CALC BC

🔬 Things to Remember AP Calc BC

! Pay attention to the part with the velocity vs speed. IF VELOCITY INCREASES and Is NEGATIVE, Speed is DECREASING

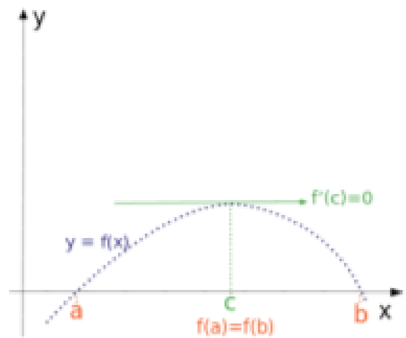
https://apcentral.collegeboard.org/pdf/ap-calculus-ab-bc-course-a-glance-0.pdf?course=ap-calculus-bc

https://www.dvusd.org/cms/lib/AZ01901092/Centricity/Domain/2903/BC cram sheet.pdf

MEAN VALUE THEOREM REQUIRES DIFFERENTIABILITY AS WELL!

Rolle's Theorem (a special case of the MVT)

- ***Formal Statement:*** If a function f is continuous on a closed interval $[a, b]$, differentiable on the open interva (a, b), and $f(a) = f(b)$, then there exists a number c in the open interval (a, b) such that $f'(c) = 0$.
- ***Translation:*** If a function is continuous and differentiable, the function must have a place with a horizontal tangent if there are two places where the function takes on the same value. In other words, there must be a relative minimum or maximum between two places where the function takes on the same value.
- ***Picture:***



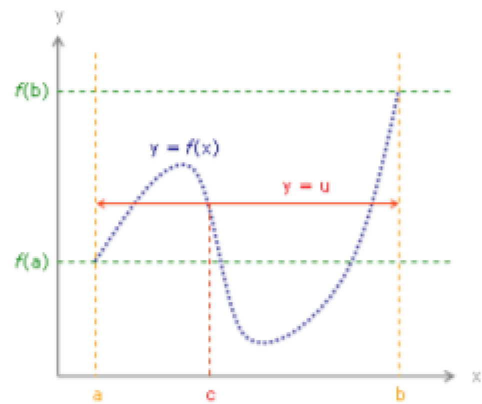
The Fundamental Theorem of Calculus (FTC)

Assume that $f(x)$, $g(x)$, and $h(x)$ are differentiable functions and that $F(x)$ is an antiderivative of $f(x)$. In other words, $F'(x) = f(x)$.

- ***The First Fundamental Theorem of Calculus (1st FTC)***
  - $\int_a^b f(x)dx = F(b) - F(a)$.
    - **Equivalently:** $\int_a^b f'(x)dx = f(b) - f(a)$ **(sometimes this is called the NET CHANGE**
    - **Equivalently:** $\int_a^x f'(t)dt = f(x) - f(a)$
  - This yields the incredibly useful formula $f(x) = f(a) + \int_a^x f'(t)dt$. This is used in MANY free response questions!
- ***The Second Fundamental Theorem of Calculus (2nd FTC)***
  - $\frac{d}{dx}\int_a^x f(t)dt = f(x)$
  - **Chain Rule Variation:** $\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)dt = f(h(x)) \cdot h'(x) - f(g(x)) \cdot g'(x)$

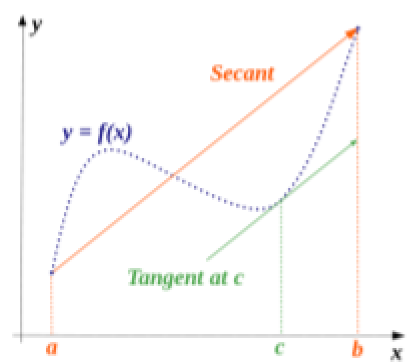
The Intermediate Value Theorem (IVT)

- ***Formal Statement:*** If a function f is continuous on a closed interval $[a, b]$ and $f(a) \neq f(b)$, then for every value of u between $f(a)$ and $f(b)$, there exist at least one value of c in the open interval (a, b) so that $f(c) = u$.
- ***Translation:*** A continuous function takes on all the values between any two of its values.
- ***Picture:***



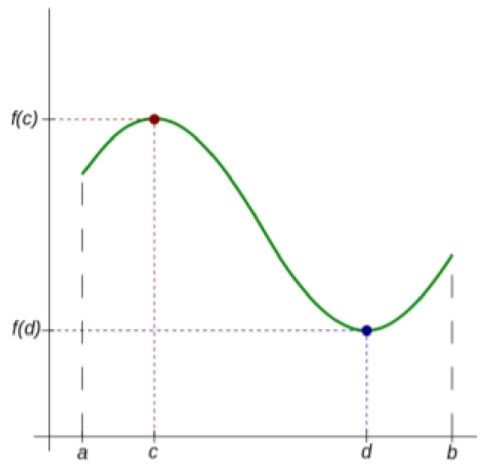
Mean Value Theorem (MVT)

- ***Formal Statement:*** If a function f is continuous on a closed interval $[a, b]$ and differentiable on the open interval (a, b), then there exists a number c in the open interval (a, b) such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.
- ***Translation:*** If a function is continuous and differentiable, somewhere in the interval the tangent line must be parallel to the secant line between the endpoints. In other words, the instantaneous rate of change is equal to the average rate of change.
- ***Picture:***



The Extreme Value Theorem (EVT)

- ***Formal Statement:*** If a function f is continuous on a closed interval $[a, b]$, then:
  1. There exists a number c in $[a, b]$ such that $f(x) \leq f(c)$ for all x in $[a, b]$.
  2. There exists a number d in $[a, b]$ such that $f(x) \geq f(d)$ for all x in $[a, b]$.
- ***Translation:*** If a function f is continuous on a closed interval $[a, b]$, then f takes on a maximum and a minimum value on that interval.
- ***Picture:***



- ***Special Notes:***
  - A function may attain its maximum and minimum value more than once. For example, the maximum value of $y = \sin(x)$ is 1 and it reaches this value many, many times.
  - The extreme values often occur at the endpoint of the domain. That's why it's so important to check the endpoints of an interval when doing a maximization/minimization problem!
  - For a constant function, the maximum and minimum values are equal (in fact, all the values are equal).

Unit 1: Limits and Continuity

1.

l'Hopital's Rule

$$\text{If } \frac{f(a)}{g(a)} = \frac{0}{0} \text{ or } = \frac{\infty}{\infty},$$

$$\text{then } \lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

2.

If f is the function defined by $f(x) = \frac{x-4}{\sqrt{x}-2}$, then $\lim_{x \to 4} f(x)$ is equivalent to which of the following?

A $\lim_{x \to 4} (\sqrt{x} - 2)$

B $\lim_{x \to 4} (\sqrt{x} + 2)$ ✓

C $\lim_{x \to 4} \left(\frac{x^2-16}{x-4}\right)$

D $\frac{\lim_{x \to 4}(x-4)}{\lim_{x \to 4}(\sqrt{x}-2)}$ ✗

Answer D

Incorrect. This would be correct if the denominator of the function f did not approach 0 as $x \to 4$.

—> You can only split the limit to numerator and denominator when the denominator limit is not = 0

3.

$\lim_{x \to 0} \frac{5x^5+3x^2+18x}{3x^5+6x}$ is → just replace w/ 0

A 0

B $\frac{5}{3}$

C 3 ✓

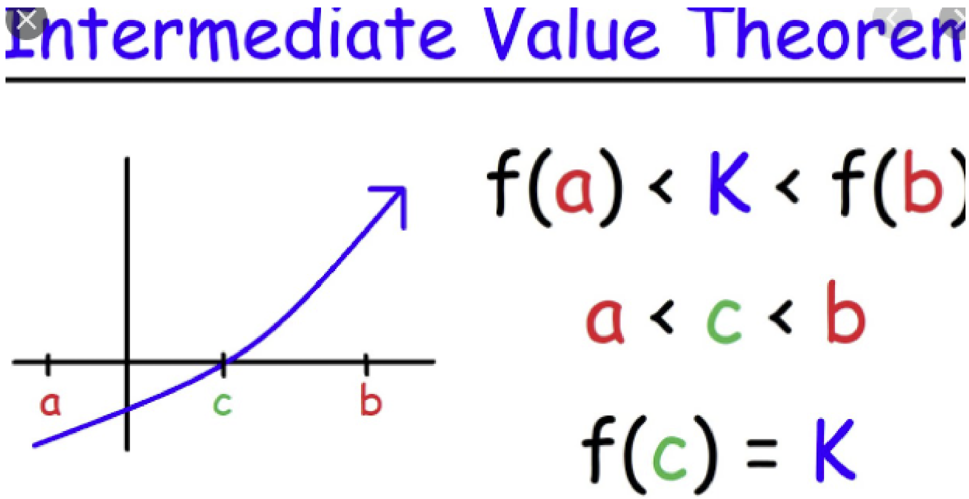
D ∞

Answer C

Correct. $\lim_{x \to 0} \frac{5x^5+3x^2+18x}{3x^5+6x} = \lim_{x \to 0} \frac{x(5x^4+3x+18)}{x(3x^4+6)} = \lim_{x \to 0} \frac{5x^4+3x+18}{3x^4+6} = \frac{0+0+18}{0+6} = 3$

You replace the limit with x=a whenever the denominator is not =0

4. Intermediate Value Theorem: (NOT CALCULUS)



continuous

5. Squeeze theorem: The squeeze (or sandwich) theorem states that if f(x)≤g(x)≤h(x) for all numbers, and at some point x=k we have f(k)=h(k), then g(k) must also be equal to them

6.

