vector spaces.

EXAMPLE (IMPORTANT)

Example The line through the origin with slope two is a vector space.

$$L = \{ \begin{pmatrix} x \\ y \end{pmatrix} \mid y = 2x \} = \{ \begin{pmatrix} t \\ 2t \end{pmatrix} \mid t \in \mathbb{R} \}$$



The parametrization

$$L = \{ \begin{pmatrix} t \\ 2t \end{pmatrix} \mid t \in \mathbb{R} \} = \{ t \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} \mid t \in \mathbb{R} \}$$

suggests that L is just like $\mathbb{R}$: there is a point on L associated with $1 \in \mathbb{R}$, a point associated with $2 \in \mathbb{R}$, etc. We will show that this function is an isomorphism between L and $\mathbb{R}^1$.

$$f(\begin{pmatrix} t \\ 2t \end{pmatrix}) = t$$

To verify that f is one-to-one suppose that f maps two members of L to the same output.

$$f(t_1 \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix}) = f(t_2 \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix})$$

By the definition of f we have $t_1 = t_2$ and so the two input members of L are equal.

To check that f is onto consider a member of the codomain, $r \in \mathbb{R}$. There is a member of the domain L that maps to it, namely this one.

$$f(r \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix}) = r$$

To finish, we combine the two structure checks, using the lemma's (2).

$$\begin{aligned} f(c_1 \cdot \begin{pmatrix} t_1 \\ 2t_1 \end{pmatrix} + c_2 \cdot \begin{pmatrix} t_2 \\ 2t_2 \end{pmatrix}) &= f((c_1 t_1 + c_2 t_2) \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix}) \\ &= c_1 t_1 + c_2 t_2 = c_1 \cdot f(\begin{pmatrix} t_1 \\ 2t_1 \end{pmatrix}) + c_2 \cdot f(\begin{pmatrix} t_2 \\ 2t_2 \end{pmatrix}) \end{aligned}$$

Questions:

✓ 1.13 For the map $f\colon \mathcal{P}_1 \to \mathbb{R}^2$ given by

$$a + bx \xmapsto{f} \begin{pmatrix} a - b \\ b \end{pmatrix}$$

Find the image of each of these elements of the domain.

(a) $3 - 2x$ (b) $2 + 2x$ (c) x

Show that this map is an isomorphism.

Three.I.1.13 These are the images.

(a) $\begin{pmatrix} 5 \\ -2 \end{pmatrix}$ (b) $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ (c) $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

To prove that f is one-to-one, assume that it maps two linear polynomials to the same image $f(a_1 + b_1x) = f(a_2 + b_2x)$. Then

$$\begin{pmatrix} a_1 - b_1 \\ b_1 \end{pmatrix} = \begin{pmatrix} a_2 - b_2 \\ b_2 \end{pmatrix}$$

and so, since column vectors are equal only when their components are equal, $b_1 = b_2$ and $a_1 = a_2$. That shows that the two linear polynomials are equal, and so f is one-to-one.

To show that f is onto, note that this member of the codomain

$$\begin{pmatrix} s \\ t \end{pmatrix}$$

is the image of this member of the domain $(s + t) + tx$.

To check that f preserves structure, we can use item (2) of Lemma 1.11.

$$\begin{aligned}
f(c_1 \cdot (a_1 + b_1x) + c_2 \cdot (a_2 + b_2x)) &= f((c_1a_1 + c_2a_2) + (c_1b_1 + c_2b_2)x) \\
&= \begin{pmatrix} (c_1a_1 + c_2a_2) - (c_1b_1 + c_2b_2) \\ c_1b_1 + c_2b_2 \end{pmatrix} \\
&= c_1 \cdot \begin{pmatrix} a_1 - b_1 \\ b_1 \end{pmatrix} + c_2 \cdot \begin{pmatrix} a_2 - b_2 \\ b_2 \end{pmatrix} \\
&= c_1 \cdot f(a_1 + b_1x) + c_2 \cdot f(a_2 + b_2x)
\end{aligned}$$

1.15 Show that the map $t\colon \mathcal{P}_2 \to \mathcal{P}_2$ given by $t(ax^2+bx+c) = bx^2-(a+c)x+a$ is an isomorphism.

Three.I.1.15 To see that the map is one-to-one suppose that $t(\vec{v}_1) = t(\vec{v}_2)$, aiming to conclude that $\vec{v}_1 = \vec{v}_2$. That is, $t(a_1x^2+b_1x+c_1) = t(a_2x^2+b_2x+c_2)$. Then $b_1x^2-(a_1+c_1)x+a_1 = b_2x^2-(a_2+c_2)x+a_2$ and because quadratic polynomials are equal only if they have have the same quadratic terms, the same constant terms, and the same linear terms we conclude that $b_1 = b_2$, that $a_1 = a_2$, and from that, $c_1 = c_2$. Therefore $a_1x^2+b_1x+c_1 = a_2x^2+b_2x+c_2$ and the function is one-to-one.

To see that the map is onto, we suppose that we are given a member $\vec{w}$ of the codomain and we find a member $\vec{v}$ of the domain that maps to it. Let the member of the codomain be $\vec{w} = px^2+qx+r$. Observe that where $\vec{v} = rx^2+px+(-q-r)$ then $t(\vec{v}) = \vec{w}$. Thus t is onto.

To see that the map is a homomorphism we show that it respects linear combinations of two elements. By Lemma 1.11 this will show that the map preserves the operations.

$$\begin{aligned}
&t(r_1(a_1x^2+b_1x+c_1)+r_2(a_2x^2+b_2x+c_2)) \\
&\quad = t((r_1a_1+r_2a_2)x^2+(r_1b_1+r_2b_2)x+(r_1c_1+r_2c_2)) \\
&\quad = (r_1b_1+r_2b_2)x^2-((r_1a_1+r_2a_2)+(r_1c_1+r_2c_2))x+(r_1a_1+r_2a_2) \\
&\quad = (r_1b_1)x^2-(r_1a_1+r_1c_1)x+r_1a_1+(r_2b_2)x^2-(r_2a_2+r_2c_2)x+r_2a_2 \\
&\quad = r_1t(a_1x^2+b_1x+c_1)+r_2t(a_2x^2+b_2x+c_2)
\end{aligned}$$

✓ 1.16 Verify that this map is an isomorphism: $h\colon \mathbb{R}^4 \to \mathcal{M}_{2\times 2}$ given by

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \mapsto \begin{pmatrix} c & a+d \\ b & d \end{pmatrix}$$

Three.I.1.16 We first verify that h is one-to-one. To do this we will show that $h(\vec{v}_1) = h(\vec{v}_2)$ implies that $\vec{v}_1 = \vec{v}_2$. So assume that

$$h(\vec{v}_1) = h(\begin{pmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \end{pmatrix}) = h(\begin{pmatrix} a_2 \\ b_2 \\ c_2 \\ d_2 \end{pmatrix}) = h(\vec{v}_2)$$

which gives

$$\begin{pmatrix} c_1 & a_1 + d_1 \\ b_1 & d_1 \end{pmatrix} = \begin{pmatrix} c_2 & a_2 + d_2 \\ b_2 & d_2 \end{pmatrix}$$

from which we conclude that $c_1 = c_2$ (by the upper-left entries), $b_1 = b_2$ (by the lower-left entries), $d_1 = d_2$ (by the lower-right entries), and with this last we get $a_1 = a_2$ (by the upper right). Therefore $\vec{v}_1 = \vec{v}_2$.

Next we will show that the map is onto, that every member of the codomain $\mathcal{M}_{2\times2}$ is the image of some four-tall member of the domain. So, given

$$\vec{w} = \begin{pmatrix} m & n \\ p & q \end{pmatrix} \in \mathcal{M}_{2\times2}$$

observe that it is the image of this domain vector.

$$\vec{v} = \begin{pmatrix} n - q \\ p \\ m \\ q \end{pmatrix}$$

To finish we verify that the map preserves linear combinations. By Lemma 1.11 this will show that the map preserves the operations.

$$\begin{aligned}
h(r_1 \cdot \begin{pmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \end{pmatrix} + r_2 \cdot \begin{pmatrix} a_2 \\ b_2 \\ c_2 \\ d_2 \end{pmatrix}) &= h(\begin{pmatrix} r_1a_1 + r_2a_2 \\ r_1b_1 + r_2b_2 \\ r_1c_1 + r_2c_2 \\ r_1d_1 + r_2d_2 \end{pmatrix}) \\
&= \begin{pmatrix} r_1c_1 + r_2c_2 & (r_1a_1 + r_2a_2) + (r_1d_1 + r_2d_2) \\ r_1b_1 + r_2b_2 & r_1d_1 + r_2d_2 \end{pmatrix} \\
&= r_1 \begin{pmatrix} c_1 & a_1 + d_1 \\ b_1 & d_1 \end{pmatrix} + r_2 \begin{pmatrix} c_2 & a_2 + d_2 \\ b_2 & d_2 \end{pmatrix} \\
&= r_1 \cdot h(\begin{pmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \end{pmatrix}) + r_2 \cdot h(\begin{pmatrix} a_2 \\ b_2 \\ c_2 \\ d_2 \end{pmatrix})
\end{aligned}$$

✓ 1.17 Decide whether each map is an isomorphism. If it is an isomorphism then prove it and if it isn't then state a condition that it fails to satisfy.

(a) $f\colon \mathcal{M}_{2\times 2} \to \mathbb{R}$ given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto ad - bc$$

(b) $f\colon \mathcal{M}_{2\times 2} \to \mathbb{R}^4$ given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a+b+c+d \\ a+b+c \\ a+b \\ a \end{pmatrix}$$

(c) $f\colon \mathcal{M}_{2\times 2} \to \mathcal{P}_3$ given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto c + (d+c)x + (b+a)x^2 + ax^3$$

(d) $f\colon \mathcal{M}_{2\times 2} \to \mathcal{P}_3$ given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto c + (d+c)x + (b+a+1)x^2 + ax^3$$

Three.I.1.17 **(a)** No; this map is not one-to-one. In particular, the matrix of all zeroes is mapped to the same image as the matrix of all ones.

(b) Yes, this is an isomorphism.

It is one-to-one:

$$\text{if } f(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}) = f(\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix})$$

$$\text{then } \begin{pmatrix} a_1+b_1+c_1+d_1 \\ a_1+b_1+c_1 \\ a_1+b_1 \\ a_1 \end{pmatrix} = \begin{pmatrix} a_2+b_2+c_2+d_2 \\ a_2+b_2+c_2 \\ a_2+b_2 \\ a_2 \end{pmatrix}$$

gives that $a_1 = a_2$, and that $b_1 = b_2$, and that $c_1 = c_2$, and that $d_1 = d_2$.

It is onto, since this shows

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = f(\begin{pmatrix} w & z-w \\ y-z & x-y \end{pmatrix})$$

that any four-tall vector is the image of a 2×2 matrix.

Finally, it preserves combinations

$$\begin{aligned}
f(r_1 \cdot \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} + r_2 \cdot \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix})
&= f(\begin{pmatrix} r_1a_1+r_2a_2 & r_1b_1+r_2b_2 \\ r_1c_1+r_2c_2 & r_1d_1+r_2d_2 \end{pmatrix}) \\
&= \begin{pmatrix} r_1a_1+\cdots+r_2d_2 \\ r_1a_1+\cdots+r_2c_2 \\ r_1a_1+\cdots+r_2b_2 \\ r_1a_1+r_2a_2 \end{pmatrix} \\
&= r_1 \cdot \begin{pmatrix} a_1+\cdots+d_1 \\ a_1+\cdots+c_1 \\ a_1+b_1 \\ a_1 \end{pmatrix} + r_2 \cdot \begin{pmatrix} a_2+\cdots+d_2 \\ a_2+\cdots+c_2 \\ a_2+b_2 \\ a_2 \end{pmatrix} \\
&= r_1 \cdot f(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}) + r_2 \cdot f(\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix})
\end{aligned}$$

and so item (2) of Lemma 1.11 shows that it preserves structure.

(c) Yes, it is an isomorphism.

To show that it is one-to-one, we suppose that two members of the domain have the same image under f.

$$f(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}) = f(\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix})$$

This gives, by the definition of f, that $c_1 + (d_1+c_1)x + (b_1+a_1)x^2 + a_1x^3 = c_2 + (d_2+c_2)x + (b_2+a_2)x^2 + a_2x^3$ and then the fact that polynomials are equal only when their coefficients are equal gives a set of linear equations

$$\begin{aligned} c_1 &= c_2 \\ d_1+c_1 &= d_2+c_2 \\ b_1+a_1 &= b_2+a_2 \\ a_1 &= a_2 \end{aligned}$$

that has only the solution $a_1 = a_2$, $b_1 = b_2$, $c_1 = c_2$, and $d_1 = d_2$.

To show that f is onto, we note that $p + qx + rx^2 + sx^3$ is the image under f of this matrix.

$$\begin{pmatrix} s & r-s \\ p & q-p \end{pmatrix}$$

We can check that f preserves structure by using item (2) of Lemma 1.11.

$$\begin{aligned}
f(r_1 \cdot \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} + r_2 \cdot \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix})
&= f(\begin{pmatrix} r_1a_1+r_2a_2 & r_1b_1+r_2b_2 \\ r_1c_1+r_2c_2 & r_1d_1+r_2d_2 \end{pmatrix}) \\
&= (r_1c_1+r_2c_2) + (r_1d_1+r_2d_2+r_1c_1+r_2c_2)x \\
&\quad + (r_1b_1+r_2b_2+r_1a_1+r_2a_2)x^2 + (r_1a_1+r_2a_2)x^3 \\
&= r_1 \cdot (c_1 + (d_1+c_1)x + (b_1+a_1)x^2 + a_1x^3) \\
&\quad + r_2 \cdot (c_2 + (d_2+c_2)x + (b_2+a_2)x^2 + a_2x^3) \\
&= r_1 \cdot f(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}) + r_2 \cdot f(\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix})
\end{aligned}$$

(d) No, this map does not preserve structure. For instance, it does not send the matrix of all zeroes to the zero polynomial.

1.24 For what k is $\mathcal{P}_k$ isomorphic to $\mathbb{R}^n$?

Three.I.1.24 If $n \geqslant 1$ then $\mathcal{P}_{n-1} \cong \mathbb{R}^n$. (If we take $\mathcal{P}_{-1}$ and $\mathbb{R}^0$ to be trivial vector spaces, then the relationship extends one dimension lower.) The natural isomorphism between them is this.

$$a_0 + a_1 x + \cdots + a_{n-1}x^{n-1} \mapsto \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{pmatrix}$$

Checking that it is an isomorphism is straightforward.

1.27 Are any two trivial spaces isomorphic?

Three.I.1.27 Yes; where the two spaces are $\{\vec{a}\}$ and $\{\vec{b}\}$, the map sending $\vec{a}$ to $\vec{b}$ is clearly one-to-one and onto, and also preserves what little structure there is.

!

1.29 Show that any isomorphism $f\colon \mathcal{P}_0 \to \mathbb{R}^1$ has the form $a \mapsto ka$ for some nonzero real number k.

Three.I.1.29 Consider the basis $\langle 1 \rangle$ for $\mathcal{P}_0$ and let $f(1) \in \mathbb{R}$ be k. For any $a \in \mathcal{P}_0$ we have that $f(a) = f(a \cdot 1) = af(1) = ak$ and so f's action is multiplication by k. Note that $k \neq 0$ or else the map is not one-to-one. (Incidentally, any such map $a \mapsto ka$ is an isomorphism, as is easy to check.)

1.30 These prove that isomorphism is an equivalence relation.

(a) Show that the identity map $\mathrm{id}\colon V \to V$ is an isomorphism. Thus, any vector space is isomorphic to itself.

(b) Show that if $f\colon V \to W$ is an isomorphism then so is its inverse $f^{-1}\colon W \to V$. Thus, if V is isomorphic to W then also W is isomorphic to V.

(c) Show that a composition of isomorphisms is an isomorphism: if $f\colon V \to W$ is an isomorphism and $g\colon W \to U$ is an isomorphism then so also is $g \circ f\colon V \to U$. Thus, if V is isomorphic to W and W is isomorphic to U, then also V is isomorphic to U.

Three.I.1.30 In each item, following item (2) of Lemma 1.11, we show that the map preserves structure by showing that the it preserves linear combinations of two members of the domain.

(a) The identity map is clearly one-to-one and onto. For linear combinations the check is easy.

$$\mathrm{id}(c_1 \cdot \vec{v}_1 + c_2 \cdot \vec{v}_2) = c_1\vec{v}_1 + c_2\vec{v}_2 = c_1 \cdot \mathrm{id}(\vec{v}_1) + c_2 \cdot \mathrm{id}(\vec{v}_2)$$

(b) The inverse of a correspondence is also a correspondence (as stated in the appendix), so we need only check that the inverse preserves linear combinations. Assume that $\vec{w}_1 = f(\vec{v}_1)$ (so $f^{-1}(\vec{w}_1) = \vec{v}_1$) and assume that $\vec{w}_2 = f(\vec{v}_2)$.

$$\begin{aligned} f^{-1}(c_1 \cdot \vec{w}_1 + c_2 \cdot \vec{w}_2) &= f^{-1}\big(c_1 \cdot f(\vec{v}_1) + c_2 \cdot f(\vec{v}_2)\big) \\ &= f^{-1}\big(f\big(c_1\vec{v}_1 + c_2\vec{v}_2\big)\big) \\ &= c_1\vec{v}_1 + c_2\vec{v}_2 \\ &= c_1 \cdot f^{-1}(\vec{w}_1) + c_2 \cdot f^{-1}(\vec{w}_2) \end{aligned}$$

(c) The composition of two correspondences is a correspondence (as stated in the appendix), so we need only check that the composition map preserves linear combinations.

$$\begin{aligned} g \circ f\,(c_1 \cdot \vec{v}_1 + c_2 \cdot \vec{v}_2) &= g\big(f(c_1\vec{v}_1 + c_2\vec{v}_2)\big) \\ &= g\big(c_1 \cdot f(\vec{v}_1) + c_2 \cdot f(\vec{v}_2)\big) \\ &= c_1 \cdot g\big(f(\vec{v}_1)\big) + c_2 \cdot g\big(f(\vec{v}_2)\big) \\ &= c_1 \cdot g \circ f\,(\vec{v}_1) + c_2 \cdot g \circ f\,(\vec{v}_2) \end{aligned}$$

1.34 Produce an automorphism of $\mathcal{P}_2$ other than the identity map, and other than a shift map $p(x) \mapsto p(x-k)$.

Three.I.1.34 First, the map $p(x) \mapsto p(x+k)$ doesn't count because it is a version of $p(x) \mapsto p(x-k)$. Here is a correct answer (many others are also correct):

$a_0 + a_1x + a_2x^2 \mapsto a_2 + a_0x + a_1x^2$. Verification that this is an isomorphism is straightforward.

1.36 Refer to Lemma 1.10 and Lemma 1.11. Find two more things preserved by isomorphism.

Three.I.1.36 There are many answers; two are linear independence and subspaces.

First we show that if a set $\{\vec{v}_1, \ldots, \vec{v}_n\}$ is linearly independent then its image $\{f(\vec{v}_1), \ldots, f(\vec{v}_n)\}$ is also linearly independent. Consider a linear relationship among members of the image set.

$$0 = c_1 f(\vec{v}_1) + \cdots + c_n f(\vec{v_n}) = f(c_1\vec{v}_1) + \cdots + f(c_n\vec{v_n}) = f(c_1\vec{v}_1 + \cdots + c_n\vec{v_n})$$

Because this map is an isomorphism, it is one-to-one. So f maps only one vector from the domain to the zero vector in the range, that is, $c_1\vec{v}_1 + \cdots + c_n\vec{v}_n$ equals the zero vector (in the domain, of course). But, if $\{\vec{v}_1, \ldots, \vec{v}_n\}$ is linearly independent then all of the c's are zero, and so $\{f(\vec{v}_1), \ldots, f(\vec{v}_n)\}$ is linearly independent also. (*Remark.* There is a small point about this argument that is worth mention. In a set, repeats collapse, that is, strictly speaking, this is a one-element set: $\{\vec{v}, \vec{v}\}$, because the things listed as in it are the same thing. Observe, however, the use of the subscript n in the above argument. In moving from the domain set $\{\vec{v}_1, \ldots, \vec{v}_n\}$ to the image set $\{f(\vec{v}_1), \ldots, f(\vec{v}_n)\}$, there is no collapsing, because the image set does not have repeats, because the isomorphism f is one-to-one.)

To show that if $f\colon V \to W$ is an isomorphism and if U is a subspace of the domain V then the set of image vectors $f(U) = \{\vec{w} \in W \mid \vec{w} = f(\vec{u}) \text{ for some } \vec{u} \in U\}$ is a subspace of W, we need only show that it is closed under linear combinations of two of its members (it is nonempty because it contains the image of the zero vector). We have

$$c_1 \cdot f(\vec{u}_1) + c_2 \cdot f(\vec{u}_2) = f(c_1\vec{u}_1) + f(c_2\vec{u}_2) = f(c_1\vec{u}_1 + c_2\vec{u}_2)$$

and $c_1\vec{u}_1 + c_2\vec{u}_2$ is a member of U because of the closure of a subspace under combinations. Hence the combination of $f(\vec{u}_1)$ and $f(\vec{u}_2)$ is a member of $f(U)$.

prove linear independence and subspaces are preserved under isomorphism

I.2 Dimension Characterizes Isomorphism

Inverse of isomorphism is isomorphism

I.2 Dimension Characterizes Isomorphism

In the prior subsection, after stating the definition of isomorphism, we gave some results supporting our sense that such a map describes spaces as "the same." Here we will develop this intuition. When two (unequal) spaces are isomorphic we think of them as almost equal, as equivalent. We shall make that precise by proving that the relationship 'is isomorphic to' is an equivalence relation.

2.1 Lemma The inverse of an isomorphism is also an isomorphism.

Proof Suppose that V is isomorphic to W via $f\colon V \to W$. An isomorphism is a correspondence between the sets so f has an inverse function $f^{-1}\colon W \to V$ that is also a correspondence.[*]

We will show that because f preserves linear combinations, so also does f^{-1}. Suppose that $\vec{w}_1, \vec{w}_2 \in W$. Because it is an isomorphism, f is onto and there

[*] More information on inverse functions is in the appendix.

are $\vec{v}_1, \vec{v}_2 \in V$ such that $\vec{w}_1 = f(\vec{v}_1)$ and $\vec{w}_2 = f(\vec{v}_2)$. Then

$$\begin{aligned} f^{-1}(c_1 \cdot \vec{w}_1 + c_2 \cdot \vec{w}_2) &= f^{-1}\big(c_1 \cdot f(\vec{v}_1) + c_2 \cdot f(\vec{v}_2)\big) \\ &= f^{-1}\big(f(c_1\vec{v}_1 + c_2\vec{v}_2)\big) = c_1\vec{v}_1 + c_2\vec{v}_2 = c_1 \cdot f^{-1}(\vec{w}_1) + c_2 \cdot f^{-1}(\vec{w}_2) \end{aligned}$$

since $f^{-1}(\vec{w}_1) = \vec{v}_1$ and $f^{-1}(\vec{w}_2) = \vec{v}_2$. With that, by Lemma 1.11's second statement, this map preserves structure. QED

$f: V \to W \qquad f^{-1}: W \to V$

f^{-1} preserves structure

Let $\vec{w_1}, \vec{w_2} \in W$, $c_1, c_2 \in \mathbb{R}$.

Show $f^{-1}(c_1\vec{w_1} + c_2\vec{w_2}) = c_1 f^{-1}(\vec{w_1}) + c_2 f^{-1}(\vec{w_2})$

Since f is onto, $\exists\, \vec{v_1}, \vec{v_2} \in V$ s.t. $f(\vec{v_1}) = \vec{w_1}$, $f(\vec{v_2}) = \vec{w_2}$

$$\begin{aligned}
\text{LHS} &= f^{-1}(c_1 f(\vec{v_1}) + c_2 f(\vec{v_2})) \\
&= f^{-1}(f(c_1\vec{v_1}) + f(c_2\vec{v_2})) = f^{-1}(f(c_1\vec{v_1} + c_2\vec{v_2})) \\
&= c_1\vec{v_1} + c_2\vec{v_2}
\end{aligned}$$

Isomorphism is equivalence relation between vector spaces

2.2 Theorem Isomorphism is an equivalence relation between vector spaces.

PROOF We must prove that the relation is symmetric, reflexive, and transitive.

To check reflexivity, that any space is isomorphic to itself, consider the identity map. It is clearly one-to-one and onto. This shows that it preserves linear combinations.

$$\mathrm{id}(c_1 \cdot \vec{v}_1 + c_2 \cdot \vec{v}_2) = c_1\vec{v}_1 + c_2\vec{v}_2 = c_1 \cdot \mathrm{id}(\vec{v}_1) + c_2 \cdot \mathrm{id}(\vec{v}_2)$$

Symmetry, that if V is isomorphic to W then also W is isomorphic to V, holds by Lemma 2.1 since each isomorphism map from V to W is paired with an isomorphism from W to V.

To finish we must check transitivity, that if V is isomorphic to W and W is isomorphic to U then V is isomorphic to U. Let $f\colon V \to W$ and $g\colon W \to U$ be isomorphisms. Consider their composition $g \circ f\colon V \to U$. Because the composition of correspondences is a correspondence, we need only check that the composition preserves linear combinations.

$$\begin{aligned}
g \circ f\,(c_1 \cdot \vec{v}_1 + c_2 \cdot \vec{v}_2) &= g\big(\,f(\,c_1 \cdot \vec{v}_1 + c_2 \cdot \vec{v}_2\,)\,\big) \\
&= g\big(\,c_1 \cdot f(\vec{v}_1) + c_2 \cdot f(\vec{v}_2)\,\big) \\
&= c_1 \cdot g\big(f(\vec{v}_1)\big) + c_2 \cdot g\big(f(\vec{v}_2)\big) \\
&= c_1 \cdot (g \circ f)\,(\vec{v}_1) + c_2 \cdot (g \circ f)\,(\vec{v}_2)
\end{aligned}$$

Thus the composition is an isomorphism. QED

Since it is an equivalence, isomorphism partitions the universe of vector spaces into classes: each space is in one and only one isomorphism class.

All finite dimensional vector spaces:

$V \cong W$

The next result characterizes these classes by dimension. That is, we can describe each class simply by giving the number that is the dimension of all of the spaces in that class.

IMPORTANT THEOREM!!!

2.3 Theorem Vector spaces are isomorphic if and only if they have the same dimension.

In this double implication statement the proof of each half involves a significant idea so we will do the two separately.

Isomorphism and Dimensions

2.4 Lemma If spaces are isomorphic then they have the same dimension.

PROOF We shall show that an isomorphism of two spaces gives a correspondence between their bases. That is, we shall show that if $f\colon V \to W$ is an isomorphism and a basis for the domain V is $B = \langle \vec{\beta}_1, \dots, \vec{\beta}_n \rangle$ then its image $D = \langle f(\vec{\beta}_1), \dots, f(\vec{\beta}_n) \rangle$ is a basis for the codomain W. (The other half of the correspondence, that for any basis of W the inverse image is a basis for V, follows from the fact that f^{-1} is also an isomorphism and so we can apply the prior sentence to f^{-1}.)

To see that D spans W, fix any $\vec{w} \in W$. Because f is an isomorphism it is onto and so there is a $\vec{v} \in V$ with $\vec{w} = f(\vec{v})$. Expand $\vec{v}$ as a combination of basis vectors.

$$\vec{w} = f(\vec{v}) = f(v_1\vec{\beta}_1 + \cdots + v_n\vec{\beta}_n) = v_1 \cdot f(\vec{\beta}_1) + \cdots + v_n \cdot f(\vec{\beta}_n)$$

For linear independence of D, if

$$\vec{0}_W = c_1 f(\vec{\beta}_1) + \cdots + c_n f(\vec{\beta}_n) = f(c_1\vec{\beta}_1 + \cdots + c_n\vec{\beta}_n)$$

then, since f is one-to-one and so the only vector sent to $\vec{0}_W$ is $\vec{0}_V$, we have that $\vec{0}_V = c_1\vec{\beta}_1 + \cdots + c_n\vec{\beta}_n$, which implies that all of the c's are zero. QED

Reverse

2.5 Lemma If spaces have the same dimension then they are isomorphic.

Proof We will prove that any space of dimension n is isomorphic to $\mathbb{R}^n$. Then we will have that all such spaces are isomorphic to each other by transitivity, which was shown in Theorem 2.2.

Let V be n-dimensional. Fix a basis $B = \langle \vec{\beta}_1, \ldots, \vec{\beta}_n \rangle$ for the domain V. Consider the operation of representing the members of V with respect to B as a function from V to $\mathbb{R}^n$.

$$\vec{v} = v_1\vec{\beta}_1 + \cdots + v_n\vec{\beta}_n \overset{\text{Rep}_B}{\longmapsto} \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$$

It is well-defined* since every $\vec{v}$ has one and only one such representation (see Remark 2.7 following this proof).

* More information on well-defined is in the appendix.

This function is one-to-one because if

$$\text{Rep}_B(u_1\vec{\beta}_1 + \cdots + u_n\vec{\beta}_n) = \text{Rep}_B(v_1\vec{\beta}_1 + \cdots + v_n\vec{\beta}_n)$$

then

$$\begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$$

and so $u_1 = v_1, \ldots, u_n = v_n$, implying that the original arguments $u_1\vec{\beta}_1 + \cdots + u_n\vec{\beta}_n$ and $v_1\vec{\beta}_1 + \cdots + v_n\vec{\beta}_n$ are equal.

This function is onto; any member of $\mathbb{R}^n$

$$\vec{w} = \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}$$

is the image of some $\vec{v} \in V$, namely $\vec{w} = \text{Rep}_B(w_1\vec{\beta}_1 + \cdots + w_n\vec{\beta}_n)$.

Finally, this function preserves structure.

$$\begin{aligned}
\text{Rep}_B(r \cdot \vec{u} + s \cdot \vec{v}) &= \text{Rep}_B(\,(ru_1 + sv_1)\vec{\beta}_1 + \cdots + (ru_n + sv_n)\vec{\beta}_n\,) \\
&= \begin{pmatrix} ru_1 + sv_1 \\ \vdots \\ ru_n + sv_n \end{pmatrix} \\
&= r \cdot \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} + s \cdot \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \\
&= r \cdot \text{Rep}_B(\vec{u}) + s \cdot \text{Rep}_B(\vec{v})
\end{aligned}$$

Therefore Rep_B is an isomorphism. Consequently any n-dimensional space is isomorphic to $\mathbb{R}^n$. QED

QUESTIONS:

✓ 2.10 Decide if the spaces are isomorphic.

(a) $\mathbb{R}^2$, $\mathbb{R}^4$ (b) $\mathcal{P}_5$, $\mathbb{R}^5$ (c) $\mathcal{M}_{2\times3}$, $\mathbb{R}^6$ (d) $\mathcal{P}_5$, $\mathcal{M}_{2\times3}$
(e) $\mathcal{M}_{2\times k}$, $\mathcal{M}_{k\times2}$

2.11 Which of these spaces are isomorphic to each other?

(a) $\mathbb{R}^3$ (b) $\mathcal{M}_{2\times2}$ (c) $\mathcal{P}_3$ (d) $\mathbb{R}^4$ (e) $\mathcal{P}_2$

✓ 2.12 Consider the isomorphism $\mathrm{Rep}_B(\cdot)\colon \mathcal{P}_1 \to \mathbb{R}^2$ where $B = \langle 1, 1+x \rangle$. Find the image of each of these elements of the domain.

(a) $3-2x$; (b) $2+2x$; (c) x

2.13 For which n is the space isomorphic to $\mathbb{R}^n$?

(a) $\mathcal{P}_4$
(b) $\mathcal{P}_1$
(c) $\mathcal{M}_{2\times3}$
(d) the plane $2x-y+z=0$ subset of $\mathbb{R}^3$
(e) the vector space of linear combinations of three letters $\{ax+by+cz \mid a,b,c \in \mathbb{R}\}$

✓ 2.14 Show that if $m \neq n$ then $\mathbb{R}^m \ncong \mathbb{R}^n$.

