Linear Description for Stable Matching Polytope: Alternative Proof

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Abstract

In this paper we provide an alternative proof for the description.

Keywords: Stable Matching, Polytope, Extreme Points

TODO

1. itroduction
2. formulate the idea of the proofs by Rothblum and van Vate
3. citation for structure of symmetric difference, adjacencies in the stable matching polytope
4. pictures

1. Stable Matching Polytope

Let $G = (V, E)$ be a bipartite graph where the bipartition is given by node sets $\mathcal{A}$ and $\mathcal{B}$. Moreover, each node $a \in \mathcal{A}$ defines a total ordering $>_a$ on its neighbour nodes $N(a) := \{b \in \mathcal{B} : \{a, b\} \in E\}$; the same holds for nodes in $\mathcal{B}$.

Let us introduce the following notation. For $a \in \mathcal{A}$ and $b \in N(\mathcal{A})$, we can define $\delta^{>b}(a) := \{\{a, \beta\} \in E : \beta >_a b\}$. Similarly we can define $\delta^{>a}(b)$, $\delta^{\leq b}(a)$, $\delta^{\geq b}(a)$, etc. For $a \in \mathcal{A}$ let $N_{\max}(a)$ denote the maximum element in $N(a)$ with respect to $>_a$, analogously define $N_{\min}(a)$ and $N_{\max}(b)$, $N_{\min}(b)$ for $b \in \mathcal{B}$.

Given a matching M and $v \in V$, we define $M(v)$ as a neighbour of v in M if v is matched by M, otherwise $M(v) := \varnothing$. To simplify notation we understand $>_v$, $v \in V$ as a total ordering on the set $N(v) \cup \{\varnothing\}$, where $\varnothing$ is the smallest element with respect to $>_v$. A matching M in G is called a *stable matching* if for every edge $e = \{a, b\} \in E$ we have

$$M \cap \left(\delta^{>a}(b) \cup \delta^{>b}(a) \cup \{e\}\right) \neq \varnothing\,, \tag{1}$$

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i.e. no edge $\{a,b\} \in E$ is absent in M if $a >_b M(b)$ and $b >_a M(a)$.

Next lemmas study connected components of $M_1 \triangle M_2 := \{e \in E : |M_1 \cap \{e\}| + |M_2 \cap \{e\}| = 1\}$ for stable matchings M_1, M_2 in G. Here, $V(C)$ and $E(C)$ denote the node and edge sets of a graph C.

Lemma 1.1 *Let M_1 and M_2 be two stable matchings in G and let C be a connected component in $M_1 \triangle M_2$. Then, either*

$$M_1(a) >_a M_2(a)\,,\ \forall a \in V(C) \cap \mathcal{A} \quad \textit{and} \quad M_1(b) <_b M_2(b)\,,\ \forall b \in V(C) \cap \mathcal{B} \quad (2)$$

or

$$M_2(a) >_a M_1(a)\,,\ \forall a \in V(C) \cap \mathcal{A} \quad \textit{and} \quad M_2(b) <_b M_1(b)\,,\ \forall b \in V(C) \cap \mathcal{B} \quad (3)$$

Proof: Since M_1 and M_2 are matchings, C is a path or a cycle. Let $v \in V$ be an end node of C if C is a path, otherwise let v be an arbitrary node of C.

W.l.o.g. $v := a \in \mathcal{A}$, $M_1(a) >_a M_2(a)$ and $b := M_1(a) \in \mathcal{B}$. If $a = M_1(b) >_b M_2(b)$, the matching M_2 violates (1) for the edge $\{a,b\} \in E$. Thus, $a = M_1(b) <_b M_2(b)$. Thus, $M_2(b) \neq \varnothing$ and the matching M_1 satisfies (1) for the edge $e := \{b, M_2(b)\} \in E$ only if $M_1(M_2(b)) >_{M_2(b)} b$. Continuing in this way, we obtain statement (2). ■

Lemma 1.2 *Let M_1 and M_2 be two stable matchings in G. Let J_1 be the union of $E(C)$ for connected components C in $M_1 \triangle M_2$ satisfying* (2)*, let $J_2 := (M_1 \triangle M_2) \setminus C$. Then both $M_1 \triangle J_1$ and $M_1 \triangle J_2$ are stable matchings in G.*

Proof: By Lemma 1.1, J_2 is the union of $E(C)$ for connected components C in $M_1 \triangle M_2$ satisfying (3).