Limit of a Composite Function

If *f* and *g* are functions such that $\lim_{x \to c} g(x) = L$ and

$\lim_{x \to L} f(x) = f(L)$, then

$$\lim_{x \to c} f(g(x)) = f\left(\lim_{x \to c} g(x)\right) = f(L).$$

Formal Definition of Continuity

A function *f* is continuous at $x = a$ when:

(1) $f(a)$ is defined

(2) $\lim_{x \to a} f(x)$ exist

(3) $\lim_{x \to a} f(x) = f(a)$

7. What Is Removable Discontinuity? A hole in a graph. That is, a discontinuity that can be “repaired” by filling in a single point.

- Removable
- Non-removable: limit DNE
- Jump

- Vertical

8.

$$f(x) = \begin{cases} 3^x & \text{for } 0 < x < 1 \\ \frac{1}{2}x^2 - x + \frac{7}{2} & \text{for } 1 < x < 2 \end{cases}$$ → no f at 1

Let f be the function defined above. Which of the following statements is true?

A f is continuous at $x = 1$. ✗

B f is not continuous at $x = 1$ because $f(1)$ does not exist. ✓

9.

Let f be the function given by $f(x) = \frac{|x^2-2|(x+0.4)}{(x^2-2)(x+0.4)}$. On which of the following open intervals is f continuous?

you can remove $(x+0.4)$ if it's $\lim_{x \to -0.4}$

10. e^x grows faster than x^whatever, as x reaches infinity

Unit 2: Differentiation: Definition and Basic Derivative Rules

Derivatives

$$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$$

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\sec x) = \tan x \sec x$$

$$\frac{d}{dx}(\csc x) = -\cot x \csc x$$

$$\frac{d}{dx}(ln\, u) = \frac{1}{u} du$$

$$\frac{d}{dx}\left(e^u\right) = e^u du$$

$$\frac{d}{dx}(log_a x) = \frac{1}{x\, ln\, a}$$

$$\frac{d}{dx}\left(a^u\right) = a^x\,(ln\, a)\, du$$

$$|f(x)|' = \frac{f(x)}{|f(x)|} f'(x)$$

Derivatives of Inverse Trigonometric Functions

$$\frac{d}{dx}\left(\sin^{-1}x\right)=\frac{1}{\sqrt{1-x^2}}, x\neq\pm1$$

$$\frac{d}{dx}\left(\cos^{-1}x\right)=\frac{-1}{\sqrt{1-x^2}}, x\neq\pm1$$

$$\frac{d}{dx}\left(\tan^{-1}x\right)=\frac{1}{1+x^2}$$

$$\frac{d}{dx}\left(\cot^{-1}x\right)=\frac{-1}{1+x^2}$$

$$\frac{d}{dx}\left(\sec^{-1}x\right)=\frac{1}{|x|\sqrt{x^2-1}}, x\neq\pm1,0$$

$$\frac{d}{dx}\left(\csc^{-1}x\right)=\frac{-1}{|x|\sqrt{x^2-1}}, x\neq\pm1,0$$

Differentiation Rules

Chain Rule

$$\frac{d}{dx}\left[f(u)\right]=f'(u)\frac{du}{dx}$$

Product Rule

$$\frac{d}{dx}(uv)=u\frac{dv}{dx}+v\frac{du}{dx}$$

Quotient Rule

$$\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$$

1. Average Rate of Change: F(b)-F(a)/ (b-a)
2.

$$f(x)=\begin{cases}4x+2 & \text{for } x\le 2\\ 3x+4 & \text{for } x>2\end{cases}$$

Let f be the function defined above. Which of the following statements is true?

A. f is neither continuous nor differentiable at $x=2$.

B. f is continuous but not differentiable at $x=2$. ✔

C. f is differentiable but not continuous at $x=2$.

D. f is both continuous and differentiable at $x=2$.

Answer B

Correct. f is continuous at $x=2$ because $\lim_{x\to 2^-} f(x)=\lim_{x\to 2^-}(4x+2)=10$, $\lim_{x\to 2^+} f(x)=\lim_{x\to 2^+}(3x+4)=10$, and $f(2)=10$.

f is not differentiable at $x=2$ because the left-hand derivative does not equal the right-hand derivative at $x=2$, as follows.

$$\lim_{h\to 0^-}\frac{f(2+h)-f(2)}{h}=\lim_{h\to 0^-}\frac{(4(2+h)+2)-10}{h}=\lim_{h\to 0^-}\frac{4h}{h}=4$$

$$\lim_{h\to 0^+}\frac{f(2+h)-f(2)}{h}=\lim_{h\to 0^+}\frac{(3(h+2)+4)-10}{h}=\lim_{h\to 0^+}\frac{3h}{h}=3$$

it's continuous because left and right equal the same thing at x=2. It is not differentiable cus the slopes are 4 and 3 —> different.

Unit 3: Differentiation: Composite, Implicit and Inverse Functions

1. Composite Derivative: (CHAIN RULE)

Taking Derivatives Using The Chain Rule

The Chain Rule

$$[f(g(x))]' = f'(g(x)) \cdot g'(x)$$

2. Inverse derivatives

Calculus 3.1: Derivatives of Inverse Functions

A. Review: Inverse Functions:

$$f(x) = y \Leftrightarrow f^{-1}(y) = x$$

B. The Derivative of an Inverse Function:

$$(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}$$

3.

Which of the following methods can be used to find the derivative of $y = \arccos(\sqrt{x}\,)$ with respect to x ?

I. Use the quotient rule to differentiate $\frac{1}{\cos(\sqrt{x})}$.
II. Use the chain rule to differentiate $\cos\left(\arccos(\sqrt{x})\right) = \sqrt{x}$.
III. Use implicit differentiation to differentiate the function y in the relation $\cos y = \sqrt{x}$ with respect to x.

A I only

B III only ✗

C II and III only ✓

D I, II, and III

Answer B

Incorrect. While method III can be used to differentiate $\arccos(\sqrt{x})$, method II can also be used, as follows. The definition of the inverse cosine function means that $\cos\left(\arccos(\sqrt{x})\right) = \sqrt{x}$. Using the chain rule, $\frac{d}{dx}\left(\cos(\arccos(\sqrt{x}))\right) = -\sin\left(\arccos(\sqrt{x})\right) \cdot \frac{d}{dx}\left(\arccos(\sqrt{x})\right) = \frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}} \Rightarrow \frac{d}{dx}\left(\arccos\ (\sqrt{x})\right) = \frac{-1}{2\sqrt{x}\cdot\sin(\arccos(\sqrt{x}))}$.
If $\theta = \arccos(\sqrt{x})$, then $0 \leq \theta \leq \frac{\pi}{2}$ and $\cos\ \theta = \sqrt{x}$. Therefore, $\sin\ \theta$ is nonnegative and $\sin^2\theta = 1 - \cos^2\theta \Rightarrow \sin\ \theta = \sqrt{1-\cos^2\theta} = \sqrt{1-(\sqrt{x})^2} = \sqrt{1-x}$ because $0 \leq x \leq 1$.

The derivative is then $\frac{d}{dx}\arccos(\sqrt{x}) = \frac{-1}{2\sqrt{x}\ \cdot\sin(\arccos(\sqrt{x}))} = \frac{-1}{2\sqrt{x}\ \cdot\sin\theta} = \frac{-1}{2\sqrt{x}\sqrt{1-x}} = \frac{-1}{2\sqrt{x(1-x)}}$ for $0 < x < 1$.

4.

$f(1) = 4$	$f'(1) = -2$	$g(3) = 7$	$g'(3) = 1$

The point $(1, 3)$ lies on the curve in the xy-plane given by the equation $f(x)g(y) = 24 + x + y$, where f is a differentiable function of x and g is a differentiable function of y. Selected values of f, f', g, and g' are given in the table above. What is the value of $\frac{dy}{dx}$ at the point $(1, 3)$?

A -11

B 4

C 5

D 13

Answer C

Correct. The chain rule is the basis for implicit differentiation.

$f'(x)g(y) + f(x)g'(y)\frac{dy}{dx} = 1 + \frac{dy}{dx}$

$(-1 + f(x)g'(y))\frac{dy}{dx} = 1 - f'(x)g(y) \Rightarrow \frac{dy}{dx} = \frac{1-f'(x)g(y)}{-1+f(x)g'(y)}$

$\left.\frac{dy}{dx}\right|_{(1,3)} = \frac{1-f'(1)g(3)}{-1+f(1)g'(3)} = \frac{1-(-14)}{-1+4} = \frac{15}{3} = 5$

Unit 4: Contextual Applications of Differentiation

1. Linear Approximation:

Formula linear approx

$$y = f(a) + f'(a)(x - a)$$

$f(a)$ = function of curve, y=f(a)

$f'(a)$ = first derivative of curve function f(a), also equal to the slope of the tangent line at a=x

x = x value of point (x,y) where the tangent line touches the curve f(a)

y = y value of point (x,y) where the tangent line touches the curve f(a)

→ View more

2. Secant Line:

A secant line is a straight line joining two points on a function.

The slope of the secant line is equivalent to the average rate of change, or simply the slope between two points.

3.

Question 13

A 10-foot ladder is leaning straight up against a wall when a person begins pulling the base of the ladder away from the wall at the rate of 1 foot per second. Which of the following is true about the distance between the top of the ladder and the ground when the base of the ladder is 9 feet from the wall?

A. The distance is increasing at a rate of $\frac{9}{\sqrt{19}}$ feet per second.

B. The distance is decreasing at a rate of $\frac{9}{\sqrt{19}}$ feet per second.

C. The distance is increasing at a rate of $\frac{\sqrt{19}}{9}$ feet per second.

D. The distance is decreasing at a rate of $\frac{\sqrt{19}}{9}$ feet per second.

4.