✓ 2.15 Is $\mathcal{M}_{m\times n} \cong \mathcal{M}_{n\times m}$?

✓ 2.16 Are any two planes through the origin in $\mathbb{R}^3$ isomorphic?

2.17 Find a set of equivalence class representatives other than the set of $\mathbb{R}^n$'s.

2.18 True or false: between any n-dimensional space and $\mathbb{R}^n$ there is exactly one isomorphism.

2.19 Can a vector space be isomorphic to one of its proper subspaces?

✓ 2.20 This subsection shows that for any isomorphism, the inverse map is also an isomorphism. This subsection also shows that for a fixed basis B of an n-dimensional vector space V, the map $\mathrm{Rep}_B\colon V \to \mathbb{R}^n$ is an isomorphism. Find the inverse of this map.

✓ 2.21 Prove these facts about matrices.

(a) The row space of a matrix is isomorphic to the column space of its transpose.
(b) The row space of a matrix is isomorphic to its column space.

Three.I.2.10 Each pair of spaces is isomorphic if and only if the two have the same dimension. We can, when there is an isomorphism, state a map, but it isn't strictly necessary.

(a) No, they have different dimensions.

(b) No, they have different dimensions.

(c) Yes, they have the same dimension. One isomorphism is this.

$$\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} \mapsto \begin{pmatrix} a \\ \vdots \\ f \end{pmatrix}$$

(d) Yes, they have the same dimension. This is an isomorphism.

$$a + bx + \cdots + fx^5 \mapsto \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}$$

(e) Yes, both have dimension $2k$.

Three.I.2.11 Just find the dimension of each space, for instance by finding a basis, and then spaces with the same dimension are isomorphic. This lists the dimension of each space.

(a) 3 **(b)** 4 **(c)** 4 **(d)** 4 **(e)** 3

Three.I.2.12 **(a)** $\text{Rep}_B(3-2x) = \begin{pmatrix} 5 \\ -2 \end{pmatrix}$ **(b)** $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ **(c)** $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

Three.I.2.13 For each, the simplest thing is to find the dimension of the space by finding a basis. For each basis given below, We will omit the verification that it is a basis.

(a) It is isomorphic to $\mathbb{R}^5$. One basis for $\mathcal{P}_4$ is $\{x^4, x^3, x^2, x, 1\}$ so the space has dimension 5.

(b) It is isomorphic to $\mathbb{R}^2$ since one basis for the space $\mathcal{P}_1 = \{a + bx \mid a, b \in \mathbb{R}\}$ is $\{1, x\}$.

(c) It is isomorphic to $\mathbb{R}^6$. One basis has these six matrices.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ldots \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(d) It is a plane so it is isomorphic to $\mathbb{R}^2$. For a more extensive answer, parametrizing the plane gives this vector description

$$\{\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1 \\ 0 \end{pmatrix} y + \begin{pmatrix} -1/2 \\ 0 \\ 1 \end{pmatrix} z \mid y, z \in \mathbb{R}\}$$

and so it has a basis consisting of those two vectors.

(e) It is isomorphic to $\mathbb{R}^3$. One basis is the set of linear combinations $\{x, y, z\}$, that is, $\{x + 0y + 0z, 0x + y + 0z, 0x + 0y + z\}$.

Three.I.2.14 They have different dimensions.

Three.I.2.15 Yes, both are mn-dimensional.

Three.I.2.16 Yes, any two (nondegenerate) planes are both two-dimensional vector spaces.

Three.I.2.17 There are many answers, one is the set of $\mathcal{P}_k$ (taking $\mathcal{P}_{-1}$ to be the trivial vector space).

Three.I.2.18 False (except when $n = 0$). For instance, if $f\colon V \to \mathbb{R}^n$ is an isomorphism then multiplying by any nonzero scalar, gives another, different, isomorphism. (Between trivial spaces the isomorphisms are unique; the only map possible is $\vec{0}_V \mapsto 0_W$.)

Three.I.2.19 No. A proper subspace has a strictly lower dimension than it's superspace; if U is a proper subspace of V then any linearly independent subset of U must have fewer than $\dim(V)$ members or else that set would be a basis for V, and U wouldn't be proper.

Three.I.2.20 Where $B = \langle \vec{\beta}_1, \ldots, \vec{\beta}_n \rangle$, the inverse is this.

$$\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \mapsto c_1\vec{\beta}_1 + \cdots + c_n\vec{\beta}_n$$

Three.I.2.21 All three spaces have dimension equal to the rank of the matrix.

Three.II Homomorphisms

II.1 Definition

1.1 Definition A function between vector spaces $h\colon V \to W$ that preserves addition

$$\text{if } \vec{v}_1, \vec{v}_2 \in V \text{ then } h(\vec{v}_1 + \vec{v}_2) = h(\vec{v}_1) + h(\vec{v}_2)$$

and scalar multiplication

$$\text{if } \vec{v} \in V \text{ and } r \in \mathbb{R} \text{ then } h(r \cdot \vec{v}) = r \cdot h(\vec{v})$$

is a *homomorphism* or *linear map*.

How to show sth is not a homomorphism: - show it doesnt preserve structure (ex: not map the zero vector correctly) by probably using a counter example

Example Of these two maps $h, g\colon \mathbb{R}^2 \to \mathbb{R}$, the first is a homomorphism while the second is not.

$$\begin{pmatrix} x \\ y \end{pmatrix} \stackrel{h}{\longmapsto} 2x - 3y \qquad \begin{pmatrix} x \\ y \end{pmatrix} \stackrel{g}{\longmapsto} 2x - 3y + 1$$

$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

$g\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 3 \end{pmatrix}\right) = g\left(\begin{pmatrix} 3 \\ 4 \end{pmatrix}\right) = 6 - 12 + 1 = -5$

$g\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) + g\left(\begin{pmatrix} 2 \\ 3 \end{pmatrix}\right) = (2-3+1) + (4-9+1)$

$= 0 - 4 = -4$

g is not a homomorphism

Lemmas:

1.6 Lemma A linear map sends the zero vector to the zero vector.

1.7 Lemma The following are equivalent for any map $f\colon V \to W$ between vector spaces.

(1) f is a homomorphism

(2) $f(c_1 \cdot \vec{v}_1 + c_2 \cdot \vec{v}_2) = c_1 \cdot f(\vec{v}_1) + c_2 \cdot f(\vec{v}_2)$ for any $c_1, c_2 \in \mathbb{R}$ and $\vec{v}_1, \vec{v}_2 \in V$

(3) $f(c_1 \cdot \vec{v}_1 + \dots + c_n \cdot \vec{v}_n) = c_1 \cdot f(\vec{v}_1) + \dots + c_n \cdot f(\vec{v}_n)$ for any $c_1, \dots, c_n \in \mathbb{R}$ and $\vec{v}_1, \dots, \vec{v}_n \in V$

To verify a map is a homomorphism, we most often use (2)!

Example Between any two vector spaces the zero map $Z\colon V \to W$ given by $Z(\vec{v}) = \vec{0}_W$ is a linear map. Using (2): $Z(c_1\vec{v}_1 + c_2\vec{v}_2) = \vec{0}_W = \vec{0}_W + \vec{0}_W = c_1 Z(\vec{v}_1) + c_2 Z(\vec{v}_2)$.

Example The *inclusion map* $\iota\colon \mathbb{R}^2 \to \mathbb{R}^3$

$$\iota(\begin{pmatrix} x \\ y \end{pmatrix}) = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$$

is a homomorphism.

$$\begin{aligned}
\iota(c_1 \cdot \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + c_2 \cdot \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}) &= \iota(\begin{pmatrix} c_1x_1 + c_2x_2 \\ c_1y_1 + c_2y_2 \end{pmatrix}) \\
&= \begin{pmatrix} c_1x_1 + c_2x_2 \\ c_1y_1 + c_2y_2 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} c_1x_1 \\ c_1y_1 \\ 0 \end{pmatrix} + \begin{pmatrix} c_2x_2 \\ c_2y_2 \\ 0 \end{pmatrix} \\
&= c_1 \cdot \iota(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}) + c_2 \cdot \iota(\begin{pmatrix} x_2 \\ y_2 \end{pmatrix})
\end{aligned}$$

-> inclusion maps preserves everything but map it to a higher dimension

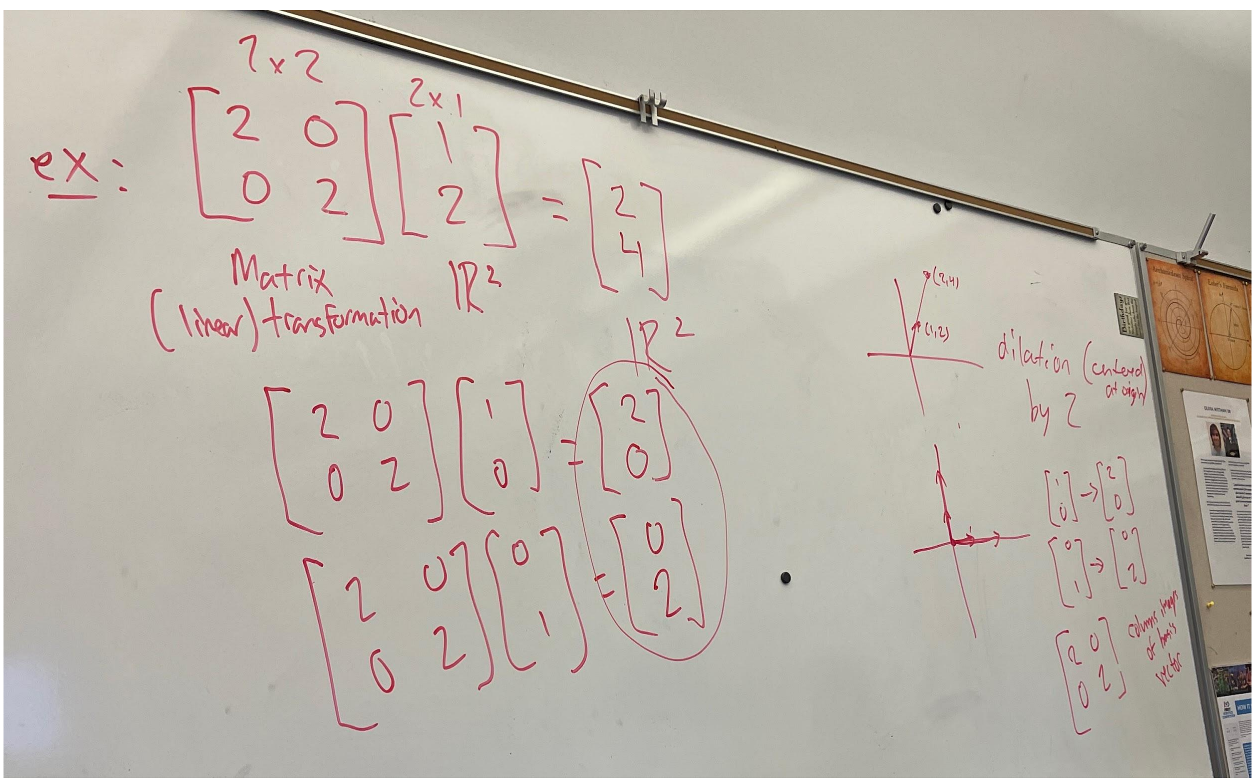
Example The *trace* of a square matrix is the sum down the upper-left to lower-right diagonal. Thus $\mathrm{Tr}\colon \mathcal{M}_{2\times 2} \to \mathbb{R}$ is this.

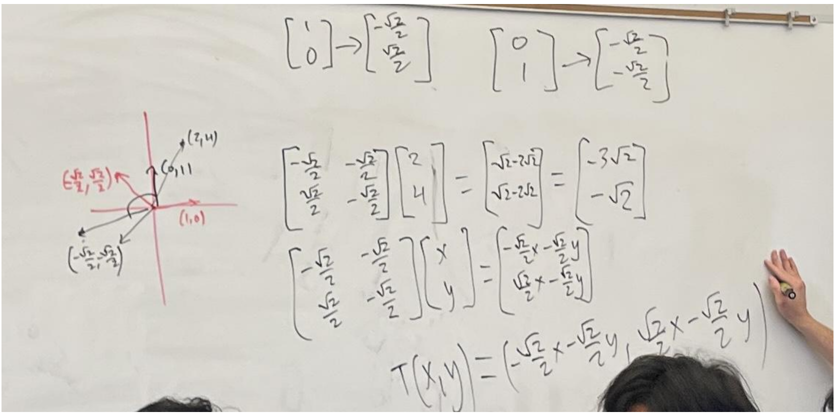
$$\mathrm{Tr}(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = a + d$$

It is linear.

$$\begin{aligned}
\mathrm{Tr}(\, r_1 \cdot \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} + r_2 \cdot \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}\,) \\
&= \mathrm{Tr}(\begin{pmatrix} r_1 a_1 + r_2 a_2 & r_1 b_1 + r_2 b_2 \\ r_1 c_1 + r_2 c_2 & r_1 d_1 + r_2 d_2 \end{pmatrix}) \\
&= (r_1 a_1 + r_2 a_2) + (r_1 d_1 + r_2 d_2) \\
&= r_1 (a_1 + d_1) + r_2 (a_2 + d_2) \\
&= r_1 \cdot \mathrm{Tr}(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}) + r_2 \cdot \mathrm{Tr}(\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix})
\end{aligned}$$

Matrix Transformation Intro





DEFINING BASIS TO DETERMINING HOMOMORPHISM FOR A VECTOR SPACE

1.9 Theorem A homomorphism is determined by its action on a basis: if V is a vector space with basis $\langle\vec{\beta}_1,\dots,\vec{\beta}_n\rangle$, if W is a vector space, and if $\vec{w}_1,\dots,\vec{w}_n\in W$ (these codomain elements need not be distinct) then there exists a homomorphism from V to W sending each $\vec{\beta}_i$ to $\vec{w}_i$, and that homomorphism is unique.

Proof For any input $\vec{v}\in V$ let its expression with respect to the basis be $\vec{v}=c_1\vec{\beta}_1+\cdots+c_n\vec{\beta}_n$. Define the associated output by using the same coordinates $h(\vec{v})=c_1\vec{w}_1+\cdots+c_n\vec{w}_n$. This is well defined because, with respect to the basis, the representation of each domain vector $\vec{v}$ is unique.

This map is a homomorphism because it preserves linear combinations: where $\vec{v_1}=c_1\vec{\beta}_1+\cdots+c_n\vec{\beta}_n$ and $\vec{v_2}=d_1\vec{\beta}_1+\cdots+d_n\vec{\beta}_n$, here is the calculation.

$$\begin{aligned}
h(r_1\vec{v}_1+r_2\vec{v}_2)&=h(\,(r_1c_1+r_2d_1)\vec{\beta}_1+\cdots+(r_1c_n+r_2d_n)\vec{\beta}_n\,)\\
&=(r_1c_1+r_2d_1)\vec{w}_1+\cdots+(r_1c_n+r_2d_n)\vec{w}_n\\
&=r_1h(\vec{v}_1)+r_2h(\vec{v}_2)
\end{aligned}$$

This map is unique because if $\hat{h}\colon V\to W$ is another homomorphism satisfying that $\hat{h}(\vec{\beta}_i)=\vec{w}_i$ for each i then h and $\hat{h}$ have the same effect on all of the vectors in the domain.

$$\begin{aligned}
\hat{h}(\vec{v})=\hat{h}(c_1\vec{\beta}_1+\cdots+c_n\vec{\beta}_n)&=c_1\hat{h}(\vec{\beta}_1)+\cdots+c_n\hat{h}(\vec{\beta}_n)\\
&=c_1\vec{w}_1+\cdots+c_n\vec{w}_n=h(\vec{v})
\end{aligned}$$

They have the same action so they are the same function. QED

=> IF WE define the homomorphism for the basis for a vector space, we can define homomorphism for the whole space.

Example The book has the proof. Here is an illustration. Consider a map $h\colon \mathbb{R}^2 \to \mathbb{R}^2$ with this action on a basis.

$$\begin{pmatrix}1\\1\end{pmatrix} \stackrel{h}{\longmapsto} \begin{pmatrix}1\\3\end{pmatrix} \quad \begin{pmatrix}1\\0\end{pmatrix} \stackrel{h}{\longmapsto} \begin{pmatrix}2\\0\end{pmatrix}$$

The effect of the map on any vector $\vec{v}$ at all is determined by those two facts. Represent that vector $\vec{v}$ with respect to the basis.

$$\begin{pmatrix}-1\\5\end{pmatrix} = 5 \cdot \begin{pmatrix}1\\1\end{pmatrix} - 6 \cdot \begin{pmatrix}1\\0\end{pmatrix}$$

Compute $h(\vec{v})$ using the definition of homomorphism.

$$h(\vec{v}) = h(\,5 \cdot \begin{pmatrix}1\\1\end{pmatrix} - 6 \cdot \begin{pmatrix}1\\0\end{pmatrix}\,) = 5 \cdot \begin{pmatrix}1\\3\end{pmatrix} - 6 \cdot \begin{pmatrix}2\\0\end{pmatrix} = \begin{pmatrix}-7\\15\end{pmatrix}$$

$$\begin{pmatrix}x\\y\end{pmatrix} = a\begin{pmatrix}1\\1\end{pmatrix} + b\begin{pmatrix}1\\0\end{pmatrix}$$

$a + b = x \qquad b = x - a = x - y$

$a + 0 = y$

$$\begin{pmatrix}x\\y\end{pmatrix} = y\begin{pmatrix}1\\1\end{pmatrix} + (x-y)\begin{pmatrix}1\\0\end{pmatrix}$$

$$h\left(\begin{pmatrix}x\\y\end{pmatrix}\right) = y\,h\left(\begin{pmatrix}1\\1\end{pmatrix}\right) + (x-y)\,h\left(\begin{pmatrix}1\\0\end{pmatrix}\right) = y\begin{pmatrix}1\\3\end{pmatrix} + (x-y)\begin{pmatrix}2\\0\end{pmatrix} = \begin{pmatrix}2x-y\\3y\end{pmatrix}$$

! JUST FROM the basis, we can figure out the homomorphism!

Example Consider $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ with this effect on the standard basis.

$$f(\vec{e}_1) = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \quad f(\vec{e}_2) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad f(\vec{e}_3) = \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}$$

Because this is the standard basis, the effect of the map on any vector $\vec{v} \in \mathbb{R}^3$ is especially easy to compute. For instance,

$$\mathrm{Rep}_{\mathcal{E}_3,\mathcal{E}_3}(\begin{pmatrix} -5 \\ 0 \\ 10 \end{pmatrix}) = \begin{pmatrix} -5 \\ 0 \\ 10 \end{pmatrix}$$

and so we have this.

$$f(\begin{pmatrix} -5 \\ 0 \\ 10 \end{pmatrix}) = -5 \cdot \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + 0 \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 10 \cdot \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 25 \\ -20 \\ 5 \end{pmatrix}$$

EXTENDED LINEARLY

1.10 Definition Let V and W be vector spaces and let $B = \langle \vec{\beta}_1, \ldots, \vec{\beta}_n \rangle$ be a basis for V. A function defined on that basis $f\colon B \to W$ is *extended linearly* to a function $\hat{f}\colon V \to W$ if for all $\vec{v} \in V$ such that $\vec{v} = c_1\vec{\beta}_1 + \cdots + c_n\vec{\beta}_n$, the action of the map is $\hat{f}(\vec{v}) = c_1 \cdot f(\vec{\beta}_1) + \cdots + c_n \cdot f(\vec{\beta}_n)$.

1.11 Example If we specify a map $h\colon \mathbb{R}^2 \to \mathbb{R}^2$ that acts on the standard basis $\mathcal{E}_2$ in this way

$$h(\begin{pmatrix} 1 \\ 0 \end{pmatrix}) = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \qquad h(\begin{pmatrix} 0 \\ 1 \end{pmatrix}) = \begin{pmatrix} -4 \\ 4 \end{pmatrix}$$

then we have also specified the action of h on any other member of the domain. For instance, the value of h on this argument

$$h(\begin{pmatrix} 3 \\ -2 \end{pmatrix}) = h(3 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} - 2 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}) = 3 \cdot h(\begin{pmatrix} 1 \\ 0 \end{pmatrix}) - 2 \cdot h(\begin{pmatrix} 0 \\ 1 \end{pmatrix}) = \begin{pmatrix} 5 \\ -5 \end{pmatrix}$$

is a direct consequence of the value of h on the basis vectors.

Later in this chapter we shall develop a convenient scheme for computations like this one, using matrices.

→ adding onto Theorem 1.9

Linear Transformation

1.12 Definition A linear map from a space into itself $t\colon V \to V$ is a *linear transformation*.

1.13 Remark In this book we use 'linear transformation' only in the case where the codomain equals the domain. Be aware that some sources instead use it as a synonym for 'linear map'. Still another synonym is 'linear operator'.

1.14 Example The map on $\mathbb{R}^2$ that projects all vectors down to the x-axis is a linear transformation.

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x \\ 0 \end{pmatrix}$$

1.15 Example The derivative map $d/dx\colon \mathcal{P}_n \to \mathcal{P}_n$

$$a_0 + a_1 x + \cdots + a_n x^n \overset{d/dx}{\longmapsto} a_1 + 2a_2 x + 3a_3 x^2 + \cdots + n a_n x^{n-1}$$

is a linear transformation as this result from calculus shows: $d(c_1 f + c_2 g)/dx = c_1\,(df/dx) + c_2\,(dg/dx)$.

1.16 Example The matrix transpose operation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

is a linear transformation of $\mathcal{M}_{2\times 2}$. (Transpose is one-to-one and onto and so is in fact an automorphism.)

We finish this subsection about maps by recalling that we can linearly combine maps. For instance, for these maps from $\mathbb{R}^2$ to itself

$$\begin{pmatrix} x \\ y \end{pmatrix} \overset{f}{\longmapsto} \begin{pmatrix} 2x \\ 3x - 2y \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x \\ y \end{pmatrix} \overset{g}{\longmapsto} \begin{pmatrix} 0 \\ 5x \end{pmatrix}$$

the linear combination $5f - 2g$ is also a transformation of $\mathbb{R}^2$.

$$\begin{pmatrix} x \\ y \end{pmatrix} \overset{5f-2g}{\longmapsto} \begin{pmatrix} 10x \\ 5x - 10y \end{pmatrix}$$

- **To show it is a linear transformation:**
  - **Check it maps a space to itself**
  - **Check it is a homomorphism**

(important) Linear Function between spaces is a Vector Space

1.17 Lemma For vector spaces V and W, the set of linear functions from V to W is itself a vector space, a subspace of the space of all functions from V to W.

We denote the space of linear maps from V to W by $\mathcal{L}(V, W)$.

PROOF This set is non-empty because it contains the zero homomorphism. So to show that it is a subspace we need only check that it is closed under the

operations. Let $f, g\colon V \to W$ be linear. Then the operation of function addition is preserved

$$\begin{aligned}(f+g)(c_1\vec{v}_1 + c_2\vec{v}_2) &= f(c_1\vec{v}_1 + c_2\vec{v}_2) + g(c_1\vec{v}_1 + c_2\vec{v}_2)\\ &= c_1 f(\vec{v}_1) + c_2 f(\vec{v}_2) + c_1 g(\vec{v}_1) + c_2 g(\vec{v}_2)\\ &= c_1\big(f+g\big)(\vec{v}_1) + c_2\big(f+g\big)(\vec{v}_2)\end{aligned}$$

as is the operation of scalar multiplication of a function.

$$\begin{aligned}(r\cdot f)(c_1\vec{v}_1 + c_2\vec{v}_2) &= r(c_1 f(\vec{v}_1) + c_2 f(\vec{v}_2))\\ &= c_1(r\cdot f)(\vec{v}_1) + c_2(r\cdot f)(\vec{v}_2)\end{aligned}$$

Hence $\mathcal{L}(V, W)$ is a subspace. QED

We started this section by defining ‘homomorphism’ as a generalization of ‘isomorphism’, by isolating the structure preservation property. Some of the points about isomorphisms carried over unchanged, while we adapted others.

Note, however, that the idea of ‘homomorphism’ is in no way somehow secondary to that of ‘isomorphism’. In the rest of this chapter we shall work mostly with homomorphisms. This is partly because any statement made about homomorphisms is automatically true about isomorphisms but more because, while the isomorphism concept is more natural, our experience will show that the homomorphism concept is more fruitful and more central to progress.

???SET OF ALL HOMOMORPHISM IS A SUBSPACE

Questions:

1.18 Decide if each $h: \mathbb{R}^3 \to \mathbb{R}^2$ is linear.

(a) $h(\begin{pmatrix} x \\ y \\ z \end{pmatrix}) = \begin{pmatrix} x \\ x+y+z \end{pmatrix}$ **(b)** $h(\begin{pmatrix} x \\ y \\ z \end{pmatrix}) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ **(c)** $h(\begin{pmatrix} x \\ y \\ z \end{pmatrix}) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

(d) $h(\begin{pmatrix} x \\ y \\ z \end{pmatrix}) = \begin{pmatrix} 2x+y \\ 3y-4z \end{pmatrix}$

(d) Yes. The verification is straightforward.

$$\begin{aligned}
h(c_1 \cdot \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + c_2 \cdot \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}) &= h(\begin{pmatrix} c_1x_1 + c_2x_2 \\ c_1y_1 + c_2y_2 \\ c_1z_1 + c_2z_2 \end{pmatrix}) \\
&= \begin{pmatrix} 2(c_1x_1 + c_2x_2) + (c_1y_1 + c_2y_2) \\ 3(c_1y_1 + c_2y_2) - 4(c_1z_1 + c_2z_2) \end{pmatrix} \\
&= c_1 \cdot \begin{pmatrix} 2x_1 + y_1 \\ 3y_1 - 4z_1 \end{pmatrix} + c_2 \cdot \begin{pmatrix} 2x_2 + y_2 \\ 3y_2 - 4z_2 \end{pmatrix} \\
&= c_1 \cdot h(\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}) + c_2 \cdot h(\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix})
\end{aligned}$$

1.19 Decide if each map $h: \mathcal{M}_{2\times 2} \to \mathbb{R}$ is linear.

(a) $h(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = a + d$

(b) $h(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = ad - bc$

(c) $h(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = 2a + 3b + c - d$

(d) $h(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = a^2 + b^2$

homomorphism = linear map

(a) Yes. The check that it preserves combinations is routine.

$$h(r_1 \cdot \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} + r_2 \cdot \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}) = h(\begin{pmatrix} r_1a_1 + r_2a_2 & r_1b_1 + r_2b_2 \\ r_1c_1 + r_2c_2 & r_1d_1 + r_2d_2 \end{pmatrix})$$
$$= (r_1a_1 + r_2a_2) + (r_1d_1 + r_2d_2)$$
$$= r_1(a_1 + d_1) + r_2(a_2 + d_2)$$
$$= r_1 \cdot h(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}) + r_2 \cdot h(\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix})$$

(b) No. For instance, not preserved is multiplication by the scalar 2.

ex

$$h(2 \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}) = h(\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}) = 4 \quad \text{while} \quad 2 \cdot h(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}) = 2 \cdot 1 = 2$$

=> Use example when it is not a homomorphism to prove

Differentiation, Integration, and INVERSE maps

1.20 Show that these are homomorphisms. Are they inverse to each other?

(a) $d/dx\colon \mathcal{P}_3 \to \mathcal{P}_2$ given by $a_0 + a_1x + a_2x^2 + a_3x^3$ maps to $a_1 + 2a_2x + 3a_3x^2$

(b) $\int\colon \mathcal{P}_2 \to \mathcal{P}_3$ given by $b_0 + b_1x + b_2x^2$ maps to $b_0x + (b_1/2)x^2 + (b_2/3)x^3$

Three.II.1.20 The check that each is a homomorphisms is routine. Here is the check for the differentiation map.

$$\frac{d}{dx}(r \cdot (a_0 + a_1x + a_2x^2 + a_3x^3) + s \cdot (b_0 + b_1x + b_2x^2 + b_3x^3))$$
$$= \frac{d}{dx}((ra_0 + sb_0) + (ra_1 + sb_1)x + (ra_2 + sb_2)x^2 + (ra_3 + sb_3)x^3)$$
$$= (ra_1 + sb_1) + 2(ra_2 + sb_2)x + 3(ra_3 + sb_3)x^2$$
$$= r \cdot (a_1 + 2a_2x + 3a_3x^2) + s \cdot (b_1 + 2b_2x + 3b_3x^2)$$
$$= r \cdot \frac{d}{dx}(a_0 + a_1x + a_2x^2 + a_3x^3) + s \cdot \frac{d}{dx}(b_0 + b_1x + b_2x^2 + b_3x^3)$$

(An alternate proof is to simply note that this is a property of differentiation that is familiar from calculus.)

These two maps are not inverses as this composition does not act as the identity map on this element of the domain.

$$1 \in \mathcal{P}_3 \overset{d/dx}{\longmapsto} 0 \in \mathcal{P}_2 \overset{\int}{\longmapsto} 0 \in \mathcal{P}_3$$

?

Projection mapping

Three.II.1.21 Each of these projections is a homomorphism. Projection to the xz-plane and to the yz-plane are these maps.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x \\ 0 \\ z \end{pmatrix} \qquad \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ y \\ z \end{pmatrix}$$

Projection to the x-axis, to the y-axis, and to the z-axis are these maps.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ y \\ 0 \end{pmatrix} \qquad \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix}$$

And projection to the origin is this map.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Homomorphism but not isomorphism

1.23 Show that, while the maps from Example 1.3 preserve linear operations, they are not isomorphisms.

1.3 Example The domain and codomain can be other than spaces of column vectors. Both of these are homomorphisms; the verifications are straightforward.

(1) $f_1 \colon \mathcal{P}_2 \to \mathcal{P}_3$ given by

$$a_0 + a_1 x + a_2 x^2 \;\mapsto\; a_0 x + (a_1/2)x^2 + (a_2/3)x^3$$

(2) $f_2 \colon \mathcal{M}_{2\times 2} \to \mathbb{R}$ given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto a + d$$

Three.II.1.23 The first is not onto; for instance, there is no polynomial that is sent the constant polynomial $p(x) = 1$. The second is not one-to-one; both of these

?

members of the domain

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

map to the same member of the codomain, $1 \in \mathbb{R}$.

Identity Map IS a linear transformation

Three.II.1.24 Yes; in any space $\mathrm{id}(c \cdot \vec{v} + d \cdot \vec{w}) = c \cdot \vec{v} + d \cdot \vec{w} = c \cdot \mathrm{id}(\vec{v}) + d \cdot \mathrm{id}(\vec{w})$.

Subtraction

1.26 Part of the definition of a linear function is that it respects addition. Does a linear function respect subtraction?

Three.II.1.26 Yes. Where $h: V \to W$ is linear, $h(\vec{u} - \vec{v}) = h(\vec{u} + (-1) \cdot \vec{v}) = h(\vec{u}) + (-1) \cdot h(\vec{v}) = h(\vec{u}) - h(\vec{v})$.

=> yea literally just multiply by c2 = -1

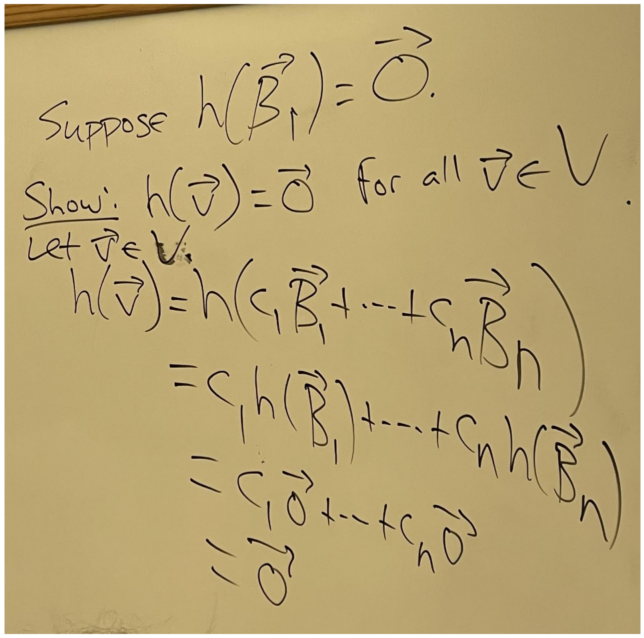
EX1:

1.27 Assume that h is a linear transformation of V and that $\langle \vec{\beta}_1, \ldots, \vec{\beta}_n \rangle$ is a basis of V. Prove each statement.