Clearly $M_1' := M_1 \triangle J_1$ and $M_2' := M_1 \triangle J_2$ are matchings in G. Let us assume that one of these matchings os not stable, w.l.o.g. assume that M_1' does not satisfy (1) for some edge $e := \{a,b\} \in E$ with $a \in \mathcal{A}$ and $b \in \mathcal{B}$.

Since M_1 and M_2 are stable, we have $|\{a,b\} \cap V(J_1)| = 1$ and $|\{a,b\} \cap V(J_2)| = 1$. If $a \in V(J_1)$ and $b \in V(J_2)$, then $M_1'(a) = M_2(a)$ and $M_1'(b) >_b M_2(b)$, and hence M_2 violates (1) for $e := \{a,b\}$. If $a \in V(J_2)$ and $b \in V(J_1)$, then $M_1'(a) = M_1(a)$ and $M_1'(b) >_b M_1(b)$, and hence M_1 violates (1) for $e := \{a,b\}$, contradiction. ■

2. Stable Matching Polytope

Let us define the *stable matching polytope* $P(G) \subseteq \mathbb{R}^E$ for graph G as follows

$$P(G) := \operatorname{conv}\{\chi(M) \in \mathbb{R}^E : M \text{ is a stable matching in } G\}\,.$$

By [], $P(G)$ is a nonempty polytope because every graph G has a stable matching.

Clearly, the vertices of $P(G)$ are in one to one correspondence with stable matchings in G. Moreover, Lemma 1.2 helps to understand what pairs of stable matchings in G do not correspond to edges of $P(G)$.

Lemma 2.1 *Let M_1 and M_2 be two stable matchings in G which define an edge of the polytope $P(G)$. Then all connected components in $M_1 \triangle M_2$ satisfy* (2) *or all of them satisfy* (3).

Proof: If the statement of the lemma does not hold, then in Lemma 1.2 the sets J_1 and J_2 are both nonempty, and hence we obtain two stable matchings $M_1 \triangle J_1$ and $M_1 \triangle J_2$ different from M_1, M_2 such that $\frac{1}{2}\chi(M_1 \triangle J_1) + \frac{1}{2}\chi(M_1 \triangle J_2) = \frac{1}{2}\chi(M_1) + \frac{1}{2}\chi(M_2)$, what condradicts the statement that M_1 and M_2 define an edge of $P(G)$. ■

Corollary 2.2 *Let M_1 and M_2 be two stable matchings in G such that*

$$M_1 \cap \delta^{>a}(b) \neq \varnothing, M_1 \cap \delta^{>b}(a) \neq \varnothing \quad \textit{and} \quad M_2 \cap \left(\delta^{>a}(b) \cup \delta^{>b}(a)\right) = \varnothing$$

for some $a \in \mathcal{A}$ and $b \in \mathcal{B}$. Then, M_1 and M_2 do not define an edge of the polytope $P(G)$.

3. Linear Description

Let us define $Q(G) \subseteq \mathbb{R}^E$ be the polytope described by the following linear constraints

$$x(\delta(v)) \leq 1 \quad \forall v \in V \qquad \text{and} \qquad x_e \geq 0 \quad \forall e \in E\,, \tag{4}$$

$$x(\delta^{>a}(b)) + x(\delta^{>b}(a)) + x_{\{a,b\}} \geq 1 \quad \forall e := \{a,b\} \in E \tag{5}$$

where $x(J) = \sum_{e \in J} x_e$ for any $J \subseteq E$.

Clearly, $P(G) \subseteq Q(G)$ because for every stable matching M in G the point $x := \chi(M)$ satisfies (4) and by (1) the point x also satisfies (5). On the other hand, every integral point in $Q(G)$ equals $\chi(M)$ for some stable matching M in G. In the remaining part of the paper we show that every vertex of $Q(G)$ is integral, thus proving the main theorem.