A particle moves along the x-axis so that at time $t \geq 0$ its position is given by $x(t) = 2t^3 - 9t^2 - 60t + 4$. What is the total distance traveled by the particle over the time interval $0 \leq t \leq 7$?

A 168

B 171

C 175

D 375

Answer D

Correct. The velocity of the particle is given by $x'(t) = 6t^2 - 18t - 60 = 6(t-5)(t+2)$. This is negative for $0 \leq t < 5$ and positive for $5 < t \leq 7$. So the particle is moving left over the time interval $0 \leq t < 5$ and moving right over the time interval $5 < t \leq 7$. Therefore, the total distance traveled by the particle over the time interval $0 \leq t \leq 7$ is given by $(x(0) - x(5)) + (x(7) - x(5)) = (4 - (-271)) + (-171 - (-271)) = 375.$

5. VELOCITY HAS DIRECTION!!! IF DISTANCE, CHECK WHERE VELOCITY CHANGES SIGNS AND CREATE SUB INTERVALS, SPEED IS A MAGNITUDE, VELOCITY INTEGRAL = DISPLACEMENT

A particle moves along the y-axis so that at time $t \geq 0$ its position is given by $y(t) = \frac{2}{3}t^3 - 5t^2 + 8t$. Over the time interval $0 < t < 5$, for what values of t is the speed of the particle increasing?

A $2.5 < t < 5$

B $4 < t < 5$ only

C $0 < t < 1$ and $4 < t < 5$

D $1 < t < 2.5$ and $4 < t < 5$

Answer D

Correct. The velocity of the particle is given by
$y'(t) = 2t^2 - 10t + 8 = 2(t-1)(t-4)$,
and the acceleration of the particle is given by
$y''(t) = 4t - 10 = 4(t - 2.5)$. The speed of the particle is increasing when the velocity and acceleration have the same sign, which occurs for $1 < t < 2.5$ and $4 < t < 5$.

Unit 5: Analytical Applications of Differentiation

Mean Value Theorem

The Mean Value Theorem Let f be a function that satisfies the following hypotheses:

1. f is continuous on the closed interval $[a, b]$.
2. f is differentiable on the open interval (a, b).

Then there is a number c in (a, b) such that

$$\boxed{1} \qquad f'(c) = \frac{f(b) - f(a)}{b - a}$$

or, equivalently,

$$\boxed{2} \qquad f(b) - f(a) = f'(c)(b - a)$$

Before proving this theorem, we can see that it is reasonable by interpreting it geomet-

—> There exist a value c such that f'(c) = average rate of change\

—> IT NEEDS TO BE CONTINUOUS AND DIFFERENTIABLE:

Example:

x	0	1	2	3	4
$f(x)$	2	-2	4	7	18

Selected values of a continuous function f are given in the table above. Which of the following statements could be false?

A. By the Intermediate Value Theorem applied to f on the interval $[0, 4]$, there is a value c such that $f(c) = 5$.

B. By the Mean Value Theorem applied to f on the interval $[0, 4]$, there is a value c such that $f'(c) = 4$. ✓

C. By the Extreme Value Theorem applied to f on the interval $[0, 4]$, there is a value c such that $f(c) \le f(x)$ for all x in $[0, 4]$.

D. By the Extreme Value Theorem applied to f on the interval $[0, 4]$, there is a value c such that $f(c) \ge f(x)$ for all x in $[0, 4]$.

Answer B

Correct. This is an attempt to use the Mean Value Theorem to conclude that there is a value c for which $f'(c) = \frac{f(4)-f(0)}{4-0} = \frac{18-2}{4} = 4$. The Mean Value Theorem requires that f is continuous on the closed interval $[0, 4]$ and differentiable on the open interval $(0, 4)$. Because we do not know that f is differentiable on the open interval $(0, 4)$, the Mean Value Theorem does not apply.

2. First Derivative Test:

FIRST FIND f'(x) = 0 —> solve for x —> these x are critical numbers

THEN YOU MUST ALSO FIND f'(x) = undefined —> get x —> x can also be CRITICAL POINTS TOO!!!

First Derivative Test

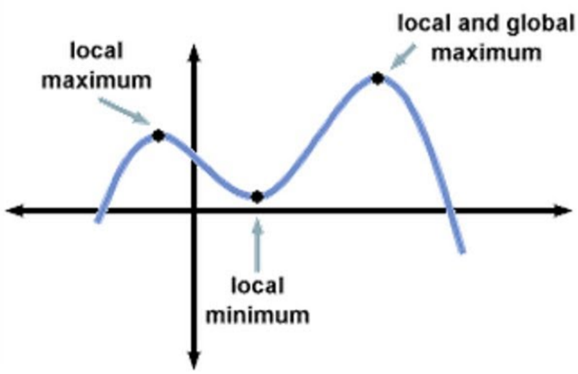
First Derivative Test:

Suppose that c is a critical number of a continuous function f:

(a) If f' changes from positive to negative at c, then f has a local maximum at c.

(b) If f' changes from negative to positive at c, then f has a local minimum at c.

(c) If f' does not change sign at c (for example, if f' is positive on both sides of c or negative on both sides), then f has no local maximum or minimum at c.

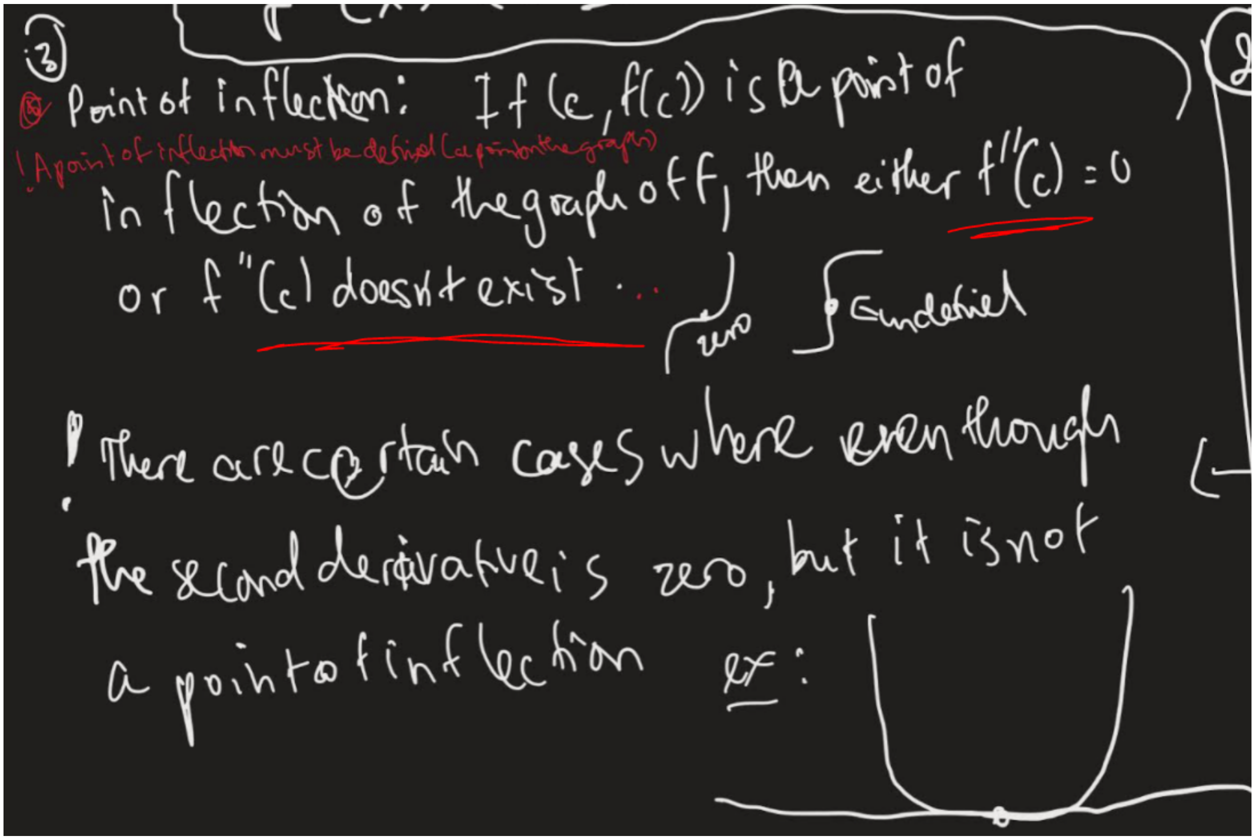


3. Second Derivative Test:

Second Derivative Test:

- If $f'(c) = 0$ and $f''(c) > 0$, then there is a local minimum at $x = c$.
- If $f'(c) = 0$ and $f''(c) < 0$, then there is a local maximum at $x = c$.
- If $f'(c) = 0$ and $f''(c) = 0$, or if $f''(c)$ doesn't exist, then the test is **inconclusive**. There might be a local maximum or minimum, or there might be a point of inflection.

4. Points of Inflection



THESE ARE ONLY POSSIBLE INFLECTION POINTS, THEN YOU HAVE TO CHOOSE POINTS AROUND THESE POINTS TO SEE IF THE SECOND DERIVATIVE CHANGES FROM POSITIVE TO NEGATIVE OR VICE VERSA - (larger than 0 is concave up, smaller than 0 is concave down)

5. Extreme Value Theorem:

The Extreme value theorem states that if a function is continuous on a closed interval [a,b], then the function must have a maximum and a minimum on the interval.