(a) If $h(\vec{\beta}_i) = \vec{0}$ for each basis vector then h is the zero map.

(b) If $h(\vec{\beta}_i) = \vec{\beta}_i$ for each basis vector then h is the identity map.

(c) If there is a scalar r such that $h(\vec{\beta}_i) = r \cdot \vec{\beta}_i$ for each basis vector then $h(\vec{v}) = r \cdot \vec{v}$ for all vectors in V.



Three.II.1.27 **(a)** Let $\vec{v} \in V$ be represented with respect to the basis as $\vec{v} = c_1\vec{\beta}_1 + \cdots + c_n\vec{\beta}_n$. Then $h(\vec{v}) = h(c_1\vec{\beta}_1 + \cdots + c_n\vec{\beta}_n) = c_1 h(\vec{\beta}_1) + \cdots + c_n h(\vec{\beta}_n) = c_1 \cdot \vec{0} + \cdots + c_n \cdot \vec{0} = \vec{0}$.

(b) This argument is similar to the prior one. Let $\vec{v} \in V$ be represented with respect to the basis as $\vec{v} = c_1\vec{\beta}_1 + \cdots + c_n\vec{\beta}_n$. Then $h(c_1\vec{\beta}_1 + \cdots + c_n\vec{\beta}_n) = c_1 h(\vec{\beta}_1) + \cdots + c_n h(\vec{\beta}_n) = c_1\vec{\beta}_1 + \cdots + c_n\vec{\beta}_n = \vec{v}$.

(c) As above, only $c_1 h(\vec{\beta}_1) + \cdots + c_n h(\vec{\beta}_n) = c_1 r\vec{\beta}_1 + \cdots + c_n r\vec{\beta}_n = r(c_1\vec{\beta}_1 + \cdots + c_n\vec{\beta}_n) = r\vec{v}$.

Identity map returns the same value as input h(X)=X

Redefined homomorphism structure

1.28 Consider the vector space $\mathbb{R}^+$ where vector addition and scalar multiplication are not the ones inherited from $\mathbb{R}$ but rather are these: $a + b$ is the product of a and b, and $r \cdot a$ is the r-th power of a. (This was shown to be a vector space in an earlier exercise.) Verify that the natural logarithm map $\ln\colon \mathbb{R}^+ \to \mathbb{R}$ is a homomorphism between these two spaces. Is it an isomorphism?

2.40 Prove that the image of a span equals the span of the images. That is, where $h\colon V \to W$ is linear, prove that if S is a subset of V then $h([S])$ equals $[h(S)]$. This generalizes Lemma 2.1 since it shows that if U is any subspace of V then its image $\{h(\vec{u}) \mid \vec{u} \in U\}$ is a subspace of W, because the span of the set U is U.

2.41 (a) Prove that for any linear map $h\colon V \to W$ and any $\vec{w} \in W$, the set $h^{-1}(\vec{w})$ has the form

$$h^{-1}(\vec{w}) = \{\vec{v} + \vec{n} \mid \vec{v}, \vec{n} \in V \text{ and } \vec{n} \in \mathscr{N}(h) \text{ and } h(\vec{v}) = \vec{w}\}$$

(if h is not onto and $\vec{w}$ is not in the range of h then this set is empty since its third condition cannot be satisfied). Such a set is a *coset* of $\mathscr{N}(h)$ and we denote it as $\vec{v} + \mathscr{N}(h)$.

(b) Consider the map $t\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$\begin{pmatrix} x \\ y \end{pmatrix} \stackrel{t}{\longmapsto} \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$$

for some scalars a, b, c, and d. Prove that t is linear.

Three.II.1.28 That it is a homomorphism follows from the familiar rules that the logarithm of a product is the sum of the logarithms $\ln(ab) = \ln(a) + \ln(b)$ and that the logarithm of a power is the multiple of the logarithm $\ln(a^r) = r\ln(a)$. This map is an isomorphism because it has an inverse, namely, the exponential map, so it is a correspondence, and therefore it is an isomorphism.

Image under set

1.29 Consider this transformation of the plane $\mathbb{R}^2$.

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x/2 \\ y/3 \end{pmatrix}$$

?

Find the image under this map of this ellipse.

$$\{\begin{pmatrix} x \\ u \end{pmatrix} \mid (x^2/4) + (y^2/9) = 1\}$$

Three.II.1.29 Where $\hat{x} = x/2$ and $\hat{y} = y/3$, the image set is

$$\{\begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} \mid \frac{(2\hat{x})^2}{4} + \frac{(3\hat{y})^2}{9} = 1\} = \{\begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} \mid \hat{x}^2 + \hat{y}^2 = 1\}$$

the unit circle in the $\hat{x}\hat{y}$-plane.

Generalization of vector multiplication functions

1.31 Verify that this map $h\colon \mathbb{R}^3 \to \mathbb{R}$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \\ z \end{pmatrix} \bullet \begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix} = 3x - y - z$$

is linear. Generalize.

Three.II.1.31 Verifying that it is linear is routine.

$$\begin{aligned} h(c_1 \cdot \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + c_2 \cdot \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}) &= h(\begin{pmatrix} c_1x_1 + c_2x_2 \\ c_1y_1 + c_2y_2 \\ c_1z_1 + c_2z_2 \end{pmatrix}) \\ &= 3(c_1x_1 + c_2x_2) - (c_1y_1 + c_2y_2) - (c_1z_1 + c_2z_2) \\ &= c_1 \cdot (3x_1 - y_1 - z_1) + c_2 \cdot (3x_2 - y_2 - z_2) \\ &= c_1 \cdot h(\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}) + c_2 \cdot h(\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}) \end{aligned}$$

The natural guess at a generalization is that for any fixed $\vec{k} \in \mathbb{R}^3$ the map $\vec{v} \mapsto \vec{v} \bullet \vec{k}$ is linear. This statement is true. It follows from properties of the dot product we have seen earlier: $(\vec{v} + \vec{u}) \bullet \vec{k} = \vec{v} \bullet \vec{k} + \vec{u} \bullet \vec{k}$ and $(r\vec{v}) \bullet \vec{k} = r(\vec{v} \bullet \vec{k})$. (The natural guess at a generalization of this generalization, that the map from $\mathbb{R}^n$ to $\mathbb{R}$ whose action consists of taking the dot product of its argument with a fixed vector $\vec{k} \in \mathbb{R}^n$ is linear, is also true.)

!

R1 to R1 confusion

1.32 Show that every homomorphism from $\mathbb{R}^1$ to $\mathbb{R}^1$ acts via multiplication by a scalar. Conclude that every nontrivial linear transformation of $\mathbb{R}^1$ is an isomorphism. Is that true for transformations of $\mathbb{R}^2$? $\mathbb{R}^n$?

?

Three.II.1.32 Let $h: \mathbb{R}^1 \to \mathbb{R}^1$ be linear. A linear map is determined by its action on a basis, so fix the basis $\langle 1 \rangle$ for $\mathbb{R}^1$. For any $r \in \mathbb{R}^1$ we have that $h(r) = h(r \cdot 1) = r \cdot h(1)$ and so h acts on any argument r by multiplying it by the constant $h(1)$. If $h(1)$ is not zero then the map is a correspondence—its inverse is division by $h(1)$—so any nontrivial transformation of $\mathbb{R}^1$ is an isomorphism.

This projection map is an example that shows that not every transformation of $\mathbb{R}^n$ acts via multiplication by a constant when $n > 1$, including when $n = 2$.

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \mapsto \begin{pmatrix} x_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Differentiation generalization

1.34 Consider the space of polynomials $\mathcal{P}_n$.

(a) Show that for each i, the i-th derivative operator d^i/dx^i is a linear transformation of that space.

(b) Conclude that for any scalars $c_k, \ldots, c_0$ this map is a linear transformation of that space.

$$f \mapsto c_k \frac{d^k}{dx^k} f + c_{k-1} \frac{d^{k-1}}{dx^{k-1}} f + \cdots + c_1 \frac{d}{dx} f + c_0 f$$

Three.II.1.34 **(a)** Each power i of the derivative operator is linear because of these rules familiar from calculus.

$$\frac{d^i}{dx^i}(\,f(x) + g(x)\,) = \frac{d^i}{dx^i} f(x) + \frac{d^i}{dx^i} g(x) \qquad \frac{d^i}{dx^i}\, r \cdot f(x) = r \cdot \frac{d^i}{dx^i} f(x)$$

(b) Any linear combination of linear maps is also a linear map. Thus, the given map is a linear transformation of $\mathcal{P}_n$.

Composition of Linear Functions

1.35 Lemma 1.17 shows that a sum of linear functions is ~~linear and~~ that a scalar multiple of a linear function is linear. Show also that a composition of linear functions is linear.

Three.II.1.35 (This argument has already appeared, as part of the proof that isomorphism is an equivalence.) Let $f\colon U \to V$ and $g\colon V \to W$ be linear. The composition preserves linear combinations

$$\begin{aligned} g \circ f(c_1\vec{u}_1 + c_2\vec{u}_2) &= g(\, f(c_1\vec{u}_1 + c_2\vec{u}_2)\,) = g(\, c_1 f(\vec{u}_1) + c_2 f(\vec{u}_2)\,) \\ &= c_1 \cdot g(f(\vec{u}_1)) + c_2 \cdot g(f(\vec{u}_2)) = c_1 \cdot g \circ f(\vec{u}_1) + c_2 \cdot g \circ f(\vec{u}_2) \end{aligned}$$

where $\vec{u}_1, \vec{u}_2 \in U$ and scalars c_1, c_2

=> composition = function in function

Independence over homomorphism

1.36 Where $f\colon V \to W$ is linear, suppose that $f(\vec{v}_1) = \vec{w}_1, \ldots, f(\vec{v}_n) = \vec{w}_n$ for some vectors $\vec{w}_1, \ldots, \vec{w}_n$ from W.

(a) If the set of $\vec{w}$'s is independent, must the set of $\vec{v}$'s also be independent?

(b) If the set of $\vec{v}$'s is independent, must the set of $\vec{w}$'s also be independent?

(c) If the set of $\vec{w}$'s spans W, must the set of $\vec{v}$'s span V?

(d) If the set of $\vec{v}$'s spans V, must the set of $\vec{w}$'s span W?

Three.II.1.36 **(a)** Yes. The set of $\vec{w}$'s cannot be linearly independent if the set of $\vec{v}$'s is linearly dependent because any nontrivial relationship in the domain $\vec{0}_V = c_1\vec{v}_1 + \cdots + c_n\vec{v}_n$ would give a nontrivial relationship in the range $f(\vec{0}_V) = \vec{0}_W = f(c_1\vec{v}_1 + \cdots + c_n\vec{v}_n) = c_1 f(\vec{v}_1) + \cdots + c_n f(\vec{v}_n) = c_1\vec{w} + \cdots + c_n\vec{w}_n$.

(b) Not necessarily. For instance, the transformation of $\mathbb{R}^2$ given by

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x+y \\ x+y \end{pmatrix}$$

sends this linearly independent set in the domain to a linearly dependent image.

$$\{\vec{v}_1, \vec{v}_2\} = \{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\} \mapsto \{\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix}\} = \{\vec{w}_1, \vec{w}_2\}$$

(c) Not necessarily. An example is the projection map $\pi\colon \mathbb{R}^3 \to \mathbb{R}^2$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \end{pmatrix}$$

and this set that does not span the domain but maps to a set that does span the codomain.

$$\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\} \stackrel{\pi}{\longmapsto} \{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\}$$

(d) Not necessarily. For instance, the injection map $\iota\colon \mathbb{R}^2 \to \mathbb{R}^3$ sends the standard basis $\mathcal{E}_2$ for the domain to a set that does not span the codomain. (*Remark.* However, the set of $\vec{w}$'s does span the range. A proof is easy.)

! independence codomain means independence domain

! independence domain DOES NOT mean independence in codomain

! span in codomain does not mean span in domain and vice versa

? Restriction to subspace is Homomorphism

1.40 Prove that the restriction of a homomorphism to a subspace of its domain is another homomorphism.

Three.II.1.40 Suppose that $h\colon V \to W$ is a homomorphism and suppose that S is a subspace of V. Consider the map $\hat{h}\colon S \to W$ defined by $\hat{h}(\vec{s}) = h(\vec{s})$. (The only difference between $\hat{h}$ and h is the difference in domain.) Then this new map is linear: $\hat{h}(c_1 \cdot \vec{s}_1 + c_2 \cdot \vec{s}_2) = h(c_1\vec{s}_1 + c_2\vec{s}_2) = c_1 h(\vec{s}_1) + c_2 h(\vec{s}_2) = c_1 \cdot \hat{h}(\vec{s}_1) + c_2 \cdot \hat{h}(\vec{s}_2)$.

Subspace trick question (ZERO VECTOR)

1.42 Consider the set of isomorphisms from a vector space to itself. Is this a subspace of the space $\mathcal{L}(V, V)$ of homomorphisms from the space to itself?

Three.II.1.42 No; the set of isomorphisms does not contain the zero map (unless the space is trivial).

Basis

1.43 Does Theorem 1.9 need that $\langle\vec{\beta}_1, \ldots, \vec{\beta}_n\rangle$ is a basis? That is, can we still get a well-defined and unique homomorphism if we drop either the condition that the set of $\vec{\beta}$'s be linearly independent, or the condition that it span the domain?

Three.II.1.43 If $\langle\vec{\beta}_1, \ldots, \vec{\beta}_n\rangle$ doesn't span the space then the map needn't be unique. For instance, if we try to define a map from $\mathbb{R}^2$ to itself by specifying only that $\vec{e}_1$ maps to itself, then there is more than one homomorphism possible; both the identity map and the projection map onto the first component fit this condition.

If we drop the condition that $\langle\vec{\beta}_1, \ldots, \vec{\beta}_n\rangle$ is linearly independent then we risk an inconsistent specification (i.e, there could be no such map). An example is if we consider $\langle\vec{e}_2, \vec{e}_1, 2\vec{e}_1\rangle$, and try to define a map from $\mathbb{R}^2$ to itself that sends $\vec{e}_2$ to itself, and sends both $\vec{e}_1$ and $2\vec{e}_1$ to $\vec{e}_1$. No homomorphism can satisfy these three conditions.

Mapping V to Rn

1.44 Let V be a vector space and assume that the maps $f_1, f_2 \colon V \to \mathbb{R}^1$ are linear.

(a) Define a map $F \colon V \to \mathbb{R}^2$ whose component functions are the given linear ones.

$$\vec{v} \mapsto \begin{pmatrix} f_1(\vec{v}) \\ f_2(\vec{v}) \end{pmatrix}$$

Show that F is linear.

(b) Does the converse hold — is any linear map from V to $\mathbb{R}^2$ made up of two linear component maps to $\mathbb{R}^1$?

(c) Generalize.

Three.II.1.44 **(a)** Briefly, the check of linearity is this.

$$F(r_1 \cdot \vec{v}_1 + r_2 \cdot \vec{v}_2) = \begin{pmatrix} f_1(r_1\vec{v}_1 + r_2\vec{v}_2) \\ f_2(r_1\vec{v}_1 + r_2\vec{v}_2) \end{pmatrix}$$

$$= r_1 \begin{pmatrix} f_1(\vec{v}_1) \\ f_2(\vec{v}_1) \end{pmatrix} + r_2 \begin{pmatrix} f_1(\vec{v}_2) \\ f_2(\vec{v}_2) \end{pmatrix} = r_1 \cdot F(\vec{v}_1) + r_2 \cdot F(\vec{v}_2)$$

(b) Yes. Let $\pi_1 \colon \mathbb{R}^2 \to \mathbb{R}^1$ and $\pi_2 \colon \mathbb{R}^2 \to \mathbb{R}^1$ be the projections

$$\begin{pmatrix} x \\ y \end{pmatrix} \xmapsto{\pi_1} x \quad \text{and} \quad \begin{pmatrix} x \\ y \end{pmatrix} \xmapsto{\pi_2} y$$

onto the two axes. Now, where $f_1(\vec{v}) = \pi_1(F(\vec{v}))$ and $f_2(\vec{v}) = \pi_2(F(\vec{v}))$ we have the desired component functions.

$$F(\vec{v}) = \begin{pmatrix} f_1(\vec{v}) \\ f_2(\vec{v}) \end{pmatrix}$$

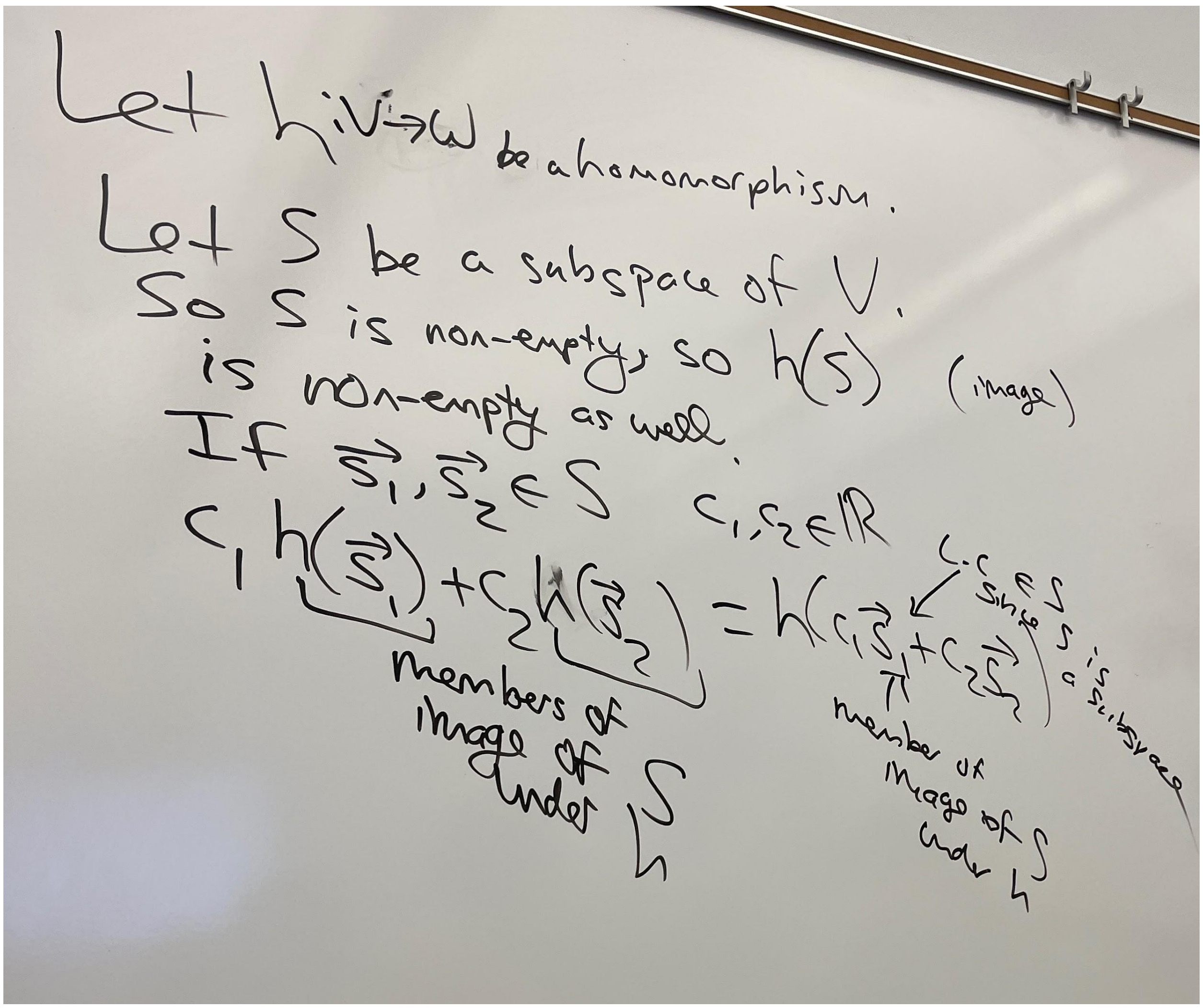
They are linear because they are the composition of linear functions, and the fact that the composition of linear functions is linear was part of the proof that isomorphism is an equivalence relation (alternatively, the check that they are linear is straightforward).

(c) In general, a map from a vector space V to an $\mathbb{R}^n$ is linear if and only if each of the component functions is linear. The verification is as in the prior item.

Range Space and Null Space

2.1 Lemma Under a homomorphism, the image of any subspace of the domain is a subspace of the codomain. In particular, the image of the entire space, the range of the homomorphism, is a subspace of the codomain.

Proof Let $h\colon V \to W$ be linear and let S be a subspace of the domain V. The image $h(S)$ is a subset of the codomain W, which is nonempty because S is nonempty. Thus, to show that $h(S)$ is a subspace of W we need only show that it is closed under linear combinations of two vectors. If $h(\vec{s}_1)$ and $h(\vec{s}_2)$ are members of $h(S)$ then $c_1 \cdot h(\vec{s}_1) + c_2 \cdot h(\vec{s}_2) = h(c_1 \cdot \vec{s}_1) + h(c_2 \cdot \vec{s}_2) = h(c_1 \cdot \vec{s}_1 + c_2 \cdot \vec{s}_2)$ is also a member of $h(S)$ because it is the image of $c_1 \cdot \vec{s}_1 + c_2 \cdot \vec{s}_2$ from S. QED



2.2 Definition The *range space* of a homomorphism $h\colon V \to W$ is

$$\mathscr{R}(h) = \{h(\vec{v}) \mid \vec{v} \in V\}$$

sometimes denoted $h(V)$. The dimension of the range space is the map's *rank*.

We shall soon see the connection between the rank of a map and the rank of a matrix.

2.3 Example For the derivative map $d/dx\colon \mathcal{P}_3 \to \mathcal{P}_3$ given by $a_0 + a_1x + a_2x^2 + a_3x^3 \mapsto a_1 + 2a_2x + 3a_3x^2$ the range space $\mathscr{R}(d/dx)$ is the set of quadratic polynomials $\{r + sx + tx^2 \mid r, s, t \in \mathbb{R}\}$. Thus, this map's rank is 3.

2.4 Example With this homomorphism $h\colon \mathcal{M}_{2\times 2} \to \mathcal{P}_3$

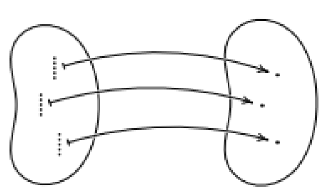
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (a + b + 2d) + cx^2 + cx^3$$

an image vector in the range can have any constant term, must have an x coefficient of zero, and must have the same coefficient of x^2 as of x^3. That is, the range space is $\mathscr{R}(h) = \{r + sx^2 + sx^3 \mid r, s \in \mathbb{R}\}$ and so the rank is 2.

The prior result shows that, in passing from the definition of isomorphism to the more general definition of homomorphism, omitting the onto requirement doesn't make an essential difference. Any homomorphism is onto some space, namely its range.

However, omitting the one-to-one condition does make a difference. A homomorphism may have many elements of the domain that map to one element of the codomain. Below is a bean sketch of a many-to-one map between sets.* It shows three elements of the codomain that are each the image of many members of the domain. (Rather than picture lots of individual $\mapsto$ arrows, each association of many inputs with one output shows only one such arrow.)

* More information on many-to-one maps is in the appendix.



Recall that for any function $h\colon V \to W$, the set of elements of V that map to $\vec{w} \in W$ is the *inverse image* $h^{-1}(\vec{w}) = \{\vec{v} \in V \mid h(\vec{v}) = \vec{w}\}$. Above, the left side shows three inverse image sets.

2.10 Lemma For any homomorphism the inverse image of a subspace of the range is a subspace of the domain. In particular, the inverse image of the trivial subspace of the range is a subspace of the domain.

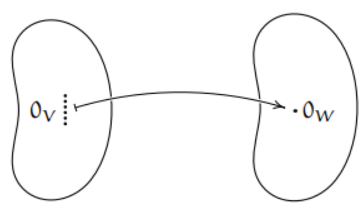
(The examples above consider inverse images of single vectors but this result is about inverse images of sets $h^{-1}(S) = \{\vec{v} \in V \mid h(\vec{v}) \in S\}$. We use the same term for both by taking the inverse image of a single element $h^{-1}(\vec{w})$ to be the inverse image of the one-element set $h^{-1}(\{\vec{w}\})$.)

Proof Let $h\colon V \to W$ be a homomorphism and let S be a subspace of the range space of h. Consider the inverse image of S. It is nonempty because it contains $\vec{0}_V$, since $h(\vec{0}_V) = \vec{0}_W$ and $\vec{0}_W$ is an element of S as S is a subspace. To finish we show that $h^{-1}(S)$ is closed under linear combinations. Let $\vec{v}_1$ and $\vec{v}_2$ be two of its elements, so that $h(\vec{v}_1)$ and $h(\vec{v}_2)$ are elements of S. Then $c_1\vec{v}_1 + c_2\vec{v}_2$ is an element of the inverse image $h^{-1}(S)$ because $h(c_1\vec{v}_1 + c_2\vec{v}_2) = c_1h(\vec{v}_1) + c_2h(\vec{v}_2)$ is a member of S. QED

2.11 Definition The *null space* or *kernel* of a linear map $h\colon V \to W$ is the inverse image of $\vec{0}_W$.

$$\mathscr{N}(h) = h^{-1}(\vec{0}_W) = \{\vec{v} \in V \mid h(\vec{v}) = \vec{0}_W\}$$

The dimension of the null space is the map's *nullity*.



cc c

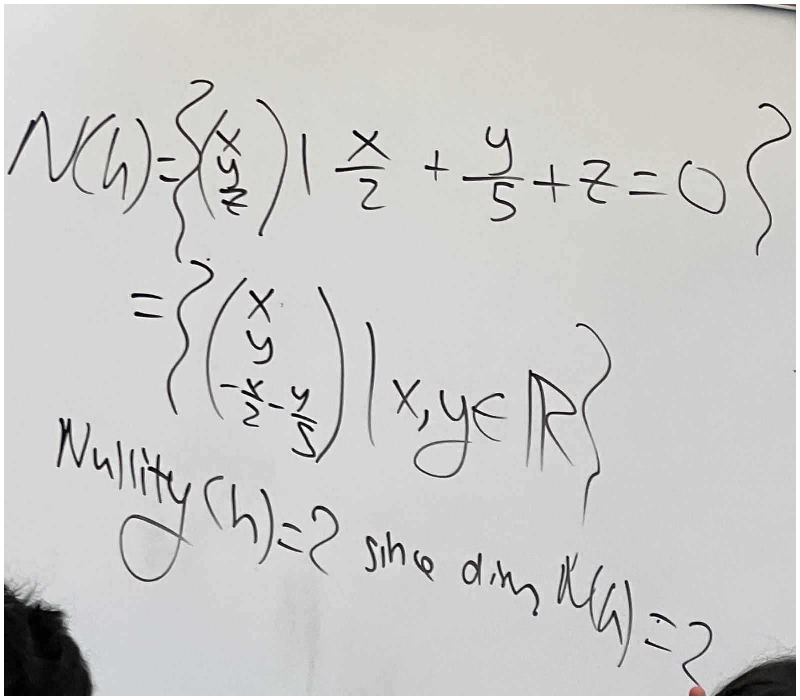
Null space = inverse map of the zero vector in the codomain!

If i have a one-to-one homomorphism, the only vector that can map into a zero vector is a zero vector → null space has to have ONLY THE ZERO VECTOR → nullity is also zero!

To find a null space: in the co-domain

Consider this map $h\colon \mathbb{R}^3 \to \mathbb{R}$.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \stackrel{h}{\longmapsto} x/2 + y/5 + z$$



-> if we need to find the inverse image for 1 in the codomain instead of 0(null space), we set equation to 1 → then write it out similar to finding the null space.

⇒ HOMOMORPHISMS ORGANIZE THE DOMAIN → each of these inverse images (including the null space) are organized sets that connects to one value in the codomain → that is why we say homomorphisms organize the domain.

Rank of range space + nullity(dimension of null space) = DIMENSION OF DOMAIN?

2.14 Theorem A linear map's rank plus its nullity equals the dimension of its domain.

PROOF Let $h\colon V \to W$ be linear and let $B_N = \langle \vec{\beta}_1, \dots, \vec{\beta}_k \rangle$ be a basis for the null space. Expand that to a basis $B_V = \langle \vec{\beta}_1, \dots, \vec{\beta}_k, \vec{\beta}_{k+1}, \dots, \vec{\beta}_n \rangle$ for the entire domain, using Corollary Two.III.2.12. We shall show that $B_R = \langle h(\vec{\beta}_{k+1}), \dots, h(\vec{\beta}_n) \rangle$ is a basis for the range space. Then counting the size of the bases gives the result.

To see that B_R is linearly independent, consider $\vec{0}_W = c_{k+1}h(\vec{\beta}_{k+1}) + \cdots + c_n h(\vec{\beta}_n)$. We have $\vec{0}_W = h(c_{k+1}\vec{\beta}_{k+1} + \cdots + c_n\vec{\beta}_n)$ and so $c_{k+1}\vec{\beta}_{k+1} + \cdots + c_n\vec{\beta}_n$ is in the null space of h. As B_N is a basis for the null space there are scalars $c_1, \dots, c_k$ satisfying this relationship.

$$c_1\vec{\beta}_1 + \cdots + c_k\vec{\beta}_k = c_{k+1}\vec{\beta}_{k+1} + \cdots + c_n\vec{\beta}_n$$

But this is an equation among members of B_V, which is a basis for V, so each c_i equals 0. Therefore B_R is linearly independent.

To show that B_R spans the range space consider a member of the range space $h(\vec{v})$. Express $\vec{v}$ as a linear combination $\vec{v} = c_1\vec{\beta}_1 + \cdots + c_n\vec{\beta}_n$ of members of B_V. This gives $h(\vec{v}) = h(c_1\vec{\beta}_1 + \cdots + c_n\vec{\beta}_n) = c_1 h(\vec{\beta}_1) + \cdots + c_k h(\vec{\beta}_k) + c_{k+1}h(\vec{\beta}_{k+1}) + \cdots + c_n h(\vec{\beta}_n)$ and since $\vec{\beta}_1, \dots, \vec{\beta}_k$ are in the null space, we have that $h(\vec{v}) = \vec{0} + \cdots + \vec{0} + c_{k+1}h(\vec{\beta}_{k+1}) + \cdots + c_n h(\vec{\beta}_n)$. Thus, $h(\vec{v})$ is a linear combination of members of B_R, and so B_R spans the range space. QED

Questions:

2.22

2.22 Find the range space and the rank of each homomorphism.

(a) $h\colon \mathcal{P}_3 \to \mathbb{R}^2$ given by

$$ax^2 + bx + c \mapsto \begin{pmatrix} a+b \\ a+c \end{pmatrix}$$

(b) $f\colon \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ x-y \\ 3y \end{pmatrix}$$

Part a)

The range space is $\mathbb{R}^2$.

Proof: Let $\begin{pmatrix} b \\ c \end{pmatrix} \in \mathbb{R}^2$.

Then consider $0x^2 + bx + c$.

$h(bx+c) = \begin{pmatrix} b \\ c \end{pmatrix}$,

So $\mathbb{R}^2$ is the range space

The rank of h is 2.

Part b) → for b: Prove by let (0 a b) be in R3
→ any vector (0 a b) is the image of the domain vector (a+b/3 and b/3)
→ so (0 a b) is the range space, the rank of that is 2

2.23

✓ 2.23 Find the range space and rank of each map.