Theorem 3.1 *For every graph G the polytope $P(G)$ equals $Q(G)$.*

Lemma 3.2 *For every graph G every vertex of the polytope $Q(G)$ is integral.*

Proof: First, let show that every vertex x of $Q(G)$ has a coordinate equal to 0 or a coordinate equal to 1. Let us assume the contrary. As every vertex, $x \in \mathbb{R}^E$ is defined as the unique point, which tightly satisfies some $|E|$ constraints describing $Q(G)$. Since x has no zero coordinate, we can assume that the tight constraints are $x(\delta(v)) \leq 1$ for $v \in V_x$ and (5) for $e \in E_x$, where $|V_x|+|E_x| = |E|$. Moreover, let us assume that we choose the $|E|$ tight constraints so that $|V_x|$ is as large as possible.

The constraints $x(\delta(v)) = 1$, $v \in V$ are linearly dependent, in particular $\sum_{a \in \mathcal{A}} \chi(\delta(a)) = \sum_{b \in \mathcal{B}} \chi(\delta(b))$. Hence, we have $V_x \subsetneq V$. On the other hand if

$a = N_{\max}(b)$ or $a = N_{\min}(b)$ then $e := \{a,b\} \notin E_x$. Indeed, $a = N_{\max}(b)$ implies $\delta^{>a}(b) = \varnothing$ then

$$1 \leq x(\delta^{>a}(b)) + x(\delta^{>b}(a)) + x_{\{a,b\}} = x(\delta^{\geq b}(a)) \leq x(\delta(a)) \leq 1\,,$$

showing that $\delta^{<b}(a) = \varnothing$ and hence $E_x \setminus e$, $V_x \cup \{a\}$ also define the vertex x. The case $a = N_{\min}(b)$ is analogous. Moreover, notice that $N_{\min}(v) \neq N_{\max}(v)$ for $v \in V_x$ since no coordinate of x equals 1.Thus,

$$|E_x| = \frac{1}{2}\sum_{v\in V} |\delta(v) \cap E_x| \leq \frac{1}{2}\sum_{v \in V_x} (|\delta(v)| - 2) + \frac{1}{2}\sum_{v\in V\setminus V_x} (|\delta(v)| - 2) = |E| - |V_x| - \frac{1}{2}|V \setminus V_x|\,,$$

what implies $|E_x| + |V_x| < |E|$, contradiction.

Now let us assume that G is a graph with the minimum number of edges such that $Q(G)$ is not an integral polytope. Let x be a nonintegral vertex of $Q(G)$.

Case $x_{\{a,b\}} = 0$ for some $a \in \mathcal{A}$, $b \in \mathcal{B}$ and $e := \{a,b\} \in E$. In this case, let x' be obtained from x by dropping the coordinate corresponding to $\{a,b\}$, and let G' be the graph with $V(G') = V$ and $E(G') \setminus \{e\}$. Define H' to be the hyperplane $\{x \in \mathbb{R}^{E(G')} : x(\delta^{>a}(b)) + x(\delta^{>b}(a)) = 1\}$. Then, x' is a vertex of the polytope $P' := P(G') \cap H'$. But every vertex of P' is either a vertex of $P(G')$ or an intersection of an edge of $P(G')$ with the hyperplane H'. Since by Corollary 2.2 there is no edge of $P(G')$ intersecting the hyperplane H', x' is a vertex of $P(G')$. By minimality of G, both x' and x are integral, contradiction.

Case $x_{\{a,b\}} = 1$ for some $a \in \mathcal{A}$, $b \in \mathcal{B}$. Let x' be obtained by dropping the coordinates corresponding to $\delta(a) \cup \delta(b)$, and let G' be the graph with $V(G') = V \setminus \{a,b\}$ and $E(G') \setminus \big(\delta(a) \cup \delta(b)\big)$. It is straightforward to see that x' is a vertex of $P(G')$. Thus by minimality of G, both x' and x are integral, contradiction. ■

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3.1. Proof of Integrality

Theorem 3.3 *$Q(G)$ is integral.*

Proof: Note that it is clear the integral points of $Q(G)$ describe stable matchings of G[**?**]. The argument proceeds by induction on $|E|$. Notice in the base case $|E| = 1$, it is immediate that $Q(G) = \{1\}$.