→ if the domain is open [x, infinity) o [x,y) doesnt work because max and min can be at the end point.

6. CHECK THE END POINTS IN CLOSED INTERVALS:

Let f be the function given by $f(x) = -x^3 + 3x^2 + 24x$. What is the absolute maximum value of f on the closed interval $[-6, 6]$?

end points!

A -6

B 36

C 80 ✕

D 180 ✓

Answer C

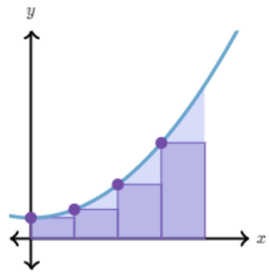
Incorrect. This results from only considering the critical points and not including the function values at the endpoints.

Unit 6: Integration and Accumulation of Change

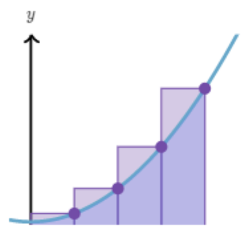
1. Left vs. Right Riemann Sum:

Left and right Riemann sums

To make a Riemann sum, we must choose how we're going to make our rectangles. One possible choice is to make our rectangles touch the curve with their top-left corners. This is called a **left Riemann sum**.



Another choice is to make our rectangles touch the curve with their top-right corners. This is a **right Riemann sum**.



Riemann Sum formulas (Left, Right, Midpoint) **FIX THIS**

Riemann Sums

There are three types of Riemann Sums

Right Riemann:

$$A = \frac{b-a}{n}[f(x_1)+f(x_2)+f(x_3)\ldots f(x_n)]$$

Left Riemann:

$$A = \frac{b-a}{n}[f(x_0)+f(x_1)+f(x_2)+f(x_3)+\ldots f(x_{n-1})]$$

Midpoint Riemann:

$$A = \frac{b-a}{n}[f(x_{1/2})+f(x_{3/2})+f(x_{5/2})+f(x_{7/2})+\ldots f(x_{n-1/2})]$$

2. Fundamental Theorem of Calculus

$$\text{Part 1: } \frac{d}{dx}\int_a^x f(x)\,dx = f(x)$$

$$\text{Part 2: } \int_a^b f(x)\,dx = F(b) - F(a)$$

3. Fundamental Theorem of Calculus with Chain rule

If $h(x) = \int_{-\frac{\pi}{4}}^{\sin^2 x} \csc(t^4+1)\,dt$ for $-\frac{\pi}{4} \le x \le \frac{\pi}{4}$, then $h'(x) =$

A) $\csc(x^4+1)$

B) $\csc(\sin^8 x + 1)$

C) $(\sin^2 x)(\csc(x^4+1))$

D) $(2\sin x\cos x)(\csc(\sin^8 x+1))$

replace $\sin^2 x$ w/ t

→ derivative of $\sin^2 x$

4. Integral to Riemann Sum

By definition, the definite integral is the limit of the Riemann sum

The above example is a specific case of the general definition for definite integrals:

The definite integral of a continuous function f over the interval $[a, b]$, denoted by $\int_a^b f(x)dx$, is the limit of a Riemann sum as the number of subdivisions approaches infinity. That is,

$$\int_a^b f(x)dx = \lim_{n \to \infty} \sum_{i=1}^{n} \Delta x \cdot f(x_i)$$

where $\Delta x = \frac{b-a}{n}$ and $x_i = a + \Delta x \cdot i$.

Reimann sum:

$$S = \sum_{i=1}^{n} f(x_i^*)(x_i - x_{i-1}) \qquad x_{i-1} \le x_i^* \le x_i$$

Left Riemann sum: $x_i^* = x_{i-1}$ for all i

Right Riemann sum: $x_i^* = x_i$ for all i

Middle Riemann sum: $x_i^* = \frac{1}{2}(x_i + x_{i-1})$ for all i

5. Using fundamental theorem of calculus to find values of functions:

Let f be a differentiable function such that $f(1) = 2$ and $f'(x) = \sqrt{x^2 + 2\cos x + 3}$. What is the value of $f(4)$?

$\int_0^1 \sqrt{x^2+2\cos x+3}\, dx$

$= f(1) - f(0) \rightarrow$ get $f(0)$

$f(4) - f(0) = \int_0^4 = 11.---$

$\rightarrow$ get $f(4)$

- **A** -6.790
- **B** 8.790
- **C** 10.790
- **D** 12.996

6. U-Substitution

x	0	1	2	3	4
$g(x)$	1	$\frac{1}{8}$	$-\frac{3}{4}$	$-\frac{13}{8}$	$-\frac{5}{2}$

Selected values of the twice-differentiable function g are given in the table above. What is the value of $\int_0^4 g'(x)\arctan^2(2g(x)+3)dx$?

$u = 2g(x) + 3$

$du = 2g'(x)\, dx$

$\rightarrow \int_{g(0)}^{g(4)} \frac{\arctan^2(u)\, du}{2}$

- **A** -14.588
- **B** -3.647
- **C** -0.721
- **D** 2.136

7. Trig And Inverse Trig Integrals

Integrals

$$\int kf(u)du = k\int f(u)du$$

$$\int du = u + C$$

$$\int u^n du = \frac{u^{n+1}}{n+1} + C, n \neq -1$$

$$\int \frac{1}{u} du = \ln|u| + C$$

$$\int e^u du = e^u + C$$

$$\int a^u du = \left(\frac{1}{\ln a}\right) a^u + C$$

$$\int \cos u du = \sin u + C$$

$$\int \sin u du = -\cos u + C$$

$$\int \tan u du = -\ln|\cos u| + C$$

$$\int \cot u du = \ln|\sin u| + C$$

$$\int \sec u du = \ln|\sec u + \tan u| + C$$

$$\int \csc u du = -\ln|\csc u + \cot u| + C$$

$$\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a} arc\sec\left(\frac{|u|}{a}\right) + C$$

$$\int \frac{du}{\sqrt{a^2 - u^2}} = \arcsin\left(\frac{u}{a}\right) + C$$

$$\int \frac{du}{a^2 + u^2} = \frac{1}{a}\arctan\left(\frac{u}{a}\right) + C$$

$$\arcsin(x) = \int_0^x \frac{1}{\sqrt{1-z^2}}\,dz\,, \qquad |x| \leq 1$$

$$\arccos(x) = \int_x^1 \frac{1}{\sqrt{1-z^2}}\,dz\,, \qquad |x| \leq 1$$

$$\arctan(x) = \int_0^x \frac{1}{z^2+1}\,dz\,,$$

$$\operatorname{arccot}(x) = \int_x^\infty \frac{1}{z^2+1}\,dz\,,$$

$$\operatorname{arcsec}(x) = \int_1^x \frac{1}{z\sqrt{z^2-1}}\,dz = \pi + \int_x^{-1} \frac{1}{z\sqrt{z^2-1}}\,dz\,, \qquad x \geq 1$$

$$\operatorname{arccsc}(x) = \int_x^\infty \frac{1}{z\sqrt{z^2-1}}\,dz = \int_{-\infty}^{x} \frac{1}{z\sqrt{z^2-1}}\,dz\,, \qquad x \geq 1$$

8. Inverse trig example:

Question 11

$$\int \frac{2}{\sqrt{-x^2-2x}}\,dx =$$

$\sqrt{1-(x-1)^2}$

complete square

A $4\sqrt{-x^2-2x}+C$

B $2\tan^{-1}(x+1)+C$

C $2\sin^{-1}(x+1)+C$

D $2\sin^{-1}(x-1)+C$

9. AN ANTIDERIVATIVE IS THE INFINITE INTEGRAL OF THE FUNCTION

For example:

Correct. The function $F(x) = \int_0^x f(t)\,dt$ is an antiderivative of f, since $F'(x) = \frac{d}{dx}\int_0^x f(t)\,dt = f(x)$ by the Fundamental Theorem of Calculus.

10. Integration by Parts:

Integration by Parts

$$\int u\,dv = uv - \int v\,du$$

Choose u in this order: **LIATE**

Logs best
Inverse
Algebraic
Trig
Exponential

11. Partial Fraction Decomposition

$$\frac{5x-7}{(x-1)^3} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3}$$

$$= \frac{0}{x-1} + \frac{5}{(x-1)^2} + \frac{-2}{(x-1)^3}$$

$$= \frac{5}{(x-1)^2} + \frac{-2}{(x-1)^3}$$

S.No.	Form of the rational function	Form of the partial fraction
1.	$\frac{px+q}{(x-a)(x-b)}$, $a \neq b$	$\frac{A}{x-a} + \frac{B}{x-b}$
2.	$\frac{px+q}{(x-a)^2}$	$\frac{A}{x-a} + \frac{B}{(x-a)^2}$
3.	$\frac{px^2+qx+r}{(x-a)(x-b)(x-c)}$	$\frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c}$
4.	$\frac{px^2+qx+r}{(x-a)^2(x-b)}$	$\frac{A}{x-a} + \frac{B}{(x-a)^2} + \frac{C}{x-b}$
5.	$\frac{px^2+qx+r}{(x-a)(x^2+bx+c)}$	$\frac{A}{x-a} + \frac{Bx+C}{x^2+bx+c}$,
	where $x^2 + bx + c$ cannot be factorised further	

Unit 7: Differential Equations

1. Separation of Variables

2. Slope fields to diff equation prediction

3. Euler's method formula:

$$y_{i+1} = y_i + hf(x_i, y_i)$$

$\Delta x \cdot f(\,i\,)$

where,

- y_{i+1} is the next estimated solution value;
- y_i is the current value;
- h is the interval between steps;
- $f(x_i, y_i)$ is the value of the derivative at the current (x_i, y_i) point.