(a) $h\colon \mathbb{R}^2 \to \mathcal{P}_3$ given by

$$\begin{pmatrix} a \\ b \end{pmatrix} \mapsto a + ax + ax^2$$

(b) $h\colon \mathcal{M}_{2\times2} \to \mathbb{R}$ given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto a + d$$

(c) $h\colon \mathcal{M}_{2\times2} \to \mathcal{P}_2$ given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto a + b + c + dx^2$$

(d) the zero map $Z\colon \mathbb{R}^3 \to \mathbb{R}^4$

Part a) range space is a(1+x+x^2)

Part b)

Part d) is it R3 or 0 rank???!!!

IMPORTANT

(b) The range space

$$\mathscr{R}(h) = \{a + d \mid a, b, c, d \in \mathbb{R}\}$$

is all of $\mathbb{R}$ (we can get any real number by taking d to be 0 and taking a to be the desired number). Thus, the rank is one.

(b) The null space is this.

$$\mathscr{N}(h) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a + d = 0\} = \{\begin{pmatrix} -d & b \\ c & d \end{pmatrix} \mid b, c, d \in \mathbb{R}\}$$

Thus the nullity is three.

(c) The null space

$$\mathscr{N}(h) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a + b + c = 0,\ d = 0\} = \{\begin{pmatrix} -b-c & b \\ c & 0 \end{pmatrix} \mid b, c \in \mathbb{R}\}$$

is a dimension 2 space, so the nullity is two.

(d) Every vector in the domain is mapped to the zero vector so the nullspace is $\mathscr{N}(h) = \mathbb{R}^3$.

2.17 Corollary The rank of a linear map is less than or equal to the dimension of the domain. Equality holds if and only if the nullity of the map is 0.

We know that an isomorphism exists between two spaces if and only if the dimension of the range equals the dimension of the domain. We have now seen that for a homomorphism to exist a necessary condition is that the dimension of the range must be less than or equal to the dimension of the domain. For instance, there is no homomorphism from $\mathbb{R}^2$ onto $\mathbb{R}^3$. There are many homomorphisms from $\mathbb{R}^2$ into $\mathbb{R}^3$, but none onto.

The range space of a linear map can be of dimension strictly less than the dimension of the domain and so linearly independent sets in the domain may map to linearly dependent sets in the range. (Example 2.3's derivative transformation on $\mathcal{P}_3$ has a domain of dimension 4 but a range of dimension 3 and the derivative sends $\{1, x, x^2, x^3\}$ to $\{0, 1, 2x, 3x^2\}$). That is, under a homomorphism independence may be lost. In contrast, dependence stays.

Image of a linearly dependent set is linearly dependent

2.18 Lemma Under a linear map, the image of a linearly dependent set is linearly dependent.

PROOF Suppose that $c_1\vec{v}_1 + \cdots + c_n\vec{v}_n = \vec{0}_V$ with some c_i nonzero. Apply h to both sides: $h(c_1\vec{v}_1 + \cdots + c_n\vec{v}_n) = c_1 h(\vec{v}_1) + \cdots + c_n h(\vec{v}_n)$ and $h(\vec{0}_V) = \vec{0}_W$. Thus we have $c_1 h(\vec{v}_1) + \cdots + c_n h(\vec{v}_n) = \vec{0}_W$ with some c_i nonzero. QED

When is independence not lost? The obvious sufficient condition is when the homomorphism is an isomorphism. This condition is also necessary; see

Exercise 37. We will finish this subsection comparing homomorphisms with isomorphisms by observing that a one-to-one homomorphism is an isomorphism from its domain onto its range.

2.19 Example This one-to-one homomorphism $\iota\colon \mathbb{R}^2 \to \mathbb{R}^3$

$$\begin{pmatrix} x \\ y \end{pmatrix} \stackrel{\iota}{\longmapsto} \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$$

gives a correspondence between $\mathbb{R}^2$ and the xy-plane subset of $\mathbb{R}^3$.

A one-to-one homomorphism is an ISOMORPHISM

2.20 Theorem Where V is an n-dimensional vector space, these are equivalent statements about a linear map $h\colon V \to W$.

(1) h is one-to-one

(2) h has an inverse from its range to its domain that is a linear map

(3) $\mathscr{N}(h) = \{\vec{0}\,\}$, that is, $\text{nullity}(h) = 0$

(4) $\text{rank}(h) = n$

(5) if $\langle \vec{\beta}_1, \ldots, \vec{\beta}_n \rangle$ is a basis for V then $\langle h(\vec{\beta}_1), \ldots, h(\vec{\beta}_n) \rangle$ is a basis for $\mathscr{R}(h)$

H being one to one doesnt mean h is an isomorphism between V & W, but h is an ISOMORPHISM between V and R(h) (its own range space)

Questions:

4 dim

(a) $h\colon \mathbb{R}^5 \to \mathbb{R}^8$ of rank five **(b)** $h\colon \underline{\mathcal{P}_3} \to \mathcal{P}_3$ of rank one

(c) $h\colon \mathbb{R}^6 \to \mathbb{R}^3$, an onto map **(d)** $h\colon \mathcal{M}_{3\times 3} \to \mathcal{M}_{3\times 3}$, onto

✓ **2.31** Each of these transformations of $\mathcal{P}_3$ is one-to-one. For each, find the inverse.

(a) $a_0 + a_1x + a_2x^2 + a_3x^3 \mapsto a_0 + a_1x + 2a_2x^2 + 3a_3x^3$

(b) $a_0 + a_1x + a_2x^2 + a_3x^3 \mapsto a_0 + a_2x + a_1x^2 + a_3x^3$

(c) $a_0 + a_1x + a_2x^2 + a_3x^3 \mapsto a_1 + a_2x + a_3x^2 + a_0x^3$

(d) $a_0 + a_1x + a_2x^2 + a_3x^3 \mapsto a_0 + (a_0 + a_1)x + (a_0 + a_1 + a_2)x^2 + (a_0 + a_1 + a_2 + a_3)x^3$

Three.II.2.31 These are the inverses.

(a) $a_0 + a_1x + a_2x^2 + a_3x^3 \mapsto a_0 + a_1x + (a_2/2)x^2 + (a_3/3)x^3$

(b) $a_0 + a_1x + a_2x^2 + a_3x^3 \mapsto a_0 + a_2x + a_1x^2 + a_3x^3$

(c) $a_0 + a_1x + a_2x^2 + a_3x^3 \mapsto a_3 + a_0x + a_1x^2 + a_2x^3$

(d) $a_0 + a_1x + a_2x^2 + a_3x^3 \mapsto a_0 + (a_1 - a_0)x + (a_2 - a_1)x^2 + (a_3 - a_2)x^3$

For instance, for the second one, the map given in the question sends $0 + 1x + 2x^2 + 3x^3 \mapsto 0 + 2x + 1x^2 + 3x^3$ and then the inverse above sends $0 + 2x + 1x^2 + 3x^3 \mapsto 0 + 1x + 2x^2 + 3x^3$. So this map is actually self-inverse.

2.33 List all pairs $(\text{rank}(h), \text{nullity}(h))$ that are possible for linear maps from $\mathbb{R}^5$ to $\mathbb{R}^3$.

Three.II.2.33 Because the rank plus the nullity equals the dimension of the domain (here, five), and the rank is at most three, the possible pairs are: $(3,2)$, $(2,3)$, $(1,4)$, and $(0,5)$. Coming up with linear maps that show that each pair is indeed possible is easy.

2.34 Does the differentiation map $d/dx\colon \mathcal{P}_n \to \mathcal{P}_n$ have an inverse?

2.35 Find the nullity of this map $h\colon \mathcal{P}_n \to \mathbb{R}$.

$$a_0 + a_1 x + \cdots + a_n x^n \mapsto \int_{x=0}^{x=1} a_0 + a_1 x + \cdots + a_n x^n \, dx$$

Three.II.2.34 No (unless $\mathcal{P}_n$ is trivial), because the two polynomials $f_0(x) = 0$ and $f_1(x) = 1$ have the same derivative; a map must be one-to-one to have an inverse.

Three.II.2.35 The null space is this.

$$\begin{aligned}\{a_0 + a_1 x + \cdots + a_n x^n \mid a_0(1) + \tfrac{a_1}{2}(1^2) + \cdots + \tfrac{a_n}{n+1}(1^{n+1}) = 0\}\\ = \{a_0 + a_1 x + \cdots + a_n x^n \mid a_0 + (a_1/2) + \cdots + (a_n/n+1) = 0\}\end{aligned}$$

Thus the nullity is n.

2.36 **(a)** Prove that a homomorphism is onto if and only if its rank equals the dimension of its codomain.

(b) Conclude that a homomorphism between vector spaces with the same dimension is one-to-one if and only if it is onto.

Three.II.2.36 **(a)** One direction is obvious: if the homomorphism is onto then its range is the codomain and so its rank equals the dimension of its codomain. For the other direction assume that the map's rank equals the dimension of the codomain. Then the map's range is a subspace of the codomain, and has dimension equal to the dimension of the codomain. Therefore, the map's range must equal the codomain, and the map is onto. (The 'therefore' is because there

is a linearly independent subset of the range that is of size equal to the dimension of the codomain, but any such linearly independent subset of the codomain must be a basis for the codomain, and so the range equals the codomain.)

(b) By Theorem 2.20, a homomorphism is one-to-one if and only if its nullity is zero. Because rank plus nullity equals the dimension of the domain, it follows that a homomorphism is one-to-one if and only if its rank equals the dimension of its domain. But this domain and codomain have the same dimension, so the map is one-to-one if and only if it is onto.

2.37 Show that a linear map is one-to-one if and only if it preserves linear independence.

Three.II.2.37 We are proving that $h\colon V \to W$ is one-to-one if and only if for every linearly independent subset S of V the subset $h(S) = \{h(\vec{s}) \mid \vec{s} \in S\}$ of W is linearly independent.

One half is easy — by Theorem 2.20, if h is not one-to-one then its null space is nontrivial, that is, it contains more than just the zero vector. So where $\vec{v} \neq \vec{0}_V$ is in that null space, the singleton set $\{\vec{v}\,\}$ is independent while its image $\{h(\vec{v})\} = \{\vec{0}_W\}$ is not.

For the other half, assume that h is one-to-one and so by Theorem 2.20 has a trivial null space. Then for any $\vec{v}_1, \ldots, \vec{v}_n \in V$, the relation

$$\vec{0}_W = c_1 \cdot h(\vec{v}_1) + \cdots + c_n \cdot h(\vec{v}_n) = h(c_1 \cdot \vec{v}_1 + \cdots + c_n \cdot \vec{v}_n)$$

implies the relation $c_1 \cdot \vec{v}_1 + \cdots + c_n \cdot \vec{v}_n = \vec{0}_V$. Hence, if a subset of V is independent then so is its image in W.

Remark. The statement is that a linear map is one-to-one if and only if it preserves independence for *all* sets (that is, if a set is independent then its image is also independent). A map that is not one-to-one may well preserve some independent sets. One example is this map from $\mathbb{R}^3$ to $\mathbb{R}^2$.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x + y + z \\ 0 \end{pmatrix}$$

Linear independence is preserved for this set

$$\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\} \mapsto \{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\}$$

and (in a somewhat more tricky example) also for this set

$$\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\} \mapsto \{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\}$$

(recall that in a set, repeated elements do not appear twice). However, there are

sets whose independence is not preserved under this map

$$\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}\} \mapsto \{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix}\}$$

and so not all sets have independence preserved.

Three.II.2.40 This is a simple calculation.

$$
\begin{aligned}
h([S]) &= \{h(c_1\vec{s}_1 + \cdots + c_n\vec{s}_n) \mid c_1, \ldots, c_n \in \mathbb{R} \text{ and } \vec{s}_1, \ldots, \vec{s}_n \in S\} \\
&= \{c_1 h(\vec{s}_1) + \cdots + c_n h(\vec{s}_n) \mid c_1, \ldots, c_n \in \mathbb{R} \text{ and } \vec{s}_1, \ldots, \vec{s}_n \in S\} \\
&= [h(S)]
\end{aligned}
$$

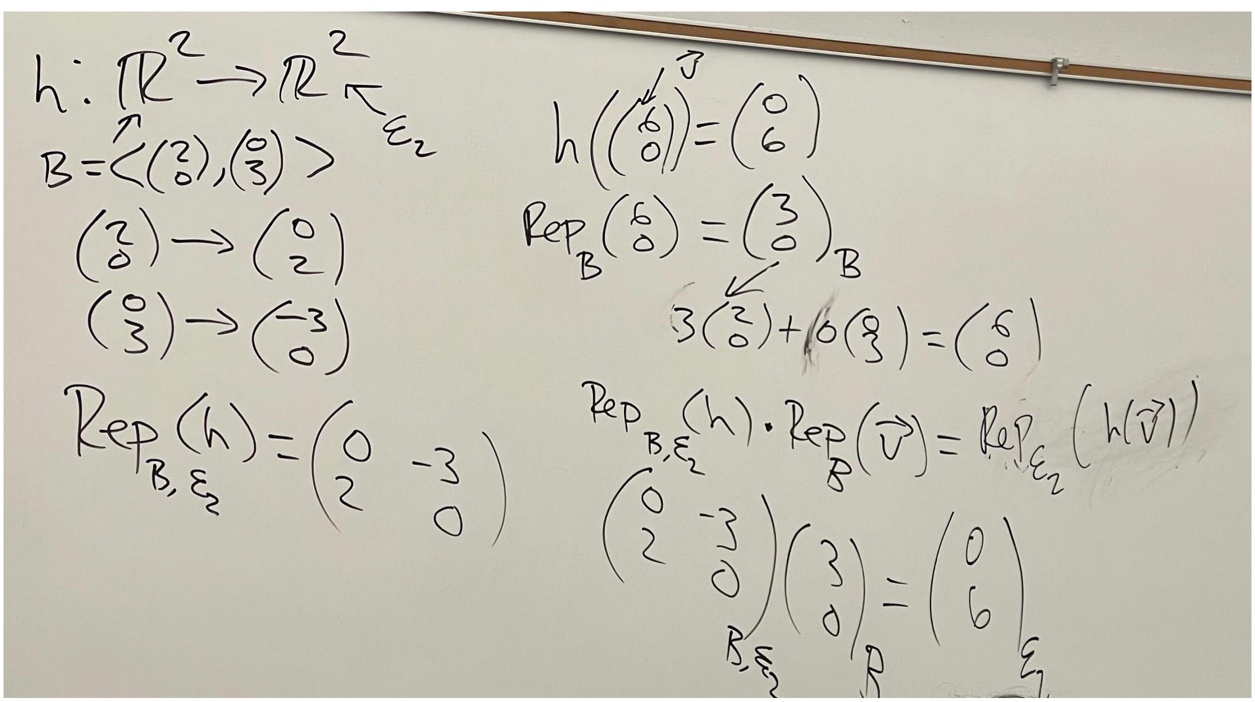
Three.II.2.41 **(a)** We will show the sets are equal $h^{-1}(\vec{w}) = \{\vec{v} + \vec{n} \mid \vec{n} \in \mathscr{N}(h)\}$ by mutual inclusion. For the $\{\vec{v} + \vec{n} \mid \vec{n} \in \mathscr{N}(h)\} \subseteq h^{-1}(\vec{w})$ direction, just note that $h(\vec{v} + \vec{n}) = h(\vec{v}) + h(\vec{n})$ equals $\vec{w}$, and so any member of the first set is a member of the second. For the $h^{-1}(\vec{w}) \subseteq \{\vec{v} + \vec{n} \mid \vec{n} \in \mathscr{N}(h)\}$ direction, consider $\vec{u} \in h^{-1}(\vec{w})$. Because h is linear, $h(\vec{u}) = h(\vec{v})$ implies that $h(\vec{u} - \vec{v}) = \vec{0}$. We can write $\vec{u} - \vec{v}$ as $\vec{n}$, and then we have that $\vec{u} \in \{\vec{v} + \vec{n} \mid \vec{n} \in \mathscr{N}(h)\}$, as desired, because $\vec{u} = \vec{v} + (\vec{u} - \vec{v})$.

(b) This check is routine.

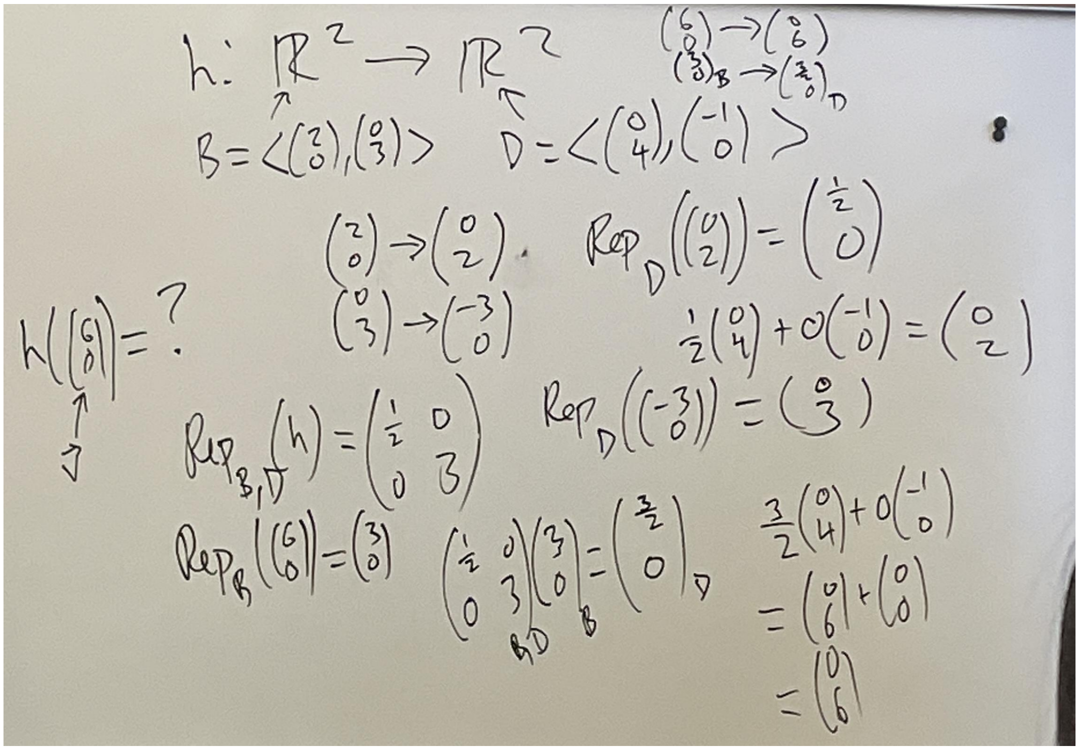
MUST BE ONE TO ONE TO HAVE AN INVERSE

Three.III Computing Linear Maps

Ex: A homomorphism that rotates a matrix:



Next step: homomorphism to another basis.



Matrix Representation of a Linear Map

1.2 Definition Suppose that V and W are vector spaces of dimensions n and m with bases B and D, and that $h\colon V \to W$ is a linear map. If

$$\mathrm{Rep}_D(h(\vec{\beta}_1)) = \begin{pmatrix} h_{1,1} \\ h_{2,1} \\ \vdots \\ h_{m,1} \end{pmatrix}_D \quad \ldots \quad \mathrm{Rep}_D(h(\vec{\beta}_n)) = \begin{pmatrix} h_{1,n} \\ h_{2,n} \\ \vdots \\ h_{m,n} \end{pmatrix}_D$$

then

$$\mathrm{Rep}_{B,D}(h) = \begin{pmatrix} h_{1,1} & h_{1,2} & \ldots & h_{1,n} \\ h_{2,1} & h_{2,2} & \ldots & h_{2,n} \\ & \vdots & & \\ h_{m,1} & h_{m,2} & \ldots & h_{m,n} \end{pmatrix}_{B,D}$$

is the *matrix representation of* h *with respect to* B, D.

In that matrix the number of columns n is the dimension of the map's domain while the number of rows m is the dimension of the codomain.

Main Formula

Basis B for V.S V Basis D for V.S. W

$$\boxed{\operatorname{Rep}_{B,D}(h)\operatorname{Rep}_B(\vec{v}) = \operatorname{Rep}_D(h(\vec{v}))}$$

→ check 1.27, 1.30 too

→ Rep B,D (h) is the REPRESENTATION MATRIX

Questions:

✓ 1.17 For a homomorphism from $\mathcal{P}_2$ to $\mathcal{P}_3$ that sends

$$1 \mapsto 1 + x, \quad x \mapsto 1 + 2x, \quad \text{and} \quad x^2 \mapsto x - x^3$$

where does $1 - 3x + 2x^2$ go?

Three.III.1.17 Here are two ways to get the answer.

First, obviously $1-3x+2x^2 = 1\cdot 1-3\cdot x+2\cdot x^2$, and so we can apply the general property of preservation of combinations to get $h(1-3x+2x^2) = h(1\cdot 1-3\cdot x+2\cdot x^2) = 1\cdot h(1)-3\cdot h(x)+2\cdot h(x^2) = 1\cdot(1+x)-3\cdot(1+2x)+2\cdot(x-x^3) = -2-3x-2x^3$.

The other way uses the computation scheme developed in this subsection. Because we know where these elements of the space go, we consider this basis $B = \langle 1, x, x^2\rangle$ for the domain. Arbitrarily, we can take $D = \langle 1, x, x^2, x^3\rangle$ as a basis for the codomain. With those choices, we have that

$$\mathrm{Rep}_{B,D}(h) = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}_{B,D}$$

and, as

$$\mathrm{Rep}_{B}(1-3x+2x^2) = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}_{B}$$

the matrix-vector multiplication calculation gives this.

$$\mathrm{Rep}_{D}(h(1-3x+2x^2)) = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}_{B,D} \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}_{B} = \begin{pmatrix} -2 \\ -3 \\ 0 \\ -2 \end{pmatrix}_{D}$$

Thus, $h(1-3x+2x^2) = -2\cdot 1-3\cdot x+0\cdot x^2-2\cdot x^3 = -2-3x-2x^3$, as above.

!

1.18 Let $h\colon \mathbb{R}^2 \to \mathcal{M}_{2\times 2}$ be the linear transformation with this action.

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

What is its effect on the general vector with entries x and y?

Three.III.1.18 Fix this natural basis for $\mathcal{M}_{2\times2}$.

$$D = \langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \rangle$$

The representation of the map h with respect to $\mathcal{E}_2, D$ is this.

$$\mathrm{Rep}_{\mathcal{E}_2,D}(h) = \begin{pmatrix} 1 & 0 \\ 2 & -1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Since the general vector is represented with respect to $\mathcal{E}_2$ by itself, and similarly every matrix is represented with respect to D by itself, we have this for the effect of the map.

$$\mathrm{Rep}_D(h(\begin{pmatrix} x \\ y \end{pmatrix})) = \begin{pmatrix} 1 & 0 \\ 2 & -1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 2x - y \\ y \\ x \end{pmatrix}$$

✓ **1.19** Assume that $h\colon \mathbb{R}^2 \to \mathbb{R}^3$ is determined by this action.

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

Using the standard bases, find

(a) the matrix representing this map;

(b) a general formula for $h(\vec{v})$.

Three.III.1.19 Again, as recalled in the subsection, with respect to $\mathcal{E}_i$, a column vector represents itself.

(a) To represent h with respect to $\mathcal{E}_2, \mathcal{E}_3$ take the images of the basis vectors from the domain, and represent them with respect to the basis for the codomain. The first is this

$$\mathrm{Rep}_{\mathcal{E}_3}(\,h(\vec{e}_1)\,) = \mathrm{Rep}_{\mathcal{E}_3}(\begin{pmatrix}2\\2\\0\end{pmatrix}) = \begin{pmatrix}2\\2\\0\end{pmatrix}$$

while the second is this.

$$\mathrm{Rep}_{\mathcal{E}_3}(\,h(\vec{e}_2)\,) = \mathrm{Rep}_{\mathcal{E}_3}(\begin{pmatrix}0\\1\\-1\end{pmatrix}) = \begin{pmatrix}0\\1\\-1\end{pmatrix}$$

Adjoin these to make the matrix.

$$\mathrm{Rep}_{\mathcal{E}_2,\mathcal{E}_3}(h) = \begin{pmatrix}2&0\\2&1\\0&-1\end{pmatrix}$$

(b) For any $\vec{v}$ in the domain $\mathbb{R}^2$,

$$\mathrm{Rep}_{\mathcal{E}_2}(\vec{v}) = \mathrm{Rep}_{\mathcal{E}_2}(\begin{pmatrix}v_1\\v_2\end{pmatrix}) = \begin{pmatrix}v_1\\v_2\end{pmatrix}$$

and so

$$\mathrm{Rep}_{\mathcal{E}_3}(\,h(\vec{v})\,) = \begin{pmatrix}2&0\\2&1\\0&-1\end{pmatrix}\begin{pmatrix}v_1\\v_2\end{pmatrix} = \begin{pmatrix}2v_1\\2v_1+v_2\\-v_2\end{pmatrix}$$

is the desired representation.

1.20 Represent the homomorphism $h\colon \mathbb{R}^3 \to \mathbb{R}^2$ given by this formula and with respect to these bases.

$$\begin{pmatrix}x\\y\\z\end{pmatrix} \mapsto \begin{pmatrix}x+y\\x+z\end{pmatrix} \qquad B = \langle \begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix} \rangle \quad D = \langle \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\2\end{pmatrix} \rangle$$

Three.III.1.20 The action of the map on the domain's basis vectors is this.

$$\begin{pmatrix}1\\1\\1\end{pmatrix}\mapsto\begin{pmatrix}2\\2\end{pmatrix}\quad\begin{pmatrix}1\\1\\0\end{pmatrix}\mapsto\begin{pmatrix}2\\1\end{pmatrix}\quad\begin{pmatrix}1\\0\\0\end{pmatrix}\mapsto\begin{pmatrix}1\\1\end{pmatrix}$$

Represent those with respect to the codomain's basis.

$$\mathrm{Rep}_D(\begin{pmatrix}2\\2\end{pmatrix})=\begin{pmatrix}2\\1\end{pmatrix}_D\quad\mathrm{Rep}_D(\begin{pmatrix}2\\1\end{pmatrix})=\begin{pmatrix}2\\1/2\end{pmatrix}_D\quad\mathrm{Rep}_D(\begin{pmatrix}1\\1\end{pmatrix})=\begin{pmatrix}1\\1/2\end{pmatrix}_D$$

Concatenate them together into a matrix.

$$\mathrm{Rep}_{B,D}(h)=\begin{pmatrix}2&2&1\\1&1/2&1/2\end{pmatrix}$$

!

✓ 1.21 Let $d/dx\colon \mathcal{P}_3\to\mathcal{P}_3$ be the derivative transformation.

(a) Represent d/dx with respect to B,B where $B=\langle 1,x,x^2,x^3\rangle$.

(b) Represent d/dx with respect to B,D where $D=\langle 1,2x,3x^2,4x^3\rangle$.

Three.III.1.21 **(a)** We must first find the image of each vector from the domain's basis, and then represent that image with respect to the codomain's basis.

$$\mathrm{Rep}_B(\frac{d\,1}{dx}) = \begin{pmatrix}0\\0\\0\\0\end{pmatrix} \quad \mathrm{Rep}_B(\frac{d\,x}{dx}) = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \quad \mathrm{Rep}_B(\frac{d\,x^2}{dx}) = \begin{pmatrix}0\\2\\0\\0\end{pmatrix}$$

$$\mathrm{Rep}_B(\frac{d\,x^3}{dx}) = \begin{pmatrix}0\\0\\3\\0\end{pmatrix}$$

Those representations are then adjoined to make the matrix representing the map.

$$\mathrm{Rep}_{B,B}(\frac{d}{dx}) = \begin{pmatrix}0&1&0&0\\0&0&2&0\\0&0&0&3\\0&0&0&0\end{pmatrix}$$

(b) Proceeding as in the prior item, we represent the images of the domain's basis vectors

$$\mathrm{Rep}_D(\frac{d\,1}{dx}) = \begin{pmatrix}0\\0\\0\\0\end{pmatrix} \quad \mathrm{Rep}_D(\frac{d\,x}{dx}) = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \quad \mathrm{Rep}_D(\frac{d\,x^2}{dx}) = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$$

$$\mathrm{Rep}_D(\frac{d\,x^3}{dx}) = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}$$

and adjoin to make the matrix.

$$\mathrm{Rep}_{B,D}(\frac{d}{dx}) = \begin{pmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\0&0&0&0\end{pmatrix}$$

1.23 Represent the identity map on any nontrivial space with respect to B, B, where B is any basis.

Three.III.1.23 Where the space is n-dimensional,

$$\mathrm{Rep}_{B,B}(\mathrm{id}) = \begin{pmatrix} 1 & 0 \ldots & 0 \\ 0 & 1 \ldots & 0 \\ & \vdots & \\ 0 & 0 \ldots & 1 \end{pmatrix}_{B,B}$$

is the $n \times n$ identity matrix.

1.24 Represent, with respect to the natural basis, the transpose transformation on the space $\mathcal{M}_{2\times 2}$ of 2×2 matrices.

?

Three.III.1.24 Taking this as the natural basis

$$B = \langle \vec{\beta}_1, \vec{\beta}_2, \vec{\beta}_3, \vec{\beta}_4 \rangle = \langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \rangle$$

the transpose map acts in this way

$$\vec{\beta}_1 \mapsto \vec{\beta}_1 \quad \vec{\beta}_2 \mapsto \vec{\beta}_3 \quad \vec{\beta}_3 \mapsto \vec{\beta}_2 \quad \vec{\beta}_4 \mapsto \vec{\beta}_4$$

so that representing the images with respect to the codomain's basis and adjoining those column vectors together gives this.

$$\mathrm{Rep}_{B,B}(\mathrm{trans}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}_{B,B}$$

!

1.29 Give a formula for the product of a matrix and $\vec{e}_i$, the column vector that is all zeroes except for a single one in the i-th position.

Three.III.1.29 The product

$$\begin{pmatrix} h_{1,1} & \ldots & h_{1,i} & \ldots & h_{1,n} \\ h_{2,1} & \ldots & h_{2,i} & \ldots & h_{2,n} \\ & \vdots & & & \\ h_{m,1} & \ldots & h_{m,i} & \ldots & h_{1,n} \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} h_{1,i} \\ h_{2,i} \\ \vdots \\ h_{m,i} \end{pmatrix}$$

gives the i-th column of the matrix.

✓ **1.30** For each vector space of functions of one real variable, represent the derivative transformation with respect to B, B.

(a) $\{a\cos x + b\sin x \mid a, b \in \mathbb{R}\}$, $B = \langle \cos x, \sin x \rangle$

(b) $\{ae^x + be^{2x} \mid a, b \in \mathbb{R}\}$, $B = \langle e^x, e^{2x} \rangle$

(c) $\{a + bx + ce^x + dxe^x \mid a, b, c, d \in \mathbb{R}\}$, $B = \langle 1, x, e^x, xe^x \rangle$

Three.III.1.30 **(a)** The images of the basis vectors for the domain are $\cos x \overset{d/dx}{\longmapsto} -\sin x$ and $\sin x \overset{d/dx}{\longmapsto} \cos x$. Representing those with respect to the codomain's basis (again, B) and adjoining the representations gives this matrix.

$$\mathrm{Rep}_{B,B}(\frac{d}{dx}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}_{B,B}$$

(b) The images of the vectors in the domain's basis are $e^x \overset{d/dx}{\longmapsto} e^x$ and $e^{2x} \overset{d/dx}{\longmapsto} 2e^{2x}$. Representing with respect to the codomain's basis and adjoining gives this matrix.

$$\mathrm{Rep}_{B,B}(\frac{d}{dx}) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}_{B,B}$$

(c) The images of the members of the domain's basis are $1 \overset{d/dx}{\longmapsto} 0$, $x \overset{d/dx}{\longmapsto} 1$, $e^x \overset{d/dx}{\longmapsto} e^x$, and $xe^x \overset{d/dx}{\longmapsto} e^x + xe^x$. Representing these images with respect to B and adjoining gives this matrix.

$$\mathrm{Rep}_{B,B}(\frac{d}{dx}) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}_{B,B}$$

✓ **1.32** Can one matrix represent two different linear maps? That is, can $\mathrm{Rep}_{B,D}(h) = \mathrm{Rep}_{\hat{B},\hat{D}}(\hat{h})$?