. Consider the inductive case. Let x be an extreme point of $Q(G)$. By Lemma **??** exists $e = vw \in E(G)$ such that $x_e = 0$ or $x_e = 1$.

. Consider the case $x_e = 0$. If x is fractional then the residual vector, z, acquired by restricting x to $\mathbb{R}^{E\setminus\{e\}}$, is fractional. Notice that $z \in Q(G-e)$. Further it is not difficult to show that x satisfies:

$$x(\delta^{>w}(v)) + x(\delta^{>v}(w)) + x_{vw} = 1$$

and so z lies on an edge of $Q(G-e)$ (the hyperplane above intersects this edge) . Let M_1 and M_2 be the stable matchings given by the extreme points of $Q(G-e)$ which describe the edge that z lies on.

. Without loss of generality we may assume

$$|M_1 \cap (\delta^{>w}(v) \cup \delta^{>v}(w))| = 2$$

and

$$|M_2 \cap (\{vw\} \cup \delta^{>w}(v) \cup \delta^{>v}(w))| = 0.$$

To see this notice that if $|M_1 \cap (\delta^{>w}(v) \cup \delta^{>v}(w))| = 1$ then M_1 would satisfy

$$x(\delta^{>w}(v)) + x(\delta^{>v}(w)) + x_{vw} = 1$$

and z would be the incidence vector of M_1, contradicting that z is fractional. Similar for M_2.

. Define J_1 and J_2 as in Lemma **??**. Since $|M_1 \cap (\delta^{>w}(v) \cup \delta^{>v}(w))| = 2$, neither J_1 nor J_2 are empty. Therefore $M_1 \triangle J_1$ and $M_2 \triangle J_2$ are distinct matchings from M_1 and M_2.

. Further it can be seen that

$$\begin{aligned}
&\chi(M_1 \triangle J_1) + \chi(M_1 \triangle J_2) \\
&= (\chi(M_1) + \chi(J_1) - 2\chi(M_1 \cap J_1)) + (\chi(M_1) + \chi(J_2) - 2\chi(M_1 \cap J_2)) \\
&= 2\chi(M_1) + \chi(J_1) + \chi(J_2) - 2(\chi(M_1 \cap J_1) + \chi(M_2 \cap J_2)) \\
&= 2\chi(M_1) + \chi(M_1 \triangle M_2) - 2(\chi(M_1) + \chi(M_1 \cap M_2)) \\
&= \chi(M_1 \triangle M_2) + 2\chi(M_1 \cap M_2) \\
&= \chi(M_1) + \chi(M_2).
\end{aligned}$$

. By Lemma **??**, $M_1 \triangle J_1$ and $M_1 \triangle J_2$ are stable. This contradicts that M_1 and M_2 are described by extreme points defining at edge of $Q(G-e)$.

. In the case $x_e = 1$, if there exists $e' \in \delta(v) \cup \delta(w) \setminus \{e\}$ then $x_{e'} = 0$ and so we may obtain a contradiction by removing e' as we did in the $x_e = 0$ case. Otherwise let z be the vector acquired by restricting x to $\mathbb{R}^{E\setminus\{e\}}$. Notice $z \in Q(G-v-w)$. If z is not a vertex then there exists $z^1, z^2 \in Q(G-v-w)$ such that $z = \frac{1}{2}(z^1 + z^2)$. Now z^1, z^2 can be extended to vectors $x^1, x^2 \in \mathbb{R}^E$ by setting $x^1_e = x^2_e = 1$. Then $x^1, x^2 \in Q(G)$ and $x = \frac{1}{2}(x^1 + x^2)$, contradicting that x is a vertex of $Q(G)$. Therefore z is a vertex of $Q(G-v-w)$, thus integral, and so x is integral. $\blacksquare$ ¿¿¿¿¿¿¿ 43ddd9458e319e07569dbe1de87ab68dbcdcaed8