4, Growth equations (exponential, logistic)

Logistics Differential Equation

$$\frac{dP}{dt} = kP(M - P)$$

We can solve this differential equation to find the logistics growth model.

Logistics Growth Model

$$P = \frac{M}{1 + Ae^{-(Mk)t}}$$

→

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{L}\right)$$

P = Population at time t

L = Limiting or Carrying Capacity
(max size of population)

k = constant of proportionality

—> population at fastest growth rate is L/2 !!!!

5.

t	W	$\frac{dW}{dt}$	$\frac{d^2W}{dt^2}$
0	10	6	3
2.56	35	12.25	0

→ 35 is when highest speed

The weight, in grams, of a population of bacteria at time t hours is modeled by the function W, the solution to a logistic differential equation. Selected values of W and its first and second derivatives are shown in the table above. Which of the following statements is true?

A $\frac{dW}{dt} = \frac{3}{125}W(35 - W)$, because the carrying capacity is 35 and the rate of change of the weight is 6 grams per hour when the weight is 10 grams.

B $\frac{dW}{dt} = \frac{3}{250}W(35 - W)$, because the carrying capacity is 35 and the rate of change of the weight is 3 grams per hour when the weight is 10 grams.

C $\frac{dW}{dt} = \frac{1}{100}W(70 - W)$, because the carrying capacity is 70 and the rate of change of the weight is 6 grams per hour when the weight is 10 grams. ✓

D $\frac{dW}{dt} = \frac{1}{200}W(70 - W)$, because the carrying capacity is 70 and the rate of change of the weight is 3 grams per hour when the weight is 10 grams.

Answer C

Correct. The form of the logistic differential equation is $\frac{dW}{dt} = kW(a - W)$, where a is the carrying capacity. If a solution has a point of inflection, that point will occur when $W = \frac{a}{2}$. Since W is a solution to a logistic differential equation and $\frac{d^2W}{dt^2} = 0$ when $W = 35$, the function W has a point of inflection at $W = 35$. Therefore, $a = 70$.

$\frac{dW}{dt} = 6$ when $W = 10$ implies that $6 = k(10)(70 - 10) = 600k$. Therefore, $k = \frac{1}{100}$.

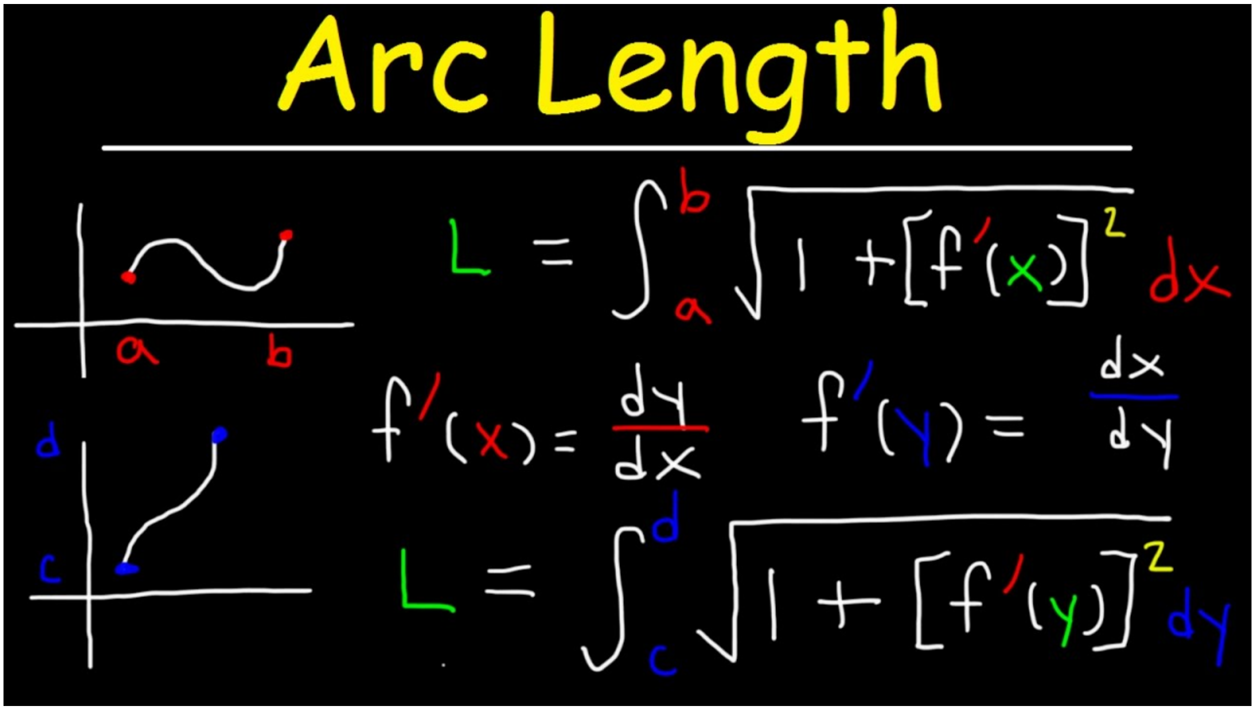
Unit 8: Applications of Integration

- Area of region bounded by two graphs

- Average value = integral/(b-a) (DIFFERENT FROM AVERAGE RATE OF CHANGE, WHICH IS JUST THE SLOPE BETWEEN THE TWO END POINTS)

- WASHER METHOD/ ROTATION INTEGRAL

- Arclength formula



Unit 9: Parametric Equations, Polar Coordinates, Vector-Valued Functions

1. FINDING SECOND DERIVATIVE OF PARAMETRIC EQUATIONS: D2y/dx2 = d/dx * dy/dx = d/dt *dy/dx / dx/dt

A curve is defined by parametric functions $x(t)$ and $y(t)$, where $\frac{dx}{dt} = 3t^2$ and $\frac{dy}{dt} = t^7 - 2t^4$.
For what positive value of t is $\frac{d^2y}{dx^2} = 1$?

- (A) 1.079
- (B) 1.149
- (C) 1.202
- (D) 1.525 ✓

Answer D

Correct. The chain rule for parametric functions is used to find the first derivative and the second derivative. The calculator is used to solve for where the second derivative equals 1.

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{t^7-2t^4}{3t^2} = \frac{1}{3}t^5 - \frac{2}{3}t^2$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\frac{5}{3}t^4 - \frac{4}{3}t}{3t^2} = \frac{5}{9}t^2 - \frac{4}{9t}$$

?

The only positive solution of $\frac{5}{9}t^2 - \frac{4}{9t} = 1$ is $t = 1.525$.

A curve is defined by the parametric functions $x(t) = e^{2t} + 3$ and $y(t) = e^{4t} + e^{t}$. What is $\frac{d^2y}{dx^2}$ in terms of t ?

A $2 - \frac{1}{4}e^{-3t}$

B $4e^{2t} - \frac{1}{2}e^{-t}$

C $4e^{2t} + \frac{1}{4}e^{-t}$

D $16e^{4t} + e^{t}$

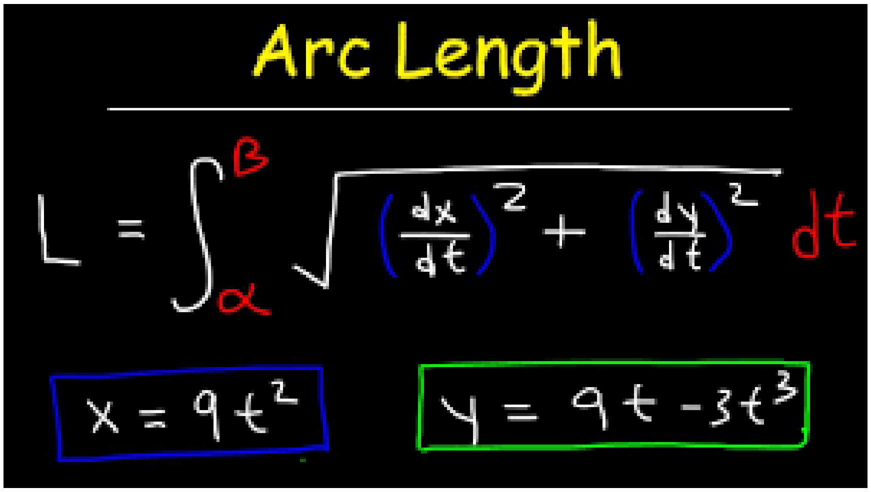
Answer A

Correct. The chain rule is used to find derivatives of parametric functions.

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{4e^{4t}+e^{t}}{2e^{2t}} = 2e^{2t} + \frac{1}{2}e^{-t}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{4e^{2t} - \frac{1}{2}e^{-t}}{2e^{2t}} = 2 - \frac{1}{4}e^{-3t}$$

2. ARCLENGTH FORMULA FOR PARAMETRIC EQUATIONS:



3.USING FUNDAMENTAL THEOREM OF CALCULUS IN PARAMETRIC EQUATION:

The instantaneous rate of change of the vector-valued function $f(t)$ is given by $r(t) = \langle e^{\sqrt{t}}, 10\cos(t + \sqrt{t}) \rangle$. If $f(2) = \langle 2e, \pi - 7 \rangle$, what is the value of $f(6)$?