Three.III.1.32 Yes, for two reasons.

First, the two maps h and $\hat{h}$ need not have the same domain and codomain. For instance,

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

represents a map $h\colon \mathbb{R}^2 \to \mathbb{R}^2$ with respect to the standard bases that sends

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} 1 \\ 3 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

and also represents a $\hat{h}\colon \mathcal{P}_1 \to \mathbb{R}^2$ with respect to $\langle 1, x\rangle$ and $\mathcal{E}_2$ that acts in this way.

$$1 \mapsto \begin{pmatrix} 1 \\ 3 \end{pmatrix} \quad \text{and} \quad x \mapsto \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

The second reason is that, even if the domain and codomain of h and $\hat{h}$ coincide, different bases produce different maps. An example is the 2×2 identity matrix

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

which represents the identity map on $\mathbb{R}^2$ with respect to $\mathcal{E}_2, \mathcal{E}_2$. However, with respect to $\mathcal{E}_2$ for the domain but the basis $D = \langle \vec{e}_2, \vec{e}_1\rangle$ for the codomain, the same matrix I represents the map that swaps the first and second components

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} y \\ x \end{pmatrix}$$

(that is, reflection about the line $y = x$).

?

1.5 Theorem Assume that V and W are vector spaces of dimensions n and m with bases B and D, and that $h\colon V \to W$ is a linear map. If h is represented by

$$\mathrm{Rep}_{B,D}(h) = \begin{pmatrix} h_{1,1} & h_{1,2} & \ldots & h_{1,n} \\ h_{2,1} & h_{2,2} & \ldots & h_{2,n} \\ & \vdots & & \\ h_{m,1} & h_{m,2} & \ldots & h_{m,n} \end{pmatrix}_{B,D}$$

and $\vec{v} \in V$ is represented by

?

$$\mathrm{Rep}_B(\vec{v}) = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}_B$$

then the representation of the image of $\vec{v}$ is this.

$$\mathrm{Rep}_D(\,h(\vec{v})\,) = \begin{pmatrix} h_{1,1}c_1 + h_{1,2}c_2 + \cdots + h_{1,n}c_n \\ h_{2,1}c_1 + h_{2,2}c_2 + \cdots + h_{2,n}c_n \\ \vdots \\ h_{m,1}c_1 + h_{m,2}c_2 + \cdots + h_{m,n}c_n \end{pmatrix}_D$$

Proof ↑

Three.III.1.33 We mimic Example 1.1, just replacing the numbers with letters.

Write B as $\langle \vec{\beta}_1, \ldots, \vec{\beta}_n \rangle$ and D as $\langle \vec{\delta}_1, \ldots, \vec{\delta}_m \rangle$. By definition of representation of a map with respect to bases, the assumption that

$$\mathrm{Rep}_{B,D}(h) = \begin{pmatrix} h_{1,1} & \ldots & h_{1,n} \\ \vdots & & \vdots \\ h_{m,1} & \ldots & h_{m,n} \end{pmatrix}$$

means that $h(\vec{\beta}_i) = h_{i,1}\vec{\delta}_1 + \cdots + h_{i,n}\vec{\delta}_n$. And, by the definition of the representation of a vector with respect to a basis, the assumption that

$$\mathrm{Rep}_B(\vec{v}) = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$$

means that $\vec{v} = c_1\vec{\beta}_1 + \cdots + c_n\vec{\beta}_n$. Substituting gives

$$\begin{aligned} h(\vec{v}) &= h(c_1 \cdot \vec{\beta}_1 + \cdots + c_n \cdot \vec{\beta}_n) \\ &= c_1 \cdot h(\vec{\beta}_1) + \cdots + c_n \cdot \vec{\beta}_n \\ &= c_1 \cdot (h_{1,1}\vec{\delta}_1 + \cdots + h_{m,1}\vec{\delta}_m) + \cdots + c_n \cdot (h_{1,n}\vec{\delta}_1 + \cdots + h_{m,n}\vec{\delta}_m) \\ &= (h_{1,1}c_1 + \cdots + h_{1,n}c_n) \cdot \vec{\delta}_1 + \cdots + (h_{m,1}c_1 + \cdots + h_{m,n}c_n) \cdot \vec{\delta}_m \end{aligned}$$

and so $h(\vec{v})$ is represented as required.

Three.III.2. Any Matrix represents a Linear Map

Any matrix you choose represents a homomorphism between some choice of vector spaces and some choices of basis.

The prior subsection shows how to start with a linear map and produce its matrix representation. What about the converse?

Example Fix a matrix

$$H = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

and also fix a domain and codomain, with bases.

$$\mathcal{E}_2 \subset \mathbb{R}^2 \qquad \langle 1-x, 1+x \rangle \subset \mathcal{P}_1$$

Is there a linear map between the spaces associated with the matrix?

Any matrix is a homomorphism between vector spaces of APPROPIATE DIMENSION with any pair of bases:

2.2 Theorem Any matrix represents a homomorphism between vector spaces of appropriate dimensions, with respect to any pair of bases.

Proof We must check that for any matrix H and any domain and codomain bases B, D, the defined map h is linear. If $\vec{v}, \vec{u} \in V$ are such that

$$\mathrm{Rep}_B(\vec{v}) = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \qquad \mathrm{Rep}_B(\vec{u}) = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$$

and $c, d \in \mathbb{R}$ then the calculation

$$\begin{aligned} h(c\vec{v} + d\vec{u}) &= \big(h_{1,1}(cv_1 + du_1) + \cdots + h_{1,n}(cv_n + du_n)\big) \cdot \vec{\delta}_1 + \\ &\qquad \cdots + \big(h_{m,1}(cv_1 + du_1) + \cdots + h_{m,n}(cv_n + du_n)\big) \cdot \vec{\delta}_m \\ &= c \cdot h(\vec{v}) + d \cdot h(\vec{u}) \end{aligned}$$

supplies that check. QED

⇒ Main Question: Given matrix that represents the homomorphism, what properties can I know about that homormorphism?

Rank of Representation Matrix = Rank of any map represented

2.4 Theorem The rank of a matrix equals the rank of any map that it represents.

PROOF Suppose that the matrix H is $m \times n$. Fix domain and codomain spaces V and W of dimension n and m with bases $B = \langle \vec{\beta}_1, \ldots, \vec{\beta}_n \rangle$ and D. Then H represents some linear map h between those spaces with respect to these bases whose range space

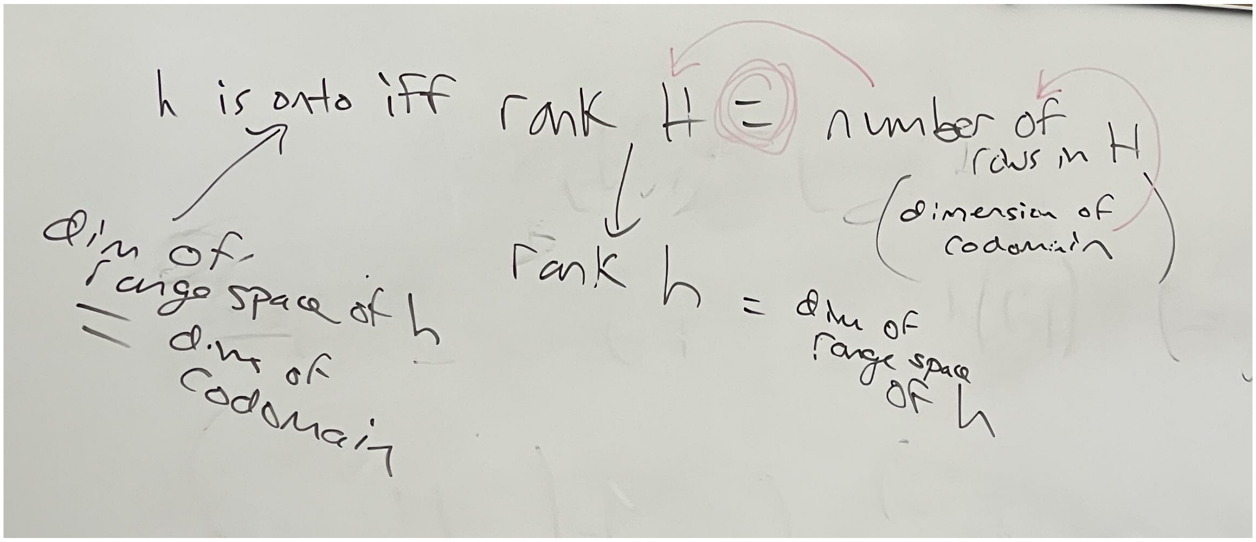
$$\begin{aligned}\{h(\vec{v}) \mid \vec{v} \in V\} &= \{h(c_1\vec{\beta}_1 + \cdots + c_n\vec{\beta}_n) \mid c_1, \ldots, c_n \in \mathbb{R}\} \\ &= \{c_1 h(\vec{\beta}_1) + \cdots + c_n h(\vec{\beta}_n) \mid c_1, \ldots, c_n \in \mathbb{R}\}\end{aligned}$$

is the span $[\{h(\vec{\beta}_1), \ldots, h(\vec{\beta}_n)\}]$. The rank of the map h is the dimension of this range space.

The rank of the matrix is the dimension of its column space, the span of the set of its columns $[\{\mathrm{Rep}_D(h(\vec{\beta}_1)), \ldots, \mathrm{Rep}_D(h(\vec{\beta}_n))\}]$.

To see that the two spans have the same dimension, recall from the proof of Lemma I.2.5 that if we fix a basis then representation with respect to that basis gives an isomorphism $\mathrm{Rep}_D \colon W \to \mathbb{R}^m$. Under this isomorphism there is a linear relationship among members of the range space if and only if the same relationship holds in the column space, e.g, $\vec{0} = c_1 \cdot h(\vec{\beta}_1) + \cdots + c_n \cdot h(\vec{\beta}_n)$ if and only if $\vec{0} = c_1 \cdot \mathrm{Rep}_D(h(\vec{\beta}_1)) + \cdots + c_n \cdot \mathrm{Rep}_D(h(\vec{\beta}_n))$. Hence, a subset of the range space is linearly independent if and only if the corresponding subset of the column space is linearly independent. Therefore the size of the largest linearly independent subset of the range space equals the size of the largest linearly independent subset of the column space, and so the two spaces have the same dimension. QED

That settles the apparent ambiguity in our use of the same word 'rank' to apply both to matrices and to maps.



=> just find the rank of the matrix → get rank of map.

Onto & number of rows, one-to-one & columns

2.6 Corollary Let h be a linear map represented by a matrix H. Then h is onto if and only if the rank of H equals the number of its rows, and h is one-to-one if and only if the rank of H equals the number of its columns.

PROOF For the onto half, the dimension of the range space of h is the rank of h, which equals the rank of H by the theorem. Since the dimension of the codomain of h equals the number of rows in H, if the rank of H equals the number of rows then the dimension of the range space equals the dimension of the codomain. But a subspace with the same dimension as its superspace must equal that superspace (because any basis for the range space is a linearly independent subset of the codomain whose size is equal to the dimension of the

codomain, and thus this basis for the range space must also be a basis for the codomain).

For the other half, a linear map is one-to-one if and only if it is an isomorphism between its domain and its range, that is, if and only if its domain has the same dimension as its range. The number of columns in H is the dimension of h's domain and by the theorem the rank of H equals the dimension of h's range. QED

Singularity Vs NonSingularity

2.7 Definition A linear map that is one-to-one and onto is *nonsingular*, otherwise it is *singular*. That is, a linear map is nonsingular if and only if it is an isomorphism.

2.8 Remark Some authors use 'nonsingular' as a synonym for one-to-one while others use it the way that we have here. The difference is slight because any map is onto its range space, so a one-to-one map is an isomorphism with its range.

In the first chapter we defined a matrix to be nonsingular if it is square and is the matrix of coefficients of a linear system with a unique solution. The next result justifies our dual use of the term.

Nonsingular linear map & Square Matrix

2.9 Lemma A nonsingular linear map is represented by a square matrix. A square matrix represents nonsingular maps if and only if it is a nonsingular matrix. Thus, a matrix represents isomorphisms if and only if it is square and nonsingular.

PROOF Assume that the map $h\colon V \to W$ is nonsingular. Corollary 2.6 says that for any matrix H representing that map, because h is onto the number of rows of H equals the rank of H, and because h is one-to-one the number of columns of H is also equal to the rank of H. Hence H is square.

Next assume that H is square, $n \times n$. The matrix H is nonsingular if and only if its row rank is n, which is true if and only if H's rank is n by Theorem Two.III.3.11, which is true if and only if h's rank is n by Theorem 2.4, which is true if and only if h is an isomorphism by Theorem I.2.3. (This last holds because the domain of h is n-dimensional as it is the number of columns in H.) QED

2.10 Example Any map from $\mathbb{R}^2$ to $\mathcal{P}_1$ represented with respect to any pair of bases by

$$\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$$

is nonsingular because this matrix has rank two.

2.11 Example Any map $g\colon V \to W$ represented by

$$\begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}$$

is singular because this matrix is singular.

We've now seen that the relationship between maps and matrices goes both ways: for a particular pair of bases, any linear map is represented by a matrix and any matrix describes a linear map. That is, by fixing spaces and bases we get a correspondence between maps and matrices. In the rest of this chapter we will explore this correspondence. For instance, we've defined for linear maps the operations of addition and scalar multiplication and we shall see what the corresponding matrix operations are. We shall also see the matrix operation that represent the map operation of composition. And, we shall see how to find the matrix that represents a map's inverse.

For maps, nonsingular means one-to-one & onto.

I think the key to solving problems is determining the rank, which will allow us to determine one to one and onto?

Computing Range and Null Spaces

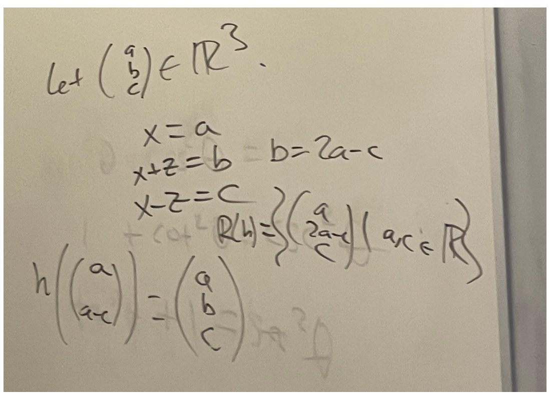
Example Consider this linear map $h\colon \mathbb{R}^3 \to \mathbb{R}^3$.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x \\ x+z \\ x-z \end{pmatrix}$$

A function is onto if every codomain member $\vec{w}$ is the image of at least one domain member $\vec{v}$. This function is not onto because there is a restriction: the first component of the output must be the average of the other two.

A function is one-to-one if every codomain member $\vec{w}$ is the image of at most one domain member $\vec{v}$. This function is not one-to-one because the output entries don't use y, so holding x and z constant and varying y gives many inputs that are all associated with the same output.

→ not onto cus its 2 dimensional.



For the calculations we use $\mathcal{E}_3 \subseteq \mathbb{R}^3$ for both the domain and codomain. Find the action of h on the domain's basis

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

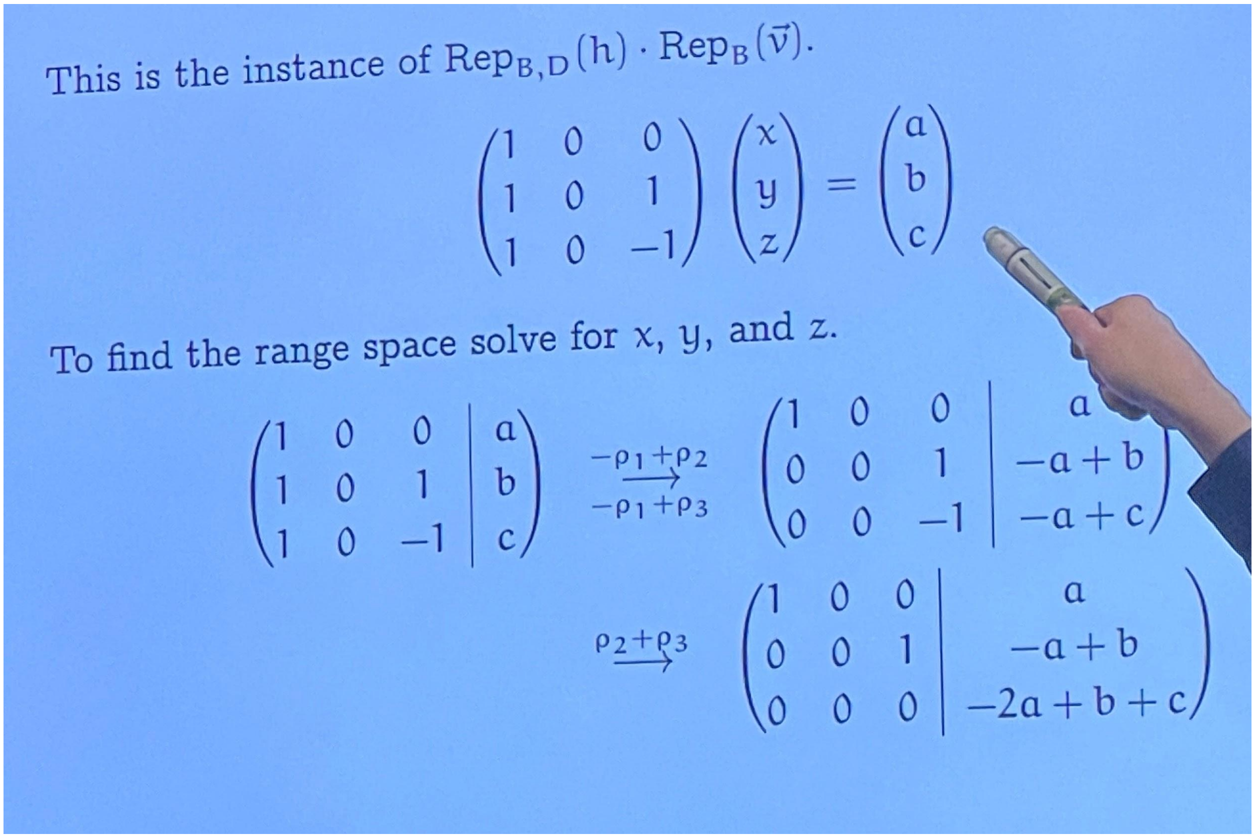
and represent those outputs with respect to the codomain's basis.

$$\mathrm{Rep}_{\mathcal{E}_3}\left(\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \mathrm{Rep}_{\mathcal{E}_3}\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \mathrm{Rep}_{\mathcal{E}_3}\left(\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

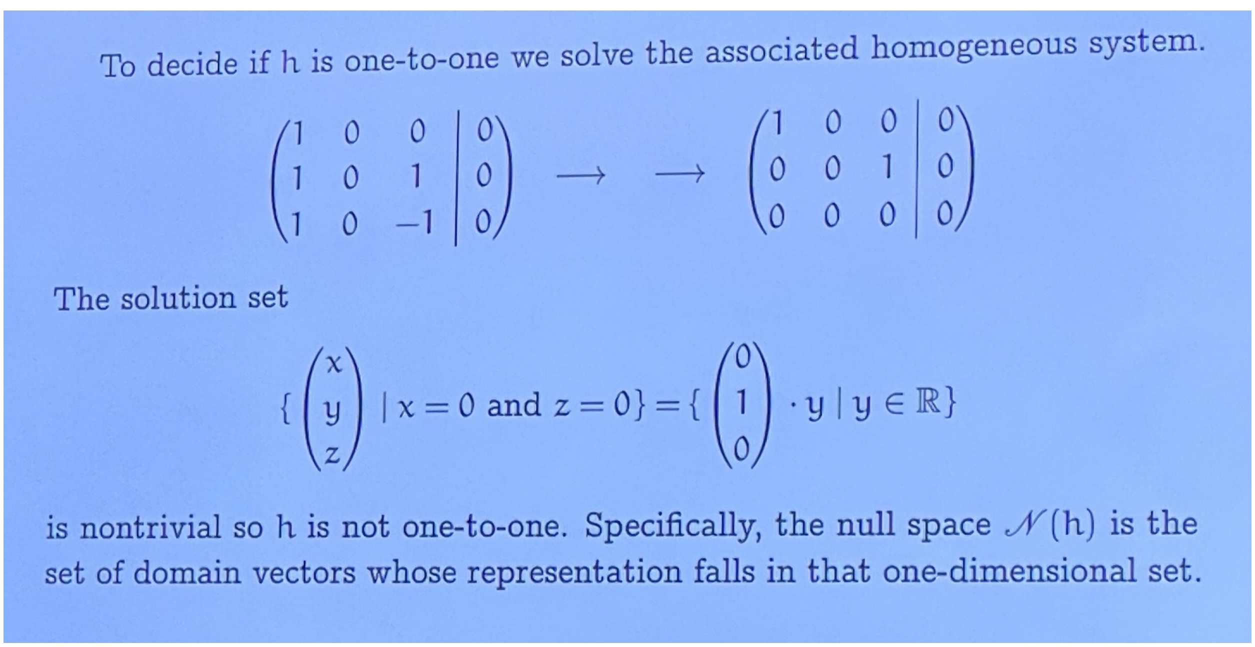
(The standard basis makes for easy calculations.)

$$H = \mathrm{Rep}_{\mathcal{E}_3,\mathcal{E}_3}(h) = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & -1 \end{pmatrix}$$

-> you can conclude that this matrix is singular cus rows are linearly dependent?



=> range space is 2 dimensional because 0 = -2a+b+c → one variable can be represented by the other two variables → not onto since range space is not all of co-domain



-> null space is not trivial → is of dimension 1 → homomorphism is not one-to-one.

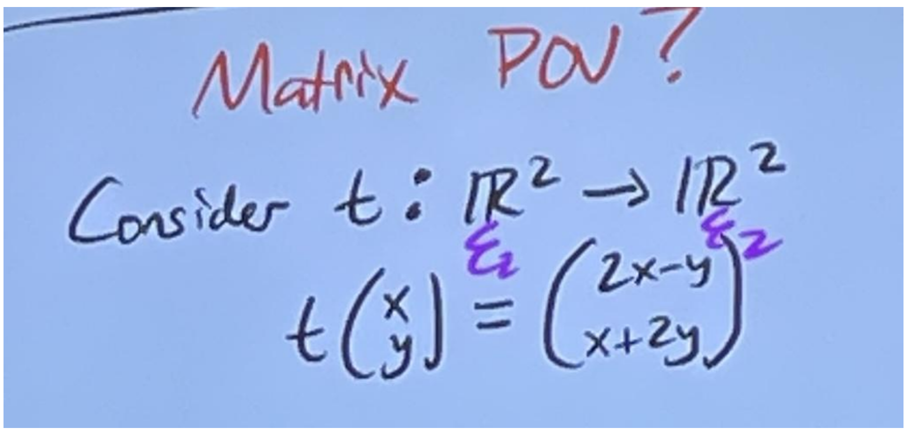
-

#columns = domain dimension ⇒ Domain = #COLUMNS OF REPRESENTATION MATRIX

#rows = codomain dimension ⇒ Codomain = #ROWS OF REPRESENTATION MATRIX

REVIEW

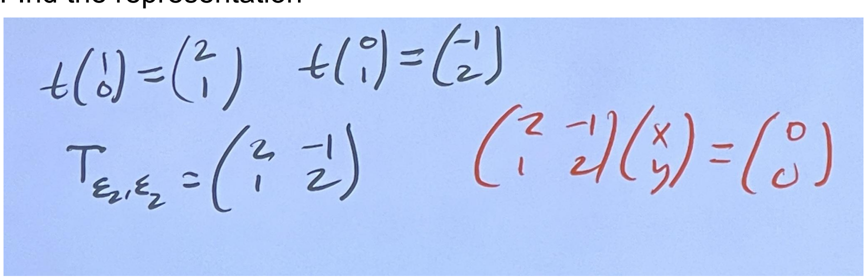
Homomorphism POV:

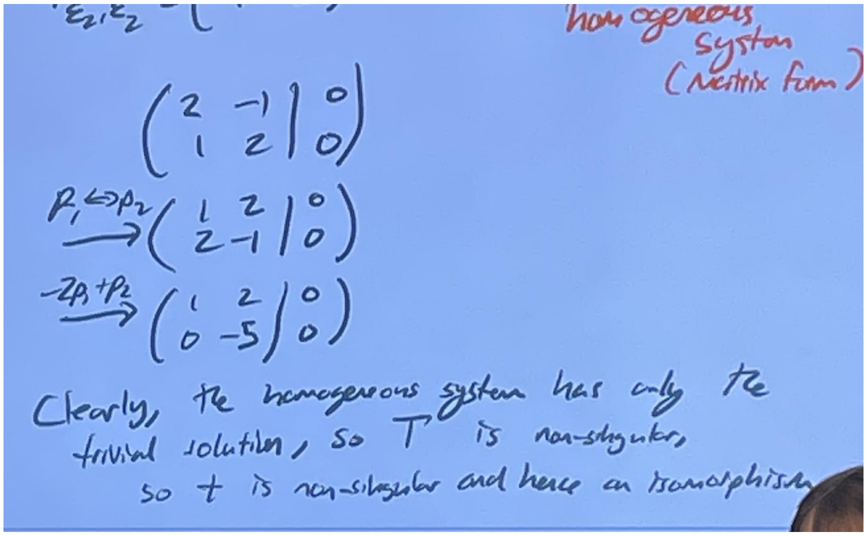


We can prove that the homomorphism is one-to-one and onto
Prove one-to-one: set t(x1,y1) = t(x2,y2) → show x1=x2, y1=y2
Prove onto: set t(x1,y1) = (a,b) → find x,y in terms of a given a,b

Matrix POV:

FInd the representation





-> Matrix T is non-singular, homomorphism t is non-singular, and hence it is an isomorphism

When you are solving a system, you are FINDING HTE NULLSPACE

The matrix reduction suggests the null space is only the zero vector → nullity = 0
→ t is **one-to-one**
→ dimension (domain) = dimension (R^2) = 2 → Rank t = 2-0 = 2 (full rank)
⇒ dimension (range t) = 2
Dimension (codomain) = dimension (R^2) = 2
→ since range t is a subspace of codomain t, the range of t = R^2 (range space IS THE CODOMAIN) → it is also **Onto**

-> also talk about row space, column space,....

=> THE EASIEST WAY TO SEE IF A HOMOMORPHISM IS ISOMORPHISM IS USING SINGULARITY THEOREM ABOVE!

=> THE POINT IS matrix POV is helpful in finding properties of both matrix and homomorphism!

Example Let h be the transformation of $\mathbb{R}^2$ represented with respect to the standard basis by this matrix.

$$H = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

This matrix has rank 2 since the second row is not a multiple of the first.

Ex: rank 2 → nullity = rank - dimension = 0 → null space is only zero vector → one-to-one

This matrix has rank 2 since the second row is not a multiple of the first.
This is the instance of equation $(*)$.

$$\mathrm{Rep}_{B,D}(h) \cdot \mathrm{Rep}_B(\vec{v}) = \mathrm{Rep}_D(\vec{w})$$

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

Do the calculation to solve this linear system for x and y.

$$\left(\begin{array}{cc|c} 1 & 2 & a \\ 3 & 4 & b \end{array}\right) \xrightarrow{-3\rho_1+\rho_2} \left(\begin{array}{cc|c} 1 & 2 & a \\ 0 & -2 & -3a+b \end{array}\right)$$

$$\xrightarrow{-(1/2)\rho_2} \left(\begin{array}{cc|c} 1 & 2 & a \\ 0 & 1 & (3/2)a-(1/2)b \end{array}\right) \xrightarrow{-2\rho_2+\rho_1} \left(\begin{array}{cc|c} 1 & 0 & -2a+b \\ 0 & 1 & (3/2)a-(1/2)b \end{array}\right)$$

-> onto & form of all domain values x & y are above.

WHAT CAN ROW SPACE AND COLUMN SPACE TELL YOU?

Questions:

2.12 For each matrix, state the dimension of the domain and codomain of any map that the matrix represents.

(a) $\begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 1 & -3 \\ 2 & 5 & 0 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & 3 \\ 1 & 4 \\ 1 & -1 \end{pmatrix}$ (d) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

(e) $\begin{pmatrix} 1 & -1 & 4 & 5 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

Cus #columns = domain dimension ⇒ Domain = #COLUMNS OF REPRESENTATION MATRIX

#rows = codomain dimension ⇒ Codomain = #ROWS OF REPRESENTATION MATRIX

Three.III.2.12 **(a)** domain: 2, codomain: 2
(b) domain: 3, codomain: 2
(c) domain: 2, codomain: 3
(d) domain: 3, codomain: 2
(e) domain: 4, codomain: 2

2.13 Consider a linear map $f\colon V \to W$ represented with respect to some bases B, D by the matrix. Decide if that map is nonsingular.

(a) $\begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 1 \\ -3 & -3 \end{pmatrix}$ (c) $\begin{pmatrix} 3 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 4 & 4 \end{pmatrix}$ (d) $\begin{pmatrix} 2 & 0 & -2 \\ 1 & 1 & 0 \\ 4 & 1 & -4 \end{pmatrix}$

Three.III.2.13 For each we just have to decide if the matrix is nonsingular, perhaps by doing Gauss's Method. (In truth, we can do each of these by eye.)
(a) This matrix is nonsingular, since the second row is not a multiple of the first, so the map is nonsingular.
(b) The matrix is singular so the map is singular.

(c) Nonsingular.
(d) Nonsingular.

✓ **2.14** Let h be the linear map defined by this matrix on the domain $\mathcal{P}_1$ and codomain $\mathbb{R}^2$ with respect to the given bases.

$$H = \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix} \qquad B = \langle 1 + x, x \rangle,\ D = \langle \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rangle$$

What is the image under h of the vector $\vec{v} = 2x - 1$?

Three.III.2.14 With respect to B the vector's representation is this.

$$\mathrm{Rep}_B(2x - 1) = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

Using the matrix-vector product we can compute $\mathrm{Rep}_D(h(\vec{v}))$

$$\mathrm{Rep}_D(h(2x-1)) = \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} -1 \\ 3 \end{pmatrix}_B = \begin{pmatrix} 1 \\ 2 \end{pmatrix}_D$$

From that representation we can compute $h(\vec{v})$.

$$h(2x - 1) = 1 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} + 2 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

✓ **2.15** Decide if each vector lies in the range of the map from $\mathbb{R}^3$ to $\mathbb{R}^2$ represented with respect to the standard bases by the matrix.

(a) $\begin{pmatrix} 1 & 1 & 3 \\ 0 & 1 & 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ **(b)** $\begin{pmatrix} 2 & 0 & 3 \\ 4 & 0 & 6 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}$?!?

Three.III.2.15 As described in the subsection, with respect to the standard bases, representations are transparent, and so, for instance, the first matrix describes this map.

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}_{\mathcal{E}_3} \mapsto \begin{pmatrix} 1 \\ 0 \end{pmatrix}_{\mathcal{E}_2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

So, for this first one, we are asking whether there are scalars such that

$$c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_3 \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

that is, whether the vector is in the column space of the matrix.

(a) Yes. We can get this conclusion by setting up the resulting linear system and applying Gauss's Method, as usual. Another way to get it is to note by inspection of the equation of columns that taking $c_3 = 3/4$, and $c_1 = -5/4$, and $c_2 = 0$ will do. Still a third way to get this conclusion is to note that the rank of the matrix is two, which equals the dimension of the codomain, and so the map is onto—the range is all of $\mathbb{R}^2$ and in particular includes the given vector.