A $\langle 11.582, -5.609 \rangle$

B $\langle 12.906, 0.163 \rangle$

C $\langle 30.170, 9.141 \rangle$

D $\langle 35.606, 5.283 \rangle$

Answer D

Correct. If the vector-valued function f has components $\langle x(t), y(t) \rangle$, then the instantaneous rate of change of f is $r(t) = \langle x'(t), y'(t) \rangle = \langle e^{\sqrt{t}}, 10\cos(t + \sqrt{t}) \rangle$. This is equivalent to the two initial value problems:

$\frac{dx}{dt} = e^{\sqrt{t}}$, $x(2) = 2e$

$\frac{dy}{dt} = 10\cos(t + \sqrt{t})$, $y(2) = \pi - 7$.

The Fundamental Theorem of Calculus is applied to the real-valued functions $x(t)$ and $y(t)$, and the evaluation of the definite integrals is done on the calculator, as follows.

$$x(6) = x(2) + \int_2^6 x'(t)dt = 2e + \int_2^6 e^{\sqrt{t}} dt = 35.606$$

$$y(6) = y(2) + \int_2^6 y'(t)dt = \pi - 7 + \int_2^6 \cos(t + \sqrt{t})\, dt = 5.283$$

Therefore, $f(6) = \langle x(6), y(6) \rangle = \langle 35.606, 5.283 \rangle$.

4. DERIVATIVE OF A VECTOR VALUED FUNCTION IS A VECTOR <x,y>

5. Polar Curves:

- Arclength

Arc Length

For a function, $f(x)$

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

For a polar graph, $r(\theta)$

$$L = \int_{\theta 1}^{\theta 2} \sqrt{[r(\theta)]^2 + [r'(\theta)]^2}\, d\theta$$

- Area inside FORMULA

Polar Curves

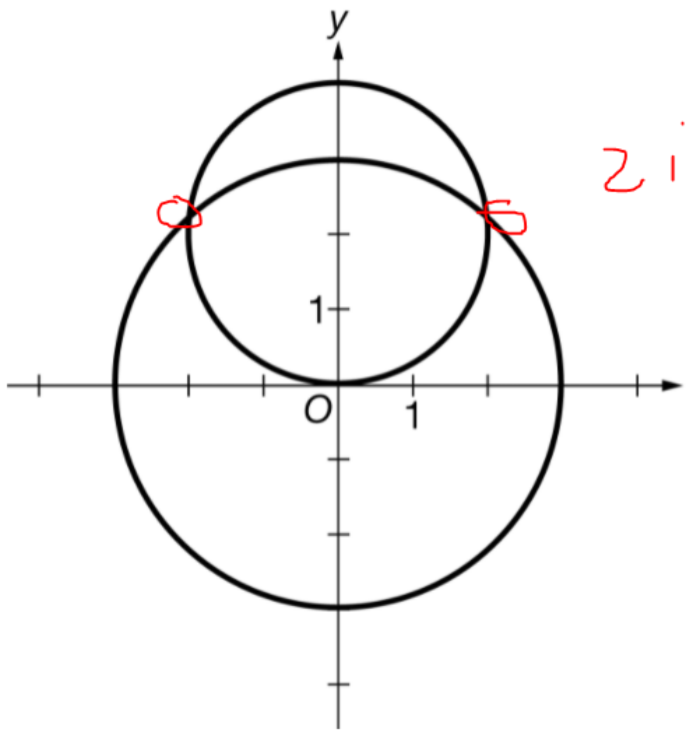
For a polar curve $r(\theta)$, the

Area inside a "leaf" is $\frac{1}{2}\int_{\theta 1}^{\theta 2} [r(\theta)]^2\, d\theta$

where $\theta 1$ and $\theta 2$ are the "first" two times that $r = 0$.

The slope of $r(\theta)$ at a given θ is

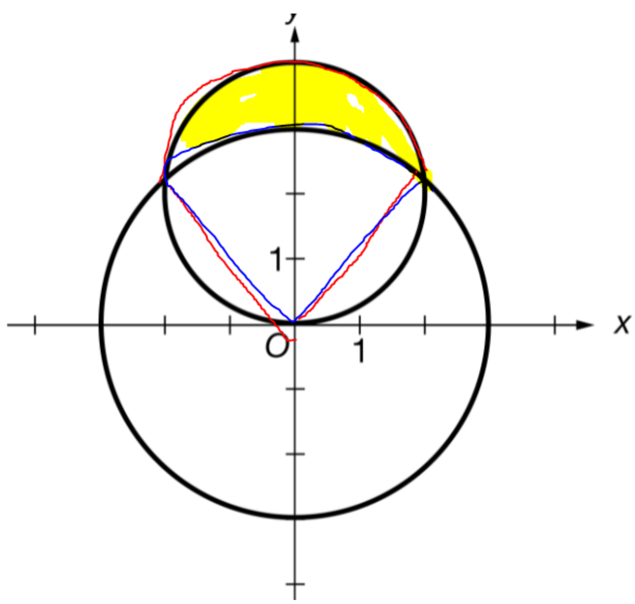
$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\frac{d}{d\theta}[r(\theta)\sin\theta]}{\frac{d}{d\theta}[r(\theta)\cos\theta]}$$



2 intersections x_1, x_2

$$\int_{x_1}^{x_2} a - b \, d\theta$$

The figure above shows the graphs of the polar curves $r = 3$ and $r = 4\sin\theta$. What is the area inside the circle $r = 4\sin\theta$ and outside the circle $r = 3$?



- CONVERTING RECTANGULAR AND POLAR COORDINATES TO FIND THE SLOPE OF TANGENT LINE

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$r = \sqrt{x^2 + y^2}$$

$$\tan\theta = \frac{y}{x}$$

Question 8

What is the slope of the line tangent to the polar curve $r = 4\theta^2$ at the point where $\theta = \frac{\pi}{4}$?

(A) $\frac{\pi}{2}$

(B) $\frac{\pi^2}{4}$

(C) $\frac{8+\pi}{8-\pi}$

(D) $\frac{\pi\sqrt{2}}{8}(8+\pi)$

$x = r\cos\theta = 4\theta^2\cos\theta$

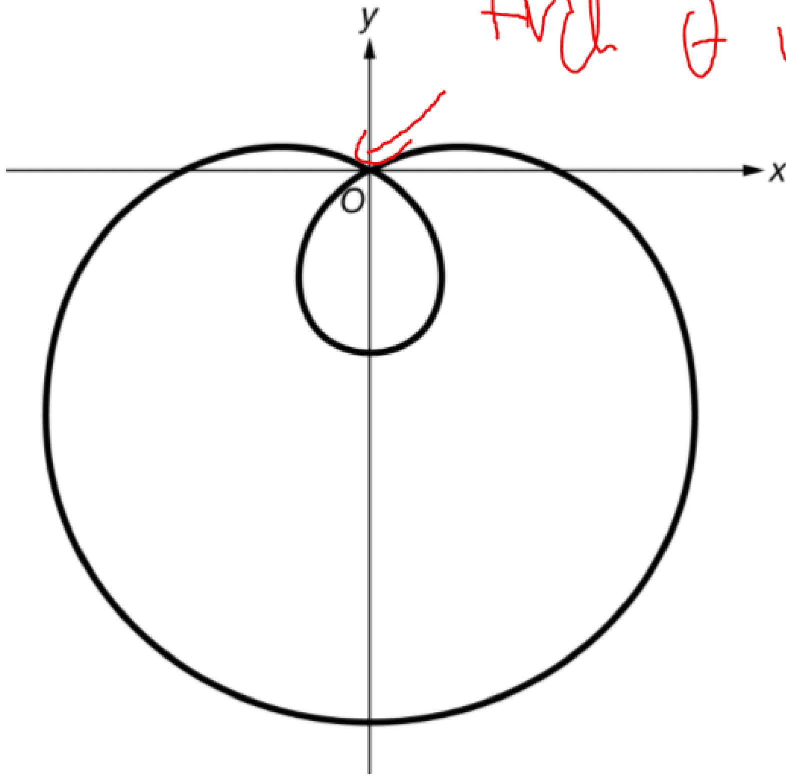
$y = r\sin\theta = 4\theta^2\sin\theta$

$\rightarrow$ Find $\frac{dy}{dx}$ (easy)