(b) No; note that all of the columns in the matrix have a second component that is twice the first, while the vector does not. Alternatively, the column space of the matrix is

$$\{c_1 \begin{pmatrix} 2 \\ 4 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 3 \\ 6 \end{pmatrix} \mid c_1, c_2, c_3 \in \mathbb{R}\} = \{c \begin{pmatrix} 1 \\ 2 \end{pmatrix} \mid c \in \mathbb{R}\}$$

(which is the fact already noted, but we got it by calculation rather than inspiration), and the given vector is not in this set.

✓ **2.19** Decide whether $1+2x$ is in the range of the map from $\mathbb{R}^3$ to $\mathcal{P}_2$ represented with respect to $\mathcal{E}_3$ and $\langle 1, 1+x^2, x\rangle$ by this matrix.

$$\begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

Three.III.2.19 Denote the given basis of $\mathcal{P}_2$ by B. Application of the linear map is represented by matrix-vector multiplication. Thus the first vector in $\mathcal{E}_3$ maps to the element of $\mathcal{P}_2$ represented with respect to B by

$$\begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

and that element is $1+x$. Calculate the other two images of basis vectors in the same way.

$$\begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} = \operatorname{Rep}_B(4+x^2) \qquad \begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \operatorname{Rep}_B(x)$$

So the range of h is the span of three polynomials $1+x$, $4+x^2$, and x. We can thus decide if $1+2x$ is in the range of the map by looking for scalars c_1, c_2, and c_3 such that

$$c_1 \cdot (1+x) + c_2 \cdot (4+x^2) + c_3 \cdot (x) = 1+2x$$

and obviously $c_1 = 1$, $c_2 = 0$, and $c_3 = 1$ suffice. Thus $1+2x$ is in the range, since it is the image of this vector.

$$1 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 0 \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 1 \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Comment. A slicker argument is to note that the matrix is nonsingular, so it has rank 3, so the range has dimension 3, and since the codomain has dimension 3 the map is onto. Thus every polynomial is the image of some vector and in particular $1+2x$ is the image of a vector in the domain.

2.20 Find the map that this matrix represents with respect to B, B.

$$\begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} \qquad B = \langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \rangle$$

Three.III.2.20 Where $B = \langle \vec{\beta}_1, \vec{\beta}_2 \rangle$ we can find $\mathrm{Rep}_B(\vec{v})$ by eye, where $\vec{v}$ is the general vector, with entries x and y.

$$\begin{pmatrix} x \\ y \end{pmatrix} = a \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + b \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{gives} \quad b = y, a = x - y$$

Thus the representation of general vector with respect to B is this.

$$\mathrm{Rep}_B(\begin{pmatrix} x \\ y \end{pmatrix}) = \begin{pmatrix} x - y \\ y \end{pmatrix}$$

Compute the effect of the map with matrix-vector multiplication.

$$\begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}_B \begin{pmatrix} x - y \\ y \end{pmatrix}_B = \begin{pmatrix} 2(x-y) + y \\ -(x-y) \end{pmatrix}_B = \begin{pmatrix} 2x - y \\ -x + y \end{pmatrix}_B$$

Finish by converting back to the standard vector representation.

$$\begin{pmatrix} 2x - y \\ -x + y \end{pmatrix}_B = (2x - y) \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + (-x + y) \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} x \\ -x + y \end{pmatrix}$$

Three.IV Matrix Operations

Three.IV.1. Sum and Scalar Products:

→ matrix representations can be added abd multiplied like normal matrix.
→ ex: adding two matrix produces a matrix that represents the sum of two homomorphism

1.4 Theorem Let $h, g \colon V \to W$ be linear maps represented with respect to bases B, D by the matrices H and G and let r be a scalar. Then with respect to B, D the map $r \cdot h \colon V \to W$ is represented by rH and the map $h + g \colon V \to W$ is represented by $H + G$.

PROOF Generalize the examples. This is Exercise 10. QED

1.5 Remark These two operations on matrices are simple, but we did not define them in this way because they are simple. We defined them this way because they represent function addition and function scalar multiplication. That is, our program is to define matrix operations by referencing function operations. Simplicity is a bonus.

We will see this again in the next subsection, where we will define the operation of multiplying matrices. Since we've just defined matrix scalar multiplication and matrix sum to be entry-by-entry operations, a naive thought is to define matrix multiplication to be the entry-by-entry product. In theory we could do whatever we please but we will instead be practical and combine the entries in the way that represents the function operation of composition.

A special case of scalar multiplication is multiplication by zero. For any map $0 \cdot h$ is the zero homomorphism and for any matrix $0 \cdot H$ is the matrix with all entries zero.

1.6 Definition A *zero matrix* has all entries 0. We write $Z_{n\times m}$ or simply Z (another common notation is $0_{n\times m}$ or just 0).

1.7 Example The zero map from any three-dimensional space to any two-dimensional space is represented by the 2×3 zero matrix

$$Z = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

DIMENSION OF DOMAIN = COLUMN, CODOMAIN = ROW!!!

1.9 Give the matrix representing the zero map from $\mathbb{R}^4$ to $\mathbb{R}^2$, with respect to the standard bases.

$$Z_{2\times 4} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

✓ 1.11 Prove each, assuming that the operations are defined, where G, H, and J are matrices, where Z is the zero matrix, and where r and s are scalars.

(a) Matrix addition is commutative $G + H = H + G$.

(b) Matrix addition is associative $G + (H + J) = (G + H) + J$.

(c) The zero matrix is an additive identity $G + Z = G$.

(d) $0 \cdot G = Z$

(e) $(r + s)G = rG + sG$

(f) Matrices have an additive inverse $G + (-1) \cdot G = Z$.

(g) $r(G + H) = rG + rH$

(h) $(rs)G = r(sG)$

Three.IV.1.11 First, each of these properties is easy to check in an entry-by-entry way. For example, writing

$$G = \begin{pmatrix} g_{1,1} & \ldots & g_{1,n} \\ \vdots & & \vdots \\ g_{m,1} & \ldots & g_{m,n} \end{pmatrix} \qquad H = \begin{pmatrix} h_{1,1} & \ldots & h_{1,n} \\ \vdots & & \vdots \\ h_{m,1} & \ldots & h_{m,n} \end{pmatrix}$$

then, by definition we have

$$G + H = \begin{pmatrix} g_{1,1} + h_{1,1} & \ldots & g_{1,n} + h_{1,n} \\ \vdots & & \vdots \\ g_{m,1} + h_{m,1} & \ldots & g_{m,n} + h_{m,n} \end{pmatrix}$$

and

$$H + G = \begin{pmatrix} h_{1,1} + g_{1,1} & \ldots & h_{1,n} + g_{1,n} \\ \vdots & & \vdots \\ h_{m,1} + g_{m,1} & \ldots & h_{m,n} + g_{m,n} \end{pmatrix}$$

and the two are equal since their entries are equal $g_{i,j} + h_{i,j} = h_{i,j} + g_{i,j}$. That is, each of these is easy to check by using Definition 1.3 alone.

However, each property is also easy to understand in terms of the represented maps, by applying Theorem 1.4 as well as the definition.

(a) The two maps $g + h$ and $h + g$ are equal because $g(\vec{v}) + h(\vec{v}) = h(\vec{v}) + g(\vec{v})$, as addition is commutative in any vector space. Because the maps are the same, they must have the same representative.
(b) As with the prior answer, except that here we apply that vector space addition is associative.
(c) As before, except that here we note that $g(\vec{v}) + z(\vec{v}) = g(\vec{v}) + \vec{0} = g(\vec{v})$.
(d) Apply that $0 \cdot g(\vec{v}) = \vec{0} = z(\vec{v})$.
(e) Apply that $(r + s) \cdot g(\vec{v}) = r \cdot g(\vec{v}) + s \cdot g(\vec{v})$.
(f) Apply the prior two items with $r = 1$ and $s = -1$.
(g) Apply that $r \cdot (g(\vec{v}) + h(\vec{v})) = r \cdot g(\vec{v}) + r \cdot h(\vec{v})$.
(h) Apply that $(rs) \cdot g(\vec{v}) = r \cdot (s \cdot g(\vec{v}))$.

Trace of Square Matrix is a HOMOMORPHISM (cus preserves structure)

1.15 The *trace* of a square matrix is the sum of the entries on the main diagonal (the $1, 1$ entry plus the $2, 2$ entry, etc.; we will see the significance of the trace in Chapter Five). Show that $\text{trace}(H + G) = \text{trace}(H) + \text{trace}(G)$. Is there a similar result for scalar multiplication?

Three.IV.1.15 That the trace of a sum is the sum of the traces holds because both $\text{trace}(H + G)$ and $\text{trace}(H) + \text{trace}(G)$ are the sum of $h_{1,1} + g_{1,1}$ with $h_{2,2} + g_{2,2}$, etc. For scalar multiplication we have $\text{trace}(r \cdot H) = r \cdot \text{trace}(H)$; the proof is easy. Thus the trace map is a homomorphism from $\mathcal{M}_{n\times n}$ to $\mathbb{R}$.

Using i,j th entries to prove identities

1.16 Recall that the *transpose* of a matrix M is another matrix, whose i, j entry is the j, i entry of M. Verify these identities.

(a) $(G + H)^T = G^T + H^T$

(b) $(r \cdot H)^T = r \cdot H^T$

Three.IV.1.16 **(a)** The i, j entry of $(G + H)^T$ is $g_{j,i} + h_{j,i}$. That is also the i, j entry of $G^T + H^T$.

(b) The i, j entry of $(r \cdot H)^T$ is $rh_{j,i}$, which is also the i, j entry of $r \cdot H^T$.

Using i,j th entries to prove identities (part 2)

✓ 1.17 A square matrix is *symmetric* if each i, j entry equals the j, i entry, that is, if the matrix equals its transpose.

(a) Prove that for any square H, the matrix $H + H^T$ is symmetric. Does every symmetric matrix have this form?

(b) Prove that the set of $n \times n$ symmetric matrices is a subspace of $\mathcal{M}_{n \times n}$.

Three.IV.1.17 (a) For $H + H^T$, the i, j entry is $h_{i,j} + h_{j,i}$ and the j, i entry of is $h_{j,i} + h_{i,j}$. The two are equal and thus $H + H^T$ is symmetric.

Every symmetric matrix does have that form, since we can write $H = (1/2) \cdot (H + H^T)$.

(b) The set of symmetric matrices is nonempty as it contains the zero matrix. Clearly a scalar multiple of a symmetric matrix is symmetric. A sum $H + G$ of two symmetric matrices is symmetric because $h_{i,j} + g_{i,j} = h_{j,i} + g_{j,i}$ (since $h_{i,j} = h_{j,i}$ and $g_{i,j} = g_{j,i}$). Thus the subset is nonempty and closed under the inherited operations, and so it is a subspace.

Scalar multiplication cannot change rank except multiplication with 0 vector

Addition can change rank (reduce rank, or increase rank)

✓ 1.18 (a) How does matrix rank interact with scalar multiplication — can a scalar product of a rank n matrix have rank less than n? Greater?

(b) How does matrix rank interact with matrix addition — can a sum of rank n matrices have rank less than n? Greater?

Three.IV.1.18 (a) Scalar multiplication leaves the rank of a matrix unchanged except that multiplication by zero leaves the matrix with rank zero. (This follows from the first theorem of the book, that multiplying a row by a nonzero scalar doesn't change the solution set of the associated linear system.)

(b) A sum of rank n matrices can have rank less than n. For instance, for any matrix H, the sum $H + (-1) \cdot H$ has rank zero.

A sum of rank n matrices can have rank greater than n. Here are rank one matrices that sum to a rank two matrix.

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Three.IV.2. Matrix Multiplication

2.1 Lemma The composition of linear maps is linear.

Proof (*Note:* this argument has already appeared, as part of the proof of Theorem I.2.2.) Let $h\colon V \to W$ and $g\colon W \to U$ be linear. The calculation

$$\begin{aligned} g \circ h\,(c_1 \cdot \vec{v}_1 + c_2 \cdot \vec{v}_2) &= g\bigl(\,h(c_1 \cdot \vec{v}_1 + c_2 \cdot \vec{v}_2)\,\bigr) = g\bigl(\,c_1 \cdot h(\vec{v}_1) + c_2 \cdot h(\vec{v}_2)\,\bigr) \\ &= c_1 \cdot g\bigl(h(\vec{v}_1)\bigr) + c_2 \cdot g\bigl(h(\vec{v}_2)\bigr) = c_1 \cdot (g \circ h)(\vec{v}_1) + c_2 \cdot (g \circ h)(\vec{v}_2) \end{aligned}$$

shows that $g \circ h\colon V \to U$ preserves linear combinations, and so is linear. QED

As we did with the operation of matrix addition and scalar multiplication, we will see how the representation of the composite relates to the representations of the compositors by first considering an example.

Applying the composition on a combined vector is the same as applying the composition on individual vectors

2.2 Example Let $h\colon \mathbb{R}^4 \to \mathbb{R}^2$ and $g\colon \mathbb{R}^2 \to \mathbb{R}^3$, fix bases $B \subset \mathbb{R}^4$, $C \subset \mathbb{R}^2$, $D \subset \mathbb{R}^3$, and let these be the representations.

$$H = \mathrm{Rep}_{B,C}(h) = \begin{pmatrix} 4 & 6 & 8 & 2 \\ 5 & 7 & 9 & 3 \end{pmatrix}_{B,C} \qquad G = \mathrm{Rep}_{C,D}(g) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}_{C,D}$$

To represent the composition $g \circ h\colon \mathbb{R}^4 \to \mathbb{R}^3$ we start with a $\vec{v}$, represent h of $\vec{v}$, and then represent g of that. The representation of $h(\vec{v})$ is the product of h's matrix and $\vec{v}$'s vector.

$$\mathrm{Rep}_C(\,h(\vec{v})\,) = \begin{pmatrix} 4 & 6 & 8 & 2 \\ 5 & 7 & 9 & 3 \end{pmatrix}_{B,C} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix}_B = \begin{pmatrix} 4v_1 + 6v_2 + 8v_3 + 2v_4 \\ 5v_1 + 7v_2 + 9v_3 + 3v_4 \end{pmatrix}_C$$

The representation of $g(\,h(\vec{v})\,)$ is the product of g's matrix and $h(\vec{v})$'s vector.

$$\begin{aligned} \mathrm{Rep}_D(\,g(h(\vec{v}))\,) &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}_{C,D} \begin{pmatrix} 4v_1 + 6v_2 + 8v_3 + 2v_4 \\ 5v_1 + 7v_2 + 9v_3 + 3v_4 \end{pmatrix}_C \\ &= \begin{pmatrix} 1\cdot(4v_1 + 6v_2 + 8v_3 + 2v_4) + 1\cdot(5v_1 + 7v_2 + 9v_3 + 3v_4) \\ 0\cdot(4v_1 + 6v_2 + 8v_3 + 2v_4) + 1\cdot(5v_1 + 7v_2 + 9v_3 + 3v_4) \\ 1\cdot(4v_1 + 6v_2 + 8v_3 + 2v_4) + 0\cdot(5v_1 + 7v_2 + 9v_3 + 3v_4) \end{pmatrix}_D \end{aligned}$$

Distributing and regrouping on the v's gives

$$= \begin{pmatrix} (1\cdot4 + 1\cdot5)v_1 + (1\cdot6 + 1\cdot7)v_2 + (1\cdot8 + 1\cdot9)v_3 + (1\cdot2 + 1\cdot3)v_4 \\ (0\cdot4 + 1\cdot5)v_1 + (0\cdot6 + 1\cdot7)v_2 + (0\cdot8 + 1\cdot9)v_3 + (0\cdot2 + 1\cdot3)v_4 \\ (1\cdot4 + 0\cdot5)v_1 + (1\cdot6 + 0\cdot7)v_2 + (1\cdot8 + 0\cdot9)v_3 + (1\cdot2 + 0\cdot3)v_4 \end{pmatrix}_D$$

which is this matrix-vector product.

$$= \begin{pmatrix} 1\cdot4+1\cdot5 & 1\cdot6+1\cdot7 & 1\cdot8+1\cdot9 & 1\cdot2+1\cdot3 \\ 0\cdot4+1\cdot5 & 0\cdot6+1\cdot7 & 0\cdot8+1\cdot9 & 0\cdot2+1\cdot3 \\ 1\cdot4+0\cdot5 & 1\cdot6+0\cdot7 & 1\cdot8+0\cdot9 & 1\cdot2+0\cdot3 \end{pmatrix}_{B,D} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix}_B$$

The matrix representing $g \circ h$ has the rows of G combined with the columns of H.

2.3 Definition The *matrix-multiplicative product* of the $m \times r$ matrix G and the $r \times n$ matrix H is the $m \times n$ matrix P, where

$$p_{i,j} = g_{i,1}h_{1,j} + g_{i,2}h_{2,j} + \cdots + g_{i,r}h_{r,j}$$

so that the i, j-th entry of the product is the dot product of the i-th row of the first matrix with the j-th column of the second.

$$GH = \begin{pmatrix} & \vdots & & \\ g_{i,1} & g_{i,2} & \cdots & g_{i,r} \\ & \vdots & & \end{pmatrix} \begin{pmatrix} & h_{1,j} & \\ \cdots & h_{2,j} & \cdots \\ & \vdots & \\ & h_{r,j} & \end{pmatrix} = \begin{pmatrix} & \vdots & \\ \cdots & p_{i,j} & \cdots \\ & \vdots & \end{pmatrix}$$

2.4 Example

$$\begin{pmatrix} 2 & 0 \\ 4 & 6 \\ 8 & 2 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 5 & 7 \end{pmatrix} = \begin{pmatrix} 2 \cdot 1 + 0 \cdot 5 & 2 \cdot 3 + 0 \cdot 7 \\ 4 \cdot 1 + 6 \cdot 5 & 4 \cdot 3 + 6 \cdot 7 \\ 8 \cdot 1 + 2 \cdot 5 & 8 \cdot 3 + 2 \cdot 7 \end{pmatrix} = \begin{pmatrix} 2 & 6 \\ 34 & 54 \\ 18 & 38 \end{pmatrix}$$

2.5 Example Some products are not defined, such as the product of a 2×3 matrix with a 2×2, because the number of columns in the first matrix must equal the number of rows in the second. But the product of two $n\times n$ matrices is always defined. Here are two 2×2's.

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 2 & -2 \end{pmatrix} = \begin{pmatrix} 1 \cdot (-1) + 2 \cdot 2 & 1 \cdot 0 + 2 \cdot (-2) \\ 3 \cdot (-1) + 4 \cdot 2 & 3 \cdot 0 + 4 \cdot (-2) \end{pmatrix} = \begin{pmatrix} 3 & -4 \\ 5 & -8 \end{pmatrix}$$

2.6 Example The matrices from Example 2.2 combine in this way.

$$\begin{aligned} &\begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 4 & 6 & 8 & 2 \\ 5 & 7 & 9 & 3 \end{pmatrix} \\ &\qquad = \begin{pmatrix} 1 \cdot 4 + 1 \cdot 5 & 1 \cdot 6 + 1 \cdot 7 & 1 \cdot 8 + 1 \cdot 9 & 1 \cdot 2 + 1 \cdot 3 \\ 0 \cdot 4 + 1 \cdot 5 & 0 \cdot 6 + 1 \cdot 7 & 0 \cdot 8 + 1 \cdot 9 & 0 \cdot 2 + 1 \cdot 3 \\ 1 \cdot 4 + 0 \cdot 5 & 1 \cdot 6 + 0 \cdot 7 & 1 \cdot 8 + 0 \cdot 9 & 1 \cdot 2 + 0 \cdot 3 \end{pmatrix} \\ &\qquad = \begin{pmatrix} 9 & 13 & 17 & 5 \\ 5 & 7 & 9 & 3 \\ 4 & 6 & 8 & 2 \end{pmatrix} \end{aligned}$$

2.7 Theorem A composition of linear maps is represented by the matrix product of the representatives.

This *arrow diagram* pictures the relationship between maps and matrices ('wrt' abbreviates 'with respect to').

$$\begin{tikzcd} & W_{\textit{wrt}\ C} \arrow[dr, "g", "G"'] & \\ V_{\textit{wrt}\ B} \arrow[ur, "h", "H"'] \arrow[rr, "g\circ h", "GH"'] & & X_{\textit{wrt}\ D} \end{tikzcd}$$

Above the arrows, the maps show that the two ways of going from V to X, straight over via the composition or else in two steps by way of W, have the same effect

$$\vec{v} \xmapsto{g\circ h} g(h(\vec{v})) \qquad \vec{v} \xmapsto{h} h(\vec{v}) \xmapsto{g} g(h(\vec{v}))$$

(this is just the definition of composition). Below the arrows, the matrices indicate that multiplying GH into the column vector $\operatorname{Rep}_B(\vec{v})$ has the same effect as multiplying the column vector first by H and then multiplying the result by G.

$$\operatorname{Rep}_{B,D}(g\circ h) = GH \qquad \operatorname{Rep}_{C,D}(g)\operatorname{Rep}_{B,C}(h) = GH$$

As mentioned in Example 2.5, because the number of columns on the left does not equal the number of rows on the right, the product as here of a 2×3 matrix with a 2×2 matrix is not defined.

$$\begin{pmatrix} -1 & 2 & 0 \\ 0 & 10 & 1.1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$$

The definition requires that the sizes match because we want that the underlying function composition is possible.

$$\text{dimension } n \text{ space} \xrightarrow{h} \text{dimension } r \text{ space} \xrightarrow{g} \text{dimension } m \text{ space} \qquad (*)$$

Thus, matrix product combines the $m\times r$ matrix G with the $r\times n$ matrix F to yield the $m\times n$ result GF. Briefly: $m\times r$ times $r\times n$ equals $m\times n$.

General process:

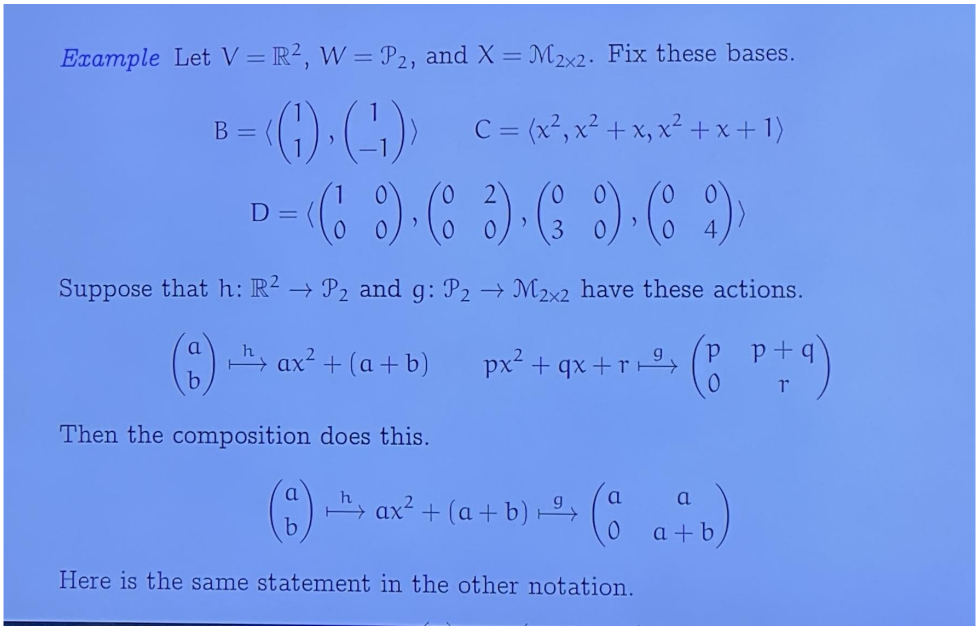
$$\mathrm{Rep}_C(h(\vec{v})) = \mathrm{Rep}_{B,C}(h) \cdot \mathrm{Rep}_B(\vec{v})$$

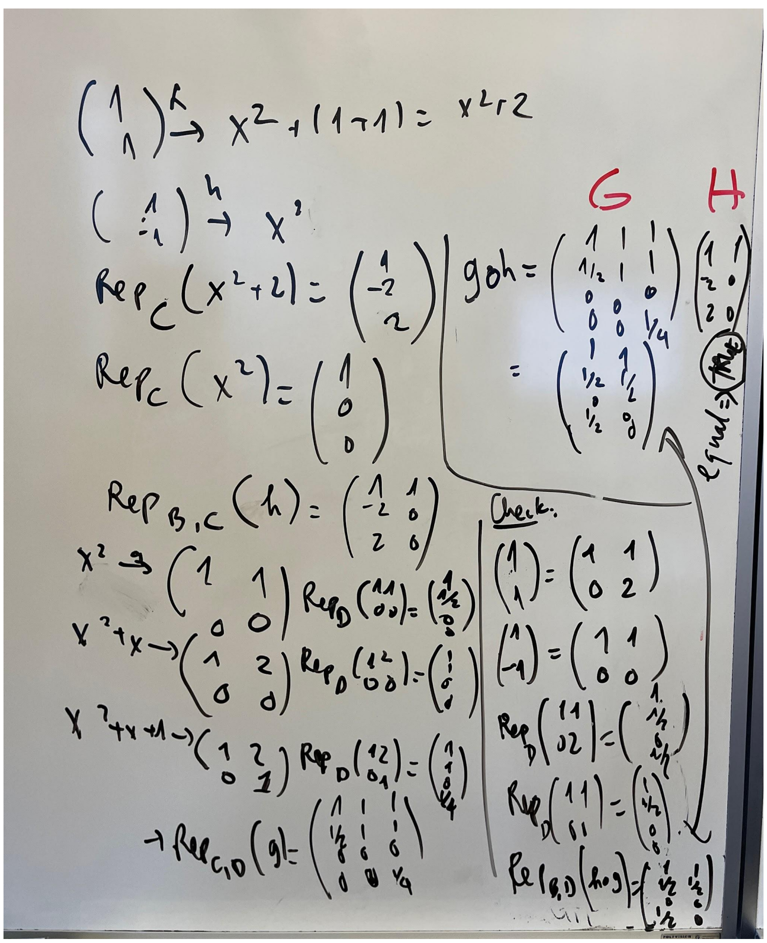
$$\mathrm{Rep}_{C,D}\, g \; \underbrace{\mathrm{Rep}_{B,C}\, h \; \mathrm{Rep}_B(\vec{v})}$$

$$G H \vec{v}$$

First RepB,C(h(v))*RepB(v) = RepC(h(v))
RepC(h(v)) is then multiplied with RepC,D(g)

THIS PROBLEM BASICALLY SUMMARIZES ALL THAT THEORY

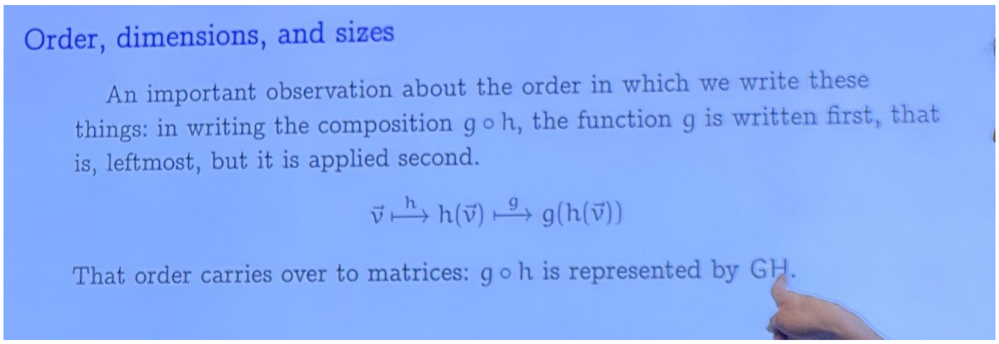




Also for matrix multiplication don't put the dot for dot product, just put them next to each other.

Composing f(g(v)) vs g(f(v)) produces DIFFERENT RESULTS - NOT THE SAME RESULTING MAP!

Order, Dimensions, Sizes



Commutative doesn't work over Matrix Multiplication!

2.9 Example Matrix multiplication is not commutative.

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = \begin{pmatrix} 19 & 22 \\ 43 & 50 \end{pmatrix} \qquad \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 23 & 34 \\ 31 & 46 \end{pmatrix}$$

2.10 Example Commutativity can fail more dramatically:

$$\begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 \\ 3 & 4 & 0 \end{pmatrix} = \begin{pmatrix} 23 & 34 & 0 \\ 31 & 46 & 0 \end{pmatrix}$$

while

$$\begin{pmatrix} 1 & 2 & 0 \\ 3 & 4 & 0 \end{pmatrix} \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}$$

isn't even defined.

2.11 Remark The fact that matrix multiplication is not commutative can seem odd at first, perhaps because most mathematical operations in prior courses are commutative. But matrix multiplication represents function composition and function composition is not commutative: if $f(x) = 2x$ and $g(x) = x + 1$ then $g \circ f(x) = 2x + 1$ while $f \circ g(x) = 2(x + 1) = 2x + 2$.

Except for the lack of commutativity, matrix multiplication is algebraically well-behaved. The next result gives some nice properties and more are in Exercise 25 and Exercise 26.

Other properties work!: Associative works & Distributive works!

2.12 Theorem If F, G, and H are matrices, and the matrix products are defined, then the product is associative $(FG)H = F(GH)$ and distributes over matrix addition $F(G+H) = FG+FH$ and $(G+H)F = GF+HF$.

PROOF Associativity holds because matrix multiplication represents function composition, which is associative: the maps $(f \circ g) \circ h$ and $f \circ (g \circ h)$ are equal as both send $\vec{v}$ to $f(g(h(\vec{v})))$.

Distributivity is similar. For instance, the first one goes $f \circ (g+h)(\vec{v}) = f\big((g+h)(\vec{v})\big) = f\big(g(\vec{v}) + h(\vec{v})\big) = f(g(\vec{v})) + f(h(\vec{v})) = f \circ g(\vec{v}) + f \circ h(\vec{v})$ (the third equality uses the linearity of f). Right-distributivity goes the same way. QED

2.13 Remark We could instead prove that result by slogging through indices. For

example, for associativity the i, j entry of $(FG)H$ is

$$\begin{aligned}
&(f_{i,1}g_{1,1} + f_{i,2}g_{2,1} + \cdots + f_{i,r}g_{r,1})h_{1,j} \\
&\quad + (f_{i,1}g_{1,2} + f_{i,2}g_{2,2} + \cdots + f_{i,r}g_{r,2})h_{2,j} \\
&\quad \vdots \\
&\quad + (f_{i,1}g_{1,s} + f_{i,2}g_{2,s} + \cdots + f_{i,r}g_{r,s})h_{s,j}
\end{aligned}$$

where F, G, and H are $m \times r$, $r \times s$, and $s \times n$ matrices. Distribute

$$\begin{aligned}
&f_{i,1}g_{1,1}h_{1,j} + f_{i,2}g_{2,1}h_{1,j} + \cdots + f_{i,r}g_{r,1}h_{1,j} \\
&\quad + f_{i,1}g_{1,2}h_{2,j} + f_{i,2}g_{2,2}h_{2,j} + \cdots + f_{i,r}g_{r,2}h_{2,j} \\
&\quad \vdots \\
&\quad + f_{i,1}g_{1,s}h_{s,j} + f_{i,2}g_{2,s}h_{s,j} + \cdots + f_{i,r}g_{r,s}h_{s,j}
\end{aligned}$$

and regroup around the f's

$$\begin{aligned}
&f_{i,1}(g_{1,1}h_{1,j} + g_{1,2}h_{2,j} + \cdots + g_{1,s}h_{s,j}) \\
&\quad + f_{i,2}(g_{2,1}h_{1,j} + g_{2,2}h_{2,j} + \cdots + g_{2,s}h_{s,j}) \\
&\quad \vdots \\
&\quad + f_{i,r}(g_{r,1}h_{1,j} + g_{r,2}h_{2,j} + \cdots + g_{r,s}h_{s,j})
\end{aligned}$$

to get the i, j entry of $F(GH)$.