- Dealing with loops - Limacons

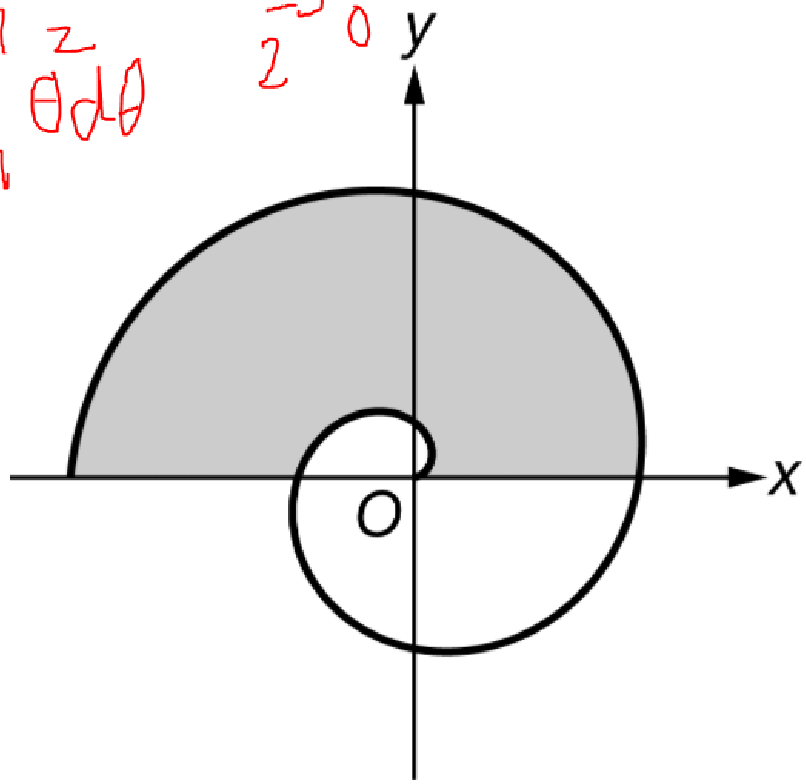
Question 9

find θ when $r=0$

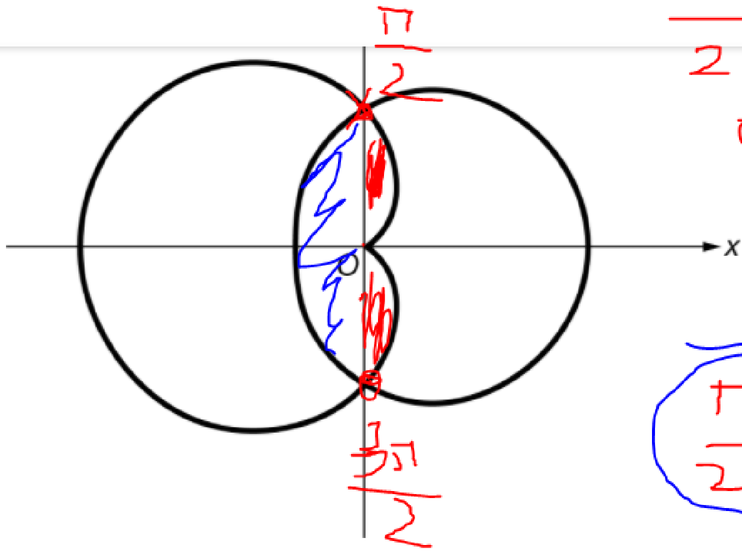


The graph of the polar curve $r = 2 - 4\sin\theta$ is shown in the figure above. Which of the following expressions gives the area of the inner loop of the polar curve?

$$\frac{1}{2}\int_{2\pi}^{3\pi} \theta^2 \, d\theta - \frac{1}{2}\int_{0}^{\pi} \theta^2 \, d\theta$$



The graph of the polar curve $r = \theta$ is shown above for $0 \le \theta \le 3\pi$. What is the area of the shaded region?



$\frac{-\pi}{2}$ to $\frac{\pi}{2}$ of left ball

+

$\frac{\pi}{2}$ to $\frac{3\pi}{2}$ of right ball

The graph above shows the polar curves $r = 2 + \cos\theta$ and $r = 2 - 2\cos\theta$. Which of the following expressions gives the area of the region that is inside both polar curves?

Unit 10: Infinite Sequences and Series

SUMMARY OF TESTS FOR SERIES

Test	Series	Condition(s) of Convergence	Condition(s) of Divergence
Integral (f is continuous, positive, and decreasing)	$\sum_{n=1}^{\infty} a_n$, $a_n = f(n) \ge 0$	$\int_1^{\infty} f(x)\,dx$ converges	$\int_1^{\infty} f(x)\,dx$ diverges
Root	$\sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} \sqrt[n]{\lvert a_n\rvert} < 1$	$\lim_{n\to\infty} \sqrt[n]{\lvert a_n\rvert} > 1$ or $= \infty$
Ratio	$\sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} \left\lvert \frac{a_{n+1}}{a_n} \right\rvert < 1$	$\lim_{n\to\infty} \left\lvert \frac{a_{n+1}}{a_n} \right\rvert > 1$ or $= \infty$
Direct Comparison ($a_n, b_n > 0$)	$\sum_{n=1}^{\infty} a_n$	$0 < a_n \le b_n$ and $\sum_{n=1}^{\infty} b_n$ converges	$0 < b_n \le a_n$ and $\sum_{n=1}^{\infty} b_n$ diverges
Limit Comparison ($a_n, b_n > 0$)	$\sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0$ and $\sum_{n=1}^{\infty} b_n$ converges	$\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0$ and $\sum_{n=1}^{\infty} b_n$ diverges

SUMMARY OF TESTS FOR SERIES

Test	Series	Condition(s) of Convergence	Condition(s) of Divergence
nth-Term	$\sum_{n=1}^{\infty} a_n$		$\lim_{n\to\infty} a_n \ne 0$
Geometric Series	$\sum_{n=0}^{\infty} ar^n$	$\lvert r\rvert < 1$	$\lvert r\rvert \ge 1$
Telescoping Series	$\sum_{n=1}^{\infty} (b_n - b_{n+1})$	$\lim_{n\to\infty} b_n = L$	
p-Series	$\sum_{n=1}^{\infty} \frac{1}{n^p}$	$p > 1$	$0 < p \le 1$
Alternating Series	$\sum_{n=1}^{\infty} (-1)^{n-1} a_n$	$0 < a_{n+1} \le a_n$ and $\lim_{n\to\infty} a_n = 0$	

Sums of Geometric Sequences and Series

Sum of finite geometric sequence:

$$S_n = \sum_{i=1}^{n} a_i r^{i-1} = a_1\left(\frac{1-r^n}{1-r}\right)$$

Sum of infinite geometric series:

$$S = \sum_{i=0}^{\infty} a_i r^i = \frac{a_1}{1-r}$$

[note: if |r|≥1, the infinite series does not have a sum]

Geometric Series Test's Absolute Value when considering Convergence

watch the absolute value

Question 1

What are all values of k for which the series $\sum_{n=0}^{\infty}\left((k^3+2)e^{-k}\right)^n$ converges?

only when $|r|<1$

A $k=-1.314$, $k=-1.193$, and $k=4.596$ only

B $k<-1.314$ and $-1.193<k<4.596$

C $-1.314<k<-1.193$ and $k>4.596$

D $k>4.596$ only

Geometric Sum Question:

Question 6

→ constant

If x and y are nonzero real numbers such that $|xy| < 1$, what is the sum of the infinite series $\sum_{n=1}^{\infty} 5x^2(xy)^{n-1}$?

sum of geometric series

- (A) $\frac{5}{1-xy}$
- (B) $\frac{5x}{y(1-xy)}$
- (C) $\frac{5x^2}{1-xy}$
- (D) $\frac{5x^3y}{1-xy}$

nth term divergence test

If $a_n = \sqrt[n]{2}$ for $n = 1, 2, 3, \ldots$, which of the following statements about $\sum_{n=1}^{\infty} a_n$ must be true?

- (A) The series converges and $\lim_{n\to\infty} a_n = 0$.
- (B) The series diverges and $\lim_{n\to\infty} a_n = 0$.
- (C) The series converges and $\lim_{n\to\infty} a_n \neq 0$.
- (D) The series diverges and $\lim_{n\to\infty} a_n \neq 0$.

$\lim \sqrt[n]{2}$

$= \lim_{n\to\infty} 2^{\frac{1}{n}}$ $\boxed{1}$

$= 2^0 = 1$

Integral test: To be usable it the term must be increasing, nonnegative , continuous.

Question 11

The integral test can be used to determine convergence or divergence for each of the following infinite series except

A $\sum_{n=1}^{\infty} \frac{2}{n^8}$

Consider ≥ 0, continuous, decreasing or not

B $\sum_{n=1}^{\infty} \frac{3}{4+6n}$

$\int e^{-n} \cdot n \rightarrow$ = integration by part

dv u

Question 12

The integral test can be used to conclude that which of the following statements about the infinite series $\sum_{n=1}^{\infty} \frac{n}{e^n}$ is true?

A The series converges, and the terms of the series have limit 0.

B The series diverges, and the terms of the series have limit 0.

C The series converges, and the terms of the series do not have limit 0.

D The series diverges, and the terms of the series do not have limit 0.

Direct comp

Question 3

Which of the following series can be used with the comparison test to determine whether the series $\sum_{n=1}^{\infty} \frac{5+2n}{n^2}$ converges or diverges?

$$\frac{5}{n^2} + \frac{2}{n} > \frac{2}{n}$$

diverge

(A) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

(B) $\sum_{n=1}^{\infty} \frac{2}{n}$

(C) $\sum_{n=1}^{\infty} \frac{5}{n}$

(D) $\sum_{n=1}^{\infty} \frac{5}{n^2}$

Which of the following statements must be true about the series $\sum_{n=0}^{\infty} a_n$ with positive terms if $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L$?

Ratio Test

(A) The series converges if $L = \frac{1}{2}$.

(B) The series converges if $L = 1$.

(C) The series converges if $L = 2$.

(D) The series converges if $L = \infty$.

If the partial sum with three terms is used to approximate the value of the convergent series $\sum_{n=3}^{\infty}(-1)^{n+1}\frac{n}{2^n}$, what is the alternating series error bound?

A $\frac{3}{32}$ ✓

B $\frac{5}{32}$

C $\frac{1}{4}$

D $\frac{3}{8}$

Answer A

Correct. Let $S=\sum_{n=3}^{\infty}(-1)^{n+1}\frac{n}{2^n}=\frac{3}{8}-\frac{4}{16}+\frac{5}{32}-\frac{6}{64}+\cdots$. This is a convergent alternating series for which $0\leq a_{n+1}\leq a_n$ for all n. The partial sum using three terms is $S_3=\frac{3}{8}-\frac{4}{16}+\frac{5}{32}$. The alternating series error bound using the first three terms of the series is the absolute value of the fourth term, the first omitted term of the series. Therefore, $|S-S_3|\leq\frac{6}{64}=\frac{3}{32}$.

Ratio test:

Question 11

The interval of convergence for the series $\sum_{n=1}^{\infty} \frac{(x+3)^{3n}}{n \cdot 8^n}$ is

(A) $-5 < x < -1$

(B) $-5 \le x < -1$

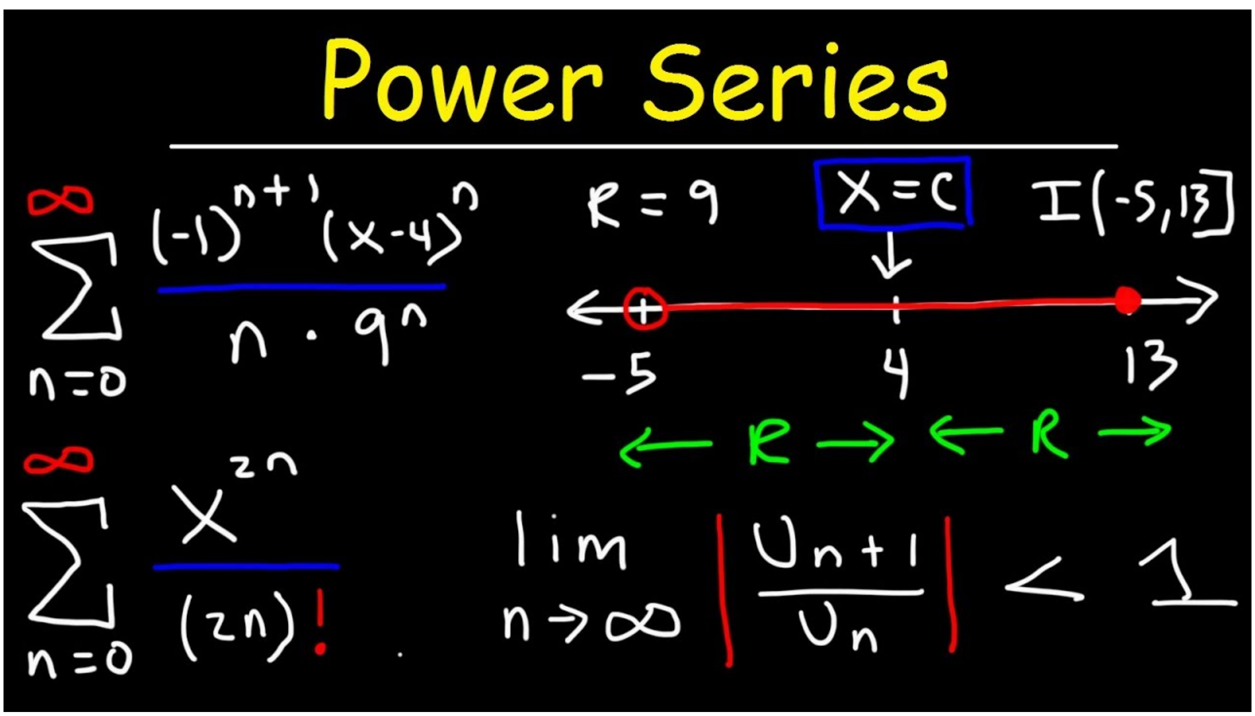
(C) $-11 < x < 5$

(D) $-11 \le x < 5$

check the endpoints

endpoint

RADIUS OF CONVERGENCE OF POWER SERIES (still have to do ratio test, it just tells you the center of convergence from the formula)



ALTERNATING SERIES ERROR BOUND

Error Bounds for Alternating Series

Let $S_n = \sum_{i=1}^{n} (-1)^{i+1} a_i$ be the n^{th} partial sum of an alternating series and let $S = \lim_{n \to \infty} S_n$. Suppose that $0 < a_{n+1} < a_n$ for all n and $\lim_{n \to \infty} a_n = 0$. Then $|S - S_n| < a_{n+1}$

This just says that you can estimate the error on the n^{th} partial sum by the $(n+1)^{\text{th}}$ term

TAYLOR/LAGRANGE ERROR BOUND

EQUATION

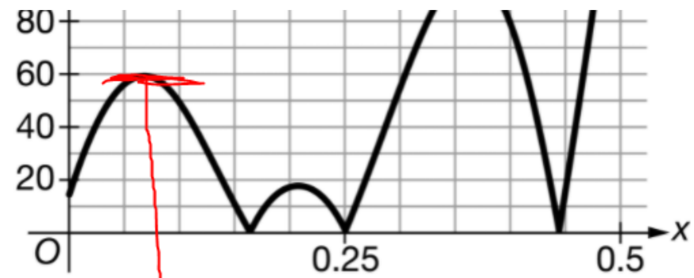
When a Taylor polynomial, $P_n(x)$, centered at $x = c$ is used to approximate a function, $f(x)$, at a value $x = a$ near the center, use the concept of a remainder as follows:

Then, Remainder = Exact Value (Function) – Polynomial Approximation

$$R_n(x) = f(x) - P_n(x)$$

$$R_n(x) = |f(x) - P_n(x)| = \left| \frac{f^{(n+1)}(z)}{(n+1!)} (x-c)^{n+1} \right|$$

11/28/2018 11:06 PM 9.7: Lagrange's Error Bound 4



Graph of $y = |f'''(x)|$

The first four terms of the Maclaurin series for the function f are $1 - 0.26x + 2.7x^2 - 2x^3$. The Maclaurin series converges to f for all values of x. The graph of $y = |f'''(x)|$, the absolute value of the third derivative of f, is shown above for $0 \le x \le 0.5$. Let $P_2(x)$ be the second-degree Taylor polynomial for f about $x = 0$. Of the following, which is the smallest value of k for which the Lagrange error bound guarantees that $\left|f\left(\frac{1}{4}\right) - P_2\left(\frac{1}{4}\right)\right| \le k$?

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A $\frac{2}{64}$

B $\frac{10}{64}$

worst case $\rightarrow \frac{60}{3!}\left(\frac{1}{4}\right)^3$

—> the maximum value for the derivative at x=z is NOT ALWAYS AT THE ENDS OF THE INTERVAL, MUST LOOK AT THE GRAPH IF GIVEN.

TAYLOR SERIES AND POLYNOMIAL

$$\begin{aligned} f(x) &= f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 \\ &\quad + \frac{f'''(x_0)}{3!}(x - x_0)^3 + \frac{f''''(x_0)}{4!}(x - x_0)^4 + \cdots \\ &= \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n. \end{aligned}$$

Important Taylor Series

Commonly Used Taylor Series

	SERIES	WHEN IS VALID/TRUE
$\frac{1}{1-x}$	$= 1 + x + x^2 + x^3 + x^4 + \dots$	NOTE THIS IS THE GEOMETRIC SERIES. JUST THINK OF x AS r
	$= \sum_{n=0}^{\infty} x^n$	$x \in (-1, 1)$
e^x	$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$	SO: $e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots$ $e^{(17x)} = \sum_{n=0}^{\infty} \frac{(17x)^n}{n!} = \sum_{n=0}^{\infty} \frac{17^n x^n}{n!}$
	$= \sum_{n=0}^{\infty} \frac{x^n}{n!}$	$x \in \mathbb{R}$
$\cos x$	$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \dots$	NOTE $y = \cos x$ IS AN <u>EVEN</u> FUNCTION (I.E., $\cos(-x) = +\cos(x)$) AND THE TAYLOR SERIS OF $y = \cos x$ HAS ONLY <u>EVEN</u> POWERS.
	$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$	$x \in \mathbb{R}$
$\sin x$	$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \dots$	NOTE $y = \sin x$ IS AN <u>ODD</u> FUNCTION (I.E., $\sin(-x) = -\sin(x)$) AND THE TAYLOR SERIS OF $y = \sin x$ HAS ONLY <u>ODD</u> POWERS.
	$= \sum_{n=1}^{\infty} (-1)^{(n-1)} \frac{x^{2n-1}}{(2n-1)!} \overset{\text{or}}{=} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$	$x \in \mathbb{R}$
$\ln(1+x)$	$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \dots$	QUESTION: IS $y = \ln(1+x)$ EVEN, ODD, OR NEITHER?
	$= \sum_{n=1}^{\infty} (-1)^{(n-1)} \frac{x^n}{n} \overset{\text{or}}{=} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$	$x \in (-1, 1]$
$\tan^{-1} x$	$= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \dots$	QUESTION: IS $y = \arctan(x)$ EVEN, ODD, OR NEITHER?
	$= \sum_{n=1}^{\infty} (-1)^{(n-1)} \frac{x^{2n-1}}{2n-1} \overset{\text{or}}{=} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$	$x \in [-1, 1]$

INTERVAL OF CONVERGENCE

Interior of interval of convergence must be absolutely convergent, if a point is conditionally convergent, it is one of the endpoints of the interval of convergence.

x	-2	-1	0	1	2	3
$\sum_{n=0}^{\infty} a_n(x-b)^n$	diverges	converges conditionally	converges absolutely	converges absolutely	converges absolutely	diverges

Consider the power series $\sum_{n=0}^{\infty} a_n(x-b)^n$, where b is an integer. The convergence or divergence of the series at various values of x is shown in the table above. What is the interval of convergence for the power series?

A $(-2, 2]$ ✗

B $(-2, 3)$

C $[-1, 2]$

D $[-1, 3)$ ✓

Answer A

Incorrect. Since the series is conditionally convergent when $x = -1$, one endpoint of the interval of convergence must be -1.

?

Answer D

Correct. A series is absolutely convergent on the interior of the interval of convergence. Since the series converges conditionally when $x = -1$, one endpoint of the interval of convergence must be -1. Since the series converges absolutely when $x = 2$ and diverges when $x = 3$, the other endpoint must be a number from 2 to 3. Since b, the center of the interval of convergence, is an integer, the right endpoint must be 3. Therefore, the interval of convergence is $[-1, 3)$.