Contrast the two proofs. The index-heavy argument is hard to understand in that while the calculations are easy to check, the arithmetic seems unconnected to any idea. The argument in the proof is shorter and also says why this property "really" holds. This illustrates the comments made at the start of the chapter on vector spaces — at least sometimes an argument from higher-level constructs is clearer.

We have now seen how to represent the composition of linear maps. The next subsection will continue to explore this operation.

→ the key is you can split and distributes because homomorphisms PRESERVE STRUCTURE → Preserving structure is the key to how matrix distributes and associates with EACH OTHER!

Problems:

Crucial Problem

✓ 2.19 Consider the two linear functions $h\colon \mathbb{R}^3 \to \mathcal{P}_2$ and $g\colon \mathcal{P}_2 \to \mathcal{M}_{2\times 2}$ given as here.

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \mapsto (a+b)x^2 + (2a+2b)x + c \qquad px^2 + qx + r \mapsto \begin{pmatrix} p & p-2q \\ q & 0 \end{pmatrix}$$

Use these bases for the spaces.

$$B = \langle \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \rangle \qquad C = \langle 1+x, 1-x, x^2 \rangle$$

$$D = \langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 3 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix} \rangle$$

(a) Give the formula for the composition map $g \circ h\colon \mathbb{R}^3 \to \mathcal{M}_{2\times 2}$ derived directly from the above definition.

(b) Represent h and g with respect to the appropriate bases.

(c) Represent the map $g \circ h$ computed in the first part with respect to the appropriate bases.

(d) Check that the product of the two matrices from the second part is the matrix from the third part.

Three.IV.2.19 **(a)** Following the definitions gives this.

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \mapsto (a+b)x^2 + (2a+2b)x + c$$

$$\mapsto \begin{pmatrix} a+b & (a+b)-2(2a+2b) \\ 2a+2b & 0 \end{pmatrix} = \begin{pmatrix} a+b & -3a-3b \\ 2a+2b & 0 \end{pmatrix}$$

(b) Because

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \mapsto 2x^2+4x+1 \quad \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \mapsto x^2+2x+1 \quad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \mapsto 0x^2+0x+1$$

we get this representation for h.

$$\mathrm{Rep}_{B,C}(h) = \begin{pmatrix} 5/2 & 3/2 & 1/2 \\ -3/2 & -1/2 & 1/2 \\ 2 & 1 & 0 \end{pmatrix}$$

Similarly, because

$$1+x \mapsto \begin{pmatrix} 0 & -2 \\ 1 & 0 \end{pmatrix} \quad 1-x \mapsto \begin{pmatrix} 0 & 2 \\ -1 & 0 \end{pmatrix} \quad x^2 \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

this is the representation of g.

$$\mathrm{Rep}_{C,D}(g) = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 1 & 1/2 \\ 1/3 & -1/3 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

(c) The action of $g \circ h$ on the domain basis is this.

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} 2 & -6 \\ 4 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & -3 \\ 2 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

We have this.

$$\mathrm{Rep}_{B,D}(g \circ h) = \begin{pmatrix} 2 & 1 & 0 \\ -3 & -3/2 & 0 \\ 4/3 & 2/3 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

(d) The matrix multiplication is routine, just take care with the order.

$$\begin{pmatrix} 0 & 0 & 1 \\ -1 & 1 & 1/2 \\ 1/3 & -1/3 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 5/2 & 3/2 & 1/2 \\ -3/2 & -1/2 & 1/2 \\ 2 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 0 \\ -3 & -3/2 & 0 \\ 4/3 & 2/3 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Matrix Multiplication yields 1x1 matrix NOT LIKE dot product which yields a scalar value

2.20 As Definition 2.3 points out, the matrix product operation generalizes the dot product. Is the dot product of a $1\times n$ row vector and a $n\times 1$ column vector the same as their matrix-multiplicative product?

Three.IV.2.20 Technically, no. The dot product operation yields a scalar while the matrix product yields a 1×1 matrix. However, we usually will ignore the distinction.

Commutes with is NOT equivalence relation among square matrices

2.31 Is 'commutes with' an equivalence relation among $n\times n$ matrices?

2.31: is "Commutes with" an equivalence relation among $n\times n$ matrices?

NO (not transitive)

reflexive: for any G & H

If $GH = HG$ (Commutative)

$\rightarrow HH = HH$ when $G = H$

$\rightarrow$ reflexive

symmetric:

If $GH = HG$ (commutative)

$\rightarrow GH = HG$ (proven)

commutative is same as symmetric

transitive: for H, G, I . $HG = GH$ & $HI = IH$ $\underbrace{}_{GI = IG}$

$$\begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}1&2\\2&1\end{bmatrix}=\begin{bmatrix}1&2\\2&1\end{bmatrix} \qquad \begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}1&2\\2&2\end{bmatrix}=\begin{bmatrix}1&2\\2&2\end{bmatrix}$$

$$\begin{bmatrix}1&2\\2&1\end{bmatrix}\begin{bmatrix}1&0\\0&1\end{bmatrix}=\begin{bmatrix}1&2\\2&1\end{bmatrix} \qquad \begin{bmatrix}1&2\\2&2\end{bmatrix}\begin{bmatrix}1&0\\0&1\end{bmatrix}=\begin{bmatrix}1&2\\2&2\end{bmatrix}$$

But

$$\begin{bmatrix}1&2\\2&1\end{bmatrix}\begin{bmatrix}1&2\\2&2\end{bmatrix}=\begin{bmatrix}5&6\\4&6\end{bmatrix} \neq \quad \rightarrow \text{Not transitive}$$

$$\begin{bmatrix}1&2\\2&2\end{bmatrix}\begin{bmatrix}1&2\\2&1\end{bmatrix}=\begin{bmatrix}5&4\\6&6\end{bmatrix}$$

Product of derivative map with itself is the SECOND derivative map!

✓ 2.21 Represent the derivative map on $\mathcal{P}_n$ with respect to B, B where B is the natural basis $\langle 1, x, \ldots, x^n \rangle$. Show that the product of this matrix with itself is defined; what map does it represent?

Three.IV.2.21 The action of d/dx on B is $1 \mapsto 0$, $x \mapsto 1$, $x^2 \mapsto 2x$, ... and so this is its $(n+1)\times(n+1)$ matrix representation.

$$\mathrm{Rep}_{B,B}(\frac{d}{dx}) = \begin{pmatrix} 0 & 1 & 0 & & 0 \\ 0 & 0 & 2 & & 0 \\ & & & \ddots & \\ 0 & 0 & 0 & & n \\ 0 & 0 & 0 & & 0 \end{pmatrix}$$

The product of this matrix with itself is defined because the matrix is square.

$$\begin{pmatrix} 0 & 1 & 0 & & 0 \\ 0 & 0 & 2 & & 0 \\ & & & \ddots & \\ 0 & 0 & 0 & & n \\ 0 & 0 & 0 & & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 & 2 & 0 & & 0 \\ 0 & 0 & 0 & 6 & & 0 \\ & & & & \ddots & \\ 0 & 0 & 0 & & & n(n-1) \\ 0 & 0 & 0 & & & 0 \\ 0 & 0 & 0 & & & 0 \end{pmatrix}$$

The map so represented is the composition

$$p \xmapsto{\frac{d}{dx}} \frac{d\,p}{dx} \xmapsto{\frac{d}{dx}} \frac{d^2\,p}{dx^2}$$

which is the second derivative operation.

! IMPORTANT 2.22

2.22 [Cleary] Match each type of matrix with all these descriptions that could fit, say 'None' if it applies: (i) can be multiplied by its transpose to make a 1×1 matrix, (ii) can represent a linear map from $\mathbb{R}^3$ to $\mathbb{R}^2$ that is not onto, (iii) can represent an isomorphism from $\mathbb{R}^3$ to $\mathcal{P}^2$.

- **(a)** a 2×3 matrix whose rank is 1
- **(b)** a 3×3 matrix that is nonsingular
- **(c)** a 2×2 matrix that is singular
- **(d)** an $n\times1$ column vector

Three.IV.2.22 **(a)** ii
(b) iii
(c) None

(d) None, or (i) if we include multiplication from the left.

2.25 **(a)** Prove that $H^pH^q = H^{p+q}$ and $(H^p)^q = H^{pq}$ for positive integers p, q.
(b) Prove that $(rH)^p = r^p \cdot H^p$ for any positive integer p and scalar $r \in \mathbb{R}$.

Three.IV.2.25 Each follows easily from the associated map fact. For instance, p applications of the transformation h, following q applications, is simply $p+q$ applications.

✓ 2.26 **(a)** How does matrix multiplication interact with scalar multiplication: is $r(GH) = (rG)H$? Is $G(rH) = r(GH)$?
(b) How does matrix multiplication interact with linear combinations: is $F(rG + sH) = r(FG) + s(FH)$? Is $(rF + sG)H = rFH + sGH$?

Three.IV.2.26 Although we can do these by going through the indices, they are best understood in terms of the represented maps. That is, fix spaces and bases so that the matrices represent linear maps f, g, h.

(a) Yes; we have both $r \cdot (g \circ h)\,(\vec{v}) = r \cdot g(\,h(\vec{v})\,) = (r \cdot g) \circ h\,(\vec{v})$ and $g \circ (r \cdot h)\,(\vec{v}) = g(\,r \cdot h(\vec{v})\,) = r \cdot g(h(\vec{v})) = r \cdot (g \circ h)\,(\vec{v})$ (the second equality holds because of the linearity of g).

(b) Both answers are yes. First, $f \circ (rg + sh)$ and $r \cdot (f \circ g) + s \cdot (f \circ h)$ both send $\vec{v}$ to $r \cdot f(g(\vec{v})) + s \cdot f(h(\vec{v}))$; the calculation is as in the prior item (using the linearity of f for the first one). For the other, $(rf + sg) \circ h$ and $r \cdot (f \circ h) + s \cdot (g \circ h)$ both send $\vec{v}$ to $r \cdot f(h(\vec{v})) + s \cdot g(h(\vec{v}))$.

use i, j entry method

2.27 We can ask how the matrix product operation interacts with the transpose operation.

(a) Show that $(GH)^{T} = H^{T}G^{T}$. → use to prove

(b) A square matrix is *symmetric* if each i, j entry equals the j, i entry, that is, if the matrix equals its own transpose. Show that the matrices HH^{T} and $H^{T}H$ are symmetric.

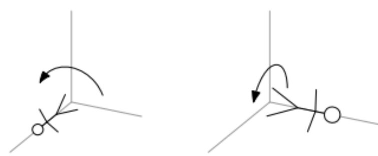
Three.IV.2.27 We have not seen a map interpretation of the transpose operation, so we will verify these by considering the entries.

(a) The i, j entry of GH^{T} is the j, i entry of GH, which is the dot product of the j-th row of G and the i-th column of H. The i, j entry of $H^{T}G^{T}$ is the dot product of the i-th row of H^{T} and the j-th column of G^{T}, which is the dot product of the i-th column of H and the j-th row of G. Dot product is commutative and so these two are equal.

(b) By the prior item each equals its transpose, e.g., $(HH^{T})^{T} = H^{T^{T}}H^{T} = HH^{T}$.

✓ **2.28** Rotation of vectors in $\mathbb{R}^3$ about an axis is a linear map. Show that linear maps do not commute by showing geometrically that rotations do not commute.

Three.IV.2.28 Consider $r_x, r_y : \mathbb{R}^3 \to \mathbb{R}^3$ rotating all vectors $\pi/2$ radians counterclockwise about the x and y axes (counterclockwise in the sense that a person whose head is at $\vec{e}_1$ or $\vec{e}_2$ and whose feet are at the origin sees, when looking toward the origin, the rotation as counterclockwise).



Rotating r_x first and then r_y is different than rotating r_y first and then r_x. In particular, $r_x(\vec{e}_3) = -\vec{e}_2$ so $r_y \circ r_x(\vec{e}_3) = -\vec{e}_2$, while $r_y(\vec{e}_3) = \vec{e}_1$ so $r_x \circ r_y(\vec{e}_3) = \vec{e}_1$, and hence the maps do not commute.

! Important?

2.30 How does matrix rank interact with matrix multiplication?

(a) Can the product of rank n matrices have rank less than n? Greater?

(b) Show that the rank of the product of two matrices is less than or equal to the minimum of the rank of each factor.

Three.IV.2.30 **(a)** The product of rank n matrices can have rank less than or equal to n but not greater than n.

To see that the rank can fall, consider the maps $\pi_x, \pi_y\colon \mathbb{R}^2 \to \mathbb{R}^2$ projecting onto the axes. Each is rank one but their composition $\pi_x \circ \pi_y$, which is the zero map, is rank zero. That translates over to matrices representing those maps in this way.

$$\mathrm{Rep}_{\mathcal{E}_2,\mathcal{E}_2}(\pi_x) \cdot \mathrm{Rep}_{\mathcal{E}_2,\mathcal{E}_2}(\pi_y) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

To prove that the product of rank n matrices cannot have rank greater than n, we can apply the map result that the image of a linearly dependent set is linearly dependent. That is, if $h\colon V \to W$ and $g\colon W \to X$ both have rank n then a set in the range $\mathscr{R}(g \circ h)$ of size larger than n is the image under g of a set in W of size larger than n and so is linearly dependent (since the rank of h is n). Now, the image of a linearly dependent set is dependent, so any set of size larger than n in the range is dependent. (By the way, observe that the rank of g was not mentioned. See the next part.)

(b) Fix spaces and bases and consider the associated linear maps f and g. Recall that the dimension of the image of a map (the map's rank) is less than or equal to the dimension of the domain, and consider the arrow diagram.

$$V \overset{f}{\longmapsto} \mathscr{R}(f) \overset{g}{\longmapsto} \mathscr{R}(g \circ f)$$

First, the image of $\mathscr{R}(f)$ must have dimension less than or equal to the dimension of $\mathscr{R}(f)$, by the prior sentence. On the other hand, $\mathscr{R}(f)$ is a subset of the domain of g, and thus its image has dimension less than or equal the dimension of the domain of g. Combining those two, the rank of a composition is less than or equal to the minimum of the two ranks.

The matrix fact follows immediately.

! Important - Zero matrix

2.32 *(We will use this exercise in the Matrix Inverses exercises.)* Here is another property of matrix multiplication that might be puzzling at first sight.

(a) Prove that the composition of the projections $\pi_x, \pi_y : \mathbb{R}^3 \to \mathbb{R}^3$ onto the x and y axes is the zero map despite that neither one is itself the zero map.

(b) Prove that the composition of the derivatives d^2/dx^2, $d^3/dx^3 : \mathcal{P}_4 \to \mathcal{P}_4$ is the zero map despite that neither is the zero map.

(c) Give a matrix equation representing the first fact.

(d) Give a matrix equation representing the second.

When two things multiply to give zero despite that neither is zero we say that each is a *zero divisor*.

Three.IV.2.32 **(a)** Either of these.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \stackrel{\pi_x}{\longmapsto} \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix} \stackrel{\pi_y}{\longmapsto} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} x \\ y \\ z \end{pmatrix} \stackrel{\pi_y}{\longmapsto} \begin{pmatrix} 0 \\ y \\ 0 \end{pmatrix} \stackrel{\pi_x}{\longmapsto} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

(b) The composition is the fifth derivative map d^5/dx^5 on the space of fourth-degree polynomials.

(c) With respect to the natural bases,

$$\text{Rep}_{\mathcal{E}_3,\mathcal{E}_3}(\pi_x) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \text{Rep}_{\mathcal{E}_3,\mathcal{E}_3}(\pi_y) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and their product (in either order) is the zero matrix.

(d) Where $B = \langle 1, x, x^2, x^3, x^4 \rangle$,

$$\text{Rep}_{B,B}(\frac{d^2}{dx^2}) = \begin{pmatrix} 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 12 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \text{Rep}_{B,B}(\frac{d^3}{dx^3}) = \begin{pmatrix} 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 24 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and their product (in either order) is the zero matrix.

! Cool Counterexample Method

2.33 Show that, for square matrices, $(S + T)(S - T)$ need not equal $S^2 - T^2$.

Three.IV.2.33 Note that $(S+T)(S-T) = S^2 - ST + TS - T^2$, so a reasonable try is to look at matrices that do not commute so that $-ST$ and TS don't cancel: with

$$S = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \quad T = \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}$$

we have the desired inequality.

$$(S+T)(S-T) = \begin{pmatrix} -56 & -56 \\ -88 & -88 \end{pmatrix} \qquad S^2 - T^2 = \begin{pmatrix} -60 & -68 \\ -76 & -84 \end{pmatrix}$$

Three.IV.3. Mechanics of Matrix Multiplication

3.2 Definition A matrix with all 0's except for a 1 in the i, j entry is an i, j *unit* matrix (or *matrix unit*).

3.3 Example This is the $1, 2$ unit matrix with three rows and two columns, multiplying from the left.

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = \begin{pmatrix} 7 & 8 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Acting from the left, an i, j unit matrix copies row j of the multiplicand into row i of the result. From the right an i, j unit matrix picks out column i of the multiplicand and copies it into column j of the result.

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 4 \\ 0 & 7 \end{pmatrix}$$

3.7 Lemma In a product of two matrices G and H, the columns of GH are formed by taking G times the columns of H

$$G \cdot \begin{pmatrix} \vdots & & \vdots \\ \vec{h}_1 & \cdots & \vec{h}_n \\ \vdots & & \vdots \end{pmatrix} = \begin{pmatrix} \vdots & & \vdots \\ G\cdot\vec{h}_1 & \cdots & G\cdot\vec{h}_n \\ \vdots & & \vdots \end{pmatrix}$$

and the rows of GH are formed by taking the rows of G times H

$$\begin{pmatrix} \cdots & \vec{g}_1 & \cdots \\ & \vdots & \\ \cdots & \vec{g}_r & \cdots \end{pmatrix} \cdot H = \begin{pmatrix} \cdots & \vec{g}_1 \cdot H & \cdots \\ & \vdots & \\ \cdots & \vec{g}_r \cdot H & \cdots \end{pmatrix}$$

(ignoring the extra parentheses).

Proof We will check that in a product of 2×2 matrices, the rows of the product equal the product of the rows of G with the entire matrix H.

$$\begin{aligned} \begin{pmatrix} g_{1,1} & g_{1,2} \\ g_{2,1} & g_{2,2} \end{pmatrix} \begin{pmatrix} h_{1,1} & h_{1,2} \\ h_{2,1} & h_{2,2} \end{pmatrix} &= \begin{pmatrix} (g_{1,1} \;\; g_{1,2})H \\ (g_{2,1} \;\; g_{2,2})H \end{pmatrix} \\ &= \begin{pmatrix} (g_{1,1}h_{1,1} + g_{1,2}h_{2,1} \;\; g_{1,1}h_{1,2} + g_{1,2}h_{2,2}) \\ (g_{2,1}h_{1,1} + g_{2,2}h_{2,1} \;\; g_{2,1}h_{1,2} + g_{2,2}h_{2,2}) \end{pmatrix} \end{aligned}$$

We leave the more general check as an exercise. QED

An application of those observations is that there is a matrix that just copies out the rows and columns.

3.8 Definition The *main diagonal* (or *principal diagonal* or simply *diagonal*) of a square matrix goes from the upper left to the lower right.

3.9 Definition An *identity matrix* is square and every entry is 0 except for 1's in the main diagonal.

$$I_{n\times n}=\begin{pmatrix}1&0&\ldots&0\\0&1&\ldots&0\\&\vdots&&\\0&0&\ldots&1\end{pmatrix}$$

3.10 Example Here is the 2×2 identity matrix leaving its multiplicand unchanged when it acts from the right.

$$\begin{pmatrix}1&-2\\0&-2\\1&-1\\4&3\end{pmatrix}\begin{pmatrix}1&0\\0&1\end{pmatrix}=\begin{pmatrix}1&-2\\0&-2\\1&-1\\4&3\end{pmatrix}$$

3.11 Example Here the 3×3 identity leaves its multiplicand unchanged both from the left

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}2&3&6\\1&3&8\\-7&1&0\end{pmatrix}=\begin{pmatrix}2&3&6\\1&3&8\\-7&1&0\end{pmatrix}$$

and from the right.

$$\begin{pmatrix}2&3&6\\1&3&8\\-7&1&0\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}=\begin{pmatrix}2&3&6\\1&3&8\\-7&1&0\end{pmatrix}$$

In short, an identity matrix is the identity element of the set of $n\times n$ matrices with respect to the operation of matrix multiplication.

We can generalize the identity matrix by relaxing the ones to arbitrary reals. The resulting matrix rescales whole rows or columns.

3.12 Definition A *diagonal matrix* is square and has 0's off the main diagonal.

$$\begin{pmatrix} a_{1,1} & 0 & \dots & 0 \\ 0 & a_{2,2} & \dots & 0 \\ & \vdots & & \\ 0 & 0 & \dots & a_{n,n} \end{pmatrix}$$

3.13 Example From the left, the action of multiplication by a diagonal matrix is to rescales the rows.

$$\begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 4 & -1 \\ -1 & 3 & 4 & 4 \end{pmatrix} = \begin{pmatrix} 4 & 2 & 8 & -2 \\ 1 & -3 & -4 & -4 \end{pmatrix}$$

From the right such a matrix rescales the columns.

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 2 & 2 \end{pmatrix} \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{pmatrix} = \begin{pmatrix} 3 & 4 & -2 \\ 6 & 4 & -4 \end{pmatrix}$$

We can also generalize identity matrices by putting a single one in each row and column in ways other than putting them down the diagonal.

\

3.14 Definition A *permutation matrix* is square and is all 0's except for a single 1 in each row and column.

3.15 Example From the left these matrices permute rows.

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = \begin{pmatrix} 7 & 8 & 9 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

From the right they permute columns.

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 3 & 1 \\ 5 & 6 & 4 \\ 8 & 9 & 7 \end{pmatrix}$$

We finish this subsection by applying these observations to get matrices that perform Gauss's Method and Gauss-Jordan reduction. We have already seen how to produce a matrix that rescales rows, and a row swapper.

3.16 Example Multiplying by this matrix rescales the second row by three.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 2 & 1 & 1 \\ 0 & 1/3 & 1 & -1 \\ 1 & 0 & 2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2 & 1 & 1 \\ 0 & 1 & 3 & -3 \\ 1 & 0 & 2 & 0 \end{pmatrix}$$

3.17 Example This multiplication swaps the first and third rows.

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 2 & 1 & 1 \\ 0 & 1 & 3 & -3 \\ 1 & 0 & 2 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & -3 \\ 0 & 2 & 1 & 1 \end{pmatrix}$$

To see how to perform a row combination, we observe something about those two examples. The matrix that rescales the second row by a factor of three arises in this way from the identity.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{3\rho_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Similarly, the matrix that swaps first and third rows arises in this way.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{\rho_1 \leftrightarrow \rho_3} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

→ **CHANGING THE IDENTITY MATRIX WILL ALLOW U TO DO THE SAME FOR OTHER MATRICES BY MULTIPLYING ALTERED MATRIX WITH THOSE OTHER MATRICES**

3.19 Definition The *elementary reduction matrices* (or just *elementary matrices*) result from applying a single Gaussian operation to an identity matrix.

(1) $I \xrightarrow{k\rho_i} M_i(k)$ for $k \neq 0$

(2) $I \xrightarrow{\rho_i \leftrightarrow \rho_j} P_{i,j}$ for $i \neq j$

(3) $I \xrightarrow{k\rho_i + \rho_j} C_{i,j}(k)$ for $i \neq j$

3.20 Lemma Matrix multiplication can do Gaussian reduction.

(1) If $H \xrightarrow{k\rho_i} G$ then $M_i(k)H = G$.

(2) If $H \xrightarrow{\rho_i \leftrightarrow \rho_j} G$ then $P_{i,j}H = G$.

(3) If $H \xrightarrow{k\rho_i + \rho_j} G$ then $C_{i,j}(k)H = G$.

Proof Clear. QED

3.21 Example This is the first system, from the first chapter, on which we performed Gauss's Method.

$$\begin{aligned} 3x_3 &= 9 \\ x_1 + 5x_2 - 2x_3 &= 2 \\ (1/3)x_1 + 2x_2 \qquad &= 3 \end{aligned}$$

We can reduce it with matrix multiplication. Swap the first and third rows,

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \left(\begin{array}{ccc|c} 0 & 0 & 3 & 9 \\ 1 & 5 & -2 & 2 \\ 1/3 & 2 & 0 & 3 \end{array}\right) = \left(\begin{array}{ccc|c} 1/3 & 2 & 0 & 3 \\ 1 & 5 & -2 & 2 \\ 0 & 0 & 3 & 9 \end{array}\right)$$

triple the first row,

$$\begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \left(\begin{array}{ccc|c} 1/3 & 2 & 0 & 3 \\ 1 & 5 & -2 & 2 \\ 0 & 0 & 3 & 9 \end{array}\right) = \left(\begin{array}{ccc|c} 1 & 6 & 0 & 9 \\ 1 & 5 & -2 & 2 \\ 0 & 0 & 3 & 9 \end{array}\right)$$

and then add -1 times the first row to the second.

$$\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \left(\begin{array}{ccc|c} 1 & 6 & 0 & 9 \\ 1 & 5 & -2 & 2 \\ 0 & 0 & 3 & 9 \end{array}\right) = \left(\begin{array}{ccc|c} 1 & 6 & 0 & 9 \\ 0 & -1 & -2 & -7 \\ 0 & 0 & 3 & 9 \end{array}\right)$$

Now back substitution will give the solution.

3.22 Example Gauss-Jordan reduction works the same way. For the matrix ending the prior example, first turn the leading entries to ones,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1/3 \end{pmatrix} \left(\begin{array}{ccc|c} 1 & 6 & 0 & 9 \\ 0 & -1 & -2 & -7 \\ 0 & 0 & 3 & 9 \end{array}\right) = \left(\begin{array}{ccc|c} 1 & 6 & 0 & 9 \\ 0 & 1 & 2 & 7 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

then clear the third column, and then the second column.

$$\begin{pmatrix} 1 & -6 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} \left(\begin{array}{ccc|c} 1 & 6 & 0 & 9 \\ 0 & 1 & 2 & 7 \\ 0 & 0 & 1 & 3 \end{array}\right) = \left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

3.23 Corollary For any matrix H there are elementary reduction matrices $R_1, \ldots, R_r$ such that $R_r \cdot R_{r-1} \cdots R_1 \cdot H$ is in reduced echelon form.

Until now we have taken the point of view that our primary objects of study are vector spaces and the maps between them, and we seemed to have adopted matrices only for computational convenience. This subsection shows that this isn't the entire story.

Understanding matrix operations by understanding the mechanics of how the entries combine is also useful. In the rest of this book we shall continue to focus on maps as the primary objects but we will be pragmatic — if the matrix point of view gives some clearer idea then we will go with it.

✓ **3.24** Predict the result of each product with a permutation matrix and then check by multiplying it out.

(a) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ **(b)** $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ **(c)** $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$

Three.IV.3.24 **(a)** Acting from the left, the $P_{1,2}$ matrix swaps the first and second rows. $\begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix}$

(b) Acting from the right, $P_{1,2}$ swaps the first and second columns. $\begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix}$

(c) From the left, $P_{2,3}$ swaps the second and third rows. $\begin{pmatrix} 1 & 2 & 3 \\ 7 & 8 & 9 \\ 4 & 5 & 6 \end{pmatrix}$

! Important

✓ **3.25** Predict the result of each multiplication by an elementary reduction matrix, and then check by multiplying it out.

(a) $\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ **(b)** $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ **(c)** $\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

(d) $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ **(e)** $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Three.IV.3.25 **(a)** The second matrix has its first row multiplied by 3.

$$\begin{pmatrix} 3 & 6 \\ 3 & 4 \end{pmatrix}$$

(b) The second matrix has its second row multiplied by 2.

$$\begin{pmatrix} 1 & 2 \\ 6 & 8 \end{pmatrix}$$

(c) The second matrix undergoes the combination operation of replacing the second row with -2 times the first row added to the second.

$$\begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$$

(d) The first matrix undergoes the column operation of: replace the second column by -1 times the first column plus the second.

$$\begin{pmatrix} 1 & 1 \\ 3 & 1 \end{pmatrix}$$

(e) The first matrix has its columns swapped.

$$\begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix}$$

3.26 Predict the result of each multiplication by a diagonal matrix, and then check by multiplying it out.

(a) $\begin{pmatrix} -3 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ **(b)** $\begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

Three.IV.3.26 **(a)** The second matrix has its first row multiplied by -3 and its second row multiplied by 0.

$$\begin{pmatrix} -3 & -6 \\ 0 & 0 \end{pmatrix}$$

(b) The second matrix has its first row multiplied by 4 and its second row multiplied by 2.

$$\begin{pmatrix} 4 & 8 \\ 6 & 8 \end{pmatrix}$$

3.27 Produce each.

(a) a 3×3 matrix that, acting from the left, swaps rows one and two

(b) a 2×2 matrix that, acting from the right, swaps column one and two

?

Three.IV.3.27 **(a)** This matrix swaps row one and row three.

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} d & e & f \\ a & b & c \\ g & h & i \end{pmatrix}$$

(b) This matrix swaps column one and two.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} b & a \\ d & c \end{pmatrix}$$

! IMPORTANT ??? what is this

✓ **3.28** Show how to use matrix multiplication to bring this matrix to echelon form.

$$\begin{pmatrix} 1 & 2 & 1 & 0 \\ 2 & 3 & 1 & -1 \\ 7 & 11 & 4 & -3 \end{pmatrix}$$

Three.IV.3.28 Multiply by $C_{1,2}(-2)$, then by $C_{1,3}(-7)$, and then by $C_{2,3}(-3)$, paying attention to the right-to-left order.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -7 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 1 & 0 \\ 2 & 3 & 1 & -1 \\ 7 & 11 & 4 & -3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

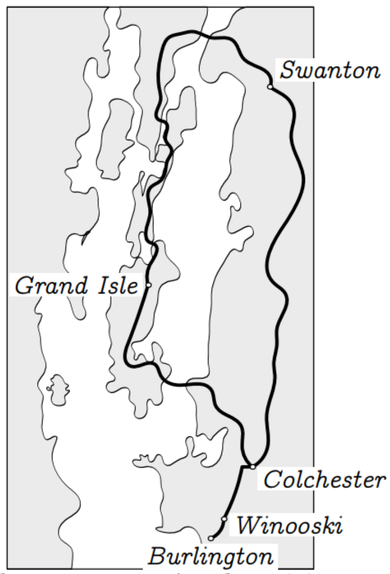
3.29 Find the product of this matrix with its transpose.

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Three.IV.3.29 The product is the identity matrix (recall that $\cos^2\theta + \sin^2\theta = 1$). An explanation is that the given matrix represents, with respect to the standard bases, a rotation in $\mathbb{R}^2$ of θ radians while the transpose represents a rotation of $-\theta$ radians. The two cancel.

3.30 The need to take linear combinations of rows and columns in tables of numbers arises often in practice. For instance, this is a map of part of Vermont and New York.

In part because of Lake Champlain, there are no roads directly connecting some pairs of towns. For instance, there is no way to go from Winooski to Grand Isle without going through Colchester. (To simplify the graph many other roads and towns have been omitted. From top to bottom of this map is about forty miles.)



(a) The *adjacency matrix* of a map is the square matrix whose i, j entry is the number of roads from city i to city j (all (i, i) entries are 0). Produce the adjacency matrix of this map, taking the cities in alphabetical order.

(b) A matrix is *symmetric* if it equals its transpose. Show that an adjacency matrix is symmetric. (These are all two-way streets. Vermont doesn't have many one-way streets.)

(c) What is the significance of the square of the incidence matrix? The cube?

Three.IV.3.30 **(a)** The adjacency matrix is this (e.g, the first row shows that there is only one connection including Burlington, the road to Winooski).

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

(b) Because these are two-way roads, any road connecting city i to city j gives a connection between city j and city i.

(c) The square of the adjacency matrix tells how cities are connected by trips involving two roads.

3.33 Express

$$\begin{pmatrix} 1 & 0 \\ -3 & 3 \end{pmatrix}$$

as the product of two elementary reduction matrices.

Three.IV.3.33 One way to produce this matrix from the identity is to use the column operations of first multiplying the second column by three, and then adding the negative of the resulting second column to the first.

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 \\ -3 & 3 \end{pmatrix}$$

In contrast with row operations, column operations are written from left to right, so this matrix product expresses doing the above two operations.

$$\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

Remark. Alternatively, we could get the required matrix with row operations. Starting with the identity, first adding the negative of the first row to the second, and then multiplying the second row by three will work. Because we write successive row operations as matrix products from right to left, doing these two row operations is expressed with: the same matrix product.

Three.IV.4. Inverses

We finish this section by considering how to represent the inverse of a linear map. We first recall some things about inverses. Where $\pi \colon \mathbb{R}^3 \to \mathbb{R}^2$ is the projection map and $\iota \colon \mathbb{R}^2 \to \mathbb{R}^3$ is the embedding

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \overset{\pi}{\longmapsto} \begin{pmatrix} x \\ y \end{pmatrix} \qquad \begin{pmatrix} x \\ y \end{pmatrix} \overset{\iota}{\longmapsto} \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$$

then the composition $\pi \circ \iota$ is the identity map $\pi \circ \iota = \text{id}$ on $\mathbb{R}^2$.

$$\begin{pmatrix} x \\ y \end{pmatrix} \overset{\iota}{\longmapsto} \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} \overset{\pi}{\longmapsto} \begin{pmatrix} x \\ y \end{pmatrix}$$

`

We say that ι is a *right inverse* of π or, what is the same thing, that π is a *left inverse* of ι. However, composition in the other order $\iota \circ \pi$ doesn't give the identity map — here is a vector that is not sent to itself under $\iota \circ \pi$.

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \stackrel{\pi}{\longmapsto} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \stackrel{\iota}{\longmapsto} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

In fact, π has no left inverse at all. For, if f were to be a left inverse of π then we would have

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \stackrel{\pi}{\longmapsto} \begin{pmatrix} x \\ y \end{pmatrix} \stackrel{f}{\longmapsto} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

for all of the infinitely many z's. But a function f cannot send a single argument $\binom{x}{y}$ to more than one value.

So a function can have a right inverse but no left inverse, or a left inverse but no right inverse. A function can also fail to have an inverse on either side; one example is the zero transformation on $\mathbb{R}^2$.

Some functions have a *two-sided inverse*, another function that is the inverse both from the left and from the right. For instance, the transformation given by $\vec{v} \mapsto 2 \cdot \vec{v}$ has the two-sided inverse $\vec{v} \mapsto (1/2) \cdot \vec{v}$. The appendix shows that a function has a two-sided inverse if and only if it is both one-to-one and onto. The appendix also shows that if a function f has a two-sided inverse then it is unique, so we call it 'the' inverse and write f^{-1}.

In addition, recall that we have shown in Theorem II.2.20 that if a linear map has a two-sided inverse then that inverse is also linear.

Thus, our goal in this subsection is, where a linear h has an inverse, to find the relationship between $\mathrm{Rep}_{B,D}(h)$ and $\mathrm{Rep}_{D,B}(h^{-1})$.

Left inverse vs Right inverse

4.1 Definition A matrix G is a *left inverse matrix* of the matrix H if GH is the identity matrix. It is a *right inverse* if HG is the identity. A matrix H with a two-sided inverse is an *invertible matrix*. That two-sided inverse is denoted H^{-1}.

Because of the correspondence between linear maps and matrices, statements about map inverses translate into statements about matrix inverses.

4.2 Lemma If a matrix has both a left inverse and a right inverse then the two are equal.

=> AND THERE CAN ONLY BE ONE → you can have some left, some right, none left, none right, infinitely many left, or infinitely many right inverses, but **if you have both left and right**

inverses, they are both the same matrix, and there can only be one matrix that is the inverse (ONLY ONE AND IT IS BOTH THE LEFT AND RIGHT INVERSE)

=> if there exist both left inverse & right inverse ⇒ they are the SAME

Nonsingular MEANS having both left & right inverses (and they are equal)

4.3 Theorem A matrix is invertible if and only if it is nonsingular.

PROOF *(For both results.)* Given a matrix H, fix spaces of appropriate dimension for the domain and codomain and fix bases for these spaces. With respect to these bases, H represents a map h. The statements are true about the map and therefore they are true about the matrix. QED

4.4 Lemma A product of invertible matrices is invertible: if G and H are invertible and GH is defined then GH is invertible and $(GH)^{-1} = H^{-1}G^{-1}$.

PROOF Because the two matrices are invertible they are square, and because their product is defined they must both be $n \times n$. Fix spaces and bases — say, $\mathbb{R}^n$ with the standard bases — to get maps $g, h\colon \mathbb{R}^n \to \mathbb{R}^n$ that are associated with the matrices, $G = \mathrm{Rep}_{\mathcal{E}_n,\mathcal{E}_n}(g)$ and $H = \mathrm{Rep}_{\mathcal{E}_n,\mathcal{E}_n}(h)$.

Consider $h^{-1}g^{-1}$. By the prior paragraph this composition is defined. This map is a two-sided inverse of gh since $(h^{-1}g^{-1})(gh) = h^{-1}(\mathrm{id})h = h^{-1}h = \mathrm{id}$ and $(gh)(h^{-1}g^{-1}) = g(\mathrm{id})g^{-1} = gg^{-1} = \mathrm{id}$. The matrices representing the maps reflect this equality. QED

This is the arrow diagram giving the relationship between map inverses and matrix inverses. It is a special case of the diagram relating function composition to matrix multiplication.

$$\begin{tikzcd}
 & W_{wrt\ C} \arrow[dr, "h^{-1}", "H^{-1}"'] & \\
V_{wrt\ B} \arrow[ur, "h", "H"'] \arrow[rr, "\mathrm{id}", "I"'] & & V_{wrt\ B}
\end{tikzcd}$$

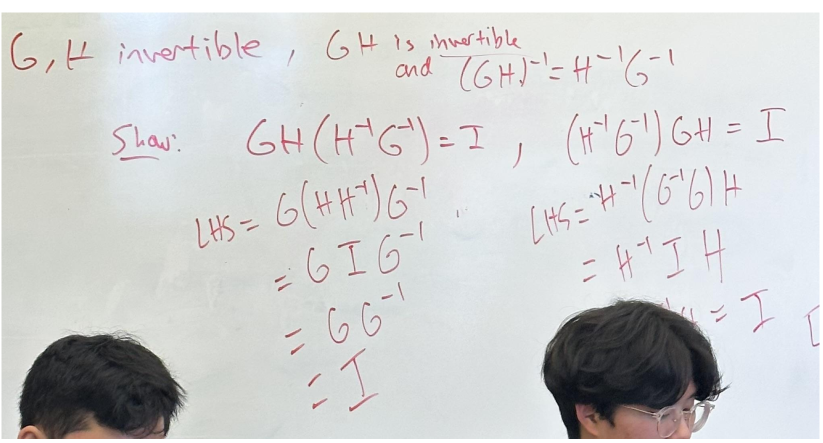
Beyond its place in our program of seeing how to represent map operations, another reason for our interest in inverses comes from linear systems. A linear system is equivalent to a matrix equation, as here.

$$\begin{aligned} x_1 + x_2 &= 3 \\ 2x_1 - x_2 &= 2 \end{aligned} \quad\Longleftrightarrow\quad \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

By fixing spaces and bases (for instance, $\mathbb{R}^2, \mathbb{R}^2$ with the standard bases), we take the matrix H to represent a map h. The matrix equation then becomes this linear map equation.

$$h(\vec{x}) = \vec{d}$$

If we had a left inverse map g then we could apply it to both sides $g \circ h(\vec{x}) = g(\vec{d})$ to get $\vec{x} = g(\vec{d})$. Restating in terms of the matrices, we want to multiply by the inverse matrix $\mathrm{Rep}_{C,B}(g) \cdot \mathrm{Rep}_C(\vec{d})$ to get $\mathrm{Rep}_B(\vec{x})$.



=>this is proof for theorem 4.4

4.7 Lemma A matrix H is invertible if and only if it can be written as the product of elementary reduction matrices. We can compute the inverse by applying to the identity matrix the same row steps, in the same order, that Gauss-Jordan reduce H.

PROOF The matrix H is invertible if and only if it is nonsingular and thus Gauss-Jordan reduces to the identity. By Corollary 3.23 we can do this reduction with elementary matrices.

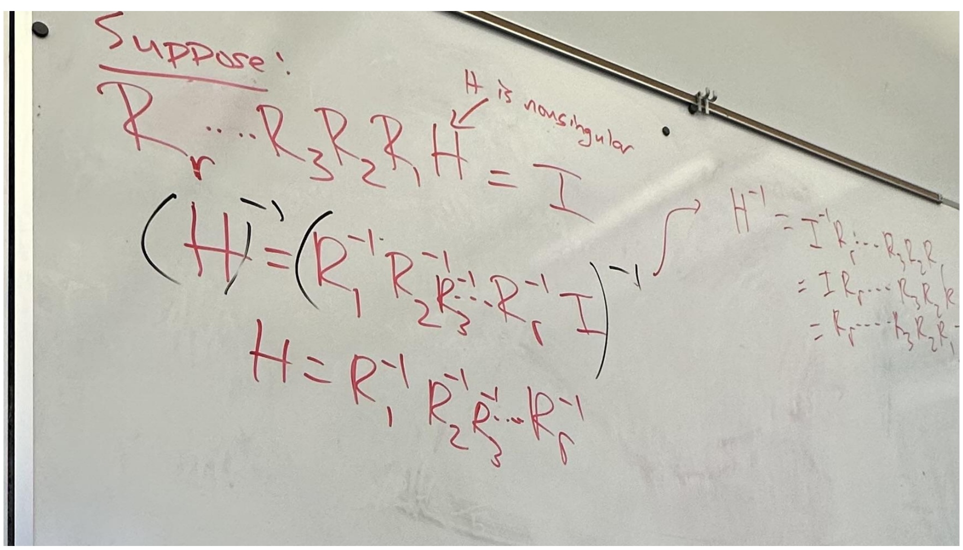
$$R_r \cdot R_{r-1} \ldots R_1 \cdot H = I \qquad (*)$$

For the first sentence of the result, note that elementary matrices are invertible because elementary row operations are reversible, and that their inverses are also elementary. Apply R_r^{-1} from the left to both sides of $(*)$. Then apply R_{r-1}^{-1}, etc. The result gives H as the product of elementary matrices $H = R_1^{-1} \cdots R_r^{-1} \cdot I$. (The I there covers the case $r = 0$.)

For the second sentence, group $(*)$ as $(R_r \cdot R_{r-1} \ldots R_1) \cdot H = I$ and recognize what's in the parentheses as the inverse $H^{-1} = R_r \cdot R_{r-1} \ldots R_1 \cdot I$. Restated: applying R_1 to the identity, followed by R_2, etc., yields the inverse of H. QED

-> basically invertible matrix means its nonsingular, meaning row reductions can bring it to the identity matrix → you can do the same row steps to get the inverse.
-> key to solving problems

Proof of theorem 4.7 (using theorem 4.4)



4.8 Example To find the inverse of

$$\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$$

do Gauss-Jordan reduction, meanwhile performing the same operations on the identity. For clerical convenience we write the matrix and the identity side-by-side and do the reduction steps together.

$$\begin{aligned} \left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 2 & -1 & 0 & 1 \end{array}\right) &\xrightarrow{-2\rho_1+\rho_2} \left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & -3 & -2 & 1 \end{array}\right) \\ &\xrightarrow{-1/3\rho_2} \left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & 2/3 & -1/3 \end{array}\right) \\ &\xrightarrow{-\rho_2+\rho_1} \left(\begin{array}{cc|cc} 1 & 0 & 1/3 & 1/3 \\ 0 & 1 & 2/3 & -1/3 \end{array}\right) \end{aligned}$$

This calculation has found the inverse.

$$\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}^{-1} = \begin{pmatrix} 1/3 & 1/3 \\ 2/3 & -1/3 \end{pmatrix}$$

4.11 Corollary The inverse for a 2×2 matrix exists and equals

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

if and only if $ad - bc \neq 0$.

PROOF This computation is Exercise 21. QED

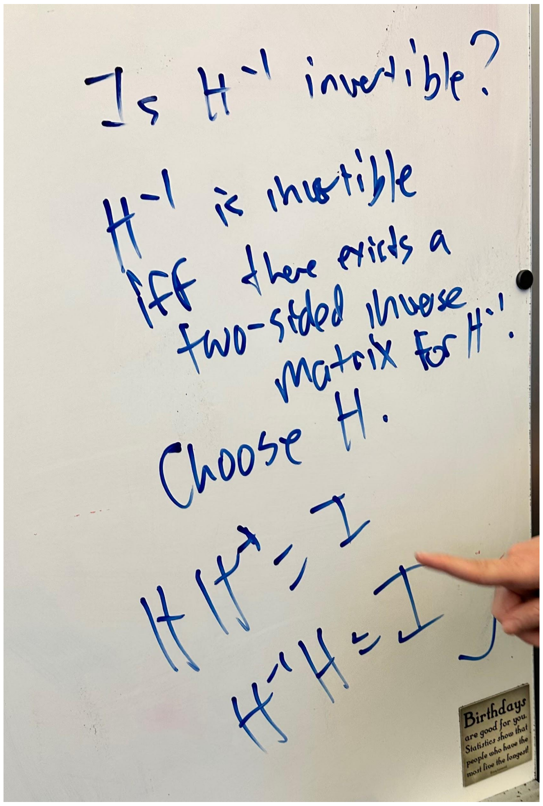
We have seen in this subsection, as in the subsection on Mechanics of Matrix Multiplication, how to exploit the correspondence between linear maps and matrices. We can fruitfully study both maps and matrices, translating back and forth to use whichever is handiest.

Over the course of this entire section we have developed an algebra system for matrices. We can compare it with the familiar algebra of real numbers. Matrix addition and subtraction work in much the same way as the real number operations except that they only combine same-sized matrices. Scalar multiplication is in some ways an extension of real number multiplication. We also have a matrix multiplication operation and its inverse that are somewhat like the familiar real number operations (associativity, and distributivity over addition,

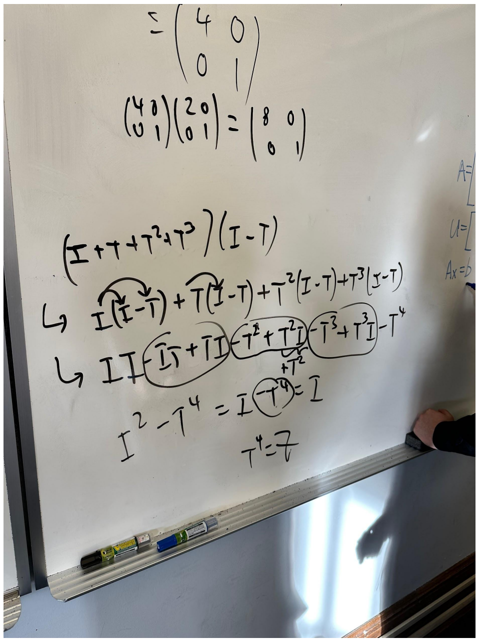
for example), but there are differences (failure of commutativity). This section provides an example that algebra systems other than the usual real number one can be interesting and useful.

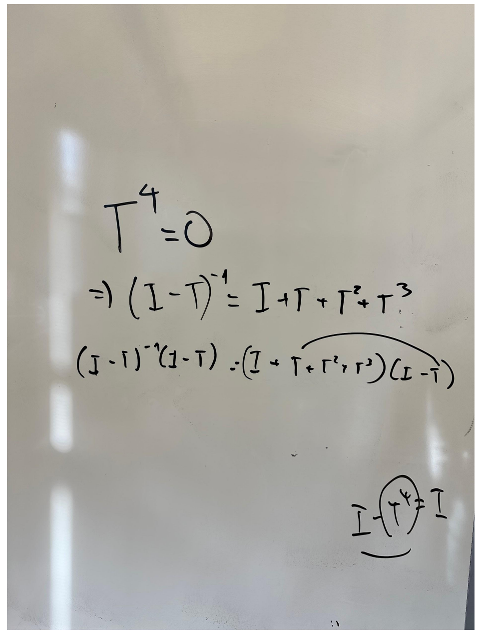
problems:

4.19.



4.28





Three.IV.1. Change of Basis

Representations vary with the bases. For instance, with respect to the bases $\mathcal{E}_2$ and

$$\mathrm{B} = \langle \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \rangle$$

$\vec{e}_1 \in \mathbb{R}^2$ has these different representations.

$$\mathrm{Rep}_{\mathcal{E}_2}(\vec{e}_1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \mathrm{Rep}_{\mathrm{B}}(\vec{e}_1) = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$$

The same holds for maps: with respect to the basis pairs $\mathcal{E}_2, \mathcal{E}_2$ and $\mathcal{E}_2, \mathrm{B}$, the identity map has these representations.

$$\mathrm{Rep}_{\mathcal{E}_2,\mathcal{E}_2}(\mathrm{id}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \mathrm{Rep}_{\mathcal{E}_2,\mathrm{B}}(\mathrm{id}) = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix}$$

This section shows how to translate among the representations. That is, we will compute how the representations vary as the bases vary.

1.1 Definition The *change of basis matrix* for bases $B, D \subset V$ is the representation of the identity map $\mathrm{id}\colon V \to V$ with respect to those bases.

$$\mathrm{Rep}_{B,D}(\mathrm{id}) = \begin{pmatrix} \vdots & & \vdots \\ \mathrm{Rep}_D(\vec{\beta}_1) & \cdots & \mathrm{Rep}_D(\vec{\beta}_n) \\ \vdots & & \vdots \end{pmatrix}$$

1.2 Remark A better name would be 'change of representation matrix' but the above name is standard.

The next result supports the definition.

1.3 Lemma To convert from the representation of a vector $\vec{v}$ with respect to B to its representation with respect to D use the change of basis matrix.

$$\mathrm{Rep}_{B,D}(\mathrm{id})\,\mathrm{Rep}_B(\vec{v}) = \mathrm{Rep}_D(\vec{v})$$

Conversely, if left-multiplication by a matrix changes bases $M \cdot \mathrm{Rep}_B(\vec{v}) = \mathrm{Rep}_D(\vec{v})$ then M is a change of basis matrix.

Proof The first sentence holds because matrix-vector multiplication represents a map application and so $\mathrm{Rep}_{B,D}(\mathrm{id}) \cdot \mathrm{Rep}_B(\vec{v}) = \mathrm{Rep}_D(\,\mathrm{id}(\vec{v})\,) = \mathrm{Rep}_D(\vec{v})$ for each $\vec{v}$. For the second sentence, with respect to B, D the matrix M represents a linear map whose action is to map each vector to itself, and is therefore the identity map. QED

1.4 Example With these bases for $\mathbb{R}^2$,

$$B = \langle \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rangle \qquad D = \langle \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \rangle$$

because

$$\mathrm{Rep}_D(\,\mathrm{id}(\begin{pmatrix} 2 \\ 1 \end{pmatrix})) = \begin{pmatrix} -1/2 \\ 3/2 \end{pmatrix}_D \qquad \mathrm{Rep}_D(\,\mathrm{id}(\begin{pmatrix} 1 \\ 0 \end{pmatrix})) = \begin{pmatrix} -1/2 \\ 1/2 \end{pmatrix}_D$$

the change of basis matrix is this.

$$\mathrm{Rep}_{B,D}(\mathrm{id}) = \begin{pmatrix} -1/2 & -1/2 \\ 3/2 & 1/2 \end{pmatrix}$$

For instance, this is the representation of $\vec{e}_2$

$$\mathrm{Rep}_B(\begin{pmatrix} 0 \\ 1 \end{pmatrix}) = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

and the matrix does the conversion.

$$\begin{pmatrix} -1/2 & -1/2 \\ 3/2 & 1/2 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$$

Checking that vector on the right is $\mathrm{Rep}_D(\vec{e}_2)$ is easy.

We finish this subsection by recognizing the change of basis matrices as a familiar set.

1.5 Lemma A matrix changes bases if and only if it is nonsingular.

1.6 Corollary A matrix is nonsingular if and only if it represents the identity map with respect to some pair of bases.

Three.V.2. Changing Map Representations

The first subsection shows how to convert the representation of a vector with respect to one basis to the representation of that same vector with respect to another basis. We next convert the representation of a map with respect to one pair of bases to the representation with respect to a different pair — we convert from $\mathrm{Rep}_{B,D}(h)$ to $\mathrm{Rep}_{\hat{B},\hat{D}}(h)$. Here is the arrow diagram.

$$\begin{array}{ccc} V_{\mathit{wrt}\ B} & \xrightarrow[H]{h} & W_{\mathit{wrt}\ D} \\ \mathrm{id}\big\downarrow & & \mathrm{id}\big\downarrow \\ V_{\mathit{wrt}\ \hat{B}} & \xrightarrow[\hat{H}]{h} & W_{\mathit{wrt}\ \hat{D}} \end{array}$$

To move from the lower-left to the lower-right we can either go straight over, or else up to V_B then over to W_D and then down. So we can calculate $\hat{H} = \mathrm{Rep}_{\hat{B},\hat{D}}(h)$ either by directly using $\hat{B}$ and $\hat{D}$, or else by first changing bases with $\mathrm{Rep}_{\hat{B},B}(\mathrm{id})$ then multiplying by $H = \mathrm{Rep}_{B,D}(h)$ and then changing bases with $\mathrm{Rep}_{D,\hat{D}}(\mathrm{id})$.

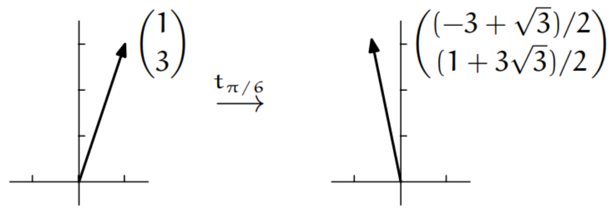
2.1 Theorem To convert from the matrix H representing a map h with respect to B, D to the matrix $\hat{H}$ representing it with respect to $\hat{B}, \hat{D}$ use this formula.

$$\hat{H} = \mathrm{Rep}_{D,\hat{D}}(\mathrm{id}) \cdot H \cdot \mathrm{Rep}_{\hat{B},B}(\mathrm{id}) \qquad (*)$$

2.2 Example The matrix

$$T = \begin{pmatrix} \cos(\pi/6) & -\sin(\pi/6) \\ \sin(\pi/6) & \cos(\pi/6) \end{pmatrix} = \begin{pmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{pmatrix}$$

represents, with respect to $\mathcal{E}_2, \mathcal{E}_2$, the transformation $t\colon \mathbb{R}^2 \to \mathbb{R}^2$ that rotates vectors through the counterclockwise angle of $\pi/6$ radians.



We can translate T to a representation with respect to these

$$\hat{B} = \langle \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \end{pmatrix} \rangle \qquad \hat{D} = \langle \begin{pmatrix} -1 \\ 0 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} \rangle$$

$$\begin{tikzcd} \cdot \arrow[r, "T"] & \cdot \arrow[d, "P"] \\ \cdot \arrow[u, "Q"] \arrow[r, "\hat{T}"'] & \cdot \end{tikzcd}$$

$\mathbb{R}^2$

$\frac{\pi}{6}$ rad c.c. within $\mathcal{E}_2$.

$$\begin{tikzcd} \mathcal{E}_2 \arrow[r, "T"] \arrow[d] & C \arrow[d] \\ \hat{B} \arrow[r, "\hat{T}"'] & \hat{D} \end{tikzcd}$$

$$\begin{tikzcd} \cdot \arrow[r, "T"] \arrow[d, "Q^{-1}"'] & \cdot \arrow[d, "P"] \\ \cdot \arrow[r, "\hat{T}"'] & \cdot \end{tikzcd}$$

$$\hat{T} = PTQ$$

$$Id\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}_{\mathcal{E}_2}$$

$$Id\begin{pmatrix}0\\2\end{pmatrix} = \begin{pmatrix}0\\2\end{pmatrix}_{\mathcal{E}_2}$$

$$Q = \mathrm{Rep}_{\hat{B},\mathcal{E}_2}(Id) = \begin{pmatrix}1 & 0\\ 1 & 2\end{pmatrix}$$

$$P = \mathrm{Rep}_{\mathcal{E}_2,\hat{D}}(Id) = \begin{pmatrix}-1 & 2\\ 0 & 3\end{pmatrix}^{-1}$$

$\mathrm{Rep}_{D,\mathcal{E}_2}(Id)$

2.3 Example Changing bases can make the matrix simpler. On $\mathbb{R}^3$ the map

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \overset{t}{\longmapsto} \begin{pmatrix} y+z \\ x+z \\ x+y \end{pmatrix}$$

is represented with respect to the standard basis in this way.

$$\mathrm{Rep}_{\mathcal{E}_3,\mathcal{E}_3}(t) = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

Representing it with respect to

$$B = \langle \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \rangle$$

gives a matrix that is diagonal.

$$\mathrm{Rep}_{B,B}(t) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Naturally we usually prefer representations that are easier to understand. We say that a map or matrix has been *diagonalized* when we find a basis B such that the representation is diagonal with respect to B, B, that is, with respect to the same starting basis as ending basis. Chapter Five finds which maps and matrices are diagonalizable.

The rest of this subsection develops the easier case of finding two bases B, D such that a representation is simple. Recall that the prior subsection shows that a matrix is a change of basis matrix if and only if it is nonsingular.

Matrix Equivalence Definition

→ the two P & Q matrices must be NONSINGULAR

-> P will bring you to echelon form, but to totally get rid of every nonzero values that are not on the diagonal (to get to the block-partial identity matrix), you need to use column operations (Q) -> multiplies from the right.

→ BTW, canonical form for matrix equivalence IS the block partial identity matrix

2.4 Definition Same-sized matrices H and $\hat{\mathrm{H}}$ are *matrix equivalent* if there are nonsingular matrices P and Q such that $\hat{\mathrm{H}} = \mathrm{PHQ}$.

2.5 Corollary Matrix equivalent matrices represent the same map, with respect to appropriate pairs of bases.

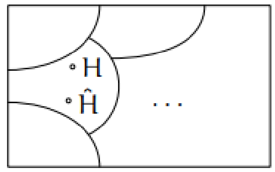
=> Matrix equivalent matrices represent the SAME MAP with the <u>appropriate</u> pair of bases

-> same mapping but different bases

Proof This is immediate from equation (∗) above. QED

Exercise 24 checks that matrix equivalence is an equivalence relation. Thus it partitions the set of matrices into matrix equivalence classes.

All matrices:



H matrix equivalent to $\hat{H}$

We can get insight into the classes by comparing matrix equivalence with row equivalence (remember that matrices are row equivalent when they can be reduced to each other by row operations). In $\hat{H} = PHQ$, the matrices P and Q are nonsingular and thus each is a product of elementary reduction matrices by Lemma IV.4.7. Left-multiplication by the reduction matrices making up P performs row operations. Right-multiplication by the reduction matrices making up Q performs column operations. Hence, matrix equivalence is a generalization of row equivalence — two matrices are row equivalent if one can be converted to the other by a sequence of row reduction steps, while two matrices are matrix equivalent if one can be converted to the other by a sequence of row reduction steps followed by a sequence of column reduction steps.

Consequently, if matrices are row equivalent then they are also matrix equivalent since we can take Q to be the identity matrix. The converse, however, does not hold: two matrices can be matrix equivalent but not row equivalent.

Row Equivalence also results in Matrix Equivalence!!!

Proof: Given two matrices A & B that are row equivalent:

→ B = PA (where P = product of some matrices, which is a non-singular matrix)
→ B = PAI (I = identity, which is nonsingular)
→ also Matrix equivalent because B=PAI is in the form of B=PAQ (where P & Q are nonsingular matrices) → PROVEN

→ extra: does matrix equivalence mean row equivalence (disprove w example)

(10 00) & (11 00) is matrix equivalent but not row equivalent

KEY: Interpret P & Q as row operations and column operations → prove theorems like this with it

Canonical Form for Matrix Equivalence

2.7 Theorem Any $m \times n$ matrix of rank k is matrix equivalent to the $m \times n$ matrix that is all zeros except that the first k diagonal entries are ones.

$$\begin{pmatrix} 1 & 0 & \ldots & 0 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 & 0 & \ldots & 0 \\ & \vdots & & & & & \\ 0 & 0 & \ldots & 1 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 & 0 & \ldots & 0 \\ & \vdots & & & & & \\ 0 & 0 & \ldots & 0 & 0 & \ldots & 0 \end{pmatrix}$$

This is a *block partial-identity* form.

$$\left(\begin{array}{c|c} I & Z \\ \hline Z & Z \end{array}\right)$$

Proof Gauss-Jordan reduce the given matrix and combine all the row reduction matrices to make P. Then use the leading entries to do column reduction and finish by swapping the columns to put the leading ones on the diagonal. Combine the column reduction matrices into Q. QED

→ any matrices of rank k can be turned into the reduced echelon form as such

→ MATRIX EQUIVALENT MATRICES MUST HAVE THE SAME DIMENSION

-> if asked if the two matrices are matrix equivalence, as long as two matrices have different rank → DO NOT need to find P&Q that satisfy.

2.8 Example We illustrate the proof by finding P and Q for this matrix.

$$\begin{pmatrix} 1 & 2 & 1 & -1 \\ 0 & 0 & 1 & -1 \\ 2 & 4 & 2 & -2 \end{pmatrix}$$

First Gauss-Jordan row-reduce.

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 & -1 \\ 0 & 0 & 1 & -1 \\ 2 & 4 & 2 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Then column-reduce, which involves right-multiplication.

$$\begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & -2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Finish by swapping columns.

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Finally, combine the left-multipliers together as P and the right-multipliers together as Q to get PHQ.

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 & -1 \\ 0 & 0 & 1 & -1 \\ 2 & 4 & 2 & -2 \end{pmatrix} \begin{pmatrix} 1 & 0 & -2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Action of Canonical Form Matrix

- If a rank 2 canonical form matrix multiplies to left of a rank 3 matrix → PROJECTION from R3→ 2 dimension

=> under some bases for the space, EVERY LINEAR MAP is EQUIVALENT TO SOME PROJECTION when interpreted on some particular bases

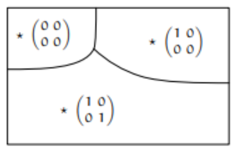
Matrix Equivalence ⇔ Rank

2.9 Corollary Matrix equivalence classes are characterized by rank: two same-sized matrices are matrix equivalent if and only if they have the same rank.

PROOF Two same-sized matrices with the same rank are equivalent to the same block partial-identity matrix. QED

2.10 Example The 2×2 matrices have only three possible ranks: zero, one, or two. Thus there are three matrix equivalence classes.

All 2×2 matrices:



Three equivalence classes

Each class consists of all of the 2×2 matrices with the same rank. There is only one rank zero matrix. The other two classes have infinitely many members; we've shown only the canonical representative.

One nice thing about the representative in Theorem 2.7 is that we can completely understand the linear map when it is expressed in this way: where the bases are $B = \langle\vec{\beta}_1,\ldots,\vec{\beta}_n\rangle$ and $D = \langle\vec{\delta}_1,\ldots,\vec{\delta}_m\rangle$ then the map's action is

$$c_1\vec{\beta}_1+\cdots+c_k\vec{\beta}_k+c_{k+1}\vec{\beta}_{k+1}+\cdots+c_n\vec{\beta}_n \;\mapsto\; c_1\vec{\delta}_1+\cdots+c_k\vec{\delta}_k+\vec{0}+\cdots+\vec{0}$$

where k is the rank. Thus we can view any linear map as a projection.

$$\begin{pmatrix} c_1\\ \vdots\\ c_k\\ c_{k+1}\\ \vdots\\ c_n \end{pmatrix}_B \longmapsto \begin{pmatrix} c_1\\ \vdots\\ c_k\\ 0\\ \vdots\\ 0 \end{pmatrix}_D